A contact problem for viscoelastic bodies with inertial effects and unilateral boundary constraints

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Abstract

We consider a viscoelastic body occupying a smooth bounded domain $\Omega \subset \mathbb{R}^3$ under the effect of a volumic traction force $\boldsymbol{g}$. The macroscopic displacement vector from the equilibrium configuration is denoted by $\mathbf{u}$. Inertial effects are considered; hence the equation for $\mathbf{u}$ contains the second order term $\mathbf{u}_{tt}$. On a part Γ_D of the boundary of Ω, the body is anchored to a support and no displacement may occur; on a second part $\Gamma_N \subset \partial\Omega$, the body can move freely; on a third portion $\Gamma_C \subset \partial\Omega$, the body is in adhesive contact with a solid support. The boundary forces acting on Γ_C due to the action of elastic stresses are responsible for delamination, i.e., progressive failure of adhesive bonds. This phenomenon is mathematically represented by a nonlinear ODE settled on Γ_C and describing the evolution of the delamination order parameter z. Following the lines of a new approach outlined in [8] and based on duality methods in Sobolev-Bochner spaces, we define a suitable concept of weak solution to the resulting PDE system. Correspondingly, we prove an existence result on finite time intervals of arbitrary length.

Key words: second order parabolic equation, viscoelasticity, weak formulation, contact problem, adhesion, mixed boundary conditions, duality.

AMS (MOS) subject classification: 35L10, 74D10, 47H05, 46A20.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain of boundary Γ. We assume Ω to be occupied, during the fixed reference interval $(0, T)$, by a viscoelastic body. No restriction is assumed on the final time $T > 0$. The displacement vector with respect to the equilibrium configuration in Ω is noted by $\mathbf{u}$. Hence, we may assume that $\mathbf{u}$ satisfies the following equation

$$\mathbf{u}_{tt} - \operatorname{div}(\mathbb{V}\varepsilon(\mathbf{u}_t) + \mathbb{E}\varepsilon(\mathbf{u})) = \mathbf{g}, \quad \text{in } \Omega, \tag{1.1}$$

where, given a vector-valued function $\boldsymbol{v}$, $\varepsilon(\boldsymbol{v})$ represents its symmetrized gradient; $\mathbf{g}$ is a volume force density; $\mathbb{V}$ and $\mathbb{E}$ are the *viscosity* and *elasticity* tensors, respectively, assumed symmetric, nondegenerate, bounded, and depending in a measurable way on the variable $x \in \Omega$ (we refer to

Section 2 below for the precise assumptions). Inertial effects may occur and are mathematically represented by the second order term $\mathbf{u}_{tt}$.

Let $\Gamma := \partial\Omega$ be the boundary of Ω and let us assume Γ to be decomposed as $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$, where the portions Γ_X, for $X = D, N, C$, are assumed to be relatively open in Γ, mutually disjoint, and to have strictly positive 2-dimensional measure. In addition to that, the distance between Γ_C and Γ_D is assumed to be strictly positive. In other words, the contact surface Γ_C and the support Γ_D need to be strictly separated by Γ_N. In the portion Γ_D of the boundary, we suppose the body to be anchored to a rigid support; hence no displacement may occur, or, in other words, $\mathbf{u}$ satisfies a homogeneous Dirichlet condition. In the part Γ_N, we assume that the body is allowed to move freely, which corresponds to asking $\mathbf{u}$ to satisfy a homogeneous Neumann boundary condition (with standard adjustments we could equally consider the nonhomogeneous case, corresponding to the physical situation where a surface traction is exerted on Γ_N). Finally, on Γ_C (here, C stands for "contact"), the body is assumed to be in adhesive contact with a hard surface, like for instance a wall. This configuration has two effects: firstly, the (trace of the) displacement $\mathbf{u}$ on Γ_C must be directed towards the interior of Ω. Namely, we ask the following constraint to be satisfied:

$$\mathbf{u} \cdot \mathbf{n} \leq 0, \quad \text{on } \Gamma_C, \tag{1.2}$$

where $\mathbf{n}$ is the outer normal unit vector to Ω. The meaning of (1.2) is that the material in Ω may not penetrate the wall, but it may well detach from it. Secondly, the contact between the body in Ω and the wall may cause "damage", i.e., loss of adhesive properties. This phenomenon is often referred to as delamination process. Mathematically, it is described by means of a second (scalar) variable z, defined on Γ_C over the same time interval $(0, T)$. The "bonding function" z takes the form of an *order parameter*; namely, $z(t, x)$ represents the fractional density of adhesive bonds that are active at the time $t \in (0, T)$ and at the point $x \in \Gamma_C$. In particular, when $z = 1$, there is total adhesion, whereas when $z = 0$ the bonds are completely broken. A value $z \in (0, 1)$ denotes a partial loss of adhesivity. In addition to that, we assume that the bonds, once broken, cannot be repaired; namely, we require the time derivative z_t to be nonpositive at each $(t, x) \in (0, T) \times \Gamma_C$.

In view of the above description, only the values $z \in [0, 1]$ and $z_t \in (-\infty, 0]$ are meaningful (or, as we will often say, are "physical"). Hence, the mathematical equation(s) for z must enforce in some way these constraints (or, equivalently, exclude the "unphysical" configurations). This scope may be reached by relying on the theory of *maximal monotone operators* (cf. the monographs [1, 4, 9]). Indeed, the equation for z, settled on Γ_C, may take the form

$$\alpha(z_t) + z_t + \beta(z) \ni a - \frac{1}{2}|\mathbf{u}|^2, \tag{1.3}$$

where α and β are *monotone graphs* in $\mathbb{R} \times \mathbb{R}$ entailing the required constraints. For instance, we may assume $\alpha = \partial I_{(-\infty,0]}$, i.e., the *subdifferential* of the *indicator function* of $(-\infty, 0]$, whereas we may take $\beta = \partial I_{[0,+\infty)}$, i.e., the *subdifferential* of the *indicator function* of $[0, +\infty)$. Hence, α provides the nonpositivity of z_t (i.e., the *irreversibility* of damage), whereas β guarantees the nonnegativity of z (i.e., the fact that when the adhesive bonds are completely broken no further damage may occur). The constraint $z \leq 1$ (i.e., the fact that the material cannot be more than completely integer) is automatically guaranteed by $z_t \leq 0$ once it is $z \leq 1$ at the initial time.

The positive constant a on the right hand side of (1.3) (more generally, we may also admit a to depend in a suitable way on $x \in \Gamma_C$) has the meaning of a threshold under which the elastic stresses are not strong enough to cause delamination. Namely, when $\frac{1}{2}|\mathbf{u}|^2$ is less than a, the right hand side of (1.3) is nonnegative; hence no damage is created. Instead, delamination may occur for $a - \frac{1}{2}|\mathbf{u}|^2 < 0$. Note also the term z_t on the left hand side of (1.3), meaning we assume here the damage evolution to be *rate-dependent*.

In order to get a closed system, one has to specify how the behavior of the damage variable influences the behavior of $\mathbf{u}$. This is made explicit by the (up to now missing) boundary condition for $\mathbf{u}$ on Γ_C, which we assume in the form

$$-\big(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})\big)\mathbf{n} = \gamma(\mathbf{u} \cdot \mathbf{n})\mathbf{n} + z\mathbf{u}. \tag{1.4}$$

Here, γ is a third maximal monotone graph enforcing the non-penetration constraint (cf. Remark 2.1 below for further comments). In particular, if we take $\gamma = \alpha = \partial I_{(-\infty,0]}$, we obtain that (the trace of) $\mathbf{u}$ is directed towards the interior of Ω, or, at most, we may have tangential displacements. In the sequel we will actually allow for more general assumptions on γ (and this is why we decided to use a different symbol to denote it).

Models for contact, delamination and damage in elastic media are becoming very popular in the recent mathematical literature. The evolution law of the adhesive damage variable z and its link with the displacement on Γ_C is based on the so-called concept of Frémond delamination (see [14]), here considered with the presence of nonnegligible viscosity of the adhesive (whose effects arise from the term z_t). Usually in the context of delamination models, the mechanical system consists of two (or more) elastic bodies attached upon an interface. Here we assume for simplicity to have only one body placed in Ω and attached to the support in Γ_C; however we observe that our results could be extended to the general case by trivial generalizations. A dynamic model for delamination without viscosity of the adhesive is also considered in [22], where the evolution equation for the variable $\mathbf{u}$ is a variant of (1.1) which takes higher order stresses into account. A dynamic model where also thermal effects are considered has been analyzed in [19]. Other related models, among many, can be found in [6, 7, 16, 17, 18, 20]. A model coupling (1.1) with an equation for z similar to (1.3) including viscosity can be found in [23]. The main difference between our model and the one in [23] stands however in the presence of the unilateral constraint (1.2), which represents also the main mathematical difficulty occurring here. Actually, when one considers mechanical models with inertial effects (i.e., containing the second order term $\mathbf{u}_{tt}$), enforcing the constraint (1.2) by the methods of monotone operator theory usually gives rise to regularity issues; namely, interpreting monotone graphs as monotone operators in the usual L^2 (or L^p) framework is generally out of reach. For this reason, in most of the related literature, this difficulty is overcome by restating the equation containing the constraint as a variational inequality. This is the case, for instance, of the recent papers [2, 11] (see also [12]). In [2], existence of a solution to a system closely related to (1.1)+(1.4) (hence, the damage variable is not explicitly considered there) is proved by discretization techniques. In [11], well posedness of a system also accounting for the evolution of z is proved by restating both equations as variational inequalities. It is worth observing that, differently from here, in [11] the evolution of z is assumed in to be *reversible*, corresponding to the choice $\alpha \equiv 0$ in the equivalent of (1.3); moreover, the quadratic term on the right hand side of the same relation is truncated (this simplifies the proof of existence compared to our case).

In the present paper, we actually prefer to follow a partially different approach, based on our recent work [8] (in collaboration with E. Bonetti and E. Rocca), where a strongly damped wave equation for a real valued variable u containing a general constraint term is analyzed. The basic idea stands in writing a weak formulation where test-functions are chosen in a "parabolic" Sobolev-Bochner space $\mathcal{V}$ (in particular, in [8], $\mathcal{V} = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$). In that setting, the monotone graph providing the constraint is restated as an operator acting in the duality between $\mathcal{V}$ and $\mathcal{V}'$. Although it can be shown rather easily that this reformulation is factually equivalent to the statement as a (suitable) variational inequality, it presents a number of notable advantages: firstly, it clarifies the regularity of the constraint term, which usually corresponds to a physical quantity; secondly, it permits us to prove further properties of solutions, like the energy inequality, or to analyze the long-time behavior, which will be the object of a forthcoming work. Finally, in this setting we can still take advantage of the basic tools of monotone operator theory in Hilbert spaces in order to prove existence.

It is worth pointing out a further mathematical difficulty of the present model compared to similar ones. Namely, the coupling term $|\mathbf{u}|^2/2$ in (1.3) has a sort of critical growth in space dimension 3. This is due to the *mixed* boundary conditions complementing (1.1), where the "Neumann" part (1.4) additionally accounts for the nonsmooth constraint term. In this setting, the best space regularity we can hope to obtain for $\mathbf{u}$ is H^1. This translates into an L^4-regularity of the trace on Γ and into a (maximal) L^2-regularity of $|\mathbf{u}|^2/2$. In view of this fact, the identification of the terms $\alpha(z_t)$ and $\beta(z)$ in the "doubly nonlinear" equation (1.3) is somehow nontrivial and requires a careful combination of monotonicity and semicontinuity tools.

We finally point out that evolution equations with inertial terms and *bilateral* constraints, like for instance $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_C, (corresponding to the case where only shear displacements are allowed) are mathematically much simpler to deal with. These models arise, for instance, in physical systems subjected to high pressure. Such a behavior is often referred to as "Mode II" evolution, in constrast to "Mode I" evolutions, that are those where the constraint is unilateral, as in (1.2).

The remainder of the paper is organized as follows: in the next section we introduce our assumptions on coefficients and data and state a mathematically rigorous weak formulation together with our main result. This states existence of at least one weak solution of suitable regularity. The proof occupies the remainder of the paper. In particular, in Section 3 a regularized problem is introduced and existence of a local solution to it is shown by means of a (Schauder) fixed point argument. Next, in Section 4, the approximation is removed by means of suitable a-priori estimates and compactness methods. Other qualitative properties of weak solutions, like the energy inequality, are also discussed there.

2 Assumptions and main results

We set $H := L^2(\Omega)$ and $\boldsymbol{H} := L^2(\Omega)^3$. Moreover, for $k \geq 1$, we introduce

$$\boldsymbol{H}^k_D := \{u \in H^k(\Omega; \mathbb{R}^3) : u = 0 \text{ on } \Gamma_D\}, \tag{2.1}$$

and we denote by $\boldsymbol{H}^{-k}_D$ the respective dual spaces. We put $\boldsymbol{V} := \boldsymbol{H}^1_D$, i.e.,

$$\boldsymbol{V} := \{u \in H^1(\Omega; \mathbb{R}^3) : u = 0 \text{ on } \Gamma_D\}. \tag{2.2}$$

We also set $\boldsymbol{V}_0 := H^1_0(\Omega; \mathbb{R}^3)$ and recall that $\boldsymbol{V}'_0 = H^{-1}(\Omega; \mathbb{R}^3)$. The spaces $\boldsymbol{V}$ and $\boldsymbol{V}_0$ are seen as (closed) subspaces of $H^1(\Omega; \mathbb{R}^3)$ (and in particular they inherit its norm). The duality between $\boldsymbol{V}$ and $\boldsymbol{V}'$ will be indicated by $\langle \cdot, \cdot \rangle$.

In the sequel we shall frequently use the continuity of the trace operator

$$\text{from } H^1(\Omega) \text{ to } H^{1/2}(\Gamma_C) \quad \text{and from } H^1(\Omega) \text{ to } L^4(\Gamma_C) \tag{2.3}$$

and its vector analogue. Moreover, the trace operator will be generally omitted in the notation; namely, functions defined in Ω and their traces on Γ_C will be indicated by the same letters.

Strong formulation. This can be stated as:

$$\mathbf{u}_{tt} - \operatorname{div}(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})) = \mathbf{g}, \quad \text{in } (0,T) \times \Omega, \tag{2.4}$$

$$\alpha(z_t) + z_t + \beta(z) \ni a - \frac{1}{2}|\mathbf{u}|^2, \quad \text{on } (0,T) \times \Gamma_C, \tag{2.5}$$

$$-\big(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})\big)\mathbf{n} \in \gamma(\mathbf{u} \cdot \mathbf{n})\mathbf{n} + z\mathbf{u}, \quad \text{on } (0,T) \times \Gamma_C, \tag{2.6}$$

complemented with the additional boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \text{on } (0,T) \times \Gamma_D, \tag{2.7}$$

$$\big(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})\big)\mathbf{n} = \mathbf{0}, \quad \text{on } (0,T) \times \Gamma_N, \tag{2.8}$$

and with the Cauchy conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{u}_t|_{t=0} = \mathbf{u}_1, \quad z|_{t=0} = z_0, \tag{2.9}$$

where the first two relations are assumed a.e. in Ω, while the third one is stated a.e. on Γ_C.

