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EMANUELA RADICI

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A PLANAR SOBOLEV EXTENSION THEOREM FOR PIECEWISE LINEAR HOMEOMORPHISMS

EMANUELA RADICI

We prove that a planar piecewise affine homeomorphism φ defined on the boundary of the square can be extended to a piecewise affine homeomorphism h of the whole square, in such a way that $\|h\|_{W^{1,p}}$ is bounded from above by $\|\varphi\|_{W^{1,p}}$ for every $p \geq 1$.

1. Introduction

Let $\mathcal{Q}$ be the unit square in $\mathbb{R}^2$ and φ be a piecewise affine homeomorphism with finitely many affine components that maps $\partial\mathcal{Q}$ to a closed curve in $\mathbb{R}^2$. We call a piecewise affine map with finitely many affine components a finitely piecewise affine map. In this work, we provide a general recipe for extending φ to a function h of the whole square which maintains the finitely piecewise affine structure and whose Sobolev $W^{1,p}$-norm is controlled from above by $\|\varphi\|_{W^{1,p}}$. That such an extension exists is well known, and its construction is not difficult, but showing the existence of an extension with good control on its norm is a substantial problem. In fact, we will establish a bound

$$\|Dh\|_{L^p(\mathcal{Q})} \leq K\|D\varphi\|_{L^p(\partial\mathcal{Q})}$$

for a suitable geometric constant K which depends only on p. It is appropriate to explain briefly the context of our work and its utility. The problem of finding approximations of a planar homeomorphism $f : \Omega \to \mathbb{R}^2$ has a long history in the literature and recently it was realized to be relevant to the study of the regularity of minimizers for standard energies in the area of nonlinear elasticity. Many important results are already available on this topic. See, for instance, [Mora-Corral 2009; Bellido and Mora-Corral 2011; Iwaniec et al. 2011; Daneri and Pratelli 2015; Hencl and Pratelli 2015] for an overview of what is known. Let us recall the approach introduced in the last two of these references, where the authors create the approximation step by step, starting from an explicit subdivision of the domain Ω that depends on the Lebesgue points of Df. Although their settings and the regularity of their approximations are very different, in both papers the strategy

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is to split the domain into a countable disjoint union of simple polygons (in general triangles or squares) by introducing a locally finite one-dimensional grid, which consists of the union of the boundaries of such polygons. A first piecewise linear approximation of f is defined on the one-dimensional grid and, only in a second step, the approximation is extended in the interior of each simple polygon, being careful to get the regularity claimed. In this work we focus on the second step of this strategy, namely we already assume the existence of a piecewise linear function f_ε defined on a locally finite grid of squares, let us call it Δ, and we give all the ingredients needed for extending f_ε to a piecewise affine function of the whole domain Ω with suitably small $W^{1,p}$-norm. In fact, the extension of f_ε into a single square $\mathcal{Q}$ involves only the values on the boundary $\partial\mathcal{Q}$ of $\mathcal{Q}$, and it is useful to have an estimate like

$$\|f_\varepsilon\|_{W^{1,p}(\mathcal{Q})} < K\|f_\varepsilon\|_{W^{1,p}(\partial\mathcal{Q})},$$

for a suitable constant K. Let us already say that our proof does not depend on the precise value of p, thus it holds true for every $p \in [1, \infty)$. An analogous result was already proved in the cases $p = \infty$ in [Daneri and Pratelli 2015] and $p = 1$ in [Hencl and Pratelli 2015], while in this work we generalize to any $p > 1$ a technique introduced in [Hencl and Pratelli 2015]. Furthermore, an extension of this result seems to be true also in the Orlicz–Sobolev spaces (see [Campbell 2015]). For us, $\mathcal{Q}$ will be the rotated square centered in the origin and with corners in $(\pm 1, 0)$, $(0, \pm 1)$. Our result is the following.

Theorem 1.1. *Let $\varphi : \partial\mathcal{Q} \to \mathbb{R}^2$ be a one-to-one piecewise affine function. Then there exists a finitely piecewise affine homeomorphism $h : \mathcal{Q} \to \mathbb{R}^2$ that satisfies $h \equiv \varphi$ on $\partial\mathcal{Q}$ and, for any $p \geq 1$, there is a constant K depending only on p such that the estimate*

$$\int_{\mathcal{Q}} |Dh(x)|^p \, dx \leq K \int_{\partial\mathcal{Q}} |D\varphi(t)|^p \, d\mathcal{H}^1(t) \tag{1-1}$$

holds.

The plan of the paper is very simple: in the following section we make a short remark about the notation, the second section is devoted to the proof of Theorem 1.1 and in the last remark we explain the case of a generic square in the plane.

Notation. Let us briefly introduce the notation we use throughout the paper. We call $\mathcal{Q} = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ the rotated square centered in the origin and $\boldsymbol{\mathcal{Q}}$ the image through h of $\mathcal{Q}$. With the capital letters A, B we always refer to points lying on the boundary of $\mathcal{Q}$, while P and R denote points in the interior of $\mathcal{Q}$. The points in $\partial\boldsymbol{\mathcal{Q}}$ and in the interior of $\boldsymbol{\mathcal{Q}}$ will be denoted similarly in bold style: $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{P}$, $\boldsymbol{R}$. When we use the same letter in normal and bold style, for example A and $\boldsymbol{A}$, this always means that $\boldsymbol{A}$ is the image of A through the mapping that we are considering in that moment. By AB and ABC we mean, respectively, the segment between A and B and the triangle of corners A, B and C (the same also for $\boldsymbol{AB}$, $\boldsymbol{ABC}$). The modulus of the horizontal and vertical derivatives of a function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted as

$$|D_1 f| = \sqrt{\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial x}\right)^2}, \qquad |D_2 f| = \sqrt{\left(\frac{\partial f_1}{\partial y}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2},$$

and with the symbol $\mathcal{H}^1$ we indicate the standard 1-dimensional Hausdorff measure. Finally, the letter K will always indicate a purely geometric constant that depends only on p. Since it is the existence, not the size, of K that matters, we do not calculate the explicit value of K but we show at each step that it remains finite and it stays independent from all the parameters but p.

