REGULARITY IN TIME FOR WEAK SOLUTIONS OF A CONTINUUM MODEL FOR EPITAXIAL GROWTH WITH ELASTICITY ON VICINAL SURFACES

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ABSTRACT. The evolution equation derived by Xiang (SIAM J. Appl. Math. 63:241–258, 2002) to describe vicinal surfaces in heteroepitaxial growth is

$$h_t = -\left[H(h_x) + \left(h_x^{-1} + h_x\right) h_{xx}\right]_{xx}, \tag{1}$$

where h denotes the surface height of the film, and H is the Hilbert transform. Existence of solutions was obtained by Dal Maso, Fonseca and Leoni (Arch. Rational Mech. Anal. 212: 1037–1064, 2014). The regularity in time was left unresolved. The aim of this paper is to prove existence, uniqueness, and Lipschitz regularity in time for weak solutions, under suitable assumptions on the initial datum.

Keywords: epitaxial growth, vicinal surfaces, evolution equations, Hilbert transform, monotone operators

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1. INTRODUCTION

Within the context of heteroepitaxial growth of a film onto a substrate, terraces and steps self-organize to accommodate misfit elasticity forces. Discrete models have been proposed by Duport, Politi and Villain [4], and Tersoff, Phang, Zhang and Lagally [7]. A continuum variant of these models has been derived by Xiang [8]. Also related are the works by Xiang and E [9], and Xu and Xiang [10]. The evolution equation derived by Xiang [8, Formula (3.62)] is (upon space inversion)

$$h_t = -\left[H(h_x) + \left(\frac{1}{h_x} + h_x\right) h_{xx}\right]_{xx}, \tag{2}$$

where h describes the height of the surface of the film, and is assumed to be monotone. The time domain is $[0,T]$ with $T > 0$ a given datum, the space domain is $I := (-\pi, \pi)$, H denotes the Hilbert transform, i.e.,

$$H(f)(x) := \frac{1}{2\pi} PV \int_I \frac{f(x-y)}{\tan(y/2)}\, \mathrm{d}y,$$

with PV denoting the Cauchy principal value. Analytical validation for the continuum model from [8] has been obtained by Dal Maso, Fonseca and Leoni in [3], where the authors transform (2) into a parabolic evolution equation

$$u_t = -\left[H(u_x) + \Phi_a'(u_{xx})\right]_{xx}, \tag{3}$$

$$\Phi_a(\xi) := \Phi(\xi + a), \qquad \Phi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}, \quad \Phi(\xi) := \begin{cases} +\infty & \text{if } \xi < 0, \\ 0 & \text{if } \xi = 0, \\ \xi \log \xi + \xi^3/6 & \text{if } \xi > 0. \end{cases}$$

Here $a > 0$ is a constant, and u is a suitable antiderivative of h. The main results in [3] is the proof of the existence of weak solutions for (3) in the sense that:

(1) ([3, Theorem 1]) for any $T, a > 0$, $u^0 \in L^2_{\mathrm{per}_0}(I)$, there exists $u \in L^3(0,T; W^{2,3}_{\mathrm{per}_0}(I))$ such that

$$\int_0^T \int_I \big[w_t(t)(w(t) - u(t)) - H(u_{xx}(t))(w_x(t) - u_x(t)) + \Phi_a(w_{xx}(t)) - \Phi_a(u_{xx}(t))\big]\, \mathrm{d}x\, \mathrm{d}t \geq 0$$

for any test function $w \in L^3(0,T;W^{2,3}_{\mathrm{per}_0}(I))$ such that $w_t \in L^{3/2}(0,T;(W^{2,3}_{\mathrm{per}_0}(I))')$ and $w(0)=u^0$. Moreover, $\log(u_{xx}+a) \in L^1(0,T;L^1(I))$;

(2) ([3, Theorem 2]) assuming, in addition, that test functions w satisfy $\log(w_{xx}+a) \in L^{3/2}(0,T;L^{3/2}(I))$, it holds

$$\int_0^T \int_I \big[w_t(t)(w(t)-u(t)) - H(u_{xx}(t))(w_x(t)-u_x(t)) \\ + \Phi'_a(w_{xx}(t))(w_{xx}(t)-u_{xx}(t))\big]\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Here

$$W^{2,3}_{\mathrm{per}_0}(I) := \left\{ f \in W^{2,3}_{\mathrm{loc}}(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic and } \int_I f\,\mathrm{d}x = 0 \right\},$$

$$L^2_{\mathrm{per}_0}(I) := \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic and } \int_I f\,\mathrm{d}x = 0 \right\}.$$

Note in that both results, the regularity in time is assumed on the test function w. Concerning the regularity in time of u, it was only proved ([3, Remark 3]) that u has finite essential pointwise variation when considered as function $u : [0,T] \longrightarrow (W^{2,\infty}_{\mathrm{per}_0}(I))'$, where

$$W^{2,\infty}_{\mathrm{per}_0}(I) := \left\{ f \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic and } \int_I f\,\mathrm{d}x = 0 \right\}.$$

The main result of this paper is:

Theorem 1. *Given $T, a > 0$, and $u^0 \in W^{2,2}_{\mathrm{per}_0}(I)$ such that*

$$\int_I z^0 v\,\mathrm{d}x - \int_I H(u^0_{xx})v_x\,\mathrm{d}x + \int_I [\Phi_a(v_{xx}) - \Phi_a(u^0_{xx})]\,\mathrm{d}x \ge 0 \tag{4}$$

for some $z^0 \in L^2_{\mathrm{per}_0}(I)$ and any $v \in W^{2,3}_{\mathrm{per}_0}(I)$, then there exists a solution $u : [0,T] \longrightarrow W^{2,3}_{\mathrm{per}_0}(I)$ of (3) *in the sense that*

$$\int_0^T \int_I u_t(t)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_I \big[H(u_{xx}(t))\varphi_x(t,x) - \Phi'_a(u_{xx}(t))\varphi_{xx}(t,x)\big]\,\mathrm{d}x\,\mathrm{d}t \tag{5}$$

for any $\varphi \in C^\infty_c((0,T)\times I;\mathbb{R})$. Moreover,

$$u \in L^\infty(0,T;W^{2,3}_{\mathrm{per}_0}(I)) \cap C^0([0,T];L^2_{\mathrm{per}_0}(I)), \qquad u_t \in L^\infty(0,T;L^2_{\mathrm{per}_0}(I)), \qquad u(0)=u^0.$$

The main argument is to first prove that the variational inequality (16) below admits a solution u, and then show that such u is also solution of (3) in the sense of Theorem 1. We remark that there is a large class of initial data u^0 satisfying (4). Assume that $u^0_{xx}+a>0$ a.e., and that $\Phi'_a(u^0_{xx})$, $\Phi_a(u^0_{xx}) \in L^1(I)$. Then the convexity of Φ gives

$$\int_I [\Phi_a(v_{xx}) - \Phi_a(u^0_{xx})]\,\mathrm{d}x \ge \int_I (v_{xx}-u^0_{xx})\Phi'_a(u^0_{xx})\,\mathrm{d}x,$$

thus a sufficient condition for (4) is that, for some $z^0 \in L^2_{\mathrm{per}_0}(I)$ and any $v \in W^{2,3}_{\mathrm{per}_0}(I)$,

$$\int_I z^0 v\,\mathrm{d}x - \int_I H(u^0_{xx})v_x\,\mathrm{d}x + \int_I v_{xx}\Phi'_a(u^0_{xx})\,\mathrm{d}x - \int_I u^0_{xx}\Phi'_a(u^0_{xx})\,\mathrm{d}x \ge 0.$$

In particular, the previous inequality holds if

$$\int_I u^0_{xx}\Phi'_a(u^0_{xx})\,\mathrm{d}x \le 0, \qquad z^0 := [-H(u^0_x) - \Phi'_a(u^0_{xx})]_{xx}.$$

Observe that if $u \in C^4(I)$, with derivatives bounded away from 0, and extended by periodicity, then such a z^0 is well defined.

