GEOMETRIC INEQUALITIES FOR FRACTIONAL LAPLACE OPERATORS AND APPLICATIONS

ELEONORA CINTI AND FAUSTO FERRARI

Abstract. We prove a weighted fractional inequality involving the solution u of a nonlocal semilinear problem in $\mathbb{R}^n$. Such inequality bounds a weighted L^2-norm of a compactly supported function ϕ by a weighted H^s-norm of ϕ. In this inequality a geometric quantity related to the level sets of u will appear. As a consequence we derive some relations between the stability of u and the validity of fractional Hardy inequalities.

1. introduction

In this paper, following the ideas contained in [17], we prove a weighted Poincaré inequality that gives us useful informations concerning the geometry of the level surfaces of stable solutions of the fractional semi-linear equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $s \in (0,1)$ and f is C^1 in the range of u.

For every locally integrable function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \frac{| u(y) |}{1+ | y |^{n+2s}} dy < +\infty,$$

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and $s \in (0,1)$, the following operator is well defined

$$(-\Delta)^s u(x) = c(n,s) \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\mid x-y\mid^{n+2s}} dy$$
$$= \lim_{\epsilon \to 0^+} c(n,s) \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{u(x)-u(y)}{\mid x-y\mid^{n+2s}} dy,$$

where $c(n,s)$ is a positive constant such that for every $x \in \mathbb{R}^n$,

$$\lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x).$$

For the definition and the main properties of the fractional Laplacian, we refer to [10], Section 3, and references therein.

Since in the sequel the constant $c(n,s)$ does not play any particular role, we omit it and simply assume that

$$(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{\mid x-y\mid^{n+2s}} dy.$$

The energy functional associated to problem (1.1) is given by

$$E(u) = \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{\mid x-y\mid^{n+2s}} dxdy - \int_{\mathbb{R}^n} F(u(y))dy, \tag{1.2}$$

where F denotes the potential, i.e. $F' = -f$.

Definition 1.1. We say that a function u satisfies (1.1) in the weak sense when for every $h \in C_0^\infty(\mathbb{R}^n)$,

$$\frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(h(x)-h(y))}{\mid x-y\mid^{n+2s}} dxdy = \int_{\mathbb{R}^n} f(u(y))h(y)dy. \tag{1.3}$$

Moreover, we say that u is a *stable* weak solution of (1.1) if the second variation of the energy is nonnegative, that is, if for every $h \in C_0^\infty(\mathbb{R}^n)$:

$$\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(h(x)-h(y))^2}{\mid x-y\mid^{n+2s}} dxdy - \int_{\mathbb{R}^n} f'(u(y))h^2(y)dy \geq 0. \tag{1.4}$$

We can state now our main result.

Theorem 1.2. *Let u be a stable weak solution of* (1.1) *and $s \in (0,1)$. Then, for every smooth and compactly supported function $\phi \in C_0^\infty(\mathbb{R}^n)$, we have*

$$
\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{\sum_{i=1}^n\left(u_{x_i}(x)-u_{x_i}(y)\right)^2-\left(|\nabla u(x)|-|\nabla u(y)|\right)^2}{|x-y|^{n+2s}}\right)\phi^2(x)dxdy
$$
$$
\leq\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla u(y)|^2\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy. \tag{1.5}
$$

One can consider (1.5) a fractional weighted Poincaré inequality since it allows to control a weighted L^2-norm of a compactly supported function ϕ by a weighted H^s-norm of the same function ϕ.

Remark 1.3. As one can see from the proof of Theorem 1.2, we can consider stable solutions u of more general integro-differential equations, and obtain a corresponding weighted Poincaré-type inequality. More precisely, if we consider a symmetric kernel $K(x,y)$ of differentiability order $s\in(0,1)$ with general possibly nonsmooth coefficients (as considered for instance in [8, 9]) and u a stable solution of

$$
\int_{\mathbb{R}^n}|u(x)-u(y)|^2K(x,y)dy=f(u),
$$

then we can deduce the analogue of inequality (1.5) with $|x-y|^{-n-2s}$ replaced by $K(x,y)$.

Inequality (1.5) can be seen as the fractional analogue of the following inequality which was studied in [15, 24, 25]:

$$
\int_{\mathbb{R}^n\setminus\{\nabla w=0\}}\left(\sum_{i=1}^{n-1}|\nabla w_{x_i}|^2-|\nabla|\nabla w||^2\right)\phi^2dx\leq\int_{\mathbb{R}^n}|\nabla w|^2|\nabla\phi|^2dx, \tag{1.6}
$$

where w is a stable solution to the local equation $\Delta w=f(w)$.

Recently inequality (1.6) has been generalized to other operators. In [17] elliptic operators of the form $\operatorname{div}(A(x)\nabla)$ have been considered. In [23] the authors take in account fractional type operators, but they prove an inequality that is related to the solution of the associated local problem obtained via the Caffarelli-Silvestre extension indeed, see [5]. As a consequence their inequality still involves weighted H^1-norms on the right-hand side.

