Trace theorems for functions of bounded variation in metric spaces *

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Abstract

In this paper we show existence of traces of functions of bounded variation on the boundary of a certain class of domains in metric measure spaces equipped with a doubling measure supporting a 1-Poincaré inequality, and obtain L^1 estimates of the trace functions. In contrast with the treatment of traces given in other papers on this subject, the traces we consider do not require knowledge of the function in the exterior of the domain. We also establish a Maz'ya-type inequality for functions of bounded variation that vanish on a set of positive capacity.

1 Introduction

The Dirichlet problem for functions of least gradient on a domain Ω is to find a function u that minimizes the energy $\int_\Omega |\nabla u|$ amongst the class of all Sobolev functions with prescribed boundary values on $\partial\Omega$. In order to make sense of the problem, one needs to know how to extend the function

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u (which is a priori defined only on Ω) to $\partial\Omega$, and one needs to know for which boundary data the problem is solvable. The focus of the current paper is to study the issue of how a function in the class $\mathrm{BV}(\Omega)$ can be extended to $\partial\Omega$, that is, whether it has a *trace* on $\partial\Omega$; the second question needs very strong geometric conditions on Ω, even in the Euclidean setting (see for example [15]), and will not be addressed in the present paper.

In the classical Euclidean setting, a standard result is that if the boundary $\partial\Omega$ can be presented locally as a Lipschitz graph, then the trace of a BV function exists. Classical treatments of boundary traces of BV functions can be found in e.g. [2, Chapter 3], [15, Chapter 2], and [5], and similar results for Carnot groups are given in [39].

In the setting of general metric measure spaces, where the standard assumptions are a doubling measure and a Poincaré inequality, traces of BV functions have only been studied recently. In [19], results on traces are obtained by making rather strong assumptions on the extendability of BV functions from the domain to the whole space. These extendability properties are satisfied by *uniform* domains, which are a natural generalization of Euclidean domains with Lipschitz boundaries. In [31], results on traces are obtained in more general domains but with stronger assumptions on the metric space. In these papers, the trace of a function in $\mathrm{BV}(\Omega)$ is influenced by how the function is extended outside Ω. In this paper, we prove the existence of traces of BV functions with fewer assumptions on the domain than is standard, and without referring to the behavior of the function in the exterior of the domain. In particular, our results hold for domains with various types of cusps, in which either uniformity or Poincaré inequalities are violated.

There are four main results in this paper, Theorem 3.4, Theorem 5.5, Theorem 6.9, and Theorem 7.2. The first two deal with traces of general BV functions on a certain class of domains, and the last two deal with BV functions with zero trace.

The structure of the paper is as follows. In Section 2 we give the necessary background definitions used throughout the paper. In Section 3 we propose a notion of traces of BV functions on an open subset Ω of X, and in the first main theorem of the paper, Theorem 3.4, we show that if the restriction of the measure to Ω is also doubling and supports a $(1,1)$-Poincaré inequality, then every function in $\mathrm{BV}(\Omega)$ has a trace on $\partial\Omega$, well-defined up to sets of $\mathcal{H}$-measure zero. In Section 4 the necessity of the hypotheses imposed on Ω in Theorem 3.4 is discussed by means of examples, and in Corollary 4.3 we describe how to relax some of these hypotheses.

The goal of Section 5 is to demonstrate that under additional geometric assumptions on the boundary of Ω (regularity of the $\mathcal{H}$-measure on $\partial\Omega$), the traces of functions in $\mathrm{BV}(\Omega)$ lie in $L^1(\partial\Omega, \mathcal{H})$. This result is given in Theorem 5.5. To provide L^1 estimates for the trace, we also prove a BV extension property of domains with such a geometric boundary; this is Proposition 5.2. Section 6 is devoted to the study of BV functions in Ω with zero trace on $\partial\Omega$. In particular, we show that if $\mathcal{H}(\partial\Omega)$ is finite for an open set $\Omega \subset X$, then every function in $\mathrm{BV}(\Omega)$ with zero trace on $\partial\Omega$ can be extended by zero to $X \setminus \Omega$ without increasing its BV energy (Theorem 6.1), and that such functions can be approximated in $\mathrm{BV}(X)$ by BV functions with compact support in Ω (Theorem 6.9). In Section 7 we prove a Maz'ya-type Sobolev inequality for functions that vanish on a set of positive capacity (Theorem 7.2). To prove Theorem 6.9 and to study certain locally Lipschitz approximations of functions in $\mathrm{BV}(\Omega)$, we use Whitney type decompositions of Ω and discrete convolutions. For a more substantial description of these tools, see also the upcoming paper [32].

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2 Preliminaries

In this section we introduce the necessary definitions and assumptions.

In this paper, (X, d, μ) is a complete metric space equipped with a Borel regular outer measure μ. The measure is assumed to be doubling, meaning that there exists a constant $C_d > 0$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B = B(x, r)$ with center $x \in X$ and radius $r > 0$. For a ball $B = B(x, r)$ we will, for brevity, sometimes use the notation $\tau B = B(x, \tau r)$, for $\tau > 0$. Note that in a metric space, a ball does not necessarily have a unique center and radius, but whenever we use the above abbreviation we will consider balls whose center and radii have been specified.

By iterating the doubling condition, we obtain that there are constants $C \geq 1$ and $Q > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C^{-1} \left(\frac{r}{R}\right)^Q \tag{2.1}$$

for every $0 < r \leq R$ and $y \in B(x,R)$. See [8] for a proof of this.

In general, $C \geq 1$ will denote a constant whose value is not necessarily the same at each occurrence. When we want to specify that a certain constant depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$. Unless otherwise specified, all constants only depend on the doubling constant C_d and the constants C_P, λ associated with the Poincaré inequality defined later.

We recall that a complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define e.g. $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ as the space of functions that are Lipschitz in every $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that Ω' is open and that $\overline{\Omega'}$ is a compact subset of Ω.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R(A) := \inf\left\{\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} :\, A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\, r_i \leq R\right\}.$$

The codimension 1 Hausdorff measure of a set $A \subset X$ is

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A). \tag{2.2}$$

The (topological) boundary ∂E of a set $E \subset X$ is defined as usual. The measure theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ for which both E and its complement have positive upper density, i.e.

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r \to 0^+} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} > 0.$$

A curve is a rectifiable continuous mapping from a compact interval to X, and is usually denoted by the symbol γ. A nonnegative Borel function g

on X is an upper gradient of an extended real-valued function u on X if for all curves γ on X, we have

$$|u(x) - u(y)| \le \int_\gamma g \, ds \tag{2.3}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. Here x and y are the end points of γ. Upper gradients were originally introduced in [25].

If g is a nonnegative μ-measurable function on X and (2.3) holds for 1-almost every curve, we say that g is a 1-weak upper gradient of u. By saying that (2.3) holds for 1-almost every curve we mean that it fails only for a curve family with zero 1-modulus. A family Γ of curves is of zero 1-modulus if there is a nonnegative Borel function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho\, ds$ is infinite.

We consider the following norm

$$\|u\|_{N^{1,1}(X)} := \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

with the infimum taken over all upper gradients g of u. The Newton-Sobolev space is defined as

$$N^{1,1}(X) := \{u : \ \|u\|_{N^{1,1}(X)} < \infty\}/\sim,$$

where the equivalence relation $\sim$ is given by $u \sim v$ if and only if

$$\|u - v\|_{N^{1,1}(X)} = 0.$$

Similarly, we can define $N^{1,1}(\Omega)$ for any open set $\Omega \subset X$. The space of Newton-Sobolev functions with zero boundary values is defined as

$$N_0^{1,1}(\Omega) := \{v|_\Omega : \ v \in N^{1,1}(X) \text{ and } v = 0 \text{ in } X \setminus \Omega\},$$

i.e. it is the subclass of $N^{1,1}(\Omega)$ consisting of those functions that can be zero extended to the whole space as Newton-Sobolev functions. For more on Newton-Sobolev spaces, we refer to [38], [22], or [8].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [36]. For $u \in L^1_{\mathrm{loc}}(X)$, we define the total variation of u as

$$\|Du\|(X) := \inf \left\{ \liminf_{i\to\infty} \int_X g_{u_i}\, d\mu : \ u_i \in \mathrm{Lip}_{\mathrm{loc}}(X),\, u_i \to u \text{ in } L^1_{\mathrm{loc}}(X) \right\},$$

where each g_{u_i} is an upper gradient of u_i. We say that a function $u \in L^1(X)$ is *of bounded variation*, and denote $u \in \mathrm{BV}(X)$, if $\|Du\|(X) < \infty$. A measurable set $E \subset X$ is said to be of *finite perimeter* if $\|D\chi_E\|(X) < \infty$. By replacing X with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. The BV norm is given by

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega). \tag{2.4}$$

It was shown in [36, Theorem 3.4] that for $u \in \mathrm{BV}(X)$, $\|Du\|$ is the restriction to the class of open sets of a finite Radon measure defined on the class of all subsets of X. This outer measure is obtained from the map $\Omega \mapsto \|Du\|(\Omega)$ on open sets $\Omega \subset X$ via the standard Carathéodory construction. Thus, for an arbitrary set $A \subset X$,

$$\|Du\|(A) := \inf\big\{\|Du\|(\Omega) : \, \Omega \supset A,\, \Omega \subset X \text{ is open}\big\}.$$

Similarly, if $u \in \mathrm{BV}(\Omega)$, then $\|Du\|(\cdot)$ is a finite Radon measure on Ω. We also denote the perimeter of a set E in Ω by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

We have the following coarea formula from [36, Proposition 4.2]: if $F \subset X$ is a Borel set and $u \in \mathrm{BV}(X)$, then

$$\|Du\|(F) = \int_{-\infty}^{\infty} P(\{u > t\}, F)\, dt. \tag{2.5}$$

If F is open, this holds for every $u \in L^1_{\mathrm{loc}}(F)$.

We will assume throughout the paper that X supports the following $(1,1)$-Poincaré inequality: there are constants $C_P > 0$ and $\lambda \ge 1$ such that for every ball $B(x,r)$, for every locally integrable function u, and for every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r \fint_{B(x,\lambda r)} g\, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u\, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u\, d\mu.$$

By approximation, we get the following $(1,1)$-Poincaré inequality for BV functions. There exists $C > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that for every ball $B(x,r)$ and every $u \in L^1_{\mathrm{loc}}(X)$, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\,d\mu \le Cr\frac{\|Du\|(B(x,\lambda r))}{\mu(B(x,\lambda r))}. \tag{2.6}$$

Given an open set $\Omega \subset X$, we can consider it as a metric space in its own right, equipped with the metric inherited from X and the restriction of μ to subsets of Ω. This restriction is a Radon measure on Ω, see [22, Lemma 2.3.15]. We say that μ is doubling on Ω if the restriction of μ to subsets of Ω is doubling, that is, if there is a constant $C \ge 1$ such that

$$0 < \mu(B(x,2r)\cap\Omega) \le C\mu(B(x,r)\cap\Omega) < \infty$$

for every $x \in \Omega$ and $r > 0$. Similarly we can require Ω to support a $(1,1)$-Poincaré inequality.