To ensure that $\int_I u^0_{xx}\Phi'_a(u^0_{xx})\,\mathrm{d}x \le 0$, the following are sufficient conditions:

(1) if $\Phi'_a(0) \ge 0$, then due to the monotonicity of Φ'_a, there exists a unique $b_0 \le 0$ such that $\Phi'_a(b_0)=0$. Thus any u^0 with $b_0 \le u^0_{xx} \le 0$ is acceptable;

(2) similarly, if $\Phi_a'(0) \le 0$, then there exists a unique $b_1 \ge 0$ such that $\Phi_a'(b_1) = 0$. Thus any u^0 with $0 \le u^0_{xx} \le b_1$ is acceptable.

2. Proof of Theorem 1

Let $T > 0$ be given, and let $I := (-\pi, \pi)$ be the space domain. Let

$$V := W^{2,3}_{\mathrm{per}_0}(I), \quad U := L^2_{\mathrm{per}_0}(I), \quad \mathcal{V} := L^2(0,T;V), \quad \mathcal{U} := L^2(0,T;U). \tag{6}$$

Note that U is an Hilbert space, V is a reflexive Banach space, and the embedding $V \hookrightarrow U$ is compact. Duality yields the pivot space structures

$$V \hookrightarrow U \hookrightarrow V', \qquad \mathcal{V} \hookrightarrow \mathcal{U} \hookrightarrow \mathcal{V}'. \tag{7}$$

For future reference, $\langle , \rangle$ (resp. $\langle , \rangle_{V',V}$) will denote the duality pairing between $L^2(I)$ and $L^2(I)$ (resp. V' and V).

Definition 2. *An operator $A : V \longrightarrow V'$ is:*

(1) **monotone** *if for any $u, v \in V$, it holds*

$$\langle Au - Av, u - v\rangle_{V',V} \ge 0.$$

*Similarly, a set $G \subseteq V \times V'$ is "*monotone*" if for any pair (u, u'), $(v, v') \in G$, it holds*

$$\langle u' - v', u - v\rangle_{V',V} \ge 0.$$

(2) **maximal monotone** *if the graph*

$$\Gamma_A := \{(u, Au) : u \in V\} \subseteq V \times V'$$

is not a proper subset of any monotone set.

(3) **pseudo-monotone** *if it is bounded, and*

$$\langle Au, u - v\rangle_{V',V} \le \liminf_n \langle Au^n, u^n - v\rangle_{V',V}$$

for every $v \in V$, $u^n, u \in V$, satisfying $u^n \rightharpoonup u$ and $\limsup_n \langle Au^n, u^n - u\rangle_{V',V} \le 0$.

(4) **hemi-continuous** *if for any $u, v \in V$ the mapping $t \longmapsto \langle A(u + tv), v\rangle_{V',V}$ is continuous.*

Remark 3. *If an operator $A : V \to V'$ is monotone and hemi-continuous, then it is maximal monotone (see* [1, Theorem 1.2]*).*

We will use the following result (see Kačur [6]).

Theorem 4. *Let V, U, $\mathcal{V}$, $\mathcal{U}$ be as defined in* (6)*. Let $A : V \longrightarrow V'$ be a maximal monotone operator, let $\phi : V \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function such that $D(\phi) := \{v \in V : \phi(v) < +\infty\} \ne \emptyset$. Let $u^0 \in U$, and suppose there exist:*

- *$v^0 \in D(\phi)$ such that*

$$\lim_{\|v\|_V \to +\infty} \frac{\langle Av, v - v^0\rangle_{V',V} + \phi(v)}{\|v\|_V} = +\infty, \tag{8}$$

- *$z^0 \in U$ such that for any $v \in V$*

$$\langle z^0, v\rangle + \langle Au^0, v\rangle_{V',V} + \phi(v) - \phi(u^0) \ge 0. \tag{9}$$

Then there exists a unique $u \in L^\infty(0,T;V) \cap C^0([0,T];U)$ such that $u_t \in L^\infty(0,T;U)$, $u(0) = u^0$, and

$$\langle u_t(t), v(t) - u(t)\rangle + \langle Au(t), v(t) - u(t)\rangle_{V',V} + \phi(v(t)) - \phi(u(t)) \ge 0$$

for a.e. time $t \in (0,T)$, and all $v \in V$.

Lemma 5. *The operator $-\mathcal{H} : V \longrightarrow V'$ given by*

$$\langle \mathcal{H}(u), v\rangle_{V',V} := \int_I H(u_{xx}) v_x \,\mathrm{d}x \tag{10}$$

is pseudo-monotone.

Proof. To prove that $\mathcal{H}$ is bounded, given $v \in V$, we observe that

$$|\langle \mathcal{H}(u), v\rangle_{V',V}| = \left| \int_I H(u_{xx}) v_x \,\mathrm{d}x \right| \le \|H(u_{xx})\|_{L^3(I)} \|v_x\|_{L^{3/2}(I)} \le c\|u\|_V \|v\|_V, \tag{11}$$

where c is a positive constant, thus $\|\mathcal{H}(u)\|_{V'} \le c\|u\|_V$.

Consider u^n, $u \in V$ such that $u^n \rightharpoonup u$ and $\limsup_n -\langle \mathcal{H}(u^n), u^n - u\rangle_{V',V} \le 0$. We need to check that $\langle \mathcal{H}(u), v-u\rangle_{V',V} \le \liminf_n \langle \mathcal{H}(u^n), v - u^n\rangle_{V',V}$ for all $v \in V$. Note that

$$\begin{aligned}(\forall n) \qquad \langle \mathcal{H}(u), v-u\rangle_{V',V} &= \langle \mathcal{H}(u-u^n), v-u\rangle_{V',V} + \langle \mathcal{H}(u^n), v-u\rangle_{V',V} \\ &= \langle \mathcal{H}(u-u^n), v-u\rangle_{V',V} + \langle \mathcal{H}(u^n), v-u^n\rangle_{V',V} + \langle \mathcal{H}(u^n), u^n-u\rangle_{V',V},\end{aligned}$$