Remark 2.1. It is worth noting that we considered the *homogeneous* condition (2.8) just for the sake of simplicity. Indeed, one could deal with the case of a nonzero boundary traction $\boldsymbol{f}$ on the right hand side with standard modifications. Moreover, it may be also worth emphasizing that, if $\boldsymbol{\sigma} := \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the stress tensor, then the boundary relation (2.6) could be split into its normal and tangential parts as follows:

$$-\sigma_{\mathbf{n}} := -(\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \in \gamma(\mathbf{u} \cdot \mathbf{n}) + z\mathbf{u} \cdot \mathbf{n}, \quad \text{on } (0,T) \times \Gamma_C, \tag{2.10}$$

$$-\boldsymbol{\sigma_t} := -\boldsymbol{\sigma}\mathbf{n} + \big((\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}\big)\mathbf{n} = z\mathbf{u}_{\boldsymbol{t}} := z\big(\mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}\big), \quad \text{on } (0,T) \times \Gamma_C. \tag{2.11}$$

This corresponds exactly to the conditions considered, e.g., in [7, (1.23-24)] for $\nu = 0$, i.e., when no friction is assumed to occur on Γ_C.

As explained in the introduction, we are not able to provide a solution to the strong (pointwise) formulation of the system. Consequently, we need a weaker notion of solution. To introduce it, we start with stating our assumptions on coefficients and data:

(a) The tensors $\mathbb{V}, \mathbb{E} \in L^\infty(\Omega; \mathbb{R}^{81})$ satisfy, a.e. in Ω, the standard symmetry properties

$$\mathbb{V}_{ijkl} = \mathbb{V}_{jikl} = \mathbb{V}_{ijlk} = \mathbb{V}_{klij}, \quad \mathbb{E}_{ijkl} = \mathbb{E}_{jikl} = \mathbb{E}_{ijlk} = \mathbb{E}_{klij}. \tag{2.12}$$

Moreover, $\mathbb{V}, \mathbb{E}$ are assumed to be (uniformly in Ω) strongly positive definite; namely, there exists a constant $\kappa > 0$ such that

$$\mathbb{E}(x)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \geq \kappa|\boldsymbol{\varepsilon}|^2, \qquad \mathbb{V}(x)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \geq \kappa|\boldsymbol{\varepsilon}|^2, \tag{2.13}$$

for a.e. $x \in \Omega$ and any symmetric matrix $\boldsymbol{\varepsilon} \in \mathbb{R}^9$. Hence, in view of Poincaré's and Korn's inequalities, for any $\boldsymbol{v} \in \boldsymbol{V}$ there holds

$$\int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \geq \kappa\|\boldsymbol{v}\|_{\boldsymbol{V}}^2, \qquad \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \geq \kappa\|\boldsymbol{v}\|_{\boldsymbol{V}}^2, \tag{2.14}$$

for a (possibly different) constant $\kappa > 0$.

(b) We set $\alpha = \partial I_{(-\infty,0]}$; moreover, we assume β and γ be *maximal monotone graphs* in $\mathbb{R} \times \mathbb{R}$ such that $\overline{D(\beta)} = [0, +\infty)$ and $\overline{D(\gamma)} = (-\infty, 0]$. In particular, an admissible choice is $\beta = \partial I_{[0,+\infty)}$ and $\gamma = \partial I_{(-\infty,0]}$. We recall that the domain $D(b)$ of a graph $b \subset \mathbb{R} \times \mathbb{R}$ is the set $\{r \in \mathbb{R} : \ b(r) \neq \emptyset\}$. We denote as $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ suitable convex and lower semicontinuous functions from $\mathbb{R}$ to $(-\infty, +\infty]$ such that $\alpha = \partial\widehat{\alpha}$, $\beta = \partial\widehat{\beta}$, $\gamma = \partial\widehat{\gamma}$. Hence, we have in particular $\widehat{\alpha} = I_{(-\infty,0]}$. We also suppose that

$$\widehat{\beta}(r), \widehat{\gamma}(r) \geq 0 \quad \text{resp. for all } \ r \in D(\widehat{\beta}), D(\widehat{\gamma}). \tag{2.15}$$

(c) The initial data satisfy

$$\mathbf{u}_0 \in \boldsymbol{V}, \quad \widehat{\gamma}(\mathbf{u}_0 \cdot \mathbf{n}) \in L^1(\Gamma_C), \quad \mathbf{u}_1 \in \boldsymbol{H}, \tag{2.16}$$

together with

$$z_0 \in L^\infty(\Gamma_C), \quad 0 \leq z_0 \leq 1 \ \text{ a.e. on } \Gamma_C, \quad \beta^0(z_0) \in L^2(\Gamma_C). \tag{2.17}$$

Here, for $r \in D(\beta)$, $\beta^0(r)$ is the element of minimum absolute value in the set $\beta(r)$ (cf. [9]). Using the definition of subdifferential one may easily prove that (2.17) implies in particular

$$\widehat{\beta}(z_0) \in L^1(\Gamma_C). \tag{2.18}$$

(d) The volumic force satisfies $\mathbf{g} \in L^2(0, T; \boldsymbol{V}')$.

(e) We let Ω be a smooth and bounded domain of $\mathbb{R}^3$. We assume that $\Gamma = \partial\Omega$ satisfies $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$, where Γ_X, for $X = D, N, C$, are relatively open in Γ, mutually disjoint, and have strictly positive 2-dimensional Hausdorff measure. Moreover we assume the distance $\mathrm{d}(\Gamma_C, \Gamma_D) > 0$ and we require that Γ_C is smooth as a subset of Γ and has at most finitely many connected components. Specifically, we assume that the boundary of Γ_C in Γ is an at most finite union of curves of class C^1.

Remark 2.2. It would also be possible to consider more general choices for α. For instance, we may ask α to be a maximal monotone graph satisfying $\overline{D(\alpha)} = (-\infty, 0]$, the analogue of (2.15), plus some additional conditions regarding the behavior near 0. However, this would give rise to a number of technical complications in the proof, whence we decided to restrict ourselves to the basic choice $\alpha = \partial I_{(-\infty,0]}$. On the other hand, our somehow general assumptions on β and γ do not require any additional technical work.

Remark 2.3. Assumption (2.15) essentially states that β and γ must have some coercivity at ∞. For what concerns β this is in fact just a technical assumption, in view of the fact that we will prove that $z \leq 1$ almost everywhere. However, it may be useful in the approximation. Note that the analogue of (2.15) also holds for $\widehat{\alpha} = I_{(-\infty,0]}$.

Energy functional. System (2.4)-(2.6) has a natural variational formulation. Namely, it can be seen as a generalized gradient flow problem for a suitable *energy functional*. It is worth pointing out this structure from the very beginning. To this aim, we will obtain the *energy estimate* directly from the system equations. Of course, such a procedure has just a formal character at this level since we have not yet specified which is our notion of solution and the related regularity. That said, we first test (2.4) by $\mathbf{u}_t$. Using also (2.12) with the boundary conditions (2.7) on Γ_D and (2.8) on Γ_N, it is easy to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}|\mathbf{u}_t|^2 + \frac{1}{2}\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})\right) + \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) : \boldsymbol{\varepsilon}(\mathbf{u}_t) = \int_{\Gamma_C} \Big(\big(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})\big)\mathbf{n}\Big) \cdot \mathbf{u}_t + \langle \boldsymbol{g}, \mathbf{u}_t\rangle. \tag{2.19}$$

Then, we test (2.5) by z_t and integrate over Γ_C. A simple integration by parts in time gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_C} \left(\widehat{\beta}(z) - az + \frac{1}{2}z|\mathbf{u}|^2\right) + \int_{\Gamma_C} \big(\alpha(z_t) + z_t\big)z_t = \int_{\Gamma_C} z(\mathbf{u}\cdot\mathbf{u}_t). \tag{2.20}$$

Next, to cancel the terms on the right hand sides of (2.19)-(2.20), one scalarly multiplies (2.6) by $-\mathbf{u}_t$ and integrates, to obtain

$$\int_{\Gamma_C} \Big(\big(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})\big)\mathbf{n}\Big)\cdot \mathbf{u}_t = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_C} \widehat{\gamma}(\mathbf{u}\cdot\mathbf{n}) - \int_{\Gamma_C} z(\mathbf{u}\cdot\mathbf{u}_t). \tag{2.21}$$

Hence, taking the sum of (2.19), (2.20), (2.21), we (formally) obtain the *energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\mathbf{u},\mathbf{u}_t,z) + \mathcal{D}(\mathbf{u}_t,z_t) = \langle \boldsymbol{g}, \mathbf{u}_t\rangle, \tag{2.22}$$

with the energy functional $\mathcal{E} = \mathcal{E}(\mathbf{u},\mathbf{u}_t,z)$ given by

$$\mathcal{E} := \int_\Omega \left(\frac{1}{2}|\mathbf{u}_t|^2 + \frac{1}{2}\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})\right) + \int_{\Gamma_C}\left(\widehat{\beta}(z) - az + \frac{1}{2}z|\mathbf{u}|^2 + \widehat{\gamma}(\mathbf{u}\cdot\mathbf{n})\right) \tag{2.23}$$

and the dissipation integral(s) $\mathcal{D} = \mathcal{D}(\mathbf{u}_t, z_t)$ defined as

$$\mathcal{D} := \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) : \boldsymbol{\varepsilon}(\mathbf{u}_t) + \int_{\Gamma_C}\big(\alpha(z_t) + z_t\big)z_t. \tag{2.24}$$

Note that, in view of assumptions (a)-(d), both $\mathcal{E}$ and $\mathcal{D}$ enjoy suitable coercivity properties. Observe also that the right hand side of (2.22) accounts for the contribution of external volumic forces.

Although relation (2.22) corresponds to a basic physical property of the model, the mathematical procedure we used is formal under many aspects. The main problem is related to the occurrence of the nonsmooth *multivalued graphs* α, β, γ. Hence, we will see in Thm. 2.5 below that, for weak solutions, we will be only able to (rigorously) prove a weak version of (2.22) in the form of an *inequality* (cf. (2.44) below).

In order to state our precise concept of weak solution we start with introducing some more functional spaces:

$$\mathcal{V} := H^1(0,T;\boldsymbol{V}), \tag{2.25}$$

$$\mathcal{H} := H^1(0,T;H^{\frac{1}{2}}(\Gamma_C)), \tag{2.26}$$

and we let $\mathcal{V}'$ and $\mathcal{H}'$ be the respective dual spaces. Moreover, for all $t \in (0,T]$, we set

$$\mathcal{V}_t := H^1(0,t;\boldsymbol{V}), \tag{2.27}$$

$$\mathcal{H}_t := H^1(0,t;H^{\frac{1}{2}}(\Gamma_C)), \tag{2.28}$$

with the dual spaces $\mathcal{V}_t'$ and $\mathcal{H}_t'$, respectively. Note that $\mathcal{H}$ is exactly the space of traces (on Γ_C) of the elements of $\mathcal{V}$ (and similarly for $\mathcal{H}_t$ and $\mathcal{V}_t$). In the sequel, we shall note as $\langle\langle \cdot, \cdot \rangle\rangle$ the duality pairings with respect to both space and time variables. For instance, that symbol may note the duality between $\mathcal{V}$ and $\mathcal{V}'$ or also that between $\mathcal{H}$ and $\mathcal{H}'$. When working on subintervals $(0,t)$, $t \leq T$, we will use the notation $\langle\langle \cdot, \cdot \rangle\rangle_t$ (e.g., that may denote the duality between $\mathcal{V}_t$ and $\mathcal{V}_t'$). Not to weight up formulas, we will use the symbol $(\cdot, \cdot)$ for the scalar product in both $\boldsymbol{H}$ and $L^2(\Gamma_C)$. The norms in $\boldsymbol{H}$ and $L^2(\Gamma_C)$ will be sometimes simply noted by $\|\cdot\|$. The double brackets $(\!(\cdot, \cdot)\!)$ will represent the L^2-scalar product in time-space variables (for instance, in $L^2(0,T;\boldsymbol{H})$ or in $L^2(0,T;L^2(\Gamma_C))$). On time subintervals, we will use the notation $(\!(\cdot, \cdot)\!)_t$.

Next, we define the convex functional

$$G : L^2(0,T;L^2(\Gamma_C)) \to [0,+\infty], \quad G(v) := \int_0^T \int_{\Gamma_C} \widehat{\gamma}(v). \tag{2.29}$$

Then, if one considers the subdifferential ∂G in the (Hilbert) space $L^2(0,T;L^2(\Gamma_C))$, it is well known that this coincides with the realization of the graph γ. Namely, for $v, \eta \in L^2(0,T;L^2(\Gamma_C))$, one has

$$\eta \in \partial G(v) \ \Leftrightarrow \ \eta(t,x) \in \gamma(v(t,x)) \ \text{ a.e. on } \ (0,T) \times \Gamma_C. \tag{2.30}$$

Hence, in particular, v complies with the constraint represented by γ. On the other hand, in our specific situation, we will not be able to interpret γ in the above sense, due to regularity lack coming from the occurrence of the term $\mathbf{u}_{tt}$. For this reason, following the lines, e.g., of [8] (cf. also [3]), we provide a suitable relaxation of γ. To this aim, we identify $L^2(0,T;L^2(\Gamma_C))$ with its dual by means of the natural scalar product, obtaining the chain of continuous and dense inclusions

$$\mathcal{H} \subset L^2(0,T;L^2(\Gamma_C)) \sim L^2(0,T;L^2(\Gamma_C))' \subset \mathcal{H}'. \tag{2.31}$$

Then, the above constructed *Hilbert triplet* permits us to relax γ in the following sense: we define as γ_w (where the subscript "w" stands for "weak") the subdifferential of the restriction of G to the space $\mathcal{H}$ with respect to the duality between $\mathcal{H}$ and $\mathcal{H}'$. Namely, for $v \in \mathcal{H}$ and $\eta \in \mathcal{H}'$, we set

$$\eta \in \gamma_w(v) \ \Leftrightarrow \ \langle\langle \eta, w - v \rangle\rangle + G(v) \leq G(w) \ \text{ for all } \ w \in \mathcal{H}. \tag{2.32}$$

A precise characterization of γ_w, which is a maximal monotone operator from $\mathcal{H}$ to $2^{\mathcal{H}'}$, is carried out in [8, Sec. 2] following the lines of results first proved in [10] (see also [3]). Here we just mention the fact that, for any $v \in \mathcal{H}$, there holds the inclusion $\gamma(v) \subset \gamma_w(v)$, which may however be strict. On the other hand, once we know that $\eta \in \gamma_w(v)$, then v is still necessarily almost everywhere nonpositive (hence, it satisfies the constraint); moreover, the inclusion $\eta \in \gamma_w(v)$ has a precise "measure-theoretic" interpretation in terms of the original graph γ (see [8, Sec. 2] for more details). The analogue of γ_w, noted with the same symbol for the sake of simplicity, may be constructed also on the space $\mathcal{H}_t$, i.e., working on time subintervals.