Given a set $E \subset X$ of locally finite perimeter, for $\mathcal{H}$-a.e. $x \in \partial^* E$ we have

$$\gamma \le \liminf_{r\to 0}\frac{\mu(E\cap B(x,r))}{\mu(B(x,r))} \le \limsup_{r\to 0}\frac{\mu(E\cap B(x,r))}{\mu(B(x,r))} \le 1-\gamma \tag{2.7}$$

where $\gamma \in (0,1/2]$ only depends on the doubling constant and the constants in the Poincaré inequality, see [1, Theorem 5.4]. We denote the collection of all points x that satisfy (2.7) by E_γ. For a Borel set $F \subset X$ and a set $E \subset X$ of finite perimeter, we know that

$$\|D\chi_E\|(F) = \int_{\partial^* E\cap F}\theta_E\,d\mathcal{H}, \tag{2.8}$$

where $\partial^* E$ is the measure-theoretic boundary of E and $\theta_E : X \to [\alpha, C_d]$, with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [3, Theorem 4.6].

In the metric setting it is not known, in general, whether the condition $\mathcal{H}(\partial^* E) < \infty$ for a μ-measurable set $E \subset X$ implies that $P(E,X) < \infty$. We say that X supports a *strong relative isoperimetric inequality* if this is true, see [27] or [29] for more on this question.

The jump set of $u \in \mathrm{BV}(X)$ is defined as

$$S_u := \{x \in X : u^\wedge(x) < u^\vee(x)\},$$

where $u^\wedge(x)$ and $u^\vee(x)$ are the lower and upper approximate limits of u defined by

$$u^\wedge(x) := \sup\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0^+} \frac{\mu(B(x,r)\cap\{u<t\})}{\mu(B(x,r))} = 0\right\} \tag{2.9}$$

and

$$u^\vee(x) := \inf\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0^+} \frac{\mu(B(x,r)\cap\{u>t\})}{\mu(B(x,r))} = 0\right\}. \tag{2.10}$$

We also set

$$\widetilde{u} := \frac{u^\wedge + u^\vee}{2}. \tag{2.11}$$

By [3, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set $\Omega \subset X$ and $u \in \mathrm{BV}(\Omega)$, we have

$$\begin{aligned} \|Du\|(\Omega) &= \|Du\|^a(\Omega) + \|Du\|^s(\Omega) \\ &= \|Du\|^a(\Omega) + \|Du\|^c(\Omega) + \|Du\|^j(\Omega) \\ &= \int_\Omega a\,d\mu + \|Du\|^c(\Omega) + \int_{\Omega\cap S_u}\int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\,dt\,d\mathcal{H}(x) \end{aligned} \tag{2.12}$$

where $a \in L^1(\Omega)$ is the density of the absolutely continuous part and the functions $\theta_{\{u>t\}}$ are as in (2.8). From this decomposition it also follows that $\mathcal{H}$ is a σ-finite measure on S_u.

3 Traces of BV functions

We give the following definition for the boundary trace, or trace for short, of a function defined on an open set.

Definition 3.1. Let $\Omega \subset X$ be an open set and let u be a μ-measurable function on Ω. A function $Tu : \partial\Omega \to \mathbb{R}$ is the trace of u if for $\mathcal{H}$-almost every $x \in \partial\Omega$ we have

$$\lim_{r\to 0^+} \fint_{\Omega\cap B(x,r)} |u - Tu(x)|\,d\mu = 0. \tag{3.1}$$

Example 3.2. Consider $X = \mathbb{C} = \mathbb{R}^2$, and set

$$\Omega = B(0,1) \setminus \{z = (x_1, x_2) : \, x_1 > 0, \, x_2 = 0\},$$

that is, the slit disk, and let $u(z) := \operatorname{Arg}(z)$. The function u does not have a trace at any boundary point in the slit, but we do have

$$\lim_{r \to 0^+} \fint_{\Omega \cap B(z,r)} u \, d\mathcal{L}^2 = \pi$$

for every $z = (x_1, 0)$ with $0 < x_1 < 1$. For this reason, it is crucial that we define traces by requiring the stronger condition (3.1).

We start by showing that for sufficiently regular domains, BV functions can be extended from the domain to its closure, which we consider as a metric space in its own right. In the following, we define $\overline{\mu}$ as the zero extension of μ from Ω to $\overline{\Omega}$, that is, for $A \subset \overline{\Omega}$ we set $\overline{\mu}(A) = \mu(A \cap \Omega)$. By [22, Lemma 2.3.20] we know that $\overline{\mu}$ is a Borel regular outer measure on $\overline{\Omega}$. Also, the zero extension of any μ-measurable function on Ω to either $\overline{\Omega}$ or the whole space X is μ-measurable, see [22, Lemma 2.3.22].

Proposition 3.3. *Assume that Ω is a bounded open set that supports a $(1,1)$-Poincaré inequality, and μ is doubling on Ω. Equip the closure $\overline{\Omega}$ with $\overline{\mu}$. If $u \in \mathrm{BV}(\Omega)$, then the zero extension of u to $\overline{\Omega}$, denoted by $\overline{u}$, satisfies $\|\overline{u}\|_{\mathrm{BV}(\overline{\Omega})} = \|u\|_{\mathrm{BV}(\Omega)}$ and thus $\|D\overline{u}\|(\partial\Omega) = 0$.*

Proof. Clearly $\|u\|_{L^1(\Omega)} = \|\overline{u}\|_{L^1(\overline{\Omega})}$. Since Ω is a metric space with a doubling measure supporting a $(1,1)$-Poincaré inequality, we know that Lipschitz functions are dense in $N^{1,1}(\Omega)$, see [8, Theorem 5.1]. Thus for $u \in \mathrm{BV}(\Omega)$ we have a sequence of Lipschitz functions u_i on Ω with $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$\|Du\|(\Omega) = \liminf_{i \to \infty} \int_\Omega g_{u_i} \, d\mu,$$

where each g_{u_i} is an upper gradient of u_i. By using the fact that Ω is bounded and by considering truncations of u, we can assume that in fact $u_i \to u$ in $L^1(\Omega)$. We can extend every Lipschitz function u_i on Ω to a Lipschitz function on $\overline{\Omega}$, still denoted by u_i. Then $u_i \to \overline{u}$ in $L^1(\overline{\Omega})$, and

$$\|Du\|(\Omega) = \liminf_{i \to \infty} \int_\Omega g_{u_i} \, d\mu = \liminf_{i \to \infty} \int_{\overline{\Omega}} g_{u_i} \, d\overline{\mu} \geq \|D\overline{u}\|(\overline{\Omega}).$$

The last inequality follows from the fact that the zero extension of g_{u_i} to $\partial\Omega$ is a 1-weak upper gradient of u_i in $\overline{\Omega}$, by [10, Lemma 5.11]. Thus we have $\overline{u} \in \mathrm{BV}(\overline{\Omega})$ with $\|D\overline{u}\|(\overline{\Omega}) = \|Du\|(\Omega)$, so it follows that $\|D\overline{u}\|(\partial\Omega) = 0$. $\square$

Recall the definition of the codimension 1 Hausdorff measure $\mathcal{H}$ from (2.2). We denote by $\overline{\mathcal{H}}$ the codimension 1 Hausdorff measure in the space $\overline{\Omega}$, defined with respect to the measure $\overline{\mu}$.

We say that an open set Ω satisfies a *measure density condition* if there is a constant $c_m > 0$ such that

$$\mu(B(x,r) \cap \Omega) \geq c_m \mu(B(x,r)) \tag{3.2}$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$ and every $r \in (0, \mathrm{diam}(\Omega))$.

Theorem 3.4. *Let Ω be a bounded open set that supports a $(1,1)$-Poincaré inequality, and μ be doubling in Ω. Then there is a linear trace operator T on $\mathrm{BV}(\Omega)$ such that given $u \in \mathrm{BV}(\Omega)$, for $\overline{\mathcal{H}}$-a.e. $x \in \partial\Omega$ we have*

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - Tu(x)|^{Q/(Q-1)}\, d\mu = 0.$$

If Ω also satisfies the measure density condition (3.2)*, the above holds for $\mathcal{H}$-a.e. $x \in \partial\Omega$.*

Proof. By [22, Lemma 7.2.3] we know that $\overline{\Omega}$ equipped with $\overline{\mu}$ also supports a $(1,1)$-Poincaré inequality, and $\overline{\mu}$ is doubling on $\overline{\Omega}$. By Proposition 3.3 we know that $\|D\overline{u}\|(\partial\Omega) = 0$ and so by the decomposition (2.12), we also know that $\overline{\mathcal{H}}(S_{\overline{u}} \cap \partial\Omega) = 0$. On the other hand, by the Lebesgue point theorem given in [28, Theorem 3.5], we have for $\overline{\mathcal{H}}$-a.e. $x \in \partial\Omega \setminus S_{\overline{u}}$

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |u - \overline{u}^\wedge(x)|^{Q/(Q-1)}\, d\mu = \lim_{r\to 0^+} \fint_{B(x,r)} |\overline{u} - \overline{u}^\wedge(x)|^{Q/(Q-1)}\, d\overline{\mu} = 0.$$

Thus we set $Tu(x) = \overline{u}^\wedge(x)$. Note that because $\overline{\mu}(\partial\Omega) = 0$, the value of $\overline{u}^\wedge$ at points in $\partial\Omega$ is unaffected by how we extend u to $\partial\Omega$. The linearity of T is immediate. If the measure density condition (3.2) holds, then $\mathcal{H}$ and $\overline{\mathcal{H}}$ are comparable for subsets of $\partial\Omega$, and so the result holds for $\mathcal{H}$-a.e. $x \in \partial\Omega$. $\square$

4 Some examples

In this section we consider examples that illustrate the function of each of the hypotheses stated in Theorem 3.4.

Theorem 3.4 is stronger than it seems, for we do *not* assume that $\mathcal{H}(\partial\Omega)$ is finite.

Example 4.1. The von Koch snowflake domain in the plane is a uniform domain, and uniform domains equipped with the restriction of the Lebesgue measure have a doubling measure supporting a $(1,1)$-Poincaré inequality, see e.g. [10]. On the other hand, the $\mathcal{H}$-measure (that is, the one-dimensional Hausdorff measure) of the boundary of the snowflake domain is infinite, so the existence of Tu at $\mathcal{H}$-almost every boundary point is a strong statement. However, we do not claim here that the trace Tu is in the class $L^1(\partial\Omega,\mathcal{H})$, and indeed this would be too much to hope for. Since constant functions are in the class $\mathrm{BV}(\Omega)$ whenever Ω is a bounded domain, and their traces are also constant functions on $\partial\Omega$, in order to consider whether traces of $\mathrm{BV}(\Omega)$-functions are in $L^1(\partial\Omega,\mathcal{H})$ it is necessary that $\mathcal{H}(\partial\Omega)$ be finite. Observe also that for the snowflake domain, $\partial^*\Omega=\partial\Omega$ so considering $L^1(\partial^*\Omega,\mathcal{H})$ does not help either.

We will consider L^1 estimates in Section 5.

Example 4.2. Let $C\subset[0,1]$ be the usual $1/3$-Cantor set, and define $\Omega:=(0,1)^2\setminus C\times C$. This is an open set from which Sobolev functions can be extended to the whole unit square, as can be seen by the characterization of Sobolev functions by means of absolute continuity on almost every line parallel to the (canonical) coordinate axes of Euclidean spaces. Thus Ω supports a $(1,1)$-Poincaré inequality, and it is clear that the Lebesgue measure is doubling in Ω and that the measure density condition (3.2) holds. Therefore by Theorem 3.4, any function $u\in\mathrm{BV}(\Omega)$ has a trace at $\mathcal{H}$-a.e. $x\in\partial\Omega$. However, the Hausdorff dimension of $C\times C$ is $2\log(2)/\log(3)>1$, and so $\mathcal{H}(\partial\Omega)=\infty$. Together with the example discussed above, this indicates that in the generality considered in Theorem 3.4 we do not have $Tu\in L^1(\partial\Omega,\mathcal{H})$.