Here, we are interested in the analogue inequality for solutions u of the nonlocal problem $(-\Delta)^s u = f(u)$ (without considering its s-harmonic extensions) which therefore will involve fractional Sobolev norms.

Moreover, making a particular choice of the function u, we establish some relations between the stability of u and the validity of a fractional Hardy inequality.

We recall now the fractional Hardy inequality with best constant. For fractional Hardy inequalities we refer to [11, 12, 13, 19, 20, 22], and in particular we refer to [19] for the best constant in the case of the entire space, and to [20] for the best constant in general domains.

More precisely, see Theorem 1.1 in [19], there exists a constant C such that: for every $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}} dxdy \geq C \int_{\mathbb{R}^n} |x|^{-2s}\varphi^2(x)dx. \tag{1.7}$$

Moreover the optimal C for which the above inequality holds is given by

$$C_{H,s} := 2\pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+2s}{4}\right)^2}{\Gamma\left(\frac{n-2s}{4}\right)^2} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{|\Gamma(-2s)|}. \tag{1.8}$$

In addition we prove the following result.

Theorem 1.4. *Let $s \in (0,1)$, and $0 < \gamma < 2s$. We have:*

i) $\mid x \mid^\gamma$ *is a stable solution of (1.1) with* $f(\rho) = -\beta(\gamma,s)\rho^{1-\frac{2s}{\gamma}}$, *if and only if there exists a contant C such that s-fractional inequality (1.7) holds;*

ii) *There exists a positive constant η, depending only on n, s, γ*

$$\eta = \eta(n,s,\gamma) = \int_0^{+\infty} \tau^{\gamma+n-2} G(\tau) d\tau, \tag{1.9}$$

where

$$G(\tau) = \int_{|y'|=1} \frac{1-\langle e, y'\rangle}{(1-2\tau\langle e,y'\rangle+\tau^2)^{\frac{n+2s}{2}}} d\mathcal{H}^{n-1}(y'),$$

$e \in \mathbb{R}^n$, $\mid e \mid = 1$, *such that : if* $\mid x \mid^\gamma$ *is a stable solution of (1.1) for* $f(\rho) = -\beta(\gamma,s)\rho^{1-\frac{2s}{\gamma}}$, *then the following weighted fractional*

Hardy ineuality holds: for every $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\eta \int_{\mathbb{R}^n} |x|^{-2s}\phi^2(x)\omega_\gamma(x)dx \\ \leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dx\right)\omega_\gamma(y)dy, \tag{1.10}$$

where $\omega_\gamma(x) = |x|^{2(\gamma-1)}$.

There is a wide literature on Hardy-type inequalities (see, for instance [22] and references therein). Concerning Hardy inequalities of fractional order and their generalizations to weights, we recall the following recent works [11, 12, 13].

To prove Theorem 1.4, we apply inequality (1.5) to a suitable radial function u, which is a stable solution of (1.1) for a certain nonlinear f. The main tool we use is a representation formula for the fractional Laplacian of radial functions, established in [18].

More precisely, in Lemma 3.1 we will establish that the function $u(x) = |x|^\gamma$ is a solution of (1.1) for the nonlinearity $f(u) = -\beta(\gamma, s)u^{1-\frac{2s}{\gamma}}$, where $\beta(\gamma, s)$ is a constant which expression is given in (3.3). Plugging this particular choice of u and f in the definition of stability (see formula (1.4)), we obtain a relation between the stability of $u(x) = |x|^\gamma$ and the validity of a fractional Hardy inequality.

We remark moreover that, in order to obtain further results, our approach could be revisited making use also of some ideas contained in the seminal paper [21] and in the successive develops, see e.g. [6], mainly applied to local linear operators.

Now , we comment on the geometric informations contained in inequality (1.5). The inequality proved in [24], [25] for stable solutions w to semi-linear equations for the classical Laplace operator $\Delta w = -f(w)$ in $\mathbb{R}^n$, can be written, see e.g. [14], as

$$\int_{\mathbb{R}^n\setminus\{\nabla w=0\}}\left(|\nabla w|^2\sum_{i=1}^{n-1}k_i^2 + |\nabla_T|\nabla w||^2\right)\phi^2 \leq \int_{\mathbb{R}^n}|\nabla w|^2|\nabla\phi|^2, \tag{1.11}$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$, where k_i, $i = 1, \dots, n-1$, are the principal curvatures of the level surfaces of the function u and ∇_T denotes the tangential gradient along the same level sets.

In the fractional case we get that (see Corollary 2.1),

$$
\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus\mathcal{C}_0}(\phi^2(x)+\phi^2(y)) \mid \nabla u(x) \mid\mid \nabla u(y) \mid \frac{\mid n(x)-n(y)\mid^2}{\mid x-y\mid^{n+2s}}dxdy\\
&\qquad\le \int_{\mathbb{R}^n\times\mathbb{R}^n}(\mid \nabla u(x)\mid^2+\mid \nabla u(y)\mid^2)\frac{\mid \phi(x)-\phi(y)\mid^2}{\mid x-y\mid^{n+2s}}dxdy,
\end{aligned}
\tag{1.12}
$$

where $n(x)=\frac{\nabla u(x)}{|\nabla u(x)|}$ is the unit normal to the level surface $\{u=c\}$ at each point $x\in\{u=c\}$, where the gradient of u does not vanish and $\mathcal{C}_0=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n\ :\ \nabla u(x)\neq 0,\ \nabla u(y)\neq 0\}$.

