A derivation of linearized Griffith energies from nonlinear models

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Abstract

We derive Griffith functionals in the framework of linearized elasticity from nonlinear and frame indifferent energies in brittle fracture via Γ-convergence. The convergence is given in terms of rescaled displacement fields measuring the distance of deformations from piecewise rigid motions. The configurations of the limiting model consist of partitions of the material, corresponding piecewise rigid deformations and displacement fields which are defined separately on each component of the cracked body. Apart from the linearized Griffith energy the limiting functional comprises also the segmentation energy which is necessary to disconnect the parts of the specimen.

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Contents

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1 Introduction

A thorough understanding of crack formation in brittle materials is of great interest in both experimental sciences and theoretical studies. Starting with the seminal contribution by Francfort and Marigo [21], where the displacements and crack paths are determined from an energy minimization principle, various variational models in the framework of free discontinuity problems have appeared in the literature over the past years. These so-called Griffith functionals comprising elastic and surface contributions generalize the original Griffith theory (see [27]) which is based on the fundamental idea that the formation of fracture may be regarded as the competition of elastic bulk and surface energies.

For the sake of a simplified mathematical description the investigation of fracture models in the realm of linearized elasticity is widely adopted (see e.g. [2, 4, 7, 12, 13, 28]) and has led to a lot of realistic applications in engineering as well as to efficient numerical approximation schemes (we refer to [5, 6, 11, 20, 31, 32, 36] making no claim to be exhaustive). On the contrary, their nonlinear counterparts are usually significantly more difficult to treat since in the regime of finite elasticity the energy density of the elastic contributions is genuinely geometrically nonlinear due to frame indifference rendering the problem highly non-convex. Consequently, in contrast to linear models already the fundamental question if minimizing configurations for given boundary data exist at all is a major difficulty. Even more challenging tasks in this context are the determination of the material behavior under expansion or compression, in particular the derivation of specific cleavage laws.

Consequently, for a deeper understanding of nonlinear models the identification of an effective linearized theory is desirable as in this way one may rigorously show that in the small displacement regime the neglection of effects arising from the non-linearities is a good approximation of the problem. Moreover, such a derivation is also interesting in the context of discrete systems. Previous investigations which were motivated by the analysis of cleavage laws for brittle crystals (see [23, 24] or the seminal paper [9]) have shown that the most interesting regime for the elastic strains is given by $\sqrt{\varepsilon}$, where ε denotes the typical interatomic distance. Consequently, a passage from discrete-to-continuum systems naturally involves a simultaneous linearization process.

In elasticity theory the nonlinear-to-linear limit is by now well understood in various different settings via Γ-convergence (cf. [10, 17, 34, 35]), where the passage is performed in terms of suitably rescaled displacement fields measuring the distance of the deformation from a rigid motion and being the fundamental quantity on which the linearized elastic energy depends. In fracture mechanics, however, the relation between the deformation of a material and corresponding displacements is more complicated since the body may be disconnected by the jump set into various components. In fact, it turns out that, without passing to rescaled configurations, in the small strain limit nonlinear Griffith energies converge to a limiting functional which is finite for piecewise rigid motions and measures the *segmentation energy* which is necessary to disconnect the body.

Obviously a major drawback of this simple limiting model appears to be the fact that it does not capture the elastic deformations which are typically present in the nonlinear models. Consequently, in order to arrive at a limiting model showing coexistence of elastic and surface contributions it is indispensable to pass to rescaled configurations similarly as in [17]. The goal of this article is to identify such an effective linearized Griffith energy as the Γ-limit of nonlinear and frame indifferent models in the small strain regime. To the best of our knowledge such a result has not yet been derived in the general setting of free discontinuity problems introduced by Ambrosio and De Giorgi [18].

The farthest reaching result in this direction seems to be a recent contribution by Negri and Toader [33] where a nonlinear-to-linear analysis is performed in the context of quasistatic evolution for a restricted class of admissible cracks. In particular, in their model the different components of the jump set are supposed to have a least positive distance rendering the problem considerably easier from an analytical point of view. In particular, the specimen cannot be separated into different parts effectively leading to a simple relation between the deformation and the rescaled displacement field. On the other hand, in [25] we have performed a simultaneous discrete-to-continuum and nonlinear-to-linear analysis for general crack geometries, but under the simplifying assumption that all deformations lie close to the identity mapping.

In the present context we establish a limiting linearized Griffith functional in a planar setting without any a priori assumptions on the deformation and the crack geometry. We identify an effective model which appears to be more general than the energies which are widely investigated in the literature. Whereas in elasticity theory, in the approaches [25, 33] mentioned before and in most linear fracture models there is a simple relation between the deformation of the material and the associated infinitesimal displacement field, in our framework the deformation is related to a triple consisting of a partition of the domain, a corresponding piecewise rigid motion being constant on each connected component of the cracked body and a displacement field which is defined separately on each piece of the specimen.

On each component of the partition the energy is of Griffith-type in the realm of linearized elasticity. In addition, the functional contains the segmentation energy which is necessary to disconnected the parts of the body. In particular, the latter contribution is a specific feature of our general model where we do not restrict the analysis to a linearization around a fixed rigid motion.

Let us briefly note that although all arguments used in the proofs of this article are valid in any space dimension, we have to restrict our analysis to two dimensions as one of the ingredients of our analysis, an SBD-rigidity result (see [22]), has only been derived in a planar setting for isotropic surface energies. However, we believe that the estimate in [22] may be generalized in the future and then the generalizations for the results in the work at hand immediately follow.

As an application of our result we present a cleavage law in a continuum setting with isotropic surface energies. As discussed before, the identification of critical loads and the investigation of crack paths is a challenging problem particularly for nonlinear models. The arguments in [23, 24, 30], where boundary value problems of uniaxial extension for brittle materials were investigated, fundamentally relied on the application of certain slicing techniques and due to the lack of convexity were not adapted to treat the case of compression. Our general Γ-limit result can now be applied to solve also boundary value problems of uniaxial compression which is as the uniaxial tension test a natural and interesting problem. Hereby we may complete the picture about the derivation of cleavage laws in [23, 24].

One essential point in our investigation is the establishing of a compactness result providing limiting configurations which consist of piecewise rigid motions and corresponding displacement fields. Similarly as in the derivation of linearized systems for elastic materials (see e.g. [17]), where the main ingredient is a quantitative geometric rigidity estimate by Friesecke, James and Müller [26], the starting point of our analysis is a quantitative SBD-rigidity result (see [22]) in the framework of special functions of bounded deformation (see [2, 4]), which is tailor-made for general Griffith models with coexistence of both both energy forms.

As there is no uniform bound on the functions, it turns out that the limiting displacements are generically not summable and we naturally end up in the space of GSBD functions (for the definition and basic properties we refer to [15]). We believe that our results are interesting also outside of this specific context as they allow to solve more general variational problems in fracture mechanics. Typically, for compactness results in function spaces as SBV (see [3] for the definition and basic properties) and SBD one needs L^∞ or L^1 bounds on the functions (see [1, 4, 15]). However, in many applications, in particular for atomistic systems and for models dealing with rescaled deformations, such estimates cannot be inferred from energy bounds. Nevertheless, we are able to treat problems without any a priori bound by passing from the deformations to displacement fields whose distance from rigid motions can be controlled.

The other essential point in our analysis is the investigation of the limiting configurations. In particular, we study the properties of the partition which disconnects the body into various parts. It turns out that an even finer segmentation may occur if on a connected component of the partition the jump set of the corresponding displacement field further separates the body. Here it becomes apparent that we treat a real multiscale model as the jump heights at the boundaries associated to the coarse partition are of order $\gg \sqrt{\varepsilon}$ ($\sqrt{\varepsilon}$ denotes the regime of the typical elastic strain), whereas the jump heights of the finer partition are of order $\sqrt{\varepsilon}$. Moreover, it is evident that the choice of the limiting partition is not unique. However, we propose a selection principle and show existence and uniqueness of a *coaresest partition*.

The paper is organized as follows. In Section 2 we state the main compactness and Γ-convergence results and discuss properties of the limiting linearized Griffith functional. Moreover, we present our application to cleavage laws for uniaxially extended or compressed brittle materials.

Section 3 is devoted to some preliminaries. We first give the definition of special functions of bounded variation and deformation and discuss basic properties. Afterwards, we recall the notion of Caccioppoli partitions which will be fundamental in our analysis to analyze the properties of limiting configurations. Moreover, we recall geometric rigidity results for elastic and brittle materials, in particular the SBD-rigidity result proved in [22].

In Section 4 we then establish the main compactness result for a sequence of deformations $(y_\varepsilon)_\varepsilon$, where ε stands for the order of the elastic energy. First, the convergence of the partitions and the corresponding rigid motions is based on compactness theorems for Caccioppoli partitions and piecewise constant functions (see [3] or Section 3.2 below).

Although the SBD-rigidity estimate is a fundamental ingredient in our analysis giving L^2 bounds for rescaled displacement fields, we still have to face major difficulties since the rigidity estimate provides a family of displacement fields $(u^\rho_\varepsilon)^\rho_\varepsilon$ with an additional parameter ρ representing a 'modification error' between y_ε and u^ρ_ε. Consequently, the goal will be to choose an appropriate diagonal sequence.

An additional challenge is the fact that the bounds in the SBD-rigidity estimate depend on ρ and explode for $\rho \to 0$. For the symmetric part of the gradient this problem can be bypassed by a Taylor expansion taking the nonlinear elastic energy ε and a higher order term into account, which shows that the constant may be chosen independently of ρ. For the function itself, however, the problem is more subtle since a uniform bound cannot be inferred by energies bounds. In particular, generically the limiting configurations are not in L^2, but only finite almost everywhere. The strategy to establish the latter assertion is to show that for fixed ε the functions $(u^\rho_\varepsilon)_\rho$ essentially coincide in a certain sense on the bulk part of the domain. Afterwards, by a careful analysis we can derive that such a property is preserved in the limit $\varepsilon \to 0$, whereby we can establish a kind of equi-integrability of the configurations.

In Section 5 we concern ourselves with the limiting configurations consisting of a partition, a corresponding piecewise rigid motion and a displacement field. Recalling that genuinely the limits provided by the compactness result are highly non-unique we introduce the notion of a *coarsest partition*. Roughly speaking, the definition states that the jump heights at the boundaries associated to this partition are of order $\gg \sqrt{\varepsilon}$ leading to a meaningful mathematical description of the observation that the size of the crack opening is a multiscale phenomenon in our model.

The fundamental point is the proof of existence and uniqueness of the coarsest partition. Uniqueness follows from the fact that under the assumption that there are two different coarsest partitions one always can find an even coarser partition. Existence is a more challenging problem. We first give an alternative characterization and identify coarsest partitions as the maximal elements of the partial order on the set of admissible partitions which is induced by subordination. We then show that each chain of the partial order has an upper bound repeating some arguments of the main compactness result. Consequently, the claim is inferred by an application of Zorn's lemma. Finally, having found the coarsest partition we can then show that the corresponding admissible displacement field is uniquely determined up to piecewise infinitesimal rigid motions.

In Section 6.1 we derive the main Γ-limit, where the elastic part can be treated as in [26, 35] and for the surface energy we separate the effects arising from the segmentation energy and the crack energy inside the components by employing a structure theorem for Caccioppoli partitions (see Theorem 3.5 below).

Finally, in Section 6.2 we prove a cleavage law and extend the results obtained in [23, 24, 30] to the case of uniaxial compression, where we essentially follow the proof in [25, 30], in particular using a piecewise rigidity result in SBD (see [14]) and a structure theorem on the boundary of sets of finite perimeter (see [19]). It turns out that in the linearized limit the behavior for compression and extension is virtually identical. We briefly note that to avoid unphysical effects such as self-penetrability further modeling assumptions would be necessary.

2 The model and main results

2.1 The nonlinear model

Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Recall the properties of the space $SBV(\Omega, \mathbb{R}^2)$, frequently abbreviated as $SBV(\Omega)$ hereafter, in Section 3.1.

For $M > 0$ we define

$$SBV_M(\Omega) = \Big\{ y \in SBV(\Omega, \mathbb{R}^2) : \|y\|_\infty + \|\nabla y\|_\infty \le M, \ \mathcal{H}^1(J_y) < +\infty \Big\}. \quad (2.1)$$

Let $W : \mathbb{R}^{2\times 2} \to [0, \infty)$ be a frame-indifferent stored energy density with $W(F) = 0$ iff $F \in SO(2)$. Assume that W is continuous, C^3 in a neighborhood of $SO(2)$ and scales quadratically at $SO(2)$ in the direction perpendicular to infinitesimal rotations. In other words, we have $W(F) \ge c\,\mathrm{dist}^2(F, SO(2))$ for all $F \in \mathbb{R}^{2\times 2}$ and a positive constant c. For $\varepsilon > 0$ define the Griffith-energy $E_\varepsilon : SBV_M(\Omega) \to [0, \infty)$ by

$$E_\varepsilon(y) = \frac{1}{\varepsilon} \int_\Omega W(\nabla y(x))\, dx + \mathcal{H}^1(J_y). \quad (2.2)$$

We briefly note that we can also treat inhomogeneous materials where the energy density has the form $W : \Omega \times \mathbb{R}^{2\times 2} \to [0, \infty)$. Moreover, it suffices to assume $W \in C^{2,\alpha}$, where $C^{2,\alpha}$ is the Hölder space with exponent $\alpha > 0$.

Observe that M may be chosen arbitrarily large (but fixed) and therefore the constraint $\|\nabla y\|_\infty \le M$ is not a real restriction as we are interested in the small displacement regime in the regions of the domain where elastic behavior occurs. For instance, the uniform bound on the absolute continuous part of the gradient is natural when dealing with discrete energies where the corresponding deformations are piecewise affine on cells of microscopic size (see e.g. [8, 25]). (Note that in discrete systems the parameter ε represents not only the order of the elastic energy, but also the typical interatomic distance.) Moreover, the uniform bound on the function is assumed only to simplify the exposition and may be dropped.

The main goal of the present work is the identification of an effective linearized Griffith energy in the small strain limit which is related to the nonlinear energies E_ε through Γ-convergence. We will also investigate the limiting model which appears to be more general than many other Griffith functionals in the realm of linearized elasticity (cf. e.g. [7, 12, 13, 28, 36]) as the limiting configuration not only consists of a displacement field, but also of a coarse partition of the domain and associated rigid motions. Moreover, it will turn out that there are various scales for the size of the crack opening occurring in the system.

2.2 The segmentation problem

As a first natural approach to the problem we concern ourselves with the question if the functionals E_ε can be related to a limiting functional for $\varepsilon \to 0$ in terms of the deformations. We observe that for configurations with uniformly bounded energy $E_\varepsilon(y_\varepsilon)$ the absolute continuous part of the gradient satisfies $\nabla y_\varepsilon \approx SO(2)$ as the stored energy density is frame-indifferent and minimized on $SO(2)$. Assuming that $y_\varepsilon \to y$ in L^1, one can show that $\nabla y \in SO(2)$ a.e. applying lower semicontinuity results for SBV functions (see [29]) and the fact that the quasiconvex envelope of W is minimized exactly on $SO(2)$ (see [37]).

A piecewise rigidity result by Chambolle, Giacomini and Ponsiglione (see Theorem 3.9 below) generalizing the classical Liouville result for smooth functions now states that an SBV function y satisfying the constraint $\nabla y \in SO(2)$ a.e. is a collection of an at most countable family of rigid deformations, i.e. the body may be divided into different components each of which subject to a different rigid motion.

Consequently, the limit of the sequence E_ε (in the sense of Γ-convergence) is given by the functional which is finite for piecewise rigid motions and measures the *segmentation energy* which is necessary to disconnect the body. The exact statement is formulated in Corollary 2.7 as a direct consequence of our main Γ-convergence result in Theorem 2.6.

Apparently it is a major drawback of this simple limiting model that it does not account for the elastic deformations which are typically present in the nonlinear models. Consequently, to obtain a better understanding of the problem it is desirable to pass to rescaled configurations and to derive a limiting linearized energy as it was performed in [17] in the framework of nonlinear elasticity theory. The main ingredient in that analysis is a quantitative rigidity result due to Friesecke, James and Müller (see Theorem 3.8). The starting point for our analysis will be a corresponding quantitative result in the SBD setting (see [22] or Theorem 3.10) adapted for Griffith functionals of the form (2.2) where both elastic bulk and surface contributions are present.

2.3 Compactness

We will now present our main compactness result for rescaled displacement fields. As a preparation, recall the notion and basic properties of a *Caccioppoli partition* in Section 3.2. For a given (ordered) Caccioppoli partition $\mathcal{P} = (P_j)_j$ of Ω let

$$\mathcal{R}(\mathcal{P}) = \Big\{T : \Omega \to \mathbb{R}^2 : T(x) = \sum\nolimits_j \chi_{P_j}(R_j\, x + c_j), R_j \in SO(2), c_j \in \mathbb{R}^2\Big\} \quad (2.3)$$

be the set of corresponding piecewise rigid motions. Likewise we define the set of piecewise infinitesimal rigid motions, denoted by $\mathcal{A}(\mathcal{P})$, replacing $R_j \in SO(2)$ by $A_j \in \mathbb{R}^{2\times 2}_{\rm skew} = \{A \in \mathbb{R}^{2\times 2} : A = -A^T\}$. Moreover, we define the triples

$$\begin{aligned}\mathcal{D} &:= \big\{(u, \mathcal{P}, T) :\ u \in SBV(\Omega),\ \mathcal{P} \text{ C.-partition of } \Omega,\ T \in \mathcal{R}(\mathcal{P})\big\},\\ \mathcal{D}_\infty &:= \big\{(u, \mathcal{P}, T) :\ \mathcal{P} \text{ C.-partition of } \Omega,\ T \in \mathcal{R}(\mathcal{P}),\ (\nabla T)^T u \in GSBD^2(\Omega, \mathbb{R}^2)\big\}.\end{aligned}$$

The space $GSBD^2(\Omega, \mathbb{R}^2)$, abbreviated by $GSBD^2(\Omega)$ hereafter, generalizes the definition of the space $SBD(\Omega)$ based on certain slicing properties, see Section 3.1. Define $e(G) = \frac{G^T + G}{2}$ for all $G \in \mathbb{R}^{2\times 2}$. We now formulate the main compactness theorem.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $M > 0$ and $\varepsilon_k \to 0$. If $E_{\varepsilon_k}(y_k) \le C$ for a sequence $y_k \in SBV_M(\Omega)$, then there exists a subsequence (not relabeled) such that the following holds:*
There are triples $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$, where $\mathcal{P}^k = (P_j^k)_j$ and

$$\begin{aligned} &(i) \;\; u_k(x) - \varepsilon_k^{-1/2}(y_k(x) - T_k(x)) \to 0 \;\; a.e., \\ &(ii) \;\; \frac{1}{\varepsilon_k}\int_\Omega W(\mathbf{Id} + \sqrt{\varepsilon_k}\nabla T_k^T \nabla u_k) \le \frac{1}{\varepsilon_k}\int_\Omega W(\nabla y_k) + o(1) \end{aligned} \tag{2.4}$$

for $\varepsilon_k \to 0$, such that we find a limiting triple $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ with

$$\begin{aligned} &(i) \;\; \chi_{P_j^k} \to \chi_{P_j} \quad \text{in measure for all } j \in \mathbb{N}, \\ &(ii) \;\; T_k \to T \text{ in } L^2(\Omega), \quad \nabla T_k \to \nabla T \text{ in } L^2(\Omega), \end{aligned} \tag{2.5}$$

for $k \to \infty$. Moreover, we get

$$\begin{aligned} &(i) \;\; u_k \to u \quad a.e. \text{ in } \Omega, \\ &(ii) \;\; e(\nabla T_k^T \nabla u_k) \rightharpoonup e(\nabla T^T \nabla u) \quad \text{weakly in } L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \\ &(iii) \;\; \|\nabla u_k\|_{L^\infty(\Omega)} \le C\varepsilon_k^{-1/8}, \end{aligned} \tag{2.6}$$

for $k \to \infty$ and for the surface energy we obtain

$$\liminf_{k\to\infty} \mathcal{H}^1(J_{y_k}) \ge \frac{1}{2}\sum\nolimits_j P(P_j, \Omega) + \mathcal{H}^1(J_u \setminus \partial P), \tag{2.7}$$

where $\partial P := \bigcup_j \partial^* P_j$.