To see why the assumptions of Theorem 3.4 are necessary at least in some form, we first note that without a Poincaré inequality, Ω can be chosen to be the slit disk from Example 3.2, where we know that traces of functions of bounded variation do not exist. On the other hand, consider a domain in $\mathbb{R}^2$

with an interior cusp:

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1 \text{ and } |x_2| > x_1^2 \text{ when } x_1 \geq 0\},$$

This does not support a $(1,1)$-Poincaré inequality, since such an inequality always implies that the space (in this case the set Ω) is quasiconvex, that is, every pair of points can be connected by a curve whose length is at most a constant times the distance between the two points (see e.g. [16, Proposition 4.4]). However, boundary traces of BV functions do exist for this domain, as can be deduced from the following result. In the above domain with an internal cusp, we can take the sets F_ε to be small closed balls centered at the origin.

Corollary 4.3. *Let Ω be an open set. The conclusions of Theorem 3.4 are true if for every $\varepsilon > 0$ there is a closed set $F_\varepsilon \subset X$ with*

$$\overline{\mathcal{H}}(\partial\Omega \cap F_\varepsilon) < \varepsilon \tag{4.1}$$

such that $\Omega \setminus F_\varepsilon$, instead of Ω itself, satisfies the hypotheses of Theorem 3.4 (apart from the last sentence).

Proof. Fix $\varepsilon > 0$. It is easy to check that $\partial\Omega \setminus F_\varepsilon \subset \partial(\Omega \setminus F_\varepsilon)$. As earlier, we can define a codimension 1 Hausdorff measure in the closure of $\Omega \setminus F_\varepsilon$ with respect to the zero extension of μ from $\Omega \setminus F_\varepsilon$, but since F_ε is closed, this agrees with $\overline{\mathcal{H}}$ on $\partial\Omega \setminus F_\varepsilon$. Thus by Theorem 3.4 we know that for $\overline{\mathcal{H}}$-a.e. $x \in \partial\Omega \setminus F_\varepsilon$, there exists $Tu(x) \in \mathbb{R}$ with

$$\lim_{r \to 0^+} \fint_{B(x,r)\cap\Omega\setminus F_\varepsilon} |u - Tu(x)|^{Q/(Q-1)}\, d\mu = 0.$$

But since F_ε is closed, this immediately implies that

$$\lim_{r \to 0^+} \fint_{B(x,r)\cap\Omega} |u - Tu(x)|^{Q/(Q-1)}\, d\mu = 0.$$

Thus the trace $Tu(x)$ exists outside a subset of $\partial\Omega$ with $\overline{\mathcal{H}}$-measure at most ε, due to (4.1). By letting $\varepsilon \to 0$, we get the result. $\square$

In the following example, we consider the distinction between the measures $\mathcal{H}$ and $\overline{\mathcal{H}}$.

Example 4.4. Without the measure density condition (3.2), the conclusion of Theorem 3.4 does not necessarily hold for $\mathcal{H}$-a.e. $x \in \partial\Omega$. As a counterexample, consider the space $X = \mathbb{R}^2$ equipped with the Euclidean metric and the weighted Lebesgue measure $d\mu := w\, d\mathcal{L}^2$ with $w = |x|^{-1}$. It can be shown that μ is doubling and X supports a $(1,1)$-Poincaré inequality, see [24, Corollary 15.34] and [7, Theorem 4]. Consider the domain

$$\Omega := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 :\, 0 < x_1 < 1,\, |x_2| < x_1^2 \right\}. \tag{4.2}$$

We can check that μ is doubling on Ω and that Ω supports a $(1,1)$-Poincaré inequality, see the upcoming Example 4.5. On the other hand, the measure density condition (3.2) clearly does not hold at the origin $(0,0)$, for $\mu(B((0,0),r)) = 2\pi r$, whereas $\mu(B((0,0),r) \cap \Omega) \le r^2$. For $(0,0) \in \partial\Omega$, if B is a ball containing $(0,0)$, then $B((0,0),2r)$ contains this ball, where r is the radius of B. Thus by the doubling property of μ, we have

$$\frac{\mu(B)}{r} \le 2\frac{\mu(B((0,0),2r))}{2r} \le C\frac{\mu(B)}{r}.$$

Therefore

$$\mathcal{H}(\{(0,0)\}) \ge \frac{1}{C} \lim_{r\to 0^+} \frac{\mu(B((0,0),r))}{r} = \frac{1}{C} \lim_{r\to 0^+} \frac{2\pi r}{r} = \frac{2\pi}{C},$$

whereas

$$\overline{\mathcal{H}}(\{(0,0)\}) \le \lim_{r\to 0^+} \frac{\mu(B((0,0),r)\cap\Omega)}{r} \le \lim_{r\to 0^+} \frac{r^2}{r} = 0.$$

Now, if we define the function $u := |x|^{-1/2}$, and denote its local Lipschitz constant by $\operatorname{Lip} u$, we have

$$\|Du\|(\Omega) = \int_\Omega \operatorname{Lip} u \, d\mu = \frac{1}{2}\int_\Omega |x|^{-3/2}\, d\mu \le \int_0^1 r^{-3/2} r^2 r^{-1}\, dr < \infty,$$

and similarly $\|u\|_{L^1(\Omega)} < \infty$, so $u \in \mathrm{BV}(\Omega)$. However, the trace of u does not exist at $(0,0)$. So without the measure density condition, we may have traces for $\overline{\mathcal{H}}$-a.e. but not $\mathcal{H}$-a.e. $x \in \partial\Omega$. Thus by allowing ourselves the flexibility of working with either $\mathcal{H}$ or $\overline{\mathcal{H}}$, and by taking Corollary 4.3 into account, we cover a wider class of domains.

Next we show that various exterior cusps do satisfy the assumptions of Theorem 3.4, apart from the measure density condition (3.2).

Example 4.5. Take the unweighted space $X = \mathbb{R}^2$, and as in (4.2), define the domain with a cusp

$$\Omega := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \,:\, 0 < x_1 < 1,\, |x_2| < x_1^2 \right\}.$$

To show that μ is doubling on Ω, it suffices to show the doubling property with respect to squares centered at points in Ω. For $x = (x_1, x_2) \in \Omega$ and $r > 0$, this can be done by an explicit computation for two separate cases, with either $r \geq x_1^2$ or $r < x_1^2$ — we leave the details to the reader.

Now we focus on showing that Ω supports a $(1,1)$-Poincaré inequality. For this, it is enough to show that Ω supports a Semmes family of curves, see [37] or [23]. Pick a pair of points $x, y \in \Omega$; here we only consider the case $x = (x_1, 0)$ and $y = (y_1, 0)$. The required condition is that there is a family $\Gamma_{x,y}$ of curves connecting x and y, and a probability measure $\alpha_{x,y}$ on this family, such that

$$\begin{aligned}
&\int_{\Gamma_{x,y}} \int_{\gamma} \chi_A(t)\, dt\, d\alpha_{x,y}(\gamma) \\
&\qquad \leq C \int_A \frac{d(z,x)}{\mathcal{L}^2|_{\Omega}(B(x, d(z,x)))} + \frac{d(z,y)}{\mathcal{L}^2|_{\Omega}(B(y, d(z,y)))}\, d\mathcal{L}^2|_{\Omega}(z)
\end{aligned} \tag{4.3}$$

for every Borel set $A \subset \Omega$. Define the Semmes family $\Gamma_{x,y}$ of curves between x and y as follows: for $s \in (-1, 1)$,

$$\gamma_s(t) := \begin{cases} (x_1 + t, st) \\ (x_1 + t, s(x_1 + t)^2) \\ (x_1 + t, -s(t + x_1 - y_1)) \end{cases} \tag{4.4}$$

for the respective cases

$$\begin{cases} t \geq 0, & y_1 - (x_1 + t) \geq (x_1 + t)^2, \quad (x_1 + t, t) \in \overline{\Omega}, \\ t \geq 0, & y_1 - (x_1 + t) \geq (x_1 + t)^2, \quad (x_1 + t, t) \notin \overline{\Omega}, \\ 0 \leq t \leq y_1 - x_1, & y_1 - (x_1 + t) < (x_1 + t)^2, \end{cases}$$

respectively. Note that the parametrization is not by arc-length, but comparable to it. Essentially, the curves spread out from x as a "pencil" of angle $\pi/2$ as long as there is space in Ω, then they become parabolas that correspond to the cusp, and finally the curves converge on y as another "pencil" of angle $\pi/2$.

Since the curves $\gamma_s \in \Gamma_{x,y}$ are parametrized by s, we can define $\alpha_{x,y} := \frac{1}{2}\mathcal{H}^1|_{(-1,1)}$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. Now, if $A \subset \Omega$ is such that every γ_s is defined in A according to the first or third possible definition given in (4.4), it is a standard result, possible to prove by the classical coarea formula, that (4.3) holds.

On the other hand, if $A \subset \Omega$ is such that every γ_s is defined by the second definition in A, for every $z \in A$, we have

$$\begin{aligned}\frac{d(z,x)}{\mathcal{L}^2(B(x,d(z,x))\cap\Omega)} &\geq \frac{1}{2}\frac{z_1-x_1}{\int_{x_1}^{x_1+2(z_1-x_1)} 2v^2\,dv} \\ &= \frac{1}{8}\left(\fint_{\{(x_1,x_1+2(z_1-x_1))\}} v^2\,dv\right)^{-1} \\ &\geq \frac{1}{8}\frac{1}{z_1^2},\end{aligned} \tag{4.5}$$

where the last inequality follows from the fact that $v \mapsto v^2$ is convex. On the other hand, by the classical coarea formula we have

$$\int_{-1}^{1}\int_{\gamma_s} \chi_A(t)\,dt\,d\mathcal{H}^1(s) \leq \int_A \frac{1}{z_1^2}\,d\mathcal{L}^2.$$

Thus condition (4.3) holds in this case as well.

Finally, it can be checked that essentially the same definition of the Semmes family works in the more general case of the space $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, with a weighted Lebesgue measure $d\mu = w\,d\mathcal{L}^n$ with $w = |x|^\alpha$, $\alpha \in \mathbb{R}$, and with a polynomial cusp of degree $\beta > 0$ rather than 2. The above computations run through with small changes at least if we require that $\alpha + \beta \geq 1$.

5 L^1 estimates for traces

In this section we consider a particular condition on the domain Ω (given below in (5.1)) which, in addition to the assumptions on Ω given in Section 3, is necessary and sufficient for obtaining L^1 estimates for traces. Such estimates are related to extensions of BV functions — recall that a domain Ω is a BV extension domain if there is a bounded operator $E : \mathrm{BV}(\Omega) \to \mathrm{BV}(X)$ such that $Eu|_\Omega = u$. By [27, Proposition 6.3] we know that if Ω is an open set with $\mathcal{H}(\partial\Omega) < \infty$ and $E \subset \Omega$ is a μ-measurable set with $P(E,\Omega) < \infty$, then $P(E,X) < \infty$. However, here we do not have an extension bound $P(E,X) \le CP(E,\Omega)$. It was shown by Burago and Maz'ya in [13] that a domain Ω is a weak BV extension domain (with control only on the total variation of the extension, not the whole BV norm) if and only if there is a constant $C \ge 1$ such that whenever $E \subset \Omega$ is of finite perimeter in Ω and of sufficiently small diameter, there is a set $F \subset X$ with $F \cap \Omega = E$ such that $P(F,X) \le CP(E,\Omega)$. See [6, Theorem 3.8] for a proof of this fact in the metric setting. In the following extension result we get a bound not only on the total variation but on the whole BV norm. Some of the assumptions we require are somewhat simpler than in [10], where Newtonian functions are extended.

