A Korn-type inequality in SBD for functions with small jump sets

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Abstract

We present a Korn-type inequality in a planar setting for special functions of bounded deformation. We prove that for each function in SBD^2 with sufficiently small jump set the distance of the function and its derivative from an infinitesimal rigid motion can be controlled in terms of the linearized elastic strain outside of a small exceptional set of finite perimeter. Particularly the result shows that each function in SBD^2 has bounded variation away from an arbitrarily small part of the domain.

Keywords. Functions of bounded deformation, Korn's inequality, Korn-Poincaré inequality, brittle materials, variational fracture.
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Contents

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1 Introduction

The space $BD(\Omega,\mathbb{R}^d)$ of functions of bounded deformation, which consists of all functions $u \in L^1(\Omega,\mathbb{R}^d)$ whose symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d\times d}_{\rm sym}$-valued Radon measure, has been introduced for the investigation of geometrically linear problems in plasticity theory and fracture mechanics (see [3, 5]). Variational damage or fracture problems are widely formulated in the subspace $SBD^2(\Omega,\mathbb{R}^d)$ (for the definition and properties of this space we refer to Section 2.1 below). In the spirit of the seminal work [23] the modeling essentially concentrates on the competition between elastic bulk contributions $\|e(u)\|_{L^2(\Omega)}$ given in terms of the linear elastic strain $e(u) := \frac{1}{2}(\nabla u^T + \nabla u)$ and surface terms that assign energy contributions on the crack paths comparable to the size of the crack $\mathcal{H}^{d-1}(J_u \cap \Omega)$, where J_u denotes the 'jump set' of u (see e.g. [6, 8, 9, 22, 29, 38]).

A major additional difficulty of these problems compared to models in SBV (see [4] for the definition and basic properties of the space of special functions of bounded variation) is the lack of control on the skew symmetric part of the distributional derivative $Du^T - Du$. In fact, it is a natural and important question to analyze in which circumstances the displacement field u or the absolutely continuous part of its derivative ∇u can be controlled by $\|e(u)\|_{L^2(\Omega)}$ and $\mathcal{H}^{d-1}(J_u)$. Apart from establishing compactness results, such properties may contribute to gain a profound understanding of the relation between SBD and SBV functions which is highly desirable since in contrast to (S)BV fine properties in BD appear not to be well understood by now. (We refer to the recent paper [14] for a thorough discussion and some results in that direction.)

The key estimate providing a relation between the symmetric and the full part of the gradient is know as Korn's inequality. In its basic version it states that for a bounded connected Lipschitz set Ω and $p \in (1,\infty)$ there is a constant $C(\Omega,p)$ depending only on p and the domain $\Omega \subset \mathbb{R}^d$ such that for all $u \in W^{1,p}(\Omega,\mathbb{R}^d)$ there is some $A \in \mathbb{R}^{d\times d}_{\rm skew}$ with

$$\|\nabla u - A\|_{L^p(\Omega)} \le C(\Omega,p)\|e(u)\|_{L^p(\Omega)}. \tag{1}$$

(See e.g. [36] for a proof and [12, 27, 28, 34] for generalizations of this result into various directions.) It turns out that the statement is false in $W^{1,1}$, i.e. one can construct functions with $e(u) \in L^1(\Omega)$, but $\nabla u \notin L^1(\Omega)$ (cf. [13, 31, 37]). On the one hand, these observations particularly show that BD is not contained in BV. On the other hand, it raises the natural question if in the space SBD^2 an estimate similar to (1) can be established due to the higher integrability for the elastic strain $e(u)$.

However, simple examples e.g. in [3] or the piecewise rigidity result proved in [11] show that (1) cannot hold for general functions in SBD^2 since the behavior of small pieces being almost or completely detached from the bulk part of the specimen might not be controlled. In the recently appeared contributions [10, 24] it has been proved that for displacement fields having small jump sets with respect to the size of the domain the distance of the function from an infinitesimal rigid motion can be estimated in terms of the linearized elastic energy outside of a small exceptional set E. However, these Korn-Poincaré-type estimates being essentially of the form

$$\|u - (A\ \cdot + c)\|_{L^2(\Omega\setminus E)} \le C(\Omega)\|e(u)\|_{L^2(\Omega)} \tag{2}$$

for $u \in SBD^2(\Omega)$ and corresponding $A \in \mathbb{R}^{d\times d}_{\rm skew}$, $c \in \mathbb{R}^d$, are significantly easier as in contrast to (1) no derivative is involved. The goal of the present article is to provide a generalization of (2) to an inequality of Korn's-type in a planar setting where one additionally controls ∇u away from a small exceptional set of finite perimeter. Our main result is the following.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a connected bounded set with Lipschitz boundary and let $p \in [1,2), q \in [1,\infty)$. Then there is a constant $C = C(\Omega,p,q)$ such that for all $u \in SBD^2(\Omega,\mathbb{R}^2)$ there is a set of finite perimeter $E \subset \Omega$ with*

$$\mathcal{H}^1(\partial E) \le C\mathcal{H}^1(J_u), \qquad |E| \le C(\mathcal{H}^1(J_u))^2 \tag{3}$$

and $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c \in \mathbb{R}^2$ such that

$$\begin{aligned} (i)\quad & \|u - (A\ \cdot + c)\|_{L^q(\Omega\setminus E)} \le C\|e(u)\|_{L^2(\Omega)}, \\ (ii)\quad & \|\nabla u - A\|_{L^p(\Omega\setminus E)} \le C\|e(u)\|_{L^2(\Omega)}, \end{aligned} \tag{4}$$

where $e(u)$ denotes the part of the strain $Eu = \frac{1}{2}(Du^T + Du)$ which is absolutely continuous with respect to $\mathcal{L}^2$.

Let us first mention that we establish the result only in two dimensions as we employ a modification technique for special functions of bounded deformation (see [24]) which was only derived in a planar setting due to technical difficulties concerning the topological structure of crack geometries in higher dimensions. Moreover, it is a natural and interesting question if the result also holds for the critical exponent $p = 2$.

Note that we can control the length of the boundary $\mathcal{H}^1(\partial E)$ of the exceptional set E which is associated to the parts of Ω being detached from the bulk part of Ω by J_u. Consequently, the result is adapted for the usage of compactness theorems for SBV and SBD functions (see [4, 5, 16]).

Although the main goal of the work at hand is the derivation of the estimate for the derivative in (4)(ii), we also provide a generalization for the integrability exponent q. In [24] the exponent was restricted to $q = 2$ due to the application of a BD Korn-Poincaré inequality and in [10] the arguments were based on slicing techniques similar to those used in the proof of Sobolev embeddings and led to an exponent $q = \frac{2d}{d-1}$. In the present context we obtain the estimate for $q < \infty = 2^*$ as in the usual Sobolev setting.

As an application we discuss that Theorem 1.1 together with an approximation result shows that SBD^2 functions have bounded variation outside an arbitrarily small exceptional set of finite perimeter (see Theorem 5.1 below). Hereby we give another contribution to the relation between SBV and SBD functions which appears to go in a slightly different direction than the results presented in [14]. We note that this statement does not immediately follow from the main theorem since a bound on ∇u does not automatically ensure that u has bounded variation due to the fact that an equivalent of Alberti's rank one property in BV (see [2]) is not known in BD.

Similarly as the previously mentioned results [10, 24] or the SBV Poincaré inequality (cf. [19]), Theorem 1.1 establishes an estimate only for functions whose jump set is small with respect to the size of the domain. Indeed, for larger jump sets the body may be separated into different parts of comparable size (cf. [11, 25, 27] for related problems). In the general case we expect a 'piecewise Korn inequality' to hold, i.e. the body may be broken into different sets and on each connected component the distance of the displacement field from a certain infinitesimal rigid motion can be controlled. We defer the analysis of this problem, for which Theorem 1.1 is a key ingredient, to a subsequent work [26].

The paper is organized as follows. In Section 2 we first recall the definition and basic properties of functions of bounded variation and deformation (Section 2.1). Then in Section 2.2 we introduce the notion of *John domains* being a class of sets with possibly highly irregular boundary (see e.g. [30, 33]). It is convenient to formulate Korn's and Poincaré's inequality for these sets since there are good criteria to obtain uniform control over the involved constants independently of the particular shape of the domain (cf. [1]). Finally, in Section 2.3 we present the modification technique proved in [24] which shows that after a small alteration of the displacement field and the jump set the jump heights of an SBD function can be controlled solely by $\|e(u)\|_{L^2(\Omega)}$ and $\mathcal{H}^1(J_u)$.

The rest of the paper then contains the proof of Theorem 1.1. We first establish a local estimate on a square and after a subsequent analysis of the problem near the boundary of the Lipschitz set the main theorem follows by a standard covering argument.

In Section 3 we concern ourselves with the local estimate and first see that by an approximation argument (cf. [9, 24]) it suffices to consider SBD functions with regular jump set. The main strategy is then to modify a function with the techniques presented in Section 2.3. Consequently, using a Korn-Poincaré inequality BD (see [32, 40]) we find good approximations of the displacement field by infinitesimal rigid motions in the neighborhoods of the jumps set. Then drawing ideas from [27] we can iteratively modify the configuration on various mesoscopic length scales to find a Sobolev function on the square which coincides with the original displacement field outside of a small exceptional set. Finally, the local estimate follows by application of the standard inequality (1).

We remark that in [27] it was not possible to gain control over the full part of the gradient as the approximating rigid motions had to be adapted after each iteration step leading to an continual increase of the involved constant. In the present context, however, the affine mappings are found a priori and are fixed during the modification procedure whereby a bound for ∇u can be established using Hölder's and a scaled Young's inequality in the case $p < 2$. Although the argument does not work in the critical case $p = 2$ and we do not have a result in that direction, the proof at least shows that, if the statement is wrong, a counterexample has to include an extremely complex crack geometry with crack pattern accumulating on an unbounded number of different mesoscopic scales.

Section 4 contains the main estimate at the boundary. We consider a Whitney covering of the domain and apply the result obtained in Section 3 on every square where the jump set J_u is small. Hereby we can again construct a Sobolev function outside a small exceptional set E. For the application of (1) now an additional difficulty occurs as we have to control the shape of domain. In this context we show that choosing E appropriately we find that the complement is a John domain for a universal *John constant* and therefore we can derive a uniform estimate of the form (1).

In Section 5 we then give the main proof and discuss an application to the relation of SBV and SBD functions. The standard examples for (S)BD functions not having bounded variation are given by configurations where small balls are cut out from the bulk part with an appropriate choice of the functions on these specific sets (see e.g. [3, 14]). We prove that each SBD function has bounded variation away from an arbitrarily small part of the body essentially showing that the mentioned construction provides the only way to obtain functions not lying in SBV.