We are now ready to introduce our concept of weak solution:

Definition 2.4. *Let $T > 0$ and let* Assumptions (a)-(e) *hold. We say that $(\mathbf{u}, \eta, z, \xi_1, \xi_2)$ is a weak solution to system* (2.4)-(2.9) *if*

$$\mathbf{u} \in W^{1,\infty}(0,T;\boldsymbol{H}) \cap H^1(0,T;\boldsymbol{V}), \tag{2.33a}$$

$$\mathbf{u}_t \in H^1(0,T;\boldsymbol{V}_0') \cap BV(0,T;\boldsymbol{H}_D^{-2}), \tag{2.33b}$$

$$z \in W^{1,\infty}(0,T;L^2(\Gamma_C)), \tag{2.33c}$$

$$\eta \in \mathcal{H}', \tag{2.33d}$$

$$\xi_1, \xi_2 \in L^\infty(0,T;L^2(\Gamma_C)) \tag{2.33e}$$

and the following properties are satisfied:

(i) For all $\boldsymbol{\varphi} \in \mathcal{V}$ it holds

$$\begin{aligned}(\mathbf{u}_t(T), \boldsymbol{\varphi}(T)) - (\!(\mathbf{u}_t, \boldsymbol{\varphi}_t)\!) + (\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!) + (\!(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!)\\ + \langle\langle \eta, \boldsymbol{\varphi} \cdot \mathbf{n} \rangle\rangle + \int_0^T \int_{\Gamma_C} z\mathbf{u} \cdot \boldsymbol{\varphi} = \langle\langle \boldsymbol{g}, \boldsymbol{\varphi} \rangle\rangle + (\mathbf{u}_1, \varphi(0)),\end{aligned} \tag{2.34}$$

with the initial conditions (2.9)*. Correspondingly, for every* $t \in [0,T)$ *there exists* $\eta_{(t)} \in \mathcal{H}_t$ *such that*

$$
\begin{aligned}
&(\mathbf{u}_t(t), \boldsymbol{\varphi}(t)) - (\!(\mathbf{u}_t, \varphi_t)\!)_t + (\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!)_t + (\!(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!)_t \\
&\qquad + \langle\!\langle \eta_{(t)}, \boldsymbol{\varphi}\cdot\mathbf{n}\rangle\!\rangle_t + \int_0^t \int_{\Gamma_C} z\mathbf{u}\cdot\boldsymbol{\varphi} = \langle\!\langle \boldsymbol{g}, \boldsymbol{\varphi}\rangle\!\rangle_t + (\mathbf{u}_1, \varphi(0)),
\end{aligned}
\tag{2.35}
$$

for all $\boldsymbol{\varphi} \in \mathcal{V}_t$*. Moreover, the functionals* η *and* $\eta_{(t)}$ *are* compatible*, namely, if* $\boldsymbol{\varphi} \in \mathcal{V}_t$ *satisfies* $\boldsymbol{\varphi}(t) = 0$ *a.e. on* Ω*, then we have*

$$
\langle\!\langle \eta_{(t)}, \boldsymbol{\varphi}\cdot\mathbf{n}\rangle\!\rangle_t = \langle\!\langle \eta, \overline{\boldsymbol{\varphi}}\cdot\mathbf{n}\rangle\!\rangle, \tag{2.36}
$$

where $\overline{\boldsymbol{\varphi}}$ *represents the trivial extension of* $\boldsymbol{\varphi}$ *to* $\mathcal{V}$ *(i.e.,* $\overline{\boldsymbol{\varphi}}(s,x) = 0$ *for* $s \in [t,T]$ *and* $x \in \Omega$*).*

(ii) For a.e. $t \in (0,T)$ *there holds*

$$
\xi_2(t) + z_t(t) + \xi_1(t) = a - \frac{1}{2}|\mathbf{u}(t)|^2, \quad \text{a.e. on } \Gamma_C. \tag{2.37}
$$

(iii) The following graphs inclusions hold true:

$$
\xi_1 \in \beta(z), \quad \text{a.e. on } (0,T)\times\Gamma_C, \tag{2.38}
$$
$$
\xi_2 \in \alpha(z_t), \quad \text{a.e. on } (0,T)\times\Gamma_C, \tag{2.39}
$$
$$
\eta \in \gamma_w(\mathbf{u}\cdot\mathbf{n}). \tag{2.40}
$$

Moreover, for all $t \in (0,T)$ *we have*

$$
\eta_{(t)} \in \gamma_w((\mathbf{u}\cdot\mathbf{n})_{\llcorner(0,t)}). \tag{2.41}
$$

Note that, since z and the term with η in (2.34) are concentrated on Γ_C, if we choose $\boldsymbol{\varphi} \in H^1(0,T;\boldsymbol{V}_0)$ in (2.34) and integrate by parts in time, we get back

$$
\langle u_{tt}, \boldsymbol{\varphi}\rangle + \int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) + \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) = \langle g, \boldsymbol{\varphi}\rangle, \tag{2.42}
$$

a.e. in $t \in [0,T]$, where the first duality product makes sense in view of the first (2.33b). Hence, in particular (2.34) implies (1.1) in the sense of distributions.

Let us now recall that the energy of the system was defined in (2.23). Moreover, analogously with (2.24), we introduce the energy dissipation as

$$
\mathcal{D} := \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) : \boldsymbol{\varepsilon}(\mathbf{u}_t) + \int_{\Gamma_C} (\xi_2 + z_t) z_t. \tag{2.43}
$$

We are now ready to state our existence theorem, constituting the main result of the present paper:

Theorem 2.5. *Let* $T > 0$ *and let* Assumptions (a)-(e) *hold. Then there exists* at least *one weak solution* $(\mathbf{u}, \eta, z, \xi_1, \xi_2)$ *to* Problem (2.4)-(2.9)*, in the sense of* Def. 2.4*. Moreover, for all times* $t_2 \in [0,T]$ *and for a.e.* $t_1 \in [0, t_2)$*, the following energy inequality holds:*

$$
\mathcal{E}(t_2) + \int_{t_1}^{t_2} \mathcal{D}(\cdot) \le \mathcal{E}(t_1) + \int_{t_1}^{t_2} \langle g, \mathbf{u}_t\rangle. \tag{2.44}
$$

The proof of the above result will occupy the remainder of the paper.

3 Regularized problem and local existence

In this section we introduce a regularized version of our problem and prove existence of a local (in time) solution by means of a suitable fixed point argument. In view of the fact that this procedure is standard under many aspects, we omit most details and just present the basic highlights.

First of all, for $\varepsilon \in (0,1)$ intended to go to 0 in the limit, we take suitable regularizations α^ε, β^ε and γ^ε of the maximal monotone graphs α, β and γ. In particular, we will consider the *Yosida approximations* defined, e.g., in [9], to which we refer the reader for details. Here we just recall that α^ε, β^ε and γ^ε are monotone and Lipschitz continuous functions defined on the whole real line. Moreover, they tend, respectively, to α, β and γ in a suitable way, usually referred to as *graph convergence* in $\mathbb{R}\times\mathbb{R}$. Namely, for any $[r;s] \in \alpha$ and any $\varepsilon \in (0,1)$, there exists $r^\varepsilon \in \mathbb{R}$ such that $r^\varepsilon \to r$ and $\alpha^\varepsilon(r^\varepsilon) \to s$ in $\mathbb{R}$ as $\varepsilon \searrow 0$, with analogous properties holding for β and γ. In view of our choice $\alpha = \partial I_{(-\infty,0]}$, we may explicitly compute

$$\alpha^\varepsilon(r) = \varepsilon^{-1}(r)^+, \quad A_\varepsilon := (\mathrm{Id} + \alpha^\varepsilon)^{-1}(r) = -(r)^- + \frac{\varepsilon}{\varepsilon+1}(r)^+. \tag{3.1}$$

Notice that A_ε is a nonexpansive operator.

Our aim will be to solve, at least locally in time, a regularized statement, which, in the strong form, can be written as follows:

$$\mathbf{u}_{tt} - \operatorname{div}\big(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})\big) = \mathbf{g}, \quad \text{in } \Omega, \tag{3.2}$$

$$\alpha^\varepsilon(z_t) + z_t + \beta^\varepsilon(z) = a - \frac{1}{2}|\mathbf{u}|^2, \quad \text{on } \Gamma_C, \tag{3.3}$$

$$-\big(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})\big)\mathbf{n} = \gamma^\varepsilon(\mathbf{u}\cdot\mathbf{n})\mathbf{n} + (z)^+\mathbf{u}, \quad \text{on } \Gamma_C, \tag{3.4}$$

coupled with the initial conditions (2.9) and the boundary conditions (2.7)-(2.8). The function $(z)^+$ in (3.4) denotes the positive part of z. Actually, z is not guaranteed to be nonnegative at the approximate level since the (smooth) function β^ε does not enforce any constraint.

In fact, we will deal with a weak formulation of the above system, to which we will apply Schauder's fixed point theorem. Hence, we start with introducing the fixed point space: for a small but otherwise arbitrary number $s \in (0,1/2)$, and for $T_0 \in (0,T]$ to be chosen at the end, we set

$$\mathcal{S}_s(T_0) := \big\{\boldsymbol{v} \in H^s(0,T_0;\boldsymbol{H}) \cap L^2(0,T_0;H^{\frac{1}{2}+s}(\Omega;\mathbb{R}^3)) : \ \boldsymbol{v}|_\Gamma \in L^4(0,T_0;L^4(\Gamma;\mathbb{R}^3))\big\}. \tag{3.5}$$

The space $\mathcal{S}_s(T_0)$ is naturally endowed with the graph norm, noted as $\|\cdot\|_{\mathcal{S}_s(T_0)}$ for brevity, which turns it into a Banach space. Notice also that the trace of $\boldsymbol{v}$ makes sense in view of the assumed $H^{\frac{1}{2}+s}$-space regularity. Then, for some $M > 0$, we consider the *closed* ball

$$B_M := \big\{\boldsymbol{v} \in \mathcal{S}_s(T_0) : \ \|\boldsymbol{v}\|_{\mathcal{S}_s(T_0)} \le M\big\}. \tag{3.6}$$

Note that the choice of $M > 0$ is essentially arbitrary. Its value will in fact influence the resulting final time T_0, but this is irrelevant at the light of the subsequent uniform estimates.

The basic steps of our fixed point argument are carried out in the next three lemmas.

Lemma 3.1. *Let $\mathbf{u}_0$ satisfy (2.16) and z_0 satisfy (2.17). More precisely, let us set*

$$U := \|\mathbf{u}_0\|_{\boldsymbol{V}} + \|\mathbf{u}_1\|_{\boldsymbol{H}} + \|\widehat{\gamma}(\mathbf{u}_0\cdot\mathbf{n})\|_{L^1(\Gamma_C)}, \tag{3.7}$$

$$Z := \|z_0\|_{L^2(\Gamma_C)} + \|\widehat{\beta}(z_0)\|_{L^1(\Gamma_C)}. \tag{3.8}$$

Let also $\overline{\mathbf{u}} \in B_M$. Then there exists one and only one function z, with

$$\|z\|_{L^\infty(0,T_0;L^2(\Gamma_C))} + \|z\|_{H^1(0,T_0;L^2(\Gamma_C))} \le Q(\varepsilon^{-1}, Z, M), \tag{3.9}$$

satisfying, a.e. on $(0,T_0)\times\Gamma_C$, the equation

$$\alpha^\varepsilon(z_t) + z_t + \beta^\varepsilon(z) = a - \frac{1}{2}|\overline{\mathbf{u}}|^2, \tag{3.10}$$

with the boundary condition $z|_{t=0} = z_0$. Here and below, Q denotes a computable nonnegative-valued function, increasingly monotone in each of its arguments, whose expression may vary on occurrence.

Proof. Using (2.17)-(2.18) with the graph convergence $\beta^\varepsilon \to \beta$, it is not difficult to prove that, at least for $\varepsilon \in (0,1)$ small enough, there holds

$$\|z_0\|_{L^2(\Gamma_C)} + \|\widehat{\beta}^\varepsilon(z_0)\|_{L^1(\Gamma_C)} \leq 2Z. \tag{3.11}$$

Then, existence of a solution z can be proved by using standard existence results for ODE's. The regularity (3.9) can be inferred simply by testing (3.10) by z_t. Actually, integrating over Γ_C, we then obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_C}\widehat{\beta}^\varepsilon(z) + \int_{\Gamma_C}\alpha^\varepsilon(z_t)z_t + \|z_t\|^2_{L^2(\Gamma_C)} &= \int_{\Gamma_C}\Big(a - \frac{1}{2}|\overline{\mathbf{u}}|^2\Big)z_t \\ &\leq \frac{1}{4}\|z_t\|^2_{L^2(\Gamma_C)} + c\big(1 + \|\overline{\mathbf{u}}\|^4_{L^4(\Gamma_C;\mathbb{R}^3)}\big).\end{aligned} \tag{3.12}$$

Observe that the integration by parts is allowed in view of the smoothness of β^ε. Then, by the first (3.1), the second integral on the left hand side is nonegative. Hence, using also (3.11), we infer

$$\|z_t\|_{L^2(0,T_0;L^2(\Gamma_C))} + \|\widehat{\beta}^\varepsilon(z)\|_{L^\infty(0,T_0;L^1(\Gamma_C))} \leq Q(M,Z,\varepsilon^{-1}), \tag{3.13}$$

whence, using again (3.11) to estimate z from z_t, we obtain (3.9). Finally, to ensure uniqueness of z one can use standard contractive arguments. For instance, if z_1 and z_2 are two solutions, one may test the difference of the corresponding equations (3.10) by the difference $(z_1 - z_2)_t$ and use monotonicity and Lipschitz continuity of α^ε and β^ε together with Gronwall's lemma. We omit details. ∎

Lemma 3.2. *Let $\mathbf{u}_0$, z_0, M, $\overline{\mathbf{u}}$ as above. Let also z be the function provided by the previous lemma. Then there exists one and only one function $\mathbf{u}$, with*

$$\|\mathbf{u}\|_{W^{1,\infty}(0,T_0;\boldsymbol{H})} + \|\mathbf{u}\|_{H^1(0,T_0;\boldsymbol{V})} + \|\mathbf{u}\|_{L^\infty(0,T_0;\boldsymbol{V})} \leq Q(\varepsilon^{-1},M,Z,U), \tag{3.14}$$

satisfying, a.e. on $(0,T_0)$ and for any $\boldsymbol{\varphi} \in \boldsymbol{V}$, the equation

$$\langle \mathbf{u}_{tt}, \boldsymbol{\varphi}\rangle + \int_\Omega (\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) + \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) + \int_{\Gamma_C}\big(\gamma^\varepsilon(\mathbf{u}\cdot\mathbf{n})\mathbf{n} + (z)^+\mathbf{u}\big)\cdot\boldsymbol{\varphi} = \langle \boldsymbol{g}, \boldsymbol{\varphi}\rangle, \tag{3.15}$$

together with the Cauchy conditions $\mathbf{u}|_{t=0} = \mathbf{u}_0$ and $\mathbf{u}_t|_{t=0} = \mathbf{u}_1$.