Remark 5.1. While we are mostly interested in applying extension results to estimates for traces, the trace operator T need not exist for a BV extension domain. This is demonstrated by the planar slit disk, which is known to be a BV extension domain by [30, Theorem 1.1]. Furthermore, we can show that every Sobolev $N^{1,1}$-extension domain is a BV extension domain, but not every BV extension domain is a Sobolev $N^{1,1}$-extension domain, as demonstrated again by the slit disk.

Proposition 5.2. *Assume that $\Omega \subset X$ is a bounded domain that supports a $(1,1)$-Poincaré inequality, that there is a constant $C_{\partial\Omega} > 0$ such that*

$$\mathcal{H}(B(x,r) \cap \partial\Omega) \le C_{\partial\Omega} \frac{\mu(B(x,r))}{r} \tag{5.1}$$

for all $r \in (0, 2\,\mathrm{diam}(\Omega))$, and that there is a constant $c_m > 0$ such that the measure density condition (3.2) holds for all $x \in \partial\Omega$ and all $r \in (0, \mathrm{diam}(\Omega))$. Then there is a constant $C_\Omega = C_\Omega(C_d, C_P, \lambda, C_{\partial\Omega}, c_m, \mathrm{diam}(\Omega))$ such that for every $u \in \mathrm{BV}(\Omega)$ the zero extension of u, denoted by $\widehat{u}$, satisfies

$$\|\widehat{u}\|_{\mathrm{BV}(X)} \le C_\Omega \|u\|_{\mathrm{BV}(\Omega)}. \tag{5.2}$$

Example 5.3. Condition (5.1) is not needed for Ω to be a BV extension domain, as demonstrated by the von Koch snowflake domain. However, this domain does not satisfy (5.2). While the boundary of the von Koch snowflake domain has infinite 1-dimensional Hausdorff measure, a modification of it will result in a domain which satisfies $\mathcal{H}(\partial\Omega) < \infty$ as well as all the hypotheses of the above theorem except for (5.1), and it can be seen directly that in such a domain, the zero extensions of BV functions are of finite perimeter in $X = \mathbb{R}^2$

but do not satisfy (5.2). One such domain can be obtained as follows. The curve obtained by the construction of the von Koch snowflake curve at the k-th step has length $(4/3)^k$. For each positive integer k we scale a copy of the k-th step of the snowflake curve by a factor $(3/4)^{2k}$, and concatenate them so that their end points lie on the x-axis in $\mathbb{R}^2$. Such a curve has total length $\sum_{k=1}^{\infty}(3/4)^k < \infty$. We replace one side of a square of length $\sum_{k=1}^{\infty}(3/4)^{2k}$ with this curve. The interior region in $\mathbb{R}^2$ surrounded by this closed curve is a uniform domain, and hence the restriction of the Lebesgue measure to this domain is doubling and supports a $(1,1)$-Poincaré inequality. Furthermore, the $\mathcal{H}$-measure of its boundary is

$$\sum_{k=1}^{\infty}(3/4)^k + 3\sum_{k=1}^{\infty}(3/4)^{2k} < \infty.$$

However, this domain fails to satisfy (5.1).

To demonstrate that we cannot discard the other assumptions of Proposition 5.2 either, we note that without the assumption of a Poincaré inequality in Ω, we could take the space to be

$$X := \{(x_1, x_2) \in \mathbb{R}^2 : \, 0 \le x_1 \le 1, \, -1 \le x_2 \le x_1^2\}$$

and then set Ω to be the domain

$$\Omega := X \setminus [0, 1/2] \times \{0\}.$$

Note that Ω satisfies the measure density condition and (5.1), but fails to support a $(1,1)$-Poincaré inequality. If we now consider the sets $E_i := \{(x_1, x_2) \in X : \, 0 \le x_1 \le 1/i, \, x_2 > 0\}$, $i \in \mathbb{N}$, then $P(E_i, \Omega) = 1/i^2$, whereas for every $F_i \subset X$ for which $F_i \cap \Omega = E_i$, we must have $P(F_i, X) \ge 1/i$. Therefore we cannot find a constant $C \ge 1$ such that $P(F_i, X) \le C P(E_i, \Omega)$ for all positive integers i.

To see that the measure density condition is also needed in the proposition, we consider X to be the Euclidean plane, with $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, \, |x_2| < x_1^2\}$ the external cusp domain we saw in Example 4.5. As shown in that example, Ω supports a $(1,1)$-Poincaré inequality and the restriction of the Lebesgue measure to Ω is doubling, and clearly (5.1) holds as well. However, Ω does not satisfy the measure density condition, and the sets $E_i := \{(x_1, x_2) \in \mathbb{R}^2 : \, 0 < x_1 < 1/i, \, |x_2| < x_1^2\}$, $i \in \mathbb{N}$, together demonstrate that we cannot find a bound $C \ge 1$ controlling the perimeter of the extensions.

Proof of Proposition 5.2. We first consider a μ-measurable set E such that $P(E,\Omega)<\infty$. By (5.1) we have $\mathcal{H}(\partial\Omega)<\infty$. Therefore by [27, Proposition 6.3] we know that $P(E,X)<\infty$. We consider the following two cases.
Case 1: $\mu(E)>\frac{1}{2}\mu(\Omega)$. Then by (2.8),

$$\begin{aligned}P(E,X)&\le P(E,\Omega)+P(E,\partial\Omega)\\&\le P(E,\Omega)+C\mathcal{H}(\partial\Omega)\\&\le P(E,\Omega)+2C\frac{\mathcal{H}(\partial\Omega)}{\mu(\Omega)}\mu(E)\le C\|\chi_E\|_{\mathrm{BV}(\Omega)}\end{aligned}$$

for some $C=C(C_d,C_P,\lambda,C_{\partial\Omega},c_m,\mathrm{diam}(\Omega))$, as desired.
Case 2: $\mu(E)\le\frac{1}{2}\mu(\Omega)$. The perimeter measure of E is carried on the set E_γ, where E_γ consists of all the points $x\in\partial E$ that satisfy (2.7), see [1, Theorem 5.4]. We therefore need to control $\mathcal{H}(E_\gamma\cap\partial\Omega)$ in terms of $P(E,\Omega)$. Fix $x\in E_\gamma\cap\partial\Omega$. Since $x\in E_\gamma$, for some $r_0>0$ we have that for all $0<r<r_0$,

$$\frac{\mu(B(x,r)\cap E)}{\mu(B(x,r)\cap\Omega)}>\frac{\gamma}{2}.$$

Fix such $r>0$. Because $\mu(E)\le\frac{1}{2}\mu(\Omega)$, we can choose the smallest $j\in\mathbb{N}$ such that

$$\frac{\mu(B(x,2^jr)\cap E)}{\mu(B(x,2^jr)\cap\Omega)}\le\frac{1}{2}.$$

If $j=1$, then by the choice of r we have

$$\frac{\mu(B(x,r)\cap E)}{\mu(B(x,r)\cap\Omega)}>\frac{\gamma}{2}.$$

If $j>1$, then we again have

$$\frac{\mu(B(x,2^{j-1}r)\cap E)}{\mu(B(x,2^{j-1}r)\cap\Omega)}>\frac{1}{2}\ge\frac{\gamma}{2},$$

and so by the doubling property of μ together with the measure density condition for Ω,

$$\frac{1}{2}\ge\frac{\mu(B(x,2^jr)\cap E)}{\mu(B(x,2^jr)\cap\Omega)}\ge\frac{\mu(B(x,2^{j-1}r)\cap E)}{\mu(B(x,2^{j-1}r)\cap\Omega)}\frac{\mu(B(x,2^{j-1}r)\cap\Omega)}{\mu(B(x,2^jr)\cap\Omega)}\ge\frac{\gamma}{2}\frac{c_m}{C_d}.$$

Set $r_x := 2^j r$. By using the relative isoperimetric inequality in Ω, the measure density condition, and the above, we get

$$CP(E,\Omega\cap B(x,\lambda r_x)) \ge \frac{\gamma}{2}\frac{c_m}{C_d}\frac{\mu(B(x,r_x)\cap\Omega)}{r_x} \ge \frac{\gamma}{2}\frac{c_m^2}{C_d^2}\frac{\mu(B(x,r_x))}{r_x}.$$

Thus we get a covering $\{B(x,\lambda r_x)\}_{x\in\partial\Omega\cap E_\gamma}$ of $\partial\Omega\cap E_\gamma$ in which every ball $B(x,r_x)$ satisfies the above inequality. By the 5-covering theorem, we can pick a countable collection $\{B_j = B(x_j,\lambda r_j)\}_{j=1}^\infty$ of pairwise disjoint balls such that $\{B(x_j,5\lambda r_j)\}_{j=1}^\infty$ is a cover of $\partial\Omega\cap E_\gamma$. Therefore by (5.1) and the doubling property of μ, together with the fact that the perimeter measure of E is comparable to $\mathcal{H}|_{E_\gamma}$ (see (2.8)), we get

$$\begin{aligned}\mathcal{H}(E_\gamma\cap\partial\Omega) &\le \sum_{j\in\mathbb{N}}\mathcal{H}(E_\gamma\cap\partial\Omega\cap 5\lambda B_j)\\ &\le C_{\partial\Omega}\sum_{j\in\mathbb{N}}\frac{\mu(5\lambda B_j)}{5r_j}\\ &\le C\sum_{j\in\mathbb{N}}\frac{\mu(B_j)}{r_j} \le C\sum_{j\in\mathbb{N}}P(E,\Omega\cap\lambda B_j) \le CP(E,\Omega).\end{aligned}$$

Thus by (2.8) again,

$$P(E,X) = P(E,\Omega)+P(E,\partial\Omega) \le P(E,\Omega)+C\mathcal{H}(E_\gamma\cap\partial\Omega) \le CP(E,\Omega).$$

Finally, for $u\in \mathrm{BV}(\Omega)$, with $\widehat{u}$ denoting the zero extension of u to $X\setminus\Omega$, by the coarea formula (2.5) we have

$$\begin{aligned}\|D\widehat{u}\|(X) &= \int_{-\infty}^{\infty} P(\{\widehat{u}>t\},X)\,dt\\ &\le C\int_{-\infty}^{\infty} P(\{u>t\},\Omega)\,dt = C\|Du\|(\Omega)\end{aligned}$$

as desired. $\square$

Suppose $\Omega\subset X$ is a bounded open set and $u\in\mathrm{BV}(\Omega)$, and suppose that the trace $Tu(x)$ exists for $\mathcal{H}$-a.e. $x\in\partial\Omega$. We wish to have the following L^1 estimate for the trace:

$$\int_{\partial\Omega}|Tu|\,d\mathcal{H} \le C_T\left(\int_\Omega |u|\,d\mu + \|Du\|(\Omega)\right) \tag{5.3}$$

for a constant $C_T>0$ which is independent of u. This kind of integral-type inequality is closely related to the condition (5.1), as we will now show.