2 Preliminaries

In this preparatory section we first recall the definition and basic properties of functions of bounded variation and state Korn's and Poincaré's inequality for *John domains*. Afterwards, we recall a result obtained in [24] providing modifications of SBD functions for which the jump heights can be controlled in terms of the linear elastic strain.

2.1 Special functions of bounded variation

In this section we collect the definitions of SBV and SBD functions. Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Recall that the space $SBV(\Omega, \mathbb{R}^d)$, abbreviated as $SBV(\Omega)$ hereafter, of *special functions of bounded variation* consists of functions $u \in L^1(\Omega, \mathbb{R}^d)$ whose distributional derivative Du is a finite Radon measure, which splits into an absolutely continuous part with density ∇u with respect to Lebesgue measure and a singular part $D^j u$. The Cantor part $D^c u$ of

$D^j u$ vanishes and thus we have

$$D^j u = [u] \otimes \xi_u \mathcal{H}^{d-1} \lfloor J_u,$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, J_u (the 'crack path') is an $\mathcal{H}^{d-1}$-rectifiable set in Ω, ξ_u is a normal of J_u and $[u] = u^+ - u^-$ (the 'crack opening') with $u^\pm$ being the one-sided limits of u at J_u. If in addition $\nabla u \in L^p(\Omega)$ for $1 < p < \infty$ and $\mathcal{H}^{d-1}(J_u) < \infty$, we write $u \in SBV^p(\Omega)$. Moreover, $SBV_{\mathrm{loc}}(\Omega)$ denotes the space of functions which belong to $SBV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Furthermore, we define the space $GSBV(\Omega)$ of *generalized special functions of bounded variation* consisting of all $\mathcal{L}^d$-measurable functions $u : \Omega \to \mathbb{R}^d$ such that for every $\phi \in C^1(\mathbb{R}^d)$ with the support of $\nabla\phi$ compact, the composition $\phi \circ u$ belongs to $SBV_{\mathrm{loc}}(\Omega)$ (see [18]). Likewise, we say $u \in GSBV^p(\Omega)$ for $u \in GSBV(\Omega)$ if $\nabla u \in L^p(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < \infty$. See [4] for the basic properties of theses function spaces.

We now state a version of Ambrosio's compactness theorem in GSBV adapted for our purposes (see e.g. [4, 17]):

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary and let $1 < p < \infty$. Let $(u_k)_k$ be a sequence in $GSBV^p(\Omega)$ such that*

$$\|\nabla u_k\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_k}) + \|u_k\|_{L^1(\Omega)} \le C$$

for some constant C not depending on k. Then there is a subsequence (not relabeled) and a function $u \in GSBV^p(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. If in addition $\|u_k\|_\infty \le C$ for all $k \in \mathbb{N}$, we find $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$.

An important subset of SBV is given by the indicator functions χ_W, where $W \subset \Omega$ is measurable with $\mathcal{H}^{d-1}(\partial W) < \infty$. Sets of this form are called *sets of finite perimeter* (cf. [4]). As a direct consequence of Theorem 2.1 we get the following compactness result.

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Let $(W_k)_k \subset \Omega$ be a sequence of measurable sets with $\mathcal{H}^{d-1}(\partial W_k) \le C$ for some constant C independent of k. Then there is a subsequence (not relabeled) and a measurable set W such that $\chi_{W_k} \to \chi_W$ in measure for $k \to \infty$.*

We say that a function $u \in L^1(\Omega, \mathbb{R}^d)$ is in $BD(\Omega, \mathbb{R}^d)$ if the symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d\times d}_{\mathrm{sym}}$-valued Radon measure. Likewise, we say u is a *special function of bounded deformation* if Eu has vanishing Cantor part $E^c u$. Then Eu can be decomposed as

$$Eu = e(u)\mathcal{L}^d + E^j u = e(u)\mathcal{L}^d + [u] \odot \xi_u \mathcal{H}^{d-1}|_{J_u}, \tag{5}$$

where $e(u)$ is the absolutely continuous part of Eu with respect to the Lebesgue measure $\mathcal{L}^d$, $[u]$, ξ_u, J_u as before and $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. If in addition $e(u) \in L^p(\Omega)$ for $1 < p < \infty$ and $\mathcal{H}^{d-1}(J_u) < \infty$, we write $u \in SBD^p(\Omega)$. For basic properties of this function space we refer to [3, 5].

We recall a Korn-Poincaré inequality in BD (see [32, 40]).

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^d$ bounded, connected with Lipschitz boundary and let $P : L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d)$ be a linear projection onto the space of infinitesimal rigid motions. Then there is a constant $C > 0$, which is invariant under rescaling of the domain, such that for all $u \in BD(\Omega, \mathbb{R}^d)$*

$$\|u - Pu\|_{L^{\frac{d}{d-1}}(\Omega)} \leq C|Eu|(\Omega),$$

where $|Eu|(\Omega)$ denotes the total variation of Eu.

We now present a density result in $SBD^2(\Omega)$ (see [24]) which is a variant of Chambolle's original result (see [9]).

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. For every $u \in SBD^2(\Omega)$ and for every $\Omega' \subset\subset \Omega$ with Lipschitz boundary there is a sequence of paraxial rectangles $R^n_1, \ldots, R^n_{m_n}$ with $|\bigcup_{j=1}^{m_n} R^n_j| \leq \frac{1}{n}$ and functions $(u_n)_n$ such that $u_n \in H^1(\Omega' \setminus \bigcup_{j=1}^{m_n} R^n_j)$ and $\|u - u_n\|_{L^1(\Omega' \setminus \bigcup_{j=1}^{m_n} R^n_j)} \leq \frac{1}{n}$ for $n \in \mathbb{N}$ as well as*

$$(i)\ \|e(u) - e(u_n)\|_{L^2(\Omega' \setminus \bigcup_{j=1}^{m_n} R^n_j)} \leq \tfrac{1}{n}, \quad (ii)\ \sum\nolimits_{j=1}^{m_n} \mathrm{diam}(\partial R^n_j) \leq \mathcal{H}^1(J_u) + \tfrac{1}{n}, \quad (6)$$

where $\mathrm{diam}(\partial R^n_j)$ denotes the diameter of R^n_j. If in addition $u \in L^\infty(\Omega)$, then $\|u_n\|_\infty \leq \|u\|_\infty$ for all $n \in \mathbb{N}$.

2.2 Poincaré's and Korn's inequality

A key idea in our analysis will be the replacement of displacement fields in SBD by suitable Sobolev functions and then the application of well know Poincaré and Korn inequalities. As the estimates will be employed on different Lipschitz sets, we need to provide uniform bounds for the constants involved in the inequalities. To this end, we introduce the notion of *John domains*.

Definition 2.5 Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set and let $x_0 \in \Omega$. We say Ω is a *c-John domain* with respect to the *John center* x_0 and with the constant c if for all $x \in \Omega$ there exists a rectifiable curve $\gamma : (0, l_\gamma) \to \Omega$, parametrized by arclength, such that $\gamma(0) = x$, $\gamma(l_\gamma) = x_0$ and $t \leq c\,\mathrm{dist}(\gamma(t), \partial\Omega)$ for all $t \in [0, l_\gamma]$.

Domains of this form were introduced by John in [30] to study problems in elasticity theory and the term was first used by Martio and Sarvas in [33]. Roughly speaking, a domain is a John domain if it is possible to connect two arbitrary points without getting too close to the boundary of the set. This class is much larger than Lipschitz domains and contains sets which may possess fractal boundaries or internal cusps (external cusps are excluded), e.g. the interior of Koch's *snow flake* is a John domain.

Although in the present work only Lipschitz sets occur, it is convenient to consider this more general notion as the constants in Poincaré's and Korn's inequalities only depend on the John constant. More precisely, we have the following statement (see e.g. [1, 7, 20]).

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^d$ be a c-John domain. Let $p \in (1,\infty)$ and $q \in (1,d)$. Then there is a constant $C = C(c,p,q) > 0$ such that for all $u \in W^{1,p}(\Omega)$ there is some $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ such that*

$$\|\nabla u - A\|_{L^p(\Omega)} \le C\|e(u)\|_{L^p(\Omega)}.$$

Moreover, for all $u \in W^{1,q}(\Omega)$ there is some $c \in \mathbb{R}^d$ such that

$$\|u - c\|_{L^{q^*}(\Omega)} \le C\|\nabla u\|_{L^q(\Omega)},$$

where $q^* = \frac{dq}{d-q}$. The constant is invariant under rescaling of the domain.

2.3 A modification with controllable jump heights

A main strategy of our proof will be the application of Theorem 2.3 in certain regions of the domain. It first appears that this inequality is not adapted for the estimates in (4) as in $|Eu|(\Omega)$ not only the elastic but also the surface energy depending on the jump height is involved. However, in [24] we have shown that one can indeed find bounds on the jump heights in terms of the elastic energy after a suitable modification of the jump set and the displacement field. Before we can recall the results obtained in [24], we have to introduce some further notations.

For $s > 0$ we partition $\mathbb{R}^2$ up to a set of measure zero into squares $Q^s(p) = p + s(-1,1)^2$ for $p \in I^s := s(1,1) + 2s\mathbb{Z}^2$. Let

$$\mathcal{U}^s := \Big\{U \subset \mathbb{R}^2 : U = \Big(\bigcup\nolimits_{p\in I} \overline{Q^s(p)}\Big)^\circ : \quad I \subset I^s\Big\}. \tag{7}$$

Here the superscript $\circ$ denotes the interior of a set. For $\mu > 0$ we define $Q_\mu := (-\mu,\mu)^2$ and consider subsets $V \subset Q_\mu$ of the form

$$\mathcal{V}^s := \{V \subset Q_\mu : V = Q_\mu \setminus \bigcup\nolimits_{l=1}^m X_l, \quad X_l \in \mathcal{U}^s, \ X_l \text{ pairwise disjoint}\}.$$

Note that each set in $V \in \mathcal{V}^s$ coincides with a set $U \in \mathcal{U}^s$ up to subtracting a set of zero Lebesgue measure, i.e. $U \subset V$, $\mathcal{L}^2(V \setminus U) = 0$. The essential difference of V and the corresponding U concerns the connected components of the complements

$Q_\mu \setminus V$ and $Q_\mu \setminus U$. In view of the density result presented in Theorem 2.4 above, the key step will be to prove the main result for configurations $u \in H^1(V)$ for some $V \in \mathcal{V}^s$.