Proof. The weak formulation (3.15) is obtained from (3.2) simply testing by $\boldsymbol{\varphi} \in \boldsymbol{V}$, integrating by parts, and using the boundary condition (3.4). Then, existence of at least one solution can be proved by adapting the procedure given, e.g., in [7]. Here we just reproduce the corresponding regularity estimate, which is needed in order to get (3.14). Namely, we take $\boldsymbol{\varphi} = \mathbf{u}_t$ in (3.15), and, proceeding as in the Energy estimate detailed before, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{1}{2}|\mathbf{u}_t|^2 + \frac{1}{2}\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u})\Big) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_C}\widehat{\gamma}^\varepsilon(\mathbf{u}\cdot\mathbf{n}) + \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) : \boldsymbol{\varepsilon}(\mathbf{u}_t) \\ &\quad = -\int_{\Gamma_C}(z)^+(\mathbf{u}\cdot\mathbf{u}_t) + \langle \boldsymbol{g}, \mathbf{u}_t\rangle \leq \|z\|_{L^2(\Gamma_C)}\|\mathbf{u}\|_{L^4(\Gamma_C)}\|\mathbf{u}_t\|_{L^4(\Gamma_C)} + \|\boldsymbol{g}\|_{\boldsymbol{V}'}\|\mathbf{u}_t\|_{\boldsymbol{V}} \\ &\quad \leq Q(\varepsilon^{-1},M,Z)\|\mathbf{u}\|_{\boldsymbol{V}}\|\mathbf{u}_t\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{V}'}\|\mathbf{u}_t\|_{\boldsymbol{V}} \leq Q(\varepsilon^{-1},M,Z)\|\mathbf{u}\|^2_{\boldsymbol{V}} + c\|\boldsymbol{g}\|^2_{\boldsymbol{V}'} + \frac{\kappa}{2}\|\mathbf{u}_t\|^2_{\boldsymbol{V}},\end{aligned} \tag{3.16}$$

where κ is the same constant as in (2.13). Note that, to deduce the last inequalities, we used (3.9) together with the trace theorem (cf. (2.3)) and Young's inequality. Now, as before, from (3.7) and the graph convergence $\gamma^\varepsilon \to \gamma$, we can easily prove, at least for $\varepsilon \in (0,1)$ small enough,

$$\|\mathbf{u}_0\|_{\boldsymbol{V}} + \|\mathbf{u}_1\|_{\boldsymbol{H}} + \|\widehat{\gamma}^\varepsilon(\mathbf{u}_0\cdot\mathbf{n})\|_{L^1(\Gamma_C)} \leq 2U. \tag{3.17}$$

Hence, integrating (3.16) in time, recalling (2.13)-(2.14), and using Gronwall's lemma, we readily obtain that $\mathbf{u}$ satisfies (3.14).

To obtain uniqueness, we can proceed similarly as above. Namely, we assume to have two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ to (3.15), take the difference of the corresponding equations, and substitute $\boldsymbol{\varphi} = (\mathbf{u}_1 - \mathbf{u}_2)_t$ therein. Then, the Lipschitz continuity of γ^ε and the properties of the trace operator permit us to get a contractive estimate via a procedure similar to (3.16). ∎

Lemma 3.3. *Let* $\mathbf{u}_0$*,* z_0*,* M*,* $\overline{\mathbf{u}}$ *as above. Let also* z *be the function provided by* Lemma 3.1 *and* $\mathbf{u}$ *be the corresponding function provided by* Lemma 3.2. *Let us consider the map*

$$\mathcal{T}: B_M \to W^{1,\infty}(0,T_0;\boldsymbol{H}) \cap L^\infty(0,T_0;\boldsymbol{V}), \quad \mathcal{T}: \overline{\mathbf{u}} \mapsto \mathbf{u}. \tag{3.18}$$

Then, at least for $\varepsilon \in (0,1)$ *sufficiently small, we can take* $T_0 \in (0,T]$*, possibly depending on* ε*,* M*,* U *and* Z*, such that the map* $\mathcal{T}$

(a) takes values into B_M*;*

(b) is continuous with respect to the (strong) topology of $\mathcal{S}_s(T_0)$*;*

(c) maps B_M *into a compact subset of* $\mathcal{S}_s(T_0)$*.*

Proof. (a) Thanks to (3.14), to (2.3), and to standard embedding and trace theorems, we have

$$\|\mathbf{u}\|_{L^4(0,T_0;L^4(\Gamma;\mathbb{R}^3))} \le cT_0^{1/4}\|\mathbf{u}\|_{L^\infty(0,T_0;L^4(\Gamma;\mathbb{R}^3))} \le cT_0^{1/4}\|\mathbf{u}\|_{L^\infty(0,T_0;\boldsymbol{V})} \le cT_0^{1/4}Q, \tag{3.19}$$

$$\|\mathbf{u}\|_{H^s(0,T_0;\boldsymbol{H})} \le c\|\mathbf{u}\|_{H^1(0,T_0;\boldsymbol{H})} \le cT_0^{1/2}\|\mathbf{u}\|_{W^{1,\infty}(0,T_0;\boldsymbol{H})} \le cT_0^{1/2}Q, \tag{3.20}$$

$$\|\mathbf{u}\|_{L^2(0,T_0;H^{\frac{1}{2}+s}(\Omega;\mathbb{R}^3))} \le c\|\mathbf{u}\|_{L^2(0,T_0;\boldsymbol{V})} \le cT_0^{1/2}\|\mathbf{u}\|_{L^\infty(0,T_0;\boldsymbol{V})} \le cT_0^{1/2}Q, \tag{3.21}$$

where we wrote Q in place of $Q(\varepsilon^{-1},M,Z,U)$, for brevity, and where $c>0$ are embedding constants *independent of* T_0. Hence, we can choose T_0 sufficiently small, possibly depending on ε, so that

$$\|\mathbf{u}\|_{\mathcal{S}_s(T_0)} \le cT_0^{1/4}Q(\varepsilon^{-1},M,Z,U) \le M, \tag{3.22}$$

as desired.

(b) Let $\{\overline{\mathbf{u}}_n\} \subset B_M$ and let $\overline{\mathbf{u}}_n \to \overline{\mathbf{u}}$ in $\mathcal{S}_s(T_0)$. Let also z_n and z be the corresponding functions given by Lemma 3.1, let $\mathbf{u}_n = \mathcal{T}(\overline{\mathbf{u}}_n)$ and let $\mathbf{u} = \mathcal{T}(\overline{\mathbf{u}})$ be the corresponding solutions given by Lemma 3.2. We have to prove that

$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in } \mathcal{S}_s(T_0). \tag{3.23}$$

First, repeating, with the proper adaptations, the uniqueness argument sketched in Lemma 3.1 (and using in particular the Lipschitz continuity of α^ε and β^ε), we can easily show that

$$\lim_{n\nearrow\infty} \|z_n - z\|_{H^1(0,T_0;L^2(\Gamma_C))} = 0. \tag{3.24}$$

Next, we work on equation (3.15). Proceeding similarly with (3.16) and performing standard manipulations, it is not difficult to obtain

$$\lim_{n\nearrow\infty} \big(\|\mathbf{u}_n - \mathbf{u}\|_{W^{1,\infty}(0,T_0;\boldsymbol{H})} + \|\mathbf{u}_n - \mathbf{u}\|_{L^\infty(0,T_0;\boldsymbol{V})}\big) = 0, \tag{3.25}$$

This relation, also on account of (2.3), implies (3.23), as desired.

(c) The proof is similar to the above one, but a bit more tricky. Indeed, we still consider a sequence $\{\overline{\mathbf{u}}_n\} \subset B_M$, but we now just assume that

$$\overline{\mathbf{u}}_n \to \overline{\mathbf{u}} \quad \text{weakly in } \mathcal{S}_s(T_0). \tag{3.26}$$

Then, with the same notation as above, we need to show that at least a *subsequence* of $\{\mathbf{u}_n\}$ satisfies (3.23). To prove this fact, we first observe that, thanks to standard interpolation and (compact) embedding results, there exists a (non-relabelled) subsequence of n such that, for some $p \in (1,2)$ depending on the choice of s,

$$\overline{\mathbf{u}}_n \to \overline{\mathbf{u}} \quad \text{strongly in } L^{2p}(0,T_0;L^{2p}(\Gamma_C;\mathbb{R}^3)), \tag{3.27}$$

whence we have in particular

$$|\overline{\mathbf{u}}_n|^2 \to |\overline{\mathbf{u}}|^2 \quad \text{strongly in } L^p(0,T_0;L^p(\Gamma_C)) \quad \text{and weakly in } L^2(0,T_0;L^2(\Gamma_C)). \tag{3.28}$$

Let us now write (3.10) for the index n and for the limit (where z is the solution corresponding to $\overline{\mathbf{u}}$), and take the difference. Rearranging term and applying the inverse operator $(\mathrm{Id} + \alpha^\varepsilon)^{-1}$, we get the relation

$$(z_n - z)_t = (\mathrm{Id} + \alpha^\varepsilon)^{-1}\Big(- \frac{1}{2}|\overline{\mathbf{u}}_n|^2 + \frac{1}{2}|\overline{\mathbf{u}}|^2 - \beta^\varepsilon(z_n) + \beta^\varepsilon(z)\Big). \tag{3.29}$$

Then, testing by $|z_n - z|^{p-1}\,\mathrm{sign}(z_n - z)$ and using the Lipschitz continuity of $(\mathrm{Id} + \alpha^\varepsilon)^{-1}$ and of β^ε together with Hölder's and Young's inequalities, it is not difficult to arrive at

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n - z\|^p_{L^p(\Gamma_C)} \le C_\varepsilon\big(\|z_n - z\|^p_{L^p(\Gamma_C)} + \|\overline{\mathbf{u}}_n - \overline{\mathbf{u}}\|^p_{L^{2p}(\Gamma_C)}\|\overline{\mathbf{u}}_n + \overline{\mathbf{u}}\|^p_{L^{2p}(\Gamma_C)}\big). \tag{3.30}$$

Hence, applying Gronwall's lemma, we obtain

$$z_n \to z \quad \text{strongly in } \ L^p(0, T_0; L^p(\Gamma_C)). \tag{3.31}$$

Moreover, by the Lipschitz continuity of $(\mathrm{Id} + \alpha^\varepsilon)^{-1}$ and of β^ε, we can easily prove that the analogue of (3.31) holds also for $(z_n)_t$. Repeating the a priori estimates given at point (b) to obtain (3.24), we then conclude that

$$z_n \to z \quad \text{weakly in } \ H^1(0, T_0; L^2(\Gamma_C)). \tag{3.32}$$

Combining (3.31) and (3.32) we also get

$$(z_n)^+ \to (z)^+ \quad \text{strongly in } \ L^r(0, T_0; L^r(\Gamma_C)) \quad \text{for every } \ r \in [1, 2). \tag{3.33}$$

Next, considering equation (3.15) with right hand side depending on z_n and repeating the usual energy estimate, it is easy to obtain

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly star in } \ W^{1,\infty}(0, T_0; \boldsymbol{H}) \cap H^1(0, T; \boldsymbol{V}) \tag{3.34}$$

for some limit function $\mathbf{u}$. In particular, due to standard compact embedding results for vector-valued Sobolev spaces, this entails

$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in } \ H^s(0, T_0; \boldsymbol{H}) \cap L^2(0, T_0; H^{\frac{1}{2}+s}(\Omega; \mathbb{R}^3)). \tag{3.35}$$

Hence, to get (3.23), it remains to show that

$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in } \ L^4(0, T_0; L^4(\Gamma; \mathbb{R}^3)). \tag{3.36}$$

This will be proved at the end. Preliminary, we may notice that $\mathbf{u}$ solves the limit equation (3.15). Actually, we have

$$(z_n)^+\mathbf{u}_n \to (z)^+\mathbf{u} \quad \text{weakly in } \ L^r(0, T_0; L^r(\Gamma_C; \mathbb{R}^3)) \quad \text{for some } \ r > 1, \tag{3.37}$$

as a consequence of (3.33), (3.35), and of the continuity of the trace operator from $H^{\frac{1}{2}+s}(\Omega)$ to $H^s(\Gamma)$ for any $s \in (0, 1/2)$. Hence, it turns out that $\mathbf{u} = \mathcal{T}(\overline{\mathbf{u}})$. To conclude the proof, we then need to show (3.36). More precisely, we will reinforce (3.34) proving that

$$\mathbf{u}_n \to \mathbf{u} \quad \textbf{strongly} \text{ in } \ W^{1,\infty}(0, T_0; \boldsymbol{H}) \cap H^1(0, T; \boldsymbol{V}). \tag{3.38}$$

Then, (3.36) will follow from (2.3). To get (3.38) we need to use a semicontinuity argument, which we just sketch. Indeed, the same procedure will be repeated in the next section under more restrictive assumptions. We actually test (3.2), written for $\mathbf{u}_n$, by $\mathbf{u}_n$, and integrate. Then, using (3.4) and performing some integration by parts, we arrive at

$$\begin{aligned}\int_0^t\int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon) + \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t)) = \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_0) : \boldsymbol{\varepsilon}(\mathbf{u}_0) - \int_0^t\int_\Omega \gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})\mathbf{u}^\varepsilon\cdot\mathbf{n} \\ + \int_0^t\int_\Omega |\mathbf{u}^\varepsilon_t|^2 - (\mathbf{u}^\varepsilon_t(t), \mathbf{u}^\varepsilon(t)) + (\mathbf{u}_1, \mathbf{u}_0) - \int_0^t\int_{\Gamma_C}(z^\varepsilon)^+|\mathbf{u}^\varepsilon|^2 + \int_0^t\langle \boldsymbol{g}, \mathbf{u}^\varepsilon\rangle.\end{aligned} \tag{3.39}$$

We also test (3.2), written for the limit $\mathbf{u}$, by $\mathbf{u}$, obtaining an analogue relation. Then, we take the lim sup of (3.39) at the level n and we compare the outcome with (3.39) written for $\mathbf{u}$. Treating the terms on the right hand side by owing to the smoothness of γ^ε and by semicontinuity tools (see the next section for details), we then obtain, for every $t \in (0, T_0]$,

$$\limsup_{\varepsilon\searrow 0} \left(\int_0^t \int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon) + \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t)) \right)$$
$$\leq \int_0^t \int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}) + \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}(t)) : \boldsymbol{\varepsilon}(\mathbf{u}(t)). \tag{3.40}$$

Thanks to the symmetry and coercivity properties of the tensors $\mathbb{E}$ and $\mathbb{V}$ (cf. assumption (a)), we then get (3.38), whence (3.36). Summarizing, we have

$$\mathcal{T}(\overline{\mathbf{u}}_n) \to \mathcal{T}(\overline{\mathbf{u}}) \quad \text{strongly in } \mathcal{S}_s(T_0), \tag{3.41}$$

which actually holds for the whole sequence $\overline{\mathbf{u}}_n$. Hence, the map $\mathcal{T}$ is compact. This concludes the proof of the lemma. ∎

As a consequence of the three lemmas, we can apply Schauder's fixed point theorem to the map $\mathcal{T}$, at least for $\varepsilon \in (0,1)$ sufficiently small. This provides existence of *at least* one local solution $(\mathbf{u}^\varepsilon, z^\varepsilon)$ to the approximate system. To be precise, $(\mathbf{u}^\varepsilon, z^\varepsilon)$ satisfies, a.e. in $(0, T_0)$, and for any $\boldsymbol{\varphi} \in \boldsymbol{V}$,

$$\langle \mathbf{u}^\varepsilon_{tt}, \boldsymbol{\varphi}\rangle + \int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) + \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon_t) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) + \int_{\Gamma_C} \gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})\boldsymbol{\varphi}\cdot\mathbf{n} + \int_{\Gamma_C} (z^\varepsilon)^+\mathbf{u}^\varepsilon\cdot\boldsymbol{\varphi} = \langle \boldsymbol{g}, \boldsymbol{\varphi}\rangle. \tag{3.42}$$

Moreover, (3.3) holds a.e. on $(0, T_0)$ and the couple $(\mathbf{u}^\varepsilon, z^\varepsilon)$ also complies with the initial conditions (2.9).