2. Proof of Theorem 1.1

We generalize to every $1 \le p < \infty$ a strategy introduced in [Hencl and Pratelli 2015] for $p = 1$. To keep this work self contained, we present also the parts of the proof stated in [Hencl and Pratelli 2015] that do not depend on the exponent p.

Proof of Theorem 1.1. Since the proof of Theorem 1.1 is long, we split it into several steps for sake of clarity.

Step I. Choice of corners. It is useful to know that $|D\varphi|^p$ does not critically accumulate around the two opposite corners of $\partial\mathcal{Q}$, which we denote as $V_1 \equiv (0, -1)$ and $V_2 \equiv (0, 1)$. More precisely, the estimate we would like to have is the following:

$$\int_{B(V_i,r)\cap\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 \le \widetilde{K} r \int_{\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 \quad \text{for all } r \in (0, \sqrt{2}),\ i \in \{1, 2\}. \tag{2-1}$$

This is not true for a generic φ, but in this step we show that there always exists a pair of opposite points P_1, P_2 on $\partial\mathcal{Q}$ that satisfies (2-1) in place of V_1, V_2 with $\widetilde{K} = 6$. Notice that it is always possible to apply a bi-Lipschitz transformation of $\partial\mathcal{Q}$ to itself which moves such P_1, P_2 to V_1, V_2 respectively. Since the bi-Lipschitz constant does not depend on the particular values of P_1 and P_2, then estimate (2-1) follows straightforwardly for a suitable constant $\widetilde{K}$. Thanks to this observation, the step is concluded once we prove the existence of such P_1, P_2. Two generic opposite points are a good choice unless at least one of them is in the set

$$\mathcal{A} = \left\{ A \in \partial\mathcal{Q} : \int_{B(A,r)\cap\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 > 6r \int_{\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 \text{ for some } r \in (0, \sqrt{2}) \right\}.$$

Thus, the existence of a good pair P_1, P_2 is established if $\mathcal{H}^1(\mathcal{A})$ is not too big. By a Vitali covering argument, it is always possible to cover $\mathcal{A}$ with a countable

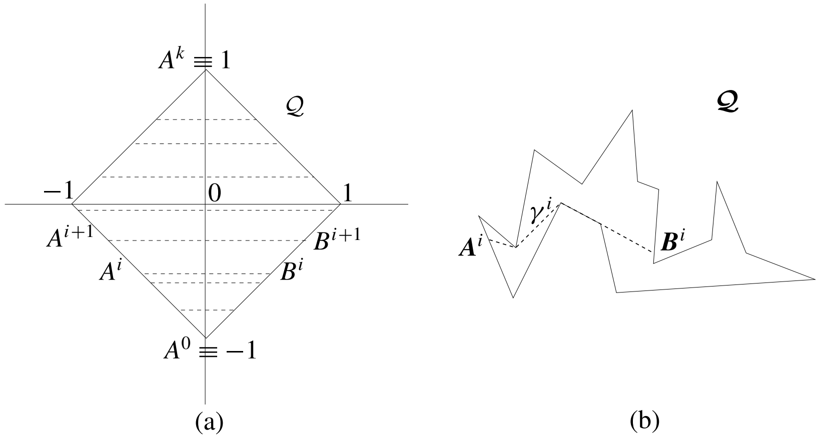

Figure 1. (a) A grid on $\mathcal{Q}$ and (b) a path γ^i in $\boldsymbol{\mathcal{Q}}$.

union of balls $B(A_i, 3r_i)$ such that $A_i \in \mathcal{A}$ and the sets $B(A_i, r_i) \cap \partial Q$ are pairwise disjoint. In particular, one has

$$\mathcal{H}^1(\mathcal{A}) \leq \sum_i 6r_i \leq \sum_i \frac{\int_{B(A,r_i)\cap\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1}{\int_{\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1} \leq 1\,,$$

thus implying the existence of a pair of opposite points, P_1 and P_2 in $\partial\mathcal{Q} \setminus \mathcal{A}$, satisfying (2-1).

Step II. Construction of one-dimensional grids in $\mathcal{Q}$ and in $\boldsymbol{\mathcal{Q}}$. Let us denote with $\boldsymbol{\mathcal{Q}}$ the bounded component of $\mathbb{R}^2 \setminus \varphi(\partial\mathcal{Q})$, which is a polygon because φ is piecewise affine. Notice that the problem of finding a piecewise affine homeomorphism h which maps $\mathcal{Q}$ into $\boldsymbol{\mathcal{Q}}$ makes sense. Since we want $h \equiv \varphi$ on $\partial\mathcal{Q}$, our approach is the following: we start extending the function φ on a suitable one-dimensional grid on $\mathcal{Q}$, we then "complete" this grid to obtain a triangulation of $\mathcal{Q}$ and, at the end, we define h inside each triangle of the triangulation as the affine function extending the values on the boundary. In this step, we introduce the one-dimensional grid in $\mathcal{Q}$ and we construct a related grid in $\boldsymbol{\mathcal{Q}}$, which will be the image through h of the grid on $\mathcal{Q}$. For the construction, we make use of several horizontal segments A^iB^i whose endpoints are in $\partial\mathcal{Q}$. We call $A^i = (A^i_1, A^i_2)$ the endpoint that has negative first component and we choose the indexes i so that A^i_2 increases with respect to i from -1 to 1 (see Figure 1(a)). It is convenient to include also V_1, V_2 in the grid, therefore we denote them consistently as $A^0 \equiv B^0 \equiv V_1 \equiv (0, -1)$ and $A^k \equiv B^k \equiv V_2 \equiv (0, 1)$. We consider many horizontal segments A^iB^i such that for every i the restriction of φ to A^iA^{i+1} or B^iB^{i+1} is linear. Notice that this property is still valid even if we take more horizontal segments A^iB^i, therefore, we are allowed to add points A^i and B^i during the construction (of course, being careful in adding only finitely many). Once the grid on $\mathcal{Q}$ is fixed, we define a second one, this time on $\boldsymbol{\mathcal{Q}}$, that is the union of the geodesics γ^i inside $\overline{\boldsymbol{\mathcal{Q}}}$ connecting $\varphi(A^i)$ and $\varphi(B^i)$. Since $\boldsymbol{\mathcal{Q}}$ is a polygon, γ^i is piecewise affine and, moreover, the junction between any two consecutive affine pieces of γ^i lies in $\partial\boldsymbol{\mathcal{Q}}$ (see Figure 1(b)). In order to simplify the notation, we write the points of $\boldsymbol{\mathcal{Q}}$ in bold style: $\boldsymbol{A}^i \equiv \varphi(A^i)$ and $\boldsymbol{B}^i \equiv \varphi(B^i)$. Up to adding points between A^0 and A^1 (we do the same also for A^{k-1} and A^k), we can always assume that γ^1 (resp. γ^{k-1}) is either a segment $\boldsymbol{A}^1\boldsymbol{B}^1$ (resp. $\boldsymbol{A}^{k-1}\boldsymbol{B}^{k-1}$), or it is the union of the two segments $\boldsymbol{A}^1\boldsymbol{A}^0$ and $\boldsymbol{A}^0\boldsymbol{B}^1$ (resp. $\boldsymbol{A}^{k-1}\boldsymbol{A}^k$ and $\boldsymbol{A}^k\boldsymbol{B}^{k-1}$) lying entirely in $\partial\boldsymbol{\mathcal{Q}}$. It is well known in the literature that the shortest path that connects two generic points inside a simply connected set is unique. In particular, this result ensures that the paths γ^i are unique and implies that the grid on $\boldsymbol{\mathcal{Q}}$ is well defined.