where $\lim_n \langle \mathcal{H}(u-u^n), v-u\rangle_{V',V} = 0$. Indeed, since $u^n \rightharpoonup u$ in V (hence in $W^{2,3}(I)$), and the embedding $W^{2,3}(I) \hookrightarrow W^{1,3}(I)$ is compact, we have that $u^n \to u$ in $W^{1,3}(I)$, and in turn

$$\begin{aligned}|\langle \mathcal{H}(u-u^n), v-u\rangle_{V',V}| &= \left| \int_I H(u-u^n)_{xx} (v-u)_x \,\mathrm{d}x \right| = \left| \int_I H(u-u^n)_x (v-u)_{xx} \,\mathrm{d}x \right| \\ &\le \|H(u-u^n)_x\|_{L^3(I)} \|(v-u)_{xx}\|_{L^{3/2}(I)} \\ &\le c\|(u-u^n)_x\|_{L^3(I)} \|(v-u)_{xx}\|_{L^{3/2}(I)} \to 0,\end{aligned}$$

for some constant $c > 0$. Moreover, $u^n \rightharpoonup u$ in V implies that $u^n_x \rightharpoonup u_x$ in $W^{1,3}(I)$, and the embedding $i : W^{1,3}(I) \hookrightarrow C^0([-\pi,\pi];\mathbb{R})$ (endowed with the sup norm) is compact. Hence $u^n_x \rightharpoonup u_x$ in $W^{1,3}(I)$ implies $\|u^n_x - u_x\|_{L^\infty(I)} \to 0$, and

$$\lim_n \langle \mathcal{H}(u^n), u^n - u\rangle_{V',V} = \lim_n \int_I H(u^n_{xx})(u^n_x - u_x) = 0$$

since $\{u^n\}$ is bounded in V, and this concludes the proof. □

Note, however, that the operator $-\mathcal{H}$ is not maximal monotone. To circumvent this difficulty, let

$$\begin{aligned} B : V \longrightarrow V', \qquad & \langle Bu, v\rangle_{V',V} := \int_I \big[u_{xx} v_{xx} - H(u_{xx}) v_x\big] \,\mathrm{d}x, \\ \Psi_a : \mathbb{R} \longrightarrow (-\infty + \infty], \qquad & \Psi_a(\xi) := \Phi_a(\xi) - \xi^2/2, \\ \psi : V \to (-\infty + \infty], \qquad & \psi(u) := \int_I \Psi_a(u_{xx}) \,\mathrm{d}x. \end{aligned}$$

Since

$$\Psi_a''(\xi) = \xi + a + \frac{1}{\xi + a} - 1 \ge 1$$

for any $\xi > -a$, Ψ_a is convex on $(-a, +\infty)$. Consequently ψ is convex.

We will use the following properties of the Hilbert transform.

(1) [2, Theorem 9.1.3] The Hilbert transform $H : L^p_{\mathrm{per}}(I) \longrightarrow L^p_{\mathrm{per}}(I)$ is a well-defined, linear, bounded operator for any $p \in (1, +\infty)$, where $L^p_{\mathrm{per}}(I) := \{f \in L^p(I) : f \text{ is } 2\pi\text{-periodic}\}$.

(2) [2, Theorem 9.1.9] The Hilbert transform $H : L^2_{\mathrm{per}}(I) \longrightarrow L^2_{\mathrm{per}}(I)$ satisfies

$$\|f\|^2_{L^2(I)} = \|H(f)\|^2_{L^2(I)} + \frac{1}{2\pi} \left(\int_I f \,\mathrm{d}x \right)^2$$

for any $f \in L^2_{\mathrm{per}}(I)$.

(3) Also, we will use the sharp Poincaré constant for $f \in W^{1,2}_{\mathrm{per}_0}(I)$. To be precise (see [5, Section 7.7]): if $f \in W^{1,2}_{\mathrm{per}_0}(I)$ then
$$\int_I f^2 \,\mathrm{d}x \le \int_I f_x^2 \,\mathrm{d}x, \tag{12}$$
where equality holds if and only if $f(\xi) = a\sin\xi + b\cos\xi$ a.e., for some $a, b \in \mathbb{R}$.

Lemma 6. *The operator $B : V \longrightarrow V'$ is maximal monotone and coercive.*

Proof. By construction B is hemi-continuous. To prove monotonicity, note that
$$|\langle \mathcal{H}u, u\rangle_{V',V}| \le \|H(u_{xx})\|_{L^2(I)}\|u_x\|_{L^2(I)} \le \|u_{xx}\|^2_{L^2(I)}, \tag{13}$$
since [2, Proposition 9.1.9] and $\int_I u_{xx}\,\mathrm{d}x = 0$ give
$$\|H(u_{xx})\|_{L^2(I)} = \|u_{xx}\|_{L^2(I)} + \frac{1}{2\pi}\left(\int_I u_{xx}\,\mathrm{d}x\right)^2 = \|u_{xx}\|_{L^2(I)},$$
while $\|u_x\|_{L^2(I)} \le \|u_{xx}\|_{L^2(I)}$ holds in view of (12). Thus B is monotone and hemi-continuous, hence maximal monotone (see Remark 3). □

Lemma 7. *The functionals $\mathcal{F}_a(v) := \int_I \Phi_a(v_{xx})\,\mathrm{d}x$ and ψ satisfy the coercivity conditions*
$$\lim_{\|v\|_V\to+\infty} \frac{-\langle \mathcal{H}(v), v\rangle_{V',V} + \psi(v)}{\|v\|_V} = \lim_{\|v\|_V\to+\infty} \frac{-\langle \mathcal{H}(v), v\rangle_{V',V} + \mathcal{F}_a(v)}{\|v\|_V} = +\infty \tag{14}$$

Proof. Note that
$$|\langle \mathcal{H}(v), v\rangle_{V',V}| \le \int_I |H(v_{xx})v_x|\,\mathrm{d}x \le \|v_{xx}\|_{L^3(I)}\|v_x\|_{L^{3/2}(I)} \le c\|v\|_V \tag{15}$$
for some $c > 0$. We consider only functions $v \in V$ such that $v_{xx} + a \ge 0$ a.e. (for the remaining v, it holds $\mathcal{F}_a(v) \equiv +\infty$ and the thesis is trivial). Periodicity, the zero-average property of functions of V, and Poincaré inequality, imply that $\|v\|_V \to +\infty$ forces $\|v_{xx}\|_{L^3(I)} \to +\infty$ (and $\|v_{xx} + a\|_{L^3(I)}/\|v_{xx}\|_{L^3(I)} \to 1$). The highest order (and the only relevant) term in $\int_I \Phi_a(v_{xx})\,\mathrm{d}x$ is the cubic term, and $\int_I (v_{xx} + a)^3\,\mathrm{d}x = \|v_{xx} + a\|^3_{L^3(I)}$. Poincaré inequality gives $\|v\|_V \le \alpha\|v_{xx}\|_{L^3(I)}$ for some constant $\alpha > 0$. Since $\langle \mathcal{H}(v), v - v_0\rangle_{V',V}$ is at most quadratic in $\|v_{xx}\|_{L^3(I)}$ (as $\|v\|_V \to +\infty$), it follows that
$$\lim_{\|v\|_V\to+\infty} \frac{-\langle \mathcal{H}(v), v - v_0\rangle_{V',V} + \mathcal{F}_a(v)}{\|v\|_V} \ge \lim_{\|v\|_V\to+\infty} \frac{\|v_{xx} + a\|^3_{L^3(I)} + \text{lower order terms}}{6\alpha\|v_{xx}\|_{L^3(I)}} = +\infty,$$
proving
$$\lim_{\|v\|_V\to+\infty} \frac{-\langle \mathcal{H}(v), v\rangle_{V',V} + \mathcal{F}_a(v)}{\|v\|_V} = +\infty.$$
The proof for
$$\lim_{\|v\|_V\to+\infty} \frac{-\langle \mathcal{H}(v), v\rangle_{V',V} + \psi(v)}{\|v\|_V} = +\infty$$
is analogous. □

For future reference, given a mapping $v : [0, T] \longrightarrow V$, with an abuse of notation we will denote by $v(t, \cdot)$ the function $v(t)$. Hence we will often write $v(t, x)$ instead of $v(t)(x)$.