Remark 3.4. It is worth remarking that $(\mathbf{u}^\varepsilon, z^\varepsilon)$ also satisfies an approximate version of the energy *equality*. Indeed, comparing terms in (3.42), one can easily prove that $\mathbf{u}^\varepsilon \in H^2(0, T_0; \boldsymbol{V}')$. In particular, this is sufficient in order for $\boldsymbol{\varphi} = \mathbf{u}^\varepsilon_t \in L^2(0, T_0; \boldsymbol{V})$ to be an admissible test function in (3.42). Hence, testing also (3.3) by z^ε_t and proceeding as in the last part of Section 2, we may infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon_t, z^\varepsilon) + \mathcal{D}^\varepsilon(\mathbf{u}^\varepsilon_t, z^\varepsilon_t) = \langle \boldsymbol{g}, \mathbf{u}^\varepsilon_t\rangle - \int_{\Gamma_C} \frac{1}{2}(z^\varepsilon)^-\mathbf{u}^\varepsilon\cdot\mathbf{u}^\varepsilon_t, \quad \text{a.e. in } (0, T_0). \tag{3.43}$$

where $\mathcal{E}^\varepsilon$ and $\mathcal{D}^\varepsilon$ are the approximate energy and dissipation functionals and the last term appears in view of the occurrence of the positive part in (3.4). Note that here all integrations by parts are fully justified in view of the fact that β and γ have been replaced by their regularized counterparts. Integrating (3.43) in time, we then also obtain the additional regularity

$$\mathbf{u}^\varepsilon \in C^1([0, T_0]; \boldsymbol{H}). \tag{3.44}$$

4 Global existence for the original system

In order to prove Theorem 2.5, we will show that, as $\varepsilon \to 0$, the regularized solutions $(\mathbf{u}^\varepsilon, z^\varepsilon)$ constructed before tend, in a suitable way and up to the extraction of a subsequence, to a weak solution to the original problem. It is worth noting from the very beginning that, at least in principle, the functions $(\mathbf{u}^\varepsilon, z^\varepsilon)$ are defined only on some subinterval $(0, T_0)$ possibly smaller than $(0, T)$ and also possibly depending on ε. However, in view of the fact that we shall derive a set of a-priori estimates that are independent of T_0, standard extension arguments imply that $(\mathbf{u}^\varepsilon, z^\varepsilon)$ can in fact be extended to the whole of $(0, T)$, and the same will hold for the limit solution $(\mathbf{u}, z)$. Hence, in order to reduce technical complications, we shall directly assume with no loss of generality that $(\mathbf{u}^\varepsilon, z^\varepsilon)$ are defined over $(0, T)$ already. In particular, the approximate energy equality (3.43) and the related regularity (3.44) turn out to hold on the whole of $(0, T)$.

That said, we proceed with the proof, which is subdivided into various steps, presented as separate lemmas.

Lemma 4.1 (Extension of boundary functions)**.** *Let $\boldsymbol{\psi} \in H^{1/2}(\Gamma_C;\mathbb{R}^3)$. Then, there exists $\boldsymbol{\varphi} =: R\boldsymbol{\psi} \in \boldsymbol{V}$ such that $\boldsymbol{\varphi}|_{\Gamma_C} = \boldsymbol{\psi}$ in the sense of traces. Moreover, the operator $R : \boldsymbol{\psi} \mapsto \boldsymbol{\varphi}$ is linear and continuous from $H^{1/2}(\Gamma_C;\mathbb{R}^3)$ to $\boldsymbol{V}$, namely there exists $c_\Omega > 0$, depending only on Ω, Γ_C, Γ_N, Γ_D, such that*

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{V}} \le c\|\boldsymbol{\psi}\|_{H^{1/2}(\Gamma_C;\mathbb{R}^3)}. \tag{4.1}$$

PROOF. We first observe that, in view of the regularity assumptions (e) on Ω, using extension by reflection and cutoff arguments, $\boldsymbol{\psi}$ can be extended to a function $\widetilde{\boldsymbol{\psi}}$ defined on the whole of Γ in such a way that $\widetilde{\boldsymbol{\psi}}$ is 0 a.e. on Γ_D and

$$\|\widetilde{\boldsymbol{\psi}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)} \le c\|\boldsymbol{\psi}\|_{H^{1/2}(\Gamma_C;\mathbb{R}^3)}. \tag{4.2}$$

Moreover, we can build the extension in such a way that the map $\boldsymbol{\psi} \mapsto \widetilde{\boldsymbol{\psi}}$ is linear. As a second step, we construct $\boldsymbol{\varphi}$ as the solution to the elliptic problem

$$-\Delta\boldsymbol{\varphi} = \mathbf{0} \ \text{ in } \Omega, \qquad \boldsymbol{\varphi} = \widetilde{\boldsymbol{\psi}} \ \text{ on } \Gamma. \tag{4.3}$$

Then, the desired properties follow from (4.2) and standard elliptic regularity results. ∎

Lemma 4.2 (Step 1: first a priori estimate)**.** *There exists a constant $M > 0$ independent of $\varepsilon \in (0,1)$ and of $T_0 \in (0,T]$ such that*

$$\|\mathbf{u}^\varepsilon\|_{W^{1,\infty}(0,T;\boldsymbol{H})} \le M, \tag{4.4a}$$
$$\|\mathbf{u}^\varepsilon\|_{H^1(0,T;\boldsymbol{V})} \le M, \tag{4.4b}$$
$$\|\mathbf{u}^\varepsilon_{tt}\|_{L^2(0,T;\boldsymbol{V}_0')} \le M, \tag{4.4c}$$
$$\|\mathbf{u}^\varepsilon_t\|_{BV(0,T;\boldsymbol{H}_D^{-2})} \le M, \tag{4.4d}$$
$$\|z^\varepsilon\|_{H^1(0,T;L^2(\Gamma_C))} \le M. \tag{4.4e}$$

Moreover, we have

$$\|\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n})\|_{L^1(0,T;L^1(\Gamma_C))} \le M, \tag{4.4f}$$
$$\|\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n})\|_{\mathcal{H}'} \le M, \tag{4.4g}$$

and, for all $t \in (0,T)$,

$$\|\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n})\|_{\mathcal{H}_t'} \le M. \tag{4.4h}$$

PROOF. Let us note as $\widehat{\beta}^\varepsilon$ and $\widehat{\gamma}^\varepsilon$ suitable antiderivatives of β^ε and γ^ε, respectively. Then, using the standard relation $|b^\varepsilon(r)| \le |b^0(r)|$ holding for any maximal monotone graph $b \subset \mathbb{R}\times\mathbb{R}$ (cf. [9]), and recalling assumption (2.15), it is not difficult to prove that, up to suitable choices of the integration constants, we can assume $\widehat{\beta}^\varepsilon$ and $\widehat{\gamma}^\varepsilon$ to be nonnegative.

That said, let us take $\boldsymbol{\varphi} = \mathbf{u}^\varepsilon_t$ in (3.42), multiply (3.3) by $\partial_t(z^\varepsilon)^+$ (i.e., the time derivative of the positive part of z^ε), sum the resulting expressions, and integrate in time. Noting as H the Heaviside function, we then observe that

$$\int_{\Gamma_C} \beta^\varepsilon(z^\varepsilon)\partial_t(z^\varepsilon)^+ = \int_{\Gamma_C} \beta^\varepsilon(z^\varepsilon)H(z^\varepsilon)\partial_t(z^\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_C} B^\varepsilon(z^\varepsilon), \tag{4.5}$$

where the function $B^\varepsilon(r)$ coincides with $\widehat{\beta}^\varepsilon(r)$ for $r > 0$ and is identically equal to $\widehat{\beta}^\varepsilon(0)$ for $r \le 0$. Hence, we infer

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^\varepsilon_t(t)\|^2 + \frac{1}{2}\int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t)) + \int_0^t\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon_t(t)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon_t(t)) + \int_{\Gamma_C} \widehat{\gamma}^\varepsilon(\mathbf{u}^\varepsilon(t)\cdot\mathbf{n}) \\
&\quad + \frac{1}{2}\int_{\Gamma_C} (z^\varepsilon)^+(t)|\mathbf{u}^\varepsilon(t)|^2 + \int_0^t \|\partial_t(z^\varepsilon)^+\|^2 + \int_0^t\int_{\Gamma_C} \alpha^\varepsilon(z^\varepsilon_t)\partial_t(z^\varepsilon)^+ + \int_{\Gamma_C} B^\varepsilon(z^\varepsilon(t)) \\
&= a\int_0^t\int_{\Gamma_C} \partial_t(z^\varepsilon)^+ + \int_0^t \langle \boldsymbol{g}, \mathbf{u}^\varepsilon_t\rangle + \frac{1}{2}\|\mathbf{u}_1\|^2 + \frac{1}{2}\int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}_0) : \boldsymbol{\varepsilon}(\mathbf{u}_0) + \int_{\Gamma_C}\widehat{\gamma}^\varepsilon(\mathbf{u}_0\cdot\mathbf{n}) + \int_{\Gamma_C} B^\varepsilon(z_0) \\
&\quad + \frac{1}{2}\int_{\Gamma_C} z_0|\mathbf{u}_0|^2 \le c + \frac{1}{2}\int_0^t \|\partial_t(z^\varepsilon)^+\|^2 + \frac{\kappa}{2}\int_0^t \|\nabla\mathbf{u}^\varepsilon_t\|^2 + c\|\boldsymbol{g}\|^2_{L^2(0,T;\boldsymbol{V}')},
\end{aligned} \tag{4.6}$$

where, in the last line, we have used Young's, Poincaré's and Korn's inequalities together with the analogue of (3.11), (3.17), and the fact that

$$\frac{1}{2}\int_{\Gamma_C} z_0|\mathbf{u}_0|^2 \le \|z_0\|_{L^2(\Gamma_C)}\|\mathbf{u}_0\|^2_{L^4(\Gamma_C;\mathbb{R}^3)} \le c\|z_0\|_{L^2(\Gamma_C)}\|\mathbf{u}_0\|^2_{\boldsymbol{V}} \le c. \tag{4.7}$$

Hence, recalling (2.13), we easily obtain (4.4a) and (4.4b).

Let us now test (3.3) by z_t^ε. Then, proceeding as before and noting that

$$\begin{aligned}\left|\int_0^t\int_{\Gamma_C} z_t^\varepsilon|\mathbf{u}^\varepsilon|^2\right| &\le \|z_t^\varepsilon\|_{L^2(0,T;L^2(\Gamma_C))}\|\mathbf{u}^\varepsilon\|^2_{L^4(0,T;L^4(\Gamma_C;\mathbb{R}^3))}\\ &\le \frac{1}{2}\|z_t^\varepsilon\|^2_{L^2(0,T;L^2(\Gamma_C))} + c\|\mathbf{u}^\varepsilon\|^4_{L^4(0,T;\boldsymbol{V})},\end{aligned} \tag{4.8}$$

using (4.4b) to estimate the last term, we readily infer (4.4e).

Next, we choose $\boldsymbol{\varphi} \in \boldsymbol{V}_0$ in (3.42). Then, the boundary integrals go away and a simple comparison of terms permits us to obtain (4.4c).

Let us now prove (4.4g). To this aim, let $\psi \in \mathcal{H}$. Then, in view of the regularity assumptions (e), we can think the outer unit normal $\mathbf{n}$ to Ω on Γ_C to be the trace of a smooth function, denoted with the same symbol, defined on an open neighbourhood $\Lambda \subset \mathbb{R}^3$ of Γ_C. As a consequence, $\boldsymbol{\psi} := \psi\mathbf{n}$ lies in $H^1(0,T;H^{1/2}(\Gamma_C;\mathbb{R}^3))$. Hence, applying Lemma 4.1, we may construct $\boldsymbol{\varphi} = R\boldsymbol{\psi} \in H^1(0,T;\boldsymbol{V})$ such that $\boldsymbol{\varphi}|_{\Gamma_C} = \boldsymbol{\psi}$) and

$$\|\boldsymbol{\varphi}\|_{H^1(0,T;\boldsymbol{V})} \le c\|\boldsymbol{\psi}\|_{H^1(0,T;H^{1/2}(\Gamma_C;\mathbb{R}^3))} \le c\|\psi\|_{\mathcal{H}}. \tag{4.9}$$

Taking such a $\boldsymbol{\varphi}$ in (3.42), integrating in time, performing suitable integrations by parts, comparing terms, and applying (2.13) and Korn's inequality, we then have

$$\begin{aligned}|\langle\langle\gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n}),\psi\rangle\rangle| &\le \|\mathbf{u}_t^\varepsilon\|_{L^2(0,T;\boldsymbol{H})}\|\boldsymbol{\varphi}_t\|_{L^2(0,T;\boldsymbol{H})} + \|\mathbf{u}_t^\varepsilon(T)\|_{\boldsymbol{H}}\|\boldsymbol{\varphi}(T)\|_{\boldsymbol{H}} + \|\mathbf{u}_1\|_{\boldsymbol{H}}\|\boldsymbol{\varphi}(0)\|_{\boldsymbol{H}}\\ &\quad + c\|\nabla\mathbf{u}_t^\varepsilon\|_{L^2(0,T;\boldsymbol{H})}\|\nabla\boldsymbol{\varphi}\|_{L^2(0,T;\boldsymbol{H})} + c\|\nabla\mathbf{u}^\varepsilon\|_{L^2(0,T;\boldsymbol{H})}\|\nabla\boldsymbol{\varphi}\|_{L^2(0,T;\boldsymbol{H})}\\ &\quad + \|(z^\varepsilon)^+\mathbf{u}^\varepsilon\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^3))}\|\boldsymbol{\varphi}\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^3))} + \|\boldsymbol{g}\|_{L^2(0,T;\boldsymbol{V}')}\|\boldsymbol{\varphi}\|_{L^2(0,T;\boldsymbol{V})}\\ &\le c\|\boldsymbol{\varphi}\|_{\mathcal{V}} \le c\|\psi\|_{\mathcal{H}},\end{aligned} \tag{4.10}$$

where the constants are provided by (4.4a)-(4.4c), (4.4e), and the continuity of the trace operator. Hence, (4.4g) follows. Repeating the same argument on subintervals, we also obtain (4.4h).