Step III. Relevant properties of the paths γ^i. In this step we present some properties of the paths γ^i. The first property is a consequence of the uniqueness: whenever two paths γ^{i+1} and γ^i intersect each other they coincide from the first to the last point of intersection. In particular, γ^{i+1} and γ^i cannot cross each other, thus allowing us to distinguish three different parts on each path

$$\gamma^i = \gamma_1^i \cup \gamma_2^i \cup \gamma_3^i, \qquad \gamma^{i+1} = \gamma_1^{i+1} \cup \gamma_2^{i+1} \cup \gamma_3^{i+1}.$$

In detail, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are the first and last points, respectively, of the common part between γ^i and γ^{i+1}, we call γ_1^i the path from $\boldsymbol{A}^i$ to $\boldsymbol{A}$ (analogous for γ_1^{i+1}), γ_2^i the path from $\boldsymbol{A}$ to $\boldsymbol{B}$ ($\gamma_2^i \equiv \gamma_2^{i+1}$) and γ_3^i the last part of the path from $\boldsymbol{B}$ to $\boldsymbol{B}^i$ (analogous for γ_3^{i+1}). When γ^i and γ^{i+1} have no intersection, we directly set $\gamma^i \equiv \gamma_1^i$ and $\gamma^{i+1} \equiv \gamma_1^{i+1}$, while γ_2^i, γ_3^i, γ_2^{i+1}, γ_3^{i+1} are empty paths. Let us observe that such subdivision of γ^i is related to the curve γ^{i+1} and there is no reason why it should match with the one related to γ^{i-1}. The last property of paths γ^i that we recall is fundamental for showing estimate (1-1) (for a formal proof see Step 5 of Theorem 2.1 in [Hencl and Pratelli 2015]). Let $\boldsymbol{P}$ be the last point of the curve γ_1^{i+1}; no matter whether $\boldsymbol{P}$ coincides with $\boldsymbol{A}$ or $\boldsymbol{B}^{i+1}$, the polygon having boundary $\gamma_1^{i+1} \cup \boldsymbol{A}^{i+1}\boldsymbol{P}$ is always convex (see Figure 2).

Step IV. Triangular grid on $\boldsymbol{\mathcal{Q}}$ and estimate on "vertical sides". In this step we are going to select finitely many points on the paths γ^i in order to get a triangular grid on $\boldsymbol{\mathcal{Q}}$. For all $i = 1, \dots, k-2$, we call $\boldsymbol{\mathcal{D}}_i$ the closure of the polygon having boundary $\boldsymbol{A}^i\boldsymbol{A}^{i+1} \cup \gamma^i \cup \gamma^{i+1} \cup \boldsymbol{B}^i\boldsymbol{B}^{i+1}$ (which lies inside the closure of $\boldsymbol{\mathcal{Q}}$), then we argue separately for each single $\boldsymbol{\mathcal{D}}_i$. For every "strip" $\boldsymbol{\mathcal{D}}_i$ we select some points on γ^i, γ^{i+1} depending on the relation between γ_1^i, γ_2^i, γ_3^i and γ_1^{i+1}, γ_2^{i+1}, γ_3^{i+1}. We argue separately for the cases $\gamma^i \cap \gamma^{i+1} \neq \varnothing$ and $\gamma^i \cap \gamma^{i+1} = \varnothing$. Both situations are depicted in Figure 3.

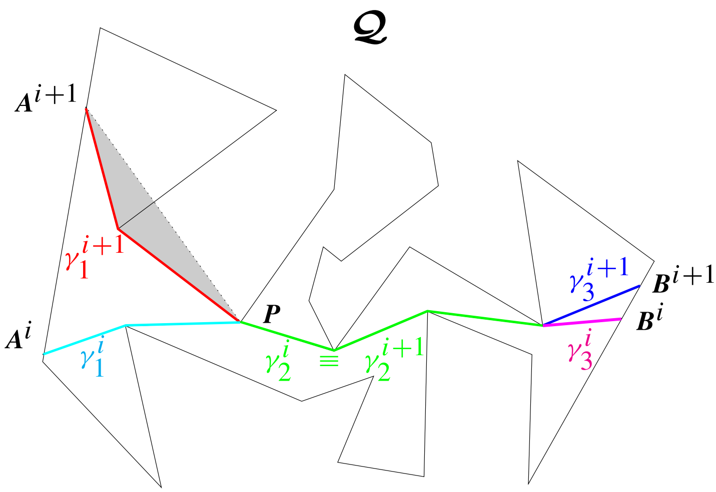

Figure 2. Convexity of the polygon delimited by $\gamma_1^{i+1} \cup A^{i+1}P$.