We remark that if Γ is a smooth regular path on the level surface $\Sigma=\{u=c\}$ such that $\Gamma'(0)=v$, and v is any unit tangent vector $v\in T_x\Sigma$ where $T_x\Sigma$ is the tangent space at x, then

$$n(\Gamma(t))-n(x)=tk(v)v+o(t)$$

as $t\to 0$, where k is the curvature along the tangent direction v. In particular

$$\mid n(\Gamma(t))-n(x)\mid^2=k(v)^2t^2+o(t^2)$$

as $t\to 0$.

For a nonlocal notion of directional curvatures of a surface we refer to [1]. We observe now that there is a close relation (at least asymptotically as $s\to\frac{1}{2}$) between the quantity $\frac{|n(x)-n(y)|^2}{|x-y|^{n+2s}}$ appearing in our inequality (1.12) and the quantity $\sum_{i=1}^{n-1}k_i^2$ involved in (1.11).

Indeed, see Lemma A.4 in [7], if Σ is a smooth hypersurface of dimension $n-1$, and ν is a smooth choice of the normal vector to Σ. Then

$$\lim_{s\to\frac{1}{2}}(1-2s)\int_\Sigma\frac{(\nu(x)-\nu(y))\cdot\nu(x)}{|x-y|^{n+2s}}dy=\frac{\omega_n}{2}\sum_{k=1}^{n-1}k_i^2(x),$$

where ω_n denote the volume of the unit ball.

Basically, the quantity $\int_\Sigma\frac{(\nu(x)-\nu(y))\cdot\nu(x)}{|x-y|^{n+2s}}dy$ describes how the normal vector varies in an average sense, taking into account also interactions "coming from far", that is interactions with all points $y\in\Sigma$. The aforementioned result tells us that, when $s\to\frac{1}{2}$, that quantity converges, up to a multiplicative constant, to the sum of the square of the principal curvatures.

In the classical case, the geometric inequality (1.11) has been used, see [14, 15] and reference therein, to prove the 1-dimensional symmetry of stable solutions to the semilinear equation $-\Delta u = f(u)$ in low dimensions. We say that a function u is $1-D$ if it depends only on one Euclidean variable, or equivalentely if its level sets are hyperplanes. In [15], by choosing a suitable test funtion ϕ with compact support in a ball of radius R in (1.11) and using an energy estimate for the solution u, the authors proved that the right-hand side of (1.11) tends to 0 for $R \to \infty$, which implies that all the principle curvature k_i of the level sets of u must be identically 0.

For the fractional case, the analogue of this $1-D$ symmetry result for stable solutions of the equation $(-\Delta u)^s = f(u)$ has been proven to be true in dimension 2 for any power $0 < s < 1$ [4, 23] and in dimension 3 for $1/2 \leq s < 1$ [2, 3]. The proofs of all these results make use of the extension established in [5], which allows to study the fractional equation (1.1) by studying a local Neumann problem in the half-space $\mathbb{R}^{n+1}_+$. In [23], the authors used a geometric inequality analogue to (1.11) for the extended problem in the half-space, while in [4, 2, 3] a different approach based on a Liouville type result is used.

One could try to see whether our fractional geometric inequality (1.5) implies $1-D$ symmetry for stable solutions of equation (1.1), at least in low dimensions. This would give an alternative proof of the above mentioned results, without using the Caffarelli-Silvestre extension.

Unfortunately, this is not the case, since in order to deduce $1-D$ symmetry from inequality (1.5) we would need some decay estimates of ∇u, that are not satisfied by the solution of our problem (see Remark 3.3).

2. Proof of Theorem 1.2

We start by a very simple observation due to the linearity of the fractional Laplacian. If u is a solution of (1.1), then its derivatives u_{x_i} satisfies the linearized problem

$$(-\Delta)^s u_{x_i} = f'(u) u_{x_i}. \tag{2.1}$$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. We start with some easy computations which will be useful in the sequel. By symmetry between x and y, we have that the two following identities hold:

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^s u(x)v(x)dx &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))}{|x-y|^{n+2s}}v(x)dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy,\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} v(x)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy \\ &\quad= \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(v^2(x)-v^2(y))}{|x-y|^{n+2s}}dxdy.\end{aligned} \tag{2.3}$$

For the convenience of the reader, we split the proof in three steps.