Finally, to get (4.4f) let us apply Lemma 4.1 to the function $\boldsymbol{\psi} = \mathbf{n}$. As noted above, we may assume $\mathbf{n} \in H^1(0,T;H^{1/2}(\Gamma_C;\mathbb{R}^3))$. Hence, we obtain an extension $\boldsymbol{m} \in H^1(0,T;\boldsymbol{V})$ of $\mathbf{n}$ to the whole domain Ω. Let us now plug $\boldsymbol{\varphi} = \mathbf{u}^\varepsilon + \boldsymbol{m}$ into (3.42). Then, estimating the resulting right hand side as in (4.10), it is not difficult to arrive at

$$\int_0^t\int_{\Gamma_C}\gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})(\mathbf{u}^\varepsilon+\boldsymbol{m})\cdot\mathbf{n} \le c\|\mathbf{u}^\varepsilon+\boldsymbol{m}\|_{\mathcal{V}} \le c, \tag{4.11}$$

the last inequality following from (4.4b) and Lemma 4.1 applied to $\mathbf{n}$. Then, we claim that the left hand side can be estimated from below as follows:

$$\int_0^t\int_{\Gamma_C}\gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})(\mathbf{u}^\varepsilon+\boldsymbol{m})\cdot\mathbf{n} = \int_0^t\int_{\Gamma_C}\gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})(\mathbf{u}^\varepsilon\cdot\mathbf{n}+1) \ge \kappa_0\|\gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})\|_{L^1(0,T:L^1(\Gamma_C))} - c. \tag{4.12}$$

with $\kappa_0 > \underline{0 \text{ and}}$ $c \ge 0$ independent of ε. To prove the last inequality in (4.12) we can use the assumption $\overline{D(\gamma)} = (-\infty, 0]$ with the coercivity (2.15). Indeed, on the one hand this implies that, for $r \ge -1/2$, there holds

$$\gamma^\varepsilon(r)(r+1) \ge \frac{1}{2}|\gamma^\varepsilon(r)|. \tag{4.13}$$

On the other hand, due to (2.15), either $\lim_{r\to-\infty}\gamma(r) < 0$ (and the same holds for γ^ε), whence we can reason as in (4.13) also for $r << 0$, or it is $\lim_{r\to-\infty}\gamma(r) = 0$ (and, again, the same holds for γ^ε), so that there is nothing to prove because in that case the L^1-norm of $\gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})$ may only explode on the set where $\mathbf{u}^\varepsilon\cdot\mathbf{n}\geq -1/2$. Hence, we have proved (4.12), which, combined with (4.11), gives (4.4f).

Finally, we take $\boldsymbol{\varphi}\in\boldsymbol{H}^2_D$ in (3.42). Then, noting that $\|\boldsymbol{\varphi}\|_{L^\infty(\Gamma_C)}\leq c\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^2_D}$, a comparison of terms in (3.42), together with estimate (4.4f), permits us to obtain (4.4d), which concludes the proof of the lemma. ∎

Lemma 4.3 (Step 2: second a priori estimate)**.** *There exists a constant $M>0$ independent of $\varepsilon\in(0,1)$ and of $T_0\in(0,T]$ such that*

$$\|\beta^\varepsilon(z^\varepsilon)\|_{L^\infty(0,T;L^2(\Gamma_C))}\leq M, \tag{4.14a}$$

$$\|\alpha^\varepsilon(z^\varepsilon_t)\|_{L^\infty(0,T;L^2(\Gamma_C))}\leq M, \tag{4.14b}$$

$$\|z^\varepsilon_t\|_{L^\infty(0,T;L^2(\Gamma_C))}\leq M. \tag{4.14c}$$

Proof. Let us multiply equation (3.10) by $\frac{\mathrm{d}}{\mathrm{d}t}\beta^\varepsilon(z^\varepsilon)=(\beta^\varepsilon)'(z^\varepsilon)z^\varepsilon_t$, so to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_C}|\beta^\varepsilon(z^\varepsilon)|^2+\int_{\Gamma_C}(\beta^\varepsilon)'(z^\varepsilon)|z^\varepsilon_t|^2+\int_{\Gamma_C}(\beta^\varepsilon)'(z^\varepsilon)\alpha^\varepsilon(z^\varepsilon_t)z^\varepsilon_t\\&\quad=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_C}\Big(a-\frac{1}{2}|\mathbf{u}^\varepsilon|^2\Big)\beta^\varepsilon(z^\varepsilon)+\int_{\Gamma_C}(\mathbf{u}^\varepsilon\cdot\mathbf{u}^\varepsilon_t)\beta^\varepsilon(z^\varepsilon),\end{aligned} \tag{4.15}$$

whence, integrating over $(0,t)$, $0<t\leq T$, and using that $|\beta^\varepsilon(\cdot)|\leq|\beta^0(\cdot)|$ together with assumption (2.17), we get

$$\begin{aligned}&\frac{1}{4}\|\beta^\varepsilon(z^\varepsilon(t))\|^2_{L^2(\Gamma_C)}+\int_0^t\int_{\Gamma_C}(\beta^\varepsilon)'(z^\varepsilon)|z^\varepsilon_t|^2+\int_0^t\int_{\Gamma_C}(\beta^\varepsilon)'(z^\varepsilon)\alpha^\varepsilon(z^\varepsilon_t)z^\varepsilon_t\\&\quad\leq\frac{1}{2}\|\beta^\varepsilon(z_0)\|^2_{L^2(\Gamma_C)}+\int_\Gamma\Big(a-\frac{1}{2}|\mathbf{u}^\varepsilon(t)|^2\Big)^2\\&\qquad+\|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;L^4(\Gamma_C;\mathbb{R}^3))}\Big(\frac{1}{2}\|\mathbf{u}^\varepsilon_t\|^2_{L^2(0,T;L^4(\Gamma_C;\mathbb{R}^3))}+\frac{1}{2}\|\beta^\varepsilon(z^\varepsilon)\|^2_{L^2(0,T;L^2(\Gamma_C))}\Big)\\&\quad\leq c\big(1+\|\beta^\varepsilon(z^\varepsilon)\|^2_{L^2(0,T;L^2(\Gamma_C))}\big),\end{aligned} \tag{4.16}$$

where we have also used Young's inequality and the estimates (4.4). Now, since $(\beta^\varepsilon)'\geq 0$ and α^ε is monotone and satisfies $\alpha^\varepsilon(0)=0$, we may use Gronwall's lemma to obtain (4.14a). Moreover, applying the nonexpansive operator $(\mathrm{Id}+\alpha^\varepsilon)^{-1}$, we may rewrite (3.10) in the form

$$z^\varepsilon_t=(\mathrm{Id}+\alpha^\varepsilon)^{-1}\Big(a-\frac{1}{2}|\mathbf{u}^\varepsilon|^2-\beta^\varepsilon(z^\varepsilon)\Big). \tag{4.17}$$

Comparing terms in (4.17), we then get (4.14c). With this information at disposal, we go back to (3.10) and a further comparison argument gives also (4.14b). ∎

Lemma 4.4 (Step 3: converging subsequence)**.** *There exist limit functions $(\mathbf{u},z,\eta,\xi_1,\xi_2)$ such that, for a (non relabelled) subsequence of $\varepsilon\to 0$, there holds*

$$\mathbf{u}^\varepsilon\rightharpoonup\mathbf{u}\quad\textit{weakly in } H^1(0,T;\boldsymbol{V})\ \textit{ and weakly star in } W^{1,\infty}(0,T;\boldsymbol{H}), \tag{4.18a}$$

$$\mathbf{u}^\varepsilon_t\rightharpoonup\mathbf{u}_t\quad\textit{weakly in } H^1(0,T;\boldsymbol{V}_0')\ \textit{ and weakly star in } BV(0,T;\boldsymbol{H}^{-2}_D), \tag{4.18b}$$

$$z^\varepsilon\to z\quad\textit{weakly star in } W^{1,\infty}(0,T;L^2(\Gamma_C)), \tag{4.18c}$$

$$\gamma^\varepsilon(\mathbf{u}^\varepsilon\cdot\mathbf{n})\rightharpoonup\eta\quad\textit{weakly in } \mathcal{H}', \tag{4.18d}$$

$$\beta^\varepsilon(z^\varepsilon)\rightharpoonup\xi_1\quad\textit{weakly star in } L^\infty(0,T;L^2(\Gamma_C)), \tag{4.18e}$$

$$\alpha^\varepsilon(z^\varepsilon_t)\rightharpoonup\xi_2\quad\textit{weakly star in } L^\infty(0,T;L^2(\Gamma_C)), \tag{4.18f}$$

together with

$$\mathbf{u}^\varepsilon_t\to\mathbf{u}_t\quad\textit{strongly in } L^2(0,T;\boldsymbol{H}), \tag{4.19}$$

$$\mathbf{u}^\varepsilon_t(t)\rightharpoonup\mathbf{u}_t(t)\quad\textit{weakly in } \boldsymbol{H},\quad\textit{for all } t\in[0,T]. \tag{4.20}$$

Moreover for all $t \in (0,T)$ there exists $\eta_{(t)} \in \mathcal{H}_t'$ such that, for the same subsequence of $\varepsilon \searrow 0$ considered before,

$$\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n})_{\llcorner(0,t)} \rightharpoonup \eta_{(t)} \quad \text{weakly in } \mathcal{H}_t'. \tag{4.21}$$

Proof. Convergences (4.18a), (4.18b), (4.18d) follow from estimates (4.4). Convergences (4.18e) and (4.18f) follow from (4.14). Moreover, (4.14c) implies (4.18c). Let us now show (4.19) and (4.20). Thanks to (4.18a) and (4.18b), we can apply the generalized form of Aubin-Lions lemma ([24, Corollary 4], [21, Corollary 7.9]) with the triple of spaces $\boldsymbol{V} \subset\subset \boldsymbol{H} \subset \boldsymbol{V}_0'$. This provides (4.19). To see (4.20), we first observe that such weak limit holds true in the space $\boldsymbol{V}_0'$ by (4.18b). Then the claim follows thanks to the fact that $\|\mathbf{u}_t(t)\|_{\boldsymbol{H}} \leq M$ for *every* $t \in [0,T]$ by (4.18a) and (3.44).

It remains to show (4.21). Let us set $v^\varepsilon := (z^\varepsilon)^+$. Then, thanks to (4.18c), there exists a nonnegative function v such that

$$v^\varepsilon \rightharpoonup v \quad \text{weakly in } L^2(0,T;L^2(\Gamma_C)). \tag{4.22}$$

Moreover, by (4.18a)-(4.18b), the Aubin-Lions lemma, and (2.3), we infer

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{strongly in } L^r(0,T;L^r(\Gamma_C;\mathbb{R}^3)), \quad \text{for all } r \in [1,4). \tag{4.23}$$

Combining (4.18a), (4.22) and (4.23), we obtain

$$v^\varepsilon \mathbf{u}^\varepsilon \to v\mathbf{u} \quad \text{weakly in } L^{4/3}(0,T;L^{4/3}(\Gamma_C;\mathbb{R}^3)). \tag{4.24}$$

Now, for all $t \in (0,T)$, let $\psi \in \mathcal{H}_t$, let $\boldsymbol{\psi} := \psi\mathbf{n}$ and let $\boldsymbol{\varphi}$ be the extension of $\boldsymbol{\psi}$ provided by Lemma 4.1. Then let us define a functional $\eta_{(t)} \in \mathcal{H}_t'$ as follows:

$$\begin{aligned}\langle\!\langle \eta_{(t)}, \psi \rangle\!\rangle_t := \int_0^t (\mathbf{u}_t, \boldsymbol{\varphi}_t) - (\mathbf{u}_t(t), \boldsymbol{\varphi}(t)) + (\mathbf{u}_1, \boldsymbol{\varphi}(0)) - \int_0^t \int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \\ - \int_0^t \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) - \int_0^T \int_{\Gamma_C} v\mathbf{u} \cdot \boldsymbol{\varphi} + \int_0^t \langle g, \boldsymbol{\varphi} \rangle.\end{aligned} \tag{4.25}$$

Now, thanks to (4.18a) and (4.22), it is seen that the value of $\langle\!\langle \eta_{(t)}, \psi \rangle\!\rangle_t$ is exactly the limit of

$$\begin{aligned}\int_0^t \int_{\Gamma_C} \gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n})\psi = \int_0^t (\mathbf{u}_t^\varepsilon, \boldsymbol{\varphi}_t) - (\mathbf{u}_t^\varepsilon(t), \boldsymbol{\varphi}(t)) + (\mathbf{u}^\varepsilon, \boldsymbol{\varphi}(0)) - \int_0^t \int_\Omega \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \\ - \int_0^t \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t^\varepsilon) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) - \int_0^T \int_{\Gamma_C} v^\varepsilon \mathbf{u}^\varepsilon \cdot \boldsymbol{\varphi} + \int_0^t \langle g, \boldsymbol{\varphi} \rangle,\end{aligned} \tag{4.26}$$

which is obtained integrating (3.42) in time, rearranging terms, and performing some integration by parts. Hence, in particular $\eta_{(t)}$ is independent of the extension map $\psi \mapsto \boldsymbol{\varphi}$. We also observe that, thanks to (4.18a)-(4.18b) and (4.24), the right hand side of (4.26) converges (to the right hand side of (4.25)) with no need of extracting further subsequences. The thesis follows. ∎

Lemma 4.5 (Step 4: refined convergence for $\mathbf{u}$)**.** *There hold the additional strong convergences:*

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{strongly in } L^2(0,T;\boldsymbol{V}), \tag{4.27}$$

$$\mathbf{u}^\varepsilon(t) \to \mathbf{u}(t) \quad \text{strongly in } \boldsymbol{V} \text{ for all } t \in [0,T]. \tag{4.28}$$

Moreover, the functions η and $\eta_{(t)}$ are identified as follows:

$$\eta \in \gamma_w(\mathbf{u} \cdot \mathbf{n}), \tag{4.29}$$

$$\eta_{(t)} \in \gamma_w((\mathbf{u} \cdot \mathbf{n})_{\llcorner(0,t)}). \tag{4.30}$$

Proof. Let us define, analogously with (2.29), the convex functional

$$G^\varepsilon : L^2(0,T;L^2(\Gamma_C)) \to [0,+\infty), \quad G^\varepsilon(v) := \int_0^T \int_{\Gamma_C} \widehat{\gamma}^\varepsilon(v). \tag{4.31}$$

Then, as observed before, the subdifferential ∂G^ε in $L^2(0,T;L^2(\Gamma_C))$ factually coincides with the graph γ^ε. On the other hand, we may interpret the function γ^ε also as a monotone operator from $\mathcal{H}$ into $\mathcal{H}'$. Indeed, if $v \in \mathcal{H}$, then it results $\gamma^\varepsilon(v) \in L^2(0,T;L^2(\Gamma_C)) \subset \mathcal{H}'$ thanks to the Lipschitz continuity of γ^ε. Moreover, for all $u, v \in \mathcal{H}$ we have

$$\langle\!\langle \gamma^\varepsilon(u) - \gamma^\varepsilon(v), u - v \rangle\!\rangle = (\!(\gamma^\varepsilon(u) - \gamma^\varepsilon(v), u - v)\!) \geq 0. \tag{4.32}$$

Also, for $u \in \mathcal{H}$, $\gamma^\varepsilon(u)$ belongs to the subdifferential γ_w^ε of (the restriction to $\mathcal{H}$ of) G^ε at the point u computed with respect to the duality pairing between $\mathcal{H}'$ and $\mathcal{H}$. Indeed, we have

$$\langle\!\langle \gamma^\varepsilon(u), v - u \rangle\!\rangle = \int_0^T \int_{\Gamma_C} \gamma^\varepsilon(u)(v-u) \leq \int_0^T \int_{\Gamma_C} \big(\widehat{\gamma}^\varepsilon(v) - \widehat{\gamma}^\varepsilon(u) \big) = G^\varepsilon(v) - G^\varepsilon(u), \tag{4.33}$$

for all $v \in \mathcal{H}$. In short terms, we have $\gamma^\varepsilon \subset \gamma_w^\varepsilon$, where γ_w^ε acts as a maximal monotone operator between $\mathcal{H}$ and $\mathcal{H}'$.

