A quantitative geometric rigidity result in SBD

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Abstract

We present a quantitative geometric rigidity estimate for special functions of bounded deformation in a planar setting generalizing a result by Friesecke, James and Müller for Sobolev functions obtained in nonlinear elasticity theory and a qualitative piecewise rigidity result by Chambolle, Giacomini and Ponsiglione for brittle materials which do not store elastic energy. We show that for each deformation there is an associated triple consisting of a partition of the domain, a corresponding piecewise rigid motion being constant on each connected component of the cracked body and a displacement field measuring the distance of the deformation from the piecewise rigid motion. We also present a related estimate in the geometrically linear setting which can be interpreted as a 'piecewise Korn-Poincaré inequality'.

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Contents

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1 Introduction

It is a subtle problem in mathematical analysis to infer global properties of a function u from conditions on its derivative ∇u given in terms of partial differential relations such as $\nabla u \in K$ or approximate relations such as $\mathrm{dist}(\nabla u, K) \ll 1$, where K denotes a specific set of matrices. In particular, constraining ∇u to be in, or close to, the set $K = SO(d)$ of rigid motions, one is led to the question to what extend such a pointwise (approximate) isometry constraint has the global consequence of rendering u itself (approximately) rigid. As will be detailed below, notably the last decades have witnessed a tremendous progress in establishing such geometric rigidity results; classical theorems for smooth functions have been extended to Sobolev functions and even sharp rigidity estimates have been derived for such functions. In this article we address the problem of deriving a quantitative rigidity estimate beyond the setting in Sobolev spaces, specifically, allowing for functions with jump discontinuities. As such a lack of regularity impedes a direct extension, our main rigidity result has to be formulated in a considerably more complex way. Moreover, major challenges arise in our framework from the fact that the distributional derivative of the mappings under consideration is barely a measure and from the necessity to gain control over both bulk and surface contributions.

Our main motivation comes from variational fracture mechanics. Since the pioneering work of Griffith [27] the propagation of crack is viewed as the result of a competition between the surface energy and the reduction of bulk energy during an infinitesimal increase of the cracked region. Based on this idea Francfort and Marigo [21] have introduced an energy functional comprising elastic bulk and surface contributions in order to tackle problems in fracture mechanics with variational methods, where the displacements and crack paths are determined from an energy minimization principle.

To simplify the mathematical description, problems in this context are often studied in the case of anti-planar shear (see e.g. [16, 20]) or in the realm of linearized elasticity (see e.g. [3, 6, 7, 19, 28, 36]) since such models are usually significantly easier to treat as their nonlinear counterparts. In fact, in the regime of finite elasticity the energy density of the elastic contributions is genuinely geometrically nonlinear due to frame indifference rendering the problem highly non-convex. Consequently, in contrast to linear models already the fundamental question if minimizing configurations for given boundary data exist at all is a challenging problem.

To gain a deeper understanding of nonlinear models in fracture mechanics it is therefore desirable to identify an effective linear theory and in this way to rigorously show that in the small displacement regime the neglection of effects arising from the non-linearities is a good approximation of the problem. Indeed, for elastic bodies not exhibiting cracks the passage from nonlinear to linearized models is by now well understood via Γ-convergence (cf. [15, 35]). It turns out that a fundamental issue in this context is the derivation of suitable rigidity estimates which, based on the deformation of a material, allow to control an associated infinitesimal displacement field measuring the distance from a rigid motion and being the essential quantity on which the linearized elastic energy depends.

Rigidity estimates have a long history going back to the fundamental result of Liouville which states that a smooth function has to be an affine mapping if its gradient is a rotation everywhere. Various generalizations of this classical qualitative theorem in the realm of nonlinear elasticity theory have appeared over the last decades (see e.g. [29, 34]). For brittle materials the problem is more subtle as additional difficulties arise from the fact that the body might be disconnected by the jump set into various components. Chambolle, Giacomini and Ponsiglione [8] recently showed that also in this setting a Liouville-type result holds and that the body behaves piecewise rigidly. In fact, under the constraint that the material does not store elastic energy the only possibility that global rigidity can fail is that the body is divided into various parts each of which subject to a different rigid motion.

However, the above mentioned results fall short of being useful for the investigation of variational models due to the restrictive constraint on the deformation gradient. The fundamental step towards quantitative results was a geometric rigidity estimate by Friesecke, James and Müller [26] which states that, loosely speaking, if the deformation gradient of an H^1-function is close to the set of rotations (e.g. in an L^2 sense), then it is in fact close to one single rotation. This result provides the essential relation between the deformation and a corresponding displacement field and allows to establish a compactness result for a sequence of displacements with uniformly bounded elastic energy.

Whereas this estimate in elasticity theory was generalized to various settings including [9, 32], to the best of our knowledge a corresponding general estimate for brittle materials has not yet been established. The farthest reaching result in this direction seems to be a recent contribution by Negri and Toader [33] where rigidity estimates are provided in the context of quasistatic evolution for a restricted class of admissible cracks. In particular, in their model the different components of the jump set are supposed to have a least positive distance rendering the problem considerably easier. In fact, one can essentially still employ the result in [26] and the specimen cannot be separated into different parts effectively leading to a simple relation between the deformation and the displacement field.

The goal of the present work is the derivation of a new kind of quantitative geometric rigidity estimate in the framework of geometric measure theory without any a priori assumptions on the deformation and the crack geometry, i.e we treat a full free discontinuity problem in the language of Ambrosio and De Giorgi [17]. We call this estimate for brittle materials, which we establish in a planar setting, an SBD-rigidity result as it is formulated in terms of *special functions of bounded deformation* (see [1, 3]). The result may be seen as a suitable combination of the aforementioned estimate for elastic materials [26] and the qualitative result in [8], being tailor-made for general Griffith models where both energy forms are coexistent.

The rigidity result provides the relation between the deformation of a brittle material and the associated displacements. Whereas in elasticity theory there is a simple connection between these two objects, in the present context the description is rather complicated since the deformation is related to a triple consisting of a partition of the domain, a corresponding piecewise rigid motion being constant on each connected component of the cracked body and a displacement field which is defined separately on each piece of the specimen. The result in the present work proves to be the fundamental ingredient to identify an effective linearized theory. For a detailed analysis of compactness results and the derivation of linearized Griffith models from nonlinear energies via Γ-convergence in a small strain limit we refer to the subsequent paper [23].

One essential point in the analysis is the derivation of an inequality for the symmetric part of the gradient. We also see that in general it is not possible to gain control over the full gradient which is not surprising as there is no analogue of Korn's inequality for SBV functions. Consequently, the result is naturally an SBD estimate. In addition, we provide an L^2-bound for the configurations measuring the distance of the deformation itself from a piecewise rigid motion. In contrast to the setting in elasticity theory this is highly nontrivial as Poincaré's inequality cannot be applied due to the possibly present complicated crack geometry. Consequently, our findings are not only interesting in the realm of finite elasticity, but also in a geometrically linear setting and can be interpreted as a 'piecewise Korn-Poincaré inequality'. Moreover, we remark that our main estimate can only be established under the additional condition that we admit an arbitrarily small modification of the deformation.

The derivation of the main result is very involved as among other things one has to face the problems that (1) the body might be disconnected by the jump set, (2) the body might be still connected but only in a small region where the elastic energy is possibly large, (3) the crack geometry might become extremely complex due to relaxation of the elastic energy by oscillating crack paths and infinite crack patterns occurring on different scales. The common difficulty of all these phenomena is the possible high irregularity of the jump set. Even if one can assume that the domain can be decomposed into different sets with Lipschitz boundary (e.g. by a density argument), there are no uniform bounds on the constants of several necessary inequalities such as the Poincaré and Korn inequality and the rigidity estimate [26].

To avoid further complicacies of technical nature concerning the topological structure of cracks in higher dimensions and to concentrate on the essential difficulties arising from the frame indifference of the energy density, we will tackle the problem in a planar setting with isotropic crack energies. However, we believe that our results can be extended to anisotropic surface terms and that the proof provides the principal techniques being necessary to establish the result in arbitrary space dimension. In fact, many arguments are valid also in dimension $d \geq 3$ and we hope that our methods, in particular the modification scheme for deformations and jump sets, may also contribute to solve related problems in the future. One of the essential reasons why we restrict ourselves to the two-dimensional framework is the usage of a Korn-Poincaré-type inequality (see [22]) which was only established in a planar setting due to a lot of technical difficulties concerning the jump set geometry.

The paper is organized as follows. In Section 2 we present the main results about geometric rigidity in SBD and also state a corresponding estimate in the geometrically linear setting which is interesting on its own and considerably simpler to prove than its nonlinear counterpart. As the proof is very long and technical, we give an overview and highlight the principal strategies for the convenience of the reader in Section 2.4.

Section 3 is devoted to some preliminaries. We first recall the definition of special functions of bounded variation and discuss basic properties. Then we recall a (local) Korn-Poincaré-type inequality in SBD (see [22] and Section 3.3) which measures the distance of the displacement field from an infinitesimal rigid motion in terms of the elastic energy. It turns out that this inequality is one of the key ingredients to derive our main result which can be compared with the fact that in elasticity theory the linearized rigidity estimate, called Korn's inequality (see [10]), is one of the fundamental steps to establish the geometrically nonlinear result in [26]. In fact, as a first approach to the main result it is convenient to replace the nonlinear problem by such a linearized version which is significantly easier since (1) the estimate only involves the function itself and not its derivative and (2) the set of infinitesimal rigid motions is a linear space in contrast to $SO(2)$.

Afterwards we recall the geometric rigidity result by Friesecke, James, Müller [26] and carry out a careful analysis how the involved constant depends on the shape of the domain. At this point we notice that easy counterexamples to rigidity estimates in SBD can be constructed if one does not admit a small modification of the deformation (see Section 3.5).

In Section 4 we introduce a procedure to modify sets. In this context, we particularly have to assure that we can control the size and the shape of the jump sets.

The rest of the paper contains the main proof of the SBD-rigidity estimate. The main strategy of the proof is to establish local rigidity results on cells of mesoscopic size (Section 5) which together with the Korn-Poincaré inequality allows to replace the deformation by a modification where the least length of the crack components has increased (Section 6). Repeating the arguments on various mesoscopic scales becoming gradually larger it is possible to show that the modified deformation behaves rigidly on each connected component of the domain (Section 7).

The fact that we analyze the problem on different length scales is indispensable to understand specific size effects correctly such as the accumulation of crack patterns on certain scales. Moreover, we briefly note that similarly as in [24] a mesoscopic localization technique proves to be useful to tackle problems in the framework of brittle materials as hereby effects arising from the bulk and the surface contributions can be separated.

Basically, this is enough the establish the requirements for compactness results in the space of SBD functions (cf. [14]). However, as we are also interested in the derivation of effective linearized models (cf. [23]), we have to assure that we do not change the total energy of the deformation during the modification procedure. In particular, for the surface energy this is a subtle problem and in Section 8 a lot of effort is needed to show that the modified configurations can be constructed in a way such that the crack length does not increase substantially.

2 The main result and overview of the proof

In this section we present our main rigidity estimates in the framework of brittle materials and give and overview of the proof strategies.

2.1 The main setting

Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary and for $M > 0$ we define

$$SBV_M(\Omega) = \Big\{ y \in SBV(\Omega, \mathbb{R}^2) : \|\nabla y\|_\infty \le M, \ \mathcal{H}^1(J_y) < +\infty \Big\}. \tag{2.1}$$

For the definition and properties of the space $SBV(\Omega, \mathbb{R}^2)$, frequently abbreviated as $SBV(\Omega)$ hereafter, we refer to Section 3.1.

Let $W : \mathbb{R}^{2\times 2} \to [0,\infty)$ be a frame-indifferent stored energy density with $W(F) = 0$ iff $F \in SO(2)$. Assume that W is continuous, C^3 in a neighborhood of $SO(2)$ and scales quadratically at $SO(2)$ in the direction perpendicular to infinitesimal rotations. In other words, we have $W(F) \geq c\,\mathrm{dist}^2(F, SO(2))$ for all $F \in \mathbb{R}^{2\times 2}$ and a positive constant c. For $\varepsilon > 0$ define the Griffith-energy $E_\varepsilon : SBV_M(\Omega) \to [0,\infty)$ by

$$E_\varepsilon(y) = \frac{1}{\varepsilon}\int_\Omega W(\nabla y(x))\,dx + \mathcal{H}^1(J_y). \tag{2.2}$$

The main goal of the work at hand is the derivation of uniform rigidity estimates for configurations with $E_\varepsilon(y) \leq C$. Performing the passage to the small strain limit $\varepsilon \to 0$ we have to face major challenges including (1) difficulties concerning the coercivity of the functionals due to the frame indifference of the energy density and (2) the possible high irregularity of the jump set rendering the problem subtle from an analytical point of view.

We briefly note that we can also treat inhomogeneous materials where the energy density has the form $W : \Omega \times \mathbb{R}^{2\times 2} \to [0,\infty)$. Moreover, it suffices to assume $W \in C^{2,\alpha}$, where $C^{2,\alpha}$ is the Hölder space with exponent $\alpha > 0$. In the context of discrete systems the small parameter ε, denoting the order of the elastic energy in our model, represents the typical interatomic distance (compare (2.2) with, e.g., the Griffith functionals in [24, 25]). Having also applications to discrete systems in mind, we will sometimes refer to ε as the 'atomic length scale'.

Observe that M may be chosen arbitrarily large (but fixed) and therefore the constraint $\|\nabla y\|_\infty \leq M$ is not a real restriction as we are interested in the small displacement regime in the regions of the domain where elastic behavior occurs. The uniform bound on the absolute continuous part of the gradient is indeed natural when dealing with discrete energies where the corresponding deformations are piecewise affine on cells of microscopic size (see e.g. [5, 25]). The condition essentially assures that the elastic energy cannot concentrate on scales being much smaller than ε. This observation already shows that the atomic length scale plays an important role in our analysis since the system shows remarkably different behavior on scales smaller and larger than the atomistic unit.

For later we also introduce a relaxed energy functional. For $\rho > 0$, $\varepsilon > 0$ and $U \subset \Omega$ define $f^\rho_\varepsilon(x) = \min\{\frac{x}{\sqrt{\varepsilon}\rho}, 1\}$ and

$$E^\rho_\varepsilon(y, U) = \frac{1}{\varepsilon}\int_U W(\nabla y(x))\,dx + \int_{J_y \cap U} f^\rho_\varepsilon(|[y](x)|)\,d\mathcal{H}^1(x). \tag{2.3}$$

Clearly, we have $E^\rho_\varepsilon(y, U) \leq E_\varepsilon(y)$ for all $y \in SBV_M(\Omega)$ and $U \subset \Omega$.

2.2 Rigidity estimates

We first observe that for configurations with uniform bounded energy $E_\varepsilon(y_\varepsilon)$ the absolute continuous part of the gradient satisfies $\nabla y_\varepsilon \approx SO(2)$ as the stored energy density is frame-indifferent and minimized on $SO(2)$. Assuming that $y_\varepsilon \to y$ in L^1, one can show that $\nabla y \in SO(2)$ a.e. applying lower semicontinuity results for SBV functions (see [31]) and the fact that the quasiconvex envelope of W is minimized exactly on $SO(2)$ (see [38]).

A classical result due to Liouville states that a smooth function y satisfying the constraint $\nabla y \in SO(2)$ is a rigid motion. In the theory of fracture mechanics global rigidity can fail if the crack disconnects the body. More precisely, Chambolle, Giacomini and Ponsiglione have proven that for configurations which do not store elastic energy (i.e. $\nabla y \in SO(2)$ a.e.) and have finite Griffith energy (i.e. $\mathcal{H}^1(J_y) < +\infty$) the only way that rigidity may fail is that the body is divided into at most countably many parts each of which subject to a different rigid motion (see [8]).

Clearly, it is desirable to establish an appropriate quantitative version of this qualitative statement. In nonlinear elasticity such quantitative estimates are available forming one of the starting points of our analysis. Friesecke, James and Müller (see [26] and Theorem 3.10 below) have extended the classical Liouville results and showed that, loosely speaking, if the deformation gradient is close to $SO(2)$ (in L^2), then it is in fact close to one single rotation $R \in SO(2)$ (in L^2).

The overall goal of this work is to 'combine' the rigidity results of the pure elastic and pure brittle regime in order to derive a rigidity estimate for general Griffith functionals (2.2) where both energy forms are coexistent. As a preparation recall the definition of the *perimeter* $P(E,\Omega)$ of a set $E \subset \mathbb{R}^2$ in Ω (see [2, Section 3.3]) and recall that we say that a partition $\mathcal{P} = (P_j)_j$ of Ω is called a *Caccioppoli partition* of Ω if $\sum_j P(P_j,\Omega) < +\infty$. Let $\Omega_\rho = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > C\rho\}$ for $\rho > 0$ and for some sufficiently large constant C.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $M > 0$ and $0 < \eta, \rho \ll 1$. Then there is a constant $C = C(\Omega, M, \eta)$ and a universal $c > 0$ such that the following holds for $\varepsilon > 0$ small enough:*
For each $y \in SBV_M(\Omega) \cap L^2(\Omega)$ with $\mathcal{H}^1(J_y) \le M$ and $\int_\Omega \mathrm{dist}^2(\nabla y, SO(2)) \le M\varepsilon$, there is an open set Ω_y with $|\Omega \setminus \Omega_y| \le C\rho$, a modification $\hat{y} \in SBV_{cM}(\Omega) \cap L^2(\Omega)$ with $\|\hat{y} - y\|^2_{L^2(\Omega_y)} + \|\nabla\hat{y} - \nabla y\|^2_{L^2(\Omega_y)} \le C\varepsilon\rho$ and

$$E^\rho_\varepsilon(\hat{y}, \Omega_\rho) \le E_\varepsilon(y) + C\rho \tag{2.4}$$

with the following properties: We find a Caccioppoli partition $\mathcal{P} = (P_j)_j$ of Ω_ρ with $\sum_j P(P_j, \Omega_\rho) \le C$ and for each P_j a corresponding rigid motion $R_j\, x + c_j$, $R_j \in SO(2)$ and $c_j \in \mathbb{R}^2$, such that the function $u : \Omega \to \mathbb{R}^2$ defined by

$$u(x) := \begin{cases} \hat{y}(x) - (R_j\ x + c_j) & \text{for } x \in P_j \\ 0 & \text{for } x \in \Omega \setminus \Omega_\rho \end{cases} \tag{2.5}$$

satisfies the estimates

$$\begin{aligned} &(i)\ \ \mathcal{H}^1(J_u) \le C, && (ii)\ \ \|u\|^2_{L^2(\Omega_\rho)} \le \hat{C}\varepsilon, \\ &(iii)\ \ \sum\nolimits_j \|e(R_j^T \nabla u)\|^2_{L^2(P_j)} \le \hat{C}\varepsilon, && (iv)\ \ \|\nabla u\|^2_{L^2(\Omega_\rho)} \le \hat{C}\varepsilon^{1-\eta} \end{aligned} \tag{2.6}$$

for some constant $\hat{C} = \hat{C}(\rho)$*, where* $e(G) = \frac{G+G^T}{2}$ *for all* $G \in \mathbb{R}^{2\times 2}$*.*

Whereas in elasticity theory there is a simple connection between the deformation y and the displacement field u, in the present context the description is rather complicated since the deformation is related to a triple $(P_j)_j$, $(R_j, c_j)_j$ and u consisting of a partition, associated piecewise rigid motion and a suitably rescaled displacement field which is defined separately on each piece of the body. The central estimate (2.6) provides the fundamental ingredients to establish a corresponding compactness result (see [23]) by employing a GSBD compactness result proved in [14].

We remark that this estimate might be wrong without allowing for a small modification of the deformation as we show by way of example in Section 3.5. Moreover, we get a sufficiently strong bound only for the symmetric part of the gradient (see (iii)) which is not surprising due to the fact that there is no analogue of Korn's inequality in SBV. However, there is at least a weaker bound on the total absolutely continuous part of the gradient (see (iv)) which will essentially be needed to derive a Γ-convergence result in the passage from nonlinear to linearized models in [23]. We emphasize that also (ii) is highly nontrivial as Poincaré's inequality cannot be applied due to the presence of discontinuity sets.

Remark 2.2 (i) The proof of Theorem 2.1 shows that the Caccioppoli partition $(P_j)_j$ is in fact a finite partition. In particular, each P_j is the union of squares of sidelength $\sim \rho$ and thus $|P_j| \ge c\rho$ for all j.

(ii) In view of (2.4) and (2.6)(i) one also has

$$E_\varepsilon(\hat{y}) \le C E_\varepsilon(y).$$

Moreover, the estimate (2.4) can even be refined. Indeed, we obtain (see (8.14) below)

$$\sum\nolimits_j \tfrac{1}{2} P(P_j, \Omega_\rho) + \int_{J_{\hat{y}} \setminus \partial P} f^\rho_\varepsilon(|[\hat{y}]|)\, d\mathcal{H}^1 \le \mathcal{H}^1(J_y) + c\rho,$$

where $\partial P := \bigcup_j \partial P_j$. Whereas on the boundary of the partition ∂P there is a sharp estimate for the surface energy, the passage to to a relaxed functional in the interior of the sets is necessary due to the possible presence of microcracks accumulating on different mesoscopic scales.

(iii) The assumption $y \in L^2(\Omega)$ may be dropped. In this case we obtain a slightly weaker approximation of the form $\|\hat{y} - y\|^2_{L^1(\Omega_y)} \le C\varepsilon\rho$ (cf. the approximation schemes in [8, Theorem 3.1], [22, Theorem 2.3]).

(iv) The approximation preserves an L^∞-bound, i.e. $\|y\|_\infty \le M$ implies $\|\hat{y}\|_\infty \le cM$.

2.3 A piecewise Korn-Poincaré inequality

We now discuss a variant of Theorem 2.1 in the geometrically linear setting which can be interpreted as a 'piecewise Korn-Poincaré-inequality in SBD'. Let $\mathbb{R}^{2\times 2}_{\rm skew} = \{A \in \mathbb{R}^{2\times 2} : A^T = -A\}$ be the set of skew symmetric matrices. Set

$$F^\rho_\varepsilon(y, U) = \frac{1}{\varepsilon}\int_U V(e(\nabla u)(x))\,dx + \int_{J_u \cap U} f^\rho_\varepsilon(|[u]|)\,d\mathcal{H}^1 \tag{2.7}$$

for a coercive quadratic form V, i.e. $V(G) \ge c|G|^2$ for $c > 0$ and $G \in \mathbb{R}^{2\times 2}_{\rm sym}$. Furthermore, define $F_\varepsilon = F^0_\varepsilon(\cdot, \Omega)$, where $f^0_\varepsilon \equiv 1$. For the definition of the space SBD we refer to Section 3.1.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $M > 0$, and $0 < \rho \ll 1$. Then there is a constant $C = C(\Omega, M)$ such that for $\varepsilon > 0$ small enough the following holds:*
For each $u \in SBD^2(\Omega, \mathbb{R}^2) \cap L^2(\Omega, \mathbb{R}^2)$ with $\mathcal{H}^1(J_u) \le M$ and

$$\int_\Omega |e(\nabla u)(x)|^2\,dx \le M\varepsilon,$$

there is an open set Ω_u with $|\Omega \setminus \Omega_u| \le C\rho$, a modification $\hat{u} : \Omega \to \mathbb{R}^2$ with $\|\hat{u} - u\|^2_{L^2(\Omega_u)} + \|e(\nabla\hat{u}) - e(\nabla u)\|^2_{L^2(\Omega_u)} \le C\rho\varepsilon$ and

$$F^\rho_\varepsilon(\hat{u}, \Omega_\rho) \le F_\varepsilon(u) + C\rho$$

with the following properties: We find a Caccioppoli partition $\mathcal{P} = (P_j)_j$ of Ω_ρ with $\sum_j P(P_j, \Omega_\rho) \le C$ and for each P_j a corresponding infinitesimal rigid motion $A_j\,x + c_j$, $A_j \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c_j \in \mathbb{R}^2$, such that $\mathcal{H}^1(J_{\hat{u}}) \le C$ and

$$(i)\ \|e(\nabla\hat{u})\|^2_{L^2(\Omega_\rho)} \le C\varepsilon, \quad (ii)\ \sum\nolimits_j \|\hat{u} - (A_j\ \cdot - c_j)\|^2_{L^2(P_j)} \le \hat{C}\varepsilon. \tag{2.8}$$

for some constant $\hat{C} = \hat{C}(\rho)$.

To prove Theorem 2.3 one may essentially follow the proof of Theorem 2.1 with some changes, where altogether the proof is considerably simpler as a lot of estimates and arguments can be skipped. We again observe that estimate (2.8) together with the result of [14] is the fundamental ingredient to establish a compactness result.

2.4 Overview of the proof

As the proof of Theorem 2.1 is very long and technical, we present here a short overview for the convenience of the reader and highlight the principle proof strategies.

The main estimates in the rigidity result (see (2.6)) provide bounds for both the displacement field u itself and its derivative. The fundamental ingredient to measure the distance of the function from a rigid motion is a (local) Korn-Poincaré-type inequality established in [22]. The other key point is then the derivation of an estimate for the symmetric part of the gradient. Using the expansion

$$|e(R^T(\nabla y - \mathbf{Id}))|^2 = \mathrm{dist}^2(\nabla y, SO(2)) + O(|\nabla y - R|^4) \tag{2.9}$$

and recalling that $\| \mathrm{dist}(\nabla y, SO(2))\|^2_{L^2(\Omega)} \sim \varepsilon$ we see that it suffices to establish an estimate of fourth order. Indeed, also in the proof of the geometric rigidity result in nonlinear elasticity (see [26]) one first derives a bound for $\|\nabla y - R\|^4_{L^4(\Omega)}$ to control the symmetric part. The control over the full gradient is then obtained by Korn's inequality.

Clearly, in our framework this rigidity result (see Theorem 3.10 below) cannot be applied due to the presence of cracks, in particular $\Omega \setminus J_y$ will generically not be a Lipschitz set. Therefore, by a density argument we again first assume that the jump set is contained in a finite number of rectangle boundaries. A careful quantitative analysis shows that the constant in Theorem 3.10 depends on the quotient of the diameter of the domain, denoted by k, and the minimal distance of two cracks, denoted by s. In particular, $C = C(k/s) \sim 1$ if $k \sim s$. Provided that $\frac{k}{s}$ is not too large, the principal strategy will be to show that possibly after a modification we get $\|\nabla y - R\|^2_{L^\infty(\Omega)} \le (C(k/s))^{-1}$ which then gives

$$\|e(R^T(\nabla y - \mathbf{Id}))\|^2_{L^2(\Omega)} \le \varepsilon + (C(k/s))^{-1}\|\nabla y - R\|^2_{L^2(\Omega)} \le C\varepsilon \tag{2.10}$$

by (2.9) and Theorem 3.10. Of course, in general we cannot suppose that $\frac{k}{s}$ is not large. Moreover, a global rigidity result may fail due to the separation of the domain by the jump set. Consequently, we will apply the presented ideas on a fine partition of the Lipschitz domain Ω consisting of squares with diameter k. This local result will be used to modify the jump set such that the minimal distance of each pair of cracks increases. Then we can repeat the arguments for a larger k. The idea is that after an iterative application of the arguments we obtain an estimate for $k \approx \rho$ which then will provide rigid motions on the connected components of the domain (see (2.5)) with the desired properties.

In Section 4 we introduce a procedure to modify sets and conduct a thorough analysis on how to control the size and the shape of the jump sets.

In Section 5 we construct piecewise constant $SO(2)$-valued mappings approximating the deformation gradient. In each square Q of diameter k we may assume that the elastic energy is bounded by $\sim \varepsilon k$ as otherwise it would be energetically favorable to introduce jumps at the boundary of the square and to replace the deformation in the interior by a rigid motion. (The same technique has been used in the proof of the Korn-Poincaré inequality.) Similarly as in [26] we pass to the harmonic part of the deformation (denoted by $\hat{y}$) and obtain by the mean value property

$$\begin{aligned}\|\nabla \hat{y} - R_Q\|^2_{L^\infty(\hat{Q})} &\le Ck^{-2}\|\nabla \hat{y} - R_Q\|^2_{L^2(Q)} \\ &\le C(k/s)k^{-2}\| \operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(Q)} \le C(k/s)k^{-1}\varepsilon\end{aligned} \tag{2.11}$$

for a suitable $R_Q \in SO(2)$, where $\hat{Q} \subset Q$ is a slightly smaller square. Consequently, if we can assure that $\frac{\varepsilon}{k} \le (C(k/s))^{-2}$ we obtain the desired L^∞-bound which allows to derive an estimate of the form (2.10). We note that for this argument we at least have to assume that $k \gg \varepsilon$ which will be denoted as the 'superatomistic regime' (recall the discussion about the signification of ε after (2.2)).

In the subsequent Section 5.2 we show that not only the distance of the derivative from a piecewise rigid motion can be controlled but also the distance of the function itself. On the one hand this is essential for (2.6), on the other hand such an estimate is crucial for establishing a modification of the deformation and the jump set. The main idea is to apply the Korn-Poincaré-type inequality proved in [22] on the function $R_Q^T y - \mathbf{id}$. Major difficulties arise from the facts that the rotation R_Q may vary from one square to another and that the inequality derived in [22] only provides a local estimate (cf. also Corollary 3.7). Consequently, the arguments have to be repeated for several shifted copies of the fine partition (see Lemma 5.4). Moreover, the projections P_Q onto the the space of infinitesimal rigid motions (see Theorem 3.3 below) have to be combined with the rotations R_Q in a suitable way to obtain appropriate rigid motions, which do not vary too much on adjacent squares (see Lemma 5.6).

Having an approximation of the deformation by piecewise rigid motions defined on squares with diameter k, we then are able to modify the function such that the minimal distance $\tilde{s}$ of two cracks of the new configuration satisfies $\tilde{s} \sim k$ (see Lemma 6.1). Now we can repeat the above procedure for some larger $\tilde{k}$ such that $\varepsilon/\tilde{k} \le (C(\tilde{k}/\tilde{s}))^{-2}$ is guaranteed and we can repeat the arguments in (2.11).

The strategy is to end up with $k \approx \rho$ after a finite number of iterations. As the number of iteration steps is not bounded but grows logarithmically with $\frac{1}{\varepsilon}$ we have to assure that in each step the surface and the elastic energy do not increase too much. The crucial point is that during the iteration process the coarseness of the partition k grows much faster than the stored elastic energy ε such that the argument in (2.11) may be repeated. The details are given in Theorem 7.3. Having an estimate for $k \approx \rho$ it is then not hard to establish the desired result up to a small exceptional set (see Theorem 7.2).

Clearly, we cannot assume that initially $s \ge \varepsilon$. In this case the argument in (2.11) can typically not be applied. As a remedy we do not employ the geometric rigidity result directly but first approximate the deformation in each square by an H^1-function, where the distance can be measured by the curl of ∇y. (See Theorem 3.1 below which was one of the essential ingredients to prove the qualitative result in [8].) We address this problem in Lemma 5.3 and subsequently we show that we may modify the configuration such that $\tilde{s} \geq \varepsilon$ (see Theorem 7.4).

Finally, by a density argument we can approximate each SBV function by a configuration where the jump set is contained in a finite number of rectangle boundaries (see proof of Theorem 7.1). Observe that standard density results as [12] cannot be applied directly in our framework since in general an L^∞ bound for the derivative is not preserved. The problem can be circumvented by using a different approximation introduced in [7] at the cost of a non exact approximation of the jump set, which suffices for our purposes.