Proof. (of Theorem 1) Lemma 6 establishes maximal monotonicity for B, while Lemma 7 ensures that (8) holds, and hypothesis (9) results from (4). Therefore, by Theorem 4 there exists a unique $u : [0, T] \longrightarrow V$ such that
$$u \in L^\infty(0, T; V) \cap C^0([0, T]; U), \qquad u_t \in L^\infty(0, T; U), \qquad u(0) = u^0,$$
and
$$\langle u_t, v - u\rangle + \langle B(u), v - u\rangle_{V',V} + \psi(v) - \psi(u) \ge 0 \tag{16}$$
for every $v \in V$ and for a.e. $t \in [0, T]$. Observe that

$$\begin{aligned}\langle B(u), v-u\rangle_{V',V} &+ \int_I [\Psi_a(v_{xx}) - \Psi_a(u_{xx})]\,\mathrm{d}x \\ &= \int_I u_{xx}(v-u)_{xx}\,\mathrm{d}x - \langle \mathcal{H}(u), v-u\rangle_{V',V} + \int_I \left[\Phi_a(v_{xx}) - \Phi_a(u_{xx}) - \frac{1}{2}(v_{xx}^2 - u_{xx}^2)\right]\mathrm{d}x, \end{aligned} \tag{17}$$

and

$$\frac{1}{2}\int_I (v_{xx}^2 - u_{xx}^2)\,\mathrm{d}x = \frac{1}{2}\int_I (v-u+2u)_{xx}(v-u)_{xx}\,\mathrm{d}x = \int_I u_{xx}(v-u)_{xx}\,\mathrm{d}x + \frac{1}{2}\|(v-u)_{xx}\|_{L^2}^2,$$

hence (17) becomes

$$\begin{aligned}\langle B(u), v-u\rangle_{V',V} &+ \int_I [\Psi_a(v_{xx}) - \Psi_a(u_{xx})]\,\mathrm{d}x \\ &= -\langle \mathcal{H}(u), v-u\rangle_{V',V} + \int_I [\Phi_a(v_{xx}) - \Phi_a(u_{xx})]\,\mathrm{d}x - \frac{1}{2}\|(v-u)_{xx}\|_{L^2}^2.\end{aligned}$$

Thus the solution u of (18) satisfies also

$$\langle u_t, v-u\rangle - \langle \mathcal{H}(u), v-u\rangle_{V',V} + \int_I [\Phi_a(v_{xx}) - \Phi_a(u_{xx})]\,\mathrm{d}x \geq 0 \tag{18}$$

for every $v \in V$ and for a.e. $t \in [0,T]$. We prove that u is also a solution of (3) in the weak sense of (5), i.e.,

$$\int_0^T \int_I u_t \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_I [H(u_{xx})\varphi_x - \Phi_a'(u_{xx})\varphi_{xx}]\,\mathrm{d}x\,\mathrm{d}t \tag{19}$$

for all $\varphi \in C_c^\infty((0,T)\times I;\mathbb{R})$. The idea is to test (16) (for all t such that it holds) with $v = u + \varepsilon(\varphi - \bar{\varphi})$ and $v = u - \varepsilon(\varphi - \bar{\varphi})$, where $\bar{\varphi}(t) := \int_I \varphi(t,x)\,\mathrm{d}x$, take the limit as $\varepsilon \to 0^+$, and integrate in t. However it is unclear whether $\Phi_a'(v_{xx}) \in L^1(I)$, or $v_{xx} + a \geq 0$ a.e. in x. An *ad hoc* construction is required to overcome these difficulties.

Step 1. Integrability of $\log(u_{xx}(t)+a)$. The first step is to prove that $\log(u_{xx}(t)+a) \in L^1(I)$ for a.e. $t \in [0,T]$, and then show that $\log(u_{xx}+a) \in L^1(0,T;L^1(I))$. Fix $\varepsilon \in (0,1)$ and let $v^\varepsilon := (1-\varepsilon)u(t)$. Using v^ε in (18), gives

$$\begin{aligned}\langle u_t(t), -\varepsilon u(t)\rangle - \langle \mathcal{H}(u(t)), -\varepsilon u(t)\rangle_{V',V} &\geq \int_I \Phi_a(u_{xx}(t)) - \Phi_a(v_{xx}^\varepsilon)\,\mathrm{d}x \\ &\geq \int_I \varepsilon u_{xx}(t)\Phi_a'((1-\varepsilon)u_{xx}(t))\,\mathrm{d}x,\end{aligned}$$

where the last inequality holds since $v_{xx}^\varepsilon = (1-\varepsilon)u_{xx}(t) \geq -(1-\varepsilon)a > -a$, hence Φ_a is differentiable in $v_{xx}^\varepsilon(x)$ for a.e. $x \in I$, and also due to the convexity of Φ_a. By Lebesgue monotone convergence theorem

$$\begin{aligned}\langle u_t(t), -u(t)\rangle - \langle \mathcal{H}(u(t)), -u(t)\rangle_{V',V} &\geq \lim_{\varepsilon\to 0^+}\int_I u_{xx}(t)\Phi_a'((1-\varepsilon)u_{xx}(t))\,\mathrm{d}x \\ &= \int_I u_{xx}(t)\Phi_a'(u_{xx}(t))\,\mathrm{d}x.\end{aligned} \tag{20}$$

Note that for $\xi > -a$, $\Phi_a'(\xi) = \log(\xi+a) + (\xi+a)^2/2 + 1$, and because $u(t) \in V$, it follows that

$$\int_I |u_{xx}(t)(u_{xx}(t)+a)^2|\,\mathrm{d}x < +\infty, \quad \int_{\{u_{xx}(t)+a\geq 1\}} |u_{xx}(t)\log(u_{xx}(t)+a)|\,\mathrm{d}x < +\infty, \tag{21}$$

$$\int_{\{u_{xx}(t)\geq -a/2\}} |u_{xx}(t)\log(u_{xx}(t)+a)|\,\mathrm{d}x < +\infty. \tag{22}$$