For $W \in \mathcal{V}^s$ with the components $X_1, \ldots, X_m$ we define $\Gamma_l(W) = \partial X_l$ for $l = 1, \ldots, m$. In the following we will refer to these sets as *boundary components* and frequently drop the subscript and write $\Gamma(W)$ or just Γ if no confusion arises. Observe that in the definition we do not require that boundary components are connected. Beside the Hausdorff-measure we define the 'diameter' of a boundary component by

$$\operatorname{diam}(\Gamma) := \sqrt{|\pi_1\Gamma|^2 + |\pi_2\Gamma|^2},$$

where π_1, π_2 denote the orthogonal projections onto the coordinate axes.

Note that by definition of $\mathcal{V}^s$ (in contrast to the definition of $\mathcal{U}^s$) two components in $(\Gamma_l)_l$ might not be disjoint. Therefore, we choose an (arbitrary) order $(\Gamma_l)_{l=1}^m = (\Gamma_l(W))_{l=1}^m$ of the boundary components of W, introduce

$$\Theta_l = \Theta_l(W) = \Gamma_l \setminus \bigcup\nolimits_{j<l} \Gamma_j$$

for $i = 1, \ldots, m$ and observe that the components $(\Theta_l)_l$ are pairwise disjoint.

We fix a sufficiently large universal constant c and let $\mathcal{W}^s \subset \mathcal{V}^s$ be the subset consisting of the sets, where for a specific ordering of the components $(\Gamma_l)_l$ we find for all Γ_l a corresponding rectangle $R_l = R(\Gamma_l) \in \mathcal{U}^s$ such that

$$(i)\ \Gamma_l \subset \overline{R_l}, \quad (ii)\ \Theta_l \subset \partial R_l, \quad (iii)\ \operatorname{diam}(\Gamma_l) \le \operatorname{diam}(\partial R_l) \le c\operatorname{diam}(\Gamma_l). \tag{8}$$

In particular, the diameter of Γ_l and the corresponding rectangle R_l are comparable. (Note that in [24, Section 5] we have defined the set $\mathcal{W}^s$ in a slightly different way. See also (3.5) and (3.6) in [24].)

We now formulate a simplified version of [24, Theorem 5.2].

Theorem 2.7 *Let $\mu, s > 0$ and $\varepsilon > 0$. Then there is a (universal) constant $C > 0$ and some $\tilde{s} \le s$ such that for all $W \in \mathcal{V}^s$, $W \subset Q_\mu$, with connected boundary components and for all $u \in H^1(W)$ there is a set $U \in \mathcal{W}^{\tilde{s}}$ with $|U \setminus W| = 0$ and letting $\tilde{Q} = (-\tilde{\mu}, \tilde{\mu})^2$ with $\tilde{\mu} = \max\{\mu - C\mathcal{H}^1(\partial U \cap Q_\mu), 0\}$ there is a modification $\bar{u}$ in SBV defined by*

$$\bar{u}(x) = \begin{cases} A_l\, x + c_l & x \in X_l(U) \quad \text{for all } \ \Gamma_l(U) \text{ with } R_l \cap \tilde{Q} \neq \emptyset, \\ u(x) & \text{else,} \end{cases} \tag{9}$$

such that for all $\Gamma_l = \Gamma_l(U)$ with $R_l \cap \tilde{Q} \neq \emptyset$

$$\int_{\Theta_l(U)} |[\bar{u}](x)|^2 \, d\mathcal{H}^1(x) \le C\varepsilon \operatorname{diam}(\partial R_l)^2. \tag{10}$$

Moreover, one has

$$\mathcal{H}^1(\partial U \cap Q_\mu) \leq C \sum\nolimits_l \mathrm{diam}(\partial R_l) \leq C\big(\mathcal{H}^1(\partial W \cap Q_\mu) + \varepsilon^{-1}\|e(u)\|^2_{L^2(W)}\big) \quad (11)$$

and $|W \setminus U| \leq C(\mathcal{H}^1(\partial U \cap Q_\mu))^2$.

Observe that (9) implies that the domain of $\bar{u}$ contains $W \cup \tilde{Q}$. Here we prefer to present an adaption of [24, Theorem 5.2] in order to avoid the introduction of additional technical notation which will not be needed in the following. For the sake of completeness we briefly indicate how Theorem 2.7 follows from [24, Theorem 5.2] (using the notation of [24]). The reader only interested in the application of the above result may readily omit the following remark.

Remark 2.8 To see Theorem 2.7 we apply [24, Theorem 5.2] for some fixed small h_* and σ. Then the definition in (9) follows from (5.9) in [24] and the fact that $R_l \cap \tilde{Q} \neq \emptyset$ implies $N^{2\hat{r}_l}(\partial R_l) \subset H(U)$ (cf. the proof of Theorem 5.8 in [24]). Moreover, (5.10) in [24] together with $|\Theta_l(U)|_* \leq c\mathrm{diam}(\partial R_l)$ (cf. (5.23) in [24]) yields (10). The second inequality in (11) follows from (5.11) in [24] and the fact that $\sum_l \mathrm{diam}(\partial R_l) \leq C\|U\|_*$ and $\|W\|_* \leq \mathcal{H}^1(\partial W \cap Q_\mu)$ (cf. definition (3.3) in [24]). Finally, (8)(ii) above implies the first inequality in (11) and the additional statement $|W \setminus U| \leq C(\mathcal{H}^1(\partial U \cap Q_\mu))^2$ holds by the isoperimetric inequality.

3 The local estimate on a square

This section is devoted to the derivation of a local estimate on a square. Recall $Q_\mu = (-\mu, \mu)^2$ for $\mu > 0$.

Theorem 3.1 *Let* $p \in [1, 2), q \in [1, \infty)$ *and* $\mu > 0$. *Then there is a constant* $C = C(p, q)$ *such that for all* $u \in SBD^2(Q_\mu)$ *there is a set of finite perimeter* $E \subset Q_\mu$ *with*

$$\mathcal{H}^1(\partial E) \leq C\mathcal{H}^1(J_u), \qquad |E| \leq C(\mathcal{H}^1(J_u))^2 \quad (12)$$

and $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $c \in \mathbb{R}^2$ *such that*

$$\begin{aligned} &(i) \quad \|u - (A\,\cdot + c)\|_{L^q(\hat{Q}\setminus E)} \leq C\mu^{\frac{2}{q}}\|e(u)\|_{L^2(Q_\mu)}, \\ &(ii) \quad \|\nabla u - A\|_{L^p(\hat{Q}\setminus E)} \leq C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \end{aligned} \quad (13)$$

where $\hat{Q} = (-\hat{\mu}, \hat{\mu})^2$ *with* $\hat{\mu} = \max\{\mu - C\mathcal{H}^1(J_u), 0\}$. *Moreover, we obtain* $u\chi_{\hat{Q}\setminus E} \in GSBV^p(Q_\mu)$. *If in addition* $u \in L^\infty(Q_\mu)$, *we find* $u\chi_{\hat{Q}\setminus E} \in SBV^p(Q_\mu)$.

Observe that the additional statement that $u\chi_{\hat{Q}\setminus E}$ lies in (G)SBV^p does not directly follow from (13)(ii) as a Korn-type inequality for ∇u does not automatically guarantee that u has bounded variation. In fact, an equivalent of Alberti's rank one property in BV (see [2]) stating $|D^c u|(\hat{Q}) \le \sqrt{2}|E^c u|(\hat{Q})$ for BV functions is not known in BD. In the present context we circumvent this problem by employing an appropriate approximation result.

We start with displacements fields $u \in H^1(W)$ for some $W \in \mathcal{V}^s$ and afterwards use a density argument to prove the general version of Theorem 3.1.

3.1 Functions with regular jump set

We first introduce some further notation. Let $\theta \in 2^{-\mathbb{N}}$ small, fixed and define $s_i = \mu\theta^i$ for $i \ge 0$. In the following proofs we will consider the coverings $\mathcal{U}^{s_i}$ (recall (7)) consisting of dyadic squares and write $\mathcal{U}^i = \mathcal{U}^{s_i}$ for shorthand. Moreover, we let

$$\mathcal{Q}_i = \{Q = Q^{s_i}(p) : p \in I^{s_i}\}. \tag{14}$$

For each $Q \in \mathcal{Q}_i$ we introduce enlarged squares $Q \subset Q''' \subset Q'' \subset Q'$ defined by $Q''' = \frac{17}{16}Q$, $Q'' = \frac{9}{8}Q$ and $Q' = \frac{5}{4}Q$, where λQ denotes the square with the same center and λ-times the sidelength of Q. Moreover, by $\mathrm{dist}(A, B)$ we denote the euclidian distance between $A, B \subset \mathbb{R}^2$. In the sequel *infinitesimal rigid motions* $A\,x + c$, $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $c \in \mathbb{R}^2$, will appear frequently and we will often write $a(x) = a_{A,c}(x) = A\,x + c$ for the sake of brevity. We first prove the following result.

Lemma 3.2 *Let $s > 0$, $\mu > 0$. Theorem 3.1 holds for all functions $u \in SBD^2(Q_\mu)$ such that $u|_W \in H^1(W)$ for some $W \in \mathcal{V}^s$ with connected boundary components and $\mathcal{H}^1(\partial W \cap Q_\mu) \le C\mathcal{H}^1(J_u)$ as well as $|Q_\mu \setminus W| \le C(\mathcal{H}^1(J_u))^2$ for some $C = C(p, q)$.*

Proof. We first apply Theorem 2.7 on u and W with $\varepsilon = (\mathcal{H}^1(\partial W \cap Q_\mu))^{-1} \cdot \|e(u)\|^2_{L^2(W)}$ to obtain sets $U \in \mathcal{W}^{\tilde{s}}$, $\tilde{Q}$ and a modification $\bar{u}$ such that (9)-(11) hold, in particular we have

$$\mathcal{H}^1(\partial U \cap Q_\mu) \le C \sum\nolimits_l \mathrm{diam}(\partial R_l) \le C\mathcal{H}^1(\partial W \cap Q_\mu) \le C\mathcal{H}^1(J_u). \tag{15}$$

Let $\hat{Q} = (-\hat{\mu}, \hat{\mu})^2$ with $\hat{\mu} = \max\{\mu - \hat{c}\mathcal{H}^1(J_u), 0\}$ for $\hat{c}$ large enough (depending on θ) such that using (15) we find

$$\hat{Q} \subset \tilde{Q}, \qquad \sum\nolimits_l \mathrm{diam}(\partial R_l) \le \tfrac{\theta}{24}\,\mathrm{dist}(\partial\tilde{Q}, \partial\hat{Q}). \tag{16}$$

We may assume that $\hat{Q} \ne \emptyset$ as otherwise the assertion of the lemma is trivial. Then (16) implies $\sum_l \mathrm{diam}(\partial R_l) \le \frac{\theta}{24}\mu = \frac{1}{24}s_1$.