The rigidity result, which we then have established, only holds up to a small exceptional set as due to the modification of the jump set the deformation might not be defined in the interior of certain rectangles. We emphasize that such an estimate is not enough to obtain good compactness and convergence results, in particular for the convergence of the surface energy further difficulties arise. Therefore, we eventually have to construct a suitable extension to the whole domain. A major challenge is to determine the surface energy correctly, at least for the relaxed functional (2.3). This problem is addressed in Section 8.

For small cracks a good extension is already provided by the Korn-Poincaré inequality [22] which is based on the derivation of a suitable modification for which jump heights can be controlled. Near large cracks we define the extension as a piecewise constant rigid motion such that the jump heights on the new jump sets are sufficiently small (see the proof of Theorem 2.1). Consequently, the length of these jumps may possibly be much larger than $\mathcal{H}^1(J_y)$, but due to the small jump height their contribution to (2.3) is considerably small. Finally, for the large cracks in the domain, in particular for the boundary $\bigcup_j \partial P_j$ of the partition $(P_j)_j$, we have to construct an appropriate jump set consisting of Jordan curves which provides the correct crack energy up to a small error (see Lemma 8.1).

3 Preliminaries

In this preparatory section we recall first the definition and basic properties of functions of bounded variation. Then we introduce the notion of *boundary components* and present the Korn-Poincaré inequality established in [22]. Finally, we recall the geometric rigidity result in nonlinear elasticity and carefully estimate the involved constant pertaining to its dependence on the shape of the domain.

3.1 Special functions of bounded variation

In this section we collect the definitions of SBV and SBD functions. Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Recall that the space $SBV(\Omega, \mathbb{R}^d)$, abbreviated as $SBV(\Omega)$ hereafter, of *special functions of bounded variation* consists of functions $y \in L^1(\Omega, \mathbb{R}^d)$ whose distributional derivative Dy is a finite Radon measure, which splits into an absolutely continuous part with density ∇y with respect to Lebesgue measure and a singular part $D^j y$ whose Cantor part vanishes and thus is of the form

$$D^j y = [y] \otimes \xi_y \mathcal{H}^{d-1} \lfloor J_y,$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, J_y (the 'crack path') is an $\mathcal{H}^{d-1}$-rectifiable set in Ω, ξ_y is a normal of J_y and $[y] = y^+ - y^-$ (the 'crack opening') with $y^\pm$ being the one-sided limits of y at J_y. If in addition $\nabla y \in L^2(\Omega)$ and $\mathcal{H}^{d-1}(J_y) < \infty$, we write $y \in SBV^2(\Omega)$. See [2] for the basic properties of this function space.

Likewise, we say that a function $y \in L^1(\Omega, \mathbb{R}^d)$ is a *special function of bounded deformation* if the symmetrized distributional derivative $Eu := \frac{(Dy)^T + Dy}{2}$ is a finite $R^{d\times d}_{\mathrm{sym}}$-valued Radon measure with vanishing Cantor part. It can be decomposed as

$$Ey = e(\nabla y)\mathcal{L}^d + E^j y = e(\nabla y)\mathcal{L}^d + [y] \odot \xi_y \mathcal{H}^{d-1}|_{J_y}, \tag{3.1}$$

where $e(\nabla y)$ is the absolutely continuous part of Ey with respect to the Lebesgue measure $\mathcal{L}^d$, $[y]$, ξ_y, J_y as before and $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. For basic properties of this function space we refer to [1, 3].

The general idea in our analysis will be to establish Theorem 2.1 for a dense subset of SBV for which we can suppose much more regularity of the jump set. For density results in the spaces SBV and SBD we refer to [12, 13] and [7], respectively. In our framework we cannot use these results directly but have to derive a slightly different variant of [12] in order to preserve an L^∞-bound for the derivative (see the proof of Theorem 7.1).

Moreover, we recall the property that the distance of an SBV function to Sobolev functions can be measured by the distribution curl ∇y (see [8, Proposition 5.1]).

Theorem 3.1 *Let $Q = (0,1)^d$. Let $y \in SBV_\infty(Q) := \{y \in SBV(Q, \mathbb{R}^d) : \|\nabla y\|_\infty < \infty, \ \ \mathcal{H}^{d-1}(J_y) < \infty\}$. Then $\mu_y := curl \nabla y$ is a measure concentrated on J_y such that*

$$|\mu_y| \le C \|\nabla y\|_\infty \mathcal{H}^{d-1}|_{J_y}.$$

Moreover, for $p < \frac{d}{d-1}$ there is a constant $C = C(p) > 0$ such that for all $y \in SBV_\infty(Q)$ there is a function $u \in H^1(Q, \mathbb{R}^d)$ such that

$$\|\nabla u - \nabla y\|_{L^p(Q)} \le C|\mu_y|(Q) \le C\|\nabla y\|_\infty \mathcal{H}^{d-1}(J_y).$$

3.2 Boundary components

Using a density result alluded to above it will suffice to prove the main result for configurations where the jump set is contained in the boundary of squares. In this section we recall the necessary notation and definitions for boundary components introduced in [22].

For $s > 0$ we partition $\mathbb{R}^2$ up to a set of measure zero into squares $Q^s(p) = p + s(-1,1)^2$ for $p \in I^s := s(1,1) + 2s\mathbb{Z}^2$. Let

$$\mathcal{U}^s := \Big\{ U \subset \mathbb{R}^2 : U = \Big(\bigcup\nolimits_{p \in I} \overline{Q^s(p)} \Big)^\circ : \quad I \subset I^s \Big\}. \tag{3.2}$$

Here the superscript $\circ$ denotes the interior of a set. Let $\mu > 0$. We will concern ourselves with subsets $V \subset Q_\mu := (-\mu,\mu)^2$ of the form

$$\mathcal{V}^s := \{ V \subset Q_\mu : V = Q_\mu \setminus \bigcup\nolimits_{i=1}^m X_i, \quad X_i \in \mathcal{U}^s, \ X_i \text{ pairwise disjoint} \} \tag{3.3}$$

for $s > 0$. Note that each set in $V \in \mathcal{V}^s$ coincides with a set $U \in \mathcal{U}^s$ up to subtracting a set of zero Lebesgue measure, i.e. $U \subset V$, $\mathcal{L}^2(V \setminus U) = 0$. The essential difference of V and the corresponding U concerns the connected components of the complements $Q_\mu \setminus V$ and $Q_\mu \setminus U$. Observe that one may have $Q_\mu \setminus \bigcup_{i=1}^m X_i = Q_\mu \setminus \bigcup_{i=1}^{\hat{m}} \hat{X}_i$ with $(X_1, \ldots, X_m) \neq (\hat{X}_1, \ldots, \hat{X}_{\hat{m}})$, e.g. by combination of different sets. In such a case we will regard $V_1 = Q_\mu \setminus \bigcup_{i=1}^m X_i$ and $V_2 = Q_\mu \setminus \bigcup_{i=1}^{\hat{m}} \hat{X}_i$ as different elements of $\mathcal{V}^s$. For this and the following sections we will always tacitly assume that all considered sets are elements of $\mathcal{V}^s$ for some small, fixed $s > 0$.

Let $W \in \mathcal{V}^s$ and arrange the components $X_1, \ldots, X_m$ of the complement such that $\partial X_i \subset Q_\mu$ for $1 \le i \le n$ and $\partial X_i \cap \partial Q_\mu \neq \emptyset$ otherwise. Define $\Gamma_i(W) = \partial X_i$ for $i = 1, \ldots, n$. In the following we will often refer to these sets as *boundary components*. Note that $\bigcup_{i=1}^n \Gamma_i(W)$ might not cover $\partial W \cap Q_\mu$ completely if $n < m$. We frequently drop the subscript and write $\Gamma(W)$ or just Γ if no confusion arises. Observe that in the definition we do not require that boundary components are connected. Therefore, we additionally introduce the subset $\mathcal{V}^s_{\mathrm{con}} \subset \mathcal{V}^s$ consisting of the sets where all $\overline{X_1}, \ldots, \overline{X_n}$ are connected.

Beside the Hausdorff-measure $|\Gamma|_{\mathcal{H}} = \mathcal{H}^1(\Gamma)$ (we will use both notations) we define the 'diameter' of a boundary component by

$$|\Gamma|_\infty := \sqrt{|\pi_1 \Gamma|^2 + |\pi_2 \Gamma|^2},$$

where π_1, π_2 denote the orthogonal projections onto the coordinate axes. We recall that many arguments in the proof of the Korn-Poincaré inequality in [22] relied on the fact that due to the strict convexity of $|\cdot|_\infty$ it is often energetically favorable if different components are combined to a larger one.

Note that by definition of $\mathcal{V}^s$ (in contrast to the definition of $\mathcal{U}^s$) two components in $(\Gamma_i)_i$ might not be disjoint. Therefore, we choose an (arbitrary) order $(\Gamma_i)_{i=1}^n = (\Gamma_i(W))_{i=1}^n$ of the boundary components of W, introduce

$$\Theta_i = \Theta_i(W) = \Gamma_i \setminus \bigcup\nolimits_{j<i} \Gamma_j \tag{3.4}$$

for $i = 1, \ldots, n$ and observe that the boundary components $(\Theta_i)_i$ are pairwise disjoint. With a slight abuse of notation we define

$$|\Theta_i|_\infty = |\Gamma_i|_\infty.$$

Again we will often drop the subscript if we consider a fixed boundary component. We now introduce a convex combination of $|\cdot|_\infty$ and $|\cdot|_{\mathcal{H}}$. For an $h_* > 0$ to be specified below we set

$$|\Theta|_* = h_* |\Theta|_{\mathcal{H}} + (1 - h_*)|\Theta|_\infty. \tag{3.5}$$

For sets $W \in \mathcal{V}^s$ we then define

$$\|W\|_Z = \sum\nolimits_{j=1}^n |\Theta_j(W)|_Z \tag{3.6}$$

for $Z = \mathcal{H}, \infty, *$. Note that $\|W\|_\infty$, $\|W\|_{\mathcal{H}}$ and thus also $\|W\|_*$ are independent of the specific order which we have chosen in (3.4). Indeed, for $\|W\|_\infty$ this is clear as $|\Theta_i|_\infty = |\Gamma_i|_\infty$, for $\|W\|_{\mathcal{H}}$ it follows from the fact that $\|W\|_{\mathcal{H}} = \mathcal{H}^1(\bigcup_{i=1}^n \Gamma_i)$.

From [22] we recall some elementary properties of $|\cdot|_*$ which will be exploited frequently in the following.

Lemma 3.2 *Let $W \subset Q_\mu$. Let $\Gamma = \Gamma(W)$ be a boundary component with $\Gamma = \partial X$ and let $\Theta \subset \Gamma$ be the corresponding set defined in* (3.4). *Moreover, let $V \in \mathcal{U}^s$ be a rectangle with $\overline{V} \cap \overline{X} \neq \emptyset$. Suppose that h_* is sufficiently small. Then*

(i) $|\Gamma|_* \geq |\partial R(\Gamma)|_*$ *if Γ is connected, where $R(\Gamma)$ denotes the smallest (closed) rectangle such that $\Gamma \subset R(\Gamma)$,*

(ii) $|\Theta|_* = |\Gamma|_* \Leftrightarrow |\Theta|_{\mathcal{H}} = |\Gamma|_{\mathcal{H}}$,

(iii) $|\partial(X \setminus \overline{V})|_\infty \leq |\Theta|_\infty$ *and* $|\Theta \setminus \overline{V}|_{\mathcal{H}} \leq |\Theta|_{\mathcal{H}}$,

(iv) $|\partial(V \cup X)|_* \leq |\partial V|_* + |\Gamma|_*$,

(v) $\frac{1}{\sqrt{2}}|\partial R|_{\mathcal{H}} \leq 2|\partial R|_\infty \leq |\partial R|_{\mathcal{H}}$ *if $R \in \mathcal{U}^s$ is are rectangle.*

As a further preparation, we define $H(W) \supset W \in \mathcal{V}^s$ as the 'variant of W without holes' by

$$H(W) = W \cup \bigcup\nolimits_{j=1}^n X_j. \tag{3.7}$$

Additionally, for $\lambda > 0$ we define $H^\lambda(W) \supset W$ as the 'variant of W without holes of size smaller than λ': We arrange the sets $(\Gamma_j)_{j=1,\ldots,n}$ in the way that $|\Gamma_j|_\infty \leq \lambda$ for $j \geq l_\lambda$ and $|\Gamma_j|_\infty > \lambda$ for $j < l_\lambda$. Define

$$H^\lambda(W) = W \cup \bigcup\nolimits_{j=l_\lambda}^n X_j. \tag{3.8}$$

3.3 A Korn-Poincaré inequality

We start this section with the formulation of the classical Korn-Poincaré inequality in BD (see [30, 37]).

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^d$ bounded, connected with Lipschitz boundary and let $P : L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d)$ be a linear projection onto the space of infinitesimal rigid motions. Then there is a constant $C > 0$, which is invariant under rescaling of the domain, such that for all $u \in BD(\Omega, \mathbb{R}^d)$*

$$\|u - Pu\|_{L^{\frac{d}{d-1}}(\Omega)} \le C|Eu|(\Omega),$$

where $Eu = \frac{Du^T + Du}{2}$ is the symmetrized distributional derivative.

There is also a corresponding trace estimate.

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^2$ bounded, connected with Lipschitz boundary. There exists a constant $C > 0$ such that the trace mapping $\gamma : BD(\Omega, \mathbb{R}^2) \to L^1(\partial\Omega, \mathbb{R}^2)$ is well defined and satisfies the estimate*

$$\|\gamma u\|_{L^1(\partial\Omega)} \le C\big(\|u\|_{L^1(\Omega)} + |Eu|(\Omega)\big)$$

for each $u \in BD(\Omega, \mathbb{R}^2)$.

It first appears that this inequality is not adapted for linearized Griffith energies of the form (2.7) (or their nonlinear counterparts (2.2)) as in $|Eu|(\Omega)$ the jump height is involved and in (2.7) we only have control over the size of the crack. However, in [22] we have shown that one can indeed find bounds on the jump heights after a suitable modification of the jump set and the displacement field. Before we can recall the results obtained in [22], we have to introduce a further notation: We fix a sufficiently large universal constant c and let $\mathcal{W}^s \subset \mathcal{V}^s$ be the subset consisting of the sets, where for a specific ordering of the boundary components $(\Gamma_l)_{l=1}^n$ we find for all components Γ_l a corresponding rectangle $R_l = R(\Gamma_l) \in \mathcal{U}^s$ such that

$$(i)\ |\Gamma_l|_\infty \le |\partial R_l|_\infty \le c|\Gamma_l|_\infty, \quad (ii)\ |\Theta_l|_\mathcal{H} \le |\partial R_l|_\mathcal{H}, \quad (iii)\ |\partial R_l|_* \le c|\Theta_l|_*. \tag{3.9}$$

In particular, the diameter of Γ_l and the corresponding rectangle R_l are comparable. (Note that in [22, Section 5] we have defined the set $\mathcal{W}^s$ in a slightly different way. See also (3.5) and (3.6) in [22].) For given $\bar\tau > 0$ and a rectangle $R_l \in \mathcal{U}^s$ we define $\tau_l = \bar\tau|\partial R_l|_\infty$ and let $N^{\tau_l}(\partial R_l) \in \mathcal{U}^s$ be the largest set in $\mathcal{U}^s$ with $N^{\tau_l}(\partial R_l) \subset \{x \in \mathbb{R}^2 \setminus \overline{R_l} : \mathrm{dist}_\infty(x, \partial R_l) \le \tau_l\}$, where $\mathrm{dist}_\infty(x, A) := \inf_{y \in A} \max_{i=1,2} |(x - y) \cdot \mathbf{e}_i|$ for $A \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$. We can now formulate [22, Theorem 5.2] as follows.

Theorem 3.5 *Let $\varepsilon > 0$ and $h_* \geq \sigma > 0$ sufficiently small. Let $C_1 = C_1(\sigma, h_*) \geq 1$ large, $0 < C_2 = C_2(\sigma, h_*) < 1$ small enough, and $\bar{\tau} > 0$ such that $C_2 \ll \bar{\tau} \ll 1$. Moreover, let $c > 0$ be a universal constant. Then for all $W \in \mathcal{V}^s_{\rm con}$ and $u \in H^1(W)$ there is a set $U \in \mathcal{W}^{C_2 s}$ with $|U \setminus W| = 0$ and an extension $\bar{u}$ in SBV defined by*

$$\bar{u}(x) = \begin{cases} A_l\, x + c_l & x \in X_l \quad \textit{for all } \Gamma_l(U) \textit{ with } N^{\tau_l}(\partial R_l) \subset H(U), \\ u(x) & \textit{else}, \end{cases} \tag{3.10}$$

such that for all $\Gamma_l(U)$ with $N^{\tau_l}(\partial R_l) \subset H(U)$

$$\int_{\Theta_l(U)} |[\bar{u}](x))|^2 \, d\mathcal{H}^1(x) \leq C_1 \varepsilon |\Theta_l(U)|^2_*. \tag{3.11}$$

Moreover, one has $|W \setminus U| \leq c\|U\|^2_\infty$ and

$$\varepsilon\|U\|_* + \|e(\nabla u)\|^2_{L^2(U)} \leq (1+\sigma)\big(\varepsilon\|W\|_* + \|e(\nabla u)\|^2_{L^2(W)}\big).$$

Remark 3.6 (i) During the modification process in Theorem 3.5 the components $X_{n+1}(W), \ldots, X_m(W)$ at the boundary of Q_μ might be changed and the corresponding components of U are given by $X_j(U) = X_j(W) \setminus \overline{H(U)}$ for $j = n+1, \ldots, m$. In particular, one has $|\partial X_j(U)|_* \leq |\partial X_j(W)|_*$ arguing as in Lemma 3.2.

(ii) Observe that $U \notin \mathcal{V}^s_{\rm con}$ is possible as components can be separated by other components in the proof of Theorem 3.5. However, we can obtain a set $U' \subset U$ with $\|U'\|_* \leq \|U\|_*$ and $|U \setminus U'| \leq C\|U'\|^2_\infty \leq C\mu\|U'\|_\infty$ such that all components of U' are pairwise disjoint and rectangular and thus particularly connected. Moreover, for each $\Gamma(U)$ the corresponding rectangle $R(U)$ given by (3.9) is contained in a component of U'. (Namely in the same component as $\Gamma(U)$.)

Recall (3.1) and define $\mathcal{E}(V) = \int_V |e(u)| + |D^j u|(V)$. Observe that $\mathcal{E}(V)$ differs from $|Eu|(V)$ as we consider the measure $D^j u$ instead of $E^j u$. We then obtain the following corollary (cf. [22, Corollary 5.7]).

Corollary 3.7 *Let $\varepsilon, \mu, h_* > 0$. Let $U \subset Q_\mu = (-\mu, \mu)^2$, $U \in \mathcal{W}^{C_2 s}$ and $u \in H^1(U)$. Assume there is a square $\tilde{Q} = (-\tilde{\mu}, \tilde{\mu})^2 \subset Q_\mu$ such that* (3.11) *is satisfied for all components $\Theta_l(U)$ having nonempty intersection with $\tilde{Q}$, where $\bar{u}$ is the extension of u defined in* (3.10). *Then there is a universal constant C such that*

$$|E\bar{u}|(\tilde{Q})^2 \leq (\mathcal{E}(\tilde{Q}))^2 \leq C\tilde{\mu}^2 \|e(\nabla u)\|^2_{L^2(U\cap\tilde{Q})} + CC_1\mu\varepsilon|\partial U \cap \tilde{Q}|_{\mathcal{H}} |\partial U \cap Q_\mu|_{\mathcal{H}},$$

where C_1 is the constant in Theorem 3.5.

Now observe that by combination of Theorem 3.5, Corollary 3.7 and Theorem 3.3 one may estimate the distance of u from an infinitesimal rigid motion. We will exploit this property in Section 5.2. [22, Lemma 6.7] provides the following estimate for the skew symmetric matrices involved in (3.10).

Lemma 3.8 *Let be given the situation of Theorem 3.5 for a function $u \in H^1(W)$ and define $y = \bar{R}\,(\mathbf{id}+u)$, where $\mathbf{id}$ denotes the identity function and $\bar{R} \in SO(2)$. Let $V \subset Q_\mu$ be a rectangle and let $\mathcal{F}(V)$ be the boundary components $(\Gamma_l)_l = (\Gamma_l(U))_l$ satisfying $N^{\tau_l}(\partial R_l) \subset V$ and (3.11). Then there is a $C_3 = C_3(\sigma, h_*)$ such that*

$$\sum\nolimits_{\Gamma_l \in \mathcal{F}(V)} |X_l|^2_\infty |A_l|^p \le C_3\big(\|\nabla y - \bar{R}\|^p_{L^p(V\cap W)} + (\varepsilon s^{-1})^{\frac{p}{2}-1}\varepsilon|\partial U \cap V|_{\mathcal{H}}\big)$$

for $p = 2, 4$, where $X_l \subset Q_\mu$, $A_l \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ is given in (3.10).

We close this section with a short remark about the constants involved in the above estimates.

Remark 3.9 (i) The constants $C_i = C_i(\sigma, h_*)$, $i = 1, 2, 3$, have polynomial growth in σ: We find $z \in \mathbb{N}$ large enough such that $C_1(\sigma, h_*), C_3(\sigma, h_*) \le C(h_*)\sigma^{-z}$ and $C_2(\sigma, h_*) \ge C(h_*)\sigma^z$.

(ii) The constant $C_2(\sigma, h_*)$ can be chosen small with respect to σ (see (5.12) in [22]). In particular, we can assume $C_2(\sigma, h_*) \ll \sigma$ as well as $\bar{C}C_2(\sigma, h_*) \le \sigma$ for constants $\bar{C} = \bar{C}(h_*)$.

(iii) We find a constant $\bar{C} = \bar{C}(h_*)$ such that $\bar{\tau} \le \bar{C}C_2$ (cf. (5.2) in [22]).

(iv) If we apply Theorem 3.5 on sets $W \in \mathcal{V}^{\bar{s}}_{\mathrm{con}}$ for some $\bar{s} \ll s$, where the length of all boundary components of W is bounded from below by s, we still obtain $U \in \mathcal{V}^{C_2 s}$.

3.4 Geometric rigidity in nonlinear elasticity

The following geometric rigidity result in nonlinear elasticity proved by Friesecke, James and Müller (see [26]) is one of the starting points for our analysis.

Theorem 3.10 *Let $\Omega \subset \mathbb{R}^d$ a (connected) Lipschitz domain and $1 < p < \infty$. Then there exists a constant $C = C(\Omega, p)$ such that for any $y \in W^{1,p}(\Omega, \mathbb{R}^d)$ there is a rotation $R \in SO(d)$ such that*

$$\|\nabla y - R\|_{L^p(\Omega)} \le C\,\|\mathrm{dist}(\nabla y, SO(d))\|_{L^p(\Omega)}\,.$$

One ingredient in the proof is the following decomposition into a harmonic and a rest part.

Theorem 3.11 *Let $\Omega \subset \mathbb{R}^2$ open and $1 < p < \infty$. There is a constant $C = C(p)$ such that all $y \in W^{1,p}(\Omega, \mathbb{R}^2)$ can be split into $y = w + z$, where w is a harmonic function and z satisfies*

$$\|\nabla y - \nabla w\|_{L^p(\Omega)} = \|\nabla z\|_{L^p(\Omega)} \leq C \| \operatorname{dist}(\nabla y, SO(2))\|_{L^p(\Omega)}.$$

Note that the constant C is independent of the domain Ω. In higher dimensions one additional needs $\|\nabla y\|_\infty \leq M$ for $M > 0$.
Proof. Following the singular-integral estimates in [11, Section 2.4] we find $\|\nabla z\|_{L^p(\Omega)} \leq c\|\mathrm{cof}\nabla y - \nabla y\|_{L^p(\Omega)}$. The assertion follows from the fact that $|\mathrm{cof}A - A|^p \leq C_p \operatorname{dist}^p(A, SO(2))$ for all $A \in \mathbb{R}^{2\times 2}$ (see also (3.11) in [26]). $\square$

For sets which are related through bi-Lipschitzian homeomorphisms with Lipschitz constants of both the homeomorphism itself and its inverse uniformly bounded the constant in Theorem 3.10 can be chosen independently of these sets, see e.g. [26].

3.5 Geometric rigidity: Dependence on the set shape

In general, the constant of the inequality stated in Section 3.4 depends crucially on the set shape. This will be discussed in detail in this section. As an introductory example we consider the deflection of a thin elastic beam.

Example 3.12 Let $U = (0,1)\times(0,\delta)$ and let $y : U \to \mathbb{R}^2$ be given by $y(x_1, x_2) = (x_2 + 1)(\sin(x_1), \cos(x_1))$. Then

$$\nabla y(x_1, x_2) = \begin{pmatrix} (x_2+1)\cos(x_1) & \sin(x_1) \\ -(x_2+1)\sin(x_1) & \cos(x_1) \end{pmatrix}$$

and therefore $\operatorname{dist}^2(\nabla y, SO(2)) = |\sqrt{\nabla y^T \nabla y} - \mathbf{Id}|^2 = x_2^2$, i.e.

$$\| \operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(U)} = \tfrac{1}{3}\delta^3.$$

Let $R_\phi \in SO(2)$, $R_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ for $\phi \in [0, 2\pi]$. Then $|\nabla y(x) - R|^2 \geq |\sin(x_1) - \sin\phi|^2 + |\cos(x_1) - \cos\phi|^2$. It is not hard to see that it exists a $C > 0$ such that $\int_0^1 |\nabla y(x) - R|^2 \, dx_1 \geq C$ for all $\phi \in [0, 2\pi]$ and $x_2 \in (0, \delta)$. We conclude that

$$\|\nabla y - R\|^2_{L^2(U)} \geq C\delta \geq \frac{C}{\delta^2} \| \operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(U)}$$

for all $R \in SO(2)$. A similar argument shows

$$\|y - (R \,\cdot + c)\|^2_{L^2(U)} \geq C\delta \geq \frac{C}{\delta^2} \| \operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(U)}$$

for all $R \in SO(2)$ and $c \in \mathbb{R}^2$.

Similar examples can be constructed in the linearized framework for the Korn-Poincaré inequality given in Theorem 3.3. As a direct consequence we get that the estimate (2.6) might be wrong without allowing for a small modification of the deformation.

Example 3.13 Let $\varepsilon > 0$. Assume without restriction that the set $U = (0,1) \times (0, \varepsilon^{\frac{1}{3}})$ considered above satisfies $\overline{U} \subset \Omega$. Define $y : \Omega \to \mathbb{R}^2$ by $y(x) = \mathbf{id} + \mathbf{e}_2$ for $x \in \Omega \backslash U$ and $y(x) = (x_2+1)(\sin(x_1), \cos(x_1))$ for $x \in U$. Then $y \in SBV^2(\Omega)$ with $J_y = (0,1) \times \{0, \varepsilon^{\frac{1}{3}}\} \cup \{1\} \times (0, \varepsilon^{\frac{1}{3}})$ and $\| \operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(\Omega)} = \frac{\varepsilon}{3}$. However, for all $R \in SO(2)$ and $c \in \mathbb{R}^2$ we have

$$\|\nabla y - R\|^2_{L^2(\Omega)} \geq C\varepsilon^{\frac{1}{3}}, \qquad \|y - (R\,\cdot + c)\|^2_{L^2(\Omega)} \geq C\varepsilon^{\frac{1}{3}}.$$

Although omitted here, a similar estimate can be derived for the symmetric part of the gradient.

Recall the definition of $\mathcal{U}^s$ in (3.2). In order to quantify how the constant in Theorem 3.10 depends on the set shape we will estimate the variation from a square $Q^s(a)$ to a neighboring square $Q^s(b)$, $b = a + 2s\nu$ for $\nu = \pm\mathbf{e}_i$, $i = 1, 2$ proceeding similarly as in [26]. Consider $y \in H^1(U)$ with $U \in \mathcal{U}^s$. On a square $Q^s(p) \subset U$ and for subsets $V \subset U$, $V \in \mathcal{U}^s$ we define for shorthand (we drop the integration variable if no confusion arises)

$$\gamma(p) = \int_{Q^s(p)} \operatorname{dist}^2(\nabla y, SO(2)), \quad \gamma(V) = \sum\nolimits_{p \in I^s(V)} \gamma(p),$$

where $I^s(V) := \{p \in I^s : Q^s(p) \subset V\}$. Applying Theorem 3.10 we obtain $R(a), R(b) \in SO(2)$ such that

$$\int_{Q^s(p)} |\nabla y - R(p)|^2 \leq C\gamma(p) \quad \text{ for } p = a, b. \tag{3.12}$$

Likewise on the rectangle $Q^s(a,b) := (\overline{Q^s(a)} \cup \overline{Q^s(b)})^\circ$ we obtain $R(a,b) \in SO(2)$ such that

$$\int_{Q^s(a,b)} |\nabla y - R(a,b)|^2 \, dx \leq C \int_{Q^s(a,b)} \operatorname{dist}^2(\nabla y, SO(2)) \leq C(\gamma(a) + \gamma(b)).$$

Combining these estimates we see $|Q^s(p)||R(p) - R(a,b)|^2 \leq C(\gamma(a) + \gamma(b))$ for $p = a, b$ and therefore

$$s^2|R(a) - R(b)|^2 \leq C(\gamma(a) + \gamma(b)). \tag{3.13}$$

More general, we consider a difference quotient with two arbitrary points $a, b \in I^s(U)$. We assume that there is a path $\xi = (\xi_0, \ldots, \xi_m)$ such that

$$\begin{aligned} &\xi_1 = a, \quad \xi_m = b, \\ &\xi_j - \xi_{j-1} = \pm 2s\mathbf{e}_i \text{ for some } i = 1, 2, \quad \forall j = 2, \ldots, m. \end{aligned} \tag{3.14}$$

Then iteratively applying the above estimate (3.13) we obtain

$$s^2|R(a)-R(b)|^2\,dx \le Cm\sum\nolimits_{j=1}^m \gamma(\xi_j). \tag{3.15}$$

We now state a first weak rigidity result.

Lemma 3.14 *Let $\mu, s > 0$ such that $l := \mu s^{-1} \in \mathbb{N}$. Then there is a constant $C > 0$ independent of μ, s such that for all connected sets $U \in \mathcal{U}^s$, $U \subset (-\mu,\mu)^2$, the following holds: For all $y \in H^1(U)$ there is a rotation $R \in SO(2)$ such that*

$$\int_U |\nabla y - R|^2 \le C(s^{-2}|U|)^2 \int_U \operatorname{dist}^2(\nabla y, SO(2)) \le Cl^4 \int_U \operatorname{dist}^2(\nabla y, SO(2)).$$

Proof. The second inequality is obvious as $|U| \le 4\mu^2$. To see the first inequality we fix $p_0 \in I^s(U)$ and consider an arbitrary $p \in I^s(U)$. As U is connected there is a path $\xi = (\xi_1 = p_0, \dots, \xi_m = p)$ with $m \le |U|(2s)^{-2}$. We first apply (3.12) on each square and then by (3.15) we obtain

$$\int_{Q^s(p)} |R(p) - R(p_0)|^2 \le C|U|s^{-2}\sum\nolimits_{j=1}^m \gamma(\xi_j) \le C|U|s^{-2}\gamma(U).$$

Then setting $R = R(p_0)$ and summing over all $p \in I^s(U)$ we derive

$$\begin{aligned}\int_U |\nabla y - R|^2 &\le C\sum\nolimits_{p\in I^s(U)} \int_{Q^s(p)} \Big(|\nabla y - R(p)|^2 + |R(p) - R(p_0)|^2\Big)\\ &\le C\sum\nolimits_{p\in I^s(U)} (\gamma(p) + |U|s^{-2}\gamma(U)) \le C\#I^s(U)\,|U|s^{-2}\gamma(U)\\ &\le C(|U|s^{-2})^2\gamma(U).\end{aligned}$$

□

Remark 3.15 (i) Let $U = (0,1)\times(0,\delta)$. If we choose $s = \frac{\delta}{2}$, Lemma 3.14 provides a constant $\sim \delta^{-2}$. Example 3.12 shows that this estimate is sharp in the sense that the exponent of δ cannot be improved.