Since $u \in L^\infty(0,T;V)$ and $u_t \in L^\infty(0,T;U)$, we have that

$$\langle u_t(t), -u(t)\rangle - \langle \mathcal{H}(u(t)), -u(t)\rangle_{V',V} < +\infty,$$

which, together with (20), (21) and (22), implies that

$$\int_J u_{xx}(t)\log(u_{xx}(t)+a)\,\mathrm{d}x < +\infty, \tag{23}$$

$$J := \{-a \le u_{xx}(t) < -a/2\} \cap \{u_{xx}(t)+a<1\}.$$

By definition of J, for all $y \in J$

$$u_{xx}(t,y) < 0, \quad \log(u_{xx}(t,y)+a) < 0,$$

i.e., the integrand $u_{xx}(t)\log(u_{xx}(t)+a)$ is nonnegative on J. Since $J \subseteq \{-a \le u_{xx}(t) < -a/2\}$, combining with (23) yields

$$\frac{a}{2}\int_J |\log(u_{xx}(t)+a)|\,\mathrm{d}x \le \int_J u_{xx}(t)\log(u_{xx}(t)+a)\,\mathrm{d}x < +\infty,$$

and so $\log(u_{xx}(t)+a) \in L^1(I)$. Integrating (20) in time gives $\log(u_{xx}+a) \in L^1(0,T;L^1(I))$, with

$$\int_0^T\int_I |u_{xx}(t)(u_{xx}(t)+a)^2|\,\mathrm{d}x\,\mathrm{d}t < +\infty, \quad \int_0^T\int_{\{u_{xx}(t)+a\ge 1\}} |u_{xx}(t)\log(u_{xx}(t)+a)|\,\mathrm{d}x\,\mathrm{d}t < +\infty,$$

$$\int_0^T\int_{\{u_{xx}(t)\ge -a/2\}} |u_{xx}(t)\log(u_{xx}(t)+a)|\,\mathrm{d}x\,\mathrm{d}t < +\infty,$$

and we conclude that $u \in L^2(0,T;V)$.

Step 2. Truncating $u_{xx}(t)$. To overcome the issue that for $\varepsilon>0$, $\varphi \in C_c^\infty((0,T)\times I;\mathbb{R})$, the function $u_{xx} + a + \varepsilon\varphi_{xx}$ may fail to be nonnegative, we construct a sequence $\{u^\delta_{xx}\}$ in the following way: let

$$u^\delta_{xx}(t,x) := \hat{u}^\delta_{xx}(t,x) - \frac{1}{2\pi}\int_I \hat{u}^\delta_{xx}(t,s)\,\mathrm{d}s, \qquad \hat{u}^\delta_{xx}(t,x) := \max\{u_{xx}(t,x)+a,\delta\} - a. \tag{24}$$

Setting

$$E^\delta(t) := \{x \in I : \hat{u}^\delta_{xx}(t,x) \ne u_{xx}(t,x)\} \qquad \text{for a.e. } t, \tag{25}$$

we have $\|\hat{u}^\delta_{xx}(t) - u_{xx}(t)\|_{L^p(I)} \le \delta\mathcal{L}^1(E^\delta(t))^{1/p}$, $p \in [1,3]$. Note that

$$\mathcal{L}^1(E^\delta(t)) \to 0 \quad \text{as } \delta \to 0^+. \tag{26}$$

Since $u(t) \in V \subseteq W^{2,3}_{\mathrm{loc}}(\mathbb{R})$, u_x is continuous and 2π-periodic, i.e. $\int_I u_{xx}(t,x)\,\mathrm{d}x = 0$ for a.e. t. Thus

$$0 = \int_I u_{xx}(t,x)\,\mathrm{d}x \le \int_I \hat{u}^\delta_{xx}(t,x)\,\mathrm{d}x \le \delta\mathcal{L}^1(E^\delta(t)), \tag{27}$$

which gives

$$\begin{aligned}\|u^\delta_{xx}(t) - u_{xx}(t)\|_{L^p(I)} &\le \|u^\delta_{xx}(t) - \hat{u}^\delta_{xx}(t)\|_{L^p(I)} + \|\hat{u}^\delta_{xx}(t) - u_{xx}(t)\|_{L^p(I)} \\ &\le 2\delta\mathcal{L}^1(E^\delta(t))^{1/p}, \qquad p \in [1,3].\end{aligned}$$

Define

$$u^\delta_x(t,x) := \int_{-\pi}^x u^\delta_{xx}(t,y)\,\mathrm{d}y + u_x(t,-\pi) - \zeta(t,\delta),$$

where $\zeta(t,\delta)$ is a constant chosen such that $\displaystyle\int_I u^\delta_x(t,y)\,\mathrm{d}y = 0$. Since $\|u^\delta_{xx}(t) - u_{xx}(t)\|_{L^1(I)} \le 2\delta\mathcal{L}^1(E^\delta(t))$, it follows that $|\zeta(t,\delta)| \le 2\delta\mathcal{L}^1(E^\delta(t))$.

Define also

$$u^\delta(t,x) := \int_{-\pi}^x u^\delta_x(t,y)\,\mathrm{d}y + u(t,-\pi) - \theta(t,\delta),$$

where $\theta(t,\delta)$ is a constant chosen such that $\int_I u^\delta(t,y)\,\mathrm{d}y = 0$. Since $\|u_x^\delta(t) - u_x(t)\|_{L^1(I)} \le 8\pi\delta\mathcal{L}^1(E^\delta(t))$, it follows $|\theta(t,\delta)| \le 8\pi\delta\mathcal{L}^1(E^\delta(t))$. With the above construction, we now have that

(i) $u_{xx}^\delta(t) \ge \delta(1 - \mathcal{L}^1(E^\delta(t))/2\pi) - a$, where we used (27);

(ii) $u_{xx}^\delta(t) \in L^3(I)$ with zero-average on I, $u_x^\delta(t) \in W^{1,3}_{\mathrm{per}_0}(I)$, and $u^\delta(t) \in V$ for a.e. t;

(iii) by Poincaré inequality, periodicity and the zero-average property of functions in V, we observe that

$$\|u^\delta(t) - u(t)\|_V \le \beta\|u_{xx}^\delta(t) - u_{xx}(t)\|_{L^3(I)} \le \beta\delta\mathcal{L}^1(E^\delta(t))^{1/3}(1 + \mathcal{L}^1(E^\delta(t))^{2/3})$$

for some constant $\beta > 0$.

Step 3. Proof of (19)**.** This will be accomplished by testing (16) with variations of the form $u^\delta(t) \pm \varepsilon\varphi(t)$. Fix $\varphi \in C_c^\infty((0,T)\times I;\mathbb{R})$, and a time t such that (16) holds. Two cases apply.