Let us start from the nonempty case (Figure 3(a)). For any endpoint of a linear piece of γ_1^{i+1} or γ_1^i we consider the corresponding point on the other path so that the segment connecting the two points is parallel to $A^i A^{i+1}$. We denote with P_j the points taken on γ_1^{i+1} and with R_j the corresponding point on γ_1^i. Notice that the convexity result introduced in Step III ensures that the segment $P_j R_j$ is always well defined and, moreover, it satisfies $\mathcal{H}^1(P_j R_j) \leq \mathcal{H}^1(A^i A^{i+1})$, thus, in particular,

$$\mathcal{H}^1(P_j R_j) \leq \max\{\mathcal{H}^1(A^i A^{i+1}), \mathcal{H}^1(B^i B^{i+1})\}. \tag{2-2}$$

With a symmetric strategy we select other points P_j and R_j on γ_3^{i+1} and γ_3^i, respectively, by taking this time $P_j R_j$ parallel to $B^i B^{i+1}$. Let us recall that (2-2) still holds true in this case, since now $\mathcal{H}^1(P_j R_j) \leq \mathcal{H}^1(B^i B^{i+1})$. Finally, in the

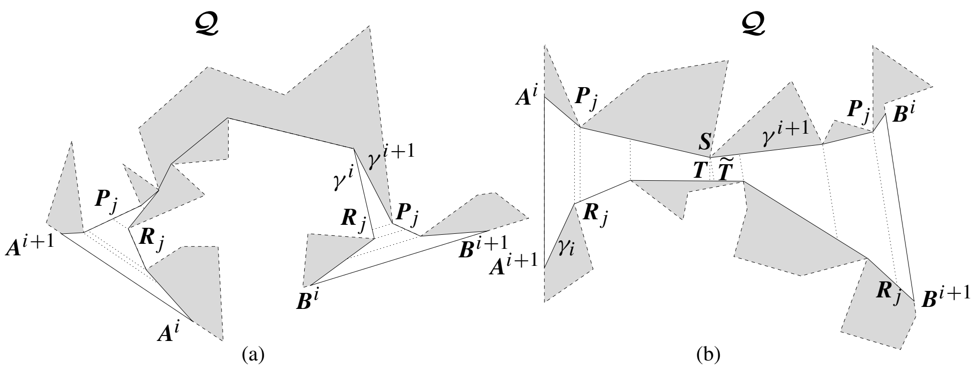

Figure 3. Selection of points on γ^i and γ^{i+1}.

common part $\gamma_2^i \equiv \gamma_2^{i+1}$, we take all the endpoints of the linear pieces of the path and by construction $\boldsymbol{P}_j \equiv \boldsymbol{R}_j$. Of course, (2-2) is trivially true because in this case $\mathcal{H}^1(\boldsymbol{P}_j\boldsymbol{R}_j) = 0$. If $\gamma^i \cap \gamma^{i+1} = \varnothing$, the strategy is a little bit different (see Figure 3(b)). In the specific case in which $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ are parallel to each other, we can argue exactly as in the previous case: therefore all the points $\boldsymbol{P}_j$ selected on γ^{i+1} can be either endpoints of linear pieces of γ^{i+1} or the corresponding point of $\boldsymbol{R}_j$, where $\boldsymbol{R}_j$ is an endpoint for γ^i. Moreover, by construction, $\boldsymbol{P}_j\boldsymbol{R}_j$ is always parallel to both $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ and (2-2) is still satisfied thanks to Step III. For generic $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ we argue as follows. By symmetry, it is not restrictive to assume that $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ is vertical and the two lines with directions $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ intersect in a point that is closer to $\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^{i+1}$ than $\boldsymbol{A}^i$ and $\boldsymbol{B}^i$ (as shown in Figure 3(b)). Let $\boldsymbol{S}$, $\boldsymbol{T}$, $\widetilde{\boldsymbol{T}}$ be three points such that $\boldsymbol{S}$ is on γ^{i+1} and $\boldsymbol{T}$, $\widetilde{\boldsymbol{T}}$ are on γ^i and $\boldsymbol{S}\boldsymbol{T}$ is the shortest segment inside $\boldsymbol{\mathcal{D}}_i$ which is parallel to $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ (notice that $\boldsymbol{S}$ can even happen to be $\boldsymbol{A}^{i+1}$ or $\boldsymbol{B}^{i+1}$) while $\widetilde{\boldsymbol{T}}$ is the intersection between γ^i and the half line starting from $\boldsymbol{S}$ with direction $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$. On one side, with the usual strategy, we select points $\boldsymbol{P}_j$, $\boldsymbol{R}_j$, with $\boldsymbol{P}_j$ between $\boldsymbol{A}^{i+1}$ and $\boldsymbol{S}$ and $\boldsymbol{R}_j$ between $\boldsymbol{A}^i$ and $\boldsymbol{T}$, so that $\boldsymbol{P}_j\boldsymbol{R}_j$ is parallel to $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$. On the other side, we take $\boldsymbol{P}_j$, $\boldsymbol{R}_j$ with $\boldsymbol{P}_j$ between $\boldsymbol{S}$ and $\boldsymbol{B}^{i+1}$, $\boldsymbol{R}_j$ between $\widetilde{\boldsymbol{T}}$ and $\boldsymbol{B}^i$, and $\boldsymbol{P}_j\boldsymbol{R}_j$ parallel to $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$. Finally, to any endpoint $\boldsymbol{R}_j$ of a linear piece of γ^i that happens to be between $\boldsymbol{T}$ and $\widetilde{\boldsymbol{T}}$, we associate $\boldsymbol{P}_j \equiv \boldsymbol{S}$. Notice that, by construction, estimate (2-2) is satisfied. We can then introduce the triangular grid on $\boldsymbol{\mathcal{D}}_i$ as the union of the boundaries of the triangles $\boldsymbol{P}_j\boldsymbol{P}_{j+1}\boldsymbol{R}_j$ and $\boldsymbol{P}_{j+1}\boldsymbol{R}_j\boldsymbol{R}_{j+1}$ that are not degenerate. Moreover, recalling Step II, the polygons delimited by $\gamma^1 \cup \boldsymbol{A}^1\boldsymbol{A}^0 \cup \boldsymbol{A}^0\boldsymbol{B}^1$ and $\gamma^{k-1} \cup \boldsymbol{A}^{k-1}\boldsymbol{A}^k \cup \boldsymbol{A}^k\boldsymbol{B}^{k-1}$ are either triangles themselves or the union of two segments lying in $\partial\boldsymbol{\mathcal{Q}}$, thus we actually defined a triangular grid on the whole $\boldsymbol{\mathcal{Q}}$ and this concludes Step IV.

