HAMILTON–JACOBI EQUATIONS AND DISTANCE FUNCTIONS ON RIEMANNIAN MANIFOLDS

CARLO MANTEGAZZA AND ANDREA CARLO MENNUCCI

ABSTRACT. The paper is concerned with the properties of the distance function from a closed subset of a Riemannian manifold, with particular attention to the set of singularities.

CONTENTS

1. INTRODUCTION

The paper is concerned with the properties of the singular set of the distance function from a closed subset of a n–dimensional Riemannian manifold (M, g), in particular with its rectifiability.

Definition 1.1. We say that a subset S of M is C^r*–rectifiable*, with $r \geq 1$, if it can be covered by a countable family of embedded C^r submanifolds of dimension $(n-1)$, with the exception of a set of $\mathcal{H}^{n-1}$ zero measure, where $\mathcal{H}^{n-1}$ is the $(n-1)$–dimensional Hausdorff measure on M (see [17, 28] for a complete discussion of the notion of rectifiability).

The distance function from a closed, not empty subset K of M is defined in the usual way,

$$d_K(x) = \inf_{y \in K} d(x, y)$$

where d is the distance of M induced by the metric tensor g. The singular set $Sing$ of $d_K : M \to \mathbb{R}$ is the set where this function fails to be differentiable.
The study of the rectifiability of $Sing$ relies on the theory of *viscosity solutions* of Hamilton–Jacobi equations. Indeed, we show that the distance function d_K is a viscosity solution of the following problem

$$\begin{cases} |\nabla u|^2 = 1 & \text{in } M \setminus K, \\ u = 0 & \text{on } \partial K \end{cases}$$

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and we use the property of semiconcavity shared by such solutions to obtain a rectifiability result for $Sing$.

Then we investigate under which hypotheses also the closure of $Sing$ is rectifiable. To deal with this problem, which turns out to be strictly connected to the analysis of the geodesic flow on M starting from the boundary of K, we employ ideas originating from the study of the *cut locus* of a point in a Riemannian manifold, which is actually the very special case when K reduces to a single point. Roughly speaking, the cut locus of $p \in M$ is the collection of the points where the geodesic curves starting at p cease to be absolute minimizers of the distance between their end points. The cut locus arises naturally in many questions related to the study of topology and geometry of manifolds.

Our results lead to the conclusion that, under some conditions on the regularity of the set K, the Hausdorff dimension of the closure of the singular set is at most $(n - 1)$ and that the gradient of the distance function from K is locally a vector field with *special bounded variation* (see [3, 4, 5]).
Moreover, we also study when the singular set shares an higher regularity and we analyse in detail its topological structure if M is a two dimensional surface and K an analytic subset.

The study of the distance function and hence of the associate eikonal equation $|\nabla u| = 1$, is a special example of a connection which can be extended to a more general class of stationary Hamilton–Jacobi equations. In the last section we discuss some problems about the structure of the singular set of viscosity solutions, suggested by geometric results for the cut locus of a point.

2. Stationary Hamilton–Jacobi Equations on Manifolds

Let M be a smooth n–dimensional differentiable manifold.
We consider the following Hamilton–Jacobi problem in $\Omega \subset M$,

$$\begin{cases} H(x, du(x), u(x)) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

where $H : T^*\Omega \times \mathbb{R} \to \mathbb{R}$ and T^*M denotes the cotangent bundle of M.

Like in the standard case $M = \mathbb{R}^n$, we begin with the definition of the following generalized differentials.

Definition 2.1. Given a continuous function $u : \Omega \to \mathbb{R}$ and a point $x \in M$, the *superdifferential* of u at x is the subset of T_x^*M defined by

$$\partial^+ u(x) = \left\{ d\varphi(x) \,|\, \varphi \in C^1(M), \varphi(x) - u(x) = \min_M \varphi - u \right\}.$$

Similarly, the set

$$\partial^- u(x) = \left\{ d\psi(x) \,|\, \psi \in C^1(M), \psi(x) - u(x) = \max_M \psi - u \right\}$$

is called the *subdifferential* of u at x. Notice that it is equivalent to replace the $\max$ $(\min)$ on all M with the maximum (minimum) in an open neighborhood of x in M.

It is easy to see that $\partial^+ u(x)$ and $\partial^- u(x)$ are both nonempty if and only if u is differentiable at x. In this case we have

$$\partial^+ u(x) = \partial^- u(x) = \{du(x)\}.$$

We list here without proof some of the properties of the sub and superdifferentials which will be needed later.

Proposition 2.2. *If $\psi : N \to M$ is a map between the smooth manifolds N and M which is C^1 around $x \in N$, then*

$$\partial^+(u \circ \psi)(x) \supset \partial^+ u(\psi(x)) \circ d\psi(x) \,=\, \{v \circ d\psi(x) \,|\, v \in \partial^+ u(\psi(x))\}.$$

If ψ is a local diffeomorphism near x, the inclusion becomes an equality. An analogous statement holds for ∂^-.

Proposition 2.3. *If $\theta : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $\dot\theta(u(x)) \geq 0$, then*

$$\partial^+(\theta \circ u)(x) \supset d\theta(u(x)) \circ \partial^+ u(x) \,=\, \{d\theta(u(x)) \circ v \,|\, v \in \partial^+ u(x)\},$$

similarly for ∂^-. If $\dot\theta(u(x)) > 0$ then the inclusion is an equality.

For a locally Lipschitz function u, $\partial^+ u(x)$ and $\partial^- u(x)$ are compact convex sets, almost everywhere coinciding with the differential of the function u, by Rademacher's Theorem.
For a generic continuous function u we prove in the next proposition that $\partial^+ u(x)$ and $\partial^- u(x)$ are not empty in a dense subset.

Proposition 2.4. *Let $u : \Omega \to \mathbb{R}$ be a continuous function on an open subset Ω of M. Then the subdifferential $\partial^- u(x)$ (superdifferential $\partial^+ u(x)$) is not empty for every x in a dense subset of Ω.*

Proof. It is always possible to endow M with a Riemannian structure giving a metric $d(\cdot\,,\cdot)$ on M which generates the same topology.
Consider a generic point $y \in \Omega$ and a geodesic ball B contained in Ω with center y. If the ball B is small enough, the function $x \mapsto d^2(x, y)$ is smooth in $\overline{B}$. Taking a large positive constant A, the function $F_A(x) = u(x) + Ad^2(x, y)$ has a local minimum at a point x_A in the interior of B. At x_A the subdifferential of the function F_A must contain the origin of $T^*_{x_A}M$, hence, being $d^2(x, y)$ differentiable in the ball B, the differential of $-d^2(x, y)$ at x_A belongs to $\partial^- u(x_A)$. As the point y and the ball B were arbitrarily chosen, the set of points where the subdifferential of u is not empty is dense in Ω.
The same argument holds for the superdifferential of u, considering the function $-u$. □

Now we introduce the notion of semiconcavity which will play a central role in the first part of the paper.

Definition 2.5. Given an open set $\Omega \subset \mathbb{R}^n$, a continuous function $u : \Omega \to \mathbb{R}$ is called *locally semiconcave* if, for any open convex set $K \subset \Omega$ with compact closure in Ω, there exists a constant C such that one of the following three equivalent conditions is satisfied,

(1) $\forall x, h$ with $x, x + h, x - h \in K$,
$$u(x + h) + u(x - h) - 2u(x) \leq C|h|^2,$$
(2) $u(x) - C|x|^2$ is a concave function in K,
(3) $D^2 u \leq 2C\, Id$ in K, as distributions (Id is the $n \times n$ identity matrix).

Remark 2.6. The third condition shows that the semiconcavity of u is a local property.

In order to give a meaning to the concept of semiconcavity when the ambient space is a differentiable manifold M, we analyse the stability of this property under composition with C^2 maps.

Proposition 2.7. *If $u : \Omega \to \mathbb{R}$ is a Lipschitz function such that $u(x) - C\,|x|^2$ is concave and $\psi : \Omega' \to \Omega$ is a C^2 function with bounded first and second derivatives, then $u \circ \psi : \Omega' \to \mathbb{R}$ is a Lipschitz function and $u \circ \psi(y) - C'|y|^2$ is concave, for a suitable constant C'.*

The proof is straightforward.
By this proposition, the following definition is well–posed.

Definition 2.8. A continuous function $u : M \to \mathbb{R}$ is called *locally semiconcave* if, for any local chart $\psi : \mathbb{R}^n \to \Omega \subset M$, the function $u \circ \psi$ is locally semiconcave in $\mathbb{R}^n$ according to Definition 2.5

The importance of semiconcave functions in connection with the generalized differentials is expressed by the following proposition (see [12]).

Proposition 2.9. *Let the function $u : M \to \mathbb{R}$ be semiconcave, then the superdifferential $\partial^+ u$ is not empty at each point, moreover, $\partial^+ v$ is* upper semicontinuous, *namely*

$$x_k \to x, \quad v_k \to v, \quad v_k \in \partial^+ u(x_k) \quad \Longrightarrow \quad v \in \partial^+ u(x).$$

In particular, if the differential du exists at every *point of M, then $u \in C^1(M)$.*

Now we introduce the definition of viscosity solution.
Let Ω be an open subset of M. We suppose H, called *Hamiltonian* function, to be a continuous, real function on $T^*\Omega \times \mathbb{R}$. We are interested in the following Hamilton–Jacobi problem

$$H(x, du(x), u(x)) = 0 \qquad \text{in } \Omega. \tag{2.1}$$

Definition 2.10. We say that a continuous function u is a *viscosity solution* of equation (2.1) if for every $x \in \Omega$,

$$\begin{cases} H(x, v, u(x)) \leq 0 & \forall v \in \partial^+ u(x), \\ H(x, v, u(x)) \geq 0 & \forall v \in \partial^- u(x). \end{cases} \tag{2.2}$$

If only the first condition is satisfied (resp. the second), u is called a *viscosity subsolution* (resp. a *viscosity supersolution*).

If Ω' is an open subset of another smooth differentiable manifold N and $\psi : \Omega' \to \Omega$ is a C^1 local diffeomorphism, we define the *pull–back* of the Hamiltonian function $\psi^* H : T^*\Omega' \times \mathbb{R} \to \mathbb{R}$ by

$$\psi^* H(y, v, r) = H(\psi(y), v \circ d\psi(y)^{-1}, r).$$

Taking into account Proposition 2.2, the following statement is obvious.