We start with the identification of regions where $J_{\bar{u}}$ is too large (Step I). Afterwards, we will use (10) to apply Theorem 2.3 on these specific sets (Step II). This Korn-Poincaré estimate will then enable us to define a suitable modification (Step III) for which Korn's inequality for Sobolev functions can be employed (Step IV).

Step I (Identification of 'bad' sets): We first identify squares of various length scales where $J_{\bar{u}}$ is too large. Recalling (14) we introduce the sets

$$\mathcal{A}_i = \{Q \in \mathcal{Q}_i : \ Q' \subset \tilde{Q}, \ \ \mathcal{H}^1(Q' \cap J_{\bar{u}}) \geq \bar{c} s_i\} \tag{17}$$

for some $\bar{c} = \bar{c}(\theta) > 0$ small to be specified below. Let $A_i = \bigcup_{Q \in \mathcal{A}_i} \overline{Q}$. Then we define

$$\mathcal{B}_i = \big\{Q \in \mathcal{A}_i : \ Q \cap \hat{Q} \neq \emptyset, \ \ Q \cap \bigcup\nolimits_{j=1}^{i-1} A_j = \emptyset \big\}$$

and accordingly let $B_i = \big(\bigcup_{Q \in \mathcal{B}_i} \overline{Q}\big)^\circ$ and $B_i' = \bigcup_{Q \in \mathcal{B}_i} \overline{Q'}$. Obviously, the sets $(B_i)_i$ are pairwise disjoint. We now show that for some $I \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned} &(i) \ \ \sum\nolimits_{R_l \cap Q' \neq \emptyset} \mathrm{diam}(\partial R_l) \leq s_i \quad \text{for all } \ Q \in \mathcal{B}_i, \, i \geq 1, \\ &(ii) \ \ B_i = \emptyset \ \text{ for } \ i > I, \\ &(iii) \ \ \bigcup\nolimits_l \Gamma_l \cap \hat{Q} \subset \bigcup\nolimits_{i \geq 1} B_i, \\ &(iv) \ \ \mathcal{H}^1\big(\partial \bigcup\nolimits_{i \geq 1} B_i'\big) \leq C\mathcal{H}^1(J_{\bar{u}} \cap \tilde{Q}), \\ &(v) \ \ |B_i'| \leq C s_i \mathcal{H}^1(J_{\bar{u}} \cap \tilde{Q}) \ \text{ for all } i \geq 1 \end{aligned} \tag{18}$$

for a constant $C = C(\theta)$. We first confirm (i). We assume $\sum_{R_l \in \mathcal{F}} \mathrm{diam}(\partial R_l) > s_i$ for some $Q \in \mathcal{B}_i$, where $\mathcal{F} = \{R_l : R_l \cap Q' \neq \emptyset\}$ and derive contradictions treating two different cases:

a) Suppose there is some $R_l \in \mathcal{F}$ with $\mathrm{diam}(\partial R_l) > \frac{1}{8} s_{i-1}$. Then choose $j \leq i-1$ such that $\frac{1}{8} s_j < \mathrm{diam}(\partial R_l) \leq \frac{1}{8} s_{j-1}$ and observe that by the remark below (16) we find $j \geq 2$. Moreover, we select $Q_* \in \mathcal{Q}_{j-1}$ such that $Q \subset Q_*$. As $s_{j-1} < 8\theta^{-1}\mathrm{diam}(\partial R_l) \leq \frac{1}{3}\,\mathrm{dist}(\partial \tilde{Q}, \partial \hat{Q})$ by (16), we find $Q_*' \subset \tilde{Q}$ for θ small enough. Then using (8)(i) we get $\Gamma_l \subset \overline{R_l} \subset Q_*'$ and therefore by (8)(iii)

$$\mathcal{H}^1(J_{\bar{u}} \cap Q_*') \geq \mathcal{H}^1(\Gamma_l) \geq \mathrm{diam}(\Gamma_l) \geq C\mathrm{diam}(\partial R_l) \geq C\theta s_{j-1}.$$

For $\bar{c}$ sufficiently small this yields $Q_* \in \mathcal{A}_{j-1}$. Consequently, as $Q \subset Q_*$, this implies $Q \notin \mathcal{B}_i$ giving a contradiction.

b) Now assume $\mathrm{diam}(\partial R_l) \leq \frac{1}{8} s_{i-1}$ for all $R_l \in \mathcal{F}$. Then it is not hard to see that $\overline{R_l} \subset Q_*'$, where $Q_* \in \mathcal{Q}_{i-1}$ such that $Q \subset Q_*$. Arguing as in a) this implies $\mathcal{H}^1(J_{\bar{u}} \cap Q_*') \geq C \sum_{R_l \in \mathcal{F}} \mathrm{diam}(\partial R_l) \geq C\theta s_{i-1}$. As before we also derive $s_{i-1} < \theta^{-1} \sum_l \mathrm{diam}(\partial R_l) \leq \frac{1}{24}\,\mathrm{dist}(\partial \tilde{Q}, \partial \hat{Q})$ and thus $Q_*' \subset \tilde{Q}$. This gives $Q_* \in \mathcal{A}_{i-1}$ for $\bar{c}$ sufficiently small leading again to a contradiction.

Likewise, we obtain (ii): Assume there was some $Q \in \mathcal{B}_i$ for i so large that $s_i \le \theta\tilde{s}$. Since $R_l \in \mathcal{U}^{\tilde{s}}$ for all R_l, we find some R_l with $R_l \cap Q' \neq \emptyset$ with $\mathrm{diam}(\partial R_l) \ge \tilde{s} \ge s_{i-1}$. Then we can argue as in a) above to derive a contradiction.

Moreover, the definition in (17) implies $\bigcup_l \Gamma_l \cap \hat{Q} \subset \bigcup_{i\ge 1} A_i \cap \hat{Q}$ and thus (iii) follows from the property $\bigcup_{i\ge1} A_i \cap \hat{Q} = \bigcup_{i\ge1} B_i \cap \hat{Q}$.

To show (iv) we define $\hat{\mathcal{B}}_i = \{Q \in \mathcal{B}_i : \overline{Q'} \not\subset \bigcup_{j=1}^{i-1} B'_j\}$ as well as $\hat{B}'_i = \bigcup_{Q\in\hat{\mathcal{B}}_i} \overline{Q'}$ and observe $\bigcup_{i\ge1} B'_i = \bigcup_{i\ge1} \hat{B}'_i$. It is elementary to see that each $x \in \tilde{Q}$ is contained in at most three different sets $(\hat{B}'_i)_i$ and thus contained in at most twelve different squares Q' with $Q \in \bigcup_{i\ge1} \hat{\mathcal{B}}_i$. Consequently, we derive using (17)

$$\mathcal{H}^1\big(\partial \bigcup\nolimits_{i\ge1} B'_i\big) \le C \sum\nolimits_{i\ge1} \sum\nolimits_{Q\in\hat{\mathcal{B}}_i} \mathcal{H}^1(Q' \cap J_{\bar{u}}) \le C\mathcal{H}^1(\tilde{Q} \cap J_{\bar{u}}).$$

Finally, (v) directly follows from (17).

Step II (Korn-Poincaré inequality): Recall that by the definition in (17) we find $Q' \subset \tilde{Q}$ for every $Q \in \mathcal{B}_i$. Thus, using Theorem 2.3 on Q' for $Q \in \mathcal{B}_i$ we obtain by (9), (10) and (18)(i) suitable infinitesimal rigid motions $a_Q = a_{A_Q,c_Q}$ such that

$$\begin{aligned}
\|\bar{u} - a_Q\|^2_{L^2(Q')} &\le C\|e(\bar{u})\|^2_{L^1(Q')} + C\Big(\int_{J_{\bar{u}}\cap Q'} |[\bar{u}]|\, d\mathcal{H}^1\Big)^2 \\
&\le C s_i^2 \|e(\bar{u})\|^2_{L^2(Q')} + C\mathcal{H}^1(J_{\bar{u}} \cap Q') \int_{J_{\bar{u}}\cap Q'} |[\bar{u}]|^2\, d\mathcal{H}^1 \\
&\le C s_i^2 \|e(u)\|^2_{L^2(Q')} + C\varepsilon\mathcal{H}^1(J_{\bar{u}} \cap Q') \sum\nolimits_{R_l\cap Q'\neq\emptyset} \mathrm{diam}(\partial R_l)^2 \\
&\le C s_i^2 \big(\|e(u)\|^2_{L^2(Q')} + \varepsilon\mathcal{H}^1(J_{\bar{u}} \cap Q')\big)
\end{aligned}$$

for all $Q \in \mathcal{B}_i$. In the second step we have used Hölder's inequality. Thus, summing over all squares in $\mathcal{B}_i$ and recalling that each $x \in \tilde{Q}$ is contained in at most four different Q', $Q \in \mathcal{B}_i$, we find

$$\sum\nolimits_{Q\in\mathcal{B}_i} \|\bar{u} - a_Q\|^2_{L^2(Q')} \le C s_i^2 \|e(u)\|^2_{L^2(Q_\mu)} \tag{19}$$

for all $i \ge 1$, where we used $\varepsilon \le C(\mathcal{H}^1(J_{\bar{u}} \cap \tilde{Q}))^{-1}\|e(u)\|^2_{L^2(W)}$ (cf. (15)).

Step III (Modification): We now show that we can 'heal' the discontinuities of $\bar{u}$ in $\hat{Q}$ drawing ideas from [27, Section 6]. The strategy is to modify the displacement field inductively. Let $1 < p < 2$ be given. Recall that $I \in \mathbb{N}$ is the largest index such that $B_I \neq \emptyset$ (see (18)). Define $\bar{u}_0 = \bar{u}$ and assume $\bar{u}_j$ has already been constructed satisfying

$$\begin{aligned}
&(i)\quad J_{\bar{u}_j} \cap \hat{Q} \subset \bigcup\nolimits_{k=1}^{I-j} B_k, && (20)\\
&(ii)\quad \sum\nolimits_{Q\in\mathcal{B}_i} \|\bar{u}_j - a_Q\|^2_{L^2(Q')} \le \bar{C} s_i^2 \prod\nolimits_{k=0}^{j} (1 + \eta^{I-i-k})\ \|e(u)\|^2_{L^2(Q_\mu)}\ \ \forall i \le I - j, \\
&(iii)\quad \|e(\bar{u}_j)\|^p_{L^p(\hat{Q})} \le \bar{C}\mu^{2-p} \prod\nolimits_{k=0}^{j} (1 + \eta^{I-k})\ \|e(u)\|^p_{L^2(Q_\mu)}
\end{aligned}$$

for some $\bar{C}$ large enough, where for shorthand $\eta = \theta^{\frac{1}{2}-\frac{p}{4}} < 1$. Clearly, by (18)(ii),(iii), (19) and Hölder's inequality together with $|\hat{Q}| \le 4\mu^2$ we find that (20) holds for $j = 0$.