(ii) Following the above arguments we find that in Lemma 3.14 one can replace $p = 2$ by any $1 < p < \infty$ replacing l^4 suitably by l^{2p}.

(iii) In view of the proof in the choice of R we have the freedom to select any of the rotations which are given on each square $Q^s(p) \subset U$ by application of (3.12).

We briefly note that similar calculations may be provided to estimate the difference of rigid motions. Consider $b_1, b_2 \in \mathbb{R}^2$, and the rectangles $B_i = b_i + (-l_i, l_i) \times (-m_i, m_i) \in \mathcal{U}^s$ for $i = 1, 2$, where we assume without restriction that

$l_1 \geq m_1 > 0$, $l_2 \geq m_2 > 0$. Suppose that there is a point $b_{12} \in \overline{B_1} \cap \overline{B_2}$. For given $R_1, R_2, R_{12} \in SO(2)$ and $c_1, c_2, c_{12} \in \mathbb{R}^2$ we set $E_i := \|y - (R_i \ \cdot + c_i)\|^2_{L^2(B_i)}$ for $i = 1, 2$ and assume that

$$\|y - (R_{12} \ \cdot + c_{12})\|^2_{L^2(B_1 \cup B_2)} \leq C(E_1 + E_2).$$

Then we find

$$|B_1 \cup B_2|(l_1 + l_2)^2 \ |R_1 - R_2|^2 \leq C\kappa(E_1 + E_2), \tag{3.16}$$

as well as

$$\|y - (R_1 \ \cdot + c_1)\|^2_{L^2(B_1 \cup B_2)} + \|y - (R_2 \ \cdot + c_2)\|^2_{L^2(B_1 \cup B_2)} \leq C\kappa(E_1 + E_2), \tag{3.17}$$

where $\kappa = \frac{|B_1 \cup B_2|}{\min_j |B_j|}\big(\frac{l_1 + l_2}{\min_j l_j}\big)^2$. This estimate follows similarly as in the geometrically linear setting treated in [22, Section 2.2] and we therefore omit the details. Indeed, in all the calculations, in particular in (2.10) of [22], one may replace $\mathbb{R}^{2\times 2}_{\rm skew}$ by $SO(2)$ since the estimates essentially rely on the fact that $|R\mathbf{e}_1| = |R\mathbf{e}_2|$ which is satisfied for both $\mathbb{R}^{2\times 2}_{\rm skew}$ and $SO(2)$. Moreover, although we stated this property only for two rectangles for the sake of simplicity, we remark that an estimate of the above form also holds for sets with more general geometries.

Similarly as in (3.14), considering two arbitrary points $a, b \in I^s(U)$ connected by a path $\xi = (\xi_1, \ldots, \xi_m)$ with corresponding estimates

$$\|(R(\xi_j) - R(\xi_{j-1})) \ \cdot + c(\xi_j) - c(\xi_{j-1})\|_{L^2(Q^s_{j-1,j})} \leq CE_{j-1,j},$$

(here we defined $Q^s_{j-1,j} = (\overline{Q^s(\xi_{j-1})} \cup \overline{Q^s(\xi_j)})^\circ$) we obtain (cf. (2.20) in [22])

$$\begin{aligned} \|y - (R(a) \ \cdot + c(a))\|^2_{L^2(Q^s(b))} &\leq Cm^2 \Big(\sum\nolimits_{j=2}^m E_{j-1,j}\Big)^2 \\ &\leq Cm^3 \sum\nolimits_{j=2}^m \big(E_{j-1,j}\big)^2. \end{aligned} \tag{3.18}$$

In the last step we used Hölder's inequality. Similarly as before, (3.18) also holds for any of the other rigid motions $R(\xi_j)\, x + c(\xi_j)$ (cf. Remark 3.15(iii)).

4 Modification of sets

Before we start with the proof of Theorem 2.1, we first introduce a procedure to modify sets. In particular, it will be fundamental to assure that during the modification process boundary components do not become too large or are separated by other components.

Recall the definition of the sets $\mathcal{U}^s$, $\mathcal{V}^s$ in Section 3.2. We consider a Lipschitz domain $\Omega \subset \mathbb{R}^2$ and choose μ_0 so large that $\overline{\Omega} \subset Q_{\mu_0} = (-\mu_0, \mu_0)^2$. We let Ω^k

be the largest set in $\mathcal{V}^{\bar{c}k}$ satisfying $\Omega^k \subset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \bar{c}k\}$ for $k \geq 0$ for some $\bar{c} \geq \sqrt{2}$ large enough.

For sets $W \subset \Omega^k$, $W \in \mathcal{V}^s$, we assume that one component in definition (3.3) is given by $X = Q_{\mu_0} \setminus \Omega^k$. In particular, all other components $X_1, \ldots, X_n$ satisfy $\partial X_i \subset Q_{\mu_0}$ as $\overline{\Omega} \subset Q_{\mu_0}$. We again choose an (arbitrary) order of $(\Gamma_j)_{j=1,\ldots,n}$ and define $(\Theta_j)_j$ as in (3.4). Recall the definition of $\|\cdot\|_X$, $X = *, \infty, \mathcal{H}$, in (3.5) and (3.6). Moreover, we recall that $\mathcal{V}^s_{\mathrm{con}} \subset \mathcal{V}^s$ was defined as the subset consisting of the sets where all $\overline{X_1}, \ldots, \overline{X_n}$ are connected.

We now introduce a modification procedure for sets. Given a set $W = Q_\mu \setminus \bigcup_{i=1}^m X_i \in \mathcal{V}^s$ and some $V \in \mathcal{U}^s$ we consider the modification

$$W' = Q_\mu \setminus \bigcup\nolimits_{i=0}^m X'_i, \tag{4.1}$$

where $X'_i = X_i \setminus \overline{V}$ for $i = 1, \ldots, m$ and $X'_0 = V$. (It is convenient to start with index 0.) We observe that $W' = (W \setminus V) \cup \partial V$ (as a subset of $\mathbb{R}^2$). Therefore, for shorthand we will write $W' = (W \setminus V) \cup \partial V$ to indicate the element of $\mathcal{V}^s$ which is given by (4.1). We briefly note that then the boundary components of W' are given by $\Gamma_0(W') = \Theta_0(W') = \partial V$ as well as by $\Gamma_j(W') = \partial(X_j \setminus \overline{V})$ and $\Theta_j(W') = \Theta_j(W) \setminus \overline{V}$ for $j \geq 1$ (cf. also Lemma 3.2(iii)).

Having several pairwise disjoint sets $(V_j)_j \subset \mathcal{U}^s$ the modification is defined analogously by $W'' = (W \setminus \bigcup_j V_j) \cup \bigcup_j \partial V_j$.

As large surfaces of general shape may not be measured adequately in terms of $|\cdot|_\infty$, in what follows we have to assure that boundary components do not become too large. For $0 < s \leq \lambda \leq k$ we introduce

$$\mathcal{V}^s_{(\lambda,k)} := \{W \in \mathcal{V}^s_{\mathrm{con}} : 2\lambda \leq \max\{|\pi_1\Gamma_j(W)|, |\pi_2\Gamma_j(W)|\} \leq 2k \ \text{ for all } \Gamma_j(W)\}.$$

By definition we have $\max\{|\pi_1\Gamma_j(W)|, |\pi_2\Gamma_j(W)|\} \geq 2s$ for all $\Gamma_j(W)$ and therefore we write for shorthand $\mathcal{V}^s_k = \mathcal{V}^s_{(s,k)}$.

Although we have to avoid that boundary components become to large, it is essential to combine small components. To this end, it is convenient to alter configurations on sets of negligible measure.

Lemma 4.1 *Let $t \geq 2k$, $t' > 0$ and $W \in \mathcal{V}^s_t$.*

(i) Then there is a set $\tilde{W} \in \mathcal{V}^s_t$ with $\tilde{W} \subset W$, $|W \setminus \tilde{W}| = 0$ and $\|\tilde{W}\|_* \leq \|W\|_*$ such that

$$\Gamma_{j_1}(\tilde{W}) \cap \Gamma_{j_2}(\tilde{W}) = \emptyset \quad \text{ if } |\Gamma_{j_i}(\tilde{W})|_\infty \leq k \ \text{ for } i = 1, 2. \tag{4.2}$$

(ii) Then there is a set $U \in \mathcal{V}^s_{t+k}$ with $U \subset W$, $|W \setminus U| = 0$ and $\|U\|_* \leq \|W\|_*$ such that

$$\Gamma(U) \cap \Gamma_j(U) = \emptyset \quad \text{ for all } \Gamma_j(U) \neq \Gamma(U) \tag{4.3}$$

for all $\Gamma(U)$ with $|\Gamma(U)|_\infty \leq k$.

Proof. (i) The strategy is to combine iteratively different boundary components. Clearly, if $|\Gamma_{j_i}(W)|_\infty \leq k$ for $i = 1, 2$ with $\Gamma_{j_1}(W) \cap \Gamma_{j_2}(W) \neq \emptyset$ we may replace W by $W' = W \setminus (\overline{X_{j_1} \cup X_{j_2}})^\circ$ and note that $W' \in \mathcal{V}^s_t$ as well as $|W \setminus W'| = 0$ and $\|W'\|_* \leq \|W\|_*$ similarly as in Lemma 3.2. (Recall that $\partial X_{j_i} = \Gamma_{j_1}(W)$ for $i = 1, 2$.) We proceed in this way until we obtain a set $\tilde{W} \in \mathcal{V}^s_t$ with $|W \setminus \tilde{W}| = 0$ and $\|\tilde{W}\|_* \leq \|W\|_*$ such that (4.2) holds.

(ii) We apply (i) and then proceed to combine two components $\Gamma_{j_1}(\tilde{W}), \Gamma_{j_2}(\tilde{W})$ if $\Gamma_{j_1}(\tilde{W}) \cap \Gamma_{j_2}(\tilde{W}) \neq \emptyset$ and $\min\{|\Gamma_{j_1}(\tilde{W})|_\infty, |\Gamma_{j_2}(\tilde{W})|_\infty\} \leq k$. Arguing as before we end up with a set U satisfying $|W \setminus U| = 0$, $\|U\|_* \leq \|W\|_*$ and (4.3). It remains to show that $U \in \mathcal{V}^s_{t+k}$. Consider some $\Gamma(U) = \partial X$ with $|\Gamma(U)|_\infty > k$ and observe that there are $\Gamma(\tilde{W}) = \partial X'$ with $|\Gamma(\tilde{W})|_\infty > k$ and $\Gamma_{j_i}(\tilde{W}) = \partial X_{j_i}$, $i = 1, \ldots, m$, with $|\Gamma_{j_i}(\tilde{W})|_\infty \leq k$, $\Gamma_{j_{i_1}}(\tilde{W}) \cap \Gamma_{j_{i_2}}(\tilde{W}) = \emptyset$ for $i_1 \neq i_2$ and $\Gamma_{j_i}(\tilde{W}) \cap \Gamma(\tilde{W}) \neq \emptyset$ such that $\overline{X} = \overline{X' \cup \bigcup_{i=1}^m X_{j_i}}$. But this implies $|\pi_i \Gamma(U)| \leq 2k + |\pi_i \Gamma(\tilde{W})| \leq 2k + 2t$ for $i = 1, 2$, as desired. □

In what follows we often modify sets by subtracting rectangular neighborhoods of boundary components. In this context it is particularly important that the components remain connected and do not become too large. By $\triangle$ we denote the symmetric difference of two sets.

Lemma 4.2 *Let $k, t, t' > 0$ with $t, t' \leq Ck$ and $\nu \geq 0$. Let $V \subset \Omega^k$ with $V \in \mathcal{V}^s_{\rm con}$.*

(i) Assume that for each component $X_j = X_j(V)$, $j = 1, \ldots, n$, there is a rectangle $Z_j \in \mathcal{U}^s$ with $X_j \subset Z_j$, $|\pi_i \partial Z_j| \leq |\pi_i \partial X_j| + \nu |\partial X_j|_\infty$ for $i = 1, 2$ and $\max_{i=1,2} |\pi_i \partial Z_j| \leq 2t'$ for all $j = 1, \ldots, n$. Moreover, assume that $Z_{j_1} \setminus Z_{j_2}$ or $Z_{j_2} \setminus Z_{j_1}$ is connected for all $1 \leq j_1 < j_2 \leq n$. Then there is a set $U \in \mathcal{V}^s_{t'}$, $U \subset \Omega^k$, with $\bigcup_{j=1}^n \overline{X_j(U)} = \bigcup_{j=1}^n \overline{Z_j} \cap \Omega^k$ and $\|U\|_* \leq (1 + c\nu)\|V\|_*$ for a universal constant $c > 0$.

(ii) In addition let $V' \in \mathcal{V}^s_t$ be given and define $\hat{W} = V' \setminus \bigcup_{j=1}^n Z_j$. Then there is a set $W \in \mathcal{V}^{s/2}_{t+2t'}$ with $|W \setminus \hat{W}| = 0$, $|\hat{W} \setminus W| \leq ct\|V'\|_*$ and $\|W\|_* \leq (1 + c\nu)\|V\|_* + \|V'\|_*$.

As the proof of this result is very technical and in principle not relevant to understand the proof of the main result in the subsequent sections, it may be omitted on first reading.

Proof. (i) Let $V \subset \Omega^k$ with components $(X_j)_{j=1}^n$ and rectangles $(Z_j)_{j=1}^n$ be given. It suffices to show the following: There are connected, pairwise disjoint $(X'_j)_{j=1}^n$ with $X'_j \subset Z_j$, $\bigcup_{j=1}^n \overline{X'_j} = \bigcup_{j=1}^n \overline{Z_j}$ and

$$\Big|\bigcup_{j=1}^n \partial X'_j\Big|_{\mathcal{H}} \leq \sum_{j=1}^n |\Theta_j(V)|_{\mathcal{H}} + c\nu \sum_{j=1}^n |\Gamma_j|_{\mathcal{H}}. \tag{4.4}$$

Moreover, we have $X'_j = R_j \setminus \overline{(A^1_j \cup A^2_j)}$. Here $R_j \in \mathcal{U}^s$ is a rectangle and $A^i_j \in \mathcal{U}^s$, $i = 1, 2$, are (if nonempty) unions of rectangles whose closure intersect the corner $c^i_j \in \partial R_j$, where c^1_j, c^2_j are adjacent corners of R_j.

Then the claim of the lemma follows for $U = \Omega^k \setminus \bigcup_{j=1}^n X_j'$. Indeed, to see $\|U\|_* \le (1+c\nu)\|V\|_*$ we first observe $\sum_j |\partial X_j'|_\infty \le \sum_j |\partial Z_j|_\infty \le (1+c\nu)\sum_j |\partial X_j|_\infty$. Moreover, by (4.4) we get

$$\|U\|_{\mathcal{H}} \le |\bigcup_{j=1}^n \partial X_j'|_{\mathcal{H}} \le (1+c\nu)\|V\|_{\mathcal{H}} = (1+c\nu)|\bigcup_{j=1}^n \partial X_j|_{\mathcal{H}}. \tag{4.5}$$

In the first inequality we also used $|\partial X_j'|_\infty \le |\partial Z_j|_\infty \le Ck$ and $\Omega^k \in \mathcal{V}^{\bar{c}k}$ for $\bar{c} \gg 1$. (Arguments of this form will be used frequently in the following and from now on we will omit the details.) Finally, we conclude $U \in \mathcal{V}^s_{t'}$ as $\max_{i=1,2} |\pi_i \partial Z_j| \le 2t'$ for $j = 1, \ldots, n$.

We prove the above assertion by induction. Clearly, the claim holds for $n = 1$ for $X_1' = Z_1$. Now assume the assertion holds for sets with at most $n-1$ components and consider $V \subset \Omega^k$ with components $(X_j)_{j=1}^n$ and corresponding $(Z_j)_{j=1}^n$. Without restriction we assume that $\max_{x \in \overline{Z_n}} x_2 = \max_{x \in \bigcup_{j=1}^n \overline{Z_j}} x_2$. By hypothesis we obtain pairwise disjoint, connected sets X_j'', $j = 1, \ldots, n-1$, fulfilling the above properties, in particular $\bigcup_{j=1}^{n-1} \overline{X_j''} = \bigcup_{j=1}^{n-1} \overline{Z_j}$.

Given $Z_n = (z_1^1, z_1^2) \times (z_2^1, z_2^2)$ we set $\tilde{Z}_n = (z_1^1, z_1^2) \times (z_2^1, z_2^2]$. For $j = 1, \ldots, n-1$ let $Z_{j,i}' \in \mathcal{U}^s$ be the largest rectangle in Z_n satisfying $Z_j \cap Z_n \subset Z_{j,i}' \subset \bigcup_{j=1}^{n-1} \overline{Z_j}$ with $z_1^i \in \overline{Z_{j,i}'}$ for $i = 1,2$. If $Z_{j,i}' \neq \emptyset$ for some i, we let $Z_j' = Z_{j,i}'$, otherwise we set $Z_j' = Z_j \cap Z_n$. (Note that $Z_{j,1}' = Z_{j,2}'$ if $Z_{j,1}', Z_{j,2}' \neq \emptyset$.)

Let $J_0 \subset \{1, \ldots, n-1\}$ such that $Z_j \cap Z_n = \emptyset$ for $j \in J_0$. Let $J_1 \subset \{1, \ldots, n-1\} \setminus J_0$ such that $(\overline{Z_j'} \setminus Z_n) \cap \{z_1^1, z_1^2\} = \emptyset$ for $j \in J_1$ and $J_2 \subset \{1, \ldots, n-1\} \setminus J_0$ such that $\tilde{Z}_n \setminus Z_j'$ is a rectangle for $j \in J_2$. (Observe that $J_1 \cap J_2 = \emptyset$.) Let $J_3 = \{1, \ldots, n-1\} \setminus (J_0 \cup J_1 \cup J_2)$. Define $X_n' = Z_n \setminus \bigcup_{j \in J_2 \cup J_3} \overline{Z_j'}$. Moreover, we let $X_j' = X_j''$ for $j \in J_0 \cup J_2 \cup J_3$ and $X_j' = X_j'' \setminus \overline{X_n'}$ for $j \in J_1$. Clearly, by construction the sets are pairwise disjoint and fulfill $\bigcup_{j=1}^n \overline{X_j'} = \bigcup_{j=1}^n \overline{Z_j}$.

Moreover, we observe that the sets are connected and have the special shape given above. In fact, for $j \in J_0 \cup J_2 \cup J_3$ this is clear. For X_n' we first note that $J_3 = J_3^1 \dot\cup J_3^2$, where $\overline{Z_j'}$ intersects the lower right and the lower left corner of Z_n for $j \in J_3^1$ and $j \in J_3^2$, respectively. (It cannot happen that $\overline{Z_j'}$ intersects only the other corners due to the choice of Z_n.) We observe $X_n' = R_n \setminus \overline{(A_n^1 \cup A_n^2)}$ is connected, where $R_n = Z_n \setminus \bigcup_{j \in J_2} \overline{Z_j'}$ and $A_n^i = \bigcup_{j \in J_3^i} Z_j'$ for $i = 1,2$.

Finally, to see the properties for $j \in J_1$ we first observe that $S_j := Z_j \setminus \overline{X_n'}$ is a rectangle. In fact, otherwise due to the special shape of X_n' it is elementary to see that $(\overline{Z_j'} \setminus Z_n) \cap \{z_1^1, z_1^2\} \neq \emptyset$ and thus $j \notin J_1$. We get $X_j' = X_j'' \cap S_j = \big(R_j \cap S_j\big) \setminus \overline{(A_j^1 \cup A_j^2)}$ is connected and $X_j' = \hat{R}_j \setminus \overline{(\hat{A}_j^1 \cup \hat{A}_j^2)}$, where $\hat{R}_j = S_j$ and $\hat{A}_j^i = A_j^i \cap S_j$ for $i = 1,2$.

It remains to confirm (4.4). We first observe that

$$\sum\nolimits_{j=1}^n |\Theta_j(V)|_{\mathcal{H}} = \tfrac{1}{2}\sum\nolimits_{j=1}^n |\Gamma_j|_{\mathcal{H}} + \tfrac{1}{2}\big|\partial\big(\bigcup_{j=1}^n \overline{X_j}\big)\big|_{\mathcal{H}}. \tag{4.6}$$

(Recall that different boundary components may have nonempty intersection.) Similarly, for the components $(X'_j)_j$ we find

$$|\bigcup_{j=1}^n \partial X'_j|_{\mathcal{H}} = \tfrac{1}{2}\sum_{j=1}^n |\partial X'_j|_{\mathcal{H}} + \tfrac{1}{2}|\partial(\bigcup_{j=1}^n \overline{X'_j})|_{\mathcal{H}}.$$

We now treat the two terms on the right separately. By $T_j \in \mathcal{U}^s$ we denote the smallest rectangle containing X_j and observe that $|\partial T_j|_\infty = |\Gamma_j|_\infty$, $|\partial T_j|_{\mathcal{H}} \le |\Gamma_j|_{\mathcal{H}}$. Recall $|\partial Z_j|_{\mathcal{H}} \le |\partial T_j|_{\mathcal{H}} + c\nu|\partial T_j|_\infty \le (1+c\nu)|\Gamma_j|_{\mathcal{H}}$ for $j = 1,\dots,n$. Due to the special shape of the components X'_j we find $|\partial X'_j|_{\mathcal{H}} \le |\partial Z_j|_{\mathcal{H}}$ and thus

$$\sum_{j=1}^n |\partial X'_j|_{\mathcal{H}} \le (1+c\nu)\sum_{j=1}^n |\Gamma_j|_{\mathcal{H}}. \tag{4.7}$$

Moreover, it is elementary to see that we can find a connected set $\tilde{X}_j \supset X_j$ such that $\tilde{\Gamma}_j := \partial\tilde{X}_j$ satisfies $|\tilde{\Gamma}_j|_{\mathcal{H}} \le (1+c\nu)|\Gamma_j|_{\mathcal{H}}$ and $Z_j \in \mathcal{U}^s$ is the smallest rectangle containing $\tilde{X}_j$. By a projection argument it is then not hard to see that

$$\begin{aligned}|\partial(\bigcup_{j=1}^n \overline{X'_j})|_{\mathcal{H}} = |\partial(\bigcup_{j=1}^n \overline{Z_j})|_{\mathcal{H}} &\le |\partial(\bigcup_{j=1}^n \overline{\tilde{X}_j})|_{\mathcal{H}} \\ &\le |\partial(\bigcup_{j=1}^n \overline{X_j})|_{\mathcal{H}} + c\nu\sum|\Gamma_j|_{\mathcal{H}}.\end{aligned}$$

Consequently, we derive by (4.6) and (4.7)

$$\begin{aligned}|\bigcup_{j=1}^n \partial X'_j|_{\mathcal{H}} &\le \tfrac{1}{2}\sum_{j=1}^n |\Gamma_j|_{\mathcal{H}} + \tfrac{1}{2}|\partial(\bigcup_{j=1}^n \overline{X_j})|_{\mathcal{H}} + c\nu\sum_{j=1}^n |\Gamma_j|_{\mathcal{H}} \\ &= \sum_{j=1}^n |\Theta_j(V)|_{\mathcal{H}} + c\nu\sum_{j=1}^n |\Gamma_j|_{\mathcal{H}},\end{aligned}$$

as desired.

(ii) Let $(Y_j)_{j=1}^{n'}$ be the components of V' and let $T_j \in \mathcal{U}^s$ be the smallest rectangle containing Y_j. It is elementary to see that $T_{j_1} \setminus T_{j_2}$ is connected for $1 \le j_1, j_2 \le n'$. Thus, by (i) we obtain pairwise disjoint, connected sets $(Y'_j)_j$ with $\bigcup_j \overline{Y'_j} = \bigcup_j \overline{T_j}$ and define $V'' = \Omega^k \setminus \bigcup_{j=1}^{n'} Y'_j$. By (i) for $\nu = 0$ we then also obtain $\|V''\|_* \le \|V'\|_*$. Moreover, the isoperimetric inequality yields $|V' \setminus V''| \le ct\|V'\|_*$ since $|\partial T_j|_\infty \le 2\sqrt{2}t$ for all $j = 1,\dots,n'$.

Let $(X'_j)_{j=1}^n$ and $U \in \mathcal{V}^s_{t'}$ as given in (i). We define $W' = (U \setminus \bigcup_{j=1}^{n'} Y'_j) \cup \bigcup_{j=1}^{n'} \partial Y'_j$. Clearly, we have $|W' \setminus \hat{W}| = 0$, $|\hat{W} \setminus W'| \le ct\|V'\|_*$ and $\|W'\|_* \le (1+c\nu)\|V\|_* + \|V'\|_*$ arguing similarly as in Lemma 3.2. Observe that possibly $W' \notin \mathcal{V}^s_{\mathrm{con}}$ as the components $(X'_j)_{j=1}^n$ of U may have become disconnected. Thus, we now construct a set $W \in \mathcal{V}^{s/2}_{\mathrm{con}}$ with $|W' \triangle W| = 0$.

By $R_j \in \mathcal{U}^s$ we denote the smallest rectangle such that $X'_j \subset R_j$ for $j = 1,\dots,n$ and observe $\bigcup_j \overline{R_j} = \bigcup_j \overline{X'_j}$. To simplify the exposition we assume that each of the components $(X'_j)_j$ has become disconnected as otherwise we do not have to alter the boundary component in the modification procedure described below. Moreover, we can suppose that for each pair Y'_{j_1}, X'_{j_2}, $1 \le j_1 \le n'$, $1 \le j_2 \le n$, with $R_{j_2} \backslash Y'_{j_1}$ is not disconnected we have $X'_{j_2} \backslash Y'_{j_1}$ is not disconnected. In fact, otherwise we can pass to some $Y^*_{j_1} \subset Y'_{j_1}$ with $|\partial Y^*_{j_1}|_* \le |\partial Y'_{j_1}|_*$ such that $X'_{j_2} \setminus Y^*_{j_1}$ is not disconnected and $\bigcup_j \overline{Y'_j} \cup \bigcup_j \overline{X'_j} = \bigcup_j \overline{Y^*_j} \cup \bigcup_j \overline{X'_j}$.

We now proceed by induction. Let $W_0 = V''$ and $T^0_j = Y'_j$ for $j = 1, \dots, n'$. Assume there are pairwise disjoint, connected sets $T^{l-1}_j \in \mathcal{U}^{\frac{s}{2}}$, $j = 1, \dots, n'$ such that

$$(i) \ \ \bigcup\nolimits_{j=1}^{n'} \overline{T^{l-1}_j} = \bigcup\nolimits_{j=1}^{n'} \overline{Y'_j} \cup \bigcup\nolimits_{j=1}^{l-1} \overline{X'_j}, \quad (ii) \ \ T^{l-1}_{j_1} \cap \overline{X'_{j_2}} = T^0_{j_1} \cap \overline{X'_{j_2}} \tag{4.8}$$

for all $1 \le j_1 \le n'$, $l \le j_2 \le n$. Moreover, assume that the set $W_{l-1} := \Omega^k \backslash \bigcup_j T^{l-1}_j$ satisfies $\|W_{l-1}\|_\infty \le \sum_j |\partial T^0_j|_\infty + \sum_{i=1}^{l-1} |\partial X'_i|_\infty$ and

$$\|W_{l-1}\|_{\mathcal{H}} \le |\bigcup\nolimits_j \partial T^0_j|_{\mathcal{H}} + |\bigcup\nolimits_{i=1}^{l-1} \partial X'_i \setminus \bigcup\nolimits_j T^0_j|_{\mathcal{H}} + \frac{1}{2}\sum_{i=1}^{l-1} |\partial X'_i \cap \bigcup\nolimits_j T^0_j|_{\mathcal{H}}. \tag{4.9}$$

We now construct W_l. Let $J^l \subset \{1, \dots, n'\}$ such that $T^{l-1}_j \cap X'_l \neq \emptyset$ with $J^l = J^l_1 \dot\cup J^l_2$, where $j \in J^l_2$ if and only if $R_l \setminus T^{l-1}_j$ is disconnected.

If $j \in J^l_1$, we define $T^l_j = T^{l-1}_j \setminus \hat{X}'_l$, where $\hat{X}'_l \in \mathcal{U}^{\frac{s}{2}}$ is the largest set with $\overline{\hat{X}'_l} \subset X'_l$. It is not hard to see that $|\partial T^l_j|_\infty \le |\partial T^{l-1}_j|_\infty$ for all $j \in J^l_1$ and $|\partial T^l_j|_{\mathcal{H}} \le |\partial T^{l-1}_j \setminus X'_l|_{\mathcal{H}} + \frac{1}{2}|\partial T^{l-1}_j \cap X'_l|_{\mathcal{H}} + \frac{1}{2}|\partial X'_l \cap T^{l-1}_j|_{\mathcal{H}}$. As each $x \in \mathbb{R}^2$ is contained in at most two different ∂T^{l-1}_j, we find $\sum_{j \in J^l_1} \frac{1}{2}|\partial T^{l-1}_j \cap X'_l|_{\mathcal{H}} \le |\bigcup_{j \in J^l_1} \partial T^{l-1}_j \cap X'_l|_{\mathcal{H}}$. Therefore, taking the union over all components we derive

$$|\bigcup\nolimits_{j \in J^l_1} \partial T^l_j \cup \bigcup\nolimits_{j \notin J^l_1} \partial T^{l-1}_j\big|_{\mathcal{H}} \le \big|\bigcup\nolimits_j \partial T^{l-1}_j\big|_{\mathcal{H}} + \tfrac{1}{2}\big|\partial X'_l \cap \bigcup\nolimits_{j \in J^l_1} T^0_j\big|_{\mathcal{H}}. \tag{4.10}$$

Here we used (4.8)(ii) and the fact that the sets $(T^{l-1}_j)_j$ are pairwise disjoint. Observe that the above construction together with (4.8)(ii) and the special shape of T^0_j (see proof of (i)) implies that the sets T^l_j, $j \in J^l_1$, are connected. Moreover, (4.8)(ii) holds for $j_1 \in J^l_1$.

We define $\tilde{X}'_l = X'_l \setminus \bigcup_{j \in J^l_1} \overline{T^l_j} \in \mathcal{U}^{\frac{s}{2}}$. Due to the fact that $\hat{X}'_l \neq \emptyset$ we observe that the number of connected components of the sets $X'_l \setminus \bigcup_{j \in J^l_2} T^{l-1}_j$ and $\tilde{X}'_l \setminus \bigcup_{j \in J^l_2} T^{l-1}_j$ coincide. Therefore, letting $A_1, \dots, A_m$ be the connected components of $\tilde{X}'_l \setminus \bigcup_{j \in J^l_2} \overline{T^{l-1}_j}$ it is elementary to see that $m = \# J^l_2 + 1$.