Case 1. Assume that there exists $\delta_1 > 0$ such that $\mathcal{L}^1(E^{\delta_1}(t)) = 0$. By (24) and (26), we have that $\mathcal{L}^1(E^\delta(t)) = 0$ for any $0 < \delta \le \delta_1$ and $u^\delta(t) = u(t)$. Therefore $u_{xx}(t) + a \ge \delta_1$. Choose $\varepsilon_1 > 0$ such that $\varepsilon|\varphi_{xx}(t)| < \delta_1/2$ for all $\varepsilon \in (0,\varepsilon_1)$. We consider the variation, for $\varepsilon \in (0,\varepsilon_1)$,

$$w^\varepsilon(t) := u(t) + \varepsilon(\varphi(t) - \bar\varphi(t)), \qquad \bar\varphi(t) := \frac{1}{2\pi}\int_I \varphi(t,x)\,\mathrm{d}x.$$

Using $w^\varepsilon(t)$ in (16) we get

$$\langle u_t(t), w^\varepsilon(t) - u(t)\rangle - \langle \mathcal{H}(u), w^\varepsilon(t) - u(t)\rangle_{V',V} + \mathcal{F}_a(w^\varepsilon(t)) - \mathcal{F}_a(u(t)) \ge 0,$$

that is,

$$\langle u_t(t), \varepsilon(\varphi(t) - \bar\varphi(t))\rangle - \langle \mathcal{H}(u), \varepsilon\varphi(t)\rangle_{V',V} + \mathcal{F}_a(w^\varepsilon(t)) - \mathcal{F}_a(u(t)) \ge 0, \tag{28}$$

where we used the fact that $\langle \mathcal{H}(u), c\rangle_{V',V} = 0$ for all constants $c \in \mathbb{R}$ (see (10)). We need to prove

$$\lim_\varepsilon \frac{\mathcal{F}_a(w^\varepsilon(t)) - \mathcal{F}_a(u(t))}{\varepsilon} = \int_I \varphi_{xx}(t)\Phi_a'(u_{xx}(t))\,\mathrm{d}x. \tag{29}$$

Note that, since $\varepsilon < \varepsilon_1$, both $u_{xx}(t) + a$ and $u_{xx}(t) + a + \varepsilon\varphi_{xx}(t)$ are uniformly bounded away from zero. We observe that

$$\begin{aligned}\frac{\mathcal{F}_a(u(t)) - \mathcal{F}_a(w^\varepsilon(t))}{\varepsilon} &= \frac{1}{\varepsilon}\int_I [\Phi_a(u_{xx}(t)) - \Phi_a(u_{xx}(t) + \varepsilon\varphi_{xx}(t))]\,\mathrm{d}x \\ &\ge -\int_I \varphi_{xx}(t)\Phi_a'(u_{xx}(t) + \varepsilon\varphi_{xx}(t))\,\mathrm{d}x.\end{aligned}$$

Clearly $\varphi_{xx}(t)\Phi_a'(u_{xx}(t) + \varepsilon\varphi_{xx}(t))$ converges to $\varphi_{xx}(t)\Phi_a'(u_{xx}(t))$ a.e.. Note also that

$$\Phi_a'(u_{xx}(t) + \varepsilon\varphi_{xx}(t)) = \log(u_{xx}(t) + \varepsilon\varphi_{xx}(t) + a) + (u_{xx}(t) + \varepsilon\varphi_{xx}(t) + a)^2/2,$$

with

$$u_{xx}(t) + \delta_1/2 + a \ge u_{xx}(t) + \varepsilon\varphi_{xx}(t) + a \ge \delta_1/2$$

due to the choice of $\delta_1, \varepsilon_1 > 0$. Thus

$$\log(u_{xx}(t) + \varepsilon\varphi_{xx}(t) + a) + (u_{xx}(t) + \varepsilon\varphi_{xx}(t) + a)^2/2 \le |\log(\delta_1/2)| + (u_{xx}(t) + \delta_1/2 + a)^2/2 \in L^1(I),$$

and, by Lebesgue dominated convergence theorem, we have

$$\limsup_\varepsilon \frac{\mathcal{F}_a(u(t)) - \mathcal{F}_a(w^\varepsilon(t))}{\varepsilon} \ge \lim_\varepsilon -\int_I \varphi_{xx}(t)\Phi_a'(u_{xx}(t) + \varepsilon\varphi_{xx}(t))\,\mathrm{d}x = -\int_I \varphi_{xx}(t)\Phi_a'(u_{xx}(t))\,\mathrm{d}x,$$

or equivalently,

$$\liminf_\varepsilon \frac{\mathcal{F}_a(w^\varepsilon(t)) - \mathcal{F}_a(u(t))}{\varepsilon} \le \int_I \varphi_{xx}(t)\Phi_a'(u_{xx}(t))\,\mathrm{d}x.$$

Dividing (28) by ε and passing to the limit $\varepsilon \to 0^+$ gives

$$0\leq\langle u_t(t),\varphi(t)-\bar\varphi(t)\rangle-\langle\mathcal{H}(u),\varphi(t)\rangle_{V',V}+\int_I\varphi_{xx}(t)\Phi_a'(u_{xx}(t))\,\mathrm{d}x.$$

Case 2. Assume $\mathcal{L}^1(E^\delta(t))>0$ for all $\delta>0$. Let $M(\varphi):=2\sup_x|\varphi_{xx}(t,x)|$,

$$\varepsilon=\varepsilon(\varphi,\delta,t):=\delta/(1+M(\varphi)),\ w^\varepsilon(t):=u^\delta(t)+\varepsilon(\varphi(t)-\bar\varphi(t)),\ \bar\varphi(t):=\frac{1}{2\pi}\int_I\varphi(t,x)\,\mathrm{d}x. \tag{30}$$

Since $\mathcal{L}^1(E^\delta(t))\to0$ as $\delta\to0^+$ (see (26)), and in view of Step 2 (iii), it follows that

$$\varepsilon=O(\delta),\qquad\|u^\delta(t)-u(t)\|_V=o(\varepsilon). \tag{31}$$

Taking $w^\varepsilon(t)$ in (16) yields

$$\begin{aligned}\langle u_t(t),u^\delta(t)-u(t)+\varepsilon(\varphi(t)-\bar\varphi(t))\rangle-\langle\mathcal{H}(u(t)),u^\delta(t)-u(t)+\varepsilon\varphi(t)\rangle_{V',V}&\\+\mathcal{F}_a(u^\delta(t)+\varepsilon\varphi(t))-\mathcal{F}_a(u(t))\geq0,&\end{aligned} \tag{32}$$

By the mean value theorem, we have

$$\begin{aligned}\mathcal{F}_a(u^\delta(t)+\varepsilon\varphi(t))-\mathcal{F}_a(u(t))&=\int_I\left[\Phi_a(u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t))-\Phi_a(u_{xx}(t))\right]\mathrm{d}x\\&=\int_{S(t,\delta)}(u^\delta_{xx}(t)-u_{xx}(t)+\varepsilon\varphi_{xx}(t))\Phi_a'(\vartheta^\varepsilon(t))\,\mathrm{d}x,\end{aligned}$$

where

$$S(t,\delta):=\{u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t)\neq u_{xx}(t)\}$$

and

$$\min\{u^\delta_{xx}(t,x)+\varepsilon\varphi_{xx}(t,x),u_{xx}(t,x)\}\leq\vartheta^\varepsilon(t,x)\leq\max\{u^\delta_{xx}(t,x)+\varepsilon\varphi_{xx}(t,x),u_{xx}(t,x)\} \tag{33}$$

for any x. Next we establish the Lebesgue measurability of $S(t,\delta)\ni x\mapsto\vartheta(t,x)$. For $x\in S(t,\delta)$ it holds

$$\Phi_a(u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t))-\Phi_a(u_{xx}(t))=(u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t)-u_{xx}(t))\Phi_a'(\vartheta^\varepsilon(t,x)),$$

hence

$$\Phi_a'(\vartheta^\varepsilon(t,x))=\frac{\Phi_a(u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t))-\Phi_a(u_{xx}(t))}{u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t)-u_{xx}(t)}.$$