Proposition 2.11. *If u is a viscosity solution of $H = 0$ in $\Omega \subset M$ and $\psi : \Omega' \to \Omega$ is a C^1 local diffeomorphism, then $u \circ \psi$ is a viscosity solution of $\psi^* H = 0$ in $\Omega' \subset N$.*

3. The Distance Function from a Subset of a Manifold

From now on, M will be a smooth Riemannian manifold without boundary and complete, with a metric tensor g.
We will study the distance function d_K from a closed and not empty subset K of M. For technical reasons, sometimes it is convenient also to consider its square,

$$d^2_K(x) = [d(x, K)]^2 : M \to \mathbb{R}^+.$$

The distance is clearly a continuous function on M but in general it is not everywhere differentiable, even if the set K is smooth. For instance, if the manifold M is compact, the distance function from any proper subset cannot be differentiable everywhere, being singular at the points of absolute maximum.
This section deals with the singular set where the gradient of d_K does not exist.

The distance between two points x and y in M is defined as the infimum of the lengths of the C^1 curves joining the points x and y, where the length of a C^1 curve $\gamma : [a, b] \to M$ is given by

$$L(\gamma) = \int_a^b \sqrt{g\left(\dot\gamma(t), \dot\gamma(t)\right)}\, dt.$$

Actually, as M is complete, such infimum is reached by at least a curve (Theorem of Hopf–Rinow) which, once reparametrized by a constant multiple of the arclength, also minimizes the following *energy* functional,

$$E(\gamma) = \int_0^1 g\left(\dot\gamma(t), \dot\gamma(t)\right)\, dt$$

in the class of C^1 curves $\gamma(t) : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$.
Then, a minimizing curve γ must satisfy the Euler equation of the functional which in local coordinates read

$$\frac{d^2\gamma_k}{dt^2} + \sum_{i,j,k} \Gamma^k_{ij}(\gamma)\frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt} = 0 \qquad \forall k$$

where the smooth coefficients $\Gamma^k_{ij}(\gamma)$ are called *Christoffel symbols* and depend on the metric tensor g of M and on its derivatives.
We will call *geodesics* the curves in M satisfying such equation. By the previous discussion, length or energy minimizing curves are geodesics.
By the existence and uniqueness theorem of solutions of ODE's and the completeness of M, for every point $x \in M$ and every initial *velocity vector* $v \in T_xM$ there exists a unique smooth geodesic curve $\gamma_v : \mathbb{R} \to M$ with $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$, moreover the pair $(\gamma_v(t), \dot\gamma_v(t))$ depends smoothly on $(x, v) \in TM$ and $t \in \mathbb{R}$.
It is hence defined the *exponential map*, $\mathrm{Exp} : TM \times \mathbb{R} \to M$,

$$\mathrm{Exp}(x, v, t) = \gamma_v(t)$$

which is a smooth map onto the manifold M and the following relation

$$\mathrm{Exp}(x, \lambda v, t) = \mathrm{Exp}(x, v, \lambda t)$$

holds for every $\lambda \in \mathbb{R}$ (see [18]).

The distance of a point $x \in M$ from a closed subset K is obviously defined as the infimum of the distances from each one of its points. An equivalent definition is the following,

$$d_K(x) = \min_\gamma \int_a^b \sqrt{g\left(\dot\gamma(t), \dot\gamma(t)\right)}\, dt$$

where now the minimum is taken over the class of C^1 curves γ starting at x and ending in a point of K. It simple to see that the minimizing curves, once reparametrized, are again geodesics.
We are now ready to show that the distance function from K is a viscosity solution of the following Hamilton–Jacobi problem

$$\begin{cases} g(\nabla u, \nabla u) - 1 = 0 & \text{in } M \setminus K, \\ u = 0 & \text{on } K. \end{cases} \tag{3.1}$$

Moreover we will prove that, under general hypotheses, it is the unique solution.

In (3.1) and sometimes in the sequel we take into account the equivalence between the differential du and the gradient ∇u of u, using the identification of T^*_xM with T_xM given by the scalar product $g_x(\cdot\,,\cdot)$.
Moreover, for the sake of simplicity, we denote with $|v|$ the modulus $\sqrt{g_x(v,v)}$ of a vector $v \in T_xM$.

Theorem 3.1. *The distance function d_K is a viscosity solution of Problem* (3.1).
The function $d^2_K/2$ is a viscosity solution of

$$\begin{cases} g(\nabla u, \nabla u) - 2u = 0 & \textit{in } M, \\ u = 0 & \textit{on } K. \end{cases} \tag{3.2}$$

Proof. First of all, notice that the distance function from K is a 1–Lipschitz function, hence d_K^2 is locally Lipschitz.
First we show that d_K is a viscosity subsolution.
Let $\widetilde{x} \in M\setminus K$ and let φ be a smooth function such that $\varphi(\widetilde{x})-d_K(\widetilde{x}) = \min\,\varphi-d_K$, that is, $d\varphi(\widetilde{x}) \in \partial^+ d_K(\widetilde{x})$; then

$$-\varphi(\widetilde{x}) + \varphi(x) \geq -d_K(\widetilde{x}) + d_K(x),$$

for every $x \in M$. Considering a C^1 curve $\gamma(t)$ with $\gamma(0) = \widetilde{x}$, parametrized by arclength, we have

$$\frac{\varphi(\gamma(t)) - \varphi(\widetilde{x})}{t} \geq \frac{d_K(\gamma(t)) - d_K(\widetilde{x})}{t} \geq -\frac{d(\gamma(t),\gamma(0))}{t} \geq -1$$

since d_K is 1–Lipschitz. Passing to the limit for $t \to 0$ we get

$$d\varphi(\widetilde{x})\dot{\gamma}(0) + 1 \geq 0$$

and being γ arbitrary with $|\dot{\gamma}| = 1$, we obtain

$$-|\nabla\varphi(\widetilde{x})| + 1 \geq 0\,,$$

hence d_K is a subsolution.
Now let γ be the geodesic arc which minimizes the distance from $\widetilde{x}$ to K, parametrized by arclength with $\gamma(0) = \widetilde{x}$. Take a test function φ such that $\varphi(\widetilde{x}) - d_K(\widetilde{x}) = \max\,\varphi - d_K$, which means $d\varphi(\widetilde{x}) \in \partial^- d_K(\widetilde{x})$, then

$$-\varphi(\widetilde{x}) + \varphi(x) \leq -d_K(\widetilde{x}) + d_K(x)$$

hence,

$$t + \varphi(\gamma(t)) - \varphi(\widetilde{x}) \leq t + d_K(\gamma(t)) - d_K(\widetilde{x}) = 0\,.$$

The last equality follows from the fact that the curve γ also minimizes the distance to K for every point $\gamma(t)$, $t > 0$. Dividing by t and passing to the limit for $t \to 0$, we get

$$d\varphi(\widetilde{x})\dot{\gamma}(0) + 1 \leq 0$$

hence,

$$-|\nabla\varphi(\widetilde{x})| + 1 \leq 0.$$

This proves that d_K is also a viscosity supersolution and solves Problem (3.1).
Now we show that the function $d_K^2/2$ is a solution of Problem (3.2).
As d_K is 1–Lipschitz, at every point of K the function d_K^2 is differentiable and its differential is zero. Hence, the definition of viscosity solution holds for points belonging to K. In order to prove the theorem, it is then sufficient to test conditions (2.2) on the generalized differentials at the points of the open set $M \setminus K$.
Since $d_K^2/2$ is positive in $M \setminus K$, applying Proposition 2.3 with the function $\theta(t) = \sqrt{2t}$, we see that the function $d_K^2/2$ is a viscosity solution of

$$g\left(\frac{\nabla u}{\sqrt{2u}}, \frac{\nabla u}{\sqrt{2u}}\right) - 1 = 0$$

in $M \setminus K$. Being there positive, it also solves

$$g(\nabla u, \nabla u) - 2u = 0$$

in $M \setminus K$. This fact together with the previous remark about the behavior of d_K^2 at the points of K gives the thesis. □

As we said, we prove also an uniqueness result for such problems.

Proposition 3.2. *The distance function d_K is the unique viscosity solution of Problem* (3.1) *in the class of continuous functions bounded from below.*
The function $d_K^2/2$ is the unique viscosity solution of Problem (3.2) *in the class of continuous functions on M such that their zero set is K.*

Remark 3.3. The restriction to lower bounded functions is necessary, $\|x\|$ and $-\|x\|$ are both viscosity solutions of Problem (3.1) with $M = \mathbb{R}^n$ and $K = \{0\}$. Moreover, the completeness of M plays an important role here, if M is the unit ball of $\mathbb{R}^n$, the same example shows that the uniqueness does not hold.
Notice also that every function $d^2_H/2$ where H is a closed subset of M with $H \supset K$, is a viscosity solution of Problem (3.2), equal to zero on K.

Proof. Suppose that u is a viscosity solution of Problem (3.1) then, u is also a solution of

$$\begin{cases} |\nabla u| - 1 = 0 & \text{in } M \setminus K, \\ u = 0 & \text{on } K. \end{cases}$$

As in the work of Kružhkov [22], we consider the function $v = -e^{-u}$ which, by Proposition 2.3, turns out to be a viscosity solution of

$$\begin{cases} |\nabla v| + v = 0 & \text{in } M \setminus K, \\ v = -1 & \text{on } K \end{cases}$$

moreover, $|v| \leq e^{-\inf u}$.
We establish an uniqueness result for this last problem in the class of bounded functions v, which clearly implies the first part of the proposition. We remark that the proof is based on similar ones in [13, 14, 19].
We argue by contradiction, suppose that u and v are two bounded solutions of (3), $|u|$, $|v| \leq C$, and that at a point $\overline{x}$ we have $u(\overline{x}) \geq 2\varepsilon + v(\overline{x})$ with $\varepsilon > 0$.
Let $b(x, y) : M \times M \to \mathbb{R}$ be a smooth function satisfying

- $b \geq 0$
- $|\nabla_x b(x, y)|$, $|\nabla_y b(x, y)| \leq 2$
- $\sup_{M\times M} |d(x, y) - b(x, y)| < \infty$

such a function can be obtained *smoothing* the distance function in $M \times M$.
We fix a point x_0 in K and we define the smooth function $B(x) = b(x, x_0)^2$. By the properties of b and the boundedness of u and v, the following function Ψ : $M \times M \to \mathbb{R}$

$$\Psi(x, y) = u(x) - v(y) - \lambda d(x, y)^2 - \delta\, B(x) - \delta\, B(y)$$

has a maximum at a point $\widehat{x}, \widehat{y}$ (dependent on the positive parameters δ and λ) and such maximum $\Psi(\widehat{x}, \widehat{y})$ is less than $2C$. Hence, the function

$$x \mapsto [v(\widehat{y}) + \lambda d(x, \widehat{y})^2 + \delta\, B(x) + \delta\, B(\widehat{y})] - u(x) \tag{3.3}$$

has a minimum at $\widehat{x}$ while

$$y \mapsto [u(\widehat{x}) - \lambda d(\widehat{x}, y)^2 - \delta\, B(\widehat{x}) - \delta\, B(y)] - v(y) \tag{3.4}$$

has a maximum at $\widehat{y}$.
If $2\delta \leq \varepsilon/B(\overline{x})$ then

$$\Psi(\widehat{x}, \widehat{y}) \geq \Psi(\overline{x}, \overline{x}) \geq 2\varepsilon - 2\delta B(\overline{x}) \geq \varepsilon$$

hence, we get

$$\delta B(\widehat{x}) + \delta B(\widehat{y}) + \lambda d(\widehat{x}, \widehat{y})^2 + \varepsilon \leq u(\widehat{x}) - v(\widehat{y}) \leq 2C\,. \tag{3.5}$$