Up to a rotation by $\frac{\pi}{2}$ we can assume that each $\overline{A_i}$ intersects the upper and lower boundary of R_l and that $\overline{A_1}$ intersects the left boundary. For convenience we denote the components $(T^{l-1}_j)_{j \in J^l_2}$ by $(T^{l-1}_{j_i})_{i=1}^{m-1}$. Suppose $R_l = (0, l_1) \times (0, l_2)$. Let $a_i = \inf_{x \in A_i} x_1$ and $d_i = a_{i+1} - a_i$, where $a_{m+1} = l_1$. Define $T^l_{j_1} = (\overline{T^{l-1}_{j_1}} \cup \overline{(A_1 \cup A_2)})^\circ$ and $T^l_{j_i} = (\overline{T^{l-1}_{j_i}} \cup \overline{A_{i+1}})^\circ$ for $i = 2, \dots, m-1$. Observe that the sets are pairwise disjoint, connected and that (4.8)(ii) holds for $j_i \in J^l_2$. It is elementary to see that $|T^l_{j_1}|_\infty \le |T^{l-1}_{j_1}|_\infty + d_1 + d_2$ and $|T^l_{j_i}|_\infty \le |T^{l-1}_{j_i}|_\infty + d_{i+1}$ for $i = 2, \ldots, m-1$. Thus, we have

$$\sum\nolimits_{i=1}^{m-1} |T^l_{j_i}|_\infty \le \sum\nolimits_{i=1}^{m-1} |T^{l-1}_{j_i}|_\infty + |X'_l|_\infty. \tag{4.11}$$

For $j \notin J^l$ we define $T^l_j = T^{l-1}_j$ and observe that (4.8)(i) holds by construction and the assumptions before (4.8). Together with (4.10) and (4.8)(ii) we then also get

$$\big|\bigcup\nolimits_j \partial T^l_j\big|_{\mathcal{H}} \le \big|\bigcup\nolimits_j \partial T^{l-1}_j\big|_{\mathcal{H}} + \big|\partial X'_l \setminus \big(\bigcup\nolimits_{i=1}^{l-1} \partial X'_i \cup \bigcup\nolimits_j T^0_j\big)\big|_{\mathcal{H}} + \tfrac{1}{2}\big|\partial X'_l \cap \bigcup\nolimits_j T^0_j\big|_{\mathcal{H}}.$$

This in conjunction with (4.9) for W_{l-1} implies that (4.9) holds for W_l. Moreover, by (4.11) it is elementary to see that $\|W_l\|_\infty \le \sum_j |\partial T^0_j|_\infty + \sum_{i=1}^l |\partial X'_i|_\infty$.

Finally, we define $W = W_n$ and observe that W has the desired properties. In fact, by (4.8)(i) we have $|W \triangle W'| = 0$ and thus $|\hat{W} \setminus W| \le ct\|V'\|_*$. Moreover, we clearly get $\|W\|_\infty \le \|U\|_\infty + \|V''\|_\infty \le (1 + c\nu)\|V\|_\infty + \|V'\|_\infty$. As each $x \in \mathbb{R}^2$ is contained in at most two different $\partial X'_l$, we find by (4.9)

$$\begin{aligned}\|W\|_{\mathcal{H}} &\le \|V''\|_{\mathcal{H}} + \big|\bigcup\nolimits_{i=1}^n \partial X'_i \setminus \bigcup\nolimits_j T^0_j\big|_{\mathcal{H}} + \big|\bigcup\nolimits_{i=1}^n \partial X'_i \cap \bigcup\nolimits_j T^0_j\big|_{\mathcal{H}} \\ &= \|V''\|_{\mathcal{H}} + \|U\|_{\mathcal{H}} \le \|V'\|_{\mathcal{H}} + (1 + c\nu)\|V\|_{\mathcal{H}},\end{aligned}$$

as desired. Finally, similarly as in Lemma 4.1(ii) we obtain $|\pi_i X_j(W)| \le 2t + 4t'$ for $i = 1, 2$ for all j and thus $W \in \mathcal{V}^{s/2}_{t+2t'}$. □

5 A local rigidity estimate

We now establish a local rigidity estimate on a fine partition of the Lipschitz domain Ω. As a preparation we introduce some further notions. Recall the point set $I^s = s(1,1) + 2s\mathbb{Z}^2$, $s > 0$, introduced in Section 3.2 and the definitions of $\mathcal{U}^s, \mathcal{V}^s$ in (3.2), (3.3) with respect to the square Q_{μ_0}. We define additional partitions. Set $z_1 = (0,0)$, $z_2 = (1,0)$, $z_3 = (0,1)$, $z_4 = (1,1)$ and let $I^s_i = sz_i + 2s\mathbb{Z}^2$ as well as $Q^s_i(p) = p + s(-1,1)^2$ for $p \in I^s_i$, $i = 1, \ldots, 4$. Moreover, for $U \subset \Omega$ let

$$I^s_i(U) = \{p \in I^s_i : Q^s_i(p) \subset U\}$$

for $i = 1, \ldots, 4$. For shorthand we also write $I^s = I^s_4$ and $Q^s = Q^s_4$.

In the following, constants which are much smaller than 1 will frequently appear. For the sake of convenience we introduce one universal parameter. For given $l \ge 1$ and $0 < s, \epsilon, m \le 1$ we let

$$\vartheta = l^9 C_m^2 s^{-1} \epsilon, \tag{5.1}$$

where $C_m = C_1(m,h_*) + C_3(m,h_*) + m^{-4}C_2^{-2}(m,h_*)$ with the constants of Theorem 3.5 and Lemma 3.8 (for fixed h_*). By Remark 3.9(i) we find some $z \in \mathbb{N}$ such that $C_m \le C(h_*)m^{-z}$. Moreover, for later let $\hat{m} = C_2(m,h_*)$ and recall that by Remark 3.9(ii) we can assume $\hat{m} \ll m$ as well as $\bar{C}\hat{m} \le m$ for constants $\bar{C} = \bar{C}(h_*)$. Using only one universal parameter the estimates we establish are often not sharp. However, this will not affect our analysis.

Remark 5.1 All the constants C in the following may depend on h_* unless they are universal constants indicated as C_u. However, to avoid further notation we drop the dependence here. Only at the end of the proof in Section 8, when we pass to the limit $h_* \to 0$, we will take the h_* dependence of the constants into account.

In the following, ϵ will represent the stored elastic energy. We first construct piecewise constant $SO(2)$-valued mappings approximating the deformation gradient. Afterwards, we employ Theorem 3.5 and Corollary 3.7 to find a piecewise rigid motion being a good approximation of the deformation.

5.1 Estimates for the derivatives

We divide our investigation into two regimes, the 'superatomistic' $k \ge \epsilon$ and the 'subatomisic' $k \le \epsilon$. Here, recall that we call the ϵ-regime the 'atomistic regime' as in discrete fracture models ϵ is of the same order as the typical interatomic distance (c.f. [24, 25]). We begin with the superatomistic regime.

Lemma 5.2 *Let $k > s \ge \epsilon > 0$ with $1 \ll l := \frac{k}{s}$. Let $m^{-1} \in \mathbb{N}$ and assume that $\frac{km}{s} \in \mathbb{N}$. Then for a constant $C > 0$ we have the following:*
For all $U \in \mathcal{V}^s_k$ with $U \subset \Omega^k$ and for all $y \in H^1(U)$ with $\Delta y = 0$ in U° and

$$\gamma := \|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(U)}, \tag{5.2}$$

there is a set $W \in \mathcal{V}^{sm}_{(s,3k)}$ with $W \subset \Omega^{3k}$, $|W \setminus U| = 0$, $|(U\setminus W)\cap\Omega^{3k}| \le C_u k\|W\|_*$ and

$$\|W\|_* \le (1 + C_u m)\|U\|_* + C\epsilon^{-1}\gamma. \tag{5.3}$$

Moreover, there are mappings $\hat{R}_i : W^\circ \to SO(2)$, $i = 1,\dots,4$, which are constant on the connected components of $Q^k_i(p) \cap W^\circ$, $p \in I^k_i(\Omega^k)$, such that

$$\begin{aligned} (i) \quad & \|\nabla y - \hat{R}_i\|^2_{L^2(W)} \le C l^4 \gamma, \\ (ii) \quad & \|\nabla y - \hat{R}_i\|^4_{L^4(W)} \le C\vartheta\gamma. \end{aligned} \tag{5.4}$$

Proof. We first construct the set W. Let $J \subset I^k(\Omega^k)$ such that

$$\| \operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(Q^k(p)\cap U)} > \epsilon k \tag{5.5}$$

for all $p \in J$. Define

$$\hat{W} = \Big(U \setminus \bigcup\nolimits_{p\in J} Q^k(p)\Big) \cup \bigcup\nolimits_{p\in J} \partial Q^k(p)$$

and note that $\hat{W} \in \mathcal{V}^s_k$. In particular, the property $\hat{W} \in \mathcal{V}^s_{\rm con}$ holds since $\max\{|\pi_1\Gamma_t(U)|, |\pi_2\Gamma_t(U)|\} \le 2k$. The fact that we add the union of the boundary on the right hand side assures that we do not 'combine' boundary components. Moreover, we derive $\|\hat{W}\|_* \le \|U\|_* + C\epsilon^{-1}\gamma$. Indeed, $\sum_{p\in J} |\partial Q^k_p|_* \le 8k \cdot \#J \le 8k\frac{\gamma}{\epsilon k}$ by (5.2). For all other $\Gamma_t(\hat{W})$ we find a corresponding $\Gamma_t(U)$ (without restriction we use the same index) such that $\Theta_t(\hat{W}) = \Theta_t(U) \setminus \bigcup_{p\in J} \overline{Q^k(p)}$ and thus $|\Theta_t(\hat{W})|_* \le |\Theta_t(U)|_*$. (Arguments of this form will be used frequently in the following and from now on we will omit the details.) Furthermore, we easily deduce $|U \setminus \hat{W}| \le C_u k \|\hat{W}\|_*$.

Then we can find a set $W \in \mathcal{V}^{sm}_{2k}$ with $\|W\|_* \le (1 + C_u m)\|\hat{W}\|_*$, $|U \setminus W| \le C_u k\|W\|_*$ and $W^\circ \subset \{x \in \Omega^{3k} \cap \hat{W} : \operatorname{dist}_\infty(x, \partial\hat{W}) \le 2sm\}$, where $\operatorname{dist}_\infty(x, A) := \inf_{y\in A} \max_{i=1,2} |(x - y) \cdot \mathbf{e}_i|$ for $A \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$.

Indeed, let $M(\Gamma_j) \in \mathcal{U}^{sm}$ be the smallest rectangle satisfying $M(\Gamma_j) \supset \{x \in \mathbb{R}^2 : \operatorname{dist}_\infty(x, X_j) \le 2sm\}$, where X_j denotes the component corresponding to $\Gamma_j(\hat{W})$. Clearly, we obtain $|\pi_i\partial M(\Gamma_j)| \le |\pi_i\Gamma_j(\hat{W})| + C_u m|\Gamma_j(\hat{W})|_\infty$ for $i = 1, 2$, $j = 1, \dots, n$ as $\hat{W} \in \mathcal{V}^s$. Moreover, it is elementary to see that $M(\Gamma_{j_1}) \setminus M(\Gamma_{j_2})$ is connected for $1 \le j_1, j_2 \le n$ since $sm \ll s$. Then by Lemma 4.2(i) with $Z_j = M(\Gamma_j)$ we obtain a set $W \in \mathcal{V}^{sm}_{2k}$ which coincides with

$$\Omega^{3k} \cap \big(\hat{W} \setminus \bigcup\nolimits_{j=1}^n M(\Gamma_j)\big) = \Omega^{3k} \setminus \bigcup\nolimits_{j=1}^n M(\Gamma_j) \tag{5.6}$$

up to a set of negligible measure. Here we used $sm \ll k$. Moreover, we have $|(U \setminus W) \cap \Omega^{3k}| \le C_u k\|W\|_*$ and $\|W\|_* \le (1 + C_u m)\|\hat{W}\|_*$.

Boundary components of W are possibly smaller than $2s$ due to the modification in (5.6). Therefore, we apply Lemma 4.1(ii) to get a (not relabeled) set $W \in \mathcal{V}^{sm}_{3k}$ such that (5.3) still holds and (4.3) is satisfied. Now the fact that $U \in \mathcal{V}^s_{(s,k)}$ and (4.3) imply $W \in \mathcal{V}^{sm}_{(s,3k)}$.

Fix $i = 1, \dots, 4$ and let $F \subset Q^k_i(p) \cap W^\circ$ be a connected component of $Q^k_i(p) \cap W^\circ$. Define $\hat{F} \in \mathcal{U}^s$ as the smallest (connected) set satisfying

$$\hat{F} \supset \{x : \operatorname{dist}_\infty(x, F) < 2sm\}.$$

Due to the construction of W we get $\hat{F} \subset \hat{W}^\circ \subset U$. As $|\hat{F}| \le C_u k^2$, Lemma 3.14 for $\mu = 2k$ implies that there is a rotation $R \in SO(2)$ such that

$$\|\nabla y - R\|^2_{L^2(\hat{F})} \le Ck^4 s^{-4}\| \operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(\hat{F})} = Cl^4\gamma(\hat{F}),$$

where for shorthand we write $\gamma(\hat{F}) = \|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(\hat{F})}$. As $\nabla y - R$ is harmonic in $\hat{F}$, the mean value property of harmonic functions for $r = sm$ and Jensen's inequality yield

$$\begin{aligned} |\nabla y(x) - R|^4 &\leq \Big|\frac{1}{|B_r(x)|}\int_{B_r(x)} (\nabla y(t) - R)\,dt\Big|^4 \\ &\leq C\Big((sm)^{-2}\int_{\hat{F}} |\nabla y - R|^2\Big)^2 \leq Cl^8 m^{-4} s^{-4} \gamma(\hat{F})^2 \end{aligned} \tag{5.7}$$

for all $x \in F$. Consequently, as $\hat{F}$ intersects at most nine squares $Q^k(p)$, $p \in I^k(\Omega^k) \setminus J$, by (5.5) and $l = \frac{k}{s}$ we get $\|\nabla y - R\|^2_{L^\infty(F)} \leq Cl^4 m^{-2} s^{-2} \cdot k\epsilon \leq Cl^{-4}\vartheta$ as well as

$$\|\nabla y - R\|^4_{L^4(F)} \leq C\vartheta l^{-4} \|\nabla y - R\|^2_{L^2(\hat{F})} \leq C\vartheta\gamma(\hat{F}).$$

Proceeding in this way for every connected component F of all $Q^k_i(p)$, $p \in I^k_i(\Omega^k)$, and noting that every $Q^s(q)$, $q \in I^s(\Omega^k)$, intersects at most four different associated enlarged sets $\hat{F}$ ($Q^s(q)$ can intersect more than one set if it lies at the boundary of some $Q^k_i(p)$) we obtain a function $\hat{R}_i : W^\circ \to SO(2)$ with the desired properties (5.4). $\square$

We now concern ourselves with the subatomistic regime.

Lemma 5.3 *Let $M \geq 0$, $\epsilon > 0$ and $s \leq k \leq \epsilon$. Then for a fixed constant $C = C(M) > 0$ we have the following:*
For all $U \in \mathcal{V}^s_k$ with $U \subset \Omega^k$ and for all $y \in H^1(U)$ with γ as defined in (5.2) and $\|\nabla y\|_\infty \leq M$ there is a set $W \in \mathcal{V}^s_k$ with $W \subset \Omega^{3k}$, $|W \setminus U| = 0$, $|(U \setminus W) \cap \Omega^{3k}| \leq C_u k \|W\|_*$ and

$$\|W\|_* \leq \|U\|_* + C\epsilon^{-1}\gamma \tag{5.8}$$

as well as mappings $\hat{R}_i : \Omega^{3k} \to SO(2)$, $i = 1, \ldots, 4$, which are constant on $Q^k_i(p) \cap W$, $p \in I^k_i(\Omega^k)$, such that

$$\|\nabla y - \hat{R}_i\|^2_{L^2(W)} \leq C\gamma + C\epsilon\|U\|_*. \tag{5.9}$$

Proof. Similarly as in (5.5) we let $J \subset I^k(\Omega^k)$ such that

$$\epsilon\mathcal{H}^1(\partial U \cap Q^k(q)) + \|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(Q^k(q)\cap U)} > c_*\epsilon k \tag{5.10}$$

for all $q \in J$. Define $W = \Omega^{3k} \cap \big(\big(U \setminus \bigcup_{p\in J} Q^k(q)\big) \cup \bigcup_{p\in J} \partial Q^k(q)\big)$ and note that the $\|W\|_* \leq \|U\|_* + C\epsilon^{-1}\gamma$ for $c_* = c_*(h_*) > 0$ sufficiently large. Indeed, for the subset $J_1 \subset J$, for which (5.5) holds, we argue as in the previous proof. Then with $J_2 = J \setminus J_1$ we note $\|W\|_\infty \leq \|U\|_\infty + C\epsilon^{-1}\gamma + 2\sqrt{2}k \cdot \#J_2$ and

$\|W\|_{\mathcal{H}} \le \|U\|_{\mathcal{H}} + C\epsilon^{-1}\gamma + 8k\cdot\#J_2 - c_*k\cdot\#J_2$. This gives the desired result for c_* large. Moreover, we get $W \in \mathcal{V}^s_k$ and $|(U\setminus W)\cap\Omega^{3k}| \le C_u k\|W\|_*$.

Consider some $\tilde{Q} := Q^k_i(q)$, $q \in I^k_i(\Omega^k)$. We extend y from $\tilde{Q}\cap W$ to $\tilde{Q}$ by setting $\bar{v} = y$ on $W\cap\tilde{Q}$ and $\bar{v}(x) = x$ for all $x \in \tilde{Q}\setminus W$. Note that $\bar{v} \in SBV(\tilde{Q})$ with $J_{\bar{v}} = \partial W \cap \tilde{Q}$. By Theorem 3.1 we obtain a function $v \in H^1(\tilde{Q})$ such that by a rescaling argument

$$\|\nabla\bar{v} - \nabla v\|_{L^p(\tilde{Q})} \le Ck^{\frac{2}{p}-1}\|\nabla\bar{v}\|_\infty\mathcal{H}^1(J_{\bar{v}}\cap\tilde{Q}) \le CMk^{\frac{2}{p}-1}k^{1-\frac{1}{p}}\beta^{\frac{1}{p}} \le CM\epsilon^{\frac{1}{p}}\beta^{\frac{1}{p}}$$

for $p<2$, where $\beta = \mathcal{H}^1(\partial W\cap\tilde{Q})$. In the second step we used $\beta \le Ck$ by (5.10) and applied $k \le \epsilon$ in the last step. Consequently, we obtain

$$\|\operatorname{dist}(\nabla v, SO(2))\|^p_{L^p(\tilde{Q})} \le C\|\operatorname{dist}(\nabla\bar{v}, SO(2))\|^p_{L^p(\tilde{Q})} + C\epsilon\beta.$$

Thus, since $\gamma(\tilde{Q}) := \|\operatorname{dist}(\nabla\bar{v}, SO(2))\|^2_{L^2(\tilde{Q})} = \|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(\tilde{Q}\cap W)}$, the rigidity estimate in Theorem 3.10 yields a rotation $R \in SO(2)$ such that

$$\begin{aligned}\|\nabla v - R\|^p_{L^p(\tilde{Q})} &\le C\|\operatorname{dist}(\nabla v, SO(2))\|^p_{L^p(\tilde{Q})} \le C|\tilde{Q}|^{1-\frac{p}{2}}\gamma(\tilde{Q})^{\frac{p}{2}} + C\epsilon\beta\\ &\le C\epsilon^{2-p}\gamma(\tilde{Q})^{\frac{p}{2}-1}\gamma(\tilde{Q}) + C\epsilon\beta \le C\epsilon^{2-p}\epsilon^{p-2}\gamma(\tilde{Q}) + C\epsilon\beta\\ &\le C\gamma(\tilde{Q}) + C\epsilon\beta.\end{aligned}$$

In the second step we used Hölder's inequality and we applied (5.10) in the fourth step. This implies $\|\nabla y - R\|^p_{L^p(W\cap\tilde{Q})} \le \|\nabla\bar{v} - R\|^p_{L^p(\tilde{Q})} \le C\gamma(\tilde{Q}) + C\epsilon\beta$ and proceeding in this way for all $Q^k_i(q)$, $q \in I^k_i(\Omega^k)$, we obtain a function $\hat{R}_i : \Omega^{3k} \to SO(2)$ such that for a constant $C = C(h_*)$

$$\|\nabla y - \hat{R}_i\|^p_{L^p(W)} \le C\gamma + C\epsilon\|U\|_*,$$

where $\hat{R}_i$ is constant on $Q^k_i(p)\cap W$, $p \in I^k_i(\Omega^k)$. Finally, by $\|\nabla y\|_\infty \le M$ we derive

$$\|\nabla y - \hat{R}_i\|^2_{L^2(W)} \le (M+\sqrt{2})^{2-p}\|\nabla y - \hat{R}_i\|^p_{L^p(W)} \le C\gamma + C\epsilon\|U\|_*,$$

as desired. $\square$

Given a deformation $y : F \to \mathbb{R}^2$ for $F \subset \mathbb{R}^2$ and a rotation $R \in SO(2)$ we define the displacement field $u_R := R^Ty - \mathbf{id}$, where $\mathbf{id}$ denotes the identity function. We introduce the linear elastic strain by

$$\bar{e}_R(\nabla y) := e(\nabla u_R) = \frac{R^T\nabla y + (\nabla y)^TR}{2} - \mathbf{Id},$$

where $\mathbf{Id}$ denotes the identity matrix. For a general function $\hat{R} : F \to SO(2)$ we then define for shorthand $\alpha_{\hat{R}}(F) = \|\bar{e}_{\hat{R}}(\nabla y)\|^2_{L^2(F)}$. Applying the linearization formula

$$\operatorname{dist}(G, SO(2)) = |\bar{e}_R(G)| + O(|G-R|^2) \tag{5.11}$$

for $R \in SO(2)$ and $G \in \mathbb{R}^{2\times 2}$ we get

$$\alpha_{\hat{R}}(F) = \int_F |\bar{e}_{\hat{R}}(\nabla y)|^2 \le C_u \int_F \mathrm{dist}^2(\nabla y, SO(2)) + C_u \int_F |\nabla y - \hat{R}|^4. \quad (5.12)$$

Here we already see that it suffices to establish a rigidity estimate of fourth order as in Lemma 5.2 in order to bound the symmetric part of the gradient. One of the main ideas in the following will be to choose $l = l(s, \epsilon, m)$ in (5.4) such that $\vartheta \le 1$ which will imply $\alpha_{\hat{R}}(W) \le C_u \gamma$.

5.2 Estimates in terms of the H^1-norm

We now show that not only the distance of the derivative from a rigid motion can be controlled as derived in (5.4) and (5.9), respectively, but also the distance of the function itself. Once we have such estimates we will be in a position to 'heal' cracks (see Section 6 below). After the modification of the deformation $\nu = sd$ will stand for the minimal distance of two different cracks, where d represents the corresponding increase factor. It will turn out that the least crack length will be given by $\lambda = \nu m^{-1}$. Moreover, $k = \lambda m^{-1}$ will denote the size of the cell on which we apply Theorem 3.5. Define

$$S_i := \bigcup\nolimits_{p \in I_i^k(\Omega^{3k})} Q_i^{\frac{5}{8}k}(p)$$

and note that $\Omega^{5k} \subset \bigcup_{i=1}^4 S_i$. Recall (3.1), (3.7), (3.8) and the definition $\hat{m} = C_2(m, h_*)$ (see below (5.1)). We will proceed in two steps first deriving an estimate for the total variation of the distributional derivative (cf. Corollary 3.7) and then employing Theorem 3.3. For shorthand we will write $\gamma(F) = \|\mathrm{dist}(\nabla y, SO(2))\|^2_{L^2(F)}$, $\delta_p(F) = \sum_{i=1}^4 \|\nabla y - \hat{R}_i\|^p_{L^p(F)}$ for $p = 2, 4$ and subsets $F \subset W$.

Lemma 5.4 *Let $k > s > 0$, $\epsilon > 0$ such that $l := \frac{k}{s} = dm^{-2}$ for $m^{-1}, d \in \mathbb{N}$ with $m^{-1}, d \gg 1$. Let $\lambda = sdm^{-1} = km$. Then for constants $C, c > 0$ we have the following:*
For all $W \in \mathcal{V}^{sm}_{(s,3k)}$ with $W \subset \Omega^{3k}$ and for all $y \in H^1(W)$ with

$$\gamma := \|\mathrm{dist}(\nabla y, SO(2))\|^2_{L^2(W)}, \quad \delta_4 := \sum\nolimits_{i=1}^4 \|\nabla y - \hat{R}_i\|^4_{L^4(W)}$$

for mappings $\hat{R}_i : W^\circ \to SO(2)$, $i = 1, \dots, 4$, which are constant on the connected components of $Q_i^k(p) \cap W^\circ$, $p \in I_i^k(\Omega^{3k})$, we obtain:
We find sets $U \in \mathcal{V}^{s\hat{m}}_{70k}$, $U_Q \in \mathcal{V}^{s\hat{m}}$ with $U \subset U_Q \subset \Omega^{5k}$, $|U_Q \setminus W| = 0$ and $|(W \setminus U) \cap \Omega^{5k}| \le C_u k \|U\|_*$ such that

$$\|U\|_* \le (1 + C_u m)\|W\|_* + C\epsilon^{-1}(\gamma + \delta_4) \quad (5.13)$$

as well as

$$|Q^\lambda(p) \cap U_Q| \geq cm\lambda^2 \quad \text{for all } p \in J(U_Q), \tag{5.14}$$

where $J(U_Q) := \{p \in I^\lambda(\Omega^{3k}) : Q^\lambda(p) \cap U_Q \neq \emptyset\}$.
Moreover, letting $U_J = \bigcup_{p \in J(U_Q)} \overline{Q^\lambda(p)}$, *for* $i = 1, \ldots, 4$ *we find extensions* $\bar{y}_i \in SBV^2(U_J \cap S_i, \mathbb{R}^2)$ *with* $\bar{y}_i = y$ *on* $U_Q \cap S_i$ *such that for all* $\tilde{Q} := Q^{3\lambda}_j(p) \cap U_J$, $p \in I^\lambda_j(\Omega^{3k})$, $j = 1, \ldots, 4$, *with* $\tilde{Q} \subset S_i$ *we have that* $R_i = \hat{R}_i|_{W^\circ \cap \tilde{Q}}$ *is constant on* $W^\circ \cap \tilde{Q}$ *and*

$$\begin{aligned}(|E(R_i^T \bar{y}_i - \mathbf{id})|(\tilde{Q}))^2 \leq Ck^2 C_m \min\Big\{&\epsilon k, \gamma(W \cap Q^{2k}_i(q)) \\ &+ \delta_4(W \cap Q^{2k}_i(q)) + \epsilon|\partial W \cap Q^{2k}_i(q)|_{\mathcal{H}}\Big\},\end{aligned} \tag{5.15}$$

where $q \in I^k_i(\Omega^{3k})$ *such that* $\tilde{Q} \subset Q^k_i(q)$.

Proof. Similarly as in the previous proof we let $J \subset I^{3k}(\Omega^{3k})$ such that

$$\begin{aligned}\epsilon\mathcal{H}^1(Q^{3k}(p) \cap \partial W) + \|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(Q^{3k}(p)\cap W)} \\ + \sum_{i=1}^4 \|\nabla y - \hat{R}_i\|^4_{L^4(Q^{3k}(p)\cap W)} > c_*\epsilon k\end{aligned} \tag{5.16}$$

for all $p \in J$. Define $\hat{W} = \big(W \setminus \bigcup_{p\in J} Q^{3k}(p)\big) \cup \bigcup_{p \in J} \partial Q^{3k}(p)$ and note that choosing c_* sufficiently large and arguing as in the previous proof

$$\|\hat{W}\|_* \leq \|W\|_* + C\epsilon^{-1}(\gamma + \delta_4), \tag{5.17}$$

$\hat{W} \in \mathcal{V}^{sm}_{(s,3k)}$ as well as $|(W \setminus \hat{W}) \cap \Omega^{5k}| \leq C_u k \|\hat{W}\|_*$. We now subsequently construct sets $\hat{U}_1 \supset \ldots \supset \hat{U}_4$ (the inclusions hold up to sets of negligible measure) by application of Theorem 3.5 on connected components of $\hat{W}$ (Step (I)). Afterwards, since in Theorem 3.5 the trace estimate cannot be derived for components near the boundary, we will further modify the sets in a neighborhood of large boundary components (Step (II)). A final modification procedure will then assure property (5.14) (Step (III)).

(I) Begin with $i = 1$ and fix $q \in I^k_1(\Omega^{3k})$. Consider a connected component F of $Q^k_1(q) \cap \hat{W}^\circ$. As $\hat{R}_1 = R$ is constant on F we obtain $\alpha_R(F) \leq C(\gamma(F) + \delta_4(F))$ by (5.12). Define $Q_\mu := Q^k_1(q)$ and recall (3.7). Passing to the closure of F (not relabeled) we can regard F as an element of $\mathcal{V}^{sm}$ with respect to Q_μ (recall (3.3)), where one component is given by $X = Q_\mu \setminus H(F) \in \mathcal{U}^{sm}$. (Observe, however, that $Q_\mu \setminus H(F)$ may intersect several components of $\hat{W}$.) We apply Theorem 3.5 on $F \subset Q_\mu$ for $\varepsilon = \epsilon$, $\sigma = m$ to obtain a set $G \in \mathcal{W}^{s\hat{m}}$ with $|G \setminus F| = 0$ and

$$\epsilon\|G\|_* + \alpha_R(G) \leq (1 + C_u m)(\epsilon\|F\|_* + \alpha_R(F)). \tag{5.18}$$

(Recall that the sum in $\|F\|_*$ runs only over the boundary components having empty intersection with ∂Q_μ.) Moreover, similarly as before we have

$$|F \setminus G| \le C_u k \|G\|_* \tag{5.19}$$

and using (3.11), (3.9)(i),(ii) for all $\Gamma_t(G) \in \mathcal{T}(G) := \{\Gamma_t : N^{\tau_t}(\partial R_t) \subset H(G)\}$

$$\int_{\Theta_t(G)} |[\bar{y}_1](x))|^2 \, d\mathcal{H}^1(x) \le C C_m \epsilon |\Gamma_t(G)|_\infty^2, \tag{5.20}$$

where $\bar{y}_1$ is the extension (cf. (3.10))

$$\bar{y}_1(x) = \begin{cases} y & x \in \hat{W}, \\ R\,(\mathbf{Id} + A_t)\,x + R\,c_t & x \in X_t \ \text{ for } \Gamma_t(G) \in \mathcal{T}(G). \end{cases} \tag{5.21}$$

Here recall that ∂R_t are the rectangles given by (3.9) as well as $\tau_t = \bar{\tau}|\partial R_t|_\infty \ll |\partial R_t|_\infty$.

We proceed in this way for every connected component $(F_j)_j$ of all $Q_1^k(q)$, $q \in I_1^k(\Omega^{3k})$ and define $\hat{U}_1 = (\hat{W} \setminus \bigcup_j F_j) \cup \bigcup_j G_j \in \mathcal{V}^{s\hat{m}}$. (Observe that one may have $H(F_{j_1}) \subset H(F_{j_2})$. In this case the above arguments can be omitted for F_{j_1}.) By $\mathcal{G}$ we denote the set of boundary components $\Gamma(\hat{U}_1)$ which do not coincide with some $\Gamma_t(G_j)$. Note that by (5.12) and (5.17)

$$\begin{aligned} \epsilon\|\hat{U}_1\|_* \le \epsilon\|\hat{U}_1\|_* + \alpha_{\hat{R}_1}(\hat{U}_1) &\le (1 + C_u m)(\epsilon\|\hat{W}\|_* + \alpha_{\hat{R}_1}(\hat{W})) \\ &\le (1 + C_u m)\epsilon\|W\|_* + C(\gamma + \delta_4). \end{aligned} \tag{5.22}$$

The second step follows as by construction for each $\Gamma(\hat{U}_1) \in \mathcal{G}$ there is a $\Gamma(\hat{W}) = \partial X$ such that $\Gamma(\hat{U}_1) \subset \overline{X}$ (recall Remark 3.6(i)). By (5.19) we also get $|\hat{W} \setminus \hat{U}_1| \le C_u k \|\hat{U}_1\|_*$. Moreover, by Remark 3.6(ii) we can replace the components of $G_j \in \mathcal{V}^{s\hat{m}}$ by rectangles such that the resulting set G_j' lies in $\mathcal{V}^{s\hat{m}}_{\mathrm{con}}$. Recall that the (rectangular) components of G_j' satisfy $\max_{i=1,2}|\pi_i\Gamma(G_j')| \le 2k$.