Now, thanks to the monotonicity with respect to ε of the functionals G^ε, owing to [1, Theorem 3.20] and [1, Theorem 3.66], the maximal monotone operators γ_w^ε converge to γ_w in the graph sense:

$$\forall\, [x;y] \in \gamma_w, \quad \exists\, [x^\varepsilon; y^\varepsilon] \in \gamma_w^\varepsilon \quad \text{such that} \quad [x^\varepsilon; y^\varepsilon] \to [x;y], \tag{4.34}$$

where the convergence is intended with respect to the strong topology of $\mathcal{H} \times \mathcal{H}'$.

In order to take the limit of the equation for $\mathbf{u}$, we first observe that (3.42), after integration in time over $(0,T)$, can be equivalently rewritten in the form

$$\begin{aligned} &(\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!) + (\!(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t^\varepsilon), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!) + (\!(\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}), \boldsymbol{\varphi} \cdot \mathbf{n})\!) \\ &\quad = (\!(\mathbf{u}_t^\varepsilon, \boldsymbol{\varphi}_t)\!) - (\mathbf{u}_t^\varepsilon(T), \boldsymbol{\varphi}(T)) + (\mathbf{u}_1, \boldsymbol{\varphi}(0)) - \int_0^T \int_{\Gamma_C} v^\varepsilon \mathbf{u}^\varepsilon \cdot \boldsymbol{\varphi} + \langle\!\langle \boldsymbol{g}, \boldsymbol{\varphi} \rangle\!\rangle, \end{aligned} \tag{4.35}$$

for any function $\boldsymbol{\varphi} \in \mathcal{V}$, where $v^\varepsilon = (z^\varepsilon)^+$. Let us also notice that the counterpart of (4.35) on subintervals $(0,t)$ could be stated analogously. We now aim to let $\varepsilon \searrow 0$. To start with, we take $\boldsymbol{\varphi} = \mathbf{u}^\varepsilon$ in (4.35). Then, integrating by parts and rearranging terms, we get

$$\begin{aligned} (\!(\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}), \mathbf{u}^\varepsilon \cdot \mathbf{n})\!) &= -(\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon), \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon))\!) - \frac{1}{2} \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(T)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(T)) + \frac{1}{2} \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_0) : \boldsymbol{\varepsilon}(\mathbf{u}_0) \\ &+ (\!(\mathbf{u}_t^\varepsilon, \mathbf{u}_t^\varepsilon)\!) - (\mathbf{u}_t^\varepsilon(T), \mathbf{u}^\varepsilon(T)) + (\mathbf{u}_1, \mathbf{u}_0) - \int_0^T \int_{\Gamma_C} v^\varepsilon |\mathbf{u}^\varepsilon|^2 + \langle\!\langle \boldsymbol{g}, \mathbf{u}^\varepsilon \rangle\!\rangle. \end{aligned} \tag{4.36}$$

Now, recalling (4.23)-(4.24), we may take the limit $\varepsilon \searrow 0$ in relation (4.35). Actually, using also (4.18a) and (4.18d), we easily arrive at

$$\begin{aligned} &(\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!) + (\!(\mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}))\!) + \langle\!\langle \eta, \boldsymbol{\varphi} \cdot \mathbf{n} \rangle\!\rangle \\ &\quad = (\!(\mathbf{u}_t, \boldsymbol{\varphi}_t)\!) - (\mathbf{u}_t(T), \boldsymbol{\varphi}(T)) + (\mathbf{u}_1, \boldsymbol{\varphi}(0)) - \int_0^T \int_{\Gamma_C} v \mathbf{u} \cdot \boldsymbol{\varphi} + \langle\!\langle \boldsymbol{g}, \boldsymbol{\varphi} \rangle\!\rangle. \end{aligned} \tag{4.37}$$

In particular, in view of the fact that $\mathbf{u} \in \mathcal{V}$, we may take $\boldsymbol{\varphi} = \mathbf{u}$. Proceeding as for (4.36), we then have

$$\begin{aligned} \langle\!\langle \eta, \mathbf{u} \cdot \mathbf{n} \rangle\!\rangle &= -(\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{u}))\!) - \frac{1}{2} \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}(T)) : \boldsymbol{\varepsilon}(\mathbf{u}(T)) + \frac{1}{2} \int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_0) : \boldsymbol{\varepsilon}(\mathbf{u}_0) \\ &+ (\!(\mathbf{u}_t, \mathbf{u}_t)\!) - (\mathbf{u}_t(T), \mathbf{u}(T)) + (\mathbf{u}_1, \mathbf{u}_0) - \int_0^T \int_{\Gamma_C} v |\mathbf{u}|^2 + \langle\!\langle \boldsymbol{g}, \mathbf{u} \rangle\!\rangle. \end{aligned} \tag{4.38}$$

Now, we use the so-called Minty's semicontinuity trick in order to identify the function η. To this aim, we take the $\limsup$ as $\varepsilon \searrow 0$ in (4.36) and we compare the outcome to (4.38). Here, the key point stands in dealing with the integral term. Actually, setting

$$\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \quad \Phi(v, \mathbf{u}) := (v)^+ |\mathbf{u}|^2, \tag{4.39}$$

we may observe that Φ is continuous and nonnegative. Moreover, it is convex in $\mathbf{u}$ for any $v \in \mathbb{R}$. Hence, Ioffe's semicontinuity theorem (cf., e.g., [15]) yields

$$\int_0^T \int_{\Gamma_C} v|\mathbf{u}|^2 = \int_0^T \int_{\Gamma_C} \Phi(v, \mathbf{u}) \le \liminf_{\varepsilon \searrow 0} \int_0^T \int_{\Gamma_C} \Phi(v^\varepsilon, \mathbf{u}^\varepsilon) = \liminf_{\varepsilon \searrow 0} \int_0^T \int_{\Gamma_C} v^\varepsilon |\mathbf{u}^\varepsilon|^2. \tag{4.40}$$

Indeed, we already know that $(v)^+ = v$ and $(v^\varepsilon)^+ = v^\varepsilon$.

With (4.40) at disposal, taking the $\limsup$ of (4.36), using (4.18a), (4.19)-(4.20), (4.23) and semicontinuity of norms with respect to weak convergence, and finally comparing with (4.38), we get

$$\limsup_{\varepsilon \searrow 0} \langle\!\langle \gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}), \mathbf{u}^\varepsilon \cdot \mathbf{n} \rangle\!\rangle \le \langle\!\langle \eta, \mathbf{u} \cdot \mathbf{n} \rangle\!\rangle. \tag{4.41}$$

Thanks to the fact that $\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}) \in \gamma_w^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n})$ and to the graph convergence (4.34), this suffices to prove (4.29), i.e., to identify η (cf. [1, Prop. 3.59], see also [9, Prop. 2.5]).

At this point, we are able to reinforce the strong convergence of $\mathbf{u}^\varepsilon$ in (4.23). To this aim, let us first notice that, in view of (4.34), there exists a sequence $[x^\varepsilon; y^\varepsilon] \in \gamma_w^\varepsilon$ such that $[x^\varepsilon; y^\varepsilon] \to [\mathbf{u}\cdot\mathbf{n}; \eta]$ strongly in $\mathcal{H} \times \mathcal{H}'$. Then, by monotonicity of γ_w^ε, we have

$$\langle\!\langle \gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}), \mathbf{u}^\varepsilon \cdot \mathbf{n} \rangle\!\rangle \ge \langle\!\langle \gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}), x^\varepsilon \rangle\!\rangle + \langle\!\langle y^\varepsilon, \mathbf{u}^\varepsilon \cdot \mathbf{n} \rangle\!\rangle - \langle\!\langle y^\varepsilon, x^\varepsilon \rangle\!\rangle. \tag{4.42}$$

Then, taking the $\liminf$, noting that all terms on the right hand side pass to the limit, and using (4.41), we may conclude that

$$\lim_{\varepsilon \searrow 0} (\!(\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}), \mathbf{u}^\varepsilon \cdot \mathbf{n})\!) = \langle\!\langle \eta, \mathbf{u} \cdot \mathbf{n} \rangle\!\rangle. \tag{4.43}$$

At this point, we rearrange terms in (4.36) rewriting it in the following way:

$$\begin{aligned} (\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon), \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon))\!) + \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(T)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(T)) &= -(\!(\gamma^\varepsilon(\mathbf{u}^\varepsilon \cdot \mathbf{n}), \mathbf{u}^\varepsilon \cdot \mathbf{n})\!) + \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}_0) : \boldsymbol{\varepsilon}(\mathbf{u}_0) \\ + (\!(\mathbf{u}_t^\varepsilon, \mathbf{u}_t^\varepsilon)\!) - (\mathbf{u}_t^\varepsilon(T), \mathbf{u}^\varepsilon(T)) + (\mathbf{u}_1, \mathbf{u}_0) &- \int_0^T \int_{\Gamma_C} v^\varepsilon |\mathbf{u}^\varepsilon|^2 + \langle\!\langle \boldsymbol{g}, \mathbf{u}^\varepsilon \rangle\!\rangle. \end{aligned} \tag{4.44}$$

We also rearrange terms in (4.38) in a similar way. Then, taking the $\limsup$ of (4.44), using (4.43), and comparing with (the rearranged) (4.38), we obtain

$$\limsup_{\varepsilon \searrow 0} \left((\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon), \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon))\!) + \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(T)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(T)) \right) \le (\!(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{u}))\!) + \frac{1}{2}\int_\Omega \mathbb{V}\boldsymbol{\varepsilon}(\mathbf{u}(T)) : \boldsymbol{\varepsilon}(\mathbf{u}(T)). \tag{4.45}$$

Using assumption (a), we then get (4.27). Then, repeating the argument on subintervals $(0, t)$, we also obtain (4.28). As a further consequence, combining (4.28) with the uniform boundedness (4.4b), we get

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{strongly in } L^4(0, T; \boldsymbol{V}), \quad \text{whence in } L^4(0, T; L^4(\Gamma_C; \mathbb{R}^3)), \tag{4.46}$$

the latter property following from (2.3). Relation (4.46) will play a key role when we take the limit of the equation for z.

To conclude the proof of the lemma, we need to show (2.41). This, however, follows simply by repeating the above argument on the subintervals $(0, t)$, for $t \le T$. ∎

Lemma 4.6 (Step 5: refined convergence for z)**.** *There hold the additional strong convergences*

$$z^\varepsilon \to z \quad \textit{strongly in } L^2(0, T; L^2(\Gamma_C)), \tag{4.47}$$

$$z^\varepsilon(t) \to z(t) \quad \textit{strongly in } L^2(\Gamma_C), \quad \textit{for all } t \in [0, T]. \tag{4.48}$$

Proof. We prove (4.47) and (4.48) by adapting an argument due to Blanchard, Damlamian and Ghidouche [5, Lemma 3.3] (see also [13]). To start with, applying A_ε (cf. (3.1)), equation (3.10) may be equivalently rewritten as

$$z_t^\varepsilon = A_\varepsilon\Big(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\varepsilon(z^\varepsilon)\Big). \tag{4.49}$$

For $\delta \in (0, \frac{1}{2})$, we subtract from (4.49) the analogue expression with δ in place of ε, then we multiply the result by $z^\varepsilon - z^\delta$. After integration on $(0,t)\times\Gamma_C$, we get

$$\begin{aligned}
&\frac{1}{2}\|z^\varepsilon(t) - z^\delta(t)\|^2_{L^2(\Gamma_C)} = \int_0^t\int_{\Gamma_C}\Big(A_\varepsilon\big(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\varepsilon(z^\varepsilon)\big) - A_\varepsilon\big(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\delta(z^\delta)\big)\Big)(z^\varepsilon - z^\delta)\\
&\quad + \int_0^t\int_{\Gamma_C}\Big(A_\varepsilon\big(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\delta(z^\delta)\big) - A_\varepsilon\big(a - \frac{1}{2}|\mathbf{u}^\delta|^2 - \beta^\delta(z^\delta)\big)\Big)(z^\varepsilon - z^\delta)\\
&\quad + \int_0^t\int_{\Gamma_C}\Big(A_\varepsilon\big(a - \frac{1}{2}|\mathbf{u}^\delta|^2 - \beta^\delta(z^\delta)\big) - A_\delta\big(a - \frac{1}{2}|\mathbf{u}^\delta|^2 - \beta^\delta(z^\delta)\big)\Big)(z^\varepsilon - z^\delta)\\
&=: I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{4.50}$$

By the nonexpansivity of A_ε, we first compute

$$\begin{aligned}
I_2(t) &\le \int_0^t\int_{\Gamma_C} |\mathbf{u}^\varepsilon + \mathbf{u}^\delta||\mathbf{u}^\varepsilon - \mathbf{u}^\delta||z^\varepsilon - z^\delta|\\
&\le \|\mathbf{u}^\varepsilon - \mathbf{u}^\delta\|_{L^2(0,t;L^4(\Gamma_C;\mathbb{R}^3))}\|\mathbf{u}^\varepsilon + \mathbf{u}^\delta\|_{L^\infty(0,t;L^4(\Gamma_C;\mathbb{R}^3))}\|z^\varepsilon - z^\delta\|_{L^2(0,t;L^2(\Gamma_C))}\\
&\le c\|\mathbf{u}^\varepsilon - \mathbf{u}^\delta\|^2_{L^2(0,t;L^4(\Gamma_C;\mathbb{R}^3))} + \|z^\varepsilon - z^\delta\|^2_{L^2(0,t;L^2(\Gamma_C))}.
\end{aligned} \tag{4.51}$$

Next, recalling (3.1),

$$\begin{aligned}
I_3(t) &= \int_0^t\int_{\Gamma_C}\Big(\frac{\varepsilon}{\varepsilon+1} - \frac{\delta}{\delta+1}\Big)\Big(a - \frac{1}{2}|\mathbf{u}^\delta|^2 - \beta^\delta(z^\delta)\Big)^+(z^\varepsilon - z^\delta)\\
&\le c(\varepsilon+\delta)\big(1 + \|\mathbf{u}^\delta\|^2_{L^\infty(0,t;L^4(\Gamma_C;\mathbb{R}^3))} + \|\beta^\delta(z^\delta)\|_{L^\infty(0,t;L^2(\Gamma_C))}\big)\|z^\varepsilon - z^\delta\|_{L^1(0,t;L^2(\Gamma_C))}\\
&\le c(\varepsilon+\delta)\|z^\varepsilon - z^\delta\|_{L^1(0,t;L^2(\Gamma_C))},
\end{aligned} \tag{4.52}$$

the last inequality following from the previous uniform estimates.