We now construct $\bar{u}_{j+1}$. We consider a partition of unity $\{\varphi_0\} \cup (\varphi_Q)_{Q\in\mathcal{B}_{I-j}} \subset C^\infty(\mathbb{R}^2)$ with the properties

$$\begin{aligned} (i) &\quad \varphi_0(x) + \sum\nolimits_{Q\in\mathcal{B}_{I-j}} \varphi_Q(x) = 1 \ \text{ for all } x \in \tilde{Q}, \\ (ii) &\quad Q \subset \operatorname{supp}(\varphi_Q) \subset Q' \ \text{ for all } Q \in \mathcal{B}_{I-j}, \\ (iii) &\quad \operatorname{supp}(\varphi_0) \subset \mathbb{R}^2 \setminus \overline{B_{I-j}}, \\ (iv) &\quad \|\nabla\varphi_0\|_\infty, \|\nabla\varphi_Q\|_\infty \le cs_{I-j}^{-1} \ \text{ for all } \ Q \in \mathcal{B}_{I-j}. \end{aligned} \tag{21}$$

Then we define

$$\bar{u}_{j+1}(x) = \bar{u}_j(x) + \sum_{Q\in\mathcal{B}_{I-j}} \varphi_Q(x)(A_Q\, x + c_Q - \bar{u}_j(x)) = \bar{u}_j + \sum_{Q\in\mathcal{B}_{I-j}} \varphi_Q(a_Q - \bar{u}_j)$$

for all $x \in \tilde{Q}$. As $\bar{u}_{j+1}$ is smooth in B_{I-j} by (21)(i),(iii), (20)(i) holds. Employing (20)(ii) for j and $i = I - j$ we obtain

$$\begin{aligned} \|\bar{u}_{j+1} - \bar{u}_j\|^2_{L^2(\tilde{Q})} &\le C \sum\nolimits_{Q\in\mathcal{B}_{I-j}} \|\bar{u}_j - a_Q\|^2_{L^2(Q')} \\ &\le C\bar{C}s_{I-j}^2 \prod\nolimits_{k=0}^{j} (1 + \eta^{j-k})\|e(u)\|^2_{L^2(Q_\mu)} \le C\bar{C}s_{I-j}^2\|e(u)\|^2_{L^2(Q_\mu)}, \end{aligned}$$

where in the first step we used that each $x \in \tilde{Q}$ is contained in at most four different enlarged squares. Using $\eta = \theta^{\frac{1}{2}-\frac{p}{4}} \ge \theta$ and thus $Cs_{I-j}^2 \le Cs_i^2\eta^{2(I-j-i)} \le \frac{1}{2}s_i^2\eta^{2(I-j-i-1)}$ for all $1 \le i \le I - j - 1$ for θ small enough we then find

$$\|\bar{u}_{j+1} - \bar{u}_j\|^2_{L^2(\tilde{Q})} \le \bar{C}\tfrac{1}{2}\eta^{2(I-i-j-1)}s_i^2\|e(u)\|^2_{L^2(Q_\mu)}.$$

The last estimate together with (20)(ii) for j and a scaled version of Young's inequality of the form $(a + b)^2 \le \big((1 + \delta)a^2 + (1 + \frac{1}{\delta})b^2\big)$ $(a, b \in \mathbb{R},\ \delta > 0)$ yields for $\delta = \frac{1}{2}\eta^{I-i-j-1}$

$$\begin{aligned} &\sum\nolimits_{Q\in\mathcal{B}_i} \|\bar{u}_{j+1} - a_Q\|^2_{L^2(Q')} \\ &\quad \le (1 + \tfrac{1}{2}\eta^{I-i-j-1}) \sum\nolimits_{Q\in\mathcal{B}_i} \|\bar{u}_j - a_Q\|^2_{L^2(Q')} + C\eta^{-(I-i-j-1)}\|\bar{u}_{j+1} - \bar{u}_j\|^2_{L^2(\tilde{Q})} \\ &\quad \le \bar{C}s_i^2 \prod\nolimits_{k=0}^{j} (1 + \eta^{I-i-k})(1 + \tfrac{1}{2}\eta^{I-i-j-1})\|e(u)\|^2_{L^2(Q_\mu)} \\ &\qquad + \bar{C}\tfrac{1}{2}\eta^{I-i-j-1}s_i^2\|e(u)\|^2_{L^2(Q_\mu)} \le \bar{C}s_i^2 \prod\nolimits_{k=0}^{j+1} (1 + \eta^{I-i-k})\, \|e(u)\|^2_{L^2(Q_\mu)} \end{aligned}$$

for all $1 \le i \le I - j - 1$. This shows (20)(ii). To confirm (20)(iii) we first note that by Hölder's inequality, (20)(ii) and the fact that $|B'_{I-j}| \le C\mu s_{I-j}$ (see (18)(v)) we obtain

$$\begin{aligned}\sum\nolimits_{Q\in\mathcal{B}_{I-j}} \|\bar{u}_j - a_Q\|^p_{L^p(Q')} &\le C|B'_{I-j}|^{1-\frac{p}{2}}\big(\sum\nolimits_{Q\in\mathcal{B}_{I-j}} \|\bar{u}_j - a_Q\|^2_{L^2(Q')}\big)^{\frac{p}{2}} \\ &\le C\bar{C}^{\frac{p}{2}}\mu^{1-\frac{p}{2}} s_{I-j}^{1+\frac{p}{2}}\|e(u)\|^p_{L^2(Q_\mu)},\end{aligned} \tag{22}$$

where we again used that each $x \in \tilde{Q}$ is contained in at most four different enlarged squares. We calculate the derivative

$$\nabla\bar{u}_{j+1} = \nabla\bar{u}_j\varphi_0 + \sum\nolimits_{Q\in\mathcal{B}_{I-j}} \varphi_Q A_Q + (a_Q - \bar{u}_j)\otimes\nabla\varphi_Q.$$

Now we again apply a scaled version of Young's inequality of the form $|a+b|^p \le \big((1+\delta)a^2 + (1+\frac{1}{\delta})b^2\big)^{\frac{p}{2}} \le (1+\delta^{\frac{p}{2}})|a|^p + (1+\delta^{-\frac{p}{2}})|b|^p$ for $a, b \in \mathbb{R}$, $\delta > 0$. Consequently, similarly as before using (20)(ii), (21)(iv) and (22) we find

$$\begin{aligned}\|e(\bar{u}_{j+1})\|^p_{L^p(\hat{Q})} &\le (1+\tfrac{1}{2}\eta^{I-j-1})\|e(\bar{u}_j)\|^p_{L^p(\hat{Q})} + C\bar{C}\eta^{-(I-j-1)}s_{I-j}^{1-\frac{p}{2}}\mu^{1-\frac{p}{2}}\|e(u)\|^p_{L^2(Q_\mu)} \\ &\le \bar{C}\mu^{2-p}\prod\nolimits_{k=0}^{j}(1+\eta^{I-k})\,(1+\tfrac{1}{2}\eta^{I-j-1})\|e(u)\|^p_{L^2(Q_\mu)} \\ &\quad + \bar{C}\tfrac{1}{2}\eta^{I-j-1}\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)} \\ &\le \bar{C}\mu^{2-p}\prod\nolimits_{k=0}^{j+1}(1+\eta^{I-k})\|e(u)\|^p_{L^2(Q_\mu)},\end{aligned}$$

where we used $\eta^2 = \theta^{1-\frac{p}{2}}$ and thus $Cs_{I-j}^{1-\frac{p}{2}} \le C\mu^{1-\frac{p}{2}}\eta^{2(I-j)} \le \frac{1}{2}\mu^{1-\frac{p}{2}}\eta^{2(I-j-1)}$ for θ sufficiently small.

Step IV (Korn's inequality): We define $\hat{u} = \bar{u}_I$ and observe that by (20)(i),(iii) we have $\hat{u}|_{\hat{Q}} \in W^{1,p}(\hat{Q})$ with $\|e(\hat{u})\|_{L^p(\hat{Q})} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}$ for some $C = C(p)$. Moreover, we define $E = \bigcup_{i=1}^I B'_i \cup (Q_\mu \setminus W)$ and observe that $\hat{u} = u$ on $\hat{Q}\setminus E$ due to the construction of the functions $(\bar{u}_j)_j$. By (15) and (18)(iv) we obtain

$$\mathcal{H}^1(\partial E) \le C\mathcal{H}^1(J_{\bar{u}}\cap\tilde{Q}) + C\mathcal{H}^1(J_u) \le C\mathcal{H}^1(J_u).$$

Moreover, by definition (17) we find some $i_0 \in \mathbb{N}$ with $s_{i_0} \le c\mathcal{H}^1(J_{\bar{u}}\cap\tilde{Q})$ for a sufficiently large c such that $B_i = \emptyset$ for all $i \le i_0$. Thus, using (15), (18)(v) we find

$$\big|\bigcup\nolimits_{i=1}^I B'_i\big| \le C\mathcal{H}^1(J_{\bar{u}}\cap\tilde{Q})\sum\nolimits_{k=i_0+1}^I s_k \le C\mathcal{H}^1(J_u)s_{i_0} \le C(\mathcal{H}^1(J_u))^2.$$

This together with $|Q_\mu\setminus W| \le C(\mathcal{H}^1(J_u))^2$ yields $|E| \le C(\mathcal{H}^1(J_u))^2$ and shows (12). We now apply Poincaré's and Korn's inequality (see Theorem 2.6) and find $A \in \mathbb{R}^{2\times2}_{\rm skew}$, $c \in \mathbb{R}^2$ such that by a standard rescaling argument

$$\begin{aligned}\|\nabla u - A\|_{L^p(\hat{Q}\setminus E)} &\le \|\nabla\hat{u} - A\|_{L^p(\hat{Q})} \le C\|e(\hat{u})\|_{L^p(\hat{Q})} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \\ \|u - (A\,\cdot + c)\|_{L^q(\hat{Q}\setminus E)} &\le \|\hat{u} - (A\,\cdot + c)\|_{L^q(\hat{Q})} \le C\mu^{\frac{2}{q}-\frac{2}{p}+1}\|e(\hat{u})\|_{L^p(\hat{Q})}\end{aligned}$$

for $q \le \frac{2p}{2-p}$. Then (13)(ii) holds for $p \in (1,2)$ and the case $p = 1$ directly follows. Likewise, (13)(i) also holds for $q \in [1,\infty)$ since $p \in [1,2)$. Clearly, we obtain $u\chi_{\hat{Q}\setminus E} \in SBV^p(Q_\mu)$ due to the regularity of E. $\square$

3.2 General case

Combining Lemma 3.2 and the density result for SBD functions (Theorem 2.4) we can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $p \in (1,2)$, $q \in [1,\infty)$ and let $u \in SBD^2(Q_\mu)$ be given. Choose $Q'_\mu \subset\subset Q_\mu$ with $\mathrm{dist}(\partial Q_\mu, \partial Q'_\mu) \le C\mathcal{H}^1(J_u)$. We apply Theorem 2.4 on Q'_μ to find sets $W_n = Q'_\mu \setminus \bigcup_{j=1}^{m_n} R^n_j$ for paraxial rectangles $(R^n_j)_j$ with

$$\mathcal{H}^1(\partial W_n \cap Q'_\mu) \le C \sum\nolimits_{j=1}^{m_n} \mathrm{diam}(\partial R^n_j) \le C\mathcal{H}^1(J_u), \quad |Q'_\mu \setminus W_n| \le C(\mathcal{H}^1(J_u))^2$$

for all n sufficiently large and a sequence $u_n \in H^1(W_n)$, $n \in \mathbb{N}$, with $\|u_n - u\|_{L^1(W_n)} \le \frac{1}{n}$ and $\|e(u_n) - e(u)\|_{L^2(W_n)} \le \frac{1}{n}$.