Proposition 5.4. *Let $\Omega \subset X$ be open and bounded. If* (5.3) *holds for every $u \in \mathrm{BV}(\Omega)$, then there is a constant $C_{\partial\Omega} = C_{\partial\Omega}(C_T, C_d, \mathrm{diam}(\Omega)) > 0$ such that the boundary of Ω satisfies* (5.1) *for all $x \in \partial\Omega$ and all $r \in (0, 2\,\mathrm{diam}(\Omega))$.*

Proof. Pick $x \in \partial\Omega$. For every $i \in \mathbb{Z}$, there exists $r_i \in [2^i, 2^{i+1}]$ such that

$$P(B(x, r_i), X) \le C_d \frac{\mu(B(x, r_i))}{r_i},$$

see [27, Lemma 6.2]. We consider $i \in \mathbb{Z}$ for which $2^i \le 4\,\mathrm{diam}(\Omega)$. Given such an i, we choose $u = \chi_{B(x,r_i)\cap\Omega}$. Then we have

$$\begin{aligned}\mathcal{H}(\partial\Omega \cap B(x, r_i)) \le \int_{\partial\Omega} |Tu|\, d\mathcal{H} &\le C_T \left(\int_\Omega |u|\, d\mu + \|Du\|(\Omega) \right) \\ &= C_T \left(\mu(B(x, r_i) \cap \Omega) + P(B(x, r_i), \Omega) \right) \\ &\le C_T \left(\mu(B(x, r_i)) + P(B(x, r_i), X) \right).\end{aligned}$$

By the choice of r_i, we now have

$$\begin{aligned}\mathcal{H}(\partial\Omega \cap B(x, r_i)) &\le C_T C_d \left(\mu(B(x, r_i)) + \frac{\mu(B(x, r_i))}{r_i} \right) \\ &\le 8 C_T C_d [1 + \mathrm{diam}(\Omega)]\, \frac{\mu(B(x, r_i))}{r_i}.\end{aligned}$$

Now by the doubling property of μ it follows that the above holds for all radii $r \le 2\,\mathrm{diam}(\Omega)$, by inserting an additional factor C_d^2 on the right-hand side. □

Now we prove the second of the two main theorems of this paper.

Theorem 5.5. *Let Ω be a bounded domain that supports a $(1,1)$-Poincaré inequality and that there is a constant $c_m > 0$ such that the measure density condition* (3.2) *holds for all $x \in \partial\Omega$, $r \in (0, \mathrm{diam}(\Omega))$, and that* (5.1) *holds. Then there is a constant $C_T = C_T(C_d, C_P, \lambda, \Omega, c_m) > 0$ such that* (5.3) *holds, that is,*

$$\int_{\partial\Omega} |Tu|\, d\mathcal{H} \le C_T \left(\int_\Omega |u|\, d\mu + \|Du\|(\Omega) \right)$$

for all $u \in \mathrm{BV}(\Omega)$.

Proof. By Theorem 3.4, the trace $Tu(x)$ exists for $\mathcal{H}$-a.e. $x \in \partial\Omega$. First we show that Tu is a Borel function on $\partial\Omega$, so that the integral in (5.3) makes sense. Since X supports a Poincaré inequality, we know that a bi-Lipschitz change in the metric results in a geodesic space, i.e. a space where every pair of points can be joined by a curve whose length is equal to the distance between the two points, see [16, Proposition 4.4]. It is easy to see that traces are invariant under this transformation because the measure density condition for Ω holds. It follows from [12, Corollary 2.2] that there are positive constants C, δ such that whenever $x \in X$, $r > 0$, and $\varepsilon > 0$,

$$\mu(B(x, r[1+\varepsilon]) \setminus B(x,r)) \le C\mu(B(x,r))\varepsilon^{\delta}.$$

It follows from the above condition and the absolute continuity of integrals of u that the function

$$\partial\Omega \ni x \mapsto u_{\Omega \cap B(x,r)}$$

for any fixed $r > 0$ is continuous. Since Tu is the limit of these functions as $r \to 0$, it is a Borel function.

By Proposition 5.2 the zero extension $\widehat{u} = Eu \in \mathrm{BV}(X)$. By [19, Proposition 5.12] we know that

$$\int_{\partial\Omega} |(Eu)^{\wedge}| + |(Eu)^{\vee}| \, d\mathcal{H} \le C\|Eu\|_{\mathrm{BV}(X)}.$$

By the measure density condition, it is clear that

$$(Eu)^{\wedge}(x) \le Tu(x) \le (Eu)^{\vee}(x)$$

for every $x \in \partial\Omega$ for which the trace $Tu(x)$ exists. Thus we get by Proposition 5.2

$$\int_{\partial\Omega} |Tu| \, d\mathcal{H} \le \int_{\partial\Omega} |(Eu)^{\wedge}| + |(Eu)^{\vee}| \, d\mathcal{H} \le C\|Eu\|_{\mathrm{BV}(X)} \le C\|u\|_{\mathrm{BV}(\Omega)}.$$

$\square$

Example 5.6. Note that once again, if we define Ω by (4.2) as an exterior cusp in the unweighted space $\mathbb{R}^2$, and consider functions $u_i := \chi_{B((0,0),1/i)\cap\Omega}$, $i \in \mathbb{N}$, we see that we do not have the L^1 estimate (5.3), and neither do we have the measure density condition (3.2). If we consider $\overline{\Omega}$ as our metric space, we obviously have the measure density condition (3.2), but then we do not have (5.1), and again the L^1 estimate fails.

On the other hand, the proof of Theorem 5.5 does not really require that Ω supports a $(1,1)$-Poincaré inequality provided a trace operator exists, as demonstrated in the following example.

Example 5.7. Let $X = \mathbb{R}^2$ and let Ω be the internal cusp domain given by

$$\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1 \text{ and } |x_2| > x_1^2 \text{ when } x_1 \geq 0\},$$

equipped with the restriction of the Euclidean metric and Lebesgue measure. Note that Ω satisfies the measure density condition and (5.1), but Ω does not support a $(1,1)$-Poincaré inequality. However, by Corollary 4.3 we know that functions in $\mathrm{BV}(\Omega)$ do have a trace, and the proof of Theorem 5.5 shows that the traces are $\mathcal{H}$-measurable on $\partial\Omega$. Furthermore, Ω is a BV extension domain, see for example [30, Theorem 1.1]. Now it follows as in the proof of Theorem 5.5 that traces of functions in $\mathrm{BV}(\Omega)$ satisfy an L^1 estimate. Thus the conclusion of Theorem 5.5 is more widely applicable than it first appears.

6 BV functions with zero trace

In this section we study functions in $\mathrm{BV}(\Omega)$ that have zero trace on $\partial\Omega$. In considering Dirichlet problems for a given domain, one is interested in minimizing certain energies of a function from amongst the class of all functions with the same boundary value; that is, the difference of the test function and the boundary data should have zero trace. Hence an understanding of BV functions with zero trace is of importance.

The following argument can be found in [27], but we give a proof for the reader's convenience. Given $f \in \mathrm{BV}(X)$, recall the definition of the representative $\widetilde{f}$ from (2.11).

Theorem 6.1. *Let $\Omega \subset X$ be an open set and let $u \in \mathrm{BV}(\Omega)$. Assume either that the space supports a strong relative isoperimetric inequality, or that $\mathcal{H}(\partial\Omega) < \infty$. Then*

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u| \, d\mu = 0 \tag{6.1}$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$ if and only if the zero extension of u into the whole space X, denoted by $\widehat{u}$, is in $\mathrm{BV}(X)$ with $\|D\widehat{u}\|(X \setminus \Omega) = 0$ and $\widetilde{\widehat{u}}(x) = 0$ for $\mathcal{H}$-a.e. $x \in \partial\Omega$.

If in addition Ω *satisfies the measure density condition* (3.2)*, then the above conditions are also equivalent with the condition of having* $Tu(x) = 0$ *for* $\mathcal{H}$*-a.e.* $x \in \partial\Omega$*.*

Observe that if u has a trace Tu on $\partial\Omega$, then u necessarily satisfies (6.1) and $Tu = 0$ $\mathcal{H}$-a.e. in $\partial\Omega$.

Proof. First assume that (6.1) holds for $\mathcal{H}$-a.e. $x \in \partial\Omega$. We denote the level sets of $\widehat{u}$ by

$$E_t := \{x \in X : \widehat{u}(x) > t\}, \qquad t \in \mathbb{R}.$$

Fix $t < 0$. Then clearly $X \setminus \Omega \subset E_t$. If $x \in \partial\Omega$ such that (6.1) holds, then because on $\Omega \setminus E_t$ we have $|u| \geq t$, we get

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap \Omega \setminus E_t)}{\mu(B(x,r))} \leq \limsup_{r\to 0^+} \frac{1}{|t|\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u|\, d\mu = 0.$$

Hence E_t has density 1 at $\mathcal{H}$-a.e. $x \in \partial\Omega$.

Now fix $t > 0$. Then $E_t \subset \Omega$, and if $x \in \partial\Omega$ such that (6.1) holds, then because on E_t we have $|u| \geq t$, we get

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap E_t)}{\mu(B(x,r))} \leq \limsup_{r\to 0^+} \frac{1}{|t|\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u|\, d\mu = 0,$$

that is, E_t has density 0 at x.

From the above argument we know that $\mathcal{H}(\partial^* E_t \cap \partial\Omega) = 0$ whenever $t \neq 0$. Since $\widehat{u}$ is a zero extension, clearly we have $\partial^* E_t \setminus \overline{\Omega} = \emptyset$ for all $t \in \mathbb{R}$. By the coarea formula (2.5) we know that $P(E_t, \Omega) < \infty$ for a.e. $t \in \mathbb{R}$. Thus for a.e. $t \in \mathbb{R}$, by (2.8) we have

$$\mathcal{H}(\partial^* E_t) = \mathcal{H}(\partial^* E_t \cap \Omega) \leq C P(E_t, \Omega) < \infty.$$

If $\mathcal{H}(\partial\Omega) < \infty$, by [27, Proposition 6.3] we know that for such t, necessarily $P(E_t, X) < \infty$. The same is true if X supports a strong relative isoperimetric inequality. By (2.8) we then have

$$P(E_t, X) \leq C\mathcal{H}(\partial^* E_t) \leq C P(E_t, \Omega),$$

and then it follows from the coarea formula (2.5) that

$$\|D\widehat{u}\|(X) = \int_{-\infty}^{\infty} P(E_t, X)\, dt \leq C \int_{-\infty}^{\infty} P(E_t, \Omega)\, dt = C\|Du\|(\Omega) < \infty$$

and

$$\|D\widehat{u}\|(\partial\Omega) = \int_{-\infty}^{\infty} P(E_t, \partial\Omega)\, dt \le C \int_{-\infty}^{\infty} \mathcal{H}(\partial^* E_t \cap \partial\Omega)\, dt = 0.$$

Hence we also have $\widehat{u} \in \mathrm{BV}(X)$.

Conversely, if we know that the zero extension $\widehat{u} \in \mathrm{BV}(X)$ and $\|D\widehat{u}\|(X \setminus \Omega) = 0$, by the decomposition (2.12) we know that $\mathcal{H}(S_{\widehat{u}} \cap \partial\Omega) = 0$, and then by the Lebesgue point theorem proved in [28, Theorem 3.5], we have

$$\lim_{r\to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u|\, d\mu = \lim_{r\to 0^+} \fint_{B(x,r)} |\widehat{u} - \widetilde{\widehat{u}}(x)|\, d\mu = 0$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$.

The last statement of the theorem is clear. $\square$

To discuss Dirichlet problems in the metric setting, one needs a criterion that determines whether a BV function has the same boundary values as a given function (the boundary data). One such criterion is as follows.