Step V. Triangular grid on $\mathcal{Q}$ and definition of $\tilde{h}$. We recall that our aim is to define a piecewise affine homeomorphism h mapping $\mathcal{Q}$ to $\boldsymbol{\mathcal{Q}}$ that matches with φ on $\partial\mathcal{Q}$ and satisfies estimate (1-1). In this step, we construct a function $\tilde{h} : \mathcal{Q} \to \boldsymbol{\mathcal{Q}}$ which is piecewise affine and coincides with φ on the boundary of $\mathcal{Q}$ and in Steps VI, VII we will prove that estimate (1-1) is satisfied by $\tilde{h}$. Let us already remark that in general $\tilde{h}$ will not be a one-to-one function, therefore we will have to suitably modify it later to obtain the homeomorphism h. We split the construction into three steps: first, we define the function $\tilde{h}$ only on $\partial\mathcal{Q}$ and on the segments A^iB^i, we then make use of the triangular grid of $\boldsymbol{\mathcal{Q}}$ defined in Step IV to find a triangular grid on $\mathcal{Q}$ and, finally, we use these triangular grids to extend $\tilde{h}$ on the whole $\mathcal{Q}$. We start by taking $\tilde{h} \equiv \varphi$ on $\partial\mathcal{Q}$ such that it maps all the horizontal segments A^iB^i to the respective path γ^i parametrized at constant speed. Notice that, in this way, $\tilde{h}$ is continuous and piecewise linear. Recalling the notation introduced in the previous step, we focus then on the polygon $\boldsymbol{\mathcal{D}}_i$: we associate to any $\boldsymbol{P}_j$ on γ^{i+1} the point P_j on $A^{i+1}B^{i+1}$ such that $\boldsymbol{P}_j = \tilde{h}(P_j)$, and to any $\boldsymbol{R}_j$ on γ^i the point R_j on A^iB^i such that $\boldsymbol{R}_j = \tilde{h}(R_j)$. For the sake of clarity we denote the endpoints $A^iA^{i+1}B^{i+1}B^i$ consistently with the notation, namely we call $P_0 \equiv A^{i+1}$, $P_N \equiv B^{i+1}$, $R_0 \equiv A^i$ and $R_N \equiv B^i$ for a suitable N. We can now define $\tilde{h}$ on the strip $\mathcal{D}_i := A^iA^{i+1} \cup A^{i+1}B^{i+1} \cup B^iB^{i+1} \cup A^iB^i$ as the function which is affine on each of the triangles $P_jP_{j+1}R_j$ or $P_{j+1}R_jR_{j+1}$ (notice that clearly $\tilde{h}$ can be degenerate on some triangles). In more detail, we define $\tilde{h}$ on $P_jP_{j+1}R_j$ as the affine function which maps $P_jP_{j+1}R_j$ onto $\boldsymbol{P}_j\boldsymbol{P}_{j+1}\boldsymbol{R}_j$ extending the values on the boundary (the very same definition is used for triangles of the form $P_{j+1}R_jR_{j+1}$). It still remains to define $\tilde{h}$ on the top and bottom triangles of $\mathcal{Q}$. Let us consider the bottom triangle $A^0A^1B^1$ (the definition is symmetric for $A^kA^{k-1}B^{k-1}$); then we know from Step II that γ^1 can be either a segment $\boldsymbol{A}^1\boldsymbol{B}^1$ or the union of two segments $\boldsymbol{A}^0\boldsymbol{A}^1$, $\boldsymbol{A}^0\boldsymbol{B}^1$ laying on $\partial\boldsymbol{\mathcal{Q}}$. In the first case we again define $\tilde{h}$ as the affine function that extends the values on the boundary. In the other case we will subdivide $A^0A^1B^1$ into two triangles A^0A^1P and A^0PB^1, where P is the point on A^1B^1 such that $\tilde{h}(P) = \boldsymbol{A}^0$, then we define $\tilde{h}$ to be constantly equal to $\boldsymbol{A}^0$ on the segment PA^0 and the degenerate affine function extending the values on the boundary on A^0A^1P and A^0PB^1. By construction, the function $\tilde{h}$ is piecewise affine, coincides with φ on $\partial\mathcal{Q}$ and it is also continuous.

Step VI. Estimate for $\int_{A^0A^1B^1}|D\tilde{h}|^p$. As mentioned above, this step and the following one are devoted to showing that the function $\tilde{h}$ satisfies the estimate (1-1). Here, in particular, we focus on the bottom triangle $\mathcal{T} := A^0A^1B^1$ (the very same argument holds also for the top triangle $A^kA^{k-1}B^{k-1}$), and we prove that

$$\int_{\mathcal{T}}|D\tilde{h}|^p \le K_1\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1\,, \tag{2-3}$$

where K_1 denotes a purely geometric constant. Recall that, by definition, $\tilde{h}(\mathcal{T})$ is either the nondegenerate triangle $\boldsymbol{A}^0\boldsymbol{A}^1\boldsymbol{B}^1$ or the union of the two segments $\boldsymbol{A}^1\boldsymbol{A}^0 \cup \boldsymbol{A}^0\boldsymbol{B}^1$. In the nondegenerate case, $D\tilde{h}$ is constant on $\mathcal{T}$, therefore we denote its constant value with $D^b\tilde{h}= (D^b_1\tilde{h},\ D^b_2\tilde{h})$. Moreover, by construction, $\tilde{h}(A^0A^1) = \boldsymbol{A}^0\boldsymbol{A}^1$ and $\tilde{h}(A^0B^1) = \boldsymbol{A}^0\boldsymbol{B}^1$, thus we can write the following relations:

$$\tfrac{\sqrt{2}}{2}|D^b_1\tilde{h} + D^b_2\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\mathcal{H}^1(A^0B^1)}\,, \qquad \tfrac{\sqrt{2}}{2}|-D^b_1\tilde{h} + D^b_2\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)}{\mathcal{H}^1(A^0A^1)}\,,$$

which, in turn, imply that

$$|D^b\tilde{h}|^p \le \frac{2^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p + 2^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p}{\mathcal{H}^1(A^0A^1)^p}\,, \tag{2-4}$$

where we used the fact that $\mathcal{H}^1(A^0A^1) = \mathcal{H}^1(A^0B^1)$. On the other hand, we have that $|D\varphi|$ is constant on each of the segments A^0A^1, A^0B^1 and this gives us an estimate on $\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p$ and $\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p$, namely

$$\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p = \mathcal{H}^1(A^0A^1)^{p-1}\int_{A^0A^1}|D\varphi|^p\,d\mathcal{H}^1\,,$$
$$\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p = \mathcal{H}^1(A^0A^1)^{p-1}\int_{A^0B^1}|D\varphi|^p\,d\mathcal{H}^1\,.$$