Step 1. We start by multiplying equation (2.1) by $u_{x_i}\phi^2$ and we get

$$(-\Delta)^s u_{x_i}(u_{x_i}\phi^2) = f'(u)u_{x_i}^2\phi^2. \tag{2.4}$$

Integrating the left-hand side of (2.4) and using (2.2) and (2.3), we deduce that

$$\int_{\mathbb{R}^n} (-\Delta)^s u_{x_i}(x)(u_{x_i}\phi^2)(x)dx$$

$$\stackrel{(2.2)}{=} \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_{x_i}(x)-u_{x_i}(y))\left((u_{x_i}\phi^2)(x)-(u_{x_i}\phi^2)(y)\right)}{|x-y|^{n+2s}}dxdy$$

$$= \quad \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_{x_i}(x)-u_{x_i}(y))\left(u_{x_i}(x)\phi^2(x)-u_{x_i}(y)\phi^2(x)+u_{x_i}(y)\phi^2(x)-u_{x_i}(y)\phi^2(y)\right)}{|x-y|^{n+2s}}$$

$$\otimes\, dxdy$$

$$= \quad \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \phi^2(x)\frac{(u_{x_i}(x)-u_{x_i}(y))^2}{|x-y|^{n+2s}}dxdy$$

$$+\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{u_{x_i}(y)(u_{x_i}(x)-u_{x_i}(y))(\phi^2(x)-\phi^2(y))}{|x-y|^{n+2s}}dxdy$$

$$\stackrel{(2.3)}{=} \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \phi^2(x)\frac{(u_{x_i}(x)-u_{x_i}(y))}{|x-y|^{n+2s}}dxdy$$

$$+\frac{1}{4}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u^2_{x_i}(x)-u^2_{x_i}(y))(\phi^2(x)-\phi^2(y))}{|x-y|^{n+2s}}dxdy. \tag{2.5}$$

We integrate now also the right-hand side of (2.4) end we sum in i to deduce

$$\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \phi^2(x)\frac{\sum_{i=1}^n \left(u^2_{x_i}(x)-u^2_{x_i}(y)\right)^2}{|x-y|^{n+2s}}dxdy+$$
$$+\frac{1}{4}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(|\nabla u(x)|^2-|\nabla u(y)|^2\right)(\phi^2(x)-\phi^2(y))}{|x-y|^{n+2s}}dxdy$$
$$=\int_{\mathbb{R}^n} f'(u)(x)|\nabla u(x)|^2\phi^2(x)dx.$$

Using again (2.3) in the second term on the left-hand side, we conclude

$$\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \phi^2(x)\frac{\sum_{i=1}^n \left(u^2_{x_i}(x)-u^2_{x_i}(y)\right)^2}{|x-y|^{n+2s}}dxdy+$$
$$+\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\nabla u(y)|\left(|\nabla u(x)|-|\nabla u(y)|\right)(\phi^2(x)-\phi^2(y))}{|x-y|^{n+2s}}dxdy$$
$$=\int_{\mathbb{R}^n} f'(u)(x)|\nabla u(x)|^2\phi^2(x)dx. \tag{2.6}$$

Step 2. In this second step, we use the stability of u. We recall that since u is a stable solution of (1.1), we have for any smooth and compactly supported function φ,

$$\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}dxdy-\int_{\mathbb{R}^n}f'(u(x))\varphi^2(x)dx\geq 0, \qquad (2.7)$$

see (1.4). We choose as a test function $\varphi=|\nabla u|\phi$ and we make some computations in the first term of (2.7):

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(|\nabla u(x)|\phi(x)-|\nabla u(y)|\phi(y)\right)^2}{|x-y|^{n+2s}}dxdy\\
&=\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(|\nabla u(x)|\phi(x)-|\nabla u(y)|\phi(x)+|\nabla u(y)|\phi(x)-|\nabla u(y)|\phi(y)\right)^2}{|x-y|^{n+2s}}dxdy\\
&=\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\phi^2(x)\frac{\left(|\nabla u(x)|-|\nabla u(y)|\right)^2}{|x-y|^{n+2s}}dxdy\\
&+\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla u(y)|^2\frac{\left(\phi(x)-\phi(y)\right)^2}{|x-y|^{n+2s}}dxdy\\
&+\int_{\mathbb{R}^n\times\mathbb{R}^n}\phi(x)|\nabla u(y)|\frac{\left(|\nabla u(x)|-|\nabla u(y)|\right)\left(\phi(x)-\phi(y)\right)}{|x-y|^{n+2s}}dxdy\\
&\overset{(2.3)}{=}\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\phi^2(x)\frac{\left(|\nabla u(x)|-|\nabla u(y)|\right)^2}{|x-y|^{n+2s}}dxdy\\
&+\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla u(y)|^2\frac{\left(\phi(x)-\phi(y)\right)^2}{|x-y|^{n+2s}}dxdy\\
&+\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla u(y)|\frac{\left(|\nabla u(x)|-|\nabla u(y)|\right)\left(\phi^2(x)-\phi^2(y)\right)}{|x-y|^{n+2s}}dxdy
\end{aligned} \qquad (2.8)$$

Step 3. Combining together (2.6), (2.7) and (2.8), we conclude

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{\sum_{i=1}^n\left(u_{x_i}(x)-u_{x_i}(y)\right)^2-\left(|\nabla u(x)|-|\nabla u(y)|\right)^2}{|x-y|^{n+2s}}\right)\phi^2(x)\,dxdy\\
&\qquad\leq\int_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla u(y)|^2\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy.
\end{aligned}$$