In order to estimate $I_1(t)$ we argue as in [13, p. 270]. Namely, we use the identities

$$z^\varepsilon - \varepsilon\beta^\varepsilon(z^\varepsilon) = R_\varepsilon(z^\varepsilon), \tag{4.53}$$

$$\beta^\varepsilon(z^\varepsilon) \in \beta(R_\varepsilon(z^\varepsilon)), \tag{4.54}$$

valid for all $\varepsilon \in (0,1)$, where R_ε is the *resolvent* of β at step ε (see [9, p. 27-28]). We have

$$\begin{aligned}
I_1(t) = I_{1,1} + I_{1,2} &:= \int_0^t\int_{\Gamma_{C,0}} \frac{A_\varepsilon(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\varepsilon(z^\varepsilon)) - A_\varepsilon(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\delta(z^\delta))}{\beta^\varepsilon(z^\varepsilon) - \beta^\delta(z^\delta)}\times\\
&\qquad \times (R_\varepsilon(z^\varepsilon) - R_\delta(z^\delta))(\beta^\varepsilon(z^\varepsilon) - \beta^\delta(z^\delta))\\
&+ \int_0^t\int_{\Gamma_C}\Big(A_\varepsilon\big(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\varepsilon(z^\varepsilon)\big) - A_\varepsilon\big(a - \frac{1}{2}|\mathbf{u}^\varepsilon|^2 - \beta^\delta(z^\delta)\big)\Big)(\varepsilon\beta^\varepsilon(z^\varepsilon) - \delta\beta^\delta(z^\delta)),
\end{aligned} \tag{4.55}$$

where $\Gamma_{C,0}$ is the set where the denominator does not vanish. Now, $I_{1,1}$ is easily seen to be nonpositive thanks to the monotonicity of A_ε and to (4.53). The second term $I_{1,2}$ is controlled as follows:

$$|I_{1,2}| \le \|\beta^\varepsilon(z^\varepsilon) - \beta^\delta(z^\delta)\|_{L^2(0,t;L^2(\Gamma_C))}\big(\|\varepsilon\beta^\varepsilon(z^\varepsilon) - \delta\beta^\delta(z^\delta)\|_{L^2(0,t;L^2(\Gamma_C))}\big) \le c(\varepsilon+\delta), \tag{4.56}$$

where we used (4.14a) twice. Gathering the estimates (4.51)-(4.56), and setting $\phi(t) := \|z^\varepsilon(t) - z^\delta(t)\|_{L^2(\Gamma_C)}$ we obtain, for every $t \in (0,T]$ and every $s \in (0,t]$, the differential inequality

$$\frac{1}{2}\phi^2(s) \le \frac{c_1^2}{2}\Big(\varepsilon + \delta + \|\mathbf{u}^\varepsilon - \mathbf{u}^\delta\|^2_{L^2(0,t;L^4(\Gamma_C;\mathbb{R}^3))} + \|z^\varepsilon - z^\delta\|^2_{L^2(0,t;L^2(\Gamma_C))}\Big) + c_2(\varepsilon+\delta)\int_0^s \phi(\cdot), \tag{4.57}$$

for suitable c_1, c_2 independent of ε and δ. Then, applying the generalized Gronwall lemma [9, Lemme A.5], we deduce

$$\phi(s) \le c_1\Big(\varepsilon+\delta+\|\mathbf{u}^\varepsilon-\mathbf{u}^\delta\|^2_{L^2(0,t;L^4(\Gamma_C;\mathbb{R}^3))}+\|z^\varepsilon-z^\delta\|^2_{L^2(0,t;L^2(\Gamma_C))}\Big)^{1/2}+c_2(\varepsilon+\delta)s, \tag{4.58}$$

whence, squaring, and choosing $s=t$,

$$\phi^2(t) \le c_3\Big(\varepsilon+\delta+\|\mathbf{u}^\varepsilon-\mathbf{u}^\delta\|^2_{L^2(0,t;L^4(\Gamma_C;\mathbb{R}^3))}+\int_0^t \phi^2(\cdot)\Big). \tag{4.59}$$

Applying once more Gronwall's lemma, now in its standard form, and recalling (4.46), we then arrive at

$$\|z^\varepsilon(t)-z^\delta(t)\|_{L^2(\Gamma_C)} \to 0 \quad \text{as } |\varepsilon+\delta|\to 0, \tag{4.60}$$

for every $t\in[0,T]$, which implies (4.47) and (4.48) in view of (4.18c). ∎

Lemma 4.7 (Step 6: limit flow rule)**.** *The limit functions provided by* Lemma 4.4 *satisfy condition* (ii) *of* Def. 2.4. *Moreover the inclusions* (2.38) *and* (2.39) *hold.*

Proof. Using (4.18c), (4.18e), (4.18f), and (4.46), we can take the limit in equation (3.3) and get back (2.37), or, in other words, condition (ii). Hence, it remains to identify ξ_1 and ξ_2. Firstly, we observe that inclusion (2.38) follows by combining (4.47) with (4.18e) and using, e.g., [4, Prop. 1.1, p. 42]. Next, to prove (2.39), we use once more Minty's trick (cf., e.g., [9, Prop. 2.5]); namely, we need to check that

$$\limsup_{\varepsilon\to 0}\int_0^T\int_{\Gamma_C}\alpha_\varepsilon(z_t^\varepsilon)z_t^\varepsilon \le \int_0^T\int_{\Gamma_C}\xi_2 z_t. \tag{4.61}$$

Then, we multiply (3.10) by z_t^ε to obtain

$$\int_0^T\int_{\Gamma_C}\alpha_\varepsilon(z_t^\varepsilon)z_t^\varepsilon = -\|z_t^\varepsilon\|^2_{L^2(0,T;L^2(\Gamma_C))}-\int_{\Gamma_C}\widehat{\beta}^\varepsilon(z^\varepsilon(T))+\int_{\Gamma_C}\widehat{\beta}^\varepsilon(z_0)+\int_0^T\int_{\Gamma_C}\Big(a-\frac{1}{2}|\mathbf{u}^\varepsilon|^2\Big)z_t^\varepsilon. \tag{4.62}$$

Then, taking the lim sup, we may observe that the four terms on the right hand side can be managed, respectively, by (4.18c) and semicontinuity, by (4.18c) with the Mosco-convergence of $\widehat{\beta}^\varepsilon$ to $\widehat{\beta}$ (cf. [1, Chap. 3]), by the monotone convergence theorem, and by combining (4.18c) with (4.46) (note that having *strong* convergence of $\mathbf{u}^\varepsilon$ in L^4 is essential at this step). As a consequence, we then infer

$$\limsup_{\varepsilon\searrow 0}\int_0^T\int_{\Gamma_C}\alpha_\varepsilon(z_t^\varepsilon)z_t^\varepsilon \le -\|z_t\|^2_{L^2(0,T;L^2(\Gamma_C))}-\int_{\Gamma_C}\widehat{\beta}(z(T))+\int_{\Gamma_C}\widehat{\beta}(z_0)+\int_0^T\int_{\Gamma_C}\big(a-\frac{1}{2}|\mathbf{u}|^2\big)z_t, \tag{4.63}$$

Now, testing (2.37) by z_t and applying the chain rule formula [9, Lemme 3.3, p. 73] to integrate the product term $\xi_1 z_t$, we see that the above right hand side is equal to $\int_0^T\int_{\Gamma_C}\xi_2 z_t$. Hence, (4.61) follows. In turn, this implies (2.39), which concludes the proof. ∎

Lemma 4.8 (Step 7: condition (i))**.** *The functions* $\mathbf{u}$*,* z*,* η*, and* $\eta_{(t)}$ *obtained in* Lemma 4.4 *satisfy* condition (i) *of* Def. 2.4.

Proof. The statement follows from all the results obtained so far. In particular, we notice that the function v in (4.22) coincides with $(z)^+$, thanks to the strong convergence (4.47). Moreover, as a consequence of (2.38), we have $z\ge 0$ a.e. on $(0,T)\times\Gamma_C$. This implies that $(z)^+=z$. Hence, we may take the limit as $\varepsilon\searrow 0$ in equation (4.35), which gives back (2.34). Analogously, repeating the procedure on a subinterval $(0,t)$, we obtain (2.35). Finally, taking a test function $\boldsymbol{\varphi}\in\mathcal{V}_t$ with $\boldsymbol{\varphi}(t)\equiv 0$ and considering its trivial extension $\overline{\boldsymbol{\varphi}}$ to the whole of $(0,T)$, plugging $\overline{\boldsymbol{\varphi}}$ in (4.35) and $\boldsymbol{\varphi}$ in the analogue of (4.35) over the interval $(0,t)$, and finally letting $\varepsilon\searrow 0$, we obtain (2.36). ∎

Lemma 4.9 (Step 8: energy inequality)**.** *The energy inequality* (2.44) *holds for almost every* $t_1\in[0,T]$ *and for every* $t_2\in(t_1,T]$.

PROOF. We know that the approximate solution $(\mathbf{u}^\varepsilon, z^\varepsilon)$ satisfies the energy equality (3.43). Integrating it in the time interval $[t_1, t_2]$, we get

$$\mathcal{E}^\varepsilon(t_2) + \int_{t_1}^{t_2} \mathcal{D}^\varepsilon(\cdot) = \mathcal{E}^\varepsilon(t_1) + \int_{t_1}^{t_2} \langle \boldsymbol{g}, \mathbf{u}_t^\varepsilon \rangle - \int_{t_1}^{t_2} \int_{\Gamma_C} (z^\varepsilon)^- \mathbf{u}^\varepsilon \cdot \mathbf{u}_t^\varepsilon, \tag{4.64}$$

for every $0 \le t_1 \le t_2 \le T$. Now, we take the lim inf on both hand sides. Then, standard semicontinuity arguments and the convergence relations proved so far permit us to prove that

$$\mathcal{E}(t_2) + \int_{t_1}^{t_2} \mathcal{D}(\cdot) \le \liminf_{\varepsilon \searrow 0} \Big(\mathcal{E}^\varepsilon(t_2) + \int_{t_1}^{t_2} \mathcal{D}^\varepsilon(\cdot) \Big) \tag{4.65}$$

for every t_1, t_2. Moreover, in view of the fact that $z \ge 0$ almost everywhere, it is easy to check that $(z^\varepsilon)^-$ tends to 0 strongly in $L^2(0, T; L^2(\Gamma_C))$, whence the last integral in (4.64) vanishes in the limit. Moreover, it is also easily seen that the term depending on $\boldsymbol{g}$ passes to the limit.

Hence, it just remains to control $\mathcal{E}^\varepsilon(t_1)$, which is the more delicate point. Indeed, what we want to do is taking the lim inf of

$$\begin{aligned}\mathcal{E}^\varepsilon(t_1) = \int_\Omega \Big(\frac{1}{2} |\mathbf{u}_t^\varepsilon(t_1)|^2 + \frac{1}{2} \mathbb{E} \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t_1)) : \boldsymbol{\varepsilon}(\mathbf{u}^\varepsilon(t_1)) \Big) + \int_{\Gamma_C} \Big(- a z^\varepsilon(t_1) + \frac{1}{2} z^\varepsilon(t_1) |\mathbf{u}^\varepsilon(t_1)|^2 \Big) \\ + \int_{\Gamma_C} \Big(\widehat{\beta}^\varepsilon(z^\varepsilon(t_1)) + \widehat{\gamma}^\varepsilon(\mathbf{u}^\varepsilon(t_1) \cdot \mathbf{n}) \Big).\end{aligned} \tag{4.66}$$

Then, in view of (4.27)-(4.28), (4.19), (4.46) and (4.18c), the lim inf of the first two integrals is in fact a true limit and is given by the same couple of integrals rewritten without the ε. On the other hand, this holds just for *almost* every t_1; indeed, to take the limit of the integral of $|\mathbf{u}_t^\varepsilon(t_1)|^2$, we need to use (4.19); in other words, we deduce a.e. (in time) convergence from L^2-convergence by extraction of a subsequence.

Finally, we need to control the last integral in (4.66). This can be done by following closely the argument devised in [8, Sec. 3], to which we refer the reader. Actually, by that method we may prove more precisely that

$$\limsup_{\varepsilon \searrow 0} \int_{\Gamma_C} \Big(\widehat{\beta}^\varepsilon(z^\varepsilon(t_1)) + \widehat{\gamma}^\varepsilon(\mathbf{u}^\varepsilon(t_1) \cdot \mathbf{n}) \Big) \le \int_{\Gamma_C} \Big(\widehat{\beta}(z(t_1)) + \widehat{\gamma}(\mathbf{u}(t_1) \cdot \mathbf{n}) \Big), \tag{4.67}$$

again, for *almost* every t_1. This concludes the proof. ∎

Remark 4.10. In addition to (2.44), we may notice that the energy functional defined in (2.23) is lower semicontinuous with respect to the variable $t \in [0, T]$. Indeed, as a consequence of (2.33a)-(2.33c), we have

$$\mathbf{u} \in C([0, T]; \boldsymbol{V}), \quad \mathbf{u}_t \in C_w([0, T]; \boldsymbol{H}), \quad z \in C([0, T]; L^2(\Gamma_C)). \tag{4.68}$$

Hence, the lower semicontinuity of $t \mapsto \mathcal{E}(\mathbf{u}(t), \mathbf{u}_t(t), z(t))$ follows easily from the convexity of $\widehat{\beta}$ and $\widehat{\gamma}$ and from lower semicontinuity of norms with respect to weak convergence.

5 Conclusions

The main result of the paper, Theorem 2.5, provides the existence of solutions to the original system (1.1)-(1.4) in the sense of Def. 2.4. Let us emphasize that the techniques of convex analysis involved to pass to the limit the approximate solutions have been originally used in [8], but, to our knowledge, have not proviously used to treat hyperbolic systems with the presence of unilateral constraints. Such techniques allow us to have an explicit weak formulation of the dynamic equation for the displacement without passing through the use of variational inequalities. Moreover, as observed in the introduction, they seem to provide an easy way to overpass the difficulty due to the presence of the critical exponent 2 of the jump appearing in the energy stored by the adhesive (the sixth term in (2.23)).

Finally, let us point out that, due to the lack of smoothness, at the present stage the solutions given by Theorem 2.5 are not shown to satisfy the energy equality (2.22). In some sense, the presence of the unilateral constraint, whose effect is accomplished by the function $\eta \in \mathcal{H}'$, lead us to give the problem a weak formulation using the duality with test functions in the whole time-space. In particular η is not giving a (time) pointwise meaning. Such complication is at the core of the difficulty to prove an energy balance. As emphasized in [8], where a problem similar to (1.1)-(1.4) is studied (and where the unilateral constraint acts on the whole body Ω), the presence of solutions which do not satisfy the energy equality cannot be, in general, ruled out. To the present stage, this aim seems out of reach and will be the object of a forthcoming work. It is worth observing that the approximation of the solution (u, z) by those of the regularized problems is unconvenient in order to obtain the energy balance at the limit, at least without stronger regularity results for the solutions at disposal, since their convergence is proved to hold in a too weak topology. On the other hand it might be of high interest to prove the existence of solutions satisfying the energy balance, because, in some sense, they should reflect the dynamics of the physically admissible evolutions (see [8]). Such expected property will better fit our setting into the general framework of energetic solutions to a mechanical system.

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