Here ∂^* denotes the *essential boundary* (see (3.8)). If we drop the condition $\|y\|_\infty \le M$ in the definition of $SBV_M(\Omega)$, then (2.5) only holds for the derivatives of the piecewise rigid motions. In the following we say a triple $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ converges to $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ and write $(u_k, \mathcal{P}^k, T_k) \to (u, \mathcal{P}, T)$ if (2.4)-(2.7) are satisfied.

In our analysis of the limiting model we have to face a major difficulty concerning the fact that the limiting triples $(u, \mathcal{P}, T)$ given by the main compactness theorem for a sequence $(y_k)_k$ are not determined uniquely, but crucially depend on the choice of the sequences $(\mathcal{P}^k)_k$ and $(T_k)_k$. To illustrate this problem we consider the following simple example.

Example 2.2 Consider $\Omega = (0,1)\times(0,1)$, $\Omega_1 = (0,1)\times(0,\frac{1}{2})$, $\Omega_2 = (0,1)\times(\frac{1}{2},1)$ and

$$y_k = \mathbf{id}\chi_{\Omega_1} + (\mathbf{id} + a\sqrt{\varepsilon_k})\chi_{\Omega_2}$$

for $a \in \mathbb{R}^2$. Then possible alternatives are e.g. (1) $P^1 = \Omega$ with $R_1^1\,x + c_1^1 = \mathbf{id}$ or (2) $P_1^2 = \Omega_1$, $P_2^2 = \Omega_2$ with $R_1^2\,x + c_1^2 = \mathbf{id}$ and $R_2^2\,x + c_2^2 = \mathbf{id} + a\sqrt{\varepsilon_k}$. Letting

$u_k^1 = {\varepsilon_k}^{-\frac{1}{2}}\big(y_k - \mathbf{id}\big)$ and $u_k^2 = {\varepsilon_k}^{-\frac{1}{2}}\big(y_k - \mathbf{id}\chi_{\Omega_1} - (\mathbf{id} + a\sqrt{\varepsilon_k})\chi_{\Omega_2}\big)$ we obtain in the limit $\varepsilon_k \to 0$ two different configurations:

$$
\begin{aligned}
&u^1 = 0 \cdot \chi_{\Omega_1} + a\chi_{\Omega_2}, \quad P_1^1 = \Omega, \\
&u^2 = 0, \quad P_1^2 = \Omega_1, P_2^2 = \Omega_2.
\end{aligned}
$$

Clearly, we can equally well consider an example where we vary the rotations, e.g.

$$
y_k(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x\ \chi_{\Omega_1}(x) + \begin{pmatrix} \cos a\sqrt{\varepsilon_k} & \sin a\sqrt{\varepsilon_k} \\ -\sin a\sqrt{\varepsilon_k} & \cos a\sqrt{\varepsilon_k} \end{pmatrix} x\ \chi_{\Omega_2}(x)
$$

for $a \in \mathbb{R}$.

We now introduce a special subclass of partitions in which uniqueness will be guaranteed. The above example already shows that different partitions are not equivalent in the sense that they may contain a different 'amount of information'. Note that on the various elements of the partition the configuration u is defined separately and the different pieces of the domain are not 'aware of each other'. In particular, the possible discontinuities of u on ∂P do not have any physically reasonable interpretation. On the contrary, in the first example where we did not split up the domain, we gain the jump height as an additional information. The observation that coarser partitions provide more information about the behavior at the jump set motivates the definition of the *coarsest partition*.

Definition 2.3 Let $(y_k)_k$ be a given (sub-)sequence as in Theorem 2.1.

(i) We say a partition $\mathcal{P}$ of Ω is *admissible* for $(y_k)_k$ and write $\mathcal{P} \in \mathcal{Z}_P((y_k)_k)$ if there are triples $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ for $k \in \mathbb{N}$ as well as u, T such that $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ and $(u_k, \mathcal{P}^k, T_k) \to (u, \mathcal{P}, T)$.

(ii) We say a piecewise rigid motion T is *admissible* for $(y_k)_k$ and $\mathcal{P}$ writing $T \in \mathcal{Z}_T((y_k)_k, \mathcal{P})$ if there are triples $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ for $k \in \mathbb{N}$ as well as u such that $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ and $(u_k, \mathcal{P}^k, T_k) \to (u, \mathcal{P}, T)$.

(iii) We say a configuration u is *admissible* for $(y_k)_k$ and $\mathcal{P}$ and write $u \in \mathcal{Z}_u((y_k)_k, \mathcal{P})$ if there are triples $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ for $k \in \mathbb{N}$ as well as T such that $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ and $(u_k, \mathcal{P}^k, T_k) \to (u, \mathcal{P}, T)$.

(iv) We say a partition $\mathcal{P}$ of Ω is a *coarsest partition* for $(y_k)_k$ if the following holds: The partition is admissible, i.e. $\mathcal{P} \in \mathcal{Z}_P((y_k)_k)$, and for all admissible $u \in \mathcal{Z}_u((y_k)_k, \mathcal{P})$ the corresponding piecewise rigid motions $T_k = \sum_j (R_j^k\, x + c_j^k)\chi_{P_j^k}$ given by (iii) satisfy

$$
\frac{|R_{j_1}^k - R_{j_2}^k| + |c_{j_1}^k - c_{j_2}^k|}{\sqrt{\varepsilon_k}} \to \infty \tag{2.8}
$$

for all $j_1, j_2 \in \mathbb{N}$, $j_1 \neq j_2$ and $k \to \infty$.

In Lemma 5.2 below we find an equivalent characterization of coarsest partitions being the maximal elements of the partial order on the sets of admissible partitions which is induced by subordination. Loosely speaking, the above definition particularly implies that given a coarsest partition a region of the domain is partitioned into different sets $(P_j)_j$ if and only if the jump height of the approximating sequence u_k tends to infinity on $(\partial^* P_j)_j$.

Recall the definition of the piecewise infinitesimal rigid motions $\mathcal{A}(\mathcal{P})$ in (2.3). We now obtain a unique characterization of the limiting configuration up to piecewise infinitesimal rigid motions.

Theorem 2.4 *Let $\varepsilon_k \to 0$ be given. Let $E_{\varepsilon_k}(y_k) \le C$ for a sequence $y_k \in SBV_M(\Omega)$ and let $(y_{k_n})_{n\in\mathbb{N}}$ be a subsequence for which the assertion of Theorem 2.1 holds. Then we have the following:*

(i) There is a unique $T \in \mathcal{Z}_T((y_{k_n})_n, \mathcal{P})$ for all $\mathcal{P} \in \mathcal{Z}_P((y_{k_n})_n)$.

(ii) There is a unique coarsest partition $\bar{\mathcal{P}}$ of Ω.

(iii) Given some $u \in \mathcal{Z}_u((y_{k_n})_n, \bar{\mathcal{P}})$ all possible limiting configurations are of the form $u + \nabla T\mathcal{A}(\bar{\mathcal{P}})$, i.e. the limiting configuration is determined uniquely up to piecewise infinitesimal rigid motions.

We finish this part about the compactness result with a remark concerning similar estimates in a geometrically linear setting.

Remark 2.5 One can also derive a compactness result similar to Theorem 2.1 for linearized Griffith energies of the form

$$\int_\Omega |e(\nabla u)(x)|^2 \, dx + \mathcal{H}^1(J_u). \tag{2.9}$$

The essential difference is that the rigid motions are elements of $\mathcal{A}(\mathcal{P})$ instead of $\mathcal{R}(\mathcal{P})$, in (2.4) we consider the linearized elastic energy and estimate (2.6)(iii) may be omitted. To see this, one can essentially follow the proof of Theorem 2.1 replacing the nonlinear rigidity estimate (see Theorem 3.10) by a corresponding estimate in a geometrically linearized setting (see [22, Theorem 2.3]). As the statements are very similar, we omit the details here for the sake of simplicity.

We note that such a result allows to solve more general variational problems for fracture mechanics in the realm of linearized elasticity. For technical reasons dealing with energy functionals with the main energy term (2.9) often an a priori L^∞ bound is imposed in the literature (see e.g. [4, 13, 25, 36]) such that compactness results in SBD can be applied. Possible alternatives are to add a term of the form $\int_{J_u} |[u] \odot \nu_u| \, d\mathcal{H}^1$ giving control over the jump height (see e.g. [4]).

Recently, the space of *generalized functions of bounded deformation* was introduced to overcome this difficulty. In this framework it suffices to assume an

L^1 bound on the function u similarly as in the compactness results for GSBV, i.e. variational problems for energy functionals of the form (2.9) with an additional term $\|u\|_{L^1(\Omega)}$ are treatable. In fact, in many situations such a lower order term is present, see e.g. [1, 28]. However, there are also applications where the existence of lower order terms can not be expected such as the work at hand which deals with the passage to rescaled configurations. Moreover, in a wide class of problems arising from discrete energies one typically does not have an L^1 bound for the functions as the energies only depend on the relative distance of the material points.

The aforementioned result sheds a new light on this problem. By subtracting suitable infinitesimal rigid motions on a partition of the domain (cf. (2.4)(i)) we indeed may derive a compactness result for energies (2.9) without any extra term.

2.4 The limiting linearized model and Γ-convergence

We now introduce the limiting linearized model, discuss briefly its properties and show that it can be identified as the Γ-limit of the nonlinear energies E_ε. Let $Q = D^2W(\mathbf{Id})$ be the Hessian of the stored energy density W at the identity. Define $E : \mathcal{D}_\infty \to [0,\infty)$ by

$$E(u,\mathcal{P},T) = \int_\Omega \frac{1}{2}Q(e(\nabla T^T \nabla u)) + \mathcal{H}^1(J_u \setminus \partial P) + \frac{1}{2}\sum\nolimits_j P(P_j,\Omega), \qquad (2.10)$$

where as before $\mathcal{P} = (P_j)_j$ and $\partial P = \bigcup_j \partial^* P_j$. Recall that a configuration of the limiting model consists of a partition, a corresponding piecewise rigid motion and a displacement field.

The surface energy of E has two parts. Similarly as discussed in Section 2.2, on the right we have the *segmentation energy* which is necessary to disconnected the components of the body. Moreover, on the left we have the *inner crack energy* associated to the discontinuity set of the displacement field in each part of the material. Whereas the first two terms of the functional typically appear in the study of linearized Griffith energies, the segmentation energy is a characteristic feature of our general model where the analysis is not restricted to a linearization around a fixed rigid motion.

We now present our Γ-convergence result. Recall that we say $(u_k,\mathcal{P}^k,T_k) \to (u,\mathcal{P},T)$ if (2.4)-(2.7) hold.

Theorem 2.6 *Let $\varepsilon_k \to 0$. Then E_{ε_k} Γ-converge to E with respect to the convergence given in Theorem 2.1, i.e.*

(i) $\Gamma-\liminf$ *inequality: For all $(u,\mathcal{P},T) \in \mathcal{D}_\infty$ and for all sequences $(y_k)_k \subset SBV_M(\Omega)$ and corresponding $(u_k,\mathcal{P}^k,T_k) \in \mathcal{D}$ as given in Theorem 2.1 such that $(u_k,\mathcal{P}^k,T_k) \to (u,\mathcal{P},T)$ we have*

$$\liminf_{k\to\infty} E_{\varepsilon_k}(y_k) \geq E(u,\mathcal{P},T).$$

(ii) Existence of recovery sequences: For every $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ with $u \in L^2(\Omega)$ we find a sequence $(y_k)_k \subset SBV_M(\Omega)$ and corresponding $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ such that $(u_k, \mathcal{P}^k, T_k) \to (u, \mathcal{P}, T)$ and

$$\lim_{k\to\infty} E_{\varepsilon_k}(y_k) = E(u, \mathcal{P}, T).$$

Observe that for configurations $(u, \bar{\mathcal{P}}, T)$ defined in terms of the coarsest partition $\bar{\mathcal{P}}$ we now have an additional interpretation for the crack opening of the approximating deformations y_ε: (i) The jumps on ∂P are associated to jump heights $\gg \sqrt{\varepsilon}$ and (ii) the jump heights corresponding to the inner crack energy are of the order $\sqrt{\varepsilon}$. In fact, (i) follows from (2.8) and (ii) is a consequence of (2.4)(i). Additionally, observe that on one component P of $\bar{\mathcal{P}}$ the body may still be disconnected by the jump set $P \cap J_u$ forming a finer partition of the specimen. However, in contrast to the boundary of $\bar{\mathcal{P}}$ the jump heights $[u]_{P\cap J_u}$ have a meaningful physical interpretation.

Finally, as a direct consequence of Theorem 2.6 we get that the Γ-limit is given by the segmentation energy if we do not pass to rescaled configurations.

Corollary 2.7 *Let $\varepsilon_k \to 0$. Then E_{ε_k} Γ-converge to $E_{\rm seg}$ with respect to the $L^1(\Omega)$-convergence, where*

$$E_{\rm seg}(y) = \begin{cases} \frac{1}{2}\sum_j P(P_j, \Omega) & y = T \in \mathcal{R}(\mathcal{P}) \ \textit{for a Caccioppoli partition } \mathcal{P}, \\ +\infty & \textit{else.} \end{cases}$$

2.5 An application to cleavage laws

In fracture mechanics it is a major challenge to identify critical loads at which a body fails and to determine the geometry of crack paths that occur in the fractured regime. As an application of the above results we now finally derive such a cleavage law. We consider a special boundary value problem of uniaxial compression/extension. Let $\Omega = (0, l) \times (0, 1)$, $\Omega' = (-\eta, l + \eta) \times (0, 1)$ for $l > 0$, $\eta > 0$ and for $a_\varepsilon \in \mathbb{R}$ define

$$\mathcal{A}(a_\varepsilon) := \{y \in SBV_M(\Omega') : y_1(x) = (1 + a_\varepsilon)x_1 \text{ for } x_1 \le 0 \text{ or } x_1 \ge l\}.$$

As usual in the theory of SBV functions the boundary values have to be imposed in small neighborhoods of the boundary. In what follows the elastic part of the energy (2.2) still only depends on $y|_\Omega$, whereas the surface energy is given by $\mathcal{H}^1(J_y)$ with $J_y \subset \Omega'$. In particular, jumps on $\{0, l\} \times (0, 1)$ contribute to the energy $E_\varepsilon(y)$. (Also compare a similar discussion before [25, Theorem 2.2].) The present problem in the framework of continuum fracture mechanics with isotropic surface energies is a slightly simplified model of the problem considered in [23, 25].

As a preparation, define α such that $\inf\{Q(F) : \mathbf{e}_1^T F \mathbf{e}_1 = 1\} = \alpha$ and observe $\inf\{Q(F) : \mathbf{e}_1^T F \mathbf{e}_1 = a\} = \alpha a^2$ for all $a \in \mathbb{R}$. Moreover, let $F^a \in \mathbb{R}^{2\times 2}_{\rm sym}$ be the unique matrix such that $\mathbf{e}_1^T F^a \mathbf{e}_1 = a$ and $Q(F^a) = \inf\{Q(F) : \mathbf{e}_1^T F \mathbf{e}_1 = a\} = \alpha a^2$.

We recall that the proof of the cleavage laws in [23, 24, 30] fundamentally relied on the application of certain slicing techniques which were not suitable to treat the case of compression. Having general compactness and Γ-convergence results we can now complete the picture about cleavage laws by extending the results to the case of uniaxial compression.

Theorem 2.8 *Suppose $a_\varepsilon/\sqrt{\varepsilon} \to a \in [-\infty, \infty]$. The limiting minimal energy is given by*

$$\lim_{\varepsilon \to 0} \inf\{E_\varepsilon(y) : y \in \mathcal{A}(a_\varepsilon)\} = \min\Big\{\frac{1}{2}\alpha l a^2, 1\Big\}. \tag{2.11}$$

Let $a_{\mathrm{crit}} := \sqrt{\frac{2\alpha}{l}}$. For every sequence $(y_\varepsilon)_\varepsilon$ of almost minimizers, up to passing to subsequences, we get $\varepsilon^{-1/2}(y_\varepsilon(x) - x) \to u(x)$ for a.e. $x \in \Omega$, where

(i) if $|a| < a_{\mathrm{crit}}$, $u(x) = (0, s) + F^a x$ for $s \in \mathbb{R}$,

(ii) if $|a| > a_{\mathrm{crit}}$, $u(x) = \begin{cases} (0, s) & x_1 < p, \\ (la, t) & x_1 > p, \end{cases}$ for $s, t \in \mathbb{R}$, $p \in (0, l)$.

3 Preliminaries

In this section we collect the definitions as well as basic properties of SBV and SBD functions and state the rigidity estimates which are necessary for the derivation of our main compactness result.