Possibly replacing the rectangles by infinitesimal larger ones we can assume that there is some sequence $(s_n)_n$ such that $W_n \in \mathcal{V}^{s_n}$ with connected boundary components. We then employ Lemma 3.2 (on Q'_μ) and find $A_n \in \mathbb{R}^{2\times 2}$, $c_n \in \mathbb{R}^2$ as well as exceptional sets $E_n \supset Q'_\mu \setminus W_n$ with $\mathcal{H}^1(\partial E_n) \le C\mathcal{H}^1(J_u)$, $|E_n| \le C(\mathcal{H}^1(J_u))^2$ such that

$$\begin{aligned} &(i) \;\; \|u_n - (A_n \,\cdot + c_n)\|_{L^q(\hat{Q}\setminus E_n)} \le C\mu^{\frac{2}{q}} \|e(u)\|_{L^2(Q_\mu)}, \\ &(ii) \;\; \|\nabla u_n - A_n\|_{L^p(\hat{Q}\setminus E_n)} \le C\mu^{\frac{2}{p}-1} \|e(u)\|_{L^2(Q_\mu)}, \end{aligned} \tag{23}$$

where $\hat{Q} = (-\hat{\mu},\hat{\mu})^2$ with $\hat{\mu} = \max\{\mu - C\mathcal{H}^1(J_u), 0\}$ independently of n. First, by Theorem 2.2 we find a set $E \subset Q'_\mu$ with $\mathcal{H}^1(\partial E) \le C\mathcal{H}^1(J_u)$, $|E| \le C(\mathcal{H}^1(J_u))^2$ such that $\chi_{E_n} \to \chi_E$ in measure for $n \to \infty$ for a not relabeled subsequence. Moreover, letting $v_n(x) := (u_n(x) - (A_n\, x + c_n))\chi_{\hat{Q}\setminus E_n}(x)$ for $x \in Q_\mu$ we apply Ambrosio's compactness result in GSBV (see Theorem 2.1) to find a function $v \in GSBV^p(Q_\mu)$ such that passing to a further (not relabeled) subsequence we obtain $v_n \to v$ in L^1 and $\nabla v_n \rightharpoonup \nabla v$ weakly in L^p. In particular, we derive

$$\begin{aligned} &(i) \;\; \|v\|_{L^q(\hat{Q}\setminus E)} \le \liminf\nolimits_{n\to\infty} \|v_n\|_{L^q(Q_\mu)} \le C\mu^{\frac{2}{q}} \|e(u)\|_{L^2(Q_\mu)}, \\ &(ii) \;\; \|\nabla v\|_{L^p(\hat{Q}\setminus E)} \le \liminf\nolimits_{n\to\infty} \|\nabla v_n\|_{L^p(Q_\mu)} \le C\mu^{\frac{2}{p}-1} \|e(u)\|_{L^2(Q_\mu)}. \end{aligned}$$

Consequently, to finish the proof it suffices to show $v = (u-a)\chi_{\hat{Q}\setminus E}$ for some infinitesimal rigid motion $a = a_{A,c}$. (Observe that as before the assertion then holds also for $p = 1$.)

Possibly passing to a further subsequence we can assume $\chi_{E_n} \to \chi_E$ pointwise a.e. and thus we find a measurable set F with $|F| > 0$ such that $F \subset \hat{Q} \setminus E$,

$F \subset \hat{Q} \setminus E_n$ (up to a set of negligible measure) for n large enough. By (23)(i) and Hölder's inequality this implies

$$\|A_n \cdot + c_n\|_{L^1(F)} \le C\|e(u)\|_{L^2(Q_\mu)} + C\|u_n\|_{L^1(F)} \le C$$

for $C = C(\mu) > 0$ large enough. Consequently, we obtain $A_n \to A$, $c_n \to c$ for some $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $c \in \mathbb{R}^2$. As $u_n \chi_{\hat{Q}\setminus E_n} \to u\chi_{\hat{Q}\setminus E}$ in L^1 and $v_n \to v = v\chi_{\hat{Q}\setminus E}$ in L^1, we conclude $v = (u - (A \cdot + c))\chi_{\hat{Q}\setminus E}$.

To see the addition statement that $u\chi_{\hat{Q}\setminus E} \in SBV^p(\Omega)$ if $u \in L^\infty(\Omega)$, we observe that $|A_n|, |c_n| \le C$ and $\|u_n\|_\infty \le \|u\|_\infty$ (see Theorem 2.4) imply $\|v_n\|_\infty \le C$ independently of $n \in \mathbb{N}$ and the claim follows from Theorem 2.1. □

Remark 3.3 We briefly note that a similar estimate can be obtained for functions in the generalized space GSBD^2 (see [16]) since the approximation result used in the proof of Theorem 3.1 is also available in the generalized setting (see [29]).

4 Estimate at the boundary

In this section we give a refined estimate which holds up to the boundary of Lipschitz sets. This together with a standard covering argument will then lead to the proof of the main theorem. We first give an elementary estimate about the difference of infinitesimal rigid motions which we state in arbitrary space dimensions.

Lemma 4.1 *Let $p \in [1, \infty)$ and $\bar{c} > 0$. Then there is a constant $C = C(p, \bar{c})$ such that for all $x \in \mathbb{R}^d$, $R > 0$ and $\Omega \subset Q^x_R := x + (-R, R)^d$ with $|\Omega| \ge \bar{c}R^d$ and all affine mappings $a : \mathbb{R}^d \to \mathbb{R}^d$ one has*

$$\|a\|_{L^p(Q^x_R)} \le C\|a\|_{L^p(\Omega)}.$$

Although similar estimates have already been used (see e.g. [10, 24, 27]) we include the elementary proof here for the sake of completeness.

Proof. We first note that by an elementary translation argument it suffices to consider cubes $Q_R := (-R, R)^d$ centered at the origin. Assume the statement was false. Then there would be sequences $(R_k)_k$, $(\Omega_k)_k$ with $\Omega_k \subset Q_{R_k}$, $|\Omega_k| \ge \bar{c}R_k^d$ and a sequence of affine mappings $(a_k)_k$ with

$$\|a_k\|_{L^p(Q_{R_k})} > k\|a_k\|_{L^p(\Omega_k)}.$$

We define $b_k(x) = a_k(R_k x)$ as well as $\Omega'_k = \frac{1}{R_k}\Omega_k$ and obtain by transformation

$$\|b_k\|_{L^p(Q_1)} > k\|b_k\|_{L^p(\Omega'_k)}.$$

Then we define the affine mappings $c_k = \frac{b_k}{\|b_k\|_{L^p(Q_1)}}$ and derive

$$1 = \|c_k\|_{L^p(Q_1)} > k\|c_k\|_{L^p(\Omega'_k)}.$$

As $(c_k)_k$ are affine, we find that $\|c_k\|_{W^{1,p}(Q_1)}$ is uniformly bounded and thus a compactness result yields (after passage to a not relabeled subsequence) $c_k \to c$ in $L^p(Q_1)$ for some affine mapping c. Moreover, there is a measurable function f with $f \geq 0$, $\|f\|_{L^1(Q_1)} \geq \bar{c}$ such that $\chi_{\Omega'_k} \rightharpoonup^* f$ weakly in $L^\infty(Q_1)$. Consequently, we find $1 = \|c\|_{L^p(Q_1)}$ and $0 = \|c \cdot f\|_{L^1(\Omega')}$ which gives a contradiction. $\square$

We are now in a position to give the boundary estimate. By $d(A)$ we denote the diameter of a set $A \subset \mathbb{R}^2$.

Theorem 4.2 *Let $p \in [1,2)$, $q \in [1,\infty)$. Let $\mu > 0$ and $\psi : (-2\mu, 2\mu) \to [\mu, \infty)$ Lipschitz with $\|\psi'\|_\infty \leq \bar{c}$ and $\inf \psi = \mu$. Let*

$$\begin{aligned} U &= \{(x_1,x_2) : -2\mu < x_1 < 2\mu,\ -2\mu \leq x_2 \leq \psi(x_1)\}, \\ U' &= \{(x_1,x_2) : -\mu < x_1 < \mu,\ -\mu \leq x_2 \leq \psi(x_1)\}. \end{aligned} \tag{24}$$

Then there is a constant $C = C(p,q,\bar{c})$ such that for all $u \in SBD^2(U)$ there is a set of finite perimeter $F \subset U$ with $\mathcal{H}^1(\partial F) \leq C\mathcal{H}^1(J_u)$, $|F| \leq C(\mathcal{H}^1(J_u))^2$ and

$$\begin{aligned} (i) \quad & \|u - (A\ \cdot + c)\|_{L^q(U' \setminus F)} \leq C\mu^{\frac{2}{q}}\|e(u)\|_{L^2(U)}, \\ (ii) \quad & \|\nabla u - A\|_{L^p(U' \setminus F)} \leq C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(U)}, \end{aligned} \tag{25}$$

for suitable $A \in \mathbb{R}^{2\times 2}_{\text{skew}}$, $c \in \mathbb{R}^2$.