For $\xi>-a$, $\Phi_a'(\xi)=(\xi+a)^2/2+\log(\xi+a)+1$ is injective, hence we have

$$\vartheta^\varepsilon(t,x)=(\Phi_a')^{-1}\left(\frac{\Phi_a(u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t))-\Phi_a(u_{xx}(t))}{u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t)-u_{xx}(t)}\right),$$

which proves the Lebesgue measurability of $x\mapsto\vartheta^\varepsilon(t,x)$ on $S(t,\delta)$.

Dividing by ε and taking the limit $\delta\to0^+$ in (32) gives

$$\begin{aligned}\langle u_t(t),\varphi(t)-\bar\varphi(t)\rangle-\langle\mathcal{H}(u(t)),\varphi(t)\rangle_{V',V}+\liminf_\delta\int_{S(t,\delta)}\varphi_{xx}(t)\Phi_a'(\vartheta^\varepsilon(t))\,\mathrm{d}x&\\+\frac{1}{\varepsilon}\int_{S(t,\delta)}(u^\delta_{xx}(t)-u_{xx}(t))\Phi_a'(\vartheta^\varepsilon(t))\,\mathrm{d}x\geq0,&\end{aligned} \tag{34}$$

where we used the fact that $\|u^\delta(t)-u(t)\|_V=o(\varepsilon)$, and u is Lipschitz in time, and by (31),

$$\lim_\delta\varepsilon^{-1}\langle u_t(t),u^\delta(t)-u(t)\rangle=0,$$

$$\lim_\delta\varepsilon^{-1}|\langle\mathcal{H}(u(t)),u^\delta(t)-u(t)\rangle_{V',V}|\leq C\lim_\varepsilon\varepsilon^{-1}\|u_{xx}(t)\|_{L^3(I)}\|u^\delta(t)-u(t)\|_V=0$$

for some $C>0$.

We claim that
$$\lim_{\delta} \frac{1}{\varepsilon} \int_{S(t,\delta)} (u^\delta_{xx}(t) - u_{xx}(t)) \Phi'_a(\vartheta^\varepsilon(t)) \, \mathrm{d}x = 0. \tag{35}$$
Note that on $I \backslash E^\delta(t)$ it holds $u_{xx} + a \geq \delta$, hence by (24) and (27), we have
$$\begin{aligned} \vartheta^\varepsilon(t) + a &\geq \min\{u^\delta_{xx}(t) + \varepsilon\varphi_{xx}(t), u_{xx}(t)\} + a \\ &\geq u_{xx}(t) + a - \delta(1/2 + \mathcal{L}^1(E^\delta(t))/2\pi) \geq (u_{xx}(t) + a)/3, \end{aligned}$$
for all δ such that $\mathcal{L}^1(E^\delta(t)) \leq \pi/3$, and
$$\vartheta^\varepsilon(t) + a \leq \max\{u^\delta_{xx}(t) + \varepsilon\varphi_{xx}(t), u_{xx}(t)\} + a \leq u_{xx}(t) + a + 1$$
for all $\delta \leq 3/2$. Hence
$$\begin{aligned} &|\Phi'_a(\vartheta^\varepsilon(t))| \\ &\quad \leq |\log(u_{xx}(t) + a)| + |\log(u_{xx}(t) + a + 1)| + \log 3 + (u_{xx}(t) + a + 1)^2 =: g(t) \in L^1(I). \end{aligned} \tag{36}$$
By (24) and (27), on $I \backslash E^\delta(t)$ it holds
$$|u^\delta_{xx}(t) - u_{xx}(t)| \leq \delta \mathcal{L}^1(E^\delta(t)), \tag{37}$$
hence
$$\frac{1}{\varepsilon} \int_{(I \backslash E^\delta(t)) \cap S(t,\delta)} |(u^\delta_{xx}(t) - u_{xx}(t)) \Phi'_a(\vartheta^\varepsilon(t))| \, \mathrm{d}x \leq 2\mathcal{L}^1(E^\delta(t)) \|g(t)\|_{L^1(I)} \to 0,$$
where we have used the definition of ε as in (30). On $E^\delta(t)$ it holds
$$u^\delta_{xx}(t) + \varepsilon\varphi_{xx}(t) + a \geq \delta \left(\frac{1}{2} - \frac{\mathcal{L}^1(E^\delta(t))}{2\pi} \right) \geq (u_{xx} + a)/3,$$
hence
$$\vartheta^\varepsilon(t) + a \geq \min\{u^\delta_{xx}(t) + \varepsilon\varphi_{xx}(t), u_{xx}(t)\} + a \geq (u_{xx} + a)/3,$$
thus (36) still holds. Since $|u^\delta_{xx}(t) - u_{xx}(t)| \leq \delta\big(\frac{1}{2} + \frac{\mathcal{L}^1(E^\delta(t))}{2\pi}\big)$, we have, under the additional assumption $\delta \leq 3/5$,
$$\frac{1}{\varepsilon} \int_{E^\delta(t) \cap S(t,\delta)} |(u^\delta_{xx}(t) - u_{xx}(t)) \Phi'_a(\vartheta^\varepsilon(t))| \, \mathrm{d}x \leq 2\|g(t)\|_{L^1(E^\delta(t))} \to 0,$$
and (35) is proven.

Now we show that
$$\lim_{\delta} \int_{S(t,\delta)} \varphi_{xx}(t) \Phi'_a(\vartheta^\varepsilon(t)) \, \mathrm{d}x = \int_I \varphi_{xx}(t) \Phi'_a(u_{xx}(t)) \, \mathrm{d}x. \tag{38}$$
By definition of $S(t,\delta)$, we have
$$I \backslash S(t,\delta) = \{x : u^\delta_{xx}(t,x) + \varepsilon\varphi_{xx}(t,x) - u_{xx}(t,x) = 0\},$$
thus for any δ it holds
$$\begin{aligned} 0 &= \frac{1}{\varepsilon} \int_{I \backslash S(t,\delta)} [\Phi_a(u^\delta_{xx}(t) + \varepsilon\varphi_{xx}(t)) - \Phi_a(u_{xx}(t))] \, \mathrm{d}x \\ &= \frac{1}{\varepsilon} \int_{I \backslash S(t,\delta)} (u^\delta_{xx}(t) + \varepsilon\varphi_{xx}(t) - u_{xx}(t)) \Phi'_a(u_{xx}(t)) \, \mathrm{d}x. \end{aligned} \tag{39}$$
From the construction of u^δ_{xx}, we get