Then we define $\hat{U}_1'' := (\hat{W} \setminus \bigcup_j F_j) \cup \bigcup_j G_j' \in \mathcal{V}^{s\hat{m}}$. We now apply Lemma 4.2(ii) for $\nu = 0$, $(Z_j)_j$ the rectangular components of $(G_j')_j$ and V' the set whose boundary components are given by the elements of $\mathcal{G}$. We obtain a set $\hat{U}_1' \in \mathcal{V}^{s\hat{m}}_{5k}$ with $\|\hat{U}_1'\|_* \le \|\hat{U}_1\|_*$ and $|\hat{U}_1'' \setminus \hat{U}_1'| \le C_u k \|\hat{U}_1'\|_*$. (Strictly speaking, we need to pass from $\mathcal{V}^{s\hat{m}}$ to $\mathcal{V}^{s\hat{m}/2}$, but do not include it in the notation for convenience.) Likewise we observe $|\hat{U}_1' \setminus \hat{W}| = 0$ and $|\hat{W} \setminus \hat{U}_1'| \le C_u k \|\hat{U}_1'\|_*$. Additionally, we apply Lemma 4.1(ii) and get a (not relabeled) set $\hat{U}_1' \in \mathcal{V}^{s\hat{m}}_{6k}$ such that (4.3) and (5.22) hold. As in the proof of Lemma 5.2 this implies $\hat{U}_1' \in \mathcal{V}^{s\hat{m}}_{(s,6k)}$ since $\hat{W} \in \mathcal{V}^{sm}_{(s,3k)}$, i.e. the least length of components is bounded from below by s.

In the following, by a slight abuse of notation, we say that a component $\Gamma_t(\hat{U}_1')$, which coincides with some $\partial X_t = \Gamma_t(G')$ for some component G', satisfies

(5.20) if all corresponding $(\Gamma_{t_s}(G))_s$ with $\Gamma_{t_s}(G) \subset \overline{X_t}$ satisfy (5.20). It is not hard to see that (5.20) is satisfied for all boundary components with (recall (3.8))

$$\Gamma_t(\hat{U}_1') \cap S_1 \neq \emptyset, \quad |\Gamma_t(\hat{U}_1')|_\infty \leq \tfrac{k}{8}, \quad N_*(\Gamma_t(\hat{U}_1)) \subset H^{\frac{k}{8}}(\hat{U}_1'),$$

where $N_*(\Gamma_t(\hat{U}_1)) = \{x : \mathrm{dist}(x, \Gamma_t(\hat{U}_1')) \leq \bar{C}\hat{m}|\Gamma_t(\hat{U}_1')|_\infty\}$ for some large constant $\bar{C} = \bar{C}(h_*) > 0$. Indeed, assume that there is some $\Gamma_s = \Gamma_{t_s}(G) \subset Q_1^k(q)$ such that for the corresponding rectangle R_s one has $N^{\tau_s}(\partial R_s) \not\subset H(G)$ although the corresponding $\Gamma_t(G') = \partial X_t$ fulfills the above three properties. First, we observe $R_s \subset X_t$ by Remark 3.6(ii) and thus $R_s \subset Q_1^{\frac{3}{4}k}(q)$. By (3.9)(i) we get $|\partial R_s|_\infty \leq C|\Gamma_s|_\infty$. Consequently, since $\tau_s \ll \frac{1}{C}|\partial R_s|_\infty$ (recall the assumption in Theorem (3.5)) we have $N^{\tau_s}(\partial R_s) \subset Q_1^{\frac{7}{8}k}(q)$. Since by assumption $N^{\tau_s}(\partial R_s) \not\subset H(G)$, this would imply $|\partial H(G) \cap Q_1^k(q)|_\infty > \frac{k}{8}$.

Consequently, there is a chain of components $(\Gamma_{t_i}(\hat{U}_1'))_{i=1}^n = (\partial X_{t_i}(\hat{U}_1'))_{i=1}^n$ such that $\Gamma_{t_1}(\hat{U}_1') \cap \partial Q_\mu \neq \emptyset$, $X_{t_n}(\hat{U}_1') \cap N^{\tau_s}(\partial R_s) \neq \emptyset$ and $\Gamma_{t_{i-1}}(\hat{U}_1') \cap \Gamma_{t_i}(\hat{U}_1') \neq \emptyset$. Thus, by (4.3) there is one $\Gamma_*(\hat{U}_1')$ with $|\Gamma_*(\hat{U}_1')|_\infty > \frac{k}{8}$ such that $N^{\tau_s}(\partial R_s) \cap X_*(\hat{U}_1') \neq \emptyset$. Recalling that $R_s \subset X_t$ and $\tau_s < \bar{C}\hat{m}|\Gamma_t(\hat{U}_1')|_\infty$ for $\bar{C}$ sufficiently large by Remark 3.9(iii) we find $N_*(\Gamma_t(\hat{U}_1)) \cap X_*(U_1') \neq \emptyset$. This, however, is a contradiction to $N_*(\Gamma_t(\hat{U}_1)) \subset H^{\frac{k}{8}}(\hat{U}_1')$.

We now iteratively repeat the above construction for $i = 2, 3, 4$ for $\hat{U}_{i-1}'$ instead of $\hat{W}$ and obtain extensions $\bar{y}_2, \bar{y}_3, \bar{y}_4$ as well as $(\hat{U}_i)_{i=1}^4$ and sets $\hat{U}_4' \subset \ldots \subset \hat{U}_1' \subset \hat{W}$ (the inclusions hold up to a set of negligible measure) with $\hat{U}_4' \in \mathcal{V}^{s\hat{m}}_{(s,15k)}$ such that (5.22) holds for a possibly larger constant replacing $\hat{U}_1$ by $\hat{U}_4$. We briefly note that the sets are elements of $\mathcal{V}^{s\hat{m}}$ due to Remark 3.9(iv). Moreover, for $i = 1, \ldots, 4$, (5.20) is satisfied for $\bar{y}_i$ and all boundary components $\Gamma_t(\hat{U}_i')$ with $\Gamma_t(\hat{U}_i') \cap S_i \neq \emptyset$, $|\Gamma_t(\hat{U}_i')|_\infty \leq \frac{k}{8}$ and $N_*(\Gamma_t(\hat{U}_i')) \subset H^{\frac{k}{8}}(\hat{U}_i')$.

For later we also observe that due to the local nature of the modification process and (5.18) we get

$$\begin{aligned} |\partial\hat{U}_i \cap Q_i^k(q)|_{\mathcal{H}} \leq C|\partial\hat{W} \cap Q_i^{2k}(q)|_{\mathcal{H}} \\ + C\epsilon^{-1}\big(\gamma(\hat{W} \cap Q_i^{2k}(q)) + \delta_4(\hat{W} \cap Q_i^{2k}(q))\big). \end{aligned} \tag{5.23}$$

Although the inclusions for $(\hat{U}_i')_{i=1}^4$ only hold up to segments, we observe that the sets are 'nested' concerning small boundary components in the following sense: Letting $\hat{U}_i^* = \hat{U}_i' \cap \overline{(H^{\frac{k}{8}}(\hat{U}_i'))^\circ}$ we obtain

$$\hat{U}_4^* \subset \ldots \subset \hat{U}_1^*. \tag{5.24}$$

Indeed, assume e.g. there was a component $X(\hat{U}_1^*)$ and components $X_1, \ldots, X_n$ of $\hat{U}_2^*$ with $X(\hat{U}_1^*) \subset \bigcup_{j=1}^n \overline{X_j}$ and $\bigcup_{j=1}^n \partial X_j \cap X(\hat{U}_1^*) \neq \emptyset$. Then by construction of the sets we clearly find some X_i with $\partial X_i \cap X(\hat{U}_1^*) \neq \emptyset$, $|X(\hat{U}_1^*) \setminus X_i| > 0$

and $|\partial X_i|_\infty \le \frac{k}{8}$. This, however, together with (4.3) gives a contradiction to $X(\hat{U}_1^*) \subset \bigcup_{j=1}^n \overline{X_j}$. In particular, (5.24) implies $H^{\frac{k}{8}}(\hat{U}_4') \subset \ldots \subset H^{\frac{k}{8}}(\hat{U}_1')$ up to sets of negligible measure and thus for $i = 1, \ldots, 4$, (5.20) is satisfied for $\bar{y}_i$ and all boundary components $\Gamma_t(\hat{U}_i')$ with $\Gamma_t(\hat{U}_i') \cap S_i \neq \emptyset$, $|\Gamma_t(\hat{U}_i')|_\infty \le \frac{k}{8}$, $N_*(\Gamma_t(\hat{U}_i')) \subset H^{\frac{k}{8}}(\hat{U}_4')$. We want to remove the third condition. For that reason, we subtract neighborhoods of large boundary components as follows.

(II) Let $U^* = H^{\frac{k}{8}}(\hat{U}_4')$ and let $\Gamma_1(U^*), \ldots, \Gamma_n(U^*)$ be the boundary components. For $\Gamma_j(U^*)$ let $M(\Gamma_j)$ be the smallest rectangle in $\mathcal{U}^{s\hat{m}}$ satisfying $M(\Gamma_j) \supset \{x \in \mathbb{R}^2 : \mathrm{dist}_\infty(x, X_j) \le \bar{C}k\hat{m}\}$ for the constant $\bar{C} > 0$ introduced above, where X_j denotes the component corresponding component to $\Gamma_j(U^*)$. Clearly, using the fact that $\bar{C}\hat{m} \le m$ (see (5.1)) one has $|\pi_i \partial M(\Gamma_j)| \le |\pi_i \Gamma_j(U^*)| + C_u m |\Gamma_j(U^*)|_\infty \le 31k$ for $i = 1, 2$. As the components $(X_j)_j$ are pairwise disjoint and connected, we obtain $Z(\Gamma_{j_1}) \setminus Z(\Gamma_{j_2})$ is connected for all $1 \le j_1, j_2 \le n$, where $Z(\Gamma_j)$ denotes the smallest rectangle containing X_j. Consequently, since the neighborhoods $M(\Gamma_j) \setminus Z(\Gamma_j)$ all have the same thickness $\sim \bar{C}k\hat{m}$, we get that $M(\Gamma_{j_1}) \setminus M(\Gamma_{j_2})$ is connected for all $1 \le j_1, j_2 \le n$.

Then by Lemma 4.2(ii) applied on $V = U^*$, $V' = \Omega^{5k} \setminus \bigcup_{|\Gamma_j(\hat{U}_i')| \le \frac{k}{8}} X_j(\hat{U}_i')$ we obtain sets $\tilde{U}_i$ with $|(\hat{U}_i' \setminus \bigcup_{j=1}^n M(\Gamma_j)) \setminus \tilde{U}_i| \le C_u k \|V'\|_*$. In particular, we set $\tilde{U} = \tilde{U}_4$ and observe that $\tilde{U} \in \mathcal{V}^{s\hat{m}}_{32k}$. Moreover, we obtain $\|\tilde{U}\|_* \le (1 + C_u m)\|V\|_* + \|V'\|_*$. As $\hat{U}_4'$ satisfies (4.3), we derive $(\partial V \cap \partial V') \cap (\Omega^{5k})^\circ = \emptyset$ and therefore $\|\tilde{U}\|_* \le (1 + C_u m)\|\hat{U}_4'\|_*$, i.e. (5.22) holds replacing $\hat{U}_1$ by $\tilde{U}$ (possibly for a larger constant). Applying Lemma 4.1(ii) we get (not relabeled) sets $\tilde{U}_i \in \mathcal{V}^{s\hat{m}}_{33k}$ satisfying (4.3). For later we note that $\tilde{U}_4 \subset \ldots \subset \tilde{U}_1$ up to sets of negligible measure. This follows from (5.24) and the fact that in Lemma 4.2(ii) the components of V' are replaced by corresponding rectangles. Arguing as in (5.24) we also find

$$\tilde{U}_4^* \subset \ldots \subset \hat{U}_1^*, \quad \text{where} \quad \tilde{U}_i^* = \tilde{U}_i \cap \overline{(H^{\frac{k}{8}}(\tilde{U}_i))^\circ} \tag{5.25}$$

In particular, this also implies $H^{\frac{k}{8}}(\tilde{U}_4) \subset \ldots \subset H^{\frac{k}{8}}(\tilde{U}_1)$ up to sets of negligible measure.

We now see that for $i = 1, \ldots, 4$, (5.20) holds for $\bar{y}_i$ for all components satisfying

$$\Gamma_t(\tilde{U}_i) \cap S_i \neq \emptyset, \quad |\Gamma_t(\tilde{U}_i)|_\infty \le \tfrac{1}{8}k. \tag{5.26}$$

(Strictly speaking (5.20) holds for the corresponding components of $\hat{U}_i$.) In fact, since $\bar{C}\hat{m}|\Gamma_t(\hat{U}_i')|_\infty \le \frac{k}{8}\bar{C}\hat{m}$ for $|\Gamma_t(\hat{U}_i')|_\infty \le \frac{k}{8}$, due to the construction of $\tilde{U}_i$ components with $|\Gamma_t(\hat{U}_i')|_\infty \le \frac{k}{8}$ and $N_*(\Gamma_t(\hat{U}_i')) \not\subset H^{\frac{k}{8}}(\hat{U}_4')$ are 'combined' with a boundary component of $\hat{U}_4'$ which is larger than $\frac{k}{8}$.

We apply Lemma 4.1(i) to obtain a (not relabeled) set $\tilde{U} \in \mathcal{V}^{s\hat{m}}_{33k}$ satisfying (4.2). For each $\Gamma_t(\tilde{U})$, $t = 1, \ldots, n$, let $N_1(\Gamma_t)$, $N_2(\Gamma_t)$ be the smallest rectangles in $\mathcal{U}^{s\hat{m}}$ satisfying

$$N_1(\Gamma_t) \supset \{x \in \mathbb{R}^2 : \mathrm{dist}_\infty(x, X_t) \le \min\{Bm|\Gamma_t(\tilde{U})|_\infty, 2\lambda\}\},$$
$$N_2(\Gamma_t) \supset \{x \in \mathbb{R}^2 : \mathrm{dist}_\infty(x, X_t) \le Bm \min\{|\Gamma_t(\tilde{U})|_\infty, \lambda\}\}$$

for some $B > 0$ (independent of h_*) and $\lambda = km$, where X_t is the component corresponding to $\Gamma_t(\tilde{U})$. It is not restrictive to assume that

$$\mathcal{H}^1\big(N_2(\Gamma_t) \cap (\partial\tilde{U} \setminus (\Gamma_t(\tilde{U}) \cup \partial H^{\frac{k}{8}}(\tilde{U}))\big) \le CBm \min\{|\Gamma_t(\tilde{U})|_\infty, \lambda\} \tag{5.27}$$

for all $\Gamma_t(\tilde{U})$ with $|\Gamma_t(\tilde{U})|_\infty \le \frac{k}{8}$. Indeed, otherwise we replace $\tilde{U}$ by $\tilde{U}' := \big(\tilde{U} \setminus N_2^*(\Gamma_t)\big) \cup \partial N_2^*(\Gamma_t)$, where $N_2^*(\Gamma_t) = (N_2(\Gamma_t) \cap H^{\frac{k}{8}}(\tilde{U}))^\circ$, and arguing similarly as in (5.16) and Lemma 3.2 we get $\|\tilde{U}'\|_* \le \|\tilde{U}\|_*$. Let $(X_{t'})_{t'}$, $X_{t'} \neq X_t$, be the components of $\tilde{U}$ having nonempty intersection with $N_2^*(\Gamma_t)$. Clearly, we have $|\partial X_{t'}|_\infty \le \frac{k}{8}$. We define $T = \overline{N_2^*(\Gamma_t)} \cup \bigcup_{t'} \overline{X_{t'}}$ and modify $\tilde{U}'$ on a set of measure zero by letting $\tilde{U}'' = (\tilde{U}' \setminus T) \cup \partial T$. Arguing similarly as in the proof of Lemma 4.1 we find $\tilde{U}'' \in \mathcal{V}^{s\hat{m}}_{33k}$ and $\|\tilde{U}''\|_* \le \|\tilde{U}\|_*$. Then by Lemma 4.1(i) we find a (not relabeled) set $\tilde{U}''$ which additionally satisfies (4.2). We continue with this iterative modification process until (5.27) is satisfied for all components smaller than $\frac{k}{8}$. Finally, by Lemma 4.1(ii) we obtain a (not relabeled) set $\tilde{U}'' \in \mathcal{V}^{s\hat{m}}_{34k}$ satisfying (4.3). Noting that during the modification procedure components larger than $\frac{k}{8}$ do not become smaller than $\frac{k}{8}$ we also find $H^{\frac{k}{8}}(\tilde{U}'') \subset H^{\frac{k}{8}}(\tilde{U})$. For convenience the set will still be denoted by $\tilde{U}$ in the following.

(III) We now finally construct the sets U_Q and U. For each $t = 1, \ldots, n$ define the rectangle

$$Z_t = \bigcup\nolimits_{p \in I^\lambda(N_1(\Gamma_t))} \overline{Q^\lambda(p)}. \tag{5.28}$$

We find $Z_t \subset N_1(\Gamma_t)$ and for sufficiently small components one has $Z_t = \emptyset$. Choosing B sufficiently large we get $X_t \subset Z_t$ if $|\partial X_t|_\infty > \frac{k}{8}$. Rearrange the components in a way that $Z_t = \emptyset$ for $t > n'$. This implies

$$\Omega^{5k} \setminus H^{\frac{k}{8}}(\tilde{U}) \subset \bigcup\nolimits_{t=1}^{n'} Z_t. \tag{5.29}$$

Let $Y_t \in \mathcal{U}^{s\hat{m}}$ be the smallest rectangle containing $Z_t \cup X_t$. By the definition of $N_1(\Gamma_t)$ and Z_t we obtain

$$|\pi_i \partial Y_t| = |\pi_i \partial(\overline{Z_t \cup X_t})| \le |\pi_i \Gamma_t(\tilde{U})| + C_u m |\Gamma_t(\tilde{U})|_\infty, \quad i = 1, 2 \tag{5.30}$$

for some $C_u = C_u(B)$ large enough. As $(X_t)_t$ are pairwise disjoint and connected, it is elementary to see that $Z_{t_1} \setminus Z_{t_2}$ or $Z_{t_2} \setminus Z_{t_1}$ is connected for all $1 \le t_1, t_2 \le n'$. In fact, assume there were $t_1 \neq t_2$ such that $\overline{\pi_1 Z_{t_2}} \subset \pi_1 Z_{t_1}$ and $\overline{\pi_2 Z_{t_1}} \subset \pi_2 Z_{t_2}$.

Then due to the definition of the neighborhoods we find $\overline{\pi_1 X_{t_2}} \subset \pi_1 X_{t_1}$ and $\overline{\pi_2 X_{t_1}} \subset \pi_2 X_{t_2}$. This, however, implies $X_{t_1} \cap X_{t_2} \neq \emptyset$ and yields a contradiction. A similar argument yields that $Y_{t_1} \setminus Y_{t_2}$ or $Y_{t_2} \setminus Y_{t_1}$ is connected for all $1 \le t_1, t_2 \le n'$.

Define $U'_Q = \tilde{U} \setminus \bigcup_{j=1}^{n'} Z_j$ and let $\hat{J} \subset I^\lambda(\Omega^{3k})$ such that (cf. also (5.16))

$$\mathcal{H}^1(Q^\lambda(p) \cap \partial U'_Q) > c_* \lambda \tag{5.31}$$

for all $p \in \hat{J}$. Then let $U_Q = (\Omega^{5k} \cap U'_Q) \setminus \bigcup_{p \in \hat{J}} Q^\lambda(p)$. Observe that possibly $U_Q \notin \mathcal{V}^{s\hat{m}}_{\mathrm{con}}$. Therefore, we now define a set $U \subset U_Q$ with connected boundary components.

By Lemma 4.2(ii) for $V = \Omega^{5k} \setminus \bigcup_{t=1}^{n'} X_t$, $V' = \Omega^{5k} \setminus \bigcup_{t=n'+1}^{n} X_t$ we obtain a set U' with $|(\tilde{U} \setminus \bigcup_{t=1}^{n'} Y_t) \setminus U'| \le C_u k \|V'\|_*$ such that $U' \in \mathcal{V}^{s\hat{m}}_{\mathrm{con}}$. Moreover, recalling (5.30) as well as $|\partial X_t|_\infty \le \frac{k}{8}$ for $t > n'$, we get $U' \in \mathcal{V}^{s\hat{m}}_{69k}$ for m sufficiently small. Using (5.30) and the fact that $\tilde{U}$ satisfies (4.3) we have $\|U'\|_* \le (1 + C_u m)\|V\|_* + \|V'\|_* \le (1 + C_u m)\|\tilde{U}\|_*$. Finally, again using Lemma 4.2(ii) we find a set $U \in \mathcal{V}^{s\hat{m}}_{70k}$ with

$$\Big| \Big(\Omega^{5k} \setminus \big(\bigcup_{t=1}^{n'} Y_t \cup \bigcup_{t=n'+1}^{n} X_t \cup \bigcup_{p \in \hat{J}} Q^\lambda(p) \big) \Big) \setminus U \Big| \le C_u k \|U\|_* \tag{5.32}$$

Arguing similarly as in (5.10), (5.16) we find $\|U\|_* \le \|U'\|_* \le (1 + C_u m)\|\tilde{U}\|_*$. This implies (5.13) since $\tilde{U}$ satisfies (5.22). Moreover, we derive $|(W \setminus U) \cap \Omega^{5k}| \le C_u k \|U\|_*$.

Define U_J as in the assertion of Lemma 5.4. We see that all $\Gamma_t(\tilde{U}_i) = \partial X_t$ with $\Gamma_t(\tilde{U}_i) \cap U_J^\circ \neq \emptyset$ satisfy $|\Gamma_t(\tilde{U}_i)|_\infty \le \frac{k}{8}$. In fact, if $|\Gamma_t(\tilde{U}_i)|_\infty > \frac{k}{8}$, we would have $X_t \subset \Omega^{5k} \setminus (H^{\frac{k}{8}}(\tilde{U}_i))^\circ$ and thus $X_t \subset \Omega^{5k} \setminus (H^{\frac{k}{8}}(\tilde{U}))^\circ$, where we used $H^{\frac{k}{8}}(\tilde{U}'') \subset H^{\frac{k}{8}}(\tilde{U}) \subset H^{\frac{k}{8}}(\tilde{U}_4) \subset H^{\frac{k}{8}}(\tilde{U}_i)$ up to a set of negligible measure (see (5.25)). (Recall that the set $\tilde{U}''$ given by the modification described below (5.27) is also denoted by $\tilde{U}$ for convenience.) Therefore, by (5.29) we get $\Gamma_t(\tilde{U}_i) \subset \overline{X_t} \subset \bigcup_{j=1}^{n'} \overline{Z_j}$ and thus $\Gamma_t(\tilde{U}_i) \cap U_J^\circ = \emptyset$ giving a contradiction. Consequently, by (5.26)

$$\text{(5.20) holds for } \bar{y}_i \text{ for all } \Gamma_t(\tilde{U}_i) \text{ with } \Gamma_t(\tilde{U}_i) \cap U_J^\circ \cap S_i \neq \emptyset. \tag{5.33}$$

For later we recall that the corresponding components $(\Gamma_{t_s}(G))_s$ with $\Gamma_{t_s}(G) \subset \overline{X_t(\tilde{U}_i)}$ (which satisfy (5.20)) also satisfy (3.9) since $G \in \mathcal{W}^{s\hat{m}}$. Consider $\tilde{Q} := Q_j^{3\lambda}(p) \cap U_J \subset S_i$. We observe that $\tilde{Q}$ consists of a bounded number of squares and that $\tilde{Q} \cap U_Q$ is contained in a connected component F of $Q_i^k(q) \cap \hat{W}^\circ$. Indeed, this follows from the fact that due to the construction of U_Q, in particular (5.28), two connected components $F_1 \neq F_2$, $F_t \cap S_i \neq \emptyset$ for $t = 1, 2$, for which $H(F_t)$ is not completely contained in another component $H(F_{t'})$, fulfill $\mathrm{dist}(F_1 \cap U_J, F_2 \cap U_J) \ge 2\lambda$. This observation also implies that $\tilde{Q}^\circ$ is connected, i.e. each $Q \subset \tilde{Q}$ shares at least one face with the rest of $\tilde{Q}$. Consequently, Corollary 3.7 together with (5.20) yield

$$(|E(R_i^T \bar{y}_i - \mathbf{id})|(\tilde{Q}))^2 \le C\lambda^2 \alpha_{R_i}(U_Q \cap \tilde{Q}) + CkC_m \epsilon |\partial \hat{U}_i \cap Q_i^k(q)|^2_{\mathcal{H}},$$

where R_i is the value of the constant function $\hat{R}_i|_F$. Then (5.23) and (5.16) imply $|\partial\hat{U}_i \cap Q_i^k(q)|_{\mathcal{H}} \le Ck$ which together with (5.12) yields (5.15). For later we note that Corollary 3.7 also yields

$$(|D^j(\bar{y}_i - R_i\,\mathbf{id})|(\tilde{Q}))^2 \le C\lambda^2\alpha_{R_i}(U_Q \cap \tilde{Q}) + CkC_m\epsilon|\partial\hat{U}_i \cap Q_i^k(q)|^2_{\mathcal{H}}. \qquad (5.34)$$

It remains to show (5.14). Consider $\hat{Q} = Q^\lambda(p)$ with $\hat{Q} \cap U_Q \neq \emptyset$ and show that $|\hat{Q} \cap U_Q| \ge cm\lambda^2$. First note that $\hat{Q} \cap U_Q = \hat{Q} \cap \tilde{U}$. Let $\Gamma = \Gamma(\tilde{U}) = \partial X$ be the boundary component maximizing $|X \cap \hat{Q}|_\infty$. If $|\Gamma|_\infty \ge \frac{k}{8}$ we get a contradiction for B large enough as then $\hat{Q} \cap U_J = \emptyset$. Assume $|X \cap \hat{Q}|_\infty \ll \lambda$. Then (5.31) and the isoperimetric inequality imply $|\hat{Q} \setminus U_Q| \le C_u \sum_t |X_t(\tilde{U}) \cap \hat{Q}|^2_\infty \ll C_u\lambda \sum_t |X_t(\tilde{U}) \cap \hat{Q}|_\infty \le C_u\lambda^2$ and thus $|\hat{Q} \cap U_Q| \ge cm\lambda^2$ for m small enough. Therefore, we may assume that

$$\tfrac{1}{8}k = \tfrac{1}{8}m^{-1}\lambda \ge |\Gamma|_\infty \ge |X \cap \hat{Q}|_\infty \ge \bar{c}\lambda \qquad (5.35)$$

for $\bar{c} > 0$ small enough. It is not hard to see that $|(N_2(\Gamma) \setminus X) \cap \hat{Q}| \ge CBm\bar{c}^2\lambda^2$. Indeed, an elementary argument yields $|N_2(\Gamma) \cap \hat{Q}| \ge CBm\bar{c}^2\lambda^2$. Moreover, if we had $|\hat{Q} \setminus X| \ll Bm\bar{c}^2\lambda^2$, we would get $\hat{Q} \subset N_1(\Gamma)$ and thus $\hat{Q} \cap U_Q = \emptyset$ by the construction of U_Q. We can assume that $N_2(\Gamma) \cap \partial H^{\frac{k}{8}}(\tilde{U}) = \emptyset$ since otherwise a component larger than $\frac{k}{8}$ intersects $\hat{Q}$ and we derive $\hat{Q} \cap U_J = \emptyset$ as before. By (4.2) this also implies that all components $X_j(\tilde{U})$ with $X_j(\tilde{U}) \cap N_2(\Gamma) \neq \emptyset$ satisfy $\overline{X_j(\tilde{U})} \cap \Gamma = \emptyset$. Thus by the isoperimetric inequality and by (5.27) we get $|N_2(\Gamma) \cap \hat{Q} \cap \tilde{U}| \ge |(N_2(\Gamma) \setminus X) \cap \hat{Q}| - C(B\lambda m)^2$. This implies

$$\begin{aligned}|\hat{Q} \cap U_Q| = |\hat{Q} \cap \tilde{U}| &\ge |\hat{Q} \cap \tilde{U} \cap N_2(\Gamma)| \ge |(N_2(\Gamma) \setminus X) \cap \hat{Q}| - C(B\lambda m)^2 \\ &\ge -CB^2\lambda^2m^2 + C\bar{c}^2Bm\lambda^2 \ge cm\lambda^2\end{aligned}$$

for m sufficiently small. $\square$

Remark 5.5 (i) For later we observe that there is a set $U^H \in \mathcal{V}^\lambda_{35k}$ with

$$(i)\ \|U^H\|_* \le (1 + C_um)\|W\|_* + C\epsilon^{-1}(\gamma + \delta_4), \quad (ii)\ \|U^H\|_{\mathcal{H}} \le C_u\|U^H\|_* \quad (5.36)$$

which coincides with the set U_J considered in the previous lemma up to a set of negligible measure. In fact, we apply Lemma 4.2(i) on the rectangles $(Z_t)_{t=1}^{n'}$ considered in (5.28) and find pairwise disjoint $(Z'_t)_{t=1}^{n'}$ with $\bigcup_{j=1}^{n'} \overline{Z_j} = \bigcup_{j=1}^{n'} \overline{Z'_j}$. We define

$$U^H := \Omega^{5k} \setminus \Big(\bigcup\nolimits_{j=1}^{n'} Z'_j \cup \bigcup\nolimits_{p \in \hat{J}} Q^\lambda(p)\Big),$$

where $\hat{J}$ as in (5.32). By Lemma 4.2(i) we get (i) and $U^H \in \mathcal{V}^\lambda_{35k}$ since $|\pi_i \partial Z_t| \le 2\cdot 34k + C_umk \le 70k$ for $i = 1, 2$. Moreover, (5.36)(ii) is a consequence of Lemma 3.2 and the fact that $(Z_t)_t$, $(Q^\lambda(p))_{p\in J}$ are rectangles.

Clearly $U_J \subset U^H$. Moreover, we see that $|U^H \setminus U_J| > 0$ can only happen if there is a square $Q^\lambda(p) \subset U^H$ and components $(X_t(\tilde{U}))_t$ of $\tilde{U}$ such that $Q^\lambda(p) \subset \overline{\bigcup_t X_t(\tilde{U})}$. Since we can suppose $|\partial X_t(\tilde{U})|_\infty \le \frac{k}{8}$ (otherwise the components are contained in some rectangle Z_t), this yields a contradiction to (4.2).

(ii) For $i = 1, \ldots, 4$ we have

$$|\partial \hat{U}_i \cap U_J^\circ|_{\mathcal{H}} \le C_u(\|W\|_* + C\epsilon^{-1}(\gamma + \delta_4)).$$

In fact, recalling (5.33) we get that all $\Gamma_t(\hat{U}_i)$ with $\Gamma_t(\hat{U}_i) \cap U_J^\circ \neq \emptyset$ fulfill (5.20) and (3.9). Thus, we obtain $|\Theta_t(\hat{U}_i)|_{\mathcal{H}} \le C_u |\Theta_t(\hat{U}_i)|_*$ and the claim follows from (5.22) replacing $\hat{U}_1$ by $\hat{U}_i$.