3.1 (G)SBV and (G)SBD functions

Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Recall that the space $SBV(\Omega, \mathbb{R}^d)$, abbreviated as $SBV(\Omega)$ hereafter, of *special functions of bounded variation* consists of functions $y \in L^1(\Omega, \mathbb{R}^d)$ whose distributional derivative Dy is a finite Radon measure, which splits into an absolutely continuous part with density ∇y with respect to Lebesgue measure and a singular part $D^j y$ whose Cantor part vanishes and thus is of the form

$$D^j y = [y] \otimes \xi_y \mathcal{H}^{d-1} \lfloor J_y,$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, J_y (the 'crack path') is an $\mathcal{H}^{d-1}$-rectifiable set in Ω, ξ_y is a normal of J_y and $[y] = y^+ - y^-$ (the 'crack opening') with $y^\pm$ being the one-sided limits of y at J_y. If in addition $\nabla y \in L^2(\Omega)$ and $\mathcal{H}^{d-1}(J_y) < \infty$, we write $y \in SBV^2(\Omega)$. See [3] for the basic properties of this function space.

Likewise, we say that a function $y \in L^1(\Omega, \mathbb{R}^d)$ is a *special function of bounded deformation* if the symmetrized distributional derivative $Ey := \frac{(Dy)^T + Dy}{2}$ is a finite $R^{d\times d}_{\rm sym}$-valued Radon measure with vanishing Cantor part. It can be decomposed as

$$Ey = e(\nabla y)\mathcal{L}^d + E^j y = e(\nabla y)\mathcal{L}^d + [y] \odot \xi_y \mathcal{H}^{d-1}|_{J_y}, \tag{3.1}$$

where $e(\nabla y)$ is the absolutely continuous part of Ey with respect to the Lebesgue measure $\mathcal{L}^d$, $[y]$, ξ_y, J_y as before and $a \odot b = \frac{1}{2}(a\otimes b + b \otimes a)$. For basic properties of this function space we refer to [2, 4].

To treat variational problems as considered in Section 2 (see in particular (2.2)) the spaces $SBV(\Omega)$ and $SBD(\Omega)$ are not adequate due to the lacking L^∞-bound being essential in the compactness theorems. To overcome this difficulty the space of $GSBV(\Omega)$ was introduced consisting of all $\mathcal{L}^d$-measurable functions $y : \Omega \to \mathbb{R}^d$ such that for every $\phi \in C^1(\mathbb{R}^d)$ with the support of $\nabla\phi$ compact, the composition $\phi \circ y$ belongs to $SBV_{\rm loc}(\Omega)$ (see [18]). In this new setting one may obtain a more general compactness result (see [3, Theorem 4.36]). Unfortunately, this approach cannot be pursued in the framework of SBD functions as for a function $y \in SBD(\Omega)$ the composite $\phi \circ y$ typically does not lie in $SBD(\Omega)$. In [15], Dal Maso suggested another approach which is based on certain properties of one-dimensional slices.

First we have to introduce some notation. For every $\nu \in \mathbb{R}^d \setminus \{0\}$, for every $s \in \mathbb{R}^d$ and for every $B \subset \Omega$ we let

$$B^{\nu,s} = \{t \in \mathbb{R} : s + t\nu \in B\}. \tag{3.2}$$

Furthermore, define the hyperplane $\Pi^\nu = \{x \in \mathbb{R}^d : x \cdot \nu = 0\}$. Moreover, for every function $y : B \to \mathbb{R}^d$ we define the function $y^{\nu,s} : B^{\nu,s} \to \mathbb{R}^d$ by

$$y^{\nu,s}(t) = y(s + t\nu) \tag{3.3}$$

and $\hat{y}^{\nu,s} : B^{\nu,s} \to \mathbb{R}$ by $\hat{y}^{\nu,s}(t) = y(s + t\nu)\cdot\nu$. If $\hat{y}^{\nu,s} \in SBV(B^{\nu,s}, \mathbb{R})$ and $J_{\hat{y}^{\nu,s}}$ denotes the the *approximate jump set* we define

$$J^1_{\hat{y}^{\nu,s}} := \{x \in J_{\hat{y}^{\nu,s}} : |[\hat{y}^{\nu,s}](x)| \geq 1\}.$$

The space $GSBD(\Omega, \mathbb{R}^d)$ of *generalized functions of bounded deformation* is the space of all $\mathcal{L}^d$-measurable functions $y : \Omega \to \mathbb{R}^d$ with the following property: There exists a nonnegative bounded Radon measure λ on Ω such that for all $\nu \in S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ we have that for $\mathcal{H}^{d-1}$-a.e. $s \in \Pi^\nu$ the function $\hat{y}^{\nu,s} = y^{\nu,s}\cdot\nu$ belongs to $SBV_{\rm loc}(\Omega^{\nu,s})$ and

$$\int_{\Pi^\nu} \Big(|D\hat{y}^{\nu,s}|(B^{\nu,s} \setminus J^1_{\hat{y}^{\nu,s}}) + \mathcal{H}^0(B^{\nu,s} \cap J^1_{\hat{y}^{\nu,s}}) \Big)\, d\mathcal{H}^{d-1}(s) \leq \lambda(B)$$

for all Borel sets $B \subset \Omega$. If in addition $e(\nabla y) \in L^2(\Omega)$ and $\mathcal{H}^1(J_y) < \infty$, we write $y \in GSBD^2(\Omega)$.

We refer to [15] for basic properties of this space. In particular, for later reference we now recall fundamental slicing, compactness and approximation results. We first briefly state the main slicing properties of GSBD functions (see [15, Section 8,9].) Recall definitions (3.2) and (3.3) and let $J_y^\nu = \{x \in J_y : [y](x) \cdot \nu \neq 0\}$.

Theorem 3.1 *Let $y \in GSBD(\Omega)$. For all $\nu \in S^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. s in $\Pi^\nu = \{x : x \cdot \nu = 0\}$ we have*

$$J_{y^{\nu,s}} = (J_y^\nu)^{\nu,s},$$
$$\int_{\Pi^\nu} \# J_{y^{\nu,s}} \, d\mathcal{H}^{d-1}(s) = \int_{J_y^\nu} |\xi_y \cdot \nu| \, d\mathcal{H}^{d-1}.$$

Moreover, the approximate symmetrized gradient $e(\nabla y)$ exists in the sense of [15, (9.1)], satisfies $e(\nabla y) \in L^1(\Omega, \mathbb{R}^{d \times d}_{\mathrm{sym}})$ and for all $\nu \in S^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. s in Π^ν we have

$$\nu^T e(\nabla y(s + t\nu))\nu = (\hat{y}^{\nu,s})'(t) \quad \textit{for a.e. } t \in \Omega^{\nu,s}.$$

Similar properties for SBV functions may be found in [3, Section 3.11]. We now state a general compactness result in GSBD proved in [15, Theorem 11.3] which we slightly adapt for our purposes.

Theorem 3.2 *Let $(y_k)_k$ be a sequence in $GSBD(\Omega)$. Suppose that there exist a constant $M > 0$ and an increasing continuous functions $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{x \to \infty} \psi(x) = +\infty$ such that*

$$\int_\Omega \psi(|y_k|) + \int_\Omega |e(\nabla y_k)|^2 + \mathcal{H}^1(J_{y_k}) \le M$$

for every $k \in \mathbb{N}$. Then there exist a subsequence, still denoted by $(y_k)_k$, and a function $y \in GSBD^2(\Omega)$ such that

$$\begin{aligned} &y_k \to y \quad \textit{pointwise a.e. in} \quad \Omega, \\ &e(\nabla y_k) \rightharpoonup e(\nabla y) \quad \textit{weakly in } L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \\ &\liminf_{k \to \infty} \mathcal{H}^1(J_{y_k}) \ge \mathcal{H}^1(J_y). \end{aligned} \tag{3.4}$$

The lower semicontinuity result for the jump set can be generalized considering one-dimensional slices. Define $\theta_\sigma : [0, \infty) \to [0, 1]$ by $\theta_\sigma(t) = \min\{\frac{t}{\sigma}, 1\}$ for $\sigma > 0$ and additionally $\theta_0 \equiv 1$. Let

$$\hat{\mu}_y^{\sigma,\nu}(B) := \int_{\Pi^\nu} \int_{B^{\nu,s} \cap J_{\hat{y}^{\nu,s}}} \theta_\sigma(|[\hat{y}^{\nu,s}](t)|) \, d\mathcal{H}^0(t) \, d\mathcal{H}^{d-1}(s) \tag{3.5}$$

for all Borel sets $B \subset \Omega$.

Lemma 3.3 *Let $(y_k)_k$ be a sequence in $GSBD(\Omega)$ converging to a function $y \in GSBD(\Omega)$ in the sense of* (3.4)*. Then*

$$\hat{\mu}^{\sigma,\nu}_y(U) \le \liminf_{k\to\infty} \hat{\mu}^{\sigma,\nu}_{y_k}(U)$$

for every $\nu \in S^{d-1}$ and for all open sets $U \subset \Omega$.

Proof. As $y_k \to y$ in the sense of (3.4) we may assume that $(y^{\nu,s})_k \to y^{\nu,s}$ in $GSBV(U^{\nu,s})$ for $\mathcal{H}^{d-1}$-a.e. $s \in U^\nu := \{s \in \Pi^\nu : U^{\nu,s} \neq \emptyset\}$. This is one of the essential steps in the proof of Theorem 3.2 (cf. [15, Theorem 11.3] or [4, Theorem 1.1] for an elaborated proof in the SBD-setting). The desired claim now follows from the corresponding lower semicontinuity result for GSBV functions (see e.g. [3, Theorem 4.36]) and Fatou's lemma. $\square$

We briefly note that using the area formula (see e.g. [3, Theorem 2.71, 2.90])) and fine properties of GSBD functions (see [15]), $\hat{\mu}^{\sigma,\nu}_y(B)$ can be written equivalently as

$$\hat{\mu}^{\sigma,\nu}_y(B) = \int_{J_y \cap B} \theta_\sigma(|[y]\cdot\nu|)|\xi_y \cdot \nu| \, d\mathcal{H}^{d-1} \tag{3.6}$$

for all $\sigma > 0$, for all $\nu \in S^{d-1}$ and all Borel sets $B \subset \Omega$ (see also [15, Remark 9.3]). Finally, we recall a density result in GSBD (see [28]).

Theorem 3.4 *Let $y \in GSBD^2(\Omega) \cap L^2(\Omega)$. Then there exists a sequence $y_k \in SBV^2(\Omega)$ such that each J_{y_k} is contained in the union of a finite number of closed connected pieces of C^1-surfaces, each y_k belongs to $W^{1,\infty}(\Omega \setminus J_{y_k}, \mathbb{R}^2)$ and the following properties hold:*

$$\begin{aligned} &(i) && \|y_k - y\|_{L^2(\Omega)} \to 0,\\ &(ii) && \|e(\nabla y_k) - e(\nabla y)\|_{L^2(\Omega)} \to 0,\\ &(iii) && \mathcal{H}^1(J_{y_k}) \to \mathcal{H}^1(J_y). \end{aligned}$$

3.2 Caccioppoli partitions

We first introduce the notions of perimeter and essential boundary. Consider $E \subset \mathbb{R}^d$ measurable and let

$$P(E,\Omega) = \sup\left\{\int_E \mathrm{div}(\varphi) : \varphi \in C^1_c(\Omega,\mathbb{R}^d), \|\varphi\|_\infty \le 1\right\} \tag{3.7}$$

be the *perimeter* of E in Ω (see [3, Section 3.3]). Moreover, we define the *essential boundary* by

$$\partial^* E = \mathbb{R}^d \setminus \bigcup\nolimits_{t=0,1} \left\{x \in \mathbb{R}^d : \lim\nolimits_{\varrho\downarrow 0} \frac{|E \cap B_\varrho(x)|}{|B_\varrho(x)|} = t\right\}. \tag{3.8}$$

By [3, (3.62)] we have
$$P(E,\Omega) = \mathcal{H}^{d-1}(\Omega \cap \partial^* E).$$
We say a partition $\mathcal{P} = (P_j)_j$ of Ω is a *Caccioppoli partition* of Ω if $\sum_j P(P_j,\Omega) < +\infty$. We say a partition is *ordered* if $|P_i| \ge |P_j|$ for $i \le j$. In the whole paper we will always tacitly assume that partitions are ordered. Given a rectifiable set S we say that a Caccioppoli partition is *subordinated* to S if (up to an $\mathcal{H}^{d-1}$-negligible set) the essential boundary $\partial^* P_j$ of P_j is contained in S for every $j \in \mathbb{N}$.

The local structure of Caccioppoli partitions can be characterized as follows (see [3, Theorem 4.17]).

Theorem 3.5 *Let $(P_j)_j$ be a Caccioppoli partition of Ω. Then*
$$\bigcup\nolimits_j (P_j)^1 \cup \bigcup\nolimits_{i \neq j} \partial^* P_i \cap \partial^* P_j$$
contains $\mathcal{H}^{d-1}$-almost all of Ω.

Here $(P)^1$ denote the points where P has density one (see [3, Definition 3.60]). Essentially, the theorem states that $\mathcal{H}^{d-1}$-a.e. point of Ω either belongs to exactly one element of the partition or to the intersection of exactly two sets $\partial^* P_i$, $\partial^* P_j$. In particular, the structure theorem implies (see [3, (4.24) and Theorem 4.23])
$$2\mathcal{H}^{d-1}\Big(\bigcup\nolimits_j \partial^* P_j \cap \Omega\Big) = \sum\nolimits_j P(P_j,\Omega). \tag{3.9}$$
We now state a compactness result for ordered Caccioppoli partitions (see [3, Theorem 4.19, Remark 4.20]).

Theorem 3.6 *Let $\Omega \subset \mathbb{R}^d$ bounded, open with Lipschitz boundary. Let $\mathcal{P}_i = (P_{j,i})_j$, $i \in \mathbb{N}$, be a sequence of ordered Caccioppoli partitions of Ω fulfilling $\sup_i \sum_j P(P_{j,i},\Omega) \le C$ independently of $i \in \mathbb{N}$. Then there exists a Caccioppoli partition $\mathcal{P} = (P_j)_j$ and a not relabeled subsequence such that $P_{j,i} \to P_j$ in measure for all $j \in \mathbb{N}$ as $i \to \infty$.*

Caccioppoli partitions are naturally associated to piecewise constant functions. We say $y : \Omega \to \mathbb{R}^d$ is *piecewiese constant in* Ω if there exists a Caccioppoli partition $(P_j)_j$ of Ω and a sequence $(t_j)_j \subset \mathbb{R}^d$ such that $y = \sum_j t_j \chi_{P_j}$. We close this section with a compactness result for piecewise constant functions (see [3, Theorem 4.25]).

Theorem 3.7 *Let $\Omega \subset \mathbb{R}^d$ bounded, open with Lipschitz boundary. Let $(y_i)_i \subset SBV(\Omega,\mathbb{R}^d)$ be a sequence of piecewise constant functions such that $\sup_i(\|y_i\|_\infty + \mathcal{H}^{d-1}(J_{y_i})) \le C$ independently of $i \in \mathbb{N}$. Then there exists a not relabeled subsequence converging in measure to a piecewise constant function y.*

3.3 Rigidity estimates

In this section we first recall a geometric rigidity result obtained in the framework of nonlinear elasticity and a piecewise rigidity estimate for brittle materials for the sake of completeness. Afterwards we introduce a quantitative result in SBD adapted for Griffith energies of the form (2.2) which will be the starting point for our analysis.

We start with the quantitative geometric rigidity result by Friesecke, James, Müller [26] generalizing the classical Liouville theorem.

Theorem 3.8 *Let $\Omega \subset \mathbb{R}^d$ a (connected) Lipschitz domain and $1 < p < \infty$. Then there exists a constant $C = C(\Omega, p)$ such that for any $y \in W^{1,p}(\Omega, \mathbb{R}^d)$ there is a rotation $R \in SO(d)$ such that*

$$\|\nabla y - R\|_{L^p(\Omega)} \le C \, \|\mathrm{dist}(\nabla y, SO(d))\|_{L^p(\Omega)} .$$

In the theory of fracture mechanics the problem is more involved as global rigidity can fail if the crack disconnects the body. Chambolle, Giacomini and Ponsiglione [14] have proven the following qualitative result for brittle materials which do not store elastic energy (i.e. $\nabla y \in SO(d)$ a.e.).

Theorem 3.9 *Let $y \in SBV(\Omega)$ such that $\mathcal{H}^1(J_y) < +\infty$ and $\nabla y \in SO(d)$ a.e. Then y is a collection of an at most countable family of rigid deformations, i.e., there exists a Caccioppoli partition $\mathcal{P} = (P_j)_j$ subordinated to J_y such that*

$$y(x) = \sum\nolimits_j (R_j \, x + c_j) \chi_{P_j}(x),$$

where $R_j \in SO(d)$ and $c_j \in \mathbb{R}^d$.

Loosely speaking, the result states that the only way that rigidity may fail is that the body is divided into at most countably many parts each of which subject to a different rigid motion. We briefly note that there is an analogous result in the geometrically linear setting (see [14, Theorem A.1]): A function $u \in SBD(\Omega)$ with $\mathcal{H}^1(J_u) < +\infty$ and $e(\nabla u) = 0$ a.e. has the form $u(x) = \sum_j (A_j \, x + c_j) \chi_{P_j}(x)$ for $A_j \in \mathbb{R}^{d \times d}_{\mathrm{skew}}$ and $c_j \in \mathbb{R}^d$.

We now introduce a quantitative SBD-rigidity result which may be seen as a suitable combination of the above estimates and is tailor-made for general Griffith functionals of the form (2.2) where both energy forms are coexistent (see [22, Theorem 2.1, Remark 2.2]). Let $\Omega_\rho = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > C\rho\}$ for $\rho > 0$ and for some sufficiently large constant C. Recall (2.1), (2.2) and introduce a relaxed energy functional

$$E^\rho_\varepsilon(y, U) = \frac{1}{\varepsilon} \int_U W(\nabla y(x)) \, dx + \int_{J_y \cap U} f^\rho_\varepsilon(|[y](x)|) \, d\mathcal{H}^1(x). \tag{3.10}$$

for $\rho > 0$, $\varepsilon > 0$ and $U \subset \Omega$, where $f^\rho_\varepsilon(x) := \min\{\frac{x}{\sqrt{\varepsilon}\rho}, 1\}$.