$$\begin{aligned}
&\int_{I\setminus S(t,\delta)} \left|\frac{u^\delta_{xx}(t,x)-u_{xx}(t,x)}{\varepsilon}\Phi'_a(u_{xx}(t))\right| \mathrm{d}x\\
&\quad= \int_{(I\setminus S(t,\delta))\cap(I\setminus E^\delta(t))} \left|\frac{u^\delta_{xx}(t,x)-u_{xx}(t,x)}{\varepsilon}\Phi'_a(u_{xx}(t))\right| \mathrm{d}x\\
&\quad\quad+ \int_{(I\setminus S(t,\delta))\cap E^\delta(t)} \left|\frac{u^\delta_{xx}(t,x)-u_{xx}(t,x)}{\varepsilon}\Phi'_a(u_{xx}(t))\right| \mathrm{d}x\\
&\quad\le (1+M(\varphi))\Big(\frac{\mathcal{L}^1(E^\delta(t))}{2\pi}\|\Phi'_a(u_{xx}(t))\|_{L^1(I)} + \Big(1+\frac{\mathcal{L}^1(E^\delta(t))}{2\pi}\Big)\|\Phi'_a(u_{xx}(t))\|_{L^1(E^\delta(t))}\Big)\to 0,
\end{aligned}$$

where we used (37) and the fact that on $E^\delta(t)$, $|u^\delta_{xx}-u_{xx}|=O(\delta)$ (see (33)).

This, together with (39), gives $\int_{I\setminus S(t,\delta)}\varphi_{xx}(t)\Phi'_a(u_{xx}(t))\,\mathrm{d}x\to 0$. Let

$$\tilde\vartheta^\varepsilon(t,x) := \begin{cases} \vartheta^\varepsilon(t,x) & \text{if } x\in S(t,\delta),\\ u_{xx}(t,x) & \text{if } x\notin S(t,\delta).\end{cases}$$

Hence

$$\begin{aligned}
\lim_\delta \int_{S(t,\delta)}\varphi_{xx}(t)\Phi'_a(\vartheta^\varepsilon(t))\,\mathrm{d}x &= \lim_\delta \int_{S(t,\delta)}\varphi_{xx}(t)\Phi'_a(\vartheta^\varepsilon(t))\,\mathrm{d}x + \lim_\delta\int_{I\setminus S(t,\delta)}\varphi_{xx}(t)\Phi'_a(u_{xx}(t))\,\mathrm{d}x\\
&= \lim_\delta\int_I \varphi_{xx}(t)\Phi'_a(\tilde\vartheta^\varepsilon(t))\,\mathrm{d}x,
\end{aligned}$$

thus (38) is equivalent to proving that

$$\lim_\delta\int_I\varphi_{xx}(t)\Phi'_a(\tilde\vartheta^\varepsilon(t))\,\mathrm{d}x = \int_I\varphi_{xx}(t)\Phi'_a(u_{xx}(t))\,\mathrm{d}x. \tag{40}$$

By construction, $u^\delta_{xx}(t)+\varepsilon\varphi_{xx}(t)\to u_{xx}(t)$ a.e., hence $\tilde\vartheta^\varepsilon(t)\to u_{xx}(t)$ a.e. Therefore, (40) follows from (36) and Lebesgue dominated convergence theorem.

In view of (35) and (38), passing to the limit $\delta\to 0^+$ in (34) we get

$$\int_I u_t(t)(\varphi(t)-\bar\varphi(t))\,\mathrm{d}x \ge \int_I\big[H(u_{xx}(t))\varphi_x(t)-\Phi'_a(u_{xx}(t))\varphi_{xx}(t)\big]\,\mathrm{d}x.$$

The above argument can be repeated for any t in

$$\{t\in(0,T): \text{(16) holds, } \log(u_{xx}(t)+a)\in L^1(I),\ u(t)\in V\},$$

which has full measure, yielding

$$\int_I u_t(t)(\varphi(t)-\bar\varphi(t))\,\mathrm{d}x \ge \int_I\big[H(u_{xx}(t))\varphi_x(t)-\Phi'_a(u_{xx}(t))\varphi_{xx}(t)\big]\,\mathrm{d}x \qquad \text{for a.e. } t.$$

Integrating in time gives

$$\int_0^T\int_I u_t(t)(\varphi(t)-\bar\varphi(t))\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T\int_I\big[H(u_{xx}(t))\varphi_x(t)-\Phi'_a(u_{xx}(t))\varphi_{xx}(t)\big]\,\mathrm{d}x\,\mathrm{d}t. \tag{41}$$

Since u is Lipschitz in time and φ is smooth, we have sufficient regularity to integrate by parts, hence

$$\int_0^T\int_I u_t(t)\bar\varphi(t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\bar\varphi_t(t)\Big(\int_I u(t)\,\mathrm{d}x\Big)\,\mathrm{d}t = 0,$$

and (41) becomes

$$\int_0^T\int_I u_t(t)\varphi(t)\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T\int_I\big[H(u_{xx}(t))\varphi_x(t)-\Phi'_a(u_{xx}(t))\varphi_{xx}(t)\big]\,\mathrm{d}x\,\mathrm{d}t. \tag{42}$$

Replacing φ with $-\varphi$ in (42), we conclude (19). □

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References

[1] Browder, F.E. (1965). Multivalued monotone nonlinear mappings and duality mappings in Banach spaces. *Transactions of the American Mathematical Society* 118:338–351.

[2] Butzer, P.L., Nessel, R.J. (1971). *Fourier analysis and approximation. Volume 1: one-dimensional theory*. New York and London: Academic Press.

[3] Dal Maso, G., Fonseca, I., Leoni, G. (2014). Analytical validation of a continuum model for epitaxial growth with elasticity on vicinal surfaces. *Archives for Rational Mechanics and Analysis* 212:1037-1064.

[4] Duport, C., Politi, P., Villain, J. (1995). Growth instabilities induced by elasticity in a vicinal surface. *Journal de Physique I* 1(5):1317–1350.

[5] Hardy, G.H., Littlewood, J.E., Pólya, G. (1988). *Inequalities*. Cambridge: Cambridge Mathematical Library.

[6] Kačur, J. (1985). *Method of Rothe in evolution equations*. Leipzig: Teubner Verlaggesellschaft.

[7] Tersoff, J., Phang, Y.H., Zhang, Z., Lagally, M.G. (1995). Step-bunching instability of vicinal surfaces under stress. *Physical Review Letters* 75:2730–2733.

[8] Xiang, Y. (2002). Derivation of a continuum model for epitaxial growth with elasticity on vicinal surface. *SIAM Journal on Applied Mathematics* 63:241-258.

[9] Xiang, Y., E, W. (2004). Misfit elastic energy and a continuum model for epitaxial growth with elasticity on vicinal surfaces. *Physical Review B* 69:035409-1–035409-16.

[10] Xu, H., Xiang, Y. (2009). Derivation of a continuum model for the long-range elastic interaction on stepped epitaxial surfaces in 2+1 dimensions. *SIAM Journal on Applied Mathematics* 69(5):1393–1414.