Proof. Recall the definition of the sets $\mathcal{Q}_i$, $i \in \mathbb{N}$, and the enlarged squares $Q \subset Q''' \subset Q'' \subset Q'$ in (14). Let $\mathcal{Q}_W \subset \bigcup_{i\geq 1}\mathcal{Q}_i$ be a Whitney-type covering of U, i.e. $\bigcup_{Q \in \mathcal{Q}_W} Q' = U$ such that (cf. e.g. [1, 21, 39])

$$\begin{aligned} (i) \quad & d(Q) \leq \operatorname{dist}(Q, \partial U) \leq Cd(Q) \quad \text{for all } \ Q \in \mathcal{Q}_W, \\ (ii) \quad & \#\{Q \in \mathcal{Q}_W : x \in Q'\} \leq N \quad \text{for all } \ x \in U, \\ (iii) \quad & Q'_1 \cap Q'_2 \neq \emptyset \ \text{ for } \ Q_1, Q_2 \in \mathcal{Q}_W \ \Rightarrow \ \tfrac{1}{C}d(Q_1) \leq d(Q_2) \leq Cd(Q_1). \end{aligned} \tag{26}$$

Moreover, we consider a corresponding partition of unity $(\varphi_Q)_{Q \in \mathcal{Q}_W} \subset C^\infty(U)$ with $\sum_{Q \in \mathcal{Q}_W} \varphi_Q(x) = 1$ for $x \in U$ and

$$\begin{aligned} (i) \quad & Q \subset \operatorname{supp}(\varphi_Q) \subset Q''' \ \text{ for all } Q \in \mathcal{Q}_W, \\ (ii) \quad & \|\nabla \varphi_Q\|_\infty \leq cd(Q)^{-1} \ \text{ for all } \ Q \in \mathcal{Q}_W \end{aligned} \tag{27}$$

for a universal constant $c > 0$. Let

$$\mathcal{B} = \{Q \in \mathcal{Q}_W : \mathcal{H}^1(Q' \cap J_u) \geq \hat{c}d(Q)\} \tag{28}$$

be the 'bad' squares for some $\hat{c}$ sufficiently small. For each enlarged square $Q' = p + (-r,r)^2$, $Q \in \mathcal{B}$, we define $P_Q = (p + (-r,r) \times (-r,\infty)) \cap U$. Employing (26)(i) it is then elementary to see that $\mathcal{H}^1(\partial P_Q) \le C\mathcal{H}^1(J_u \cap Q')$ for some $C = C(\bar{c},\hat{c})$. Letting $P = \bigcup_{Q\in\mathcal{B}} \overline{P_Q}$ we obtain by (26)(ii)

$$\mathcal{H}^1(\partial P) \le CN\mathcal{H}^1(J_u) \tag{29}$$

and using the isoperimetric inequality we also find $|P| \le C(\mathcal{H}^1(J_u))^2$. We let $V = U' \setminus P$.

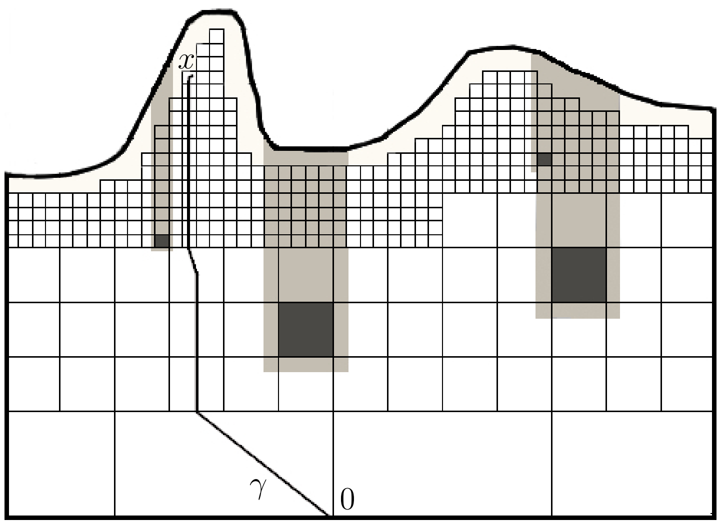

Figure 1: Illustration of a part of U and $\mathcal{Q}_W \cap \bigcup_{i=1}^3 \mathcal{Q}_i$. The squares in $\mathcal{B}$ are depicted in dark gray and the corresponding set P in light gray. Moreover, a John curve γ connecting x with 0 is sketched.

Observe that we can assume $Q \notin \mathcal{B}$ for all $Q \in \mathcal{Q}_W$ with $Q \cap (-\mu,\mu) \times \{0\} \neq \emptyset$. In fact, these squares satisfy $d(Q) \ge c\mu$. Consequently, if $Q \in \mathcal{B}$, we find $\mathcal{H}^1(J_u) \ge c\hat{c}\mu$ and in this case the assertion of the theorem holds with the choice $F = U$.

We now see that V is a John domain with center 0 and a constant only depending on $\bar{c}$. In fact, fix some $x = (x_1,x_2) \in V$ and $Q \in \mathcal{Q}_W$ such that $x \in \overline{Q}$. We consider a vertical chain $\mathcal{C}_1 = \{Q^1_1 = Q,\ldots,Q^1_{n_1}\}$ of squares in $\mathcal{Q}_W$ intersecting $\{x_1\} \times [0,x_2]$ together with a horizontal chain $\mathcal{C}_2 = \{Q^2_1 = Q^1_{n_1},\ldots,Q^2_{n_2}\}$ of squares intersecting $[x_1,0] \times \{0\}$ such that $\overline{Q^j_k} \cap \overline{Q^j_{k+1}} \neq \emptyset$ for $1 \le k \le n_j - 1$, $j = 1,2$.

Now in view of (26) we see that $d(Q^j_{k_1}) \le d(Q^j_{k_2})$ for all $1 \le k_1 \le k_2 \le n_j$, $j = 1,2$, and $d(Q^j_{k_1}) \le \theta d(Q^j_{k_2})$ for all $k_2 \ge k_1 + l$ for some $l = l(\bar{c}) \in \mathbb{N}$. Consequently, it is elementary to construct a curve γ starting in x, ending in 0 and intersecting the midpoints of the squares in $\mathcal{C}_1 \cup \mathcal{C}_2$ such that the condition given in Definition 2.5 holds (cf. Figure 1).

Let $\mathcal{G} = \mathcal{Q}_W \setminus \mathcal{B}$. For each $Q \in \mathcal{G}$ we apply Theorem 3.1 on Q' to find infinitesimal rigid motions $a_Q = a_{A_Q,c_Q}$ and exceptional sets E_Q such that by (13)

$$\begin{aligned} (i) \quad & \|u - (A_Q \, \cdot + c_Q)\|_{L^q(Q'' \setminus E_Q)} \le C d(Q)^{\frac{2}{q}} \|e(u)\|_{L^2(Q')}, \\ (ii) \quad & \|\nabla u - A_Q\|_{L^p(Q'' \setminus E_Q)} \le C d(Q)^{\frac{2}{p}-1} \|e(u)\|_{L^2(Q')}. \end{aligned} \tag{30}$$

(For $\hat{c}$ sufficiently small in (28) we can in fact assume that Q'' is contained in the shrinked square $\hat{Q}$ given by Theorem 3.1.) Moreover, by (12) and (26)(ii) we get that $E := \bigcup_{Q \in \mathcal{G}} E_Q$ fulfills $\mathcal{H}^1(\partial E) \le C\mathcal{H}^1(J_u)$ and $|E| \le C(\mathcal{H}^1(J_u))^2$.

We now estimate the difference of the infinitesimal rigid motions. Consider some $Q \in \mathcal{G}$ and let $\mathcal{N}(Q) = \{\hat{Q} \in \mathcal{G} \setminus \{Q\} : Q''' \cap \hat{Q}''' \neq \emptyset\}$. Recall $\frac{1}{C} d(\hat{Q}) \le d(Q) \le C d(\hat{Q})$ for all $\hat{Q} \in \mathcal{N}(Q)$ by (26)(iii) which also implies $\#\mathcal{N}(Q) \le C$ for some $C > 0$ large enough. Then $Q'' \cap \hat{Q}''$ contains a ball B with radius larger than $cd(Q)$ for some small $c > 0$. It is elementary to see that choosing $\hat{c}$ in (28) sufficiently small we find $|\hat{E} \cap B| \le \frac{1}{2}|B|$, where $\hat{E} = E_{\hat{Q}} \cup E_Q$. Therefore, by (30)(i) for $p = q$, (26)(iii) and the triangle inequality we derive

$$\|a_Q - a_{\hat{Q}}\|_{L^p(B \setminus \hat{E})} = \|(A_Q \, \cdot + c_Q) - (A_{\hat{Q}} \, \cdot + c_{\hat{Q}})\|_{L^p(B \setminus \hat{E})} \le C d(Q)^{\frac{2}{p}} \|e(u)\|_{L^2(Q' \cup \hat{Q}')}$$

and thus by Lemma 4.1

$$\|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le C d(Q)^2 \|e(u)\|^p_{L^2(Q' \cup \hat{Q}')} \tag{31}$$

for some $C = C(p)$. Let $N_Q = \bigcup_{\hat{Q} \in \mathcal{N}(Q)} \hat{Q}' \cup Q'$ and observe that by (26) each $x \in U$ is contained in a bounded number of different sets N_Q. Moreover, we observe that $\sum_{Q \in \mathcal{G}} d(Q)^2 \le C|V| \le C\mu^2$. Summing over all squares, recalling $\#\mathcal{N}(Q) \le C$ and using Hölder's inequality we then find

$$\begin{aligned} \sum\nolimits_{Q \in \mathcal{G}} \sum\nolimits_{\hat{Q} \in \mathcal{N}(Q)} d(Q)^{-p} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} &\le C \sum\nolimits_{Q \in \mathcal{G}} d(Q)^{2-p} \|e(u)\|^p_{L^2(N_Q)} \\ \le C \Big(\sum\nolimits_{Q \in \mathcal{G}} d(Q)^2 \Big)^{1-\frac{p}{2}} & \Big(\sum\nolimits_{Q \in \mathcal{G}} \|e(u)\|^2_{L^2(N_Q)} \Big)^{\frac{p}{2}} \le C\mu^{2-p} \|e(u)\|^p_{L^2(U)}. \end{aligned} \tag{32}$$

We observe that $\sum_{Q \in \mathcal{G}} \varphi_Q(x) = 1$ for all $x \in V$. In fact, we recall that $(\varphi_Q)_{Q \in \mathcal{Q}_W}$ is a partition of unity and $\mathrm{supp}(\varphi_Q) \subset Q''' \subset U \setminus V$ for all $Q \in \mathcal{B} = \mathcal{Q}_W \setminus \mathcal{G}$ by construction. Similarly as in the proof of Lemma 3.2 we define

$$\bar{u}(x) = \sum\nolimits_{Q \in \mathcal{G}} \varphi_Q(x)(A_Q \, x + c_Q) = \sum\nolimits_{Q \in \mathcal{G}} \varphi_Q(x) a_Q(x)$$

for all $x \in V$. Clearly, $\bar{u}$ is smooth in V. Using $\sum_{Q \in \mathcal{G}} \nabla \varphi_Q = 0$ we find that

$$\nabla \bar{u} = \sum\nolimits_{Q \in \mathcal{G}} \big((a_Q - f) \otimes \nabla \varphi_Q + \varphi_Q A_Q \big) \tag{33}$$

for all functions f. Consequently, letting $f(x) = a_Q(x)$ for $x \in Q \cap V$, $Q \in \mathcal{G}$, we derive using once more (26)(ii),(iii) and applying (27)(ii), (32)

$$\|e(\bar{u})\|^p_{L^p(V)} \leq C \sum\nolimits_{Q \in \mathcal{G}} d(Q)^{-p} \sum\nolimits_{\hat{Q} \in \mathcal{N}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q)} \leq C\mu^{2-p} \|e(u)\|^p_{L^2(U)}.$$