Theorem 3.10 *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $M > 0$ and $0 < \eta, \rho \ll 1$. Then there is a constant $C = C(\Omega, M, \eta)$ and a universal $c > 0$ such that the following holds for $\varepsilon > 0$ small enough:*
For each $y \in SBV_M(\Omega) \cap L^2(\Omega)$ with $\mathcal{H}^1(J_y) \le M$ and $\int_\Omega \operatorname{dist}^2(\nabla y, SO(2)) \le M\varepsilon$, there is an open set Ω_y with $|\Omega \setminus \Omega_y| \le C\rho$, a modification $\hat{y} \in SBV_{cM}(\Omega) \cap L^2(\Omega)$ with $\|\hat{y} - y\|^2_{L^2(\Omega_y)} + \|\nabla \hat{y} - \nabla y\|^2_{L^2(\Omega_y)} \le C\varepsilon\rho$ and

$$E^\rho_\varepsilon(\hat{y}, \Omega_\rho) \le E_\varepsilon(y) + C\rho \tag{3.11}$$

with the following properties: We find a Caccioppoli partition $\mathcal{P} = (P_j)_j$ of Ω_ρ with $\sum_j P(P_j, \Omega_\rho) \le C$ and for each P_j a corresponding rigid motion $R_j\, x + c_j$, $R_j \in SO(2)$ and $c_j \in \mathbb{R}^2$, such that the function $u : \Omega \to \mathbb{R}^2$ defined by

$$u(x) := \begin{cases} \hat{y}(x) - (R_j\ x + c_j) & \text{for } x \in P_j \\ 0 & \text{for } x \in \Omega \setminus \Omega_\rho \end{cases} \tag{3.12}$$

satisfies the estimates

$$\begin{aligned} &(i)\ \ \mathcal{H}^1(J_u) \le C, && (ii)\ \ \|u\|^2_{L^2(\Omega_\rho)} \le \hat{C}\varepsilon, \\ &(iii)\ \ \sum\nolimits_j \|e(R_j^T \nabla u)\|^2_{L^2(P_j)} \le \hat{C}\varepsilon, && (iv)\ \ \|\nabla u\|^2_{L^2(\Omega_\rho)} \le \hat{C}\varepsilon^{1-\eta} \end{aligned} \tag{3.13}$$

for some constant $\hat{C} = \hat{C}(\rho)$, where $e(G) = \frac{G+G^T}{2}$ for all $G \in \mathbb{R}^{2\times 2}$.

We remark that it is indispensable to allow for an arbitrarily small modification of the deformation (cf. [22, Section 3.5]). Moreover, we get a sufficiently strong bound only for the symmetric part of the gradient (see (iii)) which is not surprising due to the fact that there is no analogue of Korn's inequality in SBV. However, there is at least a weaker bound on the total absolutely continuous part of the gradient (see (iv)) which will essentially be needed to estimate the elastic part of the energy in the passage to the linearized theory (see e.g. (4.8), (4.9) below).

Furthermore, let as briefly note that the uniform bound on the gradient (see (2.1)) in the setting of the nonlinear model is only needed for the application of the rigidity estimate. The condition essentially assures that the elastic energy cannot concentrate on scales being much smaller than ε. In particular, this is a natural assumption in the investigation of discrete systems, where ε may be interpreted as the typical interatomic distance.

Remark 3.11 (i) The proof of Theorem 3.10 shows that the Caccioppoli partition $(P_j)_j$ is in fact a finite partition. In particular, each P_j is the union of squares of sidelength $\sim \rho$ and thus $|P_j| \ge c\rho$ for all j.

(ii) Estimate (3.11) can be refined. Indeed, we obtain

$$\sum\nolimits_j \tfrac{1}{2} P(P_j, \Omega_\rho) + \int_{J_{\hat{y}} \setminus \partial P} f^\rho_\varepsilon(|[\hat{y}]|)\, d\mathcal{H}^1 \le \mathcal{H}^1(J_y) + c\rho, \tag{3.14}$$

where $\partial P := \bigcup_j \partial^* P_j$

(iii) To derive (3.13) one essentially shows (see the proof of Theorem 3.10 in [22])

$$\|\nabla u\|^4_{L^4(\Omega_\rho)} = \sum\nolimits_j \|\nabla \hat{y} - R_j\|^4_{L^4(P_j)} \leq \hat{C}. \tag{3.15}$$

The claim then follows from $\|\operatorname{dist}(\nabla \hat{y}, SO(2))\|^2_{L^2(\Omega_\rho)} \leq M\varepsilon$ and the elementary linearization formula

$$|e(R^T G - \mathbf{Id})| \leq \operatorname{dist}(G, SO(2)) + C|G - R|^2 \tag{3.16}$$

for $G \in \mathbb{R}^{2\times 2}$ and $R \in SO(2)$, where $\mathbf{Id}$ denotes the identity matrix.

4 Compactness of rescaled configurations

This section is devoted to the proof of the main compactness result given in Theorem 2.1. Moreover, we also show that Theorem 2.1 provides an alternative proof of the piecewise rigidity result stated in Theorem 3.9.

4.1 Preparations

For the compactness theorem in GSBD (see Theorem 3.2) it is necessary that the integral for some integrand ψ with $\lim_{x\to\infty} \psi(x) = \infty$ is uniformly bounded. We first give a simple criterion for the existence of such a function which is, loosely speaking, based on the condition that the functions coincide in a certain sense on the bulk part of the domain.

Lemma 4.1 *For every increasing sequence $(b_i)_i \subset (0,\infty)$ with $b_i \to \infty$ there is an increasing concave function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{x\to\infty} \psi(x) = \infty$ and $\psi(b_i) \leq 2^i$ for all $i \in \mathbb{N}$.*

Proof. Let $f : [0,\infty) \to [0,\infty)$ be the function with $f(0) = 0$, $f(b_i) = 2^i$ which is affine on each segment $[b_i, b_{i+1}]$. Clearly, f is increasing and satisfies $f(x) \to \infty$ for $x \to \infty$, but is possibly not concave. We now construct ψ and first let $\psi = f$ on $[0, b_1]$. Assume ψ has been defined on $[0, b_i]$ and satisfies $\psi(b_i) = f(b_i) = 2^i$. If $f'(b_i-) \geq f'(b_i+)$ we set $\psi = f$ on $[b_i, b_{i+1}]$. Here, $f'(x\pm)$ denote the one-sided limits of the derivative at point x. Otherwise, we let $\psi(x) = f(b_i) + f'(b_i-)(x - b_i)$ for $x \in [b_i, \bar{x}]$, where $\bar{x}$ is the smallest value larger than b_i such that $f(\bar{x}) = f(b_i) + f'(b_i-)(\bar{x} - b_i)$. If $\bar{x}$ does not exist we are done . If $\bar{x}$ exists we assume $\bar{x} \in (b_{j-1}, b_j]$ and define $\psi = f$ on $[\bar{x}, b_j]$ noting that $\psi'(\bar{x}-) \geq \psi'(\bar{x}+)$. We end up with an increasing concave function ψ with $\psi \leq f$ and $\psi(x) \to \infty$ for $x \to \infty$, as desired.

Lemma 4.2 *Let $\Omega \subset \mathbb{R}^2$ and let $(y_l)_l \subset L^1(\Omega)$ be a sequence satisfying $|\Omega \setminus \bigcup_{n\in\mathbb{N}} \bigcap_{l\geq n}\{|y^n - y^l| \leq 1\}| = 0$. Then there is a not relabeled subsequence such that*

$$\int_\Omega \psi(|y^l|) \leq C$$

for a constant independent of l, where ψ is an increasing continuous function with $\lim_{x\to\infty}\psi(x) = +\infty$.

Proof. Define $C_l := \max_{1\leq i\leq l} \|y_i\|_{L^1(\Omega)}$ for all $l \in \mathbb{N}$. Let $A_n = \bigcap_{l\geq n}\{|y^n - y^l| \leq 1\}$ and set $B_1 = A_1$ as well as $B_n = A_n \setminus \bigcup_{m=1}^{n-1} B_m$ for all $n \in \mathbb{N}$. The sets $(B_n)_n$ are pairwise disjoint with $\sum_n |B_n| = |\Omega|$. We choose $0 = n_1 < n_2 < \ldots$ such that $\sum_{1\leq n\leq n_i} \frac{|B_n|}{|\Omega|} \geq 1 - 4^{-i}$. We let $B^i = \bigcup_{n=n_i+1}^{n_{i+1}} B_n$ and observe $|B^i| \leq 4^{-i}|\Omega|$.

We pass to the subsequence of $(n_i)_i \subset \mathbb{N}$ and choose $E^i \supset B^i$ such that $|E^i| = 4^{-i}|\Omega|$. Let $b_i = \frac{C_{n_{i+1}}}{|E^i|} + 2 = 4^i \frac{C_{n_{i+1}}}{|\Omega|} + 2$ for $i \in \mathbb{N}$ and note that $(b_i)_i$ is increasing with $b_i \to \infty$. By Lemma 4.1 we get an increasing concave function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{x\to\infty}\psi(x) = \infty$ and $\psi(b_i) \leq 2^i$ for all $i \in \mathbb{N}$. Clearly, ψ is also continuous.

For $\hat{B}^i := \Omega \setminus \bigcup_{n=1}^{n_i} B_n$ we have $|\hat{B}^i| \leq 4^{-i}|\Omega|$ and choose $\hat{E}^i \supset \hat{B}^i$ with $|\hat{E}^i| = 4^{-i}|\Omega|$. We then obtain $\frac{C_{n_i}}{|\hat{E}^i|} = 4^i \frac{C_{n_i}}{|\Omega|} \leq b_i$. Now let $l = n_i$. Using Jensen's inequality, the definition of the sets B^i, $\|y_l\|_{L^1(\Omega)} \leq C_l$ and the monotonicity of ψ we compute

$$\begin{aligned}
\int_\Omega \psi(|y^l|) &= \sum\nolimits_{1\leq j\leq i-1} \int_{B^j} \psi(|y^l|) + \int_{\hat{B}^i} \psi(|y^l|) \\
&= \sum\nolimits_{1\leq j\leq i-1} \int_{B^j} \psi(|y^{n_{j+1}}| + 2) + \int_{\hat{B}^i} \psi(|y^l|) \\
&\leq \sum\nolimits_{1\leq j\leq i-1} |E^j| \psi\Big(\fint_{E^j} |y^{n_{j+1}}| + 2\Big) + |\hat{E}^i| \psi\Big(\fint_{\hat{E}^i} |y^l|\Big) \\
&\leq \sum\nolimits_{1\leq j\leq i-1} 4^{-j}|\Omega| 2^j + 4^{-i}|\Omega| 2^i \leq |\Omega| \sum\nolimits_{j\in\mathbb{N}} 2^{-j}.
\end{aligned} \tag{4.1}$$

As the estimate is independent of $l \in (n_i)_i$, this yields $\int_\Omega \psi(|y^l|) \leq C$ uniformly in l, as desired. □

4.2 Proof of Theorem 2.1

Now we are in a position to give the proof of the main compactness result. In the first part we show that (2.4), (2.5) and (2.6) hold.

Proof of Theorem 2.1, part 1. Let $(\varepsilon_k)_k$ be an arbitrary null sequence. Let $y_k \in SBV_M(\Omega)$ with $E_{\varepsilon_k}(y_k) \leq C$ be given. The fact that $W(G) \geq c\,\mathrm{dist}^2(G, SO(2))$ for all $G \in \mathbb{R}^{2\times 2}$ implies $\|\mathrm{dist}(\nabla y_k, SO(2))\|^2_{L^2(\Omega)} \leq C\varepsilon_k$ for a constant independent of ε_k. Moreover, we have $\mathcal{H}^1(J_{y_k}) \leq C$ for all $k \in \mathbb{N}$.

Choose $\rho_0 > 0$ and let $\rho_l = 2^{-3l}\rho_0$ for all $l \in \mathbb{N}$. By Theorem 3.10 we find modifications $y_k^l \in SBV_{cM}(\Omega, \mathbb{R}^2)$ with $E_{\varepsilon_k}^{\rho_l}(y_k^l, \Omega_{\rho_l}) \le E_{\varepsilon_k}(y_k) + C\rho_l$ and

$$\|y_k^l - y_k\|^2_{L^2(\Omega_k^l)} + \|\nabla y_k^l - \nabla y_k\|^2_{L^2(\Omega_k^l)} \le C\varepsilon_k \rho_l, \tag{4.2}$$

where $\Omega_k^l := \Omega_{y_k^l}$ with $|\Omega \setminus \Omega_k^l| \le C\rho_l$. We further get Caccioppoli partitions $(P_j^{k,l})_j$ of Ω_{ρ_l} with $\sum_j P(P_j^{k,l}, \Omega_{\rho_l}) \le C$ and corresponding piecewise rigid motions $T_k^l(x) := \sum_j (R_j^{k,l}\, x + c_j^{k,l})\chi_{P_j^{k,l}}(x)$ such that the functions $v_k^l : \Omega \to \mathbb{R}^2$ defined by

$$v_k^l(x) = \begin{cases} \frac{1}{\sqrt{\varepsilon_k}} (R_j^{k,l})^T \big(y_k^l(x) - (R_j^{k,l}\, x + c_j^{k,l})\big) & \text{for } x \in P_j^{k,l},\ j \in \mathbb{N}, \\ 0 & \text{else,} \end{cases} \tag{4.3}$$

satisfy by (3.13)

$$\mathcal{H}^1(J_{v_k^l}) \le C, \quad \|v_k^l\|_{L^2(\Omega)} + \|e(\nabla v_k^l)\|_{L^2(\Omega)} \le \hat{C}_l, \quad \|\nabla v_k^l\|^2_{L^2(\Omega)} \le \hat{C}_l \varepsilon_k^{-\eta} \tag{4.4}$$

for some $\hat{C}_l = \hat{C}(\rho_l) > 0$ and $\eta > 0$ small. Observe that $|c_j^{k,l}| \le cM$ for a universal constant as $\|y_k^l\|_\infty \le cM$ for all $k \in \mathbb{N}$. Clearly, each partition may be extended to Ω by adding the element $\Omega \setminus \Omega_{\rho_l}$ and $\sum_j P(P_j^{k,l}, \Omega) \le C$ is still satisfied as $\mathcal{H}^1(\partial\Omega_{\rho_l}) \le C\mathcal{H}^1(\partial\Omega)$.

Using a diagonal argument we get a (not relabeled) subsequence of $(\varepsilon_k)_k$ such that by Theorem 3.2 for every $l \in \mathbb{N}$ we find a function $v^l \in GSBD^2(\Omega)$ with

$$v_k^l \to v^l \text{ a.e. in } \Omega \quad \text{and} \quad e(\nabla v_k^l) \rightharpoonup e(\nabla v^l) \text{ weakly in } L^2(\Omega, \mathbb{R}^{2\times 2}_{\rm sym}) \tag{4.5}$$

for $k \to \infty$. Moreover, by the compactness result for piecewise constant functions (see Theorem 3.7) we obtain an (ordered) partition $(P_j^l)_j$ of Ω with $\sum_j P(P_j^l, \Omega) \le C$ and a piecewise rigid motion $T^l(x) := \sum_j (R_j^l\, x + c_j^l)\chi_{P_j^l}(x)$ such that for all $l \in \mathbb{N}$ letting $k \to \infty$ we obtain (again up to a subsequence) $R_j^{k,l}\chi_{P_j^{k,l}} \to R_j^l \chi_{P_j^l}$ and $c_j^{k,l}\chi_{P_j^{k,l}} \to c_j^l \chi_{P_j^l}$ in measure for all $j \in \mathbb{N}$. This also implies

$$\sum\nolimits_j |P_j^{k,l} \triangle P_j^l| + \|T_k^l - T^l\|_{L^2(\Omega)} + \|\nabla T_k^l - \nabla T^l\|_{L^2(\Omega)} \to 0 \tag{4.6}$$

for $k \to \infty$, where $\triangle$ again denotes the symmetric difference of two sets. We now show that

$$\|v^l\|_{L^1(\Omega)} \le C\|v^l\|_{L^2(\Omega)} \le \hat{C}_l, \quad \mathcal{H}^1(J_{v^l}) \le C, \quad \|e(\nabla v^l)\|^2_{L^2(\Omega)} \le C. \tag{4.7}$$

The first two claims follow directly from (4.4) and (3.4). To see the third estimate we let $\chi_k^l(x) := \chi_{[0, \varepsilon_k^{-1/8}]}(|\nabla v_k^l(x)|)$. Moreover, letting $\mathbf{Id}$ be the identity matrix we define $\bar{e}_R(G) = e(R^T G - \mathbf{Id})$ for $G \in \mathbb{R}^{2\times 2}$, $R \in SO(2)$ and observe that by an elementary computation (cf. (3.16)) one has $\mathrm{dist}^2(G,SO(2)) = |\bar{e}_R(G)|^2 + \omega(G-R)$ with $\sup\{|G|^{-3}\omega(G) : |G| \le 1\} \le C$ and $\omega(RG) = \omega(G)$. We compute

$$\begin{aligned} C \ge E^{\rho_l}_{\varepsilon_k}(y^l_k, \Omega_{\rho_l}) &\ge \frac{C}{\varepsilon_k}\int_{\Omega_{\rho_l}} \mathrm{dist}^2(\nabla y^l_k, SO(2)) \\ &\ge \frac{C}{\varepsilon_k}\sum\nolimits_j \int_{P^{k,l}_j} \chi^l_k \Big(|\bar{e}_{R^{k,l}_j}(\nabla y^l_k)|^2 + \omega(\nabla y^l_k - R^{k,l}_j)\Big) \\ &= C\int_\Omega \chi^l_k\Big(|e(\nabla v^l_k)|^2 + \frac{1}{\varepsilon_k}\omega(\sqrt{\varepsilon_k}\nabla v^l_k)\Big). \end{aligned} \tag{4.8}$$

The second term of the integral can be estimated by

$$\int_\Omega \chi^l_k(x)\frac{1}{\varepsilon_k}\omega(\sqrt{\varepsilon_k}\nabla v^l_k) = \int_\Omega \chi^l_k(x)\sqrt{\varepsilon_k}|\nabla v^l_k|^3 \frac{\omega(\sqrt{\varepsilon_k}\nabla v^l_k)}{|\sqrt{\varepsilon_k}\nabla v^l_k|^3} \le C\varepsilon_k^{\frac{1}{8}} \to 0. \tag{4.9}$$

As $e(\nabla v^l_k) \rightharpoonup e(\nabla v^l)$ weakly in L^2 and $\chi^l_k \to 1$ boundedly in measure on Ω by (4.4) for η sufficiently small, it follows $\chi^l_k e(\nabla v^l_k) \rightharpoonup e(\nabla v^l)$ weakly in $L^2(\Omega)$. By lower semicontinuity we obtain $\|e(\nabla v^l)\|^2_{L^2(\Omega)} \le C$ for a constant independent of ρ_l which concludes (4.7).