We are now in a position to prove the main result of this section. Recall the definition $\lambda = sdm^{-1} = km$ and (5.1).

Lemma 5.6 *Let $k > s, \epsilon > 0$ such that $l := \frac{k}{s} = dm^{-2}$ for $m^{-1}, d \in \mathbb{N}$ with $m^{-1}, d \gg 1$. Then for a fixed constant $C > 0$ we have the following:*
For all $W \in \mathcal{V}^{sm}_{(s,3k)}$ with $W \subset \Omega^{3k}$ and for all $y \in H^1(W)$ with $\|\nabla y\|_\infty \le C$, γ as defined in (5.2) and

$$\delta_4 := \sum\nolimits_{i=1}^4 \|\nabla y - \hat{R}_i\|^4_{L^4(W)}, \quad \delta_2 := \sum\nolimits_{i=1}^4 \|\nabla y - \hat{R}_i\|^2_{L^2(W)} \tag{5.37}$$

for mappings $\hat{R}_i : W^\circ \to SO(2)$, $i = 1, \ldots, 4$, which are constant on the connected components of $Q_i^k(p) \cap W^\circ$, $p \in I_i^k(\Omega^{3k})$, we obtain:
We find sets $V \in \mathcal{V}^{s\hat{m}^2}_{71k}$, $U_J \in \mathcal{V}^\lambda$ with $V \subset U_J$ and $V \subset \Omega^{6k}$, $|V \setminus W| = 0$, $|(W \setminus V) \cap \Omega^{6k}| \le C_u k \|V\|_*$ such that

$$\|V\|_* \le (1 + C_u m)\|W\|_* + C\epsilon^{-1}(\gamma + \delta_4) \tag{5.38}$$

as well as mappings $\bar{R}_j : U_J \to SO(2)$ and $\bar{c}_j : U_J \to \mathbb{R}^2$, which are constant on $Q_j^\lambda(p)$, $p \in I_j^\lambda(\Omega^{3k})$, such that

$$\begin{aligned}
&(i) \quad \|y - (\bar{R}_j \, \cdot + \bar{c}_j)\|^2_{L^2(V)} \le CC_m^2 \lambda^2 \min\nolimits_{p=2,4}(1 + \vartheta_p)(\gamma + \delta_p + \epsilon\|W\|_*),\\
&(ii) \quad \|\nabla y - \bar{R}_j\|^p_{L^p(V)} \le CC_m^2\big(\delta_p + \vartheta_p(\gamma + \delta_4 + \epsilon\|W\|_*)\big), \ p = 2, 4,\\
&(iii) \quad \|\bar{R}_{j_1} - \bar{R}_{j_2}\|^p_{L^p(U_J)} \le CC_m^2\big(\delta_p + \vartheta_p(\gamma + \delta_4 + \epsilon\|W\|_*)\big), \ p = 2, 4,\\
&(iv) \quad \|(\bar{R}_{j_1} \, \cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2} \, \cdot + \bar{c}_{j_2})\|^2_{L^2(U_J)} \le CC_m^2 \lambda^2 \min_{p=2,4}(1 + \vartheta_p)(\gamma + \delta_p + \epsilon\|W\|_*)
\end{aligned} \tag{5.39}$$

for $j_1, j_2 = 1, \ldots, 4$, $j = 1, \ldots, 4$, where $\vartheta_4 = \vartheta$ and $\vartheta_2 = 1$. Moreover, we have

$$\lambda^{-2}\|(\bar{R}_{j_1} \, \cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2} \, \cdot + \bar{c}_{j_2})\|^2_{L^\infty(U_J)} + \|\bar{R}_{j_1} - \bar{R}_{j_2}\|^4_{L^\infty(U_J)} \le C\bar{\vartheta} \tag{5.40}$$

for $\bar{\vartheta} = \min\{\vartheta(1 + \vartheta), C_m^3\}$ and under the additional assumption that $\Delta y = 0$ in W° we obtain

$$\lambda^{-2}\|y - (\bar{R}_j \, \cdot + \bar{c}_j)\|^2_{L^\infty(V)} \le C\vartheta(1 + \vartheta). \tag{5.41}$$

Proof. Apply Lemma 5.4 to obtain $U \in \mathcal{V}^{s\hat{m}}_{70k}$, $U_Q \in \mathcal{V}^{s\hat{m}}$ with $|U_Q \setminus W| = 0$, U_J and extensions $\bar{y}_i : S_i \cap U_J \to \mathbb{R}^2$ such that (5.13), (5.14) and (5.15) hold. Consider $Q = Q^\lambda_j(p)$, $p \in I^\lambda_j(\Omega^{3k})$, $j = 1, \dots, 4$, with $Q \cap U_J \neq \emptyset$. Moreover, let $\tilde{Q} = Q^{3\lambda}_j(p) \cap U_J$. As $6\lambda < \frac{k}{4}$ by $m \ll 1$, we find some $Q^k_i(q)$ for some $i = 1, \dots, 4$ with $\tilde{Q} \subset Q^{\frac{5}{8}k}_i(q) \subset S_i$ and therefore we can apply (5.15). Recall that $\hat{R} := \hat{R}_i|_{W^\circ \cap \tilde{Q}}$ is constant due to the construction in Lemma 5.4 (see below (5.33)). By Theorem 3.3 we find $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c \in \mathbb{R}^2$ such that

$$\begin{aligned} \|\bar{y}_i - \hat{R}\,(\mathbf{Id} + A)\,\cdot - \hat{R}\,c\|^2_{L^2(\tilde{Q})} &= \|\hat{R}^T\bar{y}_i - \cdot - (A\,\cdot + c)\|^2_{L^2(\tilde{Q})} \\ &\le C(|E(\hat{R}^T\bar{y}_i - \mathbf{id})|(\tilde{Q}))^2 \le Ck^2 G, \end{aligned} \tag{5.42}$$

where

$$G := C_m \min\Big\{\epsilon k, \gamma(W \cap Q^{2k}_i(q)) + \delta_4(W \cap Q^{2k}_i(q)) + \epsilon|\partial W \cap Q^{2k}_i(q)|_{\mathcal{H}}\Big\}.$$

The constant C is independent of $\tilde{Q}$ as there are (up to rescaling) only a finite number of different shapes of $\tilde{Q}$. (Also recall that each $Q \subset \tilde{Q}$ shares at least one face with the rest of $\tilde{Q}$.)

In the proof of Lemma 5.4 we have seen that all $\Gamma_t = \Gamma_t(\tilde{U}_i)$ with $\tilde{Q} \cap \Gamma_t \neq \emptyset$ satisfy (5.20) for $\bar{y}_i$ and $|\Gamma_t|_\infty \le \frac{k}{8}$ as well as $N^{\tau_l}(\partial R_t) \subset Q^k_i(q)$ (cf. (5.33)). Thus, by Lemma 3.8 for $V = Q^k_i(q)$ we get

$$\begin{aligned} \|\nabla\bar{y}_i - \hat{R}\|^p_{L^p(\tilde{Q})} &\le C\|\nabla y - \hat{R}\|^p_{L^p(\tilde{Q}\cap\hat{W})} + C\sum\nolimits_{\Gamma_t \in \mathcal{F}(Q^k_i(q))} |X_t|^2_\infty |A_t|^p \\ &\le CC_m\delta_p(Q^k_i(q) \cap \hat{W}) + CC_m(\epsilon s^{-1})^{\frac{p}{2}-1}\epsilon|\partial\hat{U}_i \cap Q^k_i(q)|_{\mathcal{H}} \end{aligned} \tag{5.43}$$

for $p = 2, 4$, where $\hat{W}, \hat{U}_i$ as defined in the previous proof and X_t, A_t as in (5.21). Recall that the factor s^{-1} appearing in the estimate is related to the fact that the least length of boundary components of $\hat{U}_i$ is s. Thus, recalling that $\hat{U}_i$ fulfills (5.23) we obtain by the definition of G

$$\|\nabla\bar{y}_i - \hat{R}\|^p_{L^p(\tilde{Q})} \le CC_m\delta_p(Q^k_i(q) \cap \hat{W}) + C(\epsilon s^{-1})^{\frac{p}{2}-1}G =: H_p. \tag{5.44}$$

We repeat the estimate (5.42) with the Poincaré inequality in SBV (see [2, Remark 3.50]) instead of Theorem 3.3 and obtain by (5.34) and Hölder's inequality

$$\begin{aligned} \|\bar{y}_i - \hat{R}\ \cdot - \tilde{c}\|^2_{L^2(\tilde{Q})} &\le C\|\nabla\bar{y}_i - \hat{R}\|^2_{L^1(\tilde{Q})} + C(|D^j(\bar{y}_i - \hat{R}\,\mathbf{id})|(\tilde{Q}))^2 \\ &\le C\lambda^{4(1-\frac{1}{p})}H_p^{\frac{2}{p}} + Ck^2G, \end{aligned}$$

for $\tilde{c} \in \mathbb{R}^2$ for $p = 2, 4$. This together with (5.42) and an argumentation similar to (3.16) (see also (2.11) in [22], where such an estimate is derived in the geometrically linear setting) yields $\lambda^4|A|^2 \le C\lambda^{4-4/p}H_p^{2/p} + Ck^2G$ and therefore by

(5.44)

$$\begin{aligned}\lambda^2|A|^2 &\le CH_2 + Cm^{-2}G \le CC_m\delta_2(Q_i^k(q)\cap\hat{W}) + Cm^{-2}G =: \hat{H}_2,\\ \lambda^2|A|^4 &\le CH_4 + C\lambda^{-2}m^{-4}G^2 \le CH_4 + C\lambda^{-1}m^{-5}C_m\epsilon G\\ &\le CC_m\delta_4(Q_i^k(q)\cap\hat{W}) + C\vartheta G =: \hat{H}_4.\end{aligned} \tag{5.45}$$

Observe that $\hat{H}_4 \le C(1+\vartheta)G$. By (5.11) there is a rotation $\bar{R}\in SO(2)$ such that

$$\begin{aligned}|\bar{R}-\hat{R}(\mathbf{Id}+A)|^2 &= \mathrm{dist}^2(\hat{R}(\mathbf{Id}+A), SO(2))\\ &\le 0 + C|\hat{R}(\mathbf{Id}+A)-\hat{R}|^4 = C|A|^4 \le C\lambda^{-2}\hat{H}_4,\end{aligned} \tag{5.46}$$

as $\bar{e}_{\hat{R}}(\hat{R}(\mathbf{Id}+A)) = 0$. Likewise, as $|A|\le C$ by $\|\nabla y\|_\infty \le C$ we get $|\bar{R}-\hat{R}(\mathbf{Id}+A)|^2 \le C|A|^2 \le C\lambda^{-2}\hat{H}_2$. Consequently, the Poincaré inequality, (5.42) and (5.45) yield

$$\|\bar{y}_i - (\bar{R}\ \cdot + \bar{c})\|^2_{L^2(\tilde{Q})} \le Ck^2G + C\lambda^4|A|^4 \le Ck^2G + Ck^2\min\nolimits_{p=2,4}\hat{H}_p \tag{5.47}$$

for some possibly different $\bar{c}\in\mathbb{R}^2$. Moreover, we get

$$\begin{aligned}\lambda^2|\hat{R}-\bar{R}|^4 &\le C\lambda^2|\bar{R}-\hat{R}(\mathbf{Id}+A)|^4 + C\lambda^2|A|^4\\ &\le C\lambda^2|\bar{R}-\hat{R}(\mathbf{Id}+A)|^2 + C\lambda^2|A|^4 \le C\hat{H}_4.\end{aligned} \tag{5.48}$$

and likewise

$$\lambda^2|\hat{R}-\bar{R}|^2 \le C\hat{H}_2. \tag{5.49}$$

For fixed $j=1,\dots,4$ we proceed in this way on each $Q_t = Q_j^\lambda(p)$, $p\in I_j^\lambda(\Omega^{3k})$, with $Q_t\cap U_J \neq \emptyset$ and for the corresponding $\tilde{Q}_t = Q_j^{3\lambda}(p)\cap U_J$ we obtain constants $\hat{R}_t, \bar{R}_t \in SO(2)$ and $\bar{c}_t\in\mathbb{R}^2$ as given in (5.47)-(5.49). Consequently, we find mappings $\bar{R}_j : U_J \to SO(2)$ and $\bar{c}_j : U_J\to\mathbb{R}^2$ being constant on each Q_t, where on each $Q_t\subset\tilde{Q}_t$ we choose $\bar{R}_j = \bar{R}_t$ and $\bar{c}_j = \bar{c}_t$. By (5.47) and the observation that every $Q_i^{2k}(q)$ is intersected only by $\sim m^{-2}$ squares $\tilde{Q}_t$ we obtain

$$\begin{aligned}\|y-(\bar{R}_j\ \cdot + \bar{c}_j)\|^2_{L^2(U)} &\le Ck^2\min_{p=2,4}(1+\vartheta_p)m^{-2}C_m m^{-2}(\gamma+\delta_p+\epsilon\|W\|_*)\\ &\le C\lambda^2\min\nolimits_{p=2,4}(1+\vartheta_p)m^2C_m^2(\gamma+\delta_p+\epsilon\|W\|_*)\end{aligned} \tag{5.50}$$

where $\vartheta_2 = 1$ and $\vartheta_4 = \vartheta$. Here we used that $\delta_4\le C\delta_2$. Likewise, applying (5.37), (5.45), (5.48), (5.49) as well as the triangle inequality we get

$$\begin{aligned}\|\nabla y - \bar{R}_j\|^p_{L^p(U)} &\le Cm^{-2}C_m\big(\delta_p + m^{-2}\vartheta_p(\gamma+\delta_4+\epsilon\|W\|_*)\big)\\ &\le CmC_m^2\big(\delta_p + \vartheta_p(\gamma+\delta_4+\epsilon\|W\|_*)\big)\end{aligned} \tag{5.51}$$

for $p = 2, 4$. We now consider $Q_1 := Q^\lambda_{j_1}(p_1)$, $Q_2 := Q^\lambda_{j_2}(p_2)$ with $Q_1 \cap Q_2 \neq \emptyset$ and $Q_1, Q_2 \cap U_J \neq \emptyset$. Moreover, let $\tilde{Q}_i = Q^{3\lambda}_{j_i}(p_i) \cap U_J$ be the corresponding enlarged sets. It is not hard to see that there is some $Q^\lambda(p)$, $p \in J(U_Q)$, with $Q^\lambda(p) \subset \tilde{Q}_1, \tilde{Q}_2$ and therefore by the definition of U_J, in particular (5.14), we derive $|\tilde{Q}_1 \cap \tilde{Q}_2 \cap U_Q| \geq cm\lambda^2$. Let $\bar{R}_{j_i} \in SO(2)$, $\bar{c}_{j_i} \in \mathbb{R}^2$, $i = 1, 2$, be the constants constructed above. We compute

$$\begin{aligned} \lambda^2 \|\bar{R}_{j_1} - \bar{R}_{j_2}\|^p_{L^\infty(Q_1 \cap Q_2)} &\leq Cm^{-1} \|\bar{R}_{j_1} - \bar{R}_{j_2}\|^p_{L^p(\tilde{Q}_1 \cap \tilde{Q}_2 \cap U_Q)} \\ &\leq Cm^{-1} \sum\nolimits_{j=1}^4 \|\nabla y - \bar{R}_j\|^p_{L^p(\tilde{Q}_1 \cap \tilde{Q}_2 \cap U_Q)} \end{aligned} \tag{5.52}$$

and summing over all squares we get by (5.51)

$$\|\bar{R}_{j_1} - \bar{R}_{j_2}\|^p_{L^p(U_J)} \leq CC^2_m \big(\delta_p + \vartheta_p(\gamma + \delta_4 + \epsilon \|W\|_*)\big) \tag{5.53}$$

for $1 \leq j_1, j_2 \leq 4$ and $p = 2, 4$. Here we used that each $Q^{3\lambda}_j(p) \cap U_J$ only appears in a finite number of addends. Note that $\frac{|\pi_1(Q_1 \cap Q_2)| + |\pi_2(Q_1 \cap Q_2)|}{\max_{i=1,2} |\pi_i(\tilde{Q}_1 \cap \tilde{Q}_2 \cap U_Q)|} \leq Cm^{-1/2}$ and $\frac{|Q_1 \cap Q_2|}{|\tilde{Q}_1 \cap \tilde{Q}_2 \cap U_Q|} \leq Cm^{-1}$. Consequently, arguing similarly as in (3.17) we find

$$\begin{aligned} &\lambda^2 \|(\bar{R}_{j_1} \ \cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2} \ \cdot + \bar{c}_{j_2})\|^2_{L^\infty(Q_1 \cap Q_2)} \\ &\qquad \leq C(m^{-\frac{1}{2}})^2 m^{-1} \|(\bar{R}_{j_1} \ \cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2} \ \cdot + \bar{c}_{j_2})\|^2_{L^2(\tilde{Q}_1 \cap \tilde{Q}_2 \cap U_Q)}. \end{aligned} \tag{5.54}$$

Replacing (5.51) by (5.50) in the above argument we then get

$$\begin{aligned} \|(\bar{R}_{j_1} \ \cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2} \ \cdot + \bar{c}_{j_2})\|^2_{L^2(U_J)} &\leq Cm^{-2} \sum\nolimits_{j=1}^4 \|y - (\bar{R}_j \ \cdot + \bar{c}_j)\|^2_{L^2(U_Q)} \\ &\leq CC^2_m \lambda^2 \min\nolimits_{p=2,4} (1 + \vartheta_p)(\gamma + \delta_p + \epsilon \|W\|_*). \end{aligned} \tag{5.55}$$

Similarly as in the proof of Lemma 5.2 (see the construction in (5.6)) we can define $V \in \mathcal{V}^{s\hat{m}^2}_{71k}$ with $|V \setminus U| = 0$, $V^\circ \subset \{x \in U \cap \Omega^{6k} : \mathrm{dist}_\infty(x, \partial U) \geq 2s\hat{m}m\}$, $\|V\|_* \leq (1 + C_u m)\|U\|_*$ and $|(W \setminus V) \cap \Omega^{6k}| \leq C_u k \|V\|_*$. By (5.13) this implies (5.38). We note that in this case for components $\Gamma_j = \partial X_j$ with $X_j \subset U_J$ it suffices to consider a corresponding rectangle $M(\Gamma_j)$ with $M(\Gamma_j) \subset U_J$. For later we observe that this construction yields

$$V \subset U_J, \qquad \big|\big(\Omega^{6k} \setminus \bigcup M(\Gamma_j)\big) \triangle V\big| = 0. \tag{5.56}$$

We now see that (5.39) follows directly from (5.50)-(5.55).

It remains to show (5.40) and (5.41). By (5.45), (5.48) and (5.52) we find $\|\bar{R}_{j_1} - \bar{R}_{j_2}\|^4_{L^\infty(Q_1 \cap Q_2)} \leq C\lambda^{-2}(1 + \vartheta)G + C\lambda^{-2}m^{-1}G$ for sets $Q_1, Q_2 \subset U_J$ as considered above. Recalling the definition of G we then get

$$\|\bar{R}_{j_1} - \bar{R}_{j_2}\|^4_{L^\infty(Q_1 \cap Q_2)} \leq C(1 + \vartheta)\lambda^{-2} m^{-1} C_m \epsilon k \leq Cs^{-1}(1 + \vartheta) C^2_m \epsilon \leq C(1 + \vartheta)\vartheta$$

Likewise, we derive $\lambda^{-2}\|(\bar{R}_{j_1}\,\cdot+\bar{c}_{j_1})-(\bar{R}_{j_2}\,\cdot+\bar{c}_{j_2})\|^2_{L^\infty(Q_1\cap Q_2)}\le C(1+\vartheta)\vartheta$ recalling the definition of G and taking (5.54), (5.47) (for $p=4$) and the triangle inequality into account. Similarly, by (5.47) for $p=2$ and the observation that $\delta_2(Q^k_i(q)\cap\hat{W})\le Ck^2$ as $\|\nabla y\|_\infty\le C$ we find using $\epsilon\le k$

$$\begin{aligned}\lambda^{-2}\|(\bar{R}_{j_1}\,\cdot+\bar{c}_{j_1})-(\bar{R}_{j_2}\,\cdot+\bar{c}_{j_2})\|^2_{L^\infty(Q_1\cap Q_2)}&\le C\lambda^{-4}m^{-2}k^2(G+\hat{H}_2)\\&\le C\lambda^{-2}m^{-4}(m^{-2}G+C_mk^2)\le C\lambda^{-2}C_m^2k^2\le CC_m^3.\end{aligned}$$

This finishes the proof of (5.40).

Finally, to see (5.41), we repeat the argument in (5.7): Let $x\in Q\cap V\subset\tilde{Q}$ for $Q=Q^\lambda_j(p)$, $\tilde{Q}=Q^{3\lambda}_j(p)\cap U_J$ as considered above and let $\bar{R}\,\cdot+\bar{c}$ be the corresponding rigid motion as given in (5.47). Since y is assumed to be harmonic in U° the mean value property of harmonic function for $r\le s\hat{m}m$ and Jensen's inequality yield

$$\begin{aligned}|y(x)-(\bar{R}\,x+\bar{c})|^2&\le\Big|\frac{1}{|B_r(x)|}\int_{B_r(x)}(y(t)-(\bar{R}\,t+\bar{c}))\,dt\Big|^2\\&\le C|B_r(x)|^{-1}(1+\vartheta)k^2G\le C(1+\vartheta)m^{-2}\hat{m}^{-2}s^{-2}k^2G\\&\le C(1+\vartheta)C_mm^{-4}\hat{m}^{-2}l\epsilon s^{-1}\lambda^2\le C(1+\vartheta)\vartheta\lambda^2.\end{aligned}$$

Here we used (5.47) and the fact that $B_r(x)\subset U^\circ\cap\tilde{Q}$ for all $x\in Q\cap V$. $\Box$

5.3 Local rigidity for an extended function

We now state a version of Lemma 5.6 for an extension of the function y.

Corollary 5.7 *Let be given the assumptions of Lemma 5.4, Lemma 5.6 and let $U\in\mathcal{V}^{s\hat{m}}_{70k}$, $U^H\in\mathcal{V}^\lambda_{35k}$ be the sets provided by Lemma 5.4, Remark 5.5, respectively. Moreover, assume that $\vartheta\le1$. Then the estimates (5.39)(iii),(iv) hold on U^H for functions $\bar{R}_j$, $\bar{c}_j$, $j=1,\dots,4$. Moreover, we find an extension $\hat{y}\in SBV^2(U^H,\mathbb{R}^2)$ with $\hat{y}=y$ on U and $\nabla\hat{y}\in SO(2)$ on $U^H\setminus W$ a.e. such that for every $Q=Q^\lambda_j(p)$, $p\in I^\lambda_j(\Omega^{3k})$, with $Q\cap U^H\ne\emptyset$ we have*

$$\begin{aligned}&(i)\quad\|\nabla\hat{y}-\bar{R}_j\|^p_{L^p(Q)}\le CC_m^2\,(\bar{G}(N)+\delta_p(N)),\ p=2,4\\&(ii)\quad\|\hat{y}-(\bar{R}_j\,\cdot+\bar{c}_j)\|^2_{L^2(Q)}\le C\lambda^2C_m^2\,\min\{\epsilon k,\bar{G}(N)\},\\&(iii)\quad\|\hat{y}-(\bar{R}_j\,\cdot+\bar{c}_j)\|^2_{L^1(\partial Q)}\le C\lambda^2C_m^2\,\min\{\epsilon k,\bar{G}(N)\},\end{aligned}\tag{5.57}$$

where $N=N(Q)=\{x\in W:\operatorname{dist}(x,Q)\le Ck\}$ and for shorthand $\bar{G}(N)=\gamma(N)+\delta_4(N)+\epsilon\mathcal{H}^1(N\cap\partial W)$. Furthermore, we have

$$\mathcal{H}^1(J_{\hat{y}})\le C_u(\|W\|_*+C\epsilon^{-1}(\gamma+\delta_4)).\tag{5.58}$$

Proof. Recall the definition of U in (5.32) and that U_J and U^H coincide up to a set of measure zero by Remark 5.5. In Lemma 5.4 we have defined sets $(\tilde{U}_j)_{j=1}^4$, $\tilde{U}_4^* \subset \ldots \subset \tilde{U}_1^*$ (see (5.25)) and corresponding extensions $\bar{y}_i|_{U_J \cap S_i}$. Moreover, in (5.33) have seen that all $\Gamma_t(\tilde{U}_i)$ with $\Gamma_t(\tilde{U}_i) \cap U_J^\circ \cap S_i \neq \emptyset$ satisfy (5.20) for $\bar{y}_i$ and $|\Gamma_t(\tilde{U}_i)|_\infty \leq \frac{k}{8}$. By Lemma 5.6 we get that (5.39)(iii),(iv) hold.

The goal is to provide one single extension $\hat{y} : U^H \to \mathbb{R}^2$ and to confirm (5.57). Define

$$\hat{S}_i := \bigcup\nolimits_{p \in I_i^k(\Omega^{3k})} \overline{Q_i^{\frac{9}{16}k}(p)} \subset S_i$$

and let $D_i = (\tilde{U}_i \cap U_J^\circ) \cup \bigcup_{\Gamma_t(\tilde{U}_i) \subset \hat{S}_i} X_t(\tilde{U}_i)$, where $X_t(\tilde{U}_i)$ is the component corresponding to $\Gamma_t(\tilde{U}_i)$. We now show that $U_J^\circ \subset \bigcup_{i=1}^4 D_i$. To see this, it suffices to prove

$$S_i \cap U_J^\circ \subset \bigcup\nolimits_{n=1}^4 D_n, \quad i = 1, \ldots, 4. \tag{5.59}$$

Fix i and assume that (5.59) has already be established for $j > i$. As $S_i \cap U_J^\circ \subset \Omega^{5k} \subset H(\tilde{U}_i) = \tilde{U}_i \cup \bigcup_{\Gamma_t(\tilde{U}_i)} X_t(\tilde{U}_i)$ by the definition of U_J, we find $(S_i \cap U_J^\circ) \setminus D_i \subset (S_i \cap U_J^\circ) \cap \bigcup_{\Gamma_t(\tilde{U}_i) \not\subset \hat{S}_i} X_t(\tilde{U}_i)$. To see (5.59) for i, it now suffices to show that each $\Gamma_t(\tilde{U}_i)$ with $\Gamma_t(\tilde{U}_i) \cap U_J^\circ \cap S_i \neq \emptyset$ satisfies $U_J^\circ \cap X_t(\tilde{U}_i) \subset \bigcup_{n=1}^4 D_n$. Since $|\Gamma_t(\tilde{U}_i)|_\infty \leq \frac{k}{8}$ for all such components, we derive $X_t(\tilde{U}_i) \subset \hat{S}_j$ for some $j = 1, \ldots, 4$. If $j < i$, by the construction of the sets $\tilde{U}_1^* \supset \ldots \supset \tilde{U}_4^*$ we find $(X_{t_s}(\tilde{U}_j))_s$ such that

$$X_t(\tilde{U}_i) = (\tilde{U}_j \cap X_t(\tilde{U}_i)) \cup \bigcup\nolimits_s X_{t_s}(\tilde{U}_j).$$

As $X_{t_s}(\tilde{U}_j) \subset \hat{S}_j$, this implies $X_t(\tilde{U}_i) \cap U_J^\circ \subset D_j$. The case $j = i$ is clear. If $j > i$, we obtain $X_t(\tilde{U}_i) \cap U_J^\circ \subset S_j \cap U_J^\circ \subset \bigcup_{n=1}^4 D_n$ by (5.59). This yields the claim.

Set $\bar{y} = \bar{y}_4$ on $D_4 \cap U_J$, $\bar{y} = \bar{y}_j$ on $(D_j \setminus D_{j+1}) \cap U_J$ for $j = 3, 2, 1$. It is not hard to see that $\bar{y}$ is defined on U^H (as $|U^H \setminus U_J^\circ| = 0$) and $\bar{y} = y$ on U. Moreover, by construction there is a set of components $(X_t)_t$ consisting of components of $(\hat{U}_i)_i$ such that

$$J_{\bar{y}} \subset \bigcup\nolimits_t \partial X_t \subset \bigcup\nolimits_{i=1}^4 \bigcup\nolimits_t \Gamma_t(\hat{U}_i).$$

By (5.21) we have $\bar{y}(x) = \bar{y}_{i_t}(x) = R_t\,(\mathbf{Id} + A_t)\,x + R_t\,c_t$ for $x \in X_t$, where $R_t \in SO(2)$, $A_t \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c_t \in \mathbb{R}^2$ and $1 \leq i_t \leq 4$ appropriately. Note that due the the definition of the extensions in (5.21) the components X_t are associated to the sets $(\hat{U}_i)_i$, not to $(\tilde{U}_i)_i$. By Remark 5.5(ii) this yields (5.58) for $\bar{y}$.

Consider $Q = Q_j^\lambda(p)$ with $Q \cap U_J \neq \emptyset$. Let $\tilde{Q} = Q_j^{3\lambda}(p) \cap U_J$ and observe $|\tilde{Q} \cap U_J| \sim \lambda^2$. Let $\mathcal{I} \subset \{1, \ldots, 4\}$ such that for each $\iota \in \mathcal{I}$ we can select some $Q_\iota^k(q_\iota)$ such that $\tilde{Q} \subset Q_\iota^{\frac{5}{8}k}(q_\iota)$. Note that $\#\mathcal{I} > 1$ is possible. It is not hard to see that for all X_t with $X_t \cap Q \neq \emptyset$ we get $i_t \in \mathcal{I}$. This follows from the construction of the sets $(D_i)_i$ and the fact that $\tilde{Q} \not\subset S_\iota$ implies $\tilde{Q} \cap \hat{S}_\iota = \emptyset$ as $\lambda \ll k$. Following the lines of (5.44), (5.47)-(5.49) and using $\hat{H}_4 \le CG$ we find $\bar{R}^\iota \in SO(2)$, $\bar{c}^\iota \in \mathbb{R}^2$ such that

$$\|\bar{y}_\iota - (\bar{R}^\iota \, \cdot + \bar{c}^\iota)\|^2_{L^2(\tilde{Q})} \le Ck^2 G, \qquad \|\nabla \bar{y}_\iota - \bar{R}^\iota\|^p_{L^p(\tilde{Q})} \le C\hat{H}_p \tag{5.60}$$

for $\iota \in \mathcal{I}$. Note that for a special choice of $\iota \in \mathcal{I}$ (for $\iota = i$ with i as considered in (5.42)ff.) we obtain the rigid motion $\bar{R}_j\, x + \bar{c}_j$ which we defined in Lemma 5.6. Then arguing as in (5.52) and (5.54), in particular employing the triangle inequality and using (5.60), we derive

$$\begin{aligned} &\|(\bar{R}_j \, \cdot + \bar{c}_j) - (\bar{R}^\iota \, \cdot + \bar{c}^\iota)\|^2_{L^2(\tilde{Q})} \le Cm^{-2}k^2 G, \\ &\|\bar{R}_j - \bar{R}^\iota\|^p_{L^p(\tilde{Q})} \le Cm^{-1}\hat{H}_p \end{aligned} \tag{5.61}$$

for $\iota \in \mathcal{I}$. Likewise we obtain by (5.20)

$$\int_{J_{\bar{y}} \cap \overline{Q}} |[\bar{y}]|^2 \, d\mathcal{H}^1 \le C \sum_{\iota \in \mathcal{I}} \int_{J_{\bar{y}_\iota} \cap \overline{Q}} |[\bar{y}_\iota]|^2 \, d\mathcal{H}^1 \le C \sum_{\iota \in \mathcal{I}} kC_m \epsilon |\partial \hat{U}_\iota \cap Q^k_\iota(q_\iota)|_{\mathcal{H}}. \tag{5.62}$$

Here we used that all X_t with $\overline{Q} \cap X_t \neq \emptyset$ satisfy $|\partial X_t|_\infty \le \frac{k}{8}$ and thus $X_t \subset Q^k_\iota(q)$. Now we obtain

$$\begin{aligned} \|\nabla \bar{y} - \bar{R}_j\|^p_{L^p(Q)} &\le \sum\nolimits_{\iota \in \mathcal{I}} \|\nabla \bar{y}_\iota - \bar{R}_j\|^p_{L^p(Q)} \\ &\le C \sum\nolimits_{\iota \in \mathcal{I}} \big(\|\nabla \bar{y}_\iota - \bar{R}^\iota\|^p_{L^p(Q)} + \|\bar{R}^\iota - \bar{R}_j\|^p_{L^p(Q)}\big) \end{aligned}$$

for $p = 2, 4$. Choosing the constant in the definition of N sufficiently large and recalling the definition of G and $\hat{H}_p$ (see (5.45)) we obtain by (5.60) and (5.61)

$$\|\nabla \bar{y} - \bar{R}_j\|^p_{L^p(Q)} \le CC_m^2(\gamma(N) + \delta_p(N) + \epsilon |\partial W \cap N|_{\mathcal{H}}).$$

Similarly, recalling $\lambda = mk$ we derive

$$\|\bar{y} - (\bar{R}_j \, \cdot + \bar{c}_j)\|^2_{L^2(Q)} \le C\lambda^2 C_m^2 \min\{(\gamma(N) + \delta_4(N) + \epsilon |\partial W \cap N|_{\mathcal{H}}), \epsilon k\}.$$

Consequently, (5.57)(i),(ii) hold for $\bar{y}$.