By inserting both of these equations into (2-4) and using (2-1) with $r=\mathcal{H}^1(A^0A^1)$, one gets

$$|D^b\tilde{h}|^p \le \frac{2^p}{\mathcal{H}^1(A^0A^1)}\int_{B(A^0,\mathcal{H}^1(A^0A^1))\cap\partial Q}|D\varphi|^p\,d\mathcal{H}^1 \le 2^p\widetilde{K}\int_{\partial Q}|D\varphi|^p\,d\mathcal{H}^1\,.$$

Summarizing, in the nondegenerate case, we can finally deduce (2-3) from

$$\begin{aligned}\int_{\mathcal{T}}|D\tilde{h}|^p &= \frac{\mathcal{H}^1(A^0A^1)^2}{2}|D\tilde{h}|^p\\ &\le \frac{\mathcal{H}^1(A^0A^1)^2}{2}2^p\widetilde{K}\int_{\partial Q}|D\varphi|^p d\mathcal{H}^1 \le 2^p\widetilde{K}\int_{\partial Q}|D\varphi|^p d\mathcal{H}^1.\end{aligned}$$

Now, let $\tilde{h}$ be a degenerate affine function on each of the two parts A^1PA^0 and A^0PB^1, where P satisfies $\tilde{h}(P)=\boldsymbol{A}^0$, as in Step V. Let us call $|D^l\tilde{h}|$ and $|D^r\tilde{h}|$ the constant values of $|D\tilde{h}|$ on the two parts. In this case the following relations hold:

$$|D^r_1\tilde{h}| = |D^l_1\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)+\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\mathcal{H}^1(A^1B^1)} = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)+\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\sqrt{2}\mathcal{H}^1(A^1A^0)}\,,$$
$$\tfrac{\sqrt{2}}{2}|D^l_2\tilde{h}-D^l_1\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)}{\mathcal{H}^1(A^0A^1)}\,,$$
$$\tfrac{\sqrt{2}}{2}|D^r_1\tilde{h}+D^r_2\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\mathcal{H}^1(A^0B^1)}\,.$$

Therefore

$$|D^r_2\tilde{h}| \le \frac{3\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)+\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)}{\sqrt{2}\mathcal{H}^1(A^1A^0)}\,,$$

and in particular

$$\begin{aligned}|D^r\tilde{h}|^p &\le \frac{1}{(\sqrt{2}\mathcal{H}^1(A^1A^0))^p}\big(4^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p+8^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p\big)\\ &\le 2\frac{8^p}{\sqrt{2}^p}\frac{1}{\mathcal{H}^1(A^1A^0)}\int_{B(A^0,\mathcal{H}^1(A^1A^0))\cap\partial Q}|D\varphi|^p\,d\mathcal{H}^1\\ &\le 2\frac{8^p}{\sqrt{2}^p}\widetilde{K}\int_{\partial Q}|D\varphi|^p\,d\mathcal{H}^1\,,\end{aligned} \tag{2-5}$$

where the last inequality is a consequence of Step I. An estimate for $|D^l\tilde{h}|^p$ analogous to (2-5) holds by a symmetric argument, then, as a consequence, one has

$$\begin{aligned}\int_{A^0A^1B^1}|D\tilde{h}|^p &= \int_{A^0PA^1}|D^l\tilde{h}|^p + \int_{A^0PB^1}|D^r\tilde{h}|^p \\ &\le \frac{\mathcal{H}^1(A^1A^0)^2}{2}(|D^r\tilde{h}|^p + |D^l\tilde{h}|^p) \le 2^{\frac{5}{2}p+1}\widetilde{K}\int_{\partial Q}|D\varphi|^p\,d\mathcal{H}^1 .\end{aligned}$$

Therefore (2-3) holds true for the degenerate case and also in general as soon as $K_1 \ge 2^{\frac{5}{2}p+1}\widetilde{K}$.

Step VII. Estimate for $\int_{\mathcal{Q}^-}|D\tilde{h}|^p$. In this step, we show that $\tilde{h}$ satisfies (1-1) also in $\mathcal{Q}^-$, which is the square $\mathcal{Q}$ without the top and the bottom triangles. Namely, we prove that

$$\int_{\mathcal{Q}^-}|D\tilde{h}|^p \le K_2\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 .$$

To do so, we need at first a similar estimate on a generic triangle $\mathcal{T}$ of the triangulation of $\mathcal{Q}$ which is inside $\mathcal{Q}^-$. To this end, let i be the index so that $\mathcal{T}$ is included in the polygon $\mathcal{D}_i := A_iB_i \cup A_{i+1}B_{i+1} \cup A_iA_{i+1} \cup B_iB_{i+1}$. We aim to show

$$\int_{\mathcal{T}}|D\tilde{h}|^p \le K'|\mathcal{T}|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + K''\frac{|\mathcal{T}|}{|A_2^{i+1}-A_2^i|}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1. \tag{2-6}$$