$\square$

We define now the set

$$\mathcal{C}_0=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n\;:\;\nabla u(x)\neq 0,\;\nabla u(y)\neq 0\}$$

Corollary 2.1. Let u be a stable weak solution of (1.1) and $s \in (0,1)$. Then, for any smooth and compactly supported function $\phi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{C}_0} (\phi^2(x) + \phi^2(y)) \mid \nabla u(x) \mid\mid \nabla u(y) \mid \frac{\mid n(x) - n(y) \mid^2}{\mid x - y \mid^{n+2s}} dxdy$$
$$\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} (\mid \nabla u(x) \mid^2 + \mid \nabla u(y) \mid^2) \frac{\mid \phi(x) - \phi(y) \mid^2}{\mid x - y \mid^{n+2s}} dxdy, \tag{2.9}$$

Proof. By developing the computation in the left hand side of (1.5) we get, for every $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{C}_0$,

$$\begin{aligned} 2\int_{\mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{C}_0} |\nabla u(x)||\nabla u(y)| \left(\frac{1 - \langle \frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\nabla u(y)}{|\nabla u(y)|} \rangle}{|x-y|^{n+2s}} \right) \phi^2(x) dxdy \\ \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla u(y)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dxdy. \end{aligned} \tag{2.10}$$

On the other hand, for every $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{C}_0$,

$$2\left(1 - \langle \frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\nabla u(y)}{|\nabla u(y)|} \rangle\right) = \left| \frac{\nabla u(x)}{|\nabla u(x)|} - \frac{\nabla u(y)}{|\nabla u(y)|} \right|^2.$$

Thus, denoting $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{C}_0$, $n(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$, we get

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{C}_0} |\nabla u(x)||\nabla u(y)| \frac{\mid n(x) - n(y) \mid^2}{|x-y|^{n+2s}} \phi^2(x) dxdy \\ \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla u(y)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dxdy. \end{aligned} \tag{2.11}$$

Now, by symmetry, it is also true that

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{C}_0} |\nabla u(x)||\nabla u(y)| \frac{\mid n(x) - n(y) \mid^2}{|x-y|^{n+2s}} \phi^2(y) dxdy \\ \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla u(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dxdy. \end{aligned} \tag{2.12}$$

Hence, by summing respectively the right hand sides and the left hand sides of the inequality (2.11) and (2.12), we get (2.9). $\square$

3. Proof of Theorem 1.4

It is well known by a straightforward calculation that if u is a radial function, $u(x) = u(|x|)$, then

$$\Delta u(|x|) = u''(|x|) + \frac{n-1}{|x|} u'(|x|).$$

In particular if $u(r) = r^\gamma$, then

$$\Delta u(|x|) = \left(\gamma(\gamma-1) + (n-1)\gamma\right) |x|^{\gamma-2} = \gamma(\gamma-2+n)|x|^{\gamma-2}.$$

As a consequence,

if $n > 2$, whenever $\gamma \geq 0$ or $\gamma \leq -n+2$, $u(x) = c \mid x \mid^\gamma$ is sub-harmonic for $c > 0$. In particular for every $\gamma > 0$,

$$\fint_{\partial B(x,t)} |y|^\gamma d\mathcal{H}^{n-1}(y) \geq |x|^\gamma.$$

If $n = 2$ for every $\gamma \in \mathbb{R}$, $u(x) = c \mid x \mid^\gamma$ is sub-harmonic, for $c > 0$. As a consequence, recalling the characterisation remarked in [16] of the fractional operator, it follows from the representation

$$-(-\Delta)^s u(x) = n\omega_n \int_0^\infty \left(\fint_{\partial B(x,r)} (u(y) - u(x)) d\mathcal{H}^{n-1}(y) \right) r^{-1-2s} dr$$

that $|x|^\gamma$ is also $s-$subharmonic that is

$$-(-\Delta)^s |x|^\gamma \geq 0, \tag{3.1}$$

whenever

$$\int_{\mathbb{R}^n \setminus B_r(x)} \frac{|y|^\gamma}{|x-y|^{n+2s}} dy < +\infty,$$

that is whenever $n + 2s - n + 1 - \gamma > 1$, that is when $\gamma < 2s$. We recall the result in [18], it has been proved that for every radial function u

$$\begin{aligned} &-(-\Delta)^s u(r) \\ &= r^{-2s} \int_1^{+\infty} \left(u(r\tau) - u(r) + (u(\frac{r}{\tau}) - u(r))\tau^{-n+2s} \right) \tau(\tau^2-1)^{-1-2s} H(\tau) d\tau, \end{aligned} \tag{3.2}$$

where

$$H(\tau) := 2\pi\alpha_n \int_0^\pi \sin^{n-2}\theta \frac{(\sqrt{\tau^2 - \sin^2\theta} + \cos\theta)^{1+2s}}{\sqrt{\tau^2 - \sin^2\theta}} d\theta,$$

where $\alpha_n = \frac{\pi^{\frac{n-3}{2}}}{\Gamma(\frac{n-1}{2})}$. One can see that H is a positive continuous function on $[1, +\infty)$ with $H(\tau) \sim \tau^{2s}$ as $\tau \to +\infty$.