For later purposes, it is convenient to have an extension satisfying $\nabla \hat{y}(x) \in SO(2)$ for a.e. $x \in U^H \setminus W$. Arguing as in (5.46) for all components X_t we find $\tilde{R}_t \in SO(2)$ such that $|\tilde{R}_t - (R_t + R_t A_t)|^2 \le C|A_t|^4$. Therefore, by Poincaré's inequality we find for some possibly different $\tilde{c}_t \in \mathbb{R}^2$

$$\|\tilde{R}_t \, \cdot + \tilde{c}_t - (R_t\,(\mathbf{Id} + A_t) \, \cdot + R_t\, c_t)\|^2_{L^2(X_t)} \le C|\partial X_t|^2_\infty \, |X_t||A_t|^4 \tag{5.63}$$

for all X_t and likewise passing to the trace we get

$$\|\tilde{R}_t \, \cdot + \tilde{c}_t - (R_t\,(\mathbf{Id} + A_t) \, \cdot + R_t\, c_t)\|^2_{L^2(\partial X_t)} \le C|\partial X_t|^2_\infty \, |\partial X_t|_{\mathcal{H}}|A_t|^4.$$

In particular, note the the constants above do not depend on the shape of X_t as the involved functions are affine. We set $\hat{y} : U^H \to \mathbb{R}^2$ by $\hat{y}(x) = \tilde{R}_t\, x + \tilde{c}_t$ for $x \in X_t$ and $\hat{y} = y$ else. First, we see that (5.58) holds since $\mathcal{H}^1(J_{\bar{y}}) = \mathcal{H}^1(J_{\hat{y}})$. The definition together with (5.62) yields

$$\begin{aligned}\int_{J_{\bar{y}}\cap\overline{Q}} |[\hat{y}]|^2 \, d\mathcal{H}^1 &\le \int_{J_{\bar{y}}\cap\overline{Q}} |[\bar{y}]|^2 \, d\mathcal{H}^1 + C \sum\nolimits_{X_t\cap\overline{Q}\neq\emptyset} |\partial X_t|^2_\infty |\partial X_t|_{\mathcal{H}} |A_t|^4 \\ &\le \int_{J_{\bar{y}}\cap\overline{Q}} |[\bar{y}]|^2 \, d\mathcal{H}^1 + Ck \sum\nolimits_{X_t\cap\overline{Q}\neq\emptyset} |\partial X_t|^2_\infty |A_t|^4 \\ &\le CC_m k \sum\nolimits_{\iota=1}^4 \epsilon |\partial \hat{U}_\iota \cap Q^k_\iota(q_\iota)|_{\mathcal{H}}.\end{aligned}$$

In the second step we used $|\partial X_t|_{\mathcal{H}} \le Ck$ which follows from (5.23) and (5.16). In the last step we used Lemma 3.8 similarly as in the derivation of (5.43) and employed $s \ge \epsilon$. Using once more that $|J_{\bar{y}} \cap \overline{Q}|_{\mathcal{H}} \le \sum_{\iota=1}^4 |\partial \hat{U}_\iota \cap Q^k_\iota(q_\iota)|_{\mathcal{H}} \le Ck$, Hölder's inequality and (5.23) yield

$$\begin{aligned}\Big(\int_{J_{\bar{y}}\cap\overline{Q}} |[\hat{y}]| \, d\mathcal{H}^1\Big)^2 &\le |J_{\bar{y}} \cap \overline{Q}|_{\mathcal{H}} \cdot \int_{J_{\bar{y}}\cap\overline{Q}} |[\hat{y}]|^2 \, d\mathcal{H}^1 \\ &\le CC_m k^2 \sum\nolimits_{\iota=1}^4 \epsilon |\partial \hat{U}_\iota \cap Q^k_\iota(q_\iota)|_{\mathcal{H}}. \\ &\le CC_m^2 \lambda^2 \min\big\{\big(\gamma(N) + \delta_4(N) + \epsilon|\partial W \cap N|_{\mathcal{H}}\big), \epsilon k\big\}.\end{aligned} \tag{5.64}$$

Recalling $|\tilde{R}_t - (R_t + R_t A_t)|^2 \le C|A_t|^4$, $|A_t| \le C$ and again using (5.43), (5.23) we obtain

$$\|\nabla \bar{y} - \nabla \hat{y}\|^p_{L^p(Q)} \le C \sum_{X_t\cap Q\neq\emptyset} |\partial X_t|^2_\infty |A_t|^4 \le CC_m(\gamma(N) + \delta_4(N) + \epsilon|\partial W \cap N|_{\mathcal{H}})$$

for $p = 2, 4$, and analogously by (5.63) we get

$$\|\bar{y} - \hat{y}\|^2_{L^2(Q)} \le C \sum_{X_t\cap Q\neq\emptyset} |\partial X_t|^4_\infty |A_t|^4 \le CC_m^2 \lambda^2 \big(\gamma(N) + \delta_4(N) + \epsilon|\partial W \cap N|_{\mathcal{H}}\big),$$

where we employed $|\partial X_t|_\infty \le Ck = C\lambda m^{-1}$. Likewise we derive $\|\bar{y} - \hat{y}\|^2_{L^2(Q)} \le CC_m^2 \lambda^2 \epsilon k$. Together with the estimates for $\bar{y}$ this shows (5.57)(i),(ii). It remains to prove (5.57)(iii). By (5.57)(i) for $p = 4$, (5.11) and the fact that $\nabla \hat{y}(x) \in SO(2)$ for a.e. $x \in U^H \setminus W$ we find $\|\bar{e}_{\bar{R}_j}(\nabla \hat{y})\|^2_{L^2(Q)} \le CC_m^2(\gamma(N) + \delta_4(N) + \epsilon|N \cap \partial W|_{\mathcal{H}})$. This together with (5.64), $|Q| \le C\lambda^2$ and Hölder's inequality yields

$$(|E(\bar{R}_j^T \hat{y} - \mathbf{id})|(Q))^2 \le CC_m^2 \lambda^2 (\gamma(N) + \delta_4(N) + \epsilon|\partial W \cap N|_{\mathcal{H}}).$$

Then Theorem 3.4 and a rescaling argument show

$$\begin{aligned}\|\hat{y} - (\bar{R}_j \,\cdot + \bar{c}_j)\|^2_{L^1(\partial Q)} &\le C\lambda^{-2} \|\hat{y} - (\bar{R}_j \,\cdot + \bar{c}_j)\|^2_{L^1(Q)} + C(|E(\bar{R}_j^T \hat{y} - \mathbf{id})|(Q))^2 \\ &\le C\lambda^2 C_m^2 (\gamma(N) + \delta_4(N) + \epsilon|\partial W \cap N|_{\mathcal{H}}).\end{aligned}$$

In the last step we have used Hölder's inequality and (5.57)(ii). Similarly as before we also derive $\|\hat{y} - (\bar{R}_j \,\cdot + \bar{c}_j)\|^2_{L^1(\partial Q)} \le CC_m^2 \lambda^2 \epsilon k$. $\square$

6 Modification of the deformation

The goal of the section is to replace the deformation by an H^1-function on U_J. In particular, we modify the deformation in such a way that the least crack length is increased. Recall $\nu = sd = \lambda m$.

Lemma 6.1 *Let $k > s, \epsilon > 0$ such that $l := \frac{k}{s} = dm^{-2}$ for $m^{-1}, d \in \mathbb{N}$ with $m^{-1}, d \gg 1$. Then there is a constant $C > 0$ such that for all $W \in \mathcal{V}^{sm}_{(s,3k)}$ with $W \subset \Omega^{3k}$ and for all $y \in H^1(W)$ with $\|\nabla y\|_\infty \leq C$, γ as defined in* (5.2) *and δ_2, δ_4 as given in* (5.37) *we have the following:*
There are sets $U \in \mathcal{V}^{s\hat{m}^2}_{71k}$ and $U^H \in \mathcal{V}^{\nu}_{72k}$ with $U, U^H \subset \Omega^{6k}$, $|U \setminus W| = 0$, $|U^H \setminus H^\lambda(U)| = 0$, $|(W \setminus U) \cap \Omega^{6k}| + |U \setminus U^H| \leq C_u k \|U\|_*$ and

$$\|U\|_* \leq (1 + C_u m)\|W\|_* + C\epsilon^{-1}(\gamma + \delta_4) \tag{6.1}$$

as well as a function $\tilde{y} \in H^1(U^H)$ such that

$$\begin{aligned}
(i) &\quad \| \operatorname{dist}(\nabla \tilde{y}, SO(2))\|^2_{L^2(U^H)} \leq C \min_{p=2,4} (1 + \vartheta_p^3) C_m^2 (\gamma + \delta_p + \epsilon \|W\|_*),\\
(ii) &\quad \| \operatorname{dist}(\nabla \tilde{y}, SO(2))\|^2_{L^\infty(U^H)} \leq C\bar{\vartheta}(1 + \bar{\vartheta}),\\
(iii) &\quad \|\nabla y - \nabla \tilde{y}\|^2_{L^2(U)} \leq C C_m^2 (\gamma + \delta_2 + \epsilon \|W\|_*),\\
(iv) &\quad \|\tilde{y} - y\|^2_{L^2(U)} \leq C C_m^2 (1 + \vartheta)\lambda^2 (\gamma + \delta_4 + \epsilon \|W\|_*),
\end{aligned} \tag{6.2}$$

where $\bar{\vartheta} = \min\{\vartheta(1 + \vartheta), C_m^3\}$ and $\vartheta_2 = 1$, $\vartheta_4 = \vartheta$. Under the additional assumption that $\Delta y = 0$ in W° we get

$$\|\nabla y - \nabla \tilde{y}\|^4_{L^4(U)} \leq C C_m^2 \delta_4 + C C_m^2 \vartheta(1 + \vartheta)^2 (\gamma + \delta_4 + \epsilon \|W\|_*). \tag{6.3}$$

Proof. Apply Lemma 5.6 to obtain sets $V \in \mathcal{V}^{s\hat{m}^2}_{71k}$, $U_J \in \mathcal{V}^\lambda$ satisfying (5.38) and (5.39) for mappings $\bar{R}_j : U_J \to SO(2)$ and $\bar{c}_j : U_J \to \mathbb{R}^2$, $j = 1, \ldots, 4$. We first define $U = V$ and see that the estimate in (6.1). Moreover, we recall that $\Omega^{6k} \setminus U$ is the union of rectangular components (see (5.56)). For the components $\Gamma_1(H^\lambda(V)), \ldots, \Gamma_n(H^\lambda(V))$ we let $N(\Gamma_j) \in \mathcal{U}^\nu$ denote the smallest rectangle with $N(\Gamma_j) \supset X_j$, where as before X_j denotes the component corresponding to $\Gamma_j(H^\lambda(V))$.

As $\frac{\nu}{\lambda} = m$, we find $|\pi_i \partial N(\Gamma_j)| \leq |\pi_i \Gamma_j(H^\lambda(V))| + C_u m |\Gamma_j(H^\lambda(V))|_\infty$ for $i = 1, 2$. Arguing similarly as in the construction of (5.6) we have that $N(\Gamma_{j_1}) \setminus N(\Gamma_{j_2})$ is connected for $1 \leq j_1, j_2 \leq n$. We apply Lemma 4.2(i) to obtain pairwise disjoint, connected sets $(X'_j)_{j=1}^n$ such that $\bigcup_{j=1}^n \overline{N(\Gamma_j)} = \bigcup_{j=1}^n \overline{X'_j}$ and define

$$U^H = \Omega^{6k} \setminus \bigcup\nolimits_{j=1}^n X'_j.$$

It is not hard to see that $U^H \in \mathcal{V}^\nu_{72k}$. Moreover, we find $U^H \subset H^\lambda(U)$ up to a set of negligible measure and recalling (5.56) we obtain $(U^H)^\circ \subset U_J$. For later we also observe that

$$\|U^H\|_* \leq (1 + C_u m)\|H^\lambda(U)\|_*. \tag{6.4}$$

This also implies $|U \setminus U^H| \le C_u k \|U\|_*$.

Let $T_j = \bigcup_{p \in I_j^\lambda(\Omega^{3k})} Q_j^{\frac{3}{4}\lambda}(p)$ and define a partition of unity $(\eta_j)_{j=1}^4$ with $\eta_j \in C^\infty(U_J, [0,1])$, $\mathrm{supp}(\eta_j) \subset T_j$ and $\|\nabla \eta_j\|_\infty \le \frac{C}{\lambda}$. Define $\tilde{y} : U_J \to \mathbb{R}^2$ by

$$\tilde{y}(x) = \sum\nolimits_{j=1}^4 \eta_j(x)(\bar{R}_j\, x + \bar{c}_j)$$

and observe that $\tilde{y} \in H^1(U_J)$ as the functions $\bar{R}_j, \bar{c}_j$ are constant on each $Q_j^\lambda(p)$, $p \in I_j^\lambda(U_J)$. The derivative reads as

$$\nabla \tilde{y}(x) = \sum\nolimits_{j=1}^4 \big(\eta_j(x)\bar{R}_j + (\bar{R}_j\, x + \bar{c}_j) \otimes \nabla \eta_j(x)\big). \tag{6.5}$$

Since $\sum_{j=1}^4 \nabla \eta_j = 0$ we find

$$\nabla \tilde{y}(x) = \bar{R}_1 + \sum\nolimits_{j=2}^4 \big(\eta_j(x)(\bar{R}_j - \bar{R}_1) + (\bar{R}_j\, x + \bar{c}_j - (\bar{R}_1\, x + \bar{c}_1)) \otimes \nabla \eta_j(x)\big).$$

First, we compute by (5.40)

$$\begin{aligned}
\|\nabla \tilde{y} - \bar{R}_1\|^4_{L^4(U_J)} &\le C \sum_{j=2}^4 \Big(\|\bar{R}_j - \bar{R}_1\|^4_{L^4(U_J)} + \frac{1}{\lambda^4} \|\bar{R}_j\ \cdot + \bar{c}_j - (\bar{R}_1\ \cdot + \bar{c}_1)\|^4_{L^4(U_J)} \Big) \\
&\le C \sum_{j=2}^4 \Big(\|\bar{R}_j - \bar{R}_1\|^4_{L^4(U_J)} + \frac{\bar{\vartheta}}{\lambda^2} \|\bar{R}_j\ \cdot + \bar{c}_j - (\bar{R}_1\ \cdot + \bar{c}_1)\|^2_{L^2(U_J)} \Big),
\end{aligned}$$

where $\bar{\vartheta} = \min\{\vartheta(1+\vartheta), C_m^3\}$. By (5.11) we find $\bar{e}_{\bar{R}_1}(\bar{R}_j) \le C|\bar{R}_j - \bar{R}_1|^2$ and thus

$$\begin{aligned}
\|\bar{e}_{\bar{R}_1}(\nabla \tilde{y})\|^2_{L^2(U_J)} &\le C \sum_{j=2}^4 \Big(\|\bar{e}_{\bar{R}_1}(\bar{R}_j)\|^2_{L^2(U_J)} + \frac{1}{\lambda^2} \|\bar{R}_j\ \cdot + \bar{c}_j - (\bar{R}_1\ \cdot + \bar{c}_1)\|^2_{L^2(U_J)} \Big) \\
&\le C \sum_{j=2}^4 \Big(\|\bar{R}_j - \bar{R}_1\|^4_{L^4(U_J)} + \frac{1}{\lambda^2} \|\bar{R}_j\ \cdot + \bar{c}_j - (\bar{R}_1\ \cdot + \bar{c}_1)\|^2_{L^2(U_J)} \Big).
\end{aligned}$$

Again using (5.11) and (5.39)(iii),(iv) we derive

$$\|\operatorname{dist}(\nabla \tilde{y}, SO(2))\|^2_{L^2(U_J)} \le C(1+\vartheta^3)C_m^2(\gamma + \delta_4 + \epsilon\|W\|_*).$$

Similarly, we get

$$\|\nabla \tilde{y} - \bar{R}_1\|^2_{L^2(U_J)} \le C \sum_{j=2}^4 \Big(\|\bar{R}_j - \bar{R}_1\|^2_{L^2(U_J)} + \frac{1}{\lambda^2} \|\bar{R}_j\ \cdot + \bar{c}_j - (\bar{R}_1\ \cdot + \bar{c}_1)\|^2_{L^2(U_J)} \Big)$$

and thus we find by (5.39)(iii),(iv)

$$\|\operatorname{dist}(\nabla \tilde{y}, SO(2))\|^2_{L^2(U_J)} \le C C_m^2(\gamma + \delta_2 + \epsilon\|W\|_*),$$

where we used that $\delta_4 \le C\delta_2$. This gives (6.2)(i) as $(U^H)^\circ \subset U_J$. Likewise, we may replace the L^2, L^4-norms in the above estimates by the L^∞-norm. Consequently, by (5.40) we obtain $\|\nabla\tilde{y} - \bar{R}_1\|^4_{L^\infty(U_J)} \le C\bar{\vartheta}(1+\bar{\vartheta})$ and $\|\bar{e}_{\bar{R}_1}(\nabla\tilde{y})\|^2_{L^\infty(U_J)} \le C\bar{\vartheta}$ which then implies $\|\operatorname{dist}(\nabla\tilde{y}, SO(2))\|^2_{L^\infty(U_J)} \le C\bar{\vartheta}(1+\bar{\vartheta})$. It remains to show (6.2)(iii),(iv) and (6.3). By (5.39)(i) and the fact that $U = V$ we obtain

$$\|\tilde{y} - y\|^2_{L^2(U)} \le \sum\nolimits_{j=1}^4 C\|y - (\bar{R}_j\ \cdot + \bar{c}_j)\|^2_{L^2(U)} \le CC_m^2\lambda^2(1+\vartheta)(\gamma + \delta_4 + \epsilon\|W\|_*).$$

By (6.5) and the fact that $\sum_{j=1}^4 \eta_j = 1$, $\sum_{j=1}^4 \nabla\eta_j = 0$ we derive

$$\nabla y(x) - \nabla\tilde{y}(x) = \sum\nolimits_{j=1}^4 \big(\eta_j(x)(\nabla y(x) - \bar{R}_j) + (y(x) - (\bar{R}_j\, x + \bar{c}_j)) \otimes \nabla\eta_j(x)\big).$$

Therefore, by (5.39)(i)(ii) for $p = 2$ we get

$$\begin{aligned} \|\nabla\tilde{y} - \nabla y\|^2_{L^2(U)} &\le C\sum\nolimits_{j=1}^4 \Big(\|\nabla y - \bar{R}_j\|^2_{L^2(U)} + \frac{1}{\lambda^2}\|y - (\bar{R}_j\ \cdot + \bar{c}_j)\|^2_{L^2(U)}\Big) \\ &\le CC_m^2(\gamma + \delta_2 + \epsilon\|W\|_*), \end{aligned}$$

where we used that $\delta_4 \le C\delta_2$. Finally, in the case that $\Delta y = 0$ in W° we obtain by (5.39)(i)(ii) for $p = 4$ and (5.41)

$$\begin{aligned} \|\nabla\tilde{y} - \nabla y\|^4_{L^4(U)} &\le C\sum\nolimits_{j=1}^4 \Big(\|\nabla y - \bar{R}_j\|^4_{L^4(U)} + \frac{\vartheta(1+\vartheta)}{\lambda^2}\|y - (\bar{R}_j\ \cdot + \bar{c}_j)\|^2_{L^2(U)}\Big) \\ &\le CC_m^2\delta_4 + CC_m^2\vartheta(1+\vartheta)^2(\gamma + \delta_4 + \epsilon\|W\|_*). \end{aligned}$$

□

7 SBD-rigidity up to small sets

In this section we prove a slightly weaker version of the rigidity estimate given in Theorem 2.1 and postpone the proof of the general version to the next section. Recall definition (2.1).

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $M > 0$ and $0 < \eta, \rho, h_* \ll 1$. Let $q \in \mathbb{N}$ sufficiently large. Then there are constants $C_1 = C_1(\Omega, M, \eta)$, $C_2 = C_2(\Omega, M, \eta, \rho, h_*, q)$ and a universal constant $c > 0$ such that the following holds for $\varepsilon > 0$ small enough:*
For each $y \in SBV_M(\Omega)$ with $\mathcal{H}^1(J_y) \le M$ and $\int_\Omega \operatorname{dist}^2(\nabla y, SO(2)) \le M\varepsilon$, there is a set $\Omega_y \in \mathcal{V}^{\hat{s}}_{c\rho^{q-1}}$, $\hat{s} > 0$, with $\Omega_y \subset \Omega$, $|\Omega \setminus \Omega_y| \le C_1\rho$, a modification $\tilde{y} \in H^1(\Omega_y) \cap SBV_{cM}(\Omega_y)$ with $\|y - \tilde{y}\|^2_{L^2(\Omega_y)} + \|\nabla y - \nabla\tilde{y}\|^2_{L^2(\Omega_y)} \le C_1\varepsilon\rho$, a partition

$(P_i)_i$ of Ω_y and for each P_i a corresponding rigid motion $R_i\,x + c_i$, $R_i \in SO(2)$ and $c_i \in \mathbb{R}^2$, such that the function $u : \Omega \to \mathbb{R}^2$ defined by

$$u(x) := \begin{cases} \tilde{y}(x) - (R_i\,x + c_i) & \text{for } x \in P_i \\ 0 & \text{else} \end{cases} \tag{7.1}$$

satisfies

$$\begin{aligned} &(i)\ \ \|\Omega_y\|_* \le (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho, \quad &&(ii)\ \ \|u\|^2_{L^2(\Omega_y)} \le C_2\varepsilon, \\ &(iii)\ \ \sum\nolimits_i \|e(R_i^T \nabla u)\|^2_{L^2(P_i)} \le C_2\varepsilon, &&(iv)\ \ \|\nabla u\|^2_{L^2(\Omega_y)} \le C_2\varepsilon^{1-\eta}. \end{aligned} \tag{7.2}$$

We divide the proof into three steps. We begin with a version where the least crack length is almost of macroscopic size. Afterwards, we assume that the jump set consists only of a finite number of cracks of arbitrary size. Finally, we treat the general case applying a suitable approximation argument.

In what follows, constants indicated by C_1 only depend on M, η, Ω. Generic constants C may additionally depend on h_*. All constants do not depend on ρ and q unless stated otherwise. As we will eventually let $h_* \sim \rho$ in Section 8, it is essential that the constant in (7.2)(i) does not depend on h_*.

7.1 Step 1: Deformations with least crack length

We first treat the case that the least crack length is almost of macroscopic size.

Theorem 7.2 *Theorem 7.1 holds under the additional assumption that there is an $\tilde{\Omega}_y \subset \Omega^s$, $\tilde{\Omega}_y \in \mathcal{V}^s_{\rho^{q-1}}$ for some $s \ge \rho^{q-1}\varepsilon^{\frac{\eta}{8}}$ such that $y \in H^1(\tilde{\Omega}_y)$, $\|\tilde{\Omega}_y\|_* \le (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho$ and $|\Omega \setminus \tilde{\Omega}_y| \le C_1\rho$ for a constant $C_1 = C_1(\Omega, M, \eta)$.*

Proof. Let $y \in H^1(\tilde{\Omega}_y)$ be given. Let ρ and define $\varrho = \rho^q$ for some $q \in \mathbb{N}$, $q \ge 2$ large enough to be specified in the proof of Theorem 2.1 (see Section 8). Assume without restriction $\rho^{-1} \in \mathbb{N}$ large. We apply Theorem 3.11 and consider the harmonic part w of y satisfying

$$\begin{aligned} &\|\nabla y - \nabla w\|^2_{L^2(\tilde{\Omega}_y)} \le C\|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(\tilde{\Omega}_y)} \le C\varepsilon, \\ &\|\nabla y - \nabla w\|^4_{L^4(\tilde{\Omega}_y)} \le C\|\operatorname{dist}(\nabla y, SO(2))\|^4_{L^4(\tilde{\Omega}_y)} \le C\varepsilon. \end{aligned} \tag{7.3}$$

In the last inequality we used $\|\nabla y\|_\infty \le M$. Let $k = \varrho\rho^{-1} = \rho^{q-1}$. Apply Lemma 5.2 on $\tilde{\Omega}_y \cap \Omega^k$ for the function w and $\epsilon = \hat{c}\rho^{-1}\varepsilon$, $m = \rho$, where $\hat{c} > 0$ is sufficiently large. (Possibly passing to a smaller s we can assume that $k\varepsilon^{\frac{\eta}{8}} \le s \ll k = \rho^{q-1}$.) We find a set $W \subset \Omega^{3k}$, $W \in \mathcal{V}^{sm}_{(s,3k)}$ such that

$$\|W\|_* \le (1 + C_1\rho)\|\tilde{\Omega}_y\|_* + C\epsilon^{-1}\varepsilon \le (1 + C_1\rho)\|\tilde{\Omega}_y\|_* + \rho \tag{7.4}$$

by (5.3) and $|(\tilde{\Omega}_y \setminus W) \cap \Omega^{3k}| \le C_1 k \le C_1\rho$. (Here and in the following we choose the constant $\hat{c} = \hat{c}(h_*)$ always larger then the constant C.) Moreover, there are mappings $\hat{R}_i : W^\circ \to SO(2)$, $i = 1, \ldots, 4$, which are constant on the connected components of $Q^k_i(p) \cap W^\circ$, $p \in I^k_i(\Omega)$, such that by (5.4)(i) for $i = 1, \ldots, 4$

$$\|\nabla y - \hat{R}_i\|^2_{L^2(W)} \le C\varepsilon + C\|\nabla w - \hat{R}_i\|^2_{L^2(W)} \le Cl^4\varepsilon \le C\varepsilon^{1-\eta}, \tag{7.5}$$

where $l = ks^{-1} \le C\varepsilon^{-\frac{\eta}{8}}$. Moreover, as $\vartheta = l^9 C^2_m s^{-1}\varepsilon \le C(\rho)s^{-10}\varepsilon \le C(\rho)\varepsilon^{1-\frac{5}{4}\eta} \le 1$ for η, ε small enough (recall (5.1)) we also get

$$\|\nabla y - \hat{R}_i\|^4_{L^4(W)} \le C\varepsilon + C\|\nabla w - \hat{R}_i\|^4_{L^4(W)} \le C\varepsilon \tag{7.6}$$

by (5.4)(ii). Now we apply Corollary 5.7 on $W \subset \Omega^{3k}$ for $k = \rho^{q-1}$, $\lambda = 3\varrho$, $m = 3\rho$ and $\epsilon = \hat{c}\rho^{-1}\varepsilon$. We obtain a set $\Omega_y \in \mathcal{V}^{s\hat{m}}_{9k}$ with $\Omega_y \subset \Omega^{5k}$, $|\Omega_y \setminus \tilde{\Omega}_y| = 0$ such that by (5.13), (7.4) and (7.6) we find

$$\|\Omega_y\|_* \le (1 + C_1\rho)\|W\|_* + C\epsilon^{-1}\varepsilon \le (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho \tag{7.7}$$

and $|(\tilde{\Omega}_y \setminus \Omega_y) \cap \Omega^{5k}| \le C_1 k$. This together with the assumption $|\Omega \setminus \tilde{\Omega}_y| \le C_1\rho$ and the fact that $|\Omega \setminus \Omega^{5k}| \le C(\Omega)k$ yields $|\Omega \setminus \Omega_y| \le C_1\rho$. Moreover, there is a set $\Omega^H_y \in \mathcal{V}^\lambda$ with $H^\lambda(\Omega_y) \subset \Omega^H_y$ and mappings $\bar{R}_j : \Omega^H_y \to SO(2)$, $\bar{c}_j : \Omega^H_y \to \mathbb{R}^2$ being constant on $Q^{3\varrho}_j(p)$, $p \in I^{3\varrho}_j(\Omega^{3k})$, and an extension $\hat{y} \in SBV_M(\Omega^H_y, \mathbb{R}^2)$ such that by (5.57)(ii) we derive

$$\|\hat{y} - (\bar{R}_j\ \cdot + \bar{c}_j)\|^2_{L^2(\Omega^H_y)} \le C\varrho^2\rho^{-2}C^4_\rho(\varepsilon + \epsilon\|W\|_*) \le C\rho^{2q-3}C^4_\rho\varepsilon \tag{7.8}$$

where $C_\rho = C_{\frac{m}{3}}$ is the constant defined in (5.1). Here we used that each $x \in W$ is contained in at most $\sim \rho^{-2}$ different neighborhoods $N(Q)$ considered in Corollary 5.7. Moreover, the constant $\hat{c}$ was absorbed in C. Similarly, recalling $\vartheta \le 1$ we get by (5.39)(iii),(iv), (5.57)(i) and (7.5), (7.6)

$$\begin{aligned}
&\|\nabla\hat{y} - \bar{R}_j\|^2_{L^2(\Omega^H_y)} + \|\bar{R}_{j_1} - \bar{R}_{j_2}\|^2_{L^2(\Omega^H_y)} \le C\rho^{-3}C^2_\rho\varepsilon^{1-\eta},\\
&\|\nabla\hat{y} - \bar{R}_j\|^4_{L^4(\Omega^H_y)} + \|\bar{R}_{j_1} - \bar{R}_{j_2}\|^4_{L^4(\Omega^H_y)} \le C\rho^{-3}C^2_\rho\varepsilon,\\
&\|(\bar{R}_{j_1}\ \cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2}\ \cdot + \bar{c}_{j_2})\|^2_{L^2(\Omega^H_y)} \le C\rho^{2q-3}C^2_\rho\varepsilon,
\end{aligned} \tag{7.9}$$

for $j = 1, \ldots, 4$ and $1 \le j_1, j_2 \le 4$.