Definition 6.2. We say that $u \in \mathrm{BV}(\Omega)$ has *the same boundary values* as $f \in \mathrm{BV}(\Omega)$ if

$$\lim_{r\to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u - f|\, d\mu = 0 \tag{6.2}$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$.

By Theorem 6.1, the zero extension $\widehat{u-f}$ of $u - f$ is in $\mathrm{BV}(X)$ with $\|D(\widehat{u-f})\|(\partial\Omega) = 0$. Therefore, if $f \in \mathrm{BV}(X)$, then the extension Eu of u by f to $X \setminus \Omega$ is in $\mathrm{BV}(X)$ with $\|D(Eu - f)\|(\partial\Omega) = 0$. In the Euclidean setting, this is the idea behind the direct method developed by Giusti, see [15, Chapter 14].

In particular, the condition $Tu(x) = Tf|_\Omega(x)$ implies $T(u - f|_\Omega)(x) = 0$, which in turn implies (6.2), but of course the traces may fail to exist, as was the case in Example 4.4.

Definition 6.3. We define $\mathrm{BV}_0(\Omega)$ to be the class of functions $u \in \mathrm{BV}(\Omega)$ that satisfy (6.1) for $\mathcal{H}$-a.e. $x \in \partial\Omega$.

Note that in [18] and [19] a class of BV functions with zero boundary values is defined very differently, as functions $u \in \mathrm{BV}(X)$ with $u = 0$ outside

Ω. For example, if Ω is the unit ball in $\mathbb{R}^n$, then $u = \chi_\Omega$ is not in $\mathrm{BV}_0(\Omega)$, but does have zero boundary values in the sense of [18] and [19]. In minimization problems, the definition for boundary values given in [18] and [19] can be more convenient, but on the other hand it is unclear how much variation measure is concentrated on $\partial\Omega$. The situation is simpler for Newton-Sobolev functions with zero boundary values $N_0^{1,1}(\Omega)$, since whenever $u \in N^{1,1}(X)$ with $u = 0$ outside Ω, we automatically have $\|Du\|(X \setminus \Omega) = 0$; this follows e.g. from [8, Corollary 2.21]. We will not pursue the study of Dirichlet problems in the current paper, but will use the above concept of boundary values in the study of fine properties of BV functions. To do so, we will need the following lemma.

Lemma 6.4. *Let $\Omega \subset X$ be an open set, let ν be a Radon measure of finite mass on Ω, and define*

$$A := \left\{ x \in \partial\Omega : \limsup_{r \to 0^+} r \frac{\nu(B(x,r) \cap \Omega)}{\mu(B(x,r))} > 0 \right\}.$$

Then $\mathcal{H}(A) = 0$.

Proof. It suffices to show that for each $n \in \mathbb{N}$, the set

$$A_n := \left\{ x \in \partial\Omega : \limsup_{r \to 0^+} r \frac{\nu(B(x,r) \cap \Omega)}{\mu(B(x,r))} > \frac{1}{n} \right\}$$

has $\mathcal{H}$-measure zero. Fix $n \in \mathbb{N}$ and $\delta > 0$. Then for every $x \in A_n$, we can find a ball of radius $r_x < \delta/5$ such that

$$r_x \frac{\nu(B(x,r_x) \cap \Omega)}{\mu(B(x,r_x))} > \frac{1}{n}.$$

By the 5-covering theorem, we can select a countable pairwise disjoint subcollection of the collection $\{B(x,r_x)\}_{x \in A_n}$, denoted by $\{B(x_i,r_i)\}_{i \in \mathbb{N}}$, such that $A_n \subset \bigcup_{i \in \mathbb{N}} B(x_i, 5r_i)$. Now we have

$$\begin{aligned}
\mathcal{H}_\delta(A_n) \le \sum_{i \in \mathbb{N}} \frac{\mu(B(x_i,5r_i))}{5r_i} &\le \frac{C_d^3}{5} \sum_{i \in \mathbb{N}} \frac{\mu(B(x_i,r_i))}{r_i} \\
&\le \frac{C_d^3}{5} n \sum_{i \in \mathbb{N}} \nu(B(x_i,r_i) \cap \Omega) \\
&= C n \, \nu\left(\bigcup_{i \in \mathbb{N}} B(x_i,r_i) \cap \Omega \right).
\end{aligned}$$

Given that $r_i < \delta/5$, we know that

$$\bigcup_{i\in\mathbb{N}} B(x_i, r_i) \cap \Omega \subset \bigcup_{x\in\partial\Omega} B(x, \delta/5) \cap \Omega.$$

Because $\nu(\Omega) < \infty$, it follows that

$$\begin{aligned}\mathcal{H}(A_n) = \lim_{\delta\to 0^+} \mathcal{H}_\delta(A_n) &\le \lim_{\delta\to 0^+} Cn\,\nu\left(\bigcup_{x\in\partial\Omega} B(x, \delta/5)\cap\Omega\right)\\ &= Cn\,\nu\left(\bigcap_{\delta>0}\bigcup_{x\in\partial\Omega} B(x,\delta/5)\cap\Omega\right) = Cn\,\nu(\emptyset) = 0.\end{aligned}$$

Thus $\mathcal{H}(A) = 0$. $\square$

In most of the remainder of the paper, we will work with Whitney type coverings of open sets. For the construction of such coverings and their properties, see e.g. [9, Theorem 3.1]. Given any open set $\Omega \subset X$ and a scale $R > 0$, we can choose a Whitney type covering $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ of Ω such that:

1. for each $j \in \mathbb{N}$,
$$r_j := \min\{\operatorname{dist}(x_j, X\setminus\Omega)/20\lambda, R\}, \tag{6.3}$$

2. for each $k \in \mathbb{N}$, the ball $5\lambda B_k$ meets at most $C_0 = C_0(C_d, \lambda)$ balls $5\lambda B_j$ (that is, a bounded overlap property holds),

3. if $5\lambda B_j$ meets $5\lambda B_k$, then $r_j \le 2r_k$,

4. for each $j \in \mathbb{N}$,
$$\operatorname{dist}(2B_j, X\setminus\Omega) \ge 18\lambda r_j. \tag{6.4}$$

The last estimate above follows from the first estimate combined with the fact that $\lambda \ge 1$.

Given such a covering of Ω, we have a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to this covering, that is, for each $j \in \mathbb{N}$ the function ϕ_j is C/r_j-Lipschitz with $\operatorname{supp}(\phi_j) \subset 2B_j$ and $0 \le \phi_j \le 1$, such that $\sum_j \phi_j = 1$ on Ω, see e.g. [9, Theorem 3.4]). Finally, we can define a *discrete convolution* u_W of $u \in L^1_{\mathrm{loc}}(\Omega)$ with respect to the Whitney type covering by

$$u_W := \sum_{j=1}^{\infty} u_{B_j}\phi_j.$$

In general $u_W \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$. If $u \in L^1(\Omega)$, then $u_W \in L^1(\Omega)$.

In the next proposition, no assumption is made on the geometry of the open set Ω, so that in particular we do not know whether traces of BV functions exist. However, we can prove that discrete convolutions have the same boundary values as the original function in the sense of (6.2).

Proposition 6.5. *Let $\Omega \subset X$ be an open set, let $R > 0$, and let $u \in \mathrm{BV}(\Omega)$. Let $u_W \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be the discrete convolution of u with respect to a Whitney type covering $\{B_j = B(x_j, r_j)\}_{j=1}^{\infty}$ of Ω, at the scale R. Then*

$$\lim_{r\to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u_W - u|\, d\mu = 0 \tag{6.5}$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$.

Proof. Fix $x \in \partial\Omega$ and take $r > 0$. Denote by I_r the set of indices $j \in \mathbb{N}$ for which $2B_j \cap B(x,r) \neq \emptyset$, and note that by (6.4),

$$r_j \le \frac{\mathrm{dist}(2B_j, X \setminus \Omega)}{18\lambda} \le \frac{r}{18\lambda}$$

for every $j \in I_r$. Because $\sum_j \phi_j = \chi_\Omega$, we have

$$\begin{aligned}
\int_{B(x,r)\cap\Omega} |u - u_W|\, d\mu &= \int_{B(x,r)\cap\Omega} \Big| \sum_{j\in I_r} u\phi_j - \sum_{j\in I_r} u_{B_j}\phi_j \Big|\, d\mu \\
&\le \int_{B(x,r)\cap\Omega} \sum_{j\in I_r} \Big| \phi_j(u - u_{B_j}) \Big|\, d\mu \\
&\le \sum_{j\in I_r} \int_{2B_j} |u - u_{B_j}|\, d\mu \\
&\le 2C_d \sum_{j\in I_r} \int_{2B_j} |u - u_{2B_j}|\, d\mu \\
&\le 4C_d C_P \sum_{j\in I_r} r_j \|Du\|(2\lambda B_j) \\
&\le Cr\|Du\|(B(x,2r)\cap\Omega).
\end{aligned} \tag{6.6}$$

In the above, we used the fact that X supports a $(1,1)$-Poincaré inequality, and in the last inequality we used the fact that $2\lambda B_j \subset \Omega \cap B(x,2r)$ for all $j \in I_r$ and the bounded overlap of the dilated Whitney balls $2\lambda B_j$. Thus by Lemma 6.4, we have

$$\lim_{r\to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |u - u_W| \, d\mu = 0$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$. $\square$

Let $\Omega \subset X$ be an open set, $R > 0$, and let u_W be the discrete convolution of a function $u \in \mathrm{BV}(\Omega)$ with respect to a Whitney type covering $\{B_j\}_{j\in\mathbb{N}}$ of Ω at scale R. Then u_W has an upper gradient

$$g = C \sum_{j=1}^{\infty} \chi_{B_j} \frac{\|Du\|(5\lambda B_j)}{\mu(B_j)} \tag{6.7}$$

in Ω, with $C = C(C_d, C_P, \lambda)$, see e.g. the proof of [28, Proposition 4.1]. Also, if each v_i, $i \in \mathbb{N}$, is a discrete convolution of a function $u \in L^1(\Omega)$ with respect to a Whitney type covering of Ω at scale $1/i$, then

$$v_i \to u \quad \text{in } L^1(\Omega) \quad \text{as } i \to \infty, \tag{6.8}$$

as seen by the discussion in [21, Lemma 5.3].

Combining these facts with Theorem 6.1 and Proposition 6.5, we obtain the following properties for discrete convolutions.

Corollary 6.6. *Let $\Omega \subset X$ be an open set, and let $u \in \mathrm{BV}(\Omega)$. Assume either that the space supports a strong relative isoperimetric inequality, or that $\mathcal{H}(\partial\Omega) < \infty$. For each $i \in \mathbb{N}$ let each $v_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be a discrete convolution of u in Ω at scale $1/i$. Then as $i \to \infty$ we have $v_i \to u$ in $L^1(\Omega)$, $\|Dv_i\|(\Omega) \le \int_\Omega g_{v_i}\, d\mu \le C\|Du\|(\Omega)$ for some choice of upper gradients g_{v_i} of v_i, and the functions*

$$w_i := \begin{cases} v_i - u & \text{in } \Omega, \\ 0 & \text{in } X \setminus \Omega, \end{cases}$$

satisfy $w_i \in \mathrm{BV}_0(\Omega)$ with $\|Dw_i\|(\partial\Omega) = 0$.