Using this representation formula, we deduce the following result.

Lemma 3.1. *Let* $0 < \gamma < 2s$, $s \in (0,1)$ *and let* $f(\rho) = -\beta(\gamma, s)\rho^{1-\frac{2s}{\gamma}}$, *where*

$$\beta(\gamma, s) = \int_1^{+\infty} \left(\tau^\gamma - 1 + ((\frac{1}{\tau})^\gamma - 1)\tau^{-n+2s}\right) \tau(\tau^2 - 1)^{-1-2s} H(\tau) d\tau. \tag{3.3}$$

Then $v = | x |^\gamma$ *is solution of (1.1) that is, in particular,*

$$-(-\Delta)^s v(r) = \beta(\gamma, s) v^{1-\frac{2s}{\gamma}} > 0.$$

Proof.

$$\begin{aligned} &-(-\Delta)^s u(r) \\ &= r^{-2s} \int_1^{+\infty} \left((r\tau)^\gamma - r^\gamma + ((\frac{r}{\tau})^\gamma - r^\gamma)\tau^{-n+2s}\right) \tau(\tau^2 - 1)^{-1-2s} H(\tau) d\tau \\ &= r^{-2s+\gamma} \int_1^{+\infty} \left(\tau^\gamma - 1 + ((\frac{1}{\tau})^\gamma - 1)\tau^{-n+2s}\right) \tau(\tau^2 - 1)^{-1-2s} H(\tau) d\tau \\ &= \beta(\gamma, s) r^{-2s+\gamma}, \end{aligned} \tag{3.4}$$

where

$$\beta(\gamma, s) = \int_1^{+\infty} \left(\tau^\gamma - 1 + ((\frac{1}{\tau})^\gamma - 1)\tau^{-n+2s}\right) \tau(\tau^2 - 1)^{-1-2s} H(\tau) d\tau. \tag{3.5}$$

In particular

$$\beta(\gamma, s) = \int_1^{+\infty} (\tau^\gamma - 1) \left(1 - \tau^{-n+2s-\gamma}\right) \tau(\tau^2 - 1)^{-1-2s} H(\tau) d\tau. \tag{3.6}$$

Hence, if $-n + 2s - \gamma \leq 0$, and $\gamma \geq 0$, we deduce that $\beta(\gamma, s) > 0$. We remark also that if $v(|x|) = |x|^\gamma$, then keeping in mind (3.1) or previous observation and (3.4), we get

$$-(-\Delta)^s v(r) = \beta(\gamma, s) v^{1-\frac{2s}{\gamma}} \geq 0.$$

Thus v is solution of

$$(-\Delta)^s v(r) = f(v(r)) \leq 0,$$

where

$$f(v) = -\beta(\gamma, s)v^{1-\frac{2s}{\gamma}}.$$

□

We need another intermediate result summarised in the following lemma before beginning the proof of Theorem 1.4.

Lemma 3.2. *There exists a constant $\eta(n,s) > 0$ such that:*

$$\int_{\mathbb{R}^n} |x|^{\gamma-1}|y|^{\gamma-1}\frac{1-\langle\frac{x}{|x|},\frac{y}{|y|}\rangle}{|x-y|^{n+2s}}dy = \eta(n,s)|x|^{2(\gamma-1-s)}, \tag{3.7}$$

where

$$\eta = \eta(n,s,\gamma) = \int_0^{+\infty} \tau^{\gamma+n-2}G(\tau)d\tau,$$

and

$$G(\tau) = \int_{|y'|=1} \frac{1-\langle e, y'\rangle}{(1-2\tau\langle e,y'\rangle+\tau^2)^{\frac{n+2s}{2}}}d\mathcal{H}^{n-1}(y'),$$

$e \in \mathbb{R}^n$, $\mid e \mid= 1$, *is independent of* e.

Proof. First we remark that the integral depends only on $|x|$. Indeed let $x = Qx'$ where Q is a unitary matrix, that is $|\mathrm{det}Q| = 1$. Then after a change of variables,

$$\int_{\mathbb{R}^n} |Qx'|^{\gamma-1}|y|^{\gamma-1}\frac{1-\langle\frac{Qx'}{|Qx'|},\frac{y}{|y|}\rangle}{|Qx'-y|^{n+2s}}dy = \int_{\mathbb{R}^n} |Qx'|^{\gamma-1}|Q'y|^{\gamma-1}\frac{1-\langle\frac{Qx'}{|Qx'|},\frac{Qy'}{|Qy'|}\rangle}{|Qx'-Qy'|^{n+2s}}dy$$
$$= \int_{\mathbb{R}^n} |x'|^{\gamma-1}|y'|^{\gamma-1}\frac{1-\langle\frac{Qx'}{|Qx'|},\frac{Qy'}{|Qy'|}\rangle}{|x'-y'|^{n+2s}}dy = \int_{\mathbb{R}^n} |x'|^{\gamma-1}|y'|^{\gamma-1}\frac{1-\langle\frac{x'}{|x'|},Q^T\frac{Qy'}{|y'|}\rangle}{|x'-y'|^{n+2s}}dy$$
$$= \int_{\mathbb{R}^n} |x'|^{\gamma-1}|y'|^{\gamma-1}\frac{1-\langle\frac{x'}{|x'|},\frac{y'}{|y'|}\rangle}{|x'-y'|^{n+2s}}dy. \tag{3.8}$$