We compute using (26)(ii), (27)(i) and (30)(i)

$$\begin{aligned} \|\bar{u} - u\|_{L^q(V \setminus E)} &\leq C \sum\nolimits_{Q \in \mathcal{G}} \|a_Q - u\|_{L^q(Q''' \setminus E)} \leq C \sum\nolimits_{Q \in \mathcal{G}} d(Q)^{\frac{2}{q}} \|e(u)\|_{L^2(Q')} \\ &\leq C\mu^{\frac{2}{q}} \|e(u)\|_{L^2(U)}. \end{aligned} \tag{34}$$

Likewise, using (33) for $f = u$ and (30) for $q = p$ we find repeating the Hölder-type estimate in (32)

$$\begin{aligned} \|\nabla \bar{u} - \nabla u\|^p_{L^p(V \setminus E)} &\leq C \sum\nolimits_{Q \in \mathcal{G}} d(Q)^{-p} \|a_Q - u\|^p_{L^p(Q''' \setminus E)} \\ &\quad + C \sum\nolimits_{Q \in \mathcal{G}} \|\nabla u - A_Q\|^p_{L^p(Q''' \setminus E)} \\ &\leq C \sum\nolimits_{Q \in \mathcal{G}} d(Q)^{2-p} \|e(u)\|^p_{L^2(Q')} \leq C\mu^{2-p} \|e(u)\|^p_{L^2(U)}. \end{aligned} \tag{35}$$

As $\bar{u}$ is smooth in V and V is a John domain with constant only depending on $\bar{c}$, we can apply Theorem 2.6 and we find $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c \in \mathbb{R}^2$ such that by a rescaling argument

$$\mu^{-\frac{2}{q}-1+\frac{2}{p}} \|\bar{u} - (A \cdot + c)\|_{L^q(V)} + \|\nabla \bar{u} - A\|_{L^p(V)} \leq C \|e(\bar{u})\|_{L^p(V)} \leq C\mu^{\frac{2}{p}-1} \|e(u)\|_{L^2(U)}$$

for $C = C(p, q, \bar{c})$. We now define $F = E \cup P$ and by (29) and the remark below (30) we obtain $|F| \leq C(\mathcal{H}^1(J_u))^2$ as well as $\mathcal{H}^1(\partial F) \leq C\mathcal{H}^1(J_u)$. Finally, (25) follows from (34) and (35). $\square$

Remark 4.3 Similarly as in the local estimate considered in Section 3 one can show that the displacement field restricted to $U' \setminus F$ is an element of $GSBV^p$ or SBV^p, respectively. As this property will not be needed in the following, we have omitted the proof.

5 Proof of the main result and application

5.1 Proof of Theorem 3.1

We now combine the local estimate in Theorem 3.1, the boundary estimate (Theorem 4.2) and a standard covering argument to prove the main result. A similar argument may be found, e.g. in [10], where an inequality of Korn-Poincaré type is derived.

Proof of Theorem 1.1 We first choose finitely many $U_1, \dots, U_n$ being of the form given in (24) (possibly after application of an affine isometry) such that $\partial\Omega$ is covered by $U'_1, \dots, U'_n$. Moreover, we cover $\Omega \setminus \bigcup_{i=1}^n U'_i$ with squares $U'_{n+1}, \dots, U'_m$ such that the squares $2U'_{n+1}, \dots, 2U'_m$ of double size are still contained in Ω.

By a similar reasoning as in the proof of Theorem 4.2 we may suppose that $\mathcal{H}^1(J_u) \le \hat{c}$ for some $\hat{c} = \hat{c}(p,q,\Omega)$ to be specified below as otherwise we can choose $E = \Omega$ in Theorem 1.1. We now apply Theorem 3.1 and Theorem 4.2, respectively, on the sets $(U_i)_{i=1}^m$ and obtain infinitesimal rigid motions $a_i = a_{A_i,c_i}$ as well as exceptional sets $E_i \subset U_i$ such that

$$\sum_{i=1}^m \left(\|u - a_i\|_{L^q(U'_i \setminus E_i)} + \|\nabla u - A_i\|_{L^p(U'_i \setminus E_i)} \right) \le C \sum_{i=1}^m \|e(u)\|_{L^2(U_i)} \le C\|e(u)\|_{L^2(\Omega)} \tag{36}$$

for some $C = C(p,q,\Omega)$. In fact, selecting $\hat{c}$ sufficiently small we get that the shrinked squares $\hat{Q}_i$ given in Theorem 3.1 contain U'_i for $i = n+1, \dots, m$ (cf. (30) for a similar argument).

Define $E = \bigcup_{i=1}^m E_i$ and observe that $|E| \le C(\mathcal{H}^1(J_u))^2$ as well as $\mathcal{H}^1(\partial E) \le C\mathcal{H}^1(J_u)$ follow from (12) and the similar estimate for the sets at the boundary (see before (25)). Moreover, we can choose $\hat{c}$ so small such $|E| \le \frac{1}{2}\eta$, where

$$\eta := \min\{|U'_i \cap U'_j| : U'_i, U'_j, i \neq j, \text{ with } U'_i \cap U'_j \neq \emptyset\}.$$

Obviously η only depends on Ω. Consequently, we obtain $|(U'_i \cap U'_j) \setminus E| \ge \frac{1}{2}\eta$ for all U'_i, U'_j, $i \neq j$, with $U'_i \cap U'_j \neq \emptyset$. As Ω is connected, we then find by Lemma 4.1 and (36)

$$\max_{1 \le i,j \le m} \|a_i - a_j\|_{L^q(\Omega)} + \|A_i - A_j\|_{L^p(\Omega)} \le C\|e(u)\|_{L^2(\Omega)}$$

for a constant depending only on p, q, η and m. Recalling (36) and the fact that η, m only depend on Ω we finally obtain (4) for, e.g., $A = A_1$ and $c = c_1$. $\square$

5.2 Relation between SBV and SBD functions

Finally, we present a consequence of our main result concerning the relation between SBV and SBD functions. We briefly recall that the typical examples for functions lying in BD but not in BV or likewise lying in SBD^p but not in SBV^p, $p > 1$, are based on the idea to cut out small balls and to choose the displacement field appropriately on these sets (see e.g. [3, 14]). The following result shows that this construction essentially describes the only way to obtain functions of bounded deformation which do not have bounded variation. In particular, we see that for each function in $SBD^2 \cap L^\infty$ there is a modification $\tilde{u}$ in SBV such that $\{u \neq \tilde{u}\}$ is an arbitrarily small set.

Theorem 5.1 *Let $\varepsilon > 0$ and let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Then for every $u \in SBD^2(\Omega)$ we find an exceptional set E with $|E| \leq \varepsilon$ and $\mathcal{H}^1(\partial E) < +\infty$ such that $u\chi_{\Omega\setminus E} \in GSBV^p(\Omega)$ for all $p < 2$. If in addition $u \in L^\infty(\Omega)$, we obtain $u\chi_{\Omega\setminus E} \in SBV^p(\Omega)$.*

Proof. The statement follows from Theorem 3.1 by an additional covering argument. Assume first $u \in SBD^2(\Omega) \cap L^\infty(\Omega)$. Recalling (14) we cover Ω with squares in $\mathcal{Q}_s$ for $s \ll \varepsilon$ to be specified below. We define the bad squares $\mathcal{B} = \{Q \in \mathcal{Q}_s : Q' \not\subset \Omega \text{ or } \mathcal{H}^1(J_u \cap Q') \geq \bar{c}s\}$ for a constant $\bar{c} > 0$. We let $F = \Omega \cap \bigcup_{Q\in\mathcal{B}} \overline{Q}$ and observe $\mathcal{H}^1(\partial F) \leq C\mathcal{H}^1(J_u) + C\mathcal{H}^1(\partial\Omega) < +\infty$ as well as $|F| \leq Cs(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))$.

Choosing $\bar{c}$ sufficiently small we can apply Theorem 3.1 on each enlarged square $Q', Q \in \mathcal{Q}_s \setminus \mathcal{B}$, and obtain exceptional sets E_Q such that $\big(u\chi_{Q''\setminus E_Q}\big)|_{Q'} \in SBV^p(Q')$ for all $Q \in \mathcal{Q}_s \setminus \mathcal{B}$. (In fact, for $\bar{c}$ small we can assume that the shrinked square $\hat{Q}$ given in Theorem 3.1 contains Q''.)

Letting $E = \bigcup_Q E_Q \cup F$ we find by (12) that $\mathcal{H}^1(\partial E) < +\infty$ and $|E| \leq Cs$ for a constant depending only on Ω and u. Thus, $|E| \leq \varepsilon$ for s sufficiently small. Then defining $\bar{u} = u\chi_{\Omega\setminus E}$ in $\Omega \setminus F$ we derive $\bar{u} \in SBV^p(\Omega \setminus F)$ for all $p \in [1, 2)$. Observe that

$$D(u\chi_{\Omega\setminus E}) = D\bar{u} + (\bar{u} \otimes \xi_F)\, \mathcal{H}^1|_{\partial F\cap\Omega} \tag{37}$$

in Ω, where ξ_F denotes the inner normal of F (see e.g. [4, Theorem 3.87]). As $\|u\|_\infty < +\infty$, this implies $u\chi_{\Omega\setminus E} \in SBV^p(\Omega)$. Likewise, in the general case we consider $\phi \in C^1(\mathbb{R}^2)$ with the support of $\nabla\phi$ compact and find $\big(\phi(u\chi_{\Omega\setminus E})\big)|_{\Omega\setminus F} \in SBV^p(\Omega \setminus F)$. Then we repeat the argument in (37) to conclude $\phi(u\chi_{\Omega\setminus E}) \in SBV^p(\Omega)$. ☐

The above result can also be interpreted as an approximation result for SBD functions. On the one hand, it is weaker than standard density results, see e.g. [9], as it does not lead to a fine estimate for the surface energy. On the other hand, whereas in results based on interpolation arguments the approximating sequences typically only converge in L^p, in the present context we see that the functions already coincide up to a set of arbitrarily small measure.

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