Because of the lower semicontinuity of the BV-energy norm, it follows from the above corollary that

$$\|Du\|(\Omega) \le \liminf_{i\to\infty} \|Dv_i\|(\Omega) \le C\|Du\|(\Omega).$$

In the Euclidean setting (where the measure is the homogeneous Lebesgue measure), one has recourse to a better smooth approximation than the discrete convolution presented in this section, namely the smooth convolution approximation via smooth compactly supported non-negative radially symmetric convolution kernels; with such a convolution one has equality in the first inequality above. However, in the metric setting, equality is not guaranteed, see for example [19]. In order to obtain equality in the above, we need to modify the approximations v_i away from $\partial\Omega$. This is the point of the next corollary.

Corollary 6.7. *Let $\Omega \subset X$ be an open set, and let $u \in \mathrm{BV}(\Omega)$. Assume either that the space supports a strong relative isoperimetric inequality, or that $\mathcal{H}(\partial\Omega) < \infty$. Then there exist functions $\breve{v}_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, $i \in \mathbb{N}$, with $\breve{v}_i \to u$ in $L^1(\Omega)$, $\int_\Omega g_{\breve{v}_i}\, d\mu \to \|Du\|(\Omega)$ for a choice of upper gradients $g_{\breve{v}_i}$ of $\breve{v}_i$, and such that the functions*

$$\breve{w}_i := \begin{cases} \breve{v}_i - u & \text{in } \Omega, \\ 0 & \text{in } X \setminus \Omega, \end{cases}$$

satisfy $\breve{w}_i \in \mathrm{BV}_0(\Omega)$ with $\|D\breve{w}_i\|(\partial\Omega) = 0$.

Note that $\|D\breve{v}_i\|(\Omega) \le \int_\Omega g_{\breve{v}_i}\, d\mu$, and hence as a consequence we also have that $\|D\breve{v}_i\|(\Omega) \to \|Du\|(\Omega)$. When $\breve{v}_i \to u$ in $L^1(\Omega)$ and $\|D\breve{v}_i\|(\Omega) \to \|Du\|(\Omega)$, we say that the sequence of functions $\breve{v}_i$ converges *strictly* to u in $\mathrm{BV}(\Omega)$.

Proof. For every $\delta > 0$, let

$$\Omega_\delta := \{y \in \Omega :\ \mathrm{dist}(y, X \setminus \Omega) > \delta\}. \tag{6.9}$$

Fix $\varepsilon > 0$ and $x \in X$, and choose $\delta > 0$ such that

$$\|Du\|(\Omega \setminus (\Omega_\delta \cap B(x, 1/\delta))) < \varepsilon.$$

We will use the function η on X given by

$$\eta(y) := \max\left\{0, 1 - \frac{4}{\delta}\operatorname{dist}(y, \Omega_{\delta/2} \cap B(x, 2/\delta))\right\}$$

to paste discrete convolutions to good local Lipschitz approximations of u far away from $\partial\Omega$. The function η is $4/\delta$-Lipschitz.

Let each $v_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be a discrete convolution of u in Ω, at scale $1/i$. From the definition of the total variation we get a sequence of functions $u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega g_{u_i}\, d\mu \to \|Du\|(\Omega)$$

for some choice of upper gradients g_{u_i} of u_i. Now define

$$\breve{v}_i := \eta u_i + (1-\eta) v_i,$$

so that $\breve{v}_i \to u$ in $L^1(\Omega)$, and by a Leibniz rule, see e.g. [8, Lemma 2.18], each $\breve{v}_i$ has an upper gradient

$$g_i := g_\eta |u_i - v_i| + \eta g_{u_i} + (1-\eta) g_{v_i},$$

where g_η and g_{v_i} are upper gradients of η and v_i, respectively. Since we can take $g_\eta = 0$ outside $\Omega_{\delta/4} \cap B(x, 4/\delta) \Subset \Omega$, we have $g_\eta |u_i - v_i| \to 0$ in $L^1(\Omega)$, and by also using (6.7), we get

$$\begin{aligned}
\limsup_{i\to\infty} \int_\Omega g_i\, d\mu &\le \limsup_{i\to\infty} \int_\Omega g_{u_i}\, d\mu + \limsup_{i\to\infty} \int_{\Omega\setminus(\Omega_{\delta/2}\cap B(x,2/\delta))} g_{v_i}\, d\mu \\
&\le \|Du\|(\Omega) + C\|Du\|(\Omega \setminus (\Omega_\delta \cap B(x, 1/\delta))) \\
&\le \|Du\|(\Omega) + C\varepsilon.
\end{aligned}$$

The facts that $\breve{w}_i \in \mathrm{BV}(X)$ and $\|D\breve{w}_i\|(\partial\Omega) = 0$ for each $i \in \mathbb{N}$ again follow by combining Theorem 6.1 and Proposition 6.5. By a diagonalization argument, where we also let $\varepsilon \to 0$ (and hence $\delta \to 0$), we complete the proof. $\square$

We can relax the requirement that X supports a strong relative isoperimetric inequality or that $\mathcal{H}(\partial\Omega) < \infty$ and still obtain a useful, but slightly weaker, approximation as follows.

Corollary 6.8. *Let $\Omega \subset X$ be an open set. Then there is a sequence of open sets Ω_j, $j \in \mathbb{N}$, with $\Omega_j \Subset \Omega_{j+1}$ for each $j \in \mathbb{N}$ and $\Omega = \bigcup_j \Omega_j$, such that whenever $u \in \mathrm{BV}(\Omega)$, there is a sequence of functions $u_j \in \mathrm{BV}(\Omega_j)$ such that*

1. *$u_j \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_j)$ with*
$$\int_{\Omega_j} g_{u_j}\, d\mu \le \|Du\|(\Omega_j) + 2^{-j}$$
for some choice of upper gradient g_{u_j} of u_j,

2. *$\|u_j - u\|_{L^1(\Omega_j)} \le 2^{-j}$,*

3. *with $\widehat{u_j - u}$ denoting the zero extension of $u_j - u$ to $X \setminus \Omega_j$,*
$$\|D(\widehat{u_j - u})\|(\partial\Omega_j) = 0,$$

4. *$\mathcal{H}(\partial\Omega_j)$ is finite for each $j \in \mathbb{N}$.*

Proof. For unbounded sets Ω we can make modifications to the following argument along the lines of the proof of Corollary 6.7, so we assume now that Ω is bounded.

For each $\delta > 0$ let Ω_δ be as in (6.9). By the results of [11] or [4, Proposition 3.1.5] (see also [33, Inequality (2.6)]), $\mathcal{H}(\partial\Omega_\delta) < \infty$ for almost every $\delta > 0$. For each $j \in \mathbb{N}$ we choose one such $\delta_j \in [2^{-j-1}, 2^{-j}]$.

Since $\mathcal{H}(\partial\Omega_j)$ is finite, we can apply Corollary 6.7 to Ω in order to obtain a function u_j that satisfies the requirements laid out in the statement of the present corollary. This completes the proof of Corollary 6.8. □

In what follows, $\mathrm{BV}_c(\Omega)$ is the collection of all functions $u \in \mathrm{BV}(X)$ with $\mathrm{supt}(u) \Subset \Omega$. Recall the definition of $\mathrm{BV}_0(\Omega)$ given in Definition 6.3 above. The class $N_0^{1,p}(\Omega)$ has $N_c^{1,p}(\Omega)$ as a dense subclass whenever $1 \le p < \infty$, see for example [8, Theorem 5.45]. The following analogous approximation theorem for $\mathrm{BV}_0(\Omega)$ is an application of Proposition 6.5. Such an approximation is a useful tool in the study of Dirichlet problems associated with BV functions. The theorem clearly fails even for smooth bounded Euclidean domains if the notion of $\mathrm{BV}_0(\Omega)$ as given in [18] or [19] is considered.

Theorem 6.9. *Let Ω be an open set, and suppose that either the space supports a strong relative isoperimetric inequality or $\mathcal{H}(\partial\Omega) < \infty$. Suppose that*

$u \in \mathrm{BV}_0(\Omega)$ (in particular, it is enough to have $u \in \mathrm{BV}(\Omega)$ with $Tu(x) = 0$ for $\mathcal{H}$-a.e. $x \in \partial\Omega$). Then there exists a sequence $u_k \in \mathrm{BV}_c(\Omega)$, $k \in \mathbb{N}$, such that $u_k \to u$ in $\mathrm{BV}(\Omega)$.

Proof. First note that functions in $\mathrm{BV}_0(\Omega)$ can be approximated in the BV norm by bounded functions. To see this, for each positive integer n set

$$u_n = \max\{-n, \min\{u, n\}\}.$$

Then by the coarea formula and by $|u_n| \le |u|$, we see that $u_n \in \mathrm{BV}_0(\Omega)$ with $\lim_n u_n = u$ in $L^1(\Omega)$. For $t > 0$, we see that $u(x) - u_n(x) > t$ if and only if $u(x) > t + n$, and for $t < 0$, we have $u(x) - u_n(x) > t$ if and only if $u(x) > t - n$. Therefore by the coarea formula,

$$\begin{aligned}
\|D(u - u_n)\|(\Omega) &= \int_{-\infty}^{\infty} P(\{u - u_n > t\}, \Omega)\, dt \\
&= \int_{-\infty}^{0} P(\{u > t - n\}, \Omega)\, dt + \int_{0}^{\infty} P(\{u > t + n\}, \Omega)\, dt \\
&= \int_{(-\infty,-n)\cup(n,\infty)} P(\{u > t\}, \Omega)\, dt \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore $u_n \to u$ in $\mathrm{BV}(\Omega)$ as $n \to \infty$. Hence without loss of generality we will assume for the rest of the proof that $u \in \mathrm{BV}_0(\Omega)$ is bounded.

Fix $\varepsilon > 0$. Take a sequence of discrete convolutions v_i of u, each with respect to a Whitney type covering of Ω at scale $1/i$. By (6.8) we know that $v_i \to u$ in $L^1(\Omega)$ as $i \to \infty$. By Proposition 6.5 and the fact that $u \in \mathrm{BV}_0(\Omega)$, for each $i \in \mathbb{N}$ we also have

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap\Omega} |v_i|\, d\mu = 0 \tag{6.10}$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$.