We now want to pass to the limit $l \to \infty$. Similarly as in the argumentation leading to (4.6), by the compactness result for piecewise constant functions (see Theorem 3.7) we find a partition $(P_j)_j$ of Ω and a piecewise rigid motion $T(x) := \sum_j (R_j\,x + c_j)\chi_{P_j}(x)$ such that for a suitable (not relabeled) subsequence $R^l_j\chi_{P^l_j} \to R_j\chi_{P_j}$, $c^l_j\chi_{P^l_j} \to c_j\chi_{P_j}$ in measure for all $j \in \mathbb{N}$ and thus

$$\sum\nolimits_j |P^l_j \triangle P_j| + \|T^l - T\|_{L^2(\Omega)} + \|\nabla T^l - \nabla T\|_{L^2(\Omega)} \to 0 \tag{4.10}$$

for $l \to \infty$. Recalling (4.6) and using a diagonal argument we can choose a (not relabeled) subsequence of $(\rho_l)_l$ and afterwards of $(\varepsilon_k)_k$ such that for all l we have

$$\sum\nolimits_j |P^l_j \triangle P_j| \le 2^{-l}, \quad \sum\nolimits_j |P^{k,l}_j \triangle P^l_j| \le 2^{-l} \ \text{ for all } \ k \ge l. \tag{4.11}$$

We see that the compactness result in GSBD cannot be applied directly on the sequence $(v^l)_l$ as the L^2 bound in (4.7) depends on ρ_l. We now show that by choosing the rigid motions on the elements of the partitions appropriately (see (4.3)) we can construct the sequence $(v^l)_l$ such that we obtain

$$\Big|\Omega \setminus \bigcup\nolimits_{n\in\mathbb{N}} \bigcap\nolimits_{m \ge n} \{|v^n - v^m| \le 1\}\Big| = 0 \tag{4.12}$$

and thus Lemma 4.2 is applicable.

We fix $k \in \mathbb{N}$ and describe an iterative procedure to redefine $R^{k,l}_j, c^{k,l}_j$ for all $l, j \in \mathbb{N}$. Let v^1_k as defined in (4.3) and assume $\hat{R}^{k,l}_j, \hat{c}^{k,l}_j$ have been chosen for all $j \in \mathbb{N}$ (which possibly differ from $R_j^{k,l}, c_j^{k,l}$) such that (4.4) still holds possibly passing to a larger constant $\hat{C}_l$. Fix some $P_j^{k,l+1}$, $j \in \mathbb{N}$, and recall that $|P_j^{k,l+1}| \ge C(\rho_{l+1})$ as $P_j^{k,l+1}$ contains squares of size ρ (see Remark 3.11(i)). Define $D^{l+1} = P_j^{k,l+1} \cap \Omega_k^{l+1}$ and let $D_i^l = P_{j_i}^{k,l} \cap \Omega_k^l$ be the components with $P_{j_i}^{k,l} \cap P_j^{k,l+1} \neq \emptyset$ for $i = 1, \ldots, n$. Without restriction assume that $P_{j_1}^{k,l} \cap P_j^{k,l+1}$ has largest Lebesgue measure. If $|P_{j_1}^{k,l} \cap P_j^{k,l+1}| > 2|P_j^{k,l+1} \setminus (\Omega_k^l \cap \Omega_k^{l+1})|$, we define

$$\hat{R}_j^{k,l+1} = \hat{R}_{j_1}^{k,l}, \quad \hat{c}_j^{k,l+1} = \hat{c}_{j_1}^{k,l} \quad \text{on} \quad P_j^{k,l+1}.$$

Otherwise we set $\hat{R}_j^{k,l+1} = R_j^{k,l+1}$ and $\hat{c}_j^{k,l+1} = c_j^{k,l+1}$. In the first case we then obtain $|D_1^l \cap D^{l+1}| \ge \frac{1}{2}|P_{j_1}^{k,l} \cap P_j^{k,l+1}| \ge C(\rho_{l+1})$ and thus for $p = 2, 4$ we get by $\|\nabla y_k^l\|_\infty, \|\nabla y_k^{l+1}\|_\infty \le cM$

$$|\hat{R}_j^{k,l+1} - R_j^{k,l+1}|^p \le C(\rho_{l+1})\Big(\|\nabla y_k^l - \hat{R}_{j_1}^{k,l}\|^p_{L^p(D_1^l)} + \|\nabla y_k^l - \nabla y_k^{l+1}\|^2_{L^2(D_1^l \cap D^{l+1})} + \|\nabla y_k^{l+1} - R_j^{k,l+1}\|^p_{L^p(D^{l+1})}\Big).$$

The calculation may be repeated to estimate the difference of the rigid motions. Summing over all components and recalling (4.2), (4.4) (for l) as well as the estimates for the original rigid motions (for $l+1$, see (3.13)(ii),(iv) and (3.15)) we find that (4.4) still holds possibly passing to larger constants.

We define $A_{k,l} = \bigcap_{n \le m \le l} \{|v_k^m - v_k^n| \le \frac{1}{2}\}$ for all $n \in \mathbb{N}$ and $n \le l \le k$. If we show

$$|\Omega \setminus A_{k,l}| \le C 2^{-n}, \tag{4.13}$$

then (4.12) follows. Indeed, for given $l \ge n$ we can choose $K = K(l) \ge l$ so large that $|\{|v_K^m - v^m| > \frac{1}{4}\}| \le 2^{-m}$ for all $n \le m \le l$ since $v_k^m \to v^m$ in measure for $k \to \infty$. This implies

$$\big|\Omega \setminus \bigcap\nolimits_{n \le m \le l} \{|v^m - v^n| \le 1\}\big| \le |\Omega \setminus A_{K,l}| + \sum\nolimits_{n \le m \le l} |\{|v_K^m - v^m| > \tfrac{1}{4}\}| \le C2^{-n}.$$

Passing to the limit $l \to \infty$ we find $|\Omega \setminus \bigcap_{m \ge n} \{|v^m - v^n| \le 1\}| \le C2^{-n}$ and taking the union over all $n \in \mathbb{N}$ we derive (4.12).

To show (4.13) we proceed in two steps. Employing the redefinition of the piecewise rigid motions we first show that the set where $T_k^m, n \le m \le l$, differ is small. Afterwards we use (4.2) to find that the set where $y_k^m, n \le m \le l$, differ is small. We define $B_{k,l} = \bigcap_{n \le m \le l} \{T_k^m = T_k^n\}$ and prove that

$$|\Omega \setminus B_{k,l}| \le C2^{-n} \tag{4.14}$$

for all $k \ge l \ge n$. To this end, consider $\{T_k^m = T_k^{m+1}\}$ for $n \le m \le l-1$ and first note that by (4.11) we have $\sum_j |P_j^{k,m+1} \triangle P_j^{k,m}| \le 3 \cdot 2^{-m}$. Define $J_1 \subset \mathbb{N}$

such that $|P_j^{k,m} \cap P_j^{k,m+1}| \le 2|P_j^{k,m+1} \setminus (\Omega_k^m \cap \Omega_k^{m+1})|$ for all $j \in J_1$ and let $J_2 \subset \mathbb{N} \setminus J_1$ such that $|P_j^{k,m+1} \cap P_j^{k,m}| > \frac{1}{2}|P_j^{k,m+1}|$ for all $j \in J_2$. Observe that $|P_j^{k,m+1}| \le 2|P_j^{k,m+1} \setminus P_j^{k,m}|$ for $j \in J_3 := \mathbb{N} \setminus (J_1 \cup J_2)$. Due to the above construction of the rigid motions we obtain $\{T_k^m = T_k^{m+1}\} \supset \bigcup_{j \in J_2}(P_j^{k,m+1} \cap P_j^{k,m})$ and therefore recalling $|\Omega \setminus (\Omega_k^m \cap \Omega_k^{m+1})| \le C2^{-3m}$

$$
\begin{aligned}
|\Omega \setminus \{T_k^m = T_k^{m+1}\}| &\le \sum\nolimits_{j \in J_2} |P_j^{k,m+1} \setminus P_j^{k,m}| + \sum\nolimits_{j \in J_1 \cup J_3} |P_j^{k,m+1}| \\
&\le \sum\nolimits_{j \in J_2} |P_j^{k,m+1} \setminus P_j^{k,m}| + \sum\nolimits_{j \in J_1 \cup J_3} 2|P_j^{k,m+1} \setminus P_j^{k,m}| \\
&\quad + \sum\nolimits_{j \in J_1} 2|P_j^{k,m+1} \setminus (\Omega_k^m \cap \Omega_k^{m+1})| \le C2^{-m}.
\end{aligned}
$$

Summing over $n \le m \le l-1$ we establish (4.14). Now recalling (4.2), (4.14), $|\Omega \setminus \Omega_k^l| \le C\rho_l$ and the fact that $(\rho_l)_l \subset (2^{-3l}\rho_0)_l$ we find

$$
|\Omega \setminus A_{k,l}| \le |\Omega \setminus B_{k,l}| + \sum\nolimits_{n \le m \le l-1} |\{|y_k^{m+1} - y_k^m| > 2^{-m-1}\sqrt{\varepsilon_k}\}| \le C2^{-n}
$$

for all $k \ge l \ge n$, as desired.

By (4.7) and (4.12) we can apply Lemma 4.2 on the sequence $(v^l)_l$. We employ Theorem 3.2 and obtain a function $v \in GSBD(\Omega)$ and a further not relabeled subsequence with $v^l \to v$ a.e in Ω and $e(\nabla v^l) \rightharpoonup e(\nabla v)$ weakly in $L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

We now select a suitable diagonal sequence such that (2.5) and (2.6) hold. Observe that by (4.8), (4.9) the functions $\hat{v}_k^l := \chi_k^l v_k^l$ fulfill $\|e(\nabla \hat{v}_k^l)\|_{L^2(\Omega)} \le C$ and $\|\nabla \hat{v}_k^l\|_\infty \le \varepsilon_k^{-1/8}$ for a constant independent of $k, l \in \mathbb{N}$. As weak convergence in L^2 is metrizable on bounded sets and convergence in measure is metrizable (take $(f,g) \mapsto \int_\Omega \min\{|f-g|, 1\}$) we can apply a diagonal sequence argument and find a not relabeled subsequence $(y_n)_n$ and a corresponding diagonal sequence $(w_n)_{n\in\mathbb{N}} \subset (\hat{v}_k^l)_{k,l}$ with corresponding partitions $(P_j^n)_j$ and piecewise rigid motions $(T_n)_n$ such that by (4.5), (4.6) and (4.10)

$$
\begin{aligned}
&w_n \to v \ \text{ in measure on } \Omega, \qquad && e(\nabla w_n) \rightharpoonup e(\nabla v) \text{ weakly in } L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \\
&T_n \to T \text{ in } L^2(\Omega), && \nabla T_n \to \nabla T \text{ in } L^2(\Omega), \\
&\chi_{P_j^n} \to \chi_{P_j} \ \text{ in measure on } \Omega \ \text{ for all } j \in \mathbb{N},
\end{aligned}
$$

for $n \to \infty$. Up to a further subsequence we can assume $w_n \to v$ a.e. and $\nabla T_n \to \nabla T$ a.e. Finally, define $u_n = \nabla T_n w_n$ for all $n \in \mathbb{N}$ and $u = \nabla T v$ and observe that (2.5), (2.6) hold. Moreover, by (4.2), (4.3), the fact that $\|\nabla u_n\|_\infty \le \varepsilon_n^{-1/8}$, $\|\nabla y_n\|_\infty \le cM$ and the regularity of W it is not hard to see that

$$
\frac{1}{\varepsilon_n}\int_\Omega W(\mathbf{Id} + \sqrt{\varepsilon_n}\nabla T_n^T \nabla u_n) \le \frac{1}{\varepsilon_n}\int_\Omega W(\nabla y_n) + o(1).
$$

This gives (2.4)(ii). As $\chi_k^l \to 1$ boundedly in measure on Ω and $|\Omega \setminus \Omega_k^l| \to 0$

for $k, l \to \infty$ we also get (2.4)(i) recalling (4.2), (4.3) and possibly passing to a further subsequence. $\square$

It remains to show (2.7). To this end, the estimate is first carried out in terms of the relaxed functionals (see (3.10)) and passing to the limit $\rho \to 0$ we then conclude that it is also satisfied for E.

Proof of Theorem 2.1, part 2. The sets $J_v^c := \{x \in J_v : [v](x) = c\}$ for $c \in B_1(0) \setminus \{0\}$ are pairwise disjoint with $\mathcal{H}^1$-σ finite union, i.e. $\mathcal{H}^1(J_v^c) = 0$ up to at most countable values of c. Consequently, we can choose a sequence (c_j) with $0 \le |c_j| < \frac{1}{2}$ such that $\mathcal{H}^1(J_v^{c_i - c_j}) = 0$ for $i \neq j$. Replacing v by $\tilde{v} = v + \sum_j c_j \chi_{P_j}$ we thus obtain $\mathcal{H}^1(\partial P \setminus J_{\tilde{v}}) = 0$ (recall that $\partial P = \bigcup_j \partial^* P_j$, where ∂^* denotes the essential boundary.) We first show that it suffices to prove

$$\liminf_{k \to \infty} \sum\nolimits_j \mathcal{H}^1(J_{y_k}) \ge \mathcal{H}^1(J_{\tilde{v}}). \tag{4.15}$$

To see this we have to show $\mathcal{H}^1(J_{\tilde{v}}) = \mathcal{H}^1(J_u \setminus \partial P) + \frac{1}{2}\sum_j P(P_j, \Omega)$. By (3.9) we obtain $2\mathcal{H}^1(J_{\tilde{v}} \cap \partial P) = 2\mathcal{H}^1(\bigcup_j \partial^* P_j \cap \Omega) = \sum_j P(P_j, \Omega)$ and thus $\mathcal{H}^1(J_{\tilde{v}}) = \mathcal{H}^1(J_{\tilde{v}} \setminus \partial P) + \frac{1}{2}\sum_j P(P_j, \Omega) = \mathcal{H}^1(J_u \setminus \partial P) + \frac{1}{2}\sum_j P(P_j, \Omega)$, as desired.

We now show (4.15) in two steps first passing to the limit $k \to \infty$ and then letting $l \to \infty$. We replace v_k^l by $\tilde{v}_k^l = v_k^l + \sum_j c_j \chi_{P_j^{k,l}}$ and v^l by $\tilde{v}^l = v^l + \sum_j c_j \chi_{P_j^l}$ noting that $\tilde{v}_k^l \to \tilde{v}^l$ for $k \to \infty$ and $\tilde{v}^l \to \tilde{v}$ for $l \to \infty$ in the sense of (3.4). In the following we write $J_k^l = J_{\tilde{v}_k^l} \cap \Omega_{\rho_l}$ for shorthand. Recalling (4.3) we obtain by (3.14)

$$\begin{aligned} \mathcal{H}^1(J_{y_k}) &\ge \int_{J_k^l \setminus \partial P^{k,l}} \theta_{\rho_l}(|[\tilde{v}_k^l]|)\, d\mathcal{H}^1 + \frac{1}{2}\sum\nolimits_j P(P_j^{k,l}, \Omega_{\rho_l}) - C\rho_l \\ &\ge \int_{J_k^l} \theta_{\rho_l}(|[\tilde{v}_k^l]|)\, d\mathcal{H}^1 - C\rho_l \end{aligned} \tag{4.16}$$

where $\partial P^{k,l} = \bigcup_j \partial^* P_j^{k,l}$ and $\theta_\sigma(x) := \min\{\frac{x}{\sigma}, 1\}$ for $\sigma > 0$. Here we note that the passage from v_k^l to $\tilde{v}_k^l$ does not affect the estimate. We cannot directly apply lower semicontinuity results for GSBD functions due to the involved function θ_{ρ_l}. We therefore pass to the limit $k \to \infty$ on one-dimensional sections.

Recall the measure $\hat{\mu}_{\tilde{v}^l}^{\sigma,\nu}$ defined in (3.5) for $\sigma \ge 0$. By Lemma 3.3 we have

$$\hat{\mu}_{\tilde{v}^l}^{\sigma,\nu}(U) \le \liminf_{k \to \infty} \hat{\mu}_{\tilde{v}_k^l}^{\sigma,\nu}(U)$$

for all $\sigma \ge 0$, $\nu \in S^1$ and for every open set $U \subset \Omega$. Let $\kappa_1 = \int_{S^1} |\xi \cdot \nu|\, d\mathcal{H}^1(\nu)$ for some $\xi \in S^1$ which clearly does not depend on the particular choice of ξ. Using

Fatou's lemma and (3.6) we compute for l sufficiently large

$$
\begin{aligned}
\liminf_{k\to\infty} \mathcal{H}^1(J_{y_k}) + C\rho_l &\geq \liminf_{k\to\infty} \int_{J_k^l} \theta_\sigma(|[\tilde{v}_k^l]|)\, d\mathcal{H}^1 \\
&\geq \kappa_1^{-1} \int_{S^1} \liminf_{k\to\infty} \int_{J_k^l} \theta_\sigma(|[\tilde{v}_k^l](x)|) |\xi_{\tilde{v}_k^l}(x)\cdot\nu|\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(\nu) \\
&\geq \kappa_1^{-1} \int_{S^1} \liminf_{k\to\infty} \hat{\mu}^{\sigma,\nu}_{\tilde{v}_k^l}(\Omega_{\rho_l})\, d\mathcal{H}^1(\nu) \geq \kappa_1^{-1} \int_{S^1} \hat{\mu}^{\sigma,\nu}_{\tilde{v}^l}(\Omega_{\rho_l})\, d\mathcal{H}^1(\nu).
\end{aligned}
$$

We pass to the limit $l \to \infty$ (i.e. $\rho_l \to 0$) and obtain by the dominated convergence theorem

$$
\liminf_{k\to\infty} \mathcal{H}^1(J_{y_k}) \geq \kappa_1^{-1} \int_{S^1} \hat{\mu}^{\sigma,\nu}_{\tilde{v}}(\Omega)\, d\mathcal{H}^1(\nu).
$$

Recall that $\theta_\sigma \to 1$ pointwise for $\sigma \to 0$. Now letting $\sigma \to 0$ we obtain by the dominated convergence theorem and Theorem 3.1

$$
\begin{aligned}
\liminf_{k\to\infty} \mathcal{H}^1(J_{y_k}) &\geq \kappa_1^{-1} \int_{S^1} \hat{\mu}^{0,\nu}_{\tilde{v}}(\Omega)\, d\mathcal{H}^1(\nu) \\
&= \kappa_1^{-1} \int_{S^1} \int_{J^\nu_{\tilde{v}}} |\xi_{\tilde{v}}(x)\cdot\nu|\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(\nu) = \mathcal{H}^1(J_{\tilde{v}}).
\end{aligned}
$$

This gives (4.15) and completes the proof. □

At the end of this section we briefly note that our compactness result provides an alternative proof of the piecewise rigidity result given in Theorem 3.9 (at least in a planar setting).