Denote the connected components of $(\Omega^H_y)^\circ \in \mathcal{U}^{3\varrho}$ by $(P^H_i)_i$ and define $P_i = P^H_i \cap \Omega_y$. Let $J_i \subset I^\varrho(\Omega)$ be the index set such that $Q^\varrho(p) \subset P^H_i$ for all $p \in J_i$. We now estimate the variation of the rigid motions defined on these squares. Let $Q_1 = Q^\varrho(p_1)$, $Q_2 = Q^\varrho(p_2)$ for $p_1, p_2 \in J_i$ such that $\overline{Q_1} \cap \overline{Q_2} \neq \emptyset$. Let $R_t = \bar{R}_4|_{Q_t}$ and $c_t = \bar{c}_4|_{Q_t}$ for $t = 1, 2$. Then we find some $j = 1, \ldots, 4$ such that $\bar{R}_j$ is constant on $Q_1 \cup Q_2$ and thus $\varrho^2|R_1 - R_2|^p \le C\sum_{t=1,2}\|\bar{R}_j - R_t\|^p_{L^p(Q_1\cup Q_2)}$ for $p = 2, 4$. Using the arguments in (3.16) and (3.17) we get

$$\begin{aligned}
\varrho^4|R_1 - R_2|^2 + &\|(R_1 - R_2)\ \cdot + c_1 - c_2\|^2_{L^2(Q_1\cup Q_2)}\\
&\le C\sum_{t=1,2}\|(\bar{R}_j\ \cdot + \bar{c}_j) - (R_t\ \cdot + c_t)\|^2_{L^2(Q_1\cup Q_2)}.
\end{aligned} \tag{7.10}$$

Consequently, considering chains as in (3.14) and (3.18), respectively, following the arguments in the proof of Lemma 3.14 and (3.18) and recalling Remark 3.15(ii), we obtain $R_i \in SO(2)$, $c_i \in \mathbb{R}^2$ such that

$$\begin{aligned}\|\hat{y} - (R_i \,\cdot + c_i)\|^2_{L^2(P_i^H)} &\le C\|\hat{y} - (\bar{R}_4 \,\cdot + \bar{c}_4)\|^2_{L^2(P_i^H)} \\ &\quad + C\varrho^{-8} \sum\nolimits_{1 \le j_1, j_2 \le 4} \|(\bar{R}_{j_1} \,\cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2} \,\cdot + \bar{c}_{j_2})\|^2_{L^2(P_i^H)}, \\ \|\nabla \hat{y} - R_i\|^p_{L^p(P_i^H)} &\le C\|\nabla \hat{y} - \bar{R}_4\|^p_{L^p(P_i^H)} \\ &\quad + C\varrho^{-2p} \sum\nolimits_{1 \le j_1, j_2 \le 4} \|\bar{R}_{j_1} - \bar{R}_{j_2}\|^p_{L^p(P_i^H)}, \quad p = 2, 4.\end{aligned}$$

In the first estimate we used Hölder's inequality (cf. (3.18)). Summing over all connected components, (7.8) and (7.9) implies

$$\begin{aligned}&\sum\nolimits_i \|\hat{y} - (R_i \,\cdot + c_i)\|^2_{L^2(P_i^H)} \le C(\rho, q)\varepsilon, \\ &\sum\nolimits_i \|\nabla \hat{y} - R_i\|^4_{L^4(P_i^H)} \le C(\rho, q)\varepsilon, \ \sum\nolimits_j \|\nabla \hat{y} - R_i\|^2_{L^2(P_i^H)} \le C(\rho, q)\varepsilon^{1-\eta}\end{aligned} \tag{7.11}$$

for $C(\rho, q)$ large enough. Defining u as in (7.1) (for $\tilde{y} = y$) and taking also (7.7) into account, we immediately get (7.2)(i)(ii),(iv). Finally, (7.2)(iii) is a consequence of the linearization formula (5.12) and (7.11). □

7.2 Step 2: Deformations with a finite number of cracks

We now prove a version where the crack set consists of a finite number of components. We first assume that each crack is at least of atomistic size. The strategy will be to establish an estimate of the form (7.5) and (7.6) by iterative modification of y according to Lemma 6.1.

First, we introduce some notation and derive preliminary estimates. Let $\rho > 0$, set $\varrho = \rho^q$ and assume without restriction $\rho^{-1} \in \mathbb{N}$ large. As before we assume $\|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(\Omega)} \le C\varepsilon$. Choose $t^{-1} \in \mathbb{N}$ such that $t \le \rho$ and set $t_j = t^{j+1}$. By Remark 3.9(i) we can assume that $T := t^{z+18} \le C_t^{-2} t^{18}$ for $z \in \mathbb{N}$ sufficiently large (recall (5.1) for the definition of C_t). Moreover, set $T_j = T^{j+1}$. Let $\tilde{\Omega}_y \subset \Omega^s$ for some $s > 0$ be given. Let

$$B_j = \left(\|\tilde{\Omega}_y\|_* + C_*\rho\right) \cdot \sum\nolimits_{i=0}^{j-1} t^i \cdot \Pi_{i=0}^{j-1}(1 + C_* t^{i+1}) \tag{7.12}$$

and $B = \lim_{j\to\infty} B_j$ for a constant $C_* = C_*(M, \eta, \Omega) \ge 1$ to be specified below. Furthermore, let $P = \hat{c}^2(1 + \rho^{-1}B)$ for $\hat{c} = \hat{c}(h_*)$ sufficiently large. Set $s_0 = \kappa\varepsilon$ for κ sufficiently large, let $\epsilon_0 = \hat{c}^2\rho^{-1}\varepsilon$ and subsequently define $\epsilon_{j+1} = PT_j^{-1}\epsilon_j$. We set $r = \frac{1}{18}$, $\omega = \frac{\eta}{36}$ for notational convenience and for $j \ge 0$ we define

$$d_j = \Big\lfloor \min\Big\{ \Big(\frac{s_j}{\epsilon_j}\Big)^r, \varepsilon^{-\omega} \Big\} \Big\rfloor, \tag{7.13}$$

where $s_j = s_0 \Pi_{i=0}^{j-1} d_i$. In accordance with Sections 5, 6 we also define

$$l_j = d_j t_j^{-2}, \qquad \lambda_j = s_j d_j t_j^{-1}, \qquad k_j = s_j l_j. \tag{7.14}$$

As noted before, d_j describes the increase of the minimal distance of different cracks and PT_j^{-1} will be the factor of energy increase. Below we will show that indeed $d_j \gg 1$ for all $0 \le j \le J^*$, where

$$J^* = \lceil \log_{1+r}(\log_T \varepsilon^\omega)) + \tfrac{1}{\omega} \rceil.$$

One of the main reasons why the iterative application of Lemma 6.1 works is the fact that d_j increases much faster than PT_j^{-1}. We define the quotient $q_j := \frac{d_j}{PT_j^{-1}}$ and observe $q_0 = \frac{d_0 T_0}{P} = TP^{-1}(s_0 \epsilon_0^{-1})^r$ for ε sufficiently small. Recalling (7.12) and the definition $s_0 = \kappa\varepsilon$, $\epsilon_0 = \hat{c}^2\rho^{-1}\varepsilon$ we can first choose $T = T(\rho, h_*)$ so small and then $\kappa = \kappa(T, \rho, h_*, \bar{z})$ so large that

$$q_0 T^{1/r} \ge T^{-\bar{z}} \ge T^{-1} \ge \hat{c}^4 P^2 > 1 \tag{7.15}$$

for $\bar{z} \in \mathbb{N}$ to be specified below. For the third inequality we used the fact that $P \le C$ for some $C = C(C_*, \rho, h_*, M)$ independent of T. We find

$$q_j = T^{-1/r}(q_0 T^{1/r})^{(1+r)^j} \tag{7.16}$$

for $j \le \hat{J}$, where $\hat{J} \in \mathbb{N}$ is the largest index such that $\frac{s_j}{\epsilon_j} \le \varepsilon^{-\frac{\eta}{2}}$ for all $j \le \hat{J}$. Indeed, we first note that the formula is trivial for $j = 0$. Assume (7.16) holds for $j \le \hat{J} - 1$, then we compute

$$q_{j+1} = \frac{T_{j+1}}{P}\Big(\frac{s_{j+1}}{\epsilon_{j+1}}\Big)^r = \frac{T_{j+1}}{P}\Big(\frac{s_j d_j}{PT_j^{-1}\epsilon_j}\Big)^r = \frac{q_j^r T_{j+1}}{P}\Big(\frac{s_j}{\epsilon_j}\Big)^r = \frac{q_j^r d_j T_j T}{P} = Tq_j^{1+r}$$

which gives (7.16) for $j+1$, as desired. In particular, taking (7.15) into account, (7.16) implies $q_j > 1$ and thus $d_j = q_j PT_j^{-1} \gg 1$ for all $j \le \hat{J}$. For $\hat{J} < j \le J^*$ we get $d_j = \varepsilon^{-\omega}$. In fact, using (7.15) and $\epsilon_0 \le \hat{c}^2 t^{-1}\varepsilon$ we observe for C sufficiently large

$$\begin{aligned} \epsilon_j &= \epsilon_0 \Pi_{i=0}^{j-1}(PT_i^{-1}) \le \hat{c}^{-2}\epsilon_0 \Pi_{i=0}^{j-1}(T^{-(i+1)}T^{-\frac{1}{2}}) \le \hat{c}^{-2}\epsilon_0 T^{-\frac{1}{2}(j+1)^2} \\ &\le \varepsilon T^{-C-[\log_{1+r}(\log_T \varepsilon^\omega)]^2} = \varepsilon o\big(T^{-\log_{T^{-1}}\varepsilon^{-\omega}}\big) = \varepsilon \cdot o(\varepsilon^{-\omega}) \end{aligned} \tag{7.17}$$

for $\varepsilon \to 0$ for all $1 \le j \le J^*$. Consequently, if $\frac{s_j}{\epsilon_j} \ge \varepsilon^{-\frac{\omega}{r}} = \varepsilon^{-\frac{\eta}{2}}$, then $d_j = \varepsilon^{-\omega}$, $PT_j^{-1} = o(\varepsilon^{-\omega})$ (see (7.17)) and thus $\frac{s_{j+1}}{\epsilon_{j+1}} = \frac{d_j s_j}{PT_j^{-1}\epsilon_j} \ge \varepsilon^{-\frac{\omega}{r}}$. This then implies $d_j = \varepsilon^{-\omega}$ for all $\hat{J} < j \le J^*$.

We introduce $\vartheta_j = s_j^{-1}\epsilon_j l_j^9 C_{t_j}^2$ (recall definition (5.1) and $l_j = d_j t_j^{-2}$) and close the preparations by showing that

$$\vartheta_j \le \frac{\epsilon_0}{\hat{c}^2 \epsilon_{j+1}} T_j \quad \text{for} \quad 0 \le j \le J^*. \tag{7.18}$$

This particularly implies $\vartheta_j \le 1$ for all j as $\epsilon_j \ge \epsilon_0$ for all j. By (7.13)-(7.16) we obtain

$$s_j \ge \epsilon_j \varepsilon^{-\frac{\eta}{2}} \quad \text{or} \quad s_j = \epsilon_j d_j^{1/r} \ge \epsilon_j q_j^{1/r} \ge \epsilon_j T^{-\frac{\bar{z}}{r}(1+r)^j} \ge \epsilon_j T^{-9(j+1)^2}. \tag{7.19}$$

for all $0 \le j \le J^*$. The last step holds for $\bar{z} \in \mathbb{N}$ sufficiently large as $\lim_{j\to\infty} \frac{1}{r}(1+r)^j(9(j+1)^2)^{-1} = \infty$. Similarly as in (7.17) we see that $T^{-9(j+1)^2} = o(\varepsilon^{-\omega})$ for $j \le J^*$ as $\varepsilon \to 0$. Since $\varepsilon^{-\omega} = o(\varepsilon^{-\frac{\eta}{2}})$, we find $s_j \ge \epsilon_j T^{-9(j+1)^2}$ for all $0 \le j \le J^*$. Therefore, we derive by (7.13), (7.15), the first line of (7.17) and $r = \frac{1}{18}$

$$\vartheta_j \epsilon_{j+1} = s_j^{-1}\epsilon_j d_j^9 t_j^{-18} C_{t_j}^2 \; P T_j^{-1}\epsilon_j \le s_j^{-\frac{1}{2}} \epsilon_j^{\frac{3}{2}} \hat{c}^{-2} T_j^{-3} \le \hat{c}^{-2}\epsilon_0 T^{4(j+1)^2} T_j^{-3} \le \hat{c}^{-2}\epsilon_0 T_j$$

for all $0 \le j \le J^*$, as desired. In the second step we used $C_{t_j}^2 t_j^{-18} \le T_j^{-1}$ and $P \le T_j^{-1}$. Recall the definition of κ and k_0 above (see (7.14) and (7.15)).

Theorem 7.3 *Theorem 7.1 holds under the additional assumption that there is an $\tilde{\Omega}_y \subset \Omega^s$, $\tilde{\Omega}_y \in \mathcal{V}_{k_0}^s$ for some $s \ge \kappa\varepsilon$, such that $y \in H^1(\tilde{\Omega}_y)$, $\|\tilde{\Omega}_y\|_* \le (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho$ and $|\Omega \setminus \tilde{\Omega}_y| \le C_1\rho$ for a constant $C_1 = C_1(\Omega, M, \eta)$.*

Proof. Let $y \in H^1(\tilde{\Omega}_y)$ be given. If $s \ge \varepsilon^{\frac{\eta}{8}}$ we can apply Theorem 7.2, so it suffices to consider $s \le \varepsilon^{\frac{\eta}{8}}$. Recall $s_0 = \kappa\varepsilon$ for some $\kappa = \kappa(T, \rho, h_*, \bar{z}) \gg 1$ and assume $s \ge s_0$. The strategy is to apply Lemma 6.1 iteratively. Set $W_0 = W_{-1}^H = W_0^H = \tilde{\Omega}_y \in \mathcal{V}_{k_0}^s$ and $y_0 = y$. Recall $\epsilon_0 = \hat{c}^2\rho^{-1}\varepsilon$ and define

$$\gamma_0 := \|\operatorname{dist}(\nabla y_0, SO(2))\|^2_{L^2(\tilde{\Omega}_y)} \le C\frac{\rho\epsilon_0}{\hat{c}^2}, \; \alpha_0 := \|\operatorname{dist}(\nabla y_0, SO(2))\|^4_{L^4(\tilde{\Omega}_y)} \le C\frac{\rho\epsilon_0}{\hat{c}^2}.$$

In the last inequality we used $\|\nabla y\|_\infty \le M$. Recall (7.14). Set $\hat{s}_j = s_j \hat{t}_j^2$ for $j \ge 0$ and $\hat{s}_{-1} = s$, where $\hat{t}_j = C_2(t_j, h_*)$ (see (5.1)). Assume $W_j \in \mathcal{V}_{k_j}^{\hat{s}_{j-1}}$, $W_j^H \in \mathcal{V}_{k_j}^{s_j}$ are given with $W_j, W_j^H \subset \Omega^{6k_{j-1}}$, $|W_j \setminus W_{j-1}^H| = 0$ and $|\tilde{\Omega}_y \setminus W_j| \le C_1 \sum_{i=0}^{j-1} k_i$, where we set $k_{-1} = s$. Recall that $|W_j \setminus W_j^H| \le C_1 k_{j-1}$ and $|W_j^H \setminus H^{\lambda_{j-1}}(W_j)| = 0$, where $\lambda_{-1} = 0$. Set $\beta_j = \|H^{\lambda_{j-1}}(W_j)\|_*$ and $\beta_j^d = \|W_j\|_* - \|H^{\lambda_{j-1}}(W_j)\|_*$. Moreover, suppose there is a function $y_j \in H^1(W_j^H)$ with

$$\gamma_j := \|\operatorname{dist}(\nabla y_j, SO(2))\|^2_{L^2(W_j^H)}, \quad \alpha_j := \|\operatorname{dist}(\nabla y_j, SO(2))\|^4_{L^4(W_j^H)}$$

such that for $j \ge 1$

$$\begin{aligned}
&(i) && \beta_j + \beta_j^d \le (1 + C_1 t_{j-1})\beta_{j-1} + C\epsilon_{j-1}^{-1}\gamma_{j-1} \le B_j,\\
&(ii) && \gamma_j \le C T_{j-1}^{-1} t_{j-1}(\gamma_{j-1} + \epsilon_{j-1}\beta_{j-1}) \le \hat{c}^{-1} t_{j-1}\rho\epsilon_j,\\
&(iii) && \alpha_j \le C\vartheta_{j-1}\gamma_j \le C\varepsilon T_{j-1},\\
&(iv) && \|\operatorname{dist}(\nabla y_j, SO(2))\|^2_{L^\infty(W_j^H)} \le C\vartheta_{j-1},\\
&(v) && \|\nabla y_j - \nabla y_{j-1}\|^4_{L^4(W_j)} \le C\varepsilon T_{j-1},\\
&(vi) && \|\nabla y_j - \nabla y_{j-1}\|^2_{L^2(W_j)} \le C T_{j-1}^{-1}(l_{j-1}^4\gamma_{j-1} + \epsilon_{j-1}\beta_{j-1}) \le C l_{j-1}^4 \epsilon_j.
\end{aligned} \tag{7.20}$$

Setting $\vartheta_{-1} = 1$ and $t_{-1} = 1$, we note that, provided $\hat{c}$ is sufficiently large, in the case $j = 0$ (iii),(iv) are clearly satisfied for $y_0 = y$ and (i),(ii) hold neglecting the second terms. We now construct y_{j+1}, $W_{j+1} \in \mathcal{V}^{\hat{s}_j}_{k_{j+1}}$ with $W_{j+1} \subset \Omega^{6k_j}$, $|W_{j+1} \setminus W^H_j| = 0$ and $|\tilde{\Omega}_y \setminus W_{j+1}| \le C_1 \sum_{i=0}^j k_i$ as well as $W^H_{j+1} \in \mathcal{V}^{s_{j+1}}_{k_{j+1}}$.

First we apply Theorem 3.11 and let $w_j \in H^1(W^H_j)$ be the harmonic part of y_j such that similarly as in (7.3)

$$\|\nabla y_j - \nabla w_j\|^2_{L^2(W^H_j)} \le C\gamma_j, \quad \|\nabla y_j - \nabla w_j\|^4_{L^4(W^H_j)} \le C\alpha_j \tag{7.21}$$

and so in particular $\|\operatorname{dist}(\nabla w_j, SO(2))\|^2_{L^2(W^H_j)} \le C\gamma_j$. Recall $W^H_j \in \mathcal{V}^{s_j}_{k_j}$, $W_j \subset \Omega^{6k_{j-1}}$ and note $\Omega^{k_j} \subset \Omega^{6k_{j-1}}$. Then apply Lemma 5.2 with $s = s_j$, $k = k_j = s_j l_j$, $m = t_j = t^{j+1}$, $\epsilon = \epsilon_j$, $U = W^H_j \cap \Omega^{k_j}$, $y = w_j$ and obtain a set $\tilde{W}^H_j \in \mathcal{V}^{s_j t_j}_{(s_j, 3k_j)}$ such that

$$\delta_4 := \sum\nolimits_{i=1}^4 \|\nabla w_j - \hat{R}_i\|^4_{L^4(\tilde{W}^H_j)} \le C\vartheta_j\gamma_j,\ \delta_2 := \sum\nolimits_{i=1}^4 \|\nabla w_j - \hat{R}_i\|^2_{L^2(\tilde{W}^H_j)} \le Cl_j^4\gamma_j$$

for mappings $\hat{R}_i : (\tilde{W}^H_j)^\circ \to SO(2)$, $i = 1, \ldots, 4$, which are constant on the connected components of $Q^k_i(p) \cap (\tilde{W}^H_j)^\circ$, $p \in I^k_i(\Omega^k)$. We now use Lemma 6.1 with $m = t_j$, $s = s_j$, $\epsilon = \epsilon_j$, $d = d_j$, $W = \tilde{W}^H_j$, $y = w_j$ and show (7.20) for $j+1$. First, we obtain $W_{j+1} \in \mathcal{V}^{\hat{s}_j}_{71k_j} \subset \mathcal{V}^{\hat{s}_j}_{k_{j+1}}$, with $W_{j+1} \subset \Omega^{6k_j}$, $|W_{j+1} \setminus W^H_j| = 0$, $|(W^H_j \setminus W_{j+1}) \cap \Omega^{6k_j}| \le Ck_j\|W_{j+1}\|_*$ and $W^H_{j+1} \in \mathcal{V}^{s_{j+1}}_{72k_j} \subset \mathcal{V}^{s_{j+1}}_{k_{j+1}}$ with $|W^H_{j+1} \setminus H^{\lambda_j}(W_{j+1})| = 0$ and $|W_{j+1} \setminus W^H_{j+1}| \le C_1 k_j$. Recall $\|W^H_j\|_* \le (1 + C_1 t_j)\beta_j$ by (6.4). Thus, we have

$$\|W_{j+1}\|_* \le (1 + C_1 t_j)\|W^H_j\|_* + C\epsilon_j^{-1}(\gamma_j + \vartheta_j\gamma_j) \le (1 + C_1 t_j)\beta_j + C\epsilon_j^{-1}\gamma_j \tag{7.22}$$

by (5.3), (6.1) and the fact that $\vartheta_j \le 1$ (see (7.18)). Moreover, we get a function $y_{j+1} \in H^1(W^H_{j+1})$ with (see (6.2), (6.3))

$$\begin{aligned}
(i) &\quad \|\operatorname{dist}(\nabla y_{j+1}, SO(2))\|^2_{L^2(W^H_{j+1})} \le CC^2_{t_j}(\gamma_j + \epsilon_j\beta_j),\\
(ii) &\quad \|\nabla w_j - \nabla y_{j+1}\|^2_{L^2(W_{j+1})} \le CC^2_{t_j}(\gamma_j + l_j^4\gamma_j + \epsilon_j\beta_j),\\
(iii) &\quad \|\nabla w_j - \nabla y_{j+1}\|^4_{L^4(W_{j+1})} \le CC^2_{t_j}\vartheta_j(\gamma_j + \epsilon_j\beta_j),\\
(iv) &\quad \|\operatorname{dist}(\nabla y_{j+1}, SO(2))\|^2_{L^\infty(W^H_{j+1})} \le C\vartheta_j,
\end{aligned} \tag{7.23}$$

where we again used that $\vartheta_j \le 1$. The first inequality in (7.20)(ii) follows directly noting that $T_j^{-1} t_j \ge C^2_{t_j}$ and for the second inequality we use (7.20)(i),(ii) for iteration step j as well as (7.12) to see

$$CT_j^{-1}(\gamma_j + \epsilon_j\beta_j) \le CT_j^{-1}\rho\epsilon_j(1 + \rho^{-1}B_j) \le \rho\hat{c}^{-1}PT_j^{-1}\epsilon_j = \hat{c}^{-1}\rho\epsilon_{j+1}, \tag{7.24}$$

where we choose $\hat{c}$ sufficiently large. Likewise, (7.20)(i) follows by (7.22), the fact that $\|W_{j+1}\|_* = \beta_{j+1} + \beta^d_{j+1}$ and

$$\begin{aligned}\beta_{j+1} + \beta^d_{j+1} &\le (1 + C_1 t_j)B_j + \rho t_{j-1} \\ &\le \big(\|\tilde{\Omega}_y\|_* + C_*\rho\big) \cdot \textstyle\sum_{i=0}^{j-1} t^i \cdot \pi_{t=0}^{j}(1 + C_* t^{i+1}) + \rho t^j \\ &\le \big(\|\tilde{\Omega}_y\|_* + C_*\rho\big) \cdot \textstyle\sum_{i=0}^{j} t^i \cdot \pi_{t=0}^{j}(1 + C_* t^{i+1}) = B_{j+1}.\end{aligned}$$

Here we have again chosen $\hat{c}$ and C_* large enough (with respect to C and C_1, respectively). This also implies $|(W_j \setminus W_{j+1}) \cap \Omega^{6k_j}| \le Ck_j$ by (7.20)(i) and thus $|(\tilde{\Omega}_y \setminus W_{j+1})| \le C\sum_{i=0}^{j} k_i + |\Omega \setminus \Omega^{6k_j}| \le C\sum_{i=0}^{j} k_i$.

Estimate (7.20)(iv) follows from (7.23)(iv). The first inequality in (7.20)(iii) is a consequence of (7.20)(iv), the second inequality is implied by the fact that $\varepsilon = \hat{c}^{-2}\rho\epsilon_0$, (7.20)(ii) and (7.18). Moreover, (7.20)(v) follows from (7.20)(iii), (7.21), (7.23)(iii) and the fact that $\vartheta_j C^2_{t_j}(\gamma_j + \epsilon_j\beta_j) \le \vartheta_j \rho\epsilon_{j+1} \le C\varepsilon T_j$ by (7.18) and (7.24). Similarly, (7.20)(vi) follows from (7.23)(ii), (7.21) and (7.24).

We now choose $j^* \in \mathbb{N}$ such that

$$\varepsilon^{3\omega} \ge s_{j^*} \ge \varepsilon^{4\omega}, \qquad \epsilon_{j^*} \le C\varepsilon^{1-\omega} T^2_{j^*} \tag{7.25}$$

holds for ε sufficiently small. The first inequality is possible by (7.13) and we obtain $j^* \le J^* = \lceil \log_{1+r}(\log_T \varepsilon^\omega)) + \frac{1}{\omega}\rceil$. Indeed, by (7.19) and the fact that $\bar{z} \ge 1$ we get $s_j \ge \varepsilon^{-\frac{\omega}{r}}\epsilon_j = \varepsilon^{-\frac{\eta}{2}}\epsilon_j$ for $j > \lceil \log_{1+r}(\log_T \varepsilon^\omega))\rceil$ and therefore $\hat{J} \le \lceil \log_{1+r}(\log_T \varepsilon^\omega))\rceil$. The second inequality can be derived arguing as in (7.17). Similarly, proceeding as in (7.17) we have $t^{-2}_{j^*} = o(\varepsilon^{-\omega})$ for $\varepsilon \to 0$ and thus $k_{j^*} = s_{j^*} d_{j^*} t^{-2}_{j^*} = o(\varepsilon^\omega)$. This implies $\Omega^{6k_{j^*}} \supset \Omega^\varrho$ for ε small enough. We let

$$y_* = y_{j^*}, \quad W^H_* = W^H_{j^*} \cap \Omega^\varrho, \quad W_* = \bigcap\nolimits_{i=0}^{j^*} W_i \cap \Omega^\varrho.$$

It is not hard to see that $|\tilde{\Omega}_y \setminus W_*| \le C_1 \sum_{i=0}^{j^*} k_i \le C\varrho$. As $\hat{s}_j = s_j \hat{t}^2_j$ is increasing in j (note that $d_j \ge \hat{t}^{-2}_j$ for all j, see e.g. (7.19)), we find $W_* \in \mathcal{V}^{\hat{s}_0}$.

The strategy is now to establish an estimate of the form (7.5) and (7.6). Observe that $s_{j^*} \ge \varepsilon^{\frac{\eta}{8}}$, i.e. for the function $y_* \in H^1(W^H_*)$ we may proceed as in Theorem 7.2 (replacing s by s_{j^*}). Similarly as in (7.3), we apply Theorem 3.11 and let w_* be the harmonic part of y_* with

$$\|\nabla w_* - \nabla y_*\|^2_{L^2(W^H_*)} \le C\varepsilon^{1-\frac{\eta}{2}}, \quad \|\nabla w_* - \nabla y_*\|^4_{L^4(W^H_*)} \le C\varepsilon T^{j^*}. \tag{7.26}$$

by (7.20), (7.25) and $\omega \le \frac{\eta}{2}$. Apply Lemma 5.2 on $W^H_* \subset \Omega^\varrho$ for the function w_* and $k = \rho^{q-1} = \varrho\rho^{-1}$, $s = \varepsilon^{4\omega}$, $\epsilon = \hat{c}\rho^{-1}\varepsilon^{1-\frac{\eta}{2}}$, $m = \rho$. (Without restriction we can assume $s^{-1} \in \mathbb{N}$.) We find a set $W^H \subset \Omega^{3k}$, $W^H \in \mathcal{V}^{s_{j^*}m}_{3k}$ such that

$$\|W^H\|_* \le (1 + C_1\rho)\|W^H_*\|_* + C\hat{c}^{-1}\rho\varepsilon^{\frac{\eta}{2}-1}\varepsilon^{1-\frac{\eta}{2}} \le \|W^H_*\|_* + C_1\rho \tag{7.27}$$

by (5.3) as well as $|W^H_* \setminus W^H| \le |(W^H_* \setminus W^H) \cap \Omega^{3k}| + C_1 k \le C_1 k \le C_1 \rho$. Moreover, there are mappings $\hat{R}_i : (W^H)^\circ \to SO(2)$, $i = 1, \ldots, 4$, which are constant on the connected components of $Q^k_i(p) \cap (W^H)^\circ$, $p \in I^k_i(\Omega)$, such that by (5.4)(i) and (7.26)

$$\|\nabla y_* - \hat{R}_i\|^4_{L^4(W^H)} \le C\|\nabla w_* - \hat{R}_i\|^4_{L^4(W^H)} + C\varepsilon T^{j^*} \le C\vartheta\varepsilon^{1-\frac{\eta}{2}} + C\varepsilon \le C\varepsilon,$$

where similarly as before equation (7.6) we compute (recall (7.25) and $\omega = \frac{\eta}{36}$) $\vartheta \le C(\rho, q)s^{-10}\epsilon \le C(\rho, q)\varepsilon^{-40\omega}\varepsilon^{1-\omega} = C(\rho, q)\varepsilon^{1-\frac{41}{36}\eta} \le \varepsilon^{\frac{\eta}{2}}$ for ε, η small enough. Likewise, we derive

$$\|\nabla y_* - \hat{R}_i\|^2_{L^2(W^H)} \le C\|\nabla w_* - \hat{R}_i\|^2_{L^2(W^H)} + C\varepsilon^{1-\frac{\eta}{2}} \le C(1 + l^4)\varepsilon^{1-\frac{\eta}{2}} \le C\varepsilon^{1-\eta}$$

as $l = \frac{k}{s} \le C\varepsilon^{-4\omega} \le \varepsilon^{-\frac{\eta}{8}}$.

We now will construct a set $W \in \mathcal{V}^{\hat{s}_0}_{143k}$ which is contained in $W^H \cap W_* \cap \Omega^{3k} \in \mathcal{V}^{\hat{s}_0}$, where the two sets coincide up to a set of measure smaller than $C_1\rho$. (Similarly as before the difference of the sets is related to the definition of the boundary components.) Before we give the exact definition of W and establish an estimate of the form (7.4), we first observe $|\tilde{\Omega}_y \setminus W| \le C_1\rho$ arguing as before and derive estimates similar to (7.5) and (7.6).

We iteratively apply (7.20)(v) and derive for $i = 1, \ldots, 4$

$$\|\nabla y - \hat{R}_i\|^4_{L^4(W)} \le C\Big(\sum\nolimits_{\iota=1}^{j^*} (\varepsilon T_{\iota-1})^{\frac{1}{4}}\Big)^4 + C\|\nabla y_* - \hat{R}_i\|^4_{L^4(W)} \le C\varepsilon. \qquad (7.28)$$

Likewise, observe that by (7.13), (7.14) and (7.25) we have $l^4_{j-1}\epsilon_j \le l^4_j\epsilon_j = d^4_j t^{-8(j+1)}\epsilon_j \le \varepsilon^{-4\omega}\varepsilon^{1-\omega}T_j \le \varepsilon^{1-\eta}T_j$. We derive by (7.20)(vi)

$$\|\nabla y - \hat{R}_i\|^2_{L^2(W)} \le C\varepsilon^{1-\eta}\Big(\sum\nolimits_{