Let Ω_δ be as in (6.9) for $\delta > 0$, and as in the proof of Corollary 6.7, let $\eta_\delta \in \mathrm{Lip}_c(\Omega)$ be a $1/\delta$-Lipschitz function such that $0 \le \eta_\delta \le 1$ on X, with $\eta_\delta = 1$ in $\Omega_{2\delta}$ and $\eta_\delta = 0$ outside Ω_δ. Then define

$$w_{\delta,i} := \eta_\delta u + (1 - \eta_\delta) v_i.$$

It is clear that $w_{\delta,i} \to u$ in $L^1(\Omega)$ as $i \to \infty$, and by using first the Leibniz rule for BV functions obtained in [28, Corollary 4.2] (unlike for Newtonian functions, using the Leibniz rule for BV functions requires that we have bounded functions), and then (6.7), we get for large enough i

$$
\begin{aligned}
\|D(w_{\delta,i}-u)\|(\Omega) &= \|D\big((1-\eta_\delta)(v_i-u)\big)\|(\Omega) \\
&\le C\int_{\Omega_\delta\setminus\Omega_{2\delta}} g_{\eta_\delta}|v_i-u|\,d\mu + C\|Du\|(\Omega\setminus\Omega_{2\delta}) + C\|Dv_i\|(\Omega\setminus\Omega_{2\delta}) \\
&\le \frac{C}{\delta}\int_{\Omega_\delta\setminus\Omega_{2\delta}} |v_i-u|\,d\mu + C\|Du\|(\Omega\setminus\Omega_{3\delta}).
\end{aligned}
$$

We can make the second term smaller than $\varepsilon/3$ by taking δ small enough, and then we can simultaneously ensure that the first term is smaller than $\varepsilon/3$ and that $\|w_{\delta,i}-u\|_{L^1(\Omega)}$ is smaller than $\varepsilon/3$ by taking i large enough. We fix the value of δ so obtained, and then for such large enough $i\in\mathbb{N}$, we have

$$
\|w_{\delta,i}-u\|_{\mathrm{BV}(\Omega)} < \varepsilon. \tag{6.11}
$$

By [27, Theorem 1.1] and (6.10), we know that each v_i is in the space $N_0^{1,1}(\Omega)$, i.e. the space of Newtonian functions with zero boundary values. By [8, Theorem 5.45] we then have for each $i\in\mathbb{N}$ a Lipschitz function $\breve{v}_i$, with compact support contained in Ω, such that $\|\breve{v}_i-v_i\|_{N^{1,1}(\Omega)} < 1/i$, whence $\|\breve{v}_i-v_i\|_{\mathrm{BV}(\Omega)} < 1/i$. Then we can define for $\delta>0$ and $i\in\mathbb{N}$,

$$
\breve{w}_{\delta,i} := \eta_\delta u + (1-\eta_\delta)\breve{v}_i.
$$

Clearly $\|\breve{w}_{\delta,i}-w_{\delta,i}\|_{L^1(\Omega)}\to 0$ as $i\to\infty$, and by the Leibniz rule again,

$$
\begin{aligned}
\|D(\breve{w}_{\delta,i}-w_{\delta,i})\|(\Omega) &= \|D\big((1-\eta_\delta)(\breve{v}_i-v_i)\big)\|(\Omega) \\
&\le C\|D(\breve{v}_i-v_i)\|(\Omega) + C\int_\Omega g_{\eta_\delta}|\breve{v}_i-v_i|\,d\mu \to 0
\end{aligned}
$$

as $i\to\infty$. By combining this with (6.11), we have for the $\delta>0$ determined earlier and a large enough $i\in\mathbb{N}$ that $\|\breve{w}_{\delta,i}-u\|_{\mathrm{BV}(\Omega)}<\varepsilon$, and furthermore $\breve{w}_{\delta,i}\in\mathrm{BV}_c(\Omega)$. $\square$

7 Maz'ya-Sobolev inequalities

In the direct method of the calculus of variations, to obtain regularity of minimizers one needs a Sobolev inequality, namely, an estimate of the average value (on balls) of a power of a test function in terms of the average value of its gradient. Such an inequality does not hold for functions in general, as demonstrated by non-zero constant functions. However, if the test functions of interest vanish on a significantly large set, then such an estimate can be obtained. If the largeness of the set is in terms of the measure of the zero set, then such an estimate follows easily from the Poincaré inequality: for $u \in \mathrm{BV}(X)$ and any ball $B = B(x,r)$, we have for $A := \{y \in B : |u(y)| > 0\}$,

$$\left(\fint_B |u|^{Q/(Q-1)}\, d\mu\right)^{(Q-1)/Q} \le Cr\left(1 - \left(\frac{\mu(A)}{\mu(B)}\right)^{1/Q}\right)^{-1} \frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)},$$

where $Q > 0$ is the lower mass bound exponent from (2.1). We refer the interested reader to [26, Lemma 2.2] for proof of this. However, in general this is too much to ask for. The more natural way to measure largeness of the zero set of a function is in terms of the relative capacity of the set. Inequalities based on such capacitary measures are originally due to Maz'ya [34, 35]. For functions with zero trace on the boundary of Ω, the zero set contains the boundary of Ω, and hence one can see such an inequality as associated with $\mathrm{BV}_0(\Omega)$. Thus we are motivated to study Maz'ya's inequalities for BV functions.

First we define two notions of BV-capacity as follows:

Definition 7.1. For an arbitrary set $A \subset X$,

$$\mathrm{Cap}_{\mathrm{BV}}(A) := \inf \|u\|_{\mathrm{BV}(X)},$$

where the infimum is taken over functions $u \in \mathrm{BV}(X)$ that satisfy $u \ge 1$ on a neighborhood of A. If $A \subset B$ for a ball $B \subset X$, then the *relative capacity* of A with respect to $2B$ is given by

$$\mathrm{cap}_{\mathrm{BV}}(A, 2B) := \inf \|Du\|(X) = \inf \|Du\|(\overline{2B}),$$

where the infimum is taken over $u \in \mathrm{BV}(X)$ such that $u \ge 1$ on a neighborhood of A, with $u = 0$ in $X \setminus 2B$.

Note that if we just required $u \ge 1$ on A, we would have $\mathrm{Cap}_{\mathrm{BV}}(A) = 0$ whenever $\mu(A) = 0$. For more on BV-capacity, see [17] and [20].

For $u \in \mathrm{BV}(X)$, recall the definition of $\widetilde{u}$ from (2.11) as follows:

$$\widetilde{u}(x) = (u^\wedge(x) + u^\vee(x))/2, \quad x \in X.$$

By [20, Corollary 4.7] we know that for any compact $K \subset B$,

$$\mathrm{cap}_{\mathrm{BV}}(K, 2B) \le C \inf\{\|Du\|(X) : \widetilde{u} \ge 1 \text{ on } K,\ u = 0 \text{ in } X \setminus 2B\}.$$

Crucially, above we do not require that $\widetilde{u} \ge 1$ in a *neighborhood* of K. By slightly adapting the proof of this result, we obtain an analogous result for $\mathrm{Cap}_{\mathrm{BV}}$: for any compact set $K \subset X$,

$$\mathrm{Cap}_{\mathrm{BV}}(K) \le C \inf\{\|u\|_{\mathrm{BV}(X)} : \widetilde{u} \ge 1 \text{ on } K\}. \tag{7.1}$$

By [17, Corollary 3.8] we know that for any Borel set $A \subset X$,

$$\mathrm{Cap}_{\mathrm{BV}}(A) = \sup\{\mathrm{Cap}_{\mathrm{BV}}(K),\ K \subset A,\ K \text{ compact}\}. \tag{7.2}$$

By a direct modification of [17, Theorem 3.4] and [17, Corollary 3.8], we obtain an analogous result for $\mathrm{cap}_{\mathrm{BV}}$: for any Borel set $A \subset B$,

$$\mathrm{cap}_{\mathrm{BV}}(A, 2B) = \sup\{\mathrm{cap}_{\mathrm{BV}}(K, 2B),\ K \subset A,\ K \text{ compact}\}.$$

Adapting the proof of [8, Theorem 5.53] (the result [8, Theorem 5.53] in the Euclidean setting is originally due to Maz'ya [34], see also [35, Theorem 10.1.2]), we now show the following.

Theorem 7.2. *For $u \in \mathrm{BV}(X)$, let $S := \{y \in X \setminus S_u : \widetilde{u}(y) = 0\}$. Then for every ball $B = B(x, r) \subset X$, we have*

$$\left(\fint_{2B} |u|^{\frac{Q}{Q-1}} \, d\mu \right)^{\frac{Q-1}{Q}} \le C \frac{r+1}{\mathrm{Cap}_{\mathrm{BV}}(S \cap B)} \|Du\|(2\lambda B),$$

and

$$\left(\fint_{2B} |u|^{\frac{Q}{Q-1}} \, d\mu \right)^{\frac{Q-1}{Q}} \le \frac{C}{\mathrm{cap}_{\mathrm{BV}}(S \cap B, 2B)} \|Du\|(2\lambda B),$$

with $C = C(C_d, C_P, \lambda)$.

Proof. Set $q := \frac{Q}{Q-1}$. First we assume that u is bounded, $u \ge 0$, and that $\|u\|_{L^1(2B)} > 0$.

Let $\eta : X \to [0, 1]$ be a $1/r$-Lipschitz function with $\eta = 1$ in B and $\eta = 0$ in $X \setminus 2B$. We use the abbreviation

$$a := \left(\fint_{2B} u^q \, d\mu \right)^{1/q},$$

and define $v := \eta(1-u/a)$. Now $v \in \mathrm{BV}(X)$ such that $v = 0$ in $X \setminus 2B$ and $\widetilde{v} = 1$ on $S \cap B$. Pick an arbitrary compact set $K \subset S \cap B$, so that in particular $\widetilde{v} = 1$ on K. By (7.2) and by using the Leibniz rule for BV functions, see [28, Corollary 4.2], we get

$$\begin{aligned}\frac{1}{C}\operatorname{Cap}_{\mathrm{BV}}(K) &\le \int_X v\,d\mu + \|Dv\|(X)\\ &\le \frac{1}{a}\int_{2B}|u-a|\,d\mu + \frac{C}{a}\left(\|Du\|(2B) + \int_{2B}|u-a|g_\eta\,d\mu\right)\\ &\le \frac{1+Cr^{-1}}{a}\int_{2B}|u-a|\,d\mu + \frac{C}{a}\|Du\|(2B).\end{aligned} \tag{7.3}$$

To estimate the first term, we write

$$\int_{2B}|u-a|\,d\mu \le \int_{2B}|u-u_{2B}|\,d\mu + |u_{2B}-a|\,\mu(2B). \tag{7.4}$$

Here the first term can be estimated using the $(1,1)$-Poincaré inequality, whereas for the second term we have

$$\begin{aligned}|a-u_{2B}|\,\mu(2B)^{1/q} &= |\|u\|_{L^q(2B)} - \|u_{2B}\|_{L^q(2B)}|\\ &\le \|u-u_{2B}\|_{L^q(2B)}\\ &= \left(\fint_{2B}|u-u_{2B}|^q\,d\mu\right)^{1/q}\mu(2B)^{1/q}\\ &= Cr\frac{\|Du\|(2\lambda B)}{\mu(2B)}\mu(2B)^{1/q}.\end{aligned}$$

Inserting this into (7.4), we get

$$\int_{2B}|u-a|\,d\mu \le Cr\|Du\|(2\lambda B).$$

Inserting this into (7.3), we then get

$$\operatorname{Cap}_{\mathrm{BV}}(K) \le C\frac{r+1}{a}\|Du\|(2\lambda B).$$

Recalling the definition of a and using (7.2), this implies

$$\left(\fint_{2B}|u|^q\,d\mu\right)^{1/q} \le C\frac{r+1}{\operatorname{Cap}_{\mathrm{BV}}(S\cap B)}\|Du\|(2\lambda B).$$

A modification of the above argument (by dropping the term $\int_X v\, d\mu$ from (7.3)) yields the second inequality claimed in the theorem for bounded non-negative functions. For bounded BV functions that are allowed to take negative values, we can replace u with $|u|$ to obtain the desired inequalities by noting that for any $A \subset X$,

$$\|D|u|\|(A) \le \|Du\|(A).$$

Finally, for BV functions that may also be unbounded, as at the beginning of the proof of Theorem 6.9 we approximate u by bounded functions u_n in the $\mathrm{BV}(X)$-norm and note that

$$\lim_{n\to\infty} \int_{2B} |u_n|^q\, d\mu = \int_{2B} |u|^q\, d\mu$$

and

$$\lim_{n\to\infty} \|Du_n\|(2\lambda B) = \|Du\|(2\lambda B).$$

Thus we obtain the desired capacitary estimates for all $u \in \mathrm{BV}(X)$. □

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