Hence passing in polar coordinates we get

$$\int_{\mathbb{R}^n} |x|^{\gamma-1}|y|^{\gamma-1}\frac{1-\langle\frac{x}{|x|},\frac{y}{|y|}\rangle}{|x-y|^{n+2s}}dy = |x|^{\gamma-1}\int_0^{+\infty} \rho^{\gamma+n-2}\int_{|y'|=1}\frac{1-\langle\frac{x}{|x|},y'\rangle}{|x-\rho y'|^{n+2s}}d\mathcal{H}^{n-1}(y')d\rho$$
$$= |x|^{\gamma-1}\int_0^{+\infty} \rho^{\gamma+n-2}\int_{|y'|=1}\frac{1-\langle\frac{x}{|x|},y'\rangle}{||x|\frac{x}{|x|}-\rho y'|^{n+2s}}d\mathcal{H}^{n-1}(y')d\rho \tag{3.9}$$

The function

$$F(x,\rho)=\int_{|y'|=1}\frac{1-\langle\frac{x}{|x|},y'\rangle}{||x|\frac{x}{|x|}-\rho y'|^{n+2s}}d\mathcal{H}^{n-1}(y')$$

depends only on $|x|$. Indeed, for every $e\in\{|x|=1\}$

$$F(x,\rho)=F(|x|,\rho)=\int_{|y'|=1}\frac{1-\langle e,y'\rangle}{||x|e-\rho y'|^{n+2s}}d\mathcal{H}^{n-1}(y')$$

On the other hand,

$$|x-y|^{n+2s}=\left(|x|^2+2\langle x,y\rangle+|y|^2\right)^{\frac{n+2s}{2}},$$

thus

$$\begin{aligned}F(|x|,\rho)&=\int_{|y'|=1}\frac{1-\langle e,y'\rangle}{(|x|^2-2\rho|x|\langle e,y'\rangle+\rho^2)^{\frac{n+2s}{2}}}d\mathcal{H}^{n-1}(y')\\&=|x|^{-n-2s}\int_{|y'|=1}\frac{1-\langle e,y'\rangle}{(1-2\frac{\rho}{|x|}\langle e,y'\rangle+(\frac{\rho}{|x|})^2)^{\frac{n+2s}{2}}}d\mathcal{H}^{n-1}(y')=|x|^{-n-2s}G(\frac{\rho}{|x|}),\end{aligned} \tag{3.10}$$

where

$$G(s)=\int_{|y'|=1}\frac{1-\langle e,y'\rangle}{(1-2s\langle e,y'\rangle+s^2)^{\frac{n+2s}{2}}}d\mathcal{H}^{n-1}(y')$$

As a consequence

$$\begin{aligned}&\int_{\mathbb{R}^n}|x|^{\gamma-1}|y|^{\gamma-1}\frac{1-\langle\frac{x}{|x|},\frac{y}{|y|}\rangle}{|x-y|^{n+2s}}dy=|x|^{\gamma-1-n-2s}\int_0^{+\infty}\rho^{\gamma+n-2}G(\frac{\rho}{|x|})d\rho\\&=|x|^{\gamma-1-n-2s+\gamma+n-1}\int_0^{+\infty}\tau^{\gamma+n-2}G(\tau)d\tau=|x|^{2\gamma-2-2s}\int_0^{+\infty}\tau^{\gamma+n-2}G(\tau)d\tau.\end{aligned} \tag{3.11}$$

□

Now we are in position to give the proof of Theorem 1.4.

Proof of Theorem 1.4. i) In order to prove part i), it is enough to write the stability condition (1.4) for the specific choice $v(x)=|x|^\gamma$ and $f(v)=-\beta(\gamma,s)v^{1-\frac{2s}{\gamma}}$. Indeed, we have that

$$f'(v)=-(1-\frac{2s}{\gamma})\beta(\gamma,s)v^{-\frac{2s}{\gamma}},$$

and therefore v is stable if and only if

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2s}}dxdy + \left(1-\frac{2s}{\gamma}\right)\beta(\gamma,s)\int_{\mathbb{R}^n}\frac{\phi^2(s)}{|x|^{2s}}dx \geq 0,$$

that is, if and only if, the fractional Hardy inequality holds. We observe that here, we don't obtain the optimal constant $C_{H,s}$ indeed $1-\frac{2s}{\gamma}\to 0$ as $\gamma\to 2s$. Using inequality (1.7) and comparing the optimal constant $C_{H,s}$ given in (1.8) with the constant appearing above we deduce that $|x|^\gamma$ is a stable solution of (1.1) for any γ satisfying

$$2s\cdot\frac{\beta(\gamma,s)}{\beta(\gamma,s)+C_{H,s}} \leq \gamma < 2s. \tag{3.12}$$

ii) Suppose now that $v(x)=|x|^\gamma$ is a stable solution (this is the case for γ satisfying (3.12) above). We apply inequality (1.5) to this particular choice of v. We calculate $\nabla v = \gamma|x|^{\gamma-2}x$. Thus, inserting ∇v in the main inequality (1.5) we get:

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n}&\left(\frac{\sum_{i=1}^n\left(\gamma|x|^{\gamma-2}x_i-\gamma|y|^{\gamma-2}y_i\right)^2-\left(\gamma|x|^{\gamma-1}-\gamma|y|^{\gamma-1}\right)^2}{|x-y|^{n+2s}}\right)\phi^2(x)dxdy\\ &\leq \int_{\mathbb{R}^n\times\mathbb{R}^n}\gamma^2|y|^{2(\gamma-1)}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy.\end{aligned} \tag{3.13}$$

that is, developing the calculation,

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n}&\left(\frac{\sum_{i=1}^n\left(|x|^{\gamma-2}x_i-|y|^{\gamma-2}y_i\right)^2-\left(|x|^{\gamma-1}-|y|^{\gamma-1}\right)^2}{|x-y|^{n+2s}}\right)\phi^2(x)dxdy\\ &\leq \int_{\mathbb{R}^n\times\mathbb{R}^n}|y|^{2(\gamma-1)}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy,\end{aligned} \tag{3.14}$$

that is

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n}&\frac{|x|^{\gamma-1}|y|^{\gamma-1}-|x|^{\gamma-2}|y|^{\gamma-2}\langle x,y\rangle}{|x-y|^{n+2s}}\phi^2(x)dxdy\\ &\leq \int_{\mathbb{R}^n\times\mathbb{R}^n}|y|^{2(\gamma-1)}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy,\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} &|x|^{\gamma-1}|y|^{\gamma-1}\frac{1-\langle\frac{x}{|x|},\frac{y}{|y|}\rangle}{|x-y|^{n+2s}}\phi^2(x)dxdy\\ &\leq \int_{\mathbb{R}^n\times\mathbb{R}^n}|y|^{2(\gamma-1)}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy.\end{aligned}\tag{3.16}$$

Let us consider now

$$\int_{\mathbb{R}^n}|x|^{\gamma-1}|y|^{\gamma-1}\frac{1-\langle\frac{x}{|x|},\frac{y}{|y|}\rangle}{|x-y|^{n+2s}}dy.$$

Keeping in mind the result stated in the previous Lemma 3.2 we get:

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} &|x|^{\gamma-1}|y|^{\gamma-1}\frac{1-\langle\frac{x}{|x|},\frac{y}{|y|}\rangle}{|x-y|^{n+2s}}\phi^2(x)dxdy\\ &=\int_{\mathbb{R}^n}|x|^{\gamma-1}\left(\int_{\mathbb{R}^n}|y|^{\gamma-1}\frac{1-\langle\frac{x}{|x|},\frac{y}{|y|}\rangle}{|x-y|^{n+2s}}dy\right)\phi^2(x)dx\\ &=\eta(n,s)\int_{\mathbb{R}^n}|x|^{2(\gamma-1-s)}\phi^2(x)dx\end{aligned}$$

Analogously for the right hand side of (3.16) we get

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n}&|y|^{2(\gamma-1)}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy\\ &=\int_{\mathbb{R}^n}|y|^{2(\gamma-1)}\left(\int_{\mathbb{R}^n}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dx\right)dy.\end{aligned}$$

As a consequence, recalling now (3.16), we deduce that

$$\eta(n,s)\int_{\mathbb{R}^n}|x|^{2(\gamma-1-s)}\phi^2(x)dx\leq\int_{\mathbb{R}^n}|y|^{2(\gamma-1)}\left(\int_{\mathbb{R}^n}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dx\right)dy.\tag{3.17}$$

□

Remark 3.3. As explained in the introduction, inequality (1.11) has been used in [14, 15], to prove $1-D$ symmetry of stable solutions to the semilinear equation $-\Delta u=f(u)$ in $\mathbb{R}^n$. More precisely, choosing the test function

$$\phi_R(x)=\begin{cases}1 & x\in B(0,\sqrt{R})\\ 0 & x\in\mathbb{R}^n\setminus B(0,R)\\ \frac{2\log\frac{R}{|x|}}{\log R}, & B(0,R)\setminus B(0,\sqrt{R}).\end{cases}$$

and using an energy estimate for the solution u, after passing to the limit $R \to +\infty$, one can deduce that all the principal cuvature k_i of the level sets of u must vanish. In our case, if we try to insert the same function ϕ (which seems actually the best choice for the test function) in inequality (1.5), in order to conclude $1-D$ symmetry of stable solutions to the equation $(-\Delta)^s u = f(u)$, we would need the solution to satisfy some gradient decay estimates which do not hold in general.

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E.C., Wierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin (Germany)

E-mail address: `cinti@wias-berlin.de`

F.F., Dipartimento di Matematica dell'Università di Bologna, Piazza di Porta S. Donato, 5, 40126 Bologna, Italy

E-mail address: `fausto.ferrari@unibo.it`