Let $\mathcal{T}$ be of the form $P_jP_{j+1}R_j$ (of course for the other triangles $P_{j+1}R_jR_{j+1}$ the very same argument can be applied) and, by symmetry, let us also assume that $\mathcal{D}_i$ lays below the x-axis. To simplify the notation, we denote r the distance between A^0 and the horizontal segment $A^{i+1}B^{i+1}$, and σ the distance between $A^{i+1}B^{i+1}$ and A^iB^i (which is equal to $|A_2^{i+1}-A_2^i|$ and to the height of $\mathcal{D}_i$). Since $\tilde{h}$ is affine on $\mathcal{T}$, we also denote by $|D^{\mathcal{T}}\tilde{h}|$ the constant value of $|D\tilde{h}|$ on $\mathcal{T}$. Arguing similarly to Step VI, we would like to estimate both the components $|D_1^{\mathcal{T}}\tilde{h}|$ and $|D_2^{\mathcal{T}}\tilde{h}|$. It follows by construction that

$$|D_1^{\mathcal{T}}\tilde{h}|^p = \frac{\mathcal{H}^1(\gamma^{i+1})^p}{(2r)^p} .$$

Since γ^{i+1} is defined to be the shortest path in $\overline{\boldsymbol{\mathcal{Q}}}$ connecting $\boldsymbol{A}^{i+1}$ to $\boldsymbol{B}^{i+1}$, then in particular it is shorter than the image through φ of the curve connecting A^{i+1} to B^{i+1} on $\partial\mathcal{Q}$ passing through A^0. Therefore, it satisfies the inequality

$$\mathcal{H}^1(\gamma^{i+1}) \le \int_{B(A^0,r\sqrt{2})\cap\partial\mathcal{Q}}|D\varphi|\,d\mathcal{H}^1 .$$

By Hölder's inequality and (2-1) we obtain

$$\mathcal{H}^1(\gamma^{i+1})^p \le (2r\sqrt{2})^{p/p'}\int_{B(A^0,r\sqrt{2})\cap\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 ,$$

and

$$|D_1^{\mathcal{T}}\tilde{h}|^p \le \frac{\sqrt{2}^{p/p'}}{2}\frac{1}{r}\int_{B(A^0,r\sqrt{2})\cap\partial Q}|D\varphi|^p\,d\mathcal{H}^1 \le \frac{\sqrt{2}^{p/p'}}{2}\widetilde{K}\int_{\partial Q}|D\varphi|^p\,d\mathcal{H}^1\,, \tag{2-7}$$

where p' is such that $\frac{1}{p}+\frac{1}{p'}=1$. Estimating $D_2^{\mathcal{T}}\tilde{h}$ is less straightforward. Let us call

$$d := \mathcal{H}^1(A^{i+1}P_j),\quad d' := \mathcal{H}^1(A^iQ_j)\,,\quad \ell := \max\{\mathcal{H}^1(\boldsymbol{A}^i\boldsymbol{A}^{i+1}), \mathcal{H}^1(\boldsymbol{B}^i\boldsymbol{B}^{i+1})\}\,.$$

Then we can write

$$|(d'+\sigma-d)D_1^{\mathcal{T}}\tilde{h}+\sigma D_2^{\mathcal{T}}\tilde{h}| = \mathcal{H}^1(\boldsymbol{P}_j\boldsymbol{Q}_j)\le \ell\,, \tag{2-8}$$

and some geometrical considerations lead to an estimate of the term $|d-d'|\,|D_1^{\mathcal{T}}\tilde{h}|$. Indeed, the path γ^{i+1} is shorter than $\boldsymbol{A}^i\boldsymbol{A}^{i+1}\cup\gamma^i|_{\boldsymbol{A}^i\boldsymbol{Q}_j}\cup\boldsymbol{Q}_j\boldsymbol{P}_j\cup\gamma^{i+1}|_{\boldsymbol{P}_j\boldsymbol{B}^{i+1}}$, providing that

$$\mathcal{H}^1(\gamma^{i+1})\le 2\ell+\mathcal{H}^1(\gamma^i)\frac{d'}{\mathcal{H}^1(A^iB^i)}+\mathcal{H}^1(\gamma^{i+1})\left(1-\frac{d}{\mathcal{H}^1(A^{i+1}B^{i+1})}\right),$$

which gives in particular

$$d\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^1(A^{i+1}B^{i+1})}-d'\frac{\mathcal{H}^1(\gamma^i)}{\mathcal{H}^1(A^iB^i)}\le 2\ell\,.$$

Since the symmetric argument involving γ^i gives the opposite inequality (this time we use that γ^i is shorter than $\boldsymbol{A}^i\boldsymbol{A}^{i+1}\cup\gamma^{i+1}|_{\boldsymbol{A}^{i+1}\boldsymbol{P}_j}\cup\boldsymbol{P}_j\boldsymbol{Q}_j\cup\gamma^i|_{\boldsymbol{Q}_j\boldsymbol{B}^i}$), we get

$$\left|d\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^1(A^{i+1}B^{i+1})}-d'\frac{\mathcal{H}^1(\gamma^i)}{\mathcal{H}^1(A^iB^i)}\right|\le 2\ell\,.$$

Moreover, recalling that γ^{i+1} is parametrized at constant speed, it follows directly that

$$\begin{aligned}
|d-d'|\,|D_1^{\mathcal{T}}\tilde{h}| &\le 2\ell+d'\left|\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^(A^{i+1}B^{i+1})}-\frac{\mathcal{H}^1(\gamma^i)}{\mathcal{H}^(A^iB^i)}\right| \\
&\le 2\ell+\left|\frac{\mathcal{H}^(A^iB^i)}{\mathcal{H}^(A^{i+1}B^{i+1})}\mathcal{H}^1(\gamma^{i+1})-\mathcal{H}^1(\gamma^i)\right| \\
&\le 2\ell+\left|\mathcal{H}^1(\gamma^{i+1})-\mathcal{H}^1(\gamma^i)\right|+2\sigma\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^(A^{i+1}B^{i+1})}\le 4\ell+2\sigma|D_1^{\mathcal{T}}\tilde{h}|.
\end{aligned}$$

Inserting, then, the above estimate into (2-8), we get

$$|D_2^{\mathcal{T}}\tilde{h}|\le\frac{5}{\sigma}\ell+3|D_1^{\mathcal{T}}\tilde{h}|\,,$$

which in turn implies

$$|D_2^{\mathcal{T}}\tilde{h}|^p \le \left(\frac{10l}{\sigma}\right)^p + 6^p|D_1^{\mathcal{T}}\tilde{h}|^p . \tag{2-9}$$