Proof of Theorem 3.9 for $d = 2$. Let $y \in SBV(\Omega)$ with $\mathcal{H}^1(J_y) < \infty$ as well as $\int_\Omega \mathrm{dist}^2(\nabla y, SO(2)) = 0$ be given. Define an arbitrary null sequence $(\varepsilon_k)_k$ and the sequence $y_k = y$ for all $k \in \mathbb{N}$. Applying Theorem 2.1 we obtain a subsequence and configurations u_k converging almost everywhere by (2.6). Moreover, we obtain piecewise rigid motions T, T_k such that $T_k \to T$, $\nabla T_k \to \nabla T$ in $L^2(\Omega)$ by (2.5) and $y_k - T_k \to 0$ a.e. for $k \to \infty$ by (2.4)(i). This implies $y = T$ is a piecewise rigid motion, as desired. □

5 Admissible & coarsest partitions and limiting configurations

In this section we will prove Theorem 2.4. We begin with some preliminary observations. In the following let $(y_k)_k$ be a (sub-)sequence as considered in Theorem 2.1. Recall Definition 2.3. For notational convenience we will drop the dependence of $(y_k)_k$ in the sets $\mathcal{Z}_P, \mathcal{Z}_u, \mathcal{Z}_T$. Moreover, recall the definition of the set of piecewise infinitesimal rigid motions $\mathcal{A}((P_j)_j)$ in (2.3). We introduce a partial order on the admissible partitions $\mathcal{Z}_P$: Given two partitions $\mathcal{P}^1 := (P_j^1)_j, \mathcal{P}^2 := (P_j^2)_j$ in $\mathcal{Z}_P$ we say $\mathcal{P}^1 \geq \mathcal{P}^2$ if $\mathcal{P}^1$ is subordinated to $\mathcal{P}^2$, i.e. $\bigcup_j \partial^* P_j^1 \subset \bigcup_j \partial^* P_j^2$ up to an $\mathcal{H}^1$-negligible set. We observe that if $\mathcal{P}^1 \geq \mathcal{P}^2$ and $\mathcal{P}^2 \geq \mathcal{P}^1$, abbreviated by $\mathcal{P}^1 = \mathcal{P}^2$ hereafter, then the Caccioppoli partitions coincide: After a possible reordering of the sets we find $|P_j^1 \triangle P_j^2| = 0$ for all $j \in \mathbb{N}$.

We begin with the observation that the piecewise rigid motion is uniquely determined in the limit.

Lemma 5.1 *Let $(y_k)_k$ be a (sub-)sequence as considered in Theorem 2.1. Then there is a unique $T \in \mathcal{Z}_T(P)$ for all $\mathcal{P} \in \mathcal{Z}_P$.*

Proof. Assume there are $\mathcal{P}$, $\hat{\mathcal{P}} \in \mathcal{Z}_P$ and $T \in \mathcal{Z}_T(\mathcal{P})$, $\hat{T} \in \mathcal{Z}_T(\hat{\mathcal{P}})$. Let $(u_k, \mathcal{P}^k, T_k), (\hat{u}_k, \hat{\mathcal{P}}^k, \hat{T}_k) \in \mathcal{D}$ for $k \in \mathbb{N}$ be the triples according to Definition 2.3(ii). As $u_k - \hat{u}_k - (\frac{1}{\sqrt{\varepsilon_k}}(T_k - \hat{T}_k)) \to 0$ a.e. by (2.4)(i) and $u_k - \hat{u}_k$ converges pointwise a.e. (and the limits lie in $\mathbb{R}$ a.e.) by (2.6) we get $T_k - \hat{T}_k \to 0$ pointwise almost everywhere. This implies $T = \hat{T}$, as desired. $\square$

From now on T will always denote the rigid motion given by Lemma 5.1.

5.1 Equivalent characterization of the coarsest partition

We state a lemma giving an equivalent characterization of the coarsest partition (recall Definition 2.3(iv)).

Lemma 5.2 *Let $(y_k)_k$ be a (sub-)sequence as considered in Theorem 2.1. Then $\mathcal{P} \in \mathcal{Z}_P$ is coarsest if and only if it is a maximal element in the partial order $(\mathcal{Z}_P, \geq)$, i.e. $\hat{\mathcal{P}} \geq \mathcal{P}$ implies $\hat{\mathcal{P}} = \mathcal{P}$.*

Proof. (1) Assume $\mathcal{P} = (P_j)_j$ was not coarsest. According to Definition 2.3 let be $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ and u be given such that $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ and (2.4)-(2.7) hold. Without restriction possibly passing to a subsequence we assume that $\frac{1}{\sqrt{\varepsilon_k}}\big(|R_1^k - R_2^k| + |c_1^k - c_2^k|\big) \leq C$ for all $k \in \mathbb{N}$ (cf. (2.8)). By (3.16) we obtain $A^k \in \mathbb{R}^{2\times 2}_{\rm skew}$ with $|A^k| \leq C$ such that $R_1^k - R_2^k = R_1^k(\mathbf{Id} - (R_1^k)^T R_2^k) = R_1^k(\sqrt{\varepsilon_k} A^k + O(\varepsilon_k))$. Passing to a (not relabeled) subsequence we then obtain

$$\begin{aligned} S(x) &:= \lim_{k\to\infty} \frac{1}{\sqrt{\varepsilon_k}}\big((R_1^k - R_2^k)\,x + c_1^k - c_2^k\big) \\ &= \lim_{k\to\infty} \frac{1}{\sqrt{\varepsilon_k}}\big(\sqrt{\varepsilon_k} R_1^k A^k\,x + c_1^k - c_2^k\big) + O(\sqrt{\varepsilon_k}) = RA\,x + c \end{aligned} \tag{5.1}$$

for some $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c \in \mathbb{R}^2$ and $R = \lim_{k\to\infty} R_1^k$. We now define $\hat{\mathcal{P}}^k$, $\hat{\mathcal{P}}$, $\hat{T}_k$, $\hat{u}_k$, $\hat{u}$ as follows. Let $\hat{P}_1^k = P_1^k \cup P_2^k$, $\hat{P}_2^k = \emptyset$, $\hat{P}_j^k = P_j^k$ for $j \geq 3$ and likewise for the limiting partition $\hat{\mathcal{P}}$. Let $\hat{T}_k(x) = R_1^k\,x + c_1^k$ for $x \in \hat{P}_1^k$ and $\hat{T}_k(x) = T_k(x)$ for

$x \in \Omega \setminus \hat{P}_1^k$. It is elementary to see that (2.5) holds as $|R_1^k - R_2^k| + |c_1^k - c_2^k| \to 0$ for $k \to \infty$. Furthermore, we let

$$\hat{u}_k = u_k + \frac{1}{\sqrt{\varepsilon_k}}\big((R_1^k - R_2^k)\,x + c_1^k - c_2^k\big)\chi_{P_2^k}$$

and $\hat{u} = u + (RA\,x + c)\chi_{P_2}$ (see (5.1)). Then (2.4)(i) clearly holds as $\hat{T}_k - T_k = (R_1^k - R_2^k)\,x + c_1^k - c_2^k\big)\chi_{P_2^k}$. It is not hard so see that $\mathcal{H}^1(J_u \setminus \partial P) + \frac{1}{2}\sum_j P(P_j,\Omega) \geq \mathcal{H}^1(J_{\hat{u}} \setminus \partial \hat{P}) + \frac{1}{2}\sum_j P(\hat{P}_j,\Omega)$, where $\hat{P} = \bigcup_j \partial^* \hat{P}_j$ and thus also (2.7) is satisfied. It remains to verify (2.6) and (2.4)(ii). First, (2.6)(iii) is obvious and (2.6)(i) follows from (5.1) and the definition of $\hat{u}$. To see (2.6)(ii) we use $R_1^k = R_2^k + \sqrt{\varepsilon_k}R_1^k A^k + O(\varepsilon_k)$, $|A^k| \leq C$ and observe

$$\begin{aligned}
\chi_{\hat{P}_1^k} e((R_1^k)^T \nabla \hat{u}_k) &= \sum\nolimits_{j=1,2} \chi_{P_j^k} e((R_1^k)^T \nabla u_k) + \chi_{P_2^k} e((R_1^k)^T R_1^k A^k) + O(\sqrt{\varepsilon_k}) \\
&= \sum\nolimits_{j=1,2} \chi_{P_j^k} e((R_j^k)^T \nabla u_k) + O(\sqrt{\varepsilon_k}) \\
&\rightharpoonup \sum\nolimits_{j=1,2} \chi_{P_j} e(R_j^T \nabla u) = \chi_{\hat{P}_1} e(R^T \nabla \hat{u})
\end{aligned}$$

weakly in $L^2(\Omega, \mathbb{R}^{2\times 2}_{\rm sym})$. Finally, to establish (2.4)(ii) we find by $\|\nabla u_k\|_{L^\infty(\Omega)} \leq C\varepsilon_k^{-1/8}$

$$\nabla \hat{T}_k^T \nabla \hat{u}_k = (R_1^k)^T \nabla u_k + A^k + O(\sqrt{\varepsilon_k}) = (R_2^k)^T \nabla u_k + A^k + O(\varepsilon_k^{3/8}) \tag{5.2}$$

a.e. in P_2^k. Observe that $W(G) = \frac{1}{2}Q(e(G - \mathbf{Id})) + \omega(G - \mathbf{Id})$ with $\sup\{|F|^{-3}\omega(F) : |F| \leq 1\} \leq C$ and $\omega(RG) = \omega(G)$ for $G \in \mathbb{R}^{2\times 2}$, $R \in SO(2)$ by the assumptions on W, where $Q = D^2W(\mathbf{Id})$. Thus, we obtain by (5.2)

$$\begin{aligned}
\frac{1}{\varepsilon_k}\int_{P_2^k} W(\mathbf{Id} + \sqrt{\varepsilon_k}\nabla \hat{T}_k^T \nabla \hat{u}_k) &= \int_{P_2^k} \Big(\frac{1}{2}Q(e(\nabla \hat{T}_k^T \nabla \hat{u}_k)) + \frac{1}{\varepsilon_k}\omega(\sqrt{\varepsilon_k}\nabla \hat{T}_k^T \nabla \hat{u}_k)\Big) \\
&= \int_{P_2^k} \Big(\frac{1}{2}Q(e(\nabla T_k^T \nabla u_k)) + \frac{\omega(\sqrt{\varepsilon_k}\nabla \hat{u}_k)}{\varepsilon_k}\Big) + O(\varepsilon_k^{\frac{3}{4}})
\end{aligned}$$

and likewise

$$\frac{1}{\varepsilon_k}\int_{P_2^k} W(\mathbf{Id} + \sqrt{\varepsilon_k}\nabla T_k^T \nabla u_k) = \int_{P_2^k} \Big(\frac{1}{2}Q(e(\nabla T_k^T \nabla u_k)) + \frac{1}{\varepsilon_k}\omega(\sqrt{\varepsilon_k}\nabla u_k)\Big).$$

In both estimates the second terms converge to 0 arguing as in (4.9) and using $\|\nabla u_k\|_\infty, \|\nabla \hat{u}_k\|_\infty \leq C\varepsilon_k^{-1/8}$. Consequently, we get

$$\frac{1}{\varepsilon_k}\int_{P_2^k} W(\mathbf{Id} + \sqrt{\varepsilon_k}\nabla \hat{T}_k^T \nabla \hat{u}_k) = \frac{1}{\varepsilon_k}\int_{P_2^k} W(\mathbf{Id} + \sqrt{\varepsilon_k}\nabla T_k^T \nabla u_k) + o(1) \tag{5.3}$$

for $\varepsilon_k \to 0$. Thus, $\hat{\mathcal{P}}$ is an admissible partition and thus $\mathcal{P}$ is not maximal.

(2) Conversely, assume that $\mathcal{P} = (P_j)_j$ was not maximal, i.e. we find $\hat{\mathcal{P}} = (\hat{P}_j)_j$ with $\hat{\mathcal{P}} \geq \mathcal{P}$, $\hat{\mathcal{P}} \neq \mathcal{P}$, i.e. $\bigcup_j \partial^* P_j$ and $\bigcup_j \partial^* \hat{P}_j$ differ by a set of positive $\mathcal{H}^1$-measure. We may assume without restriction that $P_1 \cap \hat{P}_1$ and $P_2 \cap \hat{P}_1$ have positive $\mathcal{L}^2$-measure. According to Definition 2.3(i) let $(u_k, \mathcal{P}^k, T_k), (\hat{u}_k, \hat{\mathcal{P}}^k, \hat{T}_k) \in \mathcal{D}$ and $u, \hat{u}$ be given such that $(u, \mathcal{P}, T), (\hat{u}, \hat{\mathcal{P}}, T) \in \mathcal{D}_\infty$ and (2.4)-(2.7) hold. As by (2.6) u_k and $\hat{u}_k$ convergence pointwise a.e., by (2.4) also $\frac{1}{\sqrt{\varepsilon_k}}(T_k - \hat{T}_k)$ converges pointwise a.e. (and the limits lie in $\mathbb{R}$ a.e.). But this implies $\frac{1}{\sqrt{\varepsilon_k}}\big(|R_j^k - \hat{R}_1^k| + |c_j^k - \hat{c}_1^k|\big) \leq C$ for $j = 1, 2$ and $k \in \mathbb{N}$. Then the triangle inequality shows that (2.8) is violated and thus $\mathcal{P}$ is not a coarsest partition. $\square$

The alternative characterization now directly implies that there is at most one coarsest partition.

Lemma 5.3 *Let $(y_k)_k$ be a (sub-)sequence as considered in Theorem 2.1. Then there is at most one maximal element in $(\mathcal{Z}_P, \geq)$.*

Proof. Assume there are two maximal elements $\mathcal{P}^1, \mathcal{P}^2 \in \mathcal{Z}_P$ with $\mathcal{P}^1 \neq \mathcal{P}^2$. As before, without restriction we may assume that $P_1^1 \cap P_1^2$ and $P_2^1 \cap P_1^2$ have positive $\mathcal{L}^2$-measure. We proceed as in the proof of Lemma 5.2(2) to see that the partition $\mathcal{P}^1$ is not coarsest and thus not a maximal element in $(\mathcal{Z}_P, \geq)$. $\square$

5.2 Admissible configurations

We now analyze the admissible configurations if the partitions are given. Recall that T is uniquely determined by Lemma 5.5.

Lemma 5.4 *Let $(y_k)_k$ be a (sub-)sequence as considered in Theorem 2.1 and $\mathcal{P}, \hat{\mathcal{P}} \in \mathcal{Z}_P$ such that $\mathcal{P} \leq \hat{\mathcal{P}}$. Let $\hat{u} \in \mathcal{Z}_u(\hat{\mathcal{P}})$. Then $\mathcal{Z}_u(\mathcal{P}) = \hat{u} + \nabla T \mathcal{A}(\mathcal{P})$.*

Proof. (1) To see $\mathcal{Z}_u(\mathcal{P}) \subset \hat{u} + \nabla T \mathcal{A}(\mathcal{P})$ we have to show that $u - \hat{u} \in \nabla T \mathcal{A}(\mathcal{P})$ for all $u \in \mathcal{Z}_u(\mathcal{P})$. To this end, consider $P_{j_1} \in \mathcal{P}$, $\hat{P}_{j_2} \in \hat{\mathcal{P}}$ such that $|P_{j_1} \setminus \hat{P}_{j_2}| = 0$. Let u_k, $\hat{u}_k$ and T_k, $\hat{T}_k$ be given according to Definition 2.3. As $u_k - \hat{u}_k$ and thus $\frac{1}{\sqrt{\varepsilon_k}}(T_k - \hat{T}_k)$ converge pointwise a.e. we find $|R_{j_1}^k - \hat{R}_{j_2}^k| + |c_{j_1}^k - \hat{c}_{j_2}^k| \leq C\sqrt{\varepsilon_k}$. Repeating the argument in (5.1) we derive $u(x) - \hat{u}(x) = \lim_{k\to\infty} u_k(x) - \hat{u}_k(x) = \lim_{k\to\infty} \frac{1}{\sqrt{\varepsilon_k}}(T_k(x) - \hat{T}_k(x)) = \nabla T(x)(A\,x + c)$ for a.e. $x \in P_{j_1}$ for some $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $c \in \mathbb{R}^2$.

(2) Conversely, to see $\mathcal{Z}_u(\mathcal{P}) \supset \hat{u} + \nabla T \mathcal{A}(\mathcal{P})$ we first consider the special case $\mathcal{P} = \hat{\mathcal{P}} = (P_h)_h$. Let $\bar{u} \in \mathcal{Z}_u(\mathcal{P})$ and $A(x) = \sum_h (A_h\,x + c_h)\chi_{P_h}$ be given. We have to show that $u := \bar{u} + \nabla T A \in \mathcal{Z}_u(\mathcal{P})$.