This shows that, for a fixed δ, the pair $\widehat{x}, \widehat{y}$ is contained in a bounded set and, if λ goes to $+\infty$ the distance between $\widehat{x}$ and $\widehat{y}$ goes to zero. Possibly passing to a subsequence for λ going to infinity, $\widehat{x}$ and $\widehat{y}$ converge to a common limit point z which cannot belong to K, otherwise we would get $\varepsilon \leq u(z) - v(z) = 0$, thus, for some λ large enough also $\widehat{x}$ and $\widehat{y}$ do not belong to K.
As the function $d^2(x, y)$ is smooth in $B_z \times B_z \subset M \times M$, where B_z is a small

geodesic ball around z, choosing a suitable λ large enough we can differentiate the functions inside the square brackets in equations (3.3) and (3.4) obtaining

$$\widehat{v} = \delta\,\nabla B(\widehat{x}) + \lambda\nabla_x d^2(\widehat{x},\widehat{y})\ \in \partial^+ u(\widehat{x}),$$

$$\widehat{w} = -\delta\,\nabla B(\widehat{y}) - \lambda\nabla_y d^2(\widehat{x},\widehat{y})\ \in \partial^- v(\widehat{y}).$$

By Definition 2.10 we have that $|\widehat{v}| + u(\widehat{x}) \le 0$ and $|\widehat{w}| + v(\widehat{y}) \ge 0$, hence

$$u(\widehat{x}) - v(\widehat{y}) + |\widehat{v}| - |\widehat{w}| \le 0.$$

Moreover,

$$\begin{aligned}
|\widehat{v}| - |\widehat{w}| &= \left|\delta\,\nabla B(\widehat{v}) + \lambda\nabla_x d^2(\widehat{x},\widehat{y})\right| - \left|\delta\,\nabla B(\widehat{y}) + \lambda\nabla_y d^2(\widehat{x},\widehat{y})\right|\\
&\ge \left|\lambda\nabla_x d^2(\widehat{x},\widehat{y})\right| - \left|\lambda\nabla_y d^2(\widehat{x},\widehat{y})\right| - |\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})|\\
&= 2\lambda d(\widehat{x},\widehat{y})\,|\nabla_x d(\widehat{x},\widehat{y})| - 2\lambda d(\widehat{x},\widehat{y})\,|\nabla_y d(\widehat{x},\widehat{y})| - |\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})|\\
&= 2\lambda d(\widehat{x},\widehat{y}) - 2\lambda d(\widehat{x},\widehat{y}) - |\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})|\\
&= -\,|\delta\,\nabla B(\widehat{y})| - |\delta\,\nabla B(\widehat{x})|
\end{aligned}$$

which implies,

$$u(\widehat{x}) - v(\widehat{y}) - \delta|\nabla B(\widehat{y})| - \delta|\nabla B(\widehat{x})| \le 0\,.$$

Finally, we have that

$$\delta|\nabla B(\widehat{x})| = 2\delta|b(\widehat{x},x_0)\nabla b(\widehat{x},x_0)| \le 4\delta\sqrt{B(\widehat{x})}$$

and using the estimate $\delta B(\widehat{x}) \le 2C$ which follows from equation (3.5),

$$\delta|\nabla B(\widehat{x})| \le 8\sqrt{2\delta C} \le \varepsilon/4$$

if δ was chosen small enough. Holding the same for $\widehat{y}$, we conclude that

$$u(\widehat{x}) - v(\widehat{y}) - \varepsilon/2 \le 0$$

which is in contradiction with the fact that $u(\widehat{x}) - v(\widehat{y}) \ge \varepsilon$.

Now we prove the second part of the proposition. If u is a continuous viscosity solution of Problem (3.2) then, by Proposition 2.4 the superdifferential of u is not empty in a dense subset of $M\setminus K$, hence, directly by the equation and by continuity, u is non negative. By the hypothesis on its zero set we conclude that u is positive in all $M \setminus K$. Composing u with the function $t \mapsto \sqrt{2t}$, we see that $\sqrt{2u}$ is a positive, continuous viscosity solution of Problem (3.1), then it must coincide with d_K, by the previous result. It follows that $u = d_K^2/2$. □

We now study the rectifiability of $Sing$, the key result is the following.

Proposition 3.4. *The function d_K is locally semiconcave in $M \setminus K$.*

Proof. The distance function d_K is a viscosity solution of $H = 0$ in $M \setminus K$, where the Hamiltonian function is given by $H(x, v, t) = |v|^2 - 1$. We choose a smooth local chart $\psi : \mathbb{R}^n \to \Omega \subset M$ and we define $v = d_K \circ \psi$, which is a locally Lipschitz function and, by Proposition 2.11, it is a viscosity solution of $\psi^* H = 0$.
The pull–back of the Hamiltonian function on $\mathbb{R}^n$ takes the form

$$\psi^* H(y, w, s) = g_{\psi(y)}(d\psi(w), d\psi(w)) - 1 = g_{ij}(y)w_i w_j - 1$$

for $(y, w, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and where $g_{ij}(y)$ are the components of the metric tensor of M in local coordinates.
Since the matrix $g_{ij}(y)$ is positive definite $\psi^* H(y, w, s)$ is locally uniformly convex in w, hence, by Theorem 5.3 of [23], it follows that $v = d_K \circ \psi$ is locally semiconcave. Recalling Definition 2.8, this means that d_K is locally semiconcave in $M \setminus K$. □

The semiconcavity of d_K allows us to work with the superdifferentials when the gradient does not exist, that is, on the following *singular set*

$$Sing = \{x \in M \mid d_K^2 \text{ is not differentiable at } x\} .$$

Remark 3.5. We defined $Sing$ via the squared distance function instead of the distance in order to avoid to consider also the points of the boundary of K which are singular for d_K. It is obvious that outside K the distance and its square have the same regularity.

Notice that the points of $Sing$ are precisely those where the superdifferential is not a singleton, hence, by Proposition 2.9 we have the following result.

Proposition 3.6. *The function d_K^2 is of class C^1 in the open set $M \setminus \overline{Sing}$ and d_K is C^1 in $M \setminus \big(K \cup \overline{Sing}\big)$.*

The semiconcavity property also gives information about the propagation of singularities and the relations between the structure of the superdifferential at a point x and the set of minimal geodesics from x to K, see [2, 6].

Proposition 3.7. *The set $Ext(\partial^+ d_K^2(x)/2)$ of extremal points of the (convex) superdifferential set of $d_K^2/2$ at x is in one–to–one correspondence with the family $\mathcal{G}(x)$ of minimal geodesics from x to K. Precisely $\mathcal{G}(x)$ is described by*

$$\mathcal{G}(x) = \{Exp(x, -v, \cdot) : [0, 1] \to M \mid \text{for } v \in Ext(\partial^+ d_K^2(x)/2)\} . \tag{3.6}$$

The set of points of K at minimum distance from x are given by $Exp(x, -v, 1)$ for v in the set of extremal points of the superdifferential set of $d_K^2/2$ at x.

Hence, another characterization of $Sing$ is the set of points x where the distance $d(x, K)$ is given by more than one minimal geodesic between x and K.

Remark 3.8. As a particular case of this proposition, we have that if the function d_K^2 is differentiable at x, then the point of K closest to x is uniquely determined and given by $\mathrm{Exp}(x, -\nabla d_K^2(x)/2, 1)$.

We consider now the rectifiability of $Sing$.

Proposition 3.9. *The set $Sing$ is C^2–rectifiable.*

Proof. By a result proved in [1], the singular set of a locally semiconcave function in an open set of $\mathbb{R}^n$ is C^2–rectifiable. We take a countable family of local charts $\psi_i : \mathbb{R}^n \to \Omega_i$ and consider the functions $d_K \circ \psi_i$. These functions are locally semiconcave in $\mathbb{R}^n$ with singular sets $Sing_i$, hence, by the relation

$$Sing \subset \bigcup_{i=1}^{\infty} \psi_i(Sing_i)$$

the proposition follows. □

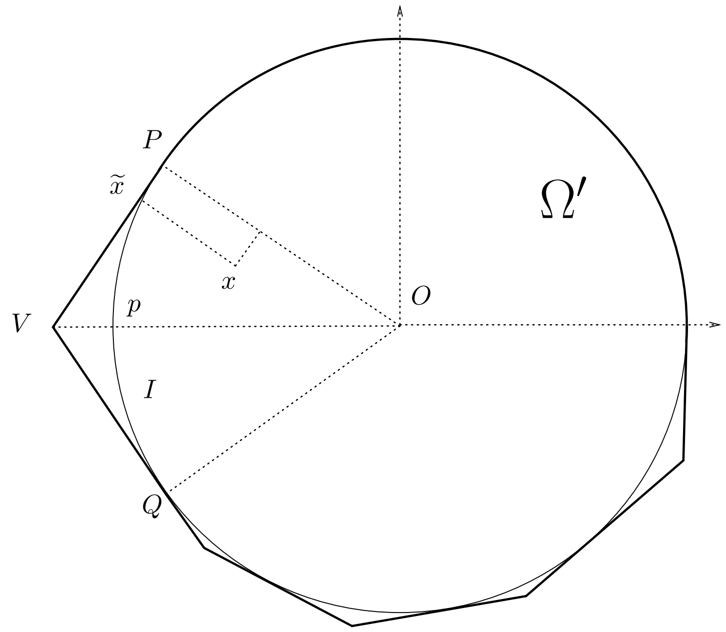
The same statement does not hold for the closure of $Sing$, for a generic closed set K. We describe a counterexample showing that indeed the set $\overline{Sing}$ is not rectifiable for a set K of regularity only $C^{1,1}$.

We want to determine a convex open set Ω with a $C^{1,1}$ boundary in $\mathbb{R}^2$, such that the closure of the singular set $Sing$ of the distance function from its boundary has nonzero Lebesgue measure, hence it is not rectifiable.