Let us notice that we can easily bound ℓ^p from above since φ is linear on A^iA^{i+1} and B^iB^{i+1}. Indeed, let ℓ be for instance equal to $\mathcal{H}^1(\boldsymbol{A}^i\boldsymbol{A}^{i+1})$; then

$$\ell^p = (\sigma\sqrt{2})^{p-1}\int_{A^iA^{i+1}}|D\varphi|^p\,d\mathcal{H}^1 \le (\sigma\sqrt{2})^{p-1}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1 , \tag{2-10}$$

where we used that $\mathcal{H}^1(A^iA^{i+1}) = \sigma\sqrt{2}$. Thus, by inserting (2-10) and (2-7) into (2-9) one gets

$$|D_2^{\mathcal{T}}\tilde{h}|^p \le 6^p\frac{\sqrt{2}^{p/p'}}{2}\widetilde{K}\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + \frac{(10\sqrt{2})^p}{2}\frac{1}{\sigma}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1 ,$$

which, together with (2-7), gives (2-6) with $K' = 6^p\frac{1}{2}\sqrt{2}^{p/p'}\widetilde{K}$ and $K'' = \frac{1}{2}(10\sqrt{2})^p$. Moreover, by summing up among all the triangles $\mathcal{T}$ in $\mathcal{D}_i$ and observing that $|\mathcal{D}_i| \le 2\sigma$ by construction, we have

$$\begin{aligned}\int_{\mathcal{D}_i}|D\tilde{h}|^p &\le K'|\mathcal{D}_i|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + K''\frac{|\mathcal{D}_i|}{\sigma}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1 \\ &\le K'|\mathcal{D}_i|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + 2K''\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1 .\end{aligned}$$

Finally, on the whole $\mathcal{Q}^-$ one gets

$$\begin{aligned}\int_{\mathcal{Q}^-}|D\tilde{h}|^p &\le K'|\mathcal{Q}^-|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + 2K''\int_{\partial\mathcal{Q}^-\cap\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 \\ &\le K_2\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 ,\end{aligned} \tag{2-11}$$

for a suitable $K_2 \ge 2\max\{K', K''\}$.

Step VIII. Definition of h and conclusion. We can now observe that, whenever $\tilde{h}$ is a homeomorphism, Theorem 1.1 follows directly. Indeed, $\tilde{h}$ coincides with φ on $\partial\mathcal{Q}$, it is finitely piecewise affine and, moreover, the estimates (2-3) and (2-11) provide that (1-1) is satisfied by $\tilde{h}$ and $K \ge \max\{K_1, K_2\}$. Unfortunately, in our construction the function $\tilde{h}$ happens to be one-to-one only when all the paths γ^i lie in the interior of $\boldsymbol{\mathcal{Q}}$ without intersecting each other. Of course in general this is not the case, but it is always possible to slightly modify the paths γ^i in order to get the one-to-one property. A possible configuration can be seen Figure 4. More precisely, it is always possible to separate a curve γ^{i+1} from either $\partial\boldsymbol{\mathcal{Q}}$ and γ^i so that the minimal distance between them is much smaller than the lengths of all the linear pieces of the paths γ and $\partial\boldsymbol{\mathcal{Q}}$. Let us notice that the minimal distance is

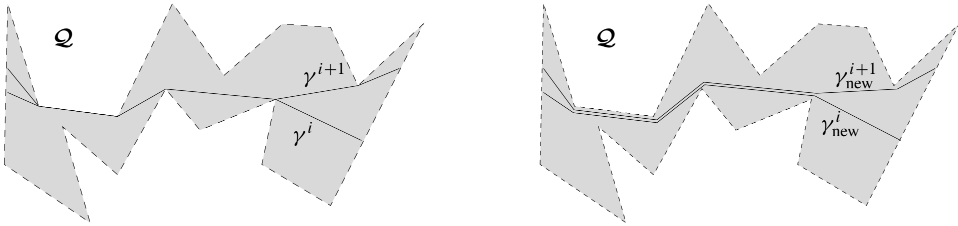

Figure 4. Modification of γ^i and γ^{i+1}.

strictly positive because there is only a finite number of paths and each of them is finitely piecewise linear. We finally define the function h in the very same way as we defined $\tilde{h}$ in Step V, but this time using the separated paths. Therefore, the function h is a homeomorphism, it is still finitely piecewise affine, it satisfies the boundary condition on $\partial\mathcal{Q}$ and, furthermore, estimate (1-1) is still valid up to increasing the geometric constant K by a quantity which is as small as we wish. This implies the validity of Theorem 1.1 and concludes the proof. □

Remark 2.1. Let $\mathcal{Q}\subset\mathbb{R}^2$ be a generic square of length side r, $p\geq 1$ and $\varphi:\partial\mathcal{Q}\to\mathbb{R}^2$ piecewise linear. Then there exists a piecewise affine function $h:\mathcal{Q}\to\mathbb{R}^2$ that coincides with φ on the boundary $\partial\mathcal{Q}$ and a geometric constant K depending only on p such that

$$\int_{\mathcal{Q}}|Dh|^p \leq K r\int_{\partial\mathcal{Q}}|D\varphi|^p\, d\mathcal{H}^1\,. \tag{2-12}$$

Indeed, there always exists an affine function ρ mapping the unit square $\mathcal{Q}_1$ onto $\mathcal{Q}$. Let us call, with a slight abuse of notation, ρ and its restriction to the boundary $\partial\mathcal{Q}_1$ with the same name. Then, by applying Theorem 1.1 to the function $\varphi\circ\rho$ and recalling that $|D\rho|=r$, it is possible to find a constant K and a piecewise affine function $\tilde{h}:\mathcal{Q}_1\to\mathbb{R}^2$ satisfying

$$\int_{\mathcal{Q}_1}|D\tilde{h}(x)|^p\,dx \leq K\int_{\partial\mathcal{Q}_1} r^p|D\varphi(\rho(t))|^p\,d\mathcal{H}^1(t)\,.$$

Finally, by defining $h:=\tilde{h}\circ\rho^{-1}$ and changing the variables, one gets

$$r^{p-2}\int_{\mathcal{Q}}|Dh|^p \leq r^{p-1}K\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1$$

and (2-12) follows.

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EMANUELA RADICI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ERLANGEN
CAUERSTRASSE 11
D-90158 ERLANGEN
GERMANY
radici@math.fau.de

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