We first note that $\mathcal{H}^1(J_u \setminus \partial P) = \mathcal{H}^1(J_{\bar{u}} \setminus \partial P)$ and thus (2.7) is satisfied. According to Definition 2.3(iii) let $(\bar{u}_k, \mathcal{P}^k, \bar{T}_k) \in \mathcal{D}$ be given such that (2.4)-(2.7) hold. Assume that $\bar{T}_k$ has the form $\bar{T}_k(x) = \bar{R}_j^k\,x + \bar{c}_j^k$ for $x \in P_j^k$. Now choose

R_j^k such that $|R_j^k - \bar{R}_j^k(\mathbf{Id} + \sqrt{\varepsilon_k}A_j)| = \mathrm{dist}(\bar{R}_j^k(\mathbf{Id} + \sqrt{\varepsilon_k}A_j), SO(2))$ and let $c_j^k = \bar{c}_j^k + \sqrt{\varepsilon_k}R_j^k c_j$. Define

$$T_k(x) = \sum\nolimits_j (R_j^k\, x + c_j^k)\chi_{P_j^k}$$

as well as $u_k = \bar{u}_k + \frac{1}{\sqrt{\varepsilon_k}}(T_k - \bar{T}_k)$. Clearly, this implies (2.4)(i). By (3.16) we have $R_j^k = \bar{R}_j^k + \sqrt{\varepsilon_k}R_j^k A_j + \omega_{j,k}$ with $\varepsilon_k^{-\frac{1}{2}}|\omega_{j,k}| \to 0$ for all $j \in \mathbb{N}$. Moreover, we find

$$T_k = \bar{T}_k + \sum\nolimits_j \big(\sqrt{\varepsilon_k}R_j^k A_j\, x + \omega_{j,k}\, x + \sqrt{\varepsilon_k}R_j^k c_j\big)\chi_{P_j^k} \to T$$

in measure for $k \to \infty$. Then it is not hard to see that $T_k \to T$ and $\nabla T_k \to \nabla T$ in L^2 which gives (2.5). Likewise, we obtain

$$\begin{aligned} u_k - \bar{u}_k = \frac{1}{\sqrt{\varepsilon_k}}\big(T_k - \bar{T}_k\big) &= \sum\nolimits_j \Big(R_j^k(A_j\, x + c_j) + \frac{1}{\sqrt{\varepsilon_k}}\omega_{j,k}\, x\Big)\chi_{P_j^k} \\ &\to \nabla T \sum\nolimits_j (A_j\, x + c_j)\chi_{P_j} = \nabla T A \end{aligned}$$

pointwise a.e. which implies $u_k \to \bar{u} + \nabla T A$ and shows (2.6)(i). We observe that there are decreasing sets V_k with $|V_k| \to 0$ for $k \to \infty$ such that $\|\nabla A\|_{L^\infty(\Omega\setminus V_k)} \le C\varepsilon_k^{-1/8}$ and $\|\sum_j \chi_{P_j^k}\varepsilon_k^{-1/2}\omega_{j,k}\|_{L^\infty(\Omega\setminus V_k)} \to 0$ for $k \to \infty$. Consequently, we obtain

$$\|\nabla u_k - \nabla \bar{u}_k\|_{L^\infty(\Omega\setminus V_k)} \le \|\nabla A\|_{L^\infty(\Omega\setminus V_k)} + \|\sum\nolimits_j \chi_{P_j^k}\varepsilon_k^{-1/2}\omega_{j,k}\|^2_{L^\infty(\Omega\setminus V_k)} \le C\varepsilon_k^{-1/8}$$

and therefore, replacing u_k by $\chi_{\Omega\setminus V_k}u_k$ we find that (2.4)(i) still holds and (2.6)(iii) is fulfilled. Arguing as in (5.2) and taking $\|\nabla \bar{u}_k\|_{L^\infty(\Omega\setminus V_k)} + \|\nabla A\|_{L^\infty(\Omega\setminus V_k)} \le C\varepsilon_k^{-1/8}$ we find

$$\begin{aligned} (R_j^k)^T \nabla u_k(x) &= (R_j^k)^T \nabla \bar{u}_k(x) + A_j + (R_j^k)^T \varepsilon_k^{-1/2} w_{j,k} \\ &= (\bar{R}_j^k)^T \nabla \bar{u}_k(x) + O(\varepsilon_k^{1/4}) + A_j + (R_j^k)^T \varepsilon_k^{-1/2} w_{j,k} \end{aligned}$$

for a.e. $x \in P_j^k \setminus V_k$. Thus, also (2.6)(ii) follows from the fact that (2.6)(ii) holds for the sequence $\bar{u}_k$ and

$$\begin{aligned} &\sum\nolimits_j \int_{P_j^k\setminus V_k} |e\big((R_j^k)^T\nabla u_k\big) - e\big((\bar{R}_j^k)^T \nabla \bar{u}_k\big)|^2 \\ &\qquad \le C\|\sum\nolimits_j \chi_{P_j^k}\varepsilon_k^{-1/2}\omega_{j,k}\|^2_{L^\infty(\Omega\setminus V_k)} + C\varepsilon_k^{1/2} \to 0. \end{aligned}$$

Finally, the above estimates together with a similar argumentation as in (5.3) yield (2.4)(ii).

In the general case we have to show $u := \hat{u} + \nabla T A \in \mathcal{Z}_u(\mathcal{P})$ for given $\hat{u} \in \mathcal{Z}_u(\hat{\mathcal{P}})$, $\hat{\mathcal{P}} \ge \mathcal{P}$ and $A \in \mathcal{A}(\mathcal{P})$. As $\mathcal{P} \in \mathcal{Z}_P$ we find some $\bar{u} \in \mathcal{Z}_u(\mathcal{P})$ which by (1) satisfies $\bar{u} - \hat{u} = \nabla T \bar{A}$ for $\bar{A} \in \mathcal{A}(\mathcal{P})$. Consequently, we get $u = \bar{u} + \nabla T(A - \bar{A})$ and by the special case in (2) we know that $u \in \mathcal{Z}_u(\mathcal{P})$, as desired. $\square$

5.3 Existence of coarsest partitions

To guarantee existence of coarsest partitions we show that each totally ordered subset has upper bounds such that afterwards we may apply Zorn's lemma.

Lemma 5.5 *Let $(y_k)_k$ be a (sub-)sequence as considered in Theorem 2.1. Let I be an arbitrary index set and let $\{\mathcal{P}_i = (P_{i,j})_j : i \in I\} \subset \mathcal{Z}_P$ be a totally ordered subset, i.e. for each $i_1, i_2 \in I$ we have $\mathcal{P}_{i_1} \le \mathcal{P}_{i_2}$ or $\mathcal{P}_{i_2} \le \mathcal{P}_{i_1}$. Then there is a partition $\mathcal{P} \in \mathcal{Z}_P$ with $\mathcal{P}_i \le \mathcal{P}$ for all $i \in I$.*

Proof. To prove the existence of an upper bound we first show that it suffices to consider a suitable countable subset of $\{\mathcal{P}_i : i \in I\}$. For notational convenience we write $i_1 \le i_2$ for $i_1, i_2 \in I$ if $\mathcal{P}_{i_1} \le \mathcal{P}_{i_2}$. Choose an arbitrary $i_0 \in I$ and note that it suffices to find an upper bound for all $i \ge i_0$. We observe that for each $i \ge i_0$ we have $\bigcup_j \partial^* P_{i,j} \subset \bigcup_j \partial^* P_{i_0,j}$ (up to an $\mathcal{H}^1$-negligible set). Thus, for each $k \in \mathbb{N}$ there are (coarsened) partitions $\mathcal{P}_i^k = (P_{i,j}^k)_j$ with $\bigcup_j \partial^* P_{i,j}^k = \bigcup_j \partial^* P_{i,j} \setminus \big(\bigcup_{j \ge k} \partial^* P_{i_0,j} \setminus \partial^*(\bigcup_{j\ge k} P_{i_0,j})\big)$ up to $\mathcal{H}^1$-negligible sets for all $i \ge i_0$. Observe that typically $\mathcal{P}_i^k$ are not elements of $\{\mathcal{P}_i : i \in I\}$, but satisfy

$$\mathcal{H}^1\Big(\bigcup\nolimits_j \partial^* P_{i,j} \setminus \bigcup\nolimits_j \partial^* P_{i,j}^k\Big) \le \omega(k)$$

with $\omega(k) \to 0$ for $k \to \infty$. After identifying partitions whose boundaries only differ by $\mathcal{H}^1$-negligible sets we find that each $\{\mathcal{P}_i^k : i \ge i_0\}$ contains only a finite number of different elements and therefore contains a maximal element $\mathcal{P}^k = (P_j^k)_j$. Now we can choose $i_0 \le i_1 \le i_2 \le \ldots$ such that $\mathcal{P}^k = \mathcal{P}_{i_k}^k$ for $k \in \mathbb{N}$. It now suffices to construct an upper bound $\mathcal{P} = (P_j)_j \in \mathcal{Z}_P$ with $\mathcal{P} \ge \mathcal{P}_{i_k}$ for all $k \in \mathbb{N}$. Indeed, we then obtain for all $i \ge i_0$

$$\begin{aligned}
\mathcal{H}^1\Big(\bigcup\nolimits_j \partial^* P_j \setminus \bigcup\nolimits_j \partial^* P_{i,j}\Big) &\le \mathcal{H}^1\Big(\bigcup\nolimits_j \partial^* P_j \setminus \bigcup\nolimits_j \partial^* P_{i,j}^k\Big) \\
&\le \mathcal{H}^1\Big(\bigcup\nolimits_j \partial^* P_j \setminus \bigcup\nolimits_j \partial^* P_j^k\Big) \\
&\le \mathcal{H}^1\Big(\bigcup\nolimits_j \partial^* P_j \setminus \bigcup\nolimits_j \partial^* P_{i_k,j}\Big) + \omega(k) = \omega(k)
\end{aligned}$$

and as $k \in \mathbb{N}$ was arbitrary, we derive $\bigcup_j \partial^* P_j \subset \bigcup_j \partial^* P_{i,j}$, as desired.

Now consider the totally ordered sequence of partitions $(\mathcal{P}_{i_k})_k$. For notational convenience we will denote the sequence by $(\mathcal{P}_i)_{i\in\mathbb{N}}$ in the following. By the compactness theorem for Caccioppoli partitions (see Theorem 3.6) we get an (ordered) Caccioppoli partition $\mathcal{P} = (P_j)_j$ such that $\chi_{P_{i,j}} \to \chi_{P_j}$ in measure for $i \to \infty$ for all $j \in \mathbb{N}$. This also implies $\mathcal{H}^1(\bigcup_j \partial^* P_j \setminus E) \le \liminf_{i\to\infty} \mathcal{H}^1(\bigcup_j \partial^* P_{i,j} \setminus E)$ for every $\mathcal{H}^1$-measurable set with $\mathcal{H}^1(E) < \infty$ (see e.g. [16, Theorem 2.8]). Consequently, we apply $\bigcup_j \partial^* P_{i,j} \subset \bigcup_j \partial^* P_{k,j}$ for $i \ge k$ to derive $\mathcal{H}^1(\bigcup_j \partial^* P_j \setminus \bigcup_j \partial^* P_{k,j}) = 0$ for all $k \in \mathbb{N}$. This implies $\mathcal{P} \ge \mathcal{P}_k$ for all $k \in \mathbb{N}$ and therefore it suffices to show that $\mathcal{P} \in \mathcal{Z}_P$. To this end, we will construct partitions $\mathcal{P}^n$, rigid motions $T_n \in \mathcal{R}(\mathcal{P}^n)$ and a limiting function u by a diagonal sequence argument.

For all $i \in \mathbb{N}$, according to Definition 2.3(i), we find $(u_i^k, \mathcal{P}_i^k, T_i^k) \in \mathcal{D}$ and a sequence of admissible limiting configurations $u_i \in \mathcal{Z}_u(\mathcal{P}_i)$ such that (2.4)-(2.7) hold as $k \to \infty$. The strategy is to select u_i in a suitable way such that we find a limiting configuration $u \in GSBD(\Omega)$ with

$$\begin{aligned} &u_i \to u \quad \text{a.e.}, \\ &e(\nabla T^T \nabla u_i) \rightharpoonup e(\nabla T^T \nabla u), \\ &\liminf_{i\to\infty} \mathcal{H}^1(J_{u_i}) \geq \mathcal{H}^1(J_u). \end{aligned} \tag{5.4}$$

Then we can choose a diagonal sequence $(\bar{u}_n) := (u_n^{k(n)})_n$ converging to the triple $(u, \mathcal{P}, T)$ in the sense of (2.4)-(2.7). Indeed, $k(n)$ can be selected such that letting $\bar{\mathcal{P}}^n = (\bar{P}_j^n)_j = \mathcal{P}_n^{k(n)}$ and $\bar{T}_n = T_n^{k(n)} \in \mathcal{R}(\mathcal{P}^n)$ we find $\chi_{\bar{P}_j^n} \to \chi_{P_j}$ in measure for all $j \in \mathbb{N}$ and $\bar{T}_n \to T$, $\nabla \bar{T}_n \to \nabla T$ in $L^2(\Omega)$ which gives (2.5). Moreover, possibly passing to a further subsequence this can be done in a way that $\bar{u}_n \to u$ a.e., $\bar{u}_n - \varepsilon_n^{-1/2}(y_n - \bar{T}_n) \to 0$ a.e. and therefore also (2.4), (2.6)(i) hold.

Likewise, (2.6)(ii) can be achieved by (5.4) and the fact the weak convergence is metrizable as $\|e(\nabla(T_i^k)^T \nabla u_i^k)\|_{L^2(\Omega)} \leq C$ for a constant independent of k, i. The last property follows from the construction of $\hat{v}_k^l$ in the proof of Theorem 2.1 (see (4.8), (4.9)). Moreover, (2.6)(iii) and (2.4)(ii) directly follow from the corresponding estimates for the functions u_i^k. Finally, to see (2.7) it suffices to prove

$$\liminf_{i\to\infty} \Big(\mathcal{H}^1(J_{u_i} \setminus \partial P_i) + \frac{1}{2}\sum\nolimits_j P(P_{i,j}, \Omega)\Big) \geq \Big(\mathcal{H}^1(J_u \setminus \partial P) + \frac{1}{2}\sum\nolimits_j P(P_j, \Omega)\Big),$$

where $\partial P_i = \bigcup_j \partial^* P_{i,j}$. This can be derived arguing as in (4.15): We may consider an infinitesimal perturbation of the form $\tilde{u}_i = u_i + \sum_j c_j \chi_{P_{i,j}}$, $\tilde{u} = u + \sum_j c_j \chi_{P_j}$ with c_j small such that $\mathcal{H}^1(\partial P_i \setminus J_{\tilde{u}_i}) = \mathcal{H}^1(\partial P \setminus J_{\tilde{u}}) = 0$ and the convergence in (5.4) still holds after replacing u_i, u by $\tilde{u}_i$, $\tilde{u}$, respectively. Then the claim follows from (5.4). Consequently, $\mathcal{P} \in \mathcal{Z}_P$ due to Definition 2.3(i).

It suffices to show (5.4). Clearly, we have $\|e(\nabla T^T \nabla u_i)\|^2_{L^2(\Omega)} \leq C$ and $\mathcal{H}^1(J_{u_i}) \leq C$ for a constant independent of $i \in \mathbb{N}$. This follows by a lower semicontinuity argument taking (2.7) and $\|e(\nabla(T_n^k)^T \nabla u_n^k)\|_{L^2(\Omega)} \leq C$ into account. Consequently, in order to apply Theorem 3.2 we have to find an increasing continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{x\to\infty} \psi(x) = +\infty$ such that $\int_\Omega \psi(|u_i|) \leq C$.

We proceed similarly as in the proof of Theorem 2.1 and define u_i iteratively. Choose $u_1 \in \mathcal{Z}_u(\mathcal{P}_1)$ arbitrarily. Given u_i we define u_{i+1} as follows. We recall $\bigcup_j \partial^* P_{i_2,j} \subset \bigcup_j \partial^* P_{i_1,j}$ for $i_1 \leq i_2$. Consider some $P_{i+1,j}$ and choose $l_{1,j} < l_{2,j} < \ldots$ such that $P_{i+1,j} = \bigcup_{k=1}^\infty P_{i,l_{k,j}}$ up to an $\mathcal{L}^2$- negligible set (observe that the union may also be finite). Without restriction assume that $P_{i,l_{1,j}}$ has largest Lebesgue measure. Choose some $\tilde{u}_{i+1} \in \mathcal{Z}_u(\mathcal{P}_{i+1})$. By Lemma 5.4 for $\mathcal{P} = \mathcal{P}_i, \hat{\mathcal{P}} = \mathcal{P}_{i+1}$ we get $(u_i - \tilde{u}_{i+1})\chi_{P_{i+1,j}} = \sum_{k=1}^\infty (A_{l_{k,j}}\, x + c_{l_{k,j}})\chi_{P_{i,l_{k,j}}}$ for $A_{l_{k,j}} \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c_{l_{k,j}} \in \mathbb{R}^2$. Now define

$$u_{i+1}(x) = \tilde{u}_{i+1}(x) + A_{l_{1,j}}\, x + c_{l_{1,j}}$$

for $x \in P_{i+1,j}$ and observe that $u_i = u_{i+1}$ on $P_{i,l_{1,j}}$. Proceeding in this way on all $P_{i+1,j}$ we find some $\tilde{A}^{i+1} \in \mathcal{A}(\mathcal{P}_{i+1})$ such that $u_{i+1} := \tilde{u}_{i+1} + \tilde{A}^{i+1} \in \mathcal{Z}_u(\mathcal{P}_{i+1})$ applying Lemma 5.4 for $\mathcal{P} = \hat{\mathcal{P}} = \mathcal{P}_{i+1}$. Moreover, there is a corresponding $A^i \in \mathcal{A}(\mathcal{P}_i)$ such that $u_{i+1} = u_i + A^i$ with $A^i = 0$ on $\bigcup_j P_{i,l_{1,j}}$.

We now show that $\sum_{i\in\mathbb{N}} |A^i| < +\infty$ a.e. To see this, we recall that $\chi_{P_{i,j}} \to \chi_{P_j}$ in measure for all $j \in \mathbb{N}$. Consequently, as due to the total order of the partitions the sets $P_{i,j}$ are increasing for fixed $j \in \mathbb{N}$, the construction of the functions $(u_i)_i$ implies $A^i = 0$ on $P_{i,j}$ for i so large that $|P_{i,j}| > \frac{1}{2}|P_j|$. Thus, for a.e. $x \in P_j$ the sum $\sum_{i\ge 1} |A^i(x)|$ is a finite sum and therefore finite. Taking the union over all $j \in \mathbb{N}$ we obtain $\sum_{i\in\mathbb{N}} |A^i| < +\infty$ a.e.

Consequently, there is an increasing continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{x\to\infty} \psi(x) = \infty$ such that $\|\psi(|u_1| + \sum_{i\in\mathbb{N}} |A^i|)\|_{L^1(\Omega)} < \infty$. Using the definition $u_{i+1} = u_i + A^i$ and the monotonicity of ψ we find $\|\psi(|u_i|)\|_{L^1(\Omega)} \le \|\psi(|u_1| + \sum_{k\in\mathbb{N}} |A^k|)\|_{L^1(\Omega)} < \infty$ for all $i \in \mathbb{N}$, as desired. $\square$

After these preparatory lemmas we are finally in a position to prove Theorem 2.4.