We start with a Cantor–like set $\mathcal{C} \subset \mathbb{S}^1$, closed with empty interior in $\mathbb{S}^1$, with no isolated points and positive $\mathcal{H}^1$ measure. Such a set clearly exists and can be described as

$$\mathcal{C} = \mathbb{S}^1 \setminus \bigcup_{i=1}^{\infty} I_i$$

where $\{I_i\}$ is a countable family of open disjoint connected arcs on $\mathbb{S}^1$ whose middle points are $p_i \in \mathbb{S}^1$.
We claim that every point of $\mathcal{C}$ is a limit point of the sequence $\{p_i\}$. If $p \in \mathcal{C}$ there must be a sequence of arcs I_{i_j} arbitrarily close to p, since the arcs are countable and the sum of their lengths is bounded by 2π we have that they shrink when j goes to infinity, hence $p_{i_j} \to p$.
We define an open convex set Ω' as the intersection of the open halfplanes, containing the origin of $\mathbb{R}^2$, determined by the tangent lines to $\mathbb{S}^1$ at the points of $\mathcal{C}$, see the following figure.



Let us take an arc I with middle point p, bounded by P, $Q \in \mathcal{C}$ and consider the associate quadrilateral $OPVQ$. If the point x is inside the open triangle OPV it is clear that the point of $\partial\Omega'$ closest to x belongs to the segment PV and it is unique ($\widetilde{x}$ in the figure). Hence, for such points the distance from the boundary of Ω coincide with the distance from the segment PV.
Applying the same argument to the open triangle OQV, we see that the segment OV consists of singular points of $d_{\partial\Omega}$, moreover, the segment OV intersects $\mathbb{S}^1$ at the point p.
It follows that the union $\mathcal{S}$ of the segments from the middle points p_i to the origin are singular points of $d_{\partial\Omega}(x)$. Being p_i dense in $\mathcal{C}$, the closure of $\mathcal{S}$ contains $\lambda\mathcal{C} \subset \mathbb{R}^2$ for every $\lambda \in [0,1]$. As $\mathcal{C}$ has $\mathcal{H}^1$ positive measure, the Lebesgue measure of $\overline{\mathcal{S}}$ is positive, by Fubini's Theorem.
Now let Ω be the set of points of $\mathbb{R}^2$ whose distance from the convex Ω' is less than 1. It is immediate to check that

$$d_{\partial\Omega}(x) = d_{\partial\Omega'}(x) + 1 \qquad \forall x \in \Omega'$$

hence for every x in the unit ball.
So the closure of the singular set of the distance function from the boundary of Ω (or from the complementary set of Ω in $\mathbb{R}^2$) has positive Lebesgue measure, moreover, by the theory of convex bodies, the boundary of Ω is of class at least $C^{1,1}$.

In the next section we show that if the boundary of K is of class at least C^3 then also the closure of $Sing$ is rectifiable. To our knowledge it is unknown even in $\mathbb{R}^2$ if the gap between such result and the previous counterexample can be filled, that is, if the C^2 regularity of the boundary of K is enough to get the rectifiability of the closure of the singular set.

4. Rectifiability of the Closure of the Singular Set

In this section we are going to show that an higher regularity of the set K implies the rectifiability of the closure of the singular set. Moreover, we determine a relation between the regularity of K and of the hypersurfaces covering $Sing$. We consider a set K which is a k–dimensional, embedded C^r submanifold of M without boundary, with $0 \leq k \leq n-1$ (the case $k = n$ is trivial) and $r \geq 2$.

Remark 4.1. Notice that the case when K is a closed subset of M with a smooth boundary ∂K is analogous. Indeed, if $x \in M \setminus K$, then the minimal geodesic from x to K ends in ∂K without touching the interior points, hence to study d_K we can simply consider the distance function from ∂K in every open connected component of $M \setminus K$. However, in this case some results like the higher smoothness of d_K^2 at the points of ∂K, expressed by Proposition 4.3, are lost.

For every $p \in K$ we consider the following three subsets of T_pM,

- T_pK, the vector subspace of tangent vectors to K at p,
- $N_pK = \{w \in T_pM \mid g_p(w, T_pK) = 0\}$, the vector subspace of normal vectors to K at p,
- $U_pK = \{w \in N_pK \mid g_p(w, w) = 1\}$, the subset of unit normal vectors to K at p,

then the bundles $NK = \{(p, v) \mid v \in N_pK\}$ and $UK = \{(p, v) \mid v \in U_pK\}$ inherit the structure of TM. Being K a C^r submanifold of M, the bundles NK and UK are respectively n–dimensional and $(n-1)$–dimensional C^{r-1} submanifolds of TM. Notice that in the special case $K = \{p\}$, we have that $NK = T_pM$ and $UK = \mathbb{S}^{n-1} \subset T_pM$.

We define the map $F : UK \times \mathbb{R}^+ \to M$ as the restriction of the exponential map of M to UK,

$$F(p, v, t) = \mathrm{Exp}(p, v, t) \qquad \forall (p, v) \in UK \text{ and } t \in \mathbb{R}^+.$$

Since UK is a C^{r-1} manifold and the exponential map of M is smooth, F and all its derivatives with respect to the variable t are of class C^{r-1}.

Remark 4.2. If a minimal geodesic, parametrized by arc length, starts at a point $p \in M$ and arrives at a point $q \in K$, its velocity vector v at q has to belong to U_qK, otherwise the condition of minimality is easily contradicted.
Since the geodesics, parametrized by arc length, ending on K are given by the family of maps $t \mapsto F(q, v, t)$ with $(q, v) \in UK$, the distance from K of a point p is given by the formula

$$d_K(p) = \inf\{t \in \mathbb{R}^+ \mid (q, v, t) \in F^{-1}(p)\}, \tag{4.1}$$

which obviously becomes $d_K(p) = \pi_{\mathbb{R}^+}\left(F^{-1}(p)\right)$ when the counterimage is a singleton (the map $\pi_{\mathbb{R}^+}$ is the projection on the second factor of the product $UK \times \mathbb{R}^+$). The study of the singularities of the squared distance function then reduces to the analysis of the (possibly set valued) map F^{-1}.
This problem, from the topological point of view, is naturally connected with the study of the singularities of continuous maps between Euclidean spaces. For instance, when K coincides with a single point of M the singular sets were shown

to be related to the classes of singularities considered by the *Theory of Catastrophes*, see [11].

Let us define the C^{r-1} map exp : $NK \to M$ by

$$\exp(p, v) = \mathrm{Exp}(p, v, 1) \qquad \forall (p, v) \in NK.$$

At the points $(p, 0) \in NK$ the map exp is differentiable and $d\exp(p, 0)$ is invertible between $T_{(p,0)}NK$ and T_pM, indeed $T_{(p,0)}NK$ can be identified with T_pM and under such identification $d\exp(p, 0)$ is the identity. Since, by hypothesis, the map exp is at least C^1, it follows that in a neighborhood of $(p, 0)$ in NK the differential of exp is invertible, hence the map exp is a C^{r-1} local diffeomorphism. Holding the relation $F(p, v, t) = \exp(p, vt)$, we conclude that for small $t > 0$, the map F is a local diffeomorphism.
Being K at least C^2, by a standard result in differential geometry, there exists an open *tubular neighborhood* Ω' of K in M formed by non intersecting, minimal geodesics starting normally from K. Hence, by the previous discussion and possibly choosing a smaller tubular neighborhood Ω of K, the map F^{-1} is well defined and C^{r-1} in $\Omega \setminus K$ (see for instance, [8]).
Then, the gradient ∇d_K^2 exists in Ω and we have, by relations (3.6) and (4.1),

$$\nabla d_K^2(p) = 2d_K(p)\frac{\partial F}{\partial t}(F^{-1}(p)).$$

Since $d_K = \pi_{\mathbb{R}^+}\left(F^{-1}(p)\right) \in C^{r-1}$ in Ω and the functions F, $\frac{\partial F}{\partial t}$ are of class C^{r-1}, it follows that ∇d_K^2 is C^{r-1} and d_K^2 is C^r in $\Omega \setminus K$. The same C^r regularity in $\Omega \setminus K$ follows immediately also for the distance function d_K.
Moreover, the function d_K^2 is C^r regular also on the set K, hence in the whole neighborhood Ω, as the square regularizes the jump of the gradient in the direction normal to K, see [7, 8]. The function d_K indeed does not share this property, it fails to be C^1 at the points of K.

We summarize these results in the following proposition.

Proposition 4.3. *If K is a regular submanifold of class C^r, with $r \geq 2$, then there exists an open subset Λ of $UK \times \mathbb{R}^+$ with the property that if $(q, v, t) \in \Lambda$ then also $(q, v, s) \in \Lambda$ for every $0 < s < t$, and an open neighborhood Ω of K in M, such that the map $F|_\Lambda : \Lambda \to \Omega \setminus K$ is a diffeomorphism.*
Moreover,

- *for every point in Ω there is an unique point of minimum distance in K (*unique projection property in Ω*),*
- *the distance function d_K is C^r in $\Omega \setminus K$,*
- *the squared distance function d_K^2 is C^r in Ω.*

Remark 4.4. It can be proved that $C^{1,1}$ is the minimal regularity of K to have the unique projection property in a neighborhood, in this case also the squared distance function turns out to be of class $C^{1,1}$.
See [15, 16] for a detailed discussion of the relation between the regularity of K and of d_K.

Remark 4.5. We underline two points in the previous discussion: the fact that around a point p the point of minimum distance is unique gives a representation of the distance function via the formula (4.1), then, since the differential of the map F is invertible, the regularity of d_K^2 follows.

Now we want to analyse what happens far from K, hence, by this remark, we need to study the subsets where these two conditions are satisfied.
A good reference for what follows is the book of Sakai [27].

Consider the geodesic curve $t \mapsto F(q, v, t)$ for $t \in [0, t_0]$ ($(q, v) \in UK$ is fixed), for small values of t_0 it minimizes the length functional between its end point and K but for large $t_0 > 0$ it could cease to be minimal between K and $p = F(q, v, t_0)$. Hence, there exists a value σ (possibly $+\infty$) such that this geodesic is minimal between q and $F(q, v, t)$ for every $t \leq \sigma$, but not on any larger interval. It is so defined a map $\sigma : UK \to \mathbb{R}^+ \cup \{+\infty\}$, giving the maximal interval $[0, \sigma(q, v)]$ such that the geodesic $F(q, v, t)$ is minimal between its end point and K.
If $\sigma(q, v) < +\infty$, we say that the point $F(q, v, \sigma(q, v))$ is the *cut point* of the geodesic $F(q, v, t)$ and we define the following set,

$$V_K = \left\{(q, v, t) \in UK \times \mathbb{R}^+ \,|\, t < \sigma(q, v)\right\}.$$

Notice that the set $F(V_K)$ clearly contains $\Omega \backslash K$, where Ω is the open neighborhood of Proposition 4.3.

Definition 4.6. The set of points $F(q, v, \sigma(q, v))$ for $(q, v) \in UK$ with $\sigma(q, v) < +\infty$ is called the *cut locus* of K, we denote it with $Cut(K)$.

The reasons why a geodesic ceases to be minimal are explained by the following proposition.

Proposition 4.7. *If for a geodesic $F(q, v, t)$ we have $\sigma(q, v) < +\infty$, at least one of the following two non exclusive conditions is satisfied:*

(1) *at the point $p = F(q, v, \sigma(q, v))$ there arrives another minimal geodesic from K,*
(2) *the differential $dF(q, v, \sigma(q, v))$ is not invertible.*

Conversely, if at least one of these conditions is satisfied the geodesic $F(q, v, t)$ cannot be minimal on an interval larger that $[0, \sigma(q, v)]$.

This result belongs to the general theory of geodesics, we refer to the books [18, 27] for the proof.
Notice that, by Proposition 3.7, if condition 1 above is satisfied then p belongs to $Sing$ and conversely, every point of $Sing$ stays in the cut locus of K.
Then we consider the following two subsets of $Cut(K)$,

- $Sing$, that is the points $p = F(q, v, \sigma(q, v))$ where more than one minimal geodesic from K arrives,
- J is the set of points $p = F(q, v, \sigma(q, v))$ such that the differential $dF(q, v, \sigma(q, v))$ is not invertible.

We call *J locus of optimal focal points*. Clearly $Sing \cup J = Cut(K)$.
Finally, in the next proposition we establish the connection between $Cut(K)$ and the distance function from K.

Proposition 4.8. *Let K be a regular submanifold of class C^r with $r \geq 2$, then*

(1) *$Cut(K) = \overline{Sing}$, that is, the cut locus of K is closed in M and $Sing$ is a dense subset.*
(2) *The set V_K is open in $UK \times \mathbb{R}^+$.*
(3) *The map $\sigma : UK \to \mathbb{R}^+$ is continuous.*
(4) *The map F is a C^{r-1} bijection between V_K and $M \setminus (K \cup Cut(K))$ with a C^{r-1} inverse.*
(5) *The cut locus $Cut(K)$ is equal to $F(\partial V_K)$ where the boundary is considered in the ambient space $UK \times \mathbb{R}^+$.*
(6) *The set $M \setminus Cut(K)$ can be continuously retracted on K, and, if $\sigma(q, v) < +\infty$ for every $(q, v) \in UK$, the set $M \setminus K$ can be continuously retracted on the cut locus $Cut(K)$.*
(7) *The open set $M \backslash Cut(K)$ has the unique projection property, moreover the squared distance function d_K^2 is of class C^r in it. The distance d_K is C^r in $M \backslash (Cut(K) \cup K)$.*

Proof. (1) First, we prove that if $p_i \in Sing$ and $p_i \to p \notin Sing$ then p has to be an optimal focal point. Suppose that $F(q_i, v_i, \sigma(q_i, v_i)) = F(r_i, w_i, \sigma(r_i, w_i)) = p_i$ with $(q_i, v_i) \neq (r_i, w_i)$ and $\sigma(q_i, v_i) = \sigma(r_i, w_i) = d_K(p_i)$, since

$$d(q_i, p) = \lim_{j\to\infty} d(q_i, p_j) \leq d(q_i, p_i) + \lim_{j\to\infty} d(p_i, p_j) = d_K(p_i) \leq d_K(p) + 1$$

definitely, the points q_i stay in a ball around p, hence we can extract a subsequence, which we call again (q_i, v_i), converging to a point $(q, v) \in UK$. By the semicontinuity af the length functional, it follows that $F(q, v, t)$ is the unique minimal geodesic from K to p. Applying the same reasoning to the points (r_i, w_i) and possibly passing again to a subsequence, we have that $(r_i, w_i) \to (q, v)$ too. Now if $dF(q, v, \sigma(q, v))$ were invertible the map F would be bijective in a neighborhood of $(q, v, d_K(p))$ but this contradicts the hypothesis $F(q_i, v_i, \sigma(q_i, v_i)) = F(r_i, w_i, \sigma(r_i, w_i))$, as $(q_i, v_i, d_K(p_i))$ and $(r_i, w_i, d_K(p_i))$ converge to $(q, v, d_K(p))$. This shows that $\overline{Sing} \subset Cut(K)$.
To prove the converse we have only to show that there are no optimal focal points outside $\overline{Sing}$.
Suppose that B is a small ball around an optimal focal point $p \in Cut(K)$ not containing any point of $\overline{Sing}$, by Proposition 3.6 the squared distance function d_p^2 is a C^1 map in B. Let $F(q, v, t)$ be the minimal geodesic from p to K, we construct the following map from B to M, see Proposition 3.6,

$$\Lambda_\varepsilon(r) = \mathrm{Exp}(r, -\nabla d_K^2(r), \varepsilon) \qquad \forall r \in B.$$

The map $\Lambda_\varepsilon : B \to M$ is a continuous map since the function d_K^2 is C^1 in B. This map is also injective because minimal geodesics cannot cross each other, hence it is an homeomorphism of B onto $\Lambda_\varepsilon(B)$. Now, if ε is small enough, the set $\Lambda_\varepsilon(B)$ is again a neighborhood of p. This shows that there must be a point $r_0 \in B$ such that $\Lambda_\varepsilon(r_0) = p$ for ε small enough. Then the minimal geodesic from r_0 to K contains p and, by the uniqueness of the minimal geodesics in B, it contains the minimal geodesic from p. This is in contradiction with the fact that the curve $F(q, v, t)$ stops to be minimal at p and gives the thesis.

(2) Let $(q, v, t) \in V_K$ then $dF(q, v, t)$ is invertible, hence this holds also in a neighborhood of (q, v, t), being F at least C^1. If $(q_i, v_i, t_i) \to (q, v, t)$ and there exists a different sequence of points (r_i, w_i, t_i) such that $F(q_i, v_i, t_i) = F(r_i, w_i, t_i)$, with the same reasoning of the previous point, we have that (r_i, w_i, t_i) also converge to (q, v, t). As before, since F is locally injective around (q, v, t), this is a contradiction.

(3) The previous point shows that σ is lower semicontinuous. On the other side, σ is upper semicontinuous since the limit curve of a sequence of minimal geodesics is again a minimal geodesic, by the semicontinuity of the length functional and by the continuity of the distance d_K.

(4) By the definition of V_K and Proposition 4.7, the function $F|_{V_K} : V_K \to M$ is injective, the differential dF is everywhere invertible in V_K and the image $F(V_K)$ is the open set $M \setminus (K \cup Cut(K))$. Hence, by the inverse function theorem, $F|_{V_K} : V_K \to M \setminus (K \cup Cut(K))$ is an invertible map with a C^{r-1} inverse.

(5) This statement is pretty obvious since ∂V_K coincides with the graph of the function σ in $UK \times \mathbb{R}^+$ which is $\{F(q, v, \sigma(q, v)) \,|\, (q, v) \in UK\} = Cut(K)$.

(6) The map $R : (M \setminus Cut(K)) \times [0,1] \to M \setminus Cut(K)$ given by

$$R(q,\alpha) = \begin{cases} q & \text{if } q \in K, \\ \mathrm{Exp}(\pi_{UK}F^{-1}(q), (1-\alpha)d_K(q)) & \text{if } q \notin Cut(K) \cup K \end{cases}$$

is the first retraction.
The second retraction $S : (M \setminus K) \times [0,1] \to M \setminus K$ is defined as

$$S(p,\alpha) = \begin{cases} p & \text{if } p \in Cut(K), \\ \mathrm{Exp}(\pi_{UK}F^{-1}(p), (1-\alpha)d_K(p) + \alpha\sigma(\pi_{UK}F^{-1}(p))) & \text{if } p \notin Cut(K) \cup K \end{cases}$$

where π_{UK} is the projection on the first factor of the product $UK \times \mathbb{R}^+$ and F^{-1} is the inverse of $F|_{V_K}$ discussed at point 4.

(7) The proof follows by the same arguments of Proposition 4.3 and the subsequent Remark 4.5.

□

We have seen in Proposition 3.9 that the set $Sing$ is always C^2–rectifiable. Now we partially improve this result in the case of higher regularity of K.

Theorem 4.9. *If K is of class C^r with $r \geq 3$ the set $Sing \setminus J$ is a C^r–rectifiable subset of M.*

To prove the theorem we need a preliminary lemma.

Lemma 4.10. *If there are infinitely many minimal geodesics from p to K, then p is an optimal focal point.*

Proof. If $F(q_i, v_i, \sigma(q_i, v_i)) = p$ for infinite distinct geodesics $F(q_i, v_i, t)$, then $\sigma(q_i, v_i) = d(q_i, p) = d_K(p)$ hence, by compactness, we may assume that $(q_i, v_i) \to (q, v)$ for some $(q, v) \in UK$. It follows (like in the proof of point 1 of Proposition 4.8) that $F(q, v, t)$ is a minimal geodesic for p and that $dF(q, v, t)$ is singular since F is not locally injective near (q, v, t). □

Proof of Theorem 4.9. Let p be a point in $Sing \setminus J$. We know that the number of minimal geodesics $F(q_i, v_i, t)$ from $p = F(q_i, v_i, \sigma(q_i, v_i))$ to K is finite by the previous lemma and greater than one, by the singularity at p. Moreover, the differential $dF(q_i, v_i, \sigma(q_i, v_i))$ is invertible for every i, then F is locally invertible in the neighborhood of every point $(q_i, v_i, \sigma(q_i, v_i)) \in UK \times \mathbb{R}^+$.
Let U_i be disjoint open neighborhoods of $(q_i, v_i, \sigma(q_i, v_i))$ such that F is a C^{r-1} diffeomorphism with its image on every U_i. We can also suppose that $F(U_i) = U$ where U is an open neighborhood of p in M. We define the functions $d_i : U \to \mathbb{R}^+$ given by

$$d_i(x) = \pi_{\mathbb{R}^+}\left(F^{-1}(x) \cap U_i\right)$$

where $\pi_{\mathbb{R}^+} : UK \times \mathbb{R}^+ \to \mathbb{R}^+$ denotes the projection on the second factor. The functions d_i are of class C^r, by the same reasoning of Proposition 4.3 and Remark 4.5.
The singular set $Sing \cap U$ is clearly contained in the union of the sets $S_{ij} = \{r \in U \mid d_i(x) = d_j(x)\}$ for $i \neq j$. We now prove that such sets are locally C^r hypersurfaces. By the implicit function theorem, it is sufficient to show that $\nabla d_i(x) - \nabla d_j(x) \neq 0$ at the points of S_{ij}. If $r = F(s_i, w_i, t) = F(s_j, w_j, t)$, for $(s_i, w_i, t) \in U_i$ and $(s_j, w_j, t) \in U_j$ then

$$\nabla d_i(x) = \nabla d_j(x) \quad \Longrightarrow \quad \frac{dF}{dt}(s_i, w_i, t) = \frac{dF}{dt}(s_j, w_j, t).$$

Hence, by the uniqueness of the geodesic from r with a certain initial velocity vector, we would get $(s_i, w_i) = (s_j, w_j)$, contradicting the hypothesis that U_i and U_j are disjoint. □

Finally we study the set of optimal focal points J.

Theorem 4.11. *If K is of class C^r with $r \geq 3$, then the set J is C^{r-2}–rectifiable.*

Since $Cut(K) = Sing \cup J$, this theorem implies the result we claimed at the beginning of this section.

Theorem 4.12. *If K is of class C^r with $r \geq 3$, the cut locus of K is C^{r-2}–rectifiable.*

The rectifiability of the cut locus has the following immediate consequence.

Corollary 4.13. *The Hausdorff dimension of the cut locus of K is at most $(n-1)$.*

This follows from a standard result of geometric measure theory:
the Hausdorff dimension of an $(n-1)$–rectifiable set is at most $(n-1)$.

To explain another consequence we need to introduce briefly the theory of functions with *bounded variation*, see [17, 28] for details. We say that a function $u : \mathbb{R}^n \to \mathbb{R}^m$ is a function with locally bounded variation $u \in BV_{\mathrm{loc}}$, if its distributional derivative Du is a Radon measure. Such notion can be easily extended to maps between manifolds using smooth local charts.
A standard result says that the derivative of a locally semiconcave function stays in BV_{loc}, in view of Proposition 3.4 this implies that the vector field ∇d_K belongs to BV_{loc} in the open set $M \setminus K$.

Now we define the subspace of BV_{loc} of functions (or vector fields, as before) with locally *special bounded variation* SBV_{loc} (see [3, 4, 5]).
The Radon measure representing the distributional derivative Du of a function $u : \mathbb{R}^n \to \mathbb{R}^m$ with locally bounded variation can be always uniquely separated in three mutually singular measures

$$Du = D^a u + Ju + Cu$$

where the first term is the part absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, Ju is a measure concentrated on a $(n-1)$–rectifiable set and Cu (called the *Cantor part*) is a measure which does not charge the subsets of Hausdorff dimension $(n-1)$.
The space SBV_{loc} is defined as the class of functions $u \in BV_{\mathrm{loc}}$ such that $Cu = 0$, that is, the Cantor part of the distributional derivative of u is zero.

Corollary 4.14. *If K is of class C^r with $r \geq 3$, the vector field ∇d_K belongs to the space $SBV_{\mathrm{loc}}(M \setminus K)$ of vector fields with locally special bounded variation.*

Proof. Being the cut locus rectifiable, hence of Hausdorff dimension $(n-1)$, the Cantor part of the distributional derivative of ∇d_K cannot be concentrated on it, so it must be concentrated in the open set $M \setminus (K \cup Cut(K))$. By point 7 of Proposition 4.8, the field ∇d_K belongs to C^{r-1} in $M \setminus (K \cup Cut(K))$ then, by the hypotheses, it is at least C^2, hence its distributional derivative coincides with the product of the classical derivative with the Lebesgue measure, this shows that $Cu(M \setminus (K \cup Cut(K)) = 0$. These two facts together prove that ∇d_K belongs to $SBV_{\mathrm{loc}}(M \setminus K)$. □

Finally we prove Theorem 4.11. To this aim we need to introduce the set $\widetilde{J}$ of the *first focal points*.
Let $F(q, v, t)$ be a geodesic from K with $(q, v) \in UK$ and $t \in \mathbb{R}^+$, considering the first value $t = c(q, v)$ such that $dF(q, v, t)$ is not invertible or setting $c(q, v) = +\infty$ if $dF(q, v, t)$ is invertible for every $t \in \mathbb{R}^+$, we define the map $c : UK \to \mathbb{R}^+ \cup \{+\infty\}$. If $c(q, v) < +\infty$ we say that $F(q, v, c(q, v))$ is the first focal point along the geodesic $F(q, v, t)$.
We consider the following set of points G in $UK \times \mathbb{R}^+$ where dF is not invertible

$$G = \big\{(q, v, c(q, v)) \in UK \times \mathbb{R}^+ \,|\, c(q, v) < +\infty\big\}$$

and we call $\widetilde{J} = F(G)$ *locus of the first focal points* of K. By Proposition 4.7, we have that $\widetilde{J} \supset J$, the set of optimal focal points.

Proof of Theorem 4.11. It is clearly sufficient to show that the set $\widetilde{J}$ is rectifiable.
At the points of the set G the rank of dF is at most $(n-1)$. We split G in two subsets,

$$G_1 = \{(q, v, c(q, v)) \in G \mid \text{ Rank } dF(q, v, c(q, v)) = n - 1\}$$
$$G_2 = \{(q, v, c(q, v)) \in G \mid \text{ Rank } dF(q, v, c(q, v)) < n - 1\}$$

which we will study separately.

The following version of Sard's Theorem can be found in the book of Federer [17, Theorem 3.4.3].

Lemma 4.15. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map of class C^l for some $l \geq 1$.*
For any $k \in \{0, 1, \dots, n-1\}$ set

$$A_k = \{x \in \mathbb{R}^n \mid \text{ Rank } dF(x) \leq k\}.$$

Then $\mathcal{H}^{k+\frac{n-k}{l}}(F(A_k)) = 0$.

Considering local charts of $UK \times \mathbb{R}^+$ and M and applying this lemma to our map F which is of class C^{r-1} we get that $\mathcal{H}^{n-2+2/(r-1)}(F(G_2)) = 0$, which implies $\mathcal{H}^{n-1}(F(G_2)) = 0$ as $r \geq 3$.
Now we want to show that the set G_1 is locally a C^{r-2} hypersurface in $UK \times \mathbb{R}^+$ using the implicit function theorem, this would prove that $F(G_1)$ is a C^{r-2}–rectifiable set in M and so $\widetilde{J}$.

Let $(q, v, c(q, v))$ be a point in G_1, by the lower semicontinuity of the rank, choosing a small neighborhood B of $(q, v, c(q, v))$ in $UK \times \mathbb{R}^+$ we can suppose that there are no points of G_2 in B.
The points of G_1 in B can be characterized as the zero set of the determinant $\det dF(q, v, t)$ which is of class C^{r-2}.
We claim that

$$\frac{\partial \det dF}{\partial t}(q, v, c(q, v)) \neq 0 \tag{4.2}$$

at the points of $G_1 \cap B$. By the implicit function theorem, this fact implies that $G_1 \cap B$ is a regular $(n-1)$–dimensional submanifold of class C^{r-2}, and we are done.

In the following we are going to assume some facts about the *covariant derivative* of M, the *Riemann curvature tensor*, the *shape operator* of K and the so called *Jacobi fields* which are standard tools in differential geometry, we refer to the book of Sakai [27] for definitions and proofs of our assertions.

We begin with a study of the tangent space to UK.
Let ∇ be the covariant derivative of M. Any vector in $T_{(q,v)}UK$ can be represented by the velocity vector $(w, u) = \nabla_s(q(s), v(s))|_{s=0}$ at $s = 0$ of a C^{r-1} curve $(q(s), v(s))$ in UK, with $(q(0), v(0)) = (q, v)$. Such a curve is given by a C^r curve $q(s)$ in K with a C^{r-1} unit vector field $v(s)$ defined along $q(s)$ and normal to K. It is then clear that $w = q'(0)$ belongs to the tangent space to K at q.
We set $u(s) = \nabla_s v(s)$ and $u(0) = u$. Suppose that $z(s)$ is an arbitrary vector field along $q(s)$ tangent to K with $z(0) = z$, then, by the orthogonality of $v(s)$ and $z(s)$, we have

$$0 = \frac{d}{ds}[g(z(s),\, v(s))] = g(z(s),\, u(s)) + g(\nabla_s z(s)\,, v(s))$$

and at the point $s = 0$ we get

$$0 = g(z,\, u) + g(\nabla_s z(s),\, v)|_{s=0}.$$

Introducing the shape operator $A_v : T_qK \to T_qK$ of K at q, relative to the unit vector v, we can rewrite this equation as

$$g(z,\, u) + g(A_v z,\, w) = 0$$

and, by the symmetry property of the shape operator,

$$g(z,\, u) + g(A_v w,\, z) = 0.$$

Hence, as z can be chosen arbitrarily in T_qK, we obtain that $u + A_v w \in N_qK$. Notice also that, since v is a unit vector, differentiating $g(v(s),\, v(s)) = 1$ we obtain $g(u,\, v) = 0$.
Resuming, the tangent space to UK at the point (q, v) is represented by the pairs of vectors $(w, u) \in T_qM \times T_qM$ such that

$$w \in T_qK, \qquad u + A_v w \in N_qK \quad \text{and} \quad g(u,\, v) = 0. \tag{4.3}$$

Consider now a vector $(w, u) \in T_{(q,v)}UK$, and the vector field

$$X(t) = \partial_{UK} F(q, v, t)(w, u)$$

along the normal geodesic $\gamma(t) = F(q, v, t)$ with unit velocity vector

$$\gamma'(t) = \partial_t F(q, v, t)\,,$$

where ∂_{UK} denotes the partial derivative with respect to the variable $(q, v) \in UK$. The field X is called Jacobi field along the geodesic $\gamma(t)$ and satisfies the following relations

$$\begin{gathered} X(0) = w, \qquad X'(0) = u, \\ X''(t) + R(X(t), \gamma'(t))\gamma'(t) = 0, \end{gathered} \tag{4.4}$$

where R is the Riemann curvature operator of M and we adopted the convention of denoting with T' the covariant derivative along the geodesic $\gamma(t)$ of any vector or tensor field T.
We take a basis $\{(w_i, u_i)\}$, for $i = 1, \dots, n-1$, of the tangent space $T_{(q,v)}UK$ and we construct an n–vector ω along γ as follows,

$$\omega(t) = X_1(t) \wedge \dots \wedge X_n(t)$$

where the fields $X_i(t) = \partial_{UK} F(q, v, t)(w_i, u_i)$ are Jacobi fields and $X_n(t) = \gamma'(t)$. Notice that the relation

$$X_n''(t) + R(X_n(t), \gamma'(t))\gamma'(t) = 0$$

is satisfied by the field X_n, since $\gamma'' = 0$. As $\{(w_i, u_i)\}$ is a basis of $T_{(q,v)}UK$, proving equation (4.2) is equivalent to show that $\omega'(c(q, v)) \neq 0$.
We argue by contradiction, by the Gauss Lemma (see [18, Chapter 3, Section E]) we have that $X_n(t)$ is orthogonal to $X_i(t)$ for every $i = 1, \dots, n-1$, so we can suppose that the nonzero vector (w_1, u_1) is the generator of the kernel of $\partial_{UK} F(q, v, c(q, v))$. Hence, $X_1(c(q, v)) = 0$ and $X_2(c(q, v)), \dots, X_n(c(q, v))$ generate a subspace of dimension $(n-1)$, by the assumption on the rank of $dF(q, v, c(q, v))$.
Computing $\omega'(c(q, v))$ we get

$$\omega'(c(q, v)) = X_1'(c(q, v)) \wedge X_2(c(q, v)) \wedge \dots \wedge X_n(c(q, v))$$

by linearity and the hypothesis that $X_1(c(q, v)) = 0$.
To conclude we need only to show that $X_1'(c(q, v))$ cannot belong to the $(n-1)$–dimensional subspace generated by $X_2(c(q, v)), \dots, X_n(c(q, v))$.
First we show that $X_1'(t)$ is orthogonal to $X_n(t)$ for every t. The derivative of the function $h(t) = g(X_1'(t),\, \gamma'(t))$ is zero by $\gamma'' = 0$ and the last equation in (4.4),

moreover the last condition of equation (4.3) shows that $h(0) = 0$, hence the function h is identically zero.
Consider now the function $f(t)$ given by

$$f(t) = g(X_1'(t),\, X_i(t)) - g(X_1(t),\, X_i'(t)).$$

We have $f(0) = g(u_1,\, w_i) - g(w_1,\, u_i)$ so, using the second relation in (4.3) and taking into account that $w_i \in T_qK$ if $i \leq n-1$, we obtain $f(0) = g(w_1,\, A_v w_i) - g(A_v w_1,\, w_i)$ which is zero since the shape operator is symmetric. Moreover,

$$\begin{aligned} f'(t) &= g(X_1''(t),\, X_i(t)) - g(X_1(t),\, X_i''(t)) \\ &= R(X_i(t), \gamma'(t), \gamma'(t), X_1(t)) - R(X_1(t), \gamma'(t), \gamma'(t), X_i(t)) = 0 \end{aligned}$$

by the properties of the curvature tensor.
Hence, the function f is identically zero and $f(c(q, v)) = 0$ gives

$$g(X_1'(c(q, v)),\, X_i(c(q, v))) = 0 \qquad \text{for } i = 2, \ldots, n-1,$$

so $X_1'(c(q, v))$ is orthogonal to every vector $X_i(c(q, v))$ and cannot belong to the subspace spanned by $\{X_i(c(q, v))\}$ for $i = 2, \ldots, n$. If $X'(c(q, v))$ would be zero, then by the differential relation (4.4), we would get $X(t) = X'(t) = 0$ for every t and in particular for $t = 0$, that is, $(w_1, u_1) = (0, 0)$ contradicting the initial hypothesis. □

5. A Special Case

In this section we study a special example which is interesting for the discussion of the next section. We assume that M is a two dimensional analytic, connected and compact surface and K is an one dimensional embedded analytic submanifold of M (a finite union of analytic curves). As before, our analysis also applies to closed sets K with analytic boundary.
We look for topological results on the structure of the cut locus of K generalizing some arguments introduced to study the special case $K = \{p\}$, see Myers [24, 25].
Our goal is to show that $Cut(K)$ is a finite graph and to connect its topological structure to the differential properties of the function d_K^2.
Clearly we have that UK, F and F^{-1}, when it exists, are analytic. Notice that the fiber of UK is $U_pK \cong \{-1, 1\}$.
The strong result given by analyticity is that the number of the optimal focal points is finite.

Lemma 5.1. *The function $c : UK \to \mathbb{R}^+ \cup \{+\infty\}$ defined in the previous section is analytic in the open set where it is not $+\infty$.*

Proof. With the same proof of Theorem 4.11, noticing that when $n = 2$ the set G coincides with G_1, we can show that the set of points $(q, v, t) \in UK \times \mathbb{R}^+$ where $\det dF(q, v, t) = 0$ is a finite union of analytic curves. Hence, as this set is the graph of the map c, we have the thesis. □

Proposition 5.2. *The set J of optimal focal points of K is finite.*

Proof. Being M compact every geodesic cannot be minimal between its end points if it is longer than the diameter of M, hence a minimal geodesic joining an optimal focal point to K has to be shorter than a fixed constant.
Consider an optimal focal point p and let $F(q, v, t)$ be a minimal geodesic from K to p which has a non invertible differential $dF(q, v, \sigma(q, v))$ (notice that in this situation we have $c(q, v) = \sigma(q, v)$), we claim that (q, v) is a critical point of the function c.

By the Gauss Lemma (see [18, Chapter 3, Section E]), the differential $dF(q,v,t)$ act on an element $(w,s)\in T_{(q,v,t)}UK\times\mathbb{R}^+$ as follows,

$$\begin{aligned} dF(q,v,t)(w,s) &= \partial_{UK}F(q,v,t)(w)+\partial_t F(q,v,t)(s) \\ &= X+sT \end{aligned} \tag{5.1}$$

where the two vectors $X,T\in T_{F(q,v,t)}M$ are mutually orthogonal and T is the unit tangent vector to the geodesic $F(q,v,t)$. Taking into account that UK is locally a curve, this shows that if $dF(q,v,t)$ is singular then $\partial_{UK}F(q,v,t)=0$.
Consider now the pull–back F^*g of the metric tensor g on $T_{(q,v,t)}UK\times\mathbb{R}^+$ via the map F. The set of points (q,v,t) where this form is not positive definite covers the graph of c. Computing this form using equation (5.1) we have,

$$\begin{aligned} (F^*g)_{(q,v,t)}((w,s),(w,s)) &= g_{F(q,v,t)}\left(dF(q,v,t)(w,s),\, dF(q,v,t)(w,s)\right) \\ &= s^2+g_{F(q,v,t)}(\partial_{UK}F(q,v,t)(w),\, \partial_{UK}F(q,v,t)(w)) \\ &= s^2+h(q,v,t)g_q(w,w) \end{aligned} \tag{5.2}$$

for a non negative function $h:UK\times\mathbb{R}^+\to\mathbb{R}$ and where $w\in T_{(q,v)}UK$ is considered as a vector in T_qM. Clearly, the set of points where the function $h:UK\times\mathbb{R}^+\to\mathbb{R}$ is equal to zero contains the graph of c by the previous discussion.
Suppose that $dc(q,v)\neq 0$, recalling that UK is a curve there exists a small neighborhood B of (q,v) in UK where the map c is invertible and c^{-1} is analytic in the open set $c(B)\subset\mathbb{R}^+$. Take a point $(r,z)\in B$ with $c(r,z)<c(q,v)$ and consider the curve $\gamma(s):[0,c(q,v)]\to M$ defined by

$$\begin{cases} \gamma(0)=r \\ \gamma(s)=F(r,z,s) & \text{for } s\in(0,c(r,z)], \\ \gamma(s)=F(c^{-1}(s),s) & \text{for } s\in(c(r,z),c(q,v)], \end{cases}$$

that is, after a piece of geodesic, we follow the locus of first focal points. This is a piecewise analytic curve in M starting from K and ending at the point p. The first piece of γ is a geodesic, hence its length is $c(r,z)$, the second piece follows the locus of first focal points. Using relation (5.2) we compute

$$\begin{aligned} |\dot\gamma(s)|^2 &= g_{\gamma(s)}(dF(c^{-1}(s),s)(w(s),1),\, dF(c^{-1}(s),s)(w(s),1)) \\ &= 1+h(c^{-1}(s),s)g(w(s),w(s)) \\ &= 1 \end{aligned}$$

for every $s\in(c(r,z),c(q,v)]$ and where $w(s)=\frac{d(c^{-1})}{ds}(s)$. In the last equality we used the fact that the point $(c^{-1}(s),s)$ belongs to the graph of c, where the function h is zero.
Then the length of the second piece of γ coincides with the variation in s, that is, $c(q,v)-c(r,z)$. Finally the total length of γ is $c(q,v)=d_K(p)$.
Such curve is C^1 since the tangent vectors of its two parts are equal at the point $F(r,z,c(r,z))$, but it is not a geodesic for $s\in(c(r,z),c(q,v)]$, otherwise (by uniqueness) γ should coincide with $F(r,z,s)$ for every $s\in(0,c(q,v)]$ and this is impossible by construction.
This fact implies that there must exist a shorter curve joining p with r and this is in contradiction with the assumption that $F(q,v,t)$ is minimal, so the claim is proved.
Arguing by contradiction, if the set of optimal focal points would be infinite then in a connected component C of $\{c(q,v)<+\infty\}\subset UK$ there would be infinite points $(q_i,v_i)\in C$ where dc is zero and $F(q_i,v_i,c(q_i,v_i))=p_i$ are distinct optimal focal points. By compactness and the initial argument on the length of a minimal geodesic from an optimal focal point, there must exists an accumulation point of

the set $\{(q_i, v_i)\}$ in $C \subset \{c(q, v) < L\}$, for a suitable constant L. Then, by the analyticity of c, this would imply that $dc(q, v)$ is identically zero and $c(q, v)$ is constant in the component C.
Defining a function $H : UK \to M$ by $H(q, v) = F(q, v, c(q, v))$ we have that

$$dH(q, v) = \partial_{UK} F(q, v, c(q, v)) + \partial_t F(q, v, c(q, v))dc(q, v) = 0,$$

as $\partial_{UK} F(q, v, c(q, v)) = 0$ and $dc(q, v) = 0$.
So the map $F(q, v, c(q, v))$ is constant in C. This implies that all the points $p_i = F(q_i, v_i, c(q_i, v_i))$ coincide contradicting the hypotheses. □

As the map $\sigma : UK \to M$ is continuous, the cut locus $Cut(K)$ is given by a finite family of curves of kind $s \mapsto F(q(s), v(s), \sigma(q(s), v(s)))$ where $(q(s), v(s))$ is a curve describing a connected component of UK. We say that $p \in Cut(K)$ is an *end point* if at the point p there arrives one and only one 1–cell of points of $Cut(K)$. We are going to prove that every end point is an optimal focal point.
First we exclude a very special case.

Lemma 5.3. *If at a point $p \in Cut(K)$ there arrive an infinite number of minimal geodesics then all these geodesics start from a unique connected component of K which is a geodesic circle around p, that is the set of points of M at a certain distance R from p. Moreover, p is an isolated point in $Cut(K)$, more precisely $Cut(K) \cap B_R(p) = \{p\}$. Conversely, if p is an isolated point in $Cut(K)$ then there is a connected component of K which is a geodesic circle around p.*

Remark 5.4. Notice that p is an optimal focal point by Lemma 4.10.
Since the connected components of K are finite, it follows that the isolated point of $Cut(K)$ are finite.

Proof. If $F(q_i, v_i, t)$ is the infinite family of minimal geodesics $F(q_i, v_i, t)$ of length R ending at p, then all the distinct points q_i belong to the geodesic circle of center p and radius R in M. The set of points $\{(q_i, v_i)\} \in UK$ clearly has an accumulation point, hence, by the analyticity of UK, the function $H(q, v) = F(q, v, R)$ is constantly equal to p in the connected component of UK containing such accumulation point. Again by the analyticity of the connected components of UK and of the curves constituting K, we can conclude that the whole circle has to be a connected component of K. Hence, from every point of this circle there is a minimal geodesic ending at p and there cannot be other points of K inside the circle, otherwise their distance from p would be less than the radius R.
Suppose now that p is isolated in $Cut(K)$ and consider the open connected component Γ of $M \setminus K$ which contains p. The boundary of Γ is a subset K' of K and every minimal geodesic starting from K' with an initial velocity vector pointing toward Γ, must necessarily cease to be minimal at p, as $Cut(K) \cap \Gamma = \{p\}$. This last assertion follows from the fact that Γ, by point 6 of Proposition 4.8, can be continuously retracted on $Cut(K) \cap \Gamma$, hence $Cut(K) \cap \Gamma$ is connected and then it coincides with $\{p\}$. This shows that there are infinite minimal geodesics from K to p and we can conclude as in the first part of the lemma. □

Suppose now that $p \in Cut(K) \setminus J$, so the number $n > 1$ of minimal geodesics $F(q_i, v_i, t)$ ending at p is finite. Consider a small ball B around p in M, then these n minimal geodesics cut the ball B in n sectors that we call S_i. Any minimal geodesic starting in a sufficiently small neighborhood of $(q_i, v_i) \in UK$ has its cut point in the ball B by continuity of the function σ, moreover this geodesic cannot cross one of the geodesics $F(q_i, v_i, t)$ before reaching its cut point, otherwise this latter ceases to be minimal. Hence, considering the continuous curve of the cut points of the geodesics starting at the points of UK locally on the right side of (q_i, v_i)

(remember that UK is one–dimensional), we have that it is all contained in one of the sectors S_i, more precisely, by continuity, in one of the two sectors adjacent to the geodesic $F(q_i, v_i, t)$. This curves gives a 1–cell of $Cut(K)$ approaching p.
With the same argument, considering the points locally on the left side of (q_i, v_i) we obtain another 1–cell, in the other sector.
Thus, we can conclude that the number of 1–cells of $Cut(K)$ arriving at a point p is at least the number of the sectors S_i, hence at least the number of minimal geodesics from p to K.
This implies that every end point of $Cut(K)$ where there arrives one and only one 1–cell, has a unique minimal geodesic to K so it has to be an optimal focal point.

Putting together these facts and Lemma 5.3, by Proposition 5.2 the end points are finite.
Following Myers [25], this result implies that the cut locus of K is a linear graph and locally a tree, moreover the points where the order of the graph is greater than two are finite.

Now we introduce the map $\#\mathcal{G}(p)$ from M to $\mathbb{N} \cup \{\infty\}$ *counting* the number of minimal geodesics from K to a point p.

Proposition 5.5. *An arc in $Cut(K)$ containing no points of J and no interior points p with $\#\mathcal{G}(p) > 2$ is a regular analytic arc.*

Proof. Let γ be such an arc in $Cut(K)$. Consider a point $p_0 \in \gamma$ with $F(q_1, v_1, \sigma(q_1, v_1)) = F(q_2, v_2, \sigma(q_2, v_2)) = p_0$, by the fact that p_0 is not an optimal focal point, applying the implicit function theorem, there is an open neighborhood B of p in $M \setminus K$ without optimal focal points and there exist analytic functions $z_1, z_2 : B \to UK$, $t_1, t_2 : B \to \mathbb{R}^+$ such that, $F(z_1(p), t_1(p)) = F(z_2(p), t_2(p)) = p$ for every $p \in B$ and $z_1(B) \cap z_2(B) = \emptyset$. If B is small enough, for every point p of B we have $\#\mathcal{G}(p) \leq 2$, then $t_1(p) = t_2(p) = d_K(p)$ if and only if $p \in Cut(K) \cap B = \gamma \cap B$.
The rest of the proof proceed as in Theorem 4.9. □

Our last goal is to show that the *order* of a point $p \in Cut(K)$, as a graph, is equal to $\#\mathcal{G}(p)$. The order of a point p of $Cut(K)$ is defined as the number of distinct 1–cells of $Cut(K)$ arriving at p. We have already seen before that that the order of p is always greater than the value $\#\mathcal{G}(p)$.
We now prove the opposite inequality.

Notice that the optimal geodesics cannot cross $Cut(K)$, otherwise they cease to be minimal since they would intersect another minimal geodesic. We can take a small ball B around a point $p \in Cut(K)$ so that the 1–cells divide it in n sectors. Let S be one of these sectors and consider a sequence of points $p_i \notin Cut(K)$ all contained in S and converging to p such that $F(q_i, v_i, t)$ are the minimal geodesics relative to p_i. By compactness, we can suppose that the points $(q_i, v_i) \in UK$ converge to a point (q, v), hence the minimal geodesics $F(q_i, v_i, t)$ converge to a minimal geodesic $F(q, v, t)$ from K to p. Being the points p_i contained in S the final part $F(q_i, v_i, t) \cap B$ of the respective minimal geodesics have to be contained in the sector S and so also the final part $F(q, v, t) \cap B$ of the minimal geodesic for p.
Taking into account the fact that such minimal geodesic cannot intersect the cut locus, we conclude that there is at least a minimal geodesic for every sector S. Being the number of the sectors equal to the order of $Cut(K)$ as a graph, we proved the opposite inequality we claimed before.
Hence, there are exactly $\#\mathcal{G}(p)$ 1–cells of the cut locus arriving at every point $p \in Cut(K)$.

We summarize all the discussion of this section in the following theorem.

Theorem 5.6. *The set $Cut(K)$ is a disjoint finite union of isolated points and linear graphs, each one locally a tree.*

The order of every point $p \in Cut(K)$ equals the function $\#\mathcal{G}(p)$, counting the number of minimal geodesic from p to K. In particular, the set of points of $Cut(K)$ with only one or more than two minimal geodesics is finite.
The set of optimal focal points in $Cut(K)$ is finite.
All the isolated points and end points of $Cut(K)$ are optimal focal points.
Considering as vertices of the graph the optimal focal points and the points of order greater than two, the arcs connecting such vertices are regular analytic arcs.

Remark 5.7. The analysis of this section also applies with small modifications (for instance, we need to allow that the circles in Lemma 5.3 could be points) to the case when K is a finite set of points, considering the set $\widetilde{K}$ consisting of a family of disjoint circles centered at the points of K with a radius R small enough.

6. SINGULARITIES OF SOLUTIONS OF HAMILTON–JACOBI EQUATIONS

An important particular case of the discussion of previous sections is given by $K = \{p\}$, that is, the distance is computed from a single fixed point $p \in M$. The cut locus of a point p in M arises naturally in various geometric problems, its definition is due to Poincaré [26] and its properties were studied by many authors, see for instance [10, 11, 20, 21, 24, 25, 29, 30]. A general introduction can be found in the books of Berger [9] and of Gallot, Hulin, Lafontaine [18]. The main reason which makes the cut locus of a point so important, is that the exponential map from the tangent space to the manifold M at the point p, gives a coordinate chart of all $M \setminus Cut\,(p)$.
We recall here some results in this special case.

Let M be a C^∞ Riemannian manifold of dimension n, then the set $U_p \equiv U\{p\}$ turns out to be C^∞ diffeomorphic to $\mathbb{S}^{n-1}$, being described by

$$U_p = \{(p, v) \in T_pM \,|\, g_p(v, v) = 1\}\,.$$

The squared distance function d_p^2 is a semiconcave function on M, hence its gradient is an SBV vector field. Moreover, d_p^2 is C^∞ in $M \setminus \overline{Sing}$, which contains an open neighborhood of p. The set $Sing \setminus J$ is locally a finite union of smooth hypersurfaces and the cut locus of p is C^∞ rectifiable in M.

Many of the ideas employed in the previous sections originated by the study of the cut locus of a point. This connection between the cut locus and the distance function could be extended to analyse also the set of singularities of viscosity solutions of more general Hamilton–Jacobi equations

$$\begin{cases} H(x, du(x), u(x)) = 0 & \text{in } \Omega \subset M, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

We discuss, for instance, a possible problem suggested by the theory of the cut locus of a point.
Suppose that $A(x)$ is an analytic map from the closure of a bounded open set $\Omega \subset \mathbb{R}^2$, homeomorphic to a ball and with an analytic boundary, to the space of positively defined 2×2–matrices.
We consider the following problem,

$$\begin{cases} \langle A(x)\nabla u(x), \nabla u(x)\rangle = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Using arguments similar to those of Section 5, it is possible to prove that the closure of the singular set of the viscosity solution is a finite graph.
If we look for the same result in the C^∞ case, a counterexample can be found. It is possible to endow the two dimensional sphere with a C^∞ metric tensor g such

that the cut locus of a certain point p is very wild, that is, it is not triangulable, hence it is not a finite graph (see [20]). Cutting away from the sphere $\mathbb{S}^2$ a small geodesic disc D around p whose intersection with the cut locus of p is empty, and mapping stereographically from p the set $\mathbb{S} \setminus D$ on $\mathbb{R}^2$, we have that the closure of the singular set of the viscosity solution of Problem (6.2) in a ball of $\mathbb{R}^2$, where A is given by the push–forward of the metric g via the stereographic projection, coincides with a homeomorphic image of the cut locus of p, hence it is not a finite graph.

Being interested in such kind of topological results, one possibility is to change point of view and to ask that the desired properties hold not in every case, but in the *generic* one. Indeed, another results on the cut locus of a point says that for a *generic* C^∞ metric (in, the cut locus of every point of a surface is triangulable and has no points of order higher than three (see [29]).

These results suggest the following conjecture.

Conjecture 6.1. For a dense subset of the C^∞ functions $A(x)$ from the closed unit ball B of $\mathbb{R}^2$ to the space of positive definite 2×2–matrices, the closure of the singular set of the viscosity solution of problem

$$\begin{cases} \langle A(x)\nabla u(x), \nabla u(x)\rangle = 1 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

is a finite graph.

More in general the same question can be asked about Problem (6.1), that is, for a *generic* function H in some class, the singular set of the solution is well behaved? Another possibility again is to keep the equation fixed and study if such property holds not for every domain Ω but for a generic one.

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(Carlo Mantegazza) SCUOLA NORMALE SUPERIORE PISA, ITALY, 56126
E-mail address, C. Mantegazza: mantegaz@sns.it

(Andrea Mennucci) SCUOLA NORMALE SUPERIORE PISA, ITALY, 56126
E-mail address, A. C. Mennucci: mennucci@sns.it