Proof of Theorem 2.4. First, (i) follows from Lemma 5.1. The uniqueness of the coarsest partition is a consequence of Lemma 5.3 and Lemma 5.2. We obtain existence by Zorn's lemma: As $(\mathcal{Z}_P, \ge)$ is a partial order and every chain has an upper bound by Lemma 5.5, there exists a maximal element $\bar{\mathcal{P}} \in \mathcal{Z}_P$. Lemma 5.2 shows that $\bar{\mathcal{P}}$ is a coarsest partition which gives (ii). Finally, assertion (iii), namely $\mathcal{Z}_u(\bar{\mathcal{P}}) = v + \nabla T \mathcal{A}(\bar{\mathcal{P}})$ for some $v \in \mathcal{Z}_u(\bar{\mathcal{P}})$, follows from Lemma 5.4 for the choice $\mathcal{P} = \hat{\mathcal{P}} = \bar{\mathcal{P}}$. $\square$

6 The effective linearized Griffith model

In this final section we identify the effective linearized Griffith functional via Γ-convergence and derive a cleavage law for the limiting model.

6.1 Derivation of linearized models via Γ-convergence

We now give the proof of Theorem 2.6.

Proof of Theorem 2.6. (i) Thanks to the preparations in the last section the lower bound is almost immediate. Let $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ be given as well as a sequence

$(y_k)_k \subset SBV_M(\Omega)$ with corresponding $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ such that (2.4)-(2.7) hold. By (2.7) it suffices to show that

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k} \int_\Omega W(\nabla y_k) \geq \int_\Omega \frac{1}{2} Q(e(\nabla T^T \nabla u)).$$

We proceed as in (4.8): Recall that $W(G) = \frac{1}{2}Q(e(G - \mathbf{Id})) + \omega(G - \mathbf{Id})$ with $\sup\{|F|^{-3}\omega(F) : |F| \leq 1\} \leq C$ by the assumptions on W, where $Q = D^2W(\mathbf{Id})$. We compute by (2.4)(ii)

$$\begin{aligned}\frac{1}{\varepsilon_k} \int_\Omega W(\nabla y_k) &\geq \frac{1}{\varepsilon_k} \int_\Omega W(\mathbf{Id} + \sqrt{\varepsilon_k} \nabla T_k^T \nabla u_k) + o(1) \\ &= \int_\Omega \frac{1}{2}\Big(Q(e(\nabla T_k^T \nabla u_k)) + \frac{1}{\varepsilon_k}\omega(\sqrt{\varepsilon_k} \nabla T_k^T \nabla u_k)\Big) + o(1)\end{aligned}$$

as $k \to \infty$. The second term converges to 0 arguing as in (4.9) and using $\|\nabla u_k\|_\infty \leq C\varepsilon_k^{-1/8}$ (see (2.6)). As $e(\nabla T_k^T \nabla u_k) \rightharpoonup e(\nabla T^T \nabla u)$ weakly in $L^2(\Omega, \mathbb{R}^{2\times 2}_{\rm sym})$ by (2.6)(ii) and Q is convex we conclude

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k} \int_\Omega W(\nabla y_k) \geq \int_\Omega \frac{1}{2} Q(e(\nabla T^T \nabla u)),$$

as desired.

(ii) By a general density result in the theory of Γ-convergence together with Theorem 3.4 and the fact that the limiting functional $E(u, \mathcal{P}, T)$ is continuous in u with respect to the convergence given in Theorem 3.4, it suffices to provide recovery sequences for functions u with $u \in W^{1,\infty}(\Omega \setminus J_u)$, where J_u is contained in the union of a finite number of closed connected pieces of C^1- curves. Moreover, as in the proof of Theorem 2.1 we may assume that $\mathcal{H}^1(\partial P \setminus J_u) = 0$ up to an infinitesimal small perturbation of u (a similar argument was used in the proof of Lemma 5.5). Let $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ and $\varepsilon_k \to 0$ be given. Define $y_k(x) = Tx + \sqrt{\varepsilon_k} u(x)$ for all $x \in \Omega$. It is not hard to see that $(y_k)_k \subset SBV_M(\Omega)$ for ε_k small enough. Moreover, define $\mathcal{P}^k = \mathcal{P}$, $T_k(x) = T\,x$ and $u_k = \frac{1}{\sqrt{\varepsilon_k}}\big(y_k - T_k\big) \equiv u$ for all $k \in \mathbb{N}$. Then (2.4),(2.5), (2.7) and the first two parts of (2.6) hold trivially. To see the (2.6)(iii) it suffices to note that $\|\nabla u_k\|_\infty = \|\nabla u\|_\infty \leq C \leq C\varepsilon_k^{-1/8}$.

We finally confirm $\lim_{k\to\infty} E_{\varepsilon_k}(y_k) = E(u, \mathcal{P}, T)$. As for all $k \in \mathbb{N}$ we have $\mathcal{H}^1(J_{u_k}) = \frac{1}{2}\sum_j P(P_j, \Omega) + \mathcal{H}^1(J_u \setminus \partial P)$, it suffices to show $\lim_{k\to\infty} \frac{1}{\varepsilon_k}\int_\Omega W(\nabla y_k) = \int_\Omega \frac{1}{2}Q(e(\nabla T^T \nabla u))$. Using again that $W(G) = \frac{1}{2}Q(e(G - \mathbf{Id})) + \omega(G - \mathbf{Id})$ we compute

$$\begin{aligned}\frac{1}{\varepsilon_k} \int_\Omega W(\nabla y_k) &= \frac{1}{\varepsilon_k} \int_\Omega W(\nabla T_k^T \nabla y_k) \\ &= \int_\Omega \Big(\frac{1}{2}Q(e(\nabla T_k^T \nabla u_k)) + \frac{1}{\varepsilon_k}\omega(\sqrt{\varepsilon_k} \nabla T_k^T \nabla u_k)\Big) \\ &= \int_\Omega \frac{1}{2}Q(e(\nabla T^T \nabla u)) + O(\sqrt{\varepsilon_k}) \to \int_\Omega \frac{1}{2}Q(e(\nabla T^T \nabla u)).\end{aligned}$$

This finishes the proof. $\square$

Remark 6.1 Due to the assumptions in the density result of Theorem 3.4 we have to suppose that $u \in L^2(\Omega)$ in Theorem 2.6(ii). A possible strategy to drop this additional assumption is to show that each limiting configuration u given by Theorem 2.1 can be approximated in the sense of (2.4)-(2.7) by a sequence $(v^l)_l \subset GSBD(\Omega) \cap L^2(\Omega)$ such that $E(u, \mathcal{P}, T) = \lim_{l\to\infty} E(v^l, \mathcal{P}, T)$. A natural candidate seems to be the sequence $(v^l)_l$ given in the proof of Theorem 2.1, but the verification of the convergence of the surface energy appears to be a subtle problem.

The proof of Corollary 2.7 is now straightforward.

Proof of Corollary 2.7. To see the liminf-inequality assume without restriction that $E_{\varepsilon_k}(y_{\varepsilon_k}) \le C$ and $y_{\varepsilon_k} \to y$ in L^1 for $k \to \infty$. By (2.4)(i), (2.5) we obtain $y = T$ for some $T \in \mathcal{R}(\mathcal{P})$ for a Caccioppoli partition $\mathcal{P}$. Moreover, Theorem 2.6 yields $\liminf_{k\to\infty} E_{\varepsilon_k}(y_k) \ge E_{\rm seg}(y)$. A recovery sequence is obviously given by $y_k = y$ for all $k \in \mathbb{N}$. $\square$

6.2 An application to cleavage laws

We are finally in a position to prove the cleavage law in Theorem 2.8. Analogous result for the case of expansive boundary values have been obtained in [30] and [25]. We thus do not repeat all the steps of these proofs but rather concentrate on the additional arguments necessary in our general setting (see (2.10)) in which we particularly can extend the aforementioned results to the case of compression.

Proof of Theorem 2.8. Let $(y_{\varepsilon_k})_k$ be a sequence of almost minimizers. Passing to a suitable subsequence, by Theorem 2.1 we obtain a triple $(u_k, \mathcal{P}^k, T_k) \in \mathcal{D}$ and a limiting triple $(u, \mathcal{P}, T) \in \mathcal{D}_\infty$ such that (2.4)-(2.7) hold and

$$E(u, \mathcal{P}, T) \le \liminf_{\varepsilon\to 0} \inf\{E_\varepsilon(y) : y \in \mathcal{A}(a_\varepsilon)\}$$

by Theorem 2.6(i). Due to the boundary conditions it is not hard to see that on each component $P_j \in \mathcal{P}$ we find $A_j \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c_j \in \mathbb{R}^2$ such that

$$\begin{aligned} u_1(x) &= \lim_{k\to\infty} \varepsilon_k^{-1/2}(\mathbf{e}_1 \cdot (\mathbf{Id} - R_j^k)\, x - \mathbf{e}_1 \cdot c_j^k + a_\varepsilon x_1) \\ &= \mathbf{e}_1 \cdot A_j\, x + \mathbf{e}_1 \cdot c_j + a x_1 \end{aligned} \tag{6.1}$$

for a.e. $x \in \Omega'$ with $x_1 < 0$ or $x_1 > l$ and $x \in P_j$. In particular, this implies

$$u_1(x_1, x_2) - u_1(\hat{x}_1, x_2) = |x_1 - \hat{x}_1| a \tag{6.2}$$

for a.e. $x \in \Omega'$ with $\hat{x}_1 < 0$, $x_1 > l$ and $(x_1, x_2), (\hat{x}_1, x_2) \in P_j$.

We first derive the limiting minimal energy and postpone the characterization of the sequence of almost minimizers to the end of the proof. The argument in (6.1) shows that $\nabla T = \mathbf{Id}$ on P_j if $|P_j \cap \{x : x_1 < 0 \text{ or } x_1 > l\}| > 0$. As in the proof of Lemma 5.5 (cf. also proof of Theorem 2.6(ii)) we may assume that $\mathcal{H}^1(\partial P \setminus J_{u_1}) = 0$ after a possible infinitesimal perturbation. Consequently, it is not restrictive to assume $\nabla T^T \nabla u = \nabla u$ a.e. Indeed, we may replace u by $\nabla T u$ in a component P_j which does not intersect the boundaries without changing the energy. By (2.10), a slicing argument in GSBD (see Theorem 3.1) and the fact that $\inf\{Q(F) : \mathbf{e}_1^T F \mathbf{e}_1 = a\} = \alpha a^2$ (see Section 2.4) we obtain

$$\begin{aligned} E(u, \mathcal{P}, T) &\geq \int_{\Omega'} \frac{1}{2} Q(e(\nabla u)) + \int_{J_u} |\nu_u \cdot \mathbf{e}_1| d\mathcal{H}^1 + \mathcal{E}(u) \\ &\geq \int_0^1 \Big(\int_0^l \frac{\alpha}{2} (\mathbf{e}_1^T \nabla u(x) \mathbf{e}_1)^2 \, dx_1 + S^{x_2}(u) \Big) \, dx_2 + \mathcal{E}(u), \end{aligned} \tag{6.3}$$

where S^{x_2} denotes the number of jumps of u_1 on a slice $(-\eta, l+\eta) \times \{x_2\}$ and $\mathcal{E}(u) = \int_{J_u} (1 - |\nu_u \cdot \mathbf{e}_1|) d\mathcal{H}^1$. If $S^{x_2} \geq 1$ the inner integral is bounded from below by 1. By the structure theorem for Caccioppoli partitions (see Theorem 3.5) we find that $((-\eta, 0) \cup (l, l+\eta)) \times \{x_2\} \subset P_j$ for some $j \in \mathbb{N}$ for $\mathcal{H}^1$-a.e. x_2 with $S^{x_2} = 0$. Consequently, if $\#S^{x_2} = 0$, by applying Jensen's inequality we derive that the term is bounded from below by $\frac{1}{2}\alpha l a^2$ due to the boundary conditions (6.2). This implies $E(u) \geq \min\{\frac{1}{2}\alpha l a^2, 1\}$.

Otherwise, it is not hard to see that the configurations $y^{\mathrm{el}}_{\varepsilon_k} = x + F^{a_{\varepsilon_k}} x$ for $x \in \Omega'$ satisfy $E_{\varepsilon_k}(y^{\mathrm{el}}_{\varepsilon_k}) \to \frac{1}{2}\alpha l a^2$ for $k \to \infty$. Likewise, we get $E_{\varepsilon_k}(y^{\mathrm{cr}}_{\varepsilon_k}) = 1$ for all $k \in \mathbb{N}$, where $y^{\mathrm{cr}}_{\varepsilon_k}(x) = x\chi_{x_1 < \frac{1}{2}} + (x + (l a_{\varepsilon_k}, 0))\chi_{x_1 > \frac{1}{2}}$ for $x \in \Omega$ and $y^{\mathrm{cr}}_{\varepsilon_k} = (x_1(1 + a_{\varepsilon_k}), x_2)$ for $x \in \Omega' \setminus \Omega$. This completes (2.11).

It remains to characterize the sequences of almost minimizers. Let u be a minimizer of E under the boundary conditions (6.1). We let first $|a| < a_{\mathrm{crit}}$ and follow the arguments in the proof of [25, Theorem 2.4]. Since $E(u, \mathcal{P}, T) = \frac{1}{2}\alpha l a^2$ we infer from (6.3) that u has no jump on a.e. slice $(-\eta, l+\eta) \times \{x_2\}$ and satisfies $\mathbf{e}_1^T \nabla u \, \mathbf{e}_1 = a$ a.e. by the imposed boundary values and the strict convexity of the mapping $t \mapsto t^2$ on $[0, \infty)$. Thus, if $J_u \neq \emptyset$, a crack normal must satisfy $\nu_u = \pm \mathbf{e}_2$ $\mathcal{H}^1$-a.e. Taking additionally $\mathcal{E}(u)$ into account we find $J_u = \emptyset$ up to an $\mathcal{H}^1$ negligible set, i.e., $u \in H^1(\Omega')$. By the strict convexity of Q on symmetric matrices and the boundary values (6.1) we see that the derivative has the form

$$\nabla u(x) = F^a + A \text{ for a.e. } x \in \Omega$$

for a suitable $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$. Since Ω is connected, we conclude

$$u(x) = F^a \, x + A \, x + c$$

for $x \in \Omega$ and some $c \in \mathbb{R}^2$. In particular, this implies $\mathcal{P}$ consists only of $P_1 = \Omega'$ and thus by (6.1) we get $A = \lim_{k\to\infty} \varepsilon_k^{-1/2}(\mathbf{Id} - R_1^k)$ and $\mathbf{e}_1 \cdot c =$

$-\lim_{k\to\infty}\varepsilon_k^{-1/2}\mathbf{e}_1\cdot c_1^k$. Letting $s=\lim_{k\to\infty}\mathbf{e}_2\cdot(\varepsilon_k^{-1/2}c_1^k+c)$ (which exists by (2.4)(i), (2.6)(i)), we now conclude by (2.4)(i) for a.e. $x\in\Omega$

$$\begin{aligned}\lim_{k\to\infty}&\varepsilon_k^{-1/2}(y_{\varepsilon_k}(x)-x)\\ &=u(x)+\lim_{k\to\infty}\varepsilon_k^{-1/2}\big((R_1^k-\mathbf{Id})\,x+c_1^k\big)\\ &=u(x)-A\,x-c+(0,s)=(0,s)+F^a\,x.\end{aligned}\tag{6.4}$$

If $|a|>a_{\rm crit}$ and u is a minimizer of E under the boundary conditions (6.1), we again consider the lower bound (6.3) and now obtain that on a.e. slice $(0,l)\times\{x_2\}$ a minimizer u has precisely one jump and that $\mathbf{e}_1^T\nabla u\,\mathbf{e}_1=0$ a.e. By the strict convexity of Q on symmetric matrices we then derive that ∇u is antisymmetric a.e. As a consequence, the linearized piecewise rigidity estimate for SBD functions (see [14, Theorem A.1] or the remark below Theorem 3.9) yields that there is a Caccioppoli partition (E_i) of Ω such that

$$u(x)=\sum\nolimits_i(A_ix+b_i)\chi_{E_i}\quad\text{and}\quad J_u=\bigcup\nolimits_i\partial^*E_i\cap\Omega,$$

where $A_i^T=-A_i\in\mathbb{R}^{2\times2}$ and $b_i\in\mathbb{R}^2$. (Note that indeed the linearized rigidity estimate can also be applied in the GSBD-setting as it relies on a slicing argument and an approximation which is also available in the generalized framework, see [28, Section 3.3]. The only difference is that the approximation does not converge in L^1 but only pointwise a.e. which does not affect the argument.)

As $\mathcal{E}(u)=0$, we also note that $\nu_u=\pm\mathbf{e}_1$ a.e. on J_u. Following the arguments in [30], in particular using regularity results for boundary curves of sets of finite perimeter and exhausting the sets ∂^*E_i with Jordan curves, we find that

$$J_u=\bigcup\nolimits_i\partial^*E_i\cap\Omega\subset(p,0)+\mathbb{R}\mathbf{e}_1$$

for some p such that $(p,0)+\mathbb{R}e_2$ intersects both segments $(0,l)\times\{0\}$ and $(0,l)\times\{1\}$. We thus obtain that (E_i) consists of only two sets and u has the form

$$u(x)=\begin{cases}A_1\,x+c_1 & \text{for } x_1<p,\\ A_2\,x+c_2 & \text{for } x_1>p,\end{cases}$$

for $A_i\in\mathbb{R}^{2\times2}_{\rm skew}$ and $c_i\in\mathbb{R}^2$, $i=1,2$. Now repeating the calculation in (6.4) for the sets $P_1=\{x\in\Omega':x_1<p\}$ and $P_2=\Omega'\setminus P_1$ we find $s,t\in\mathbb{R}$ such that for $x\in\Omega$ a.e.

$$\begin{aligned}\lim_{k\to\infty}\varepsilon_k^{-1/2}(y_{\varepsilon_k}(x)-x)&=u(x)-(A_1\,x+c_1)\chi_{x_1<p}(x)+(A_2\,x+c_2)\chi_{x_1>p}(x)\\ &\quad+(0,s)\chi_{x_1<p}(x)-((la,t))\chi_{x_1>p}(x).\end{aligned}$$

This finishes the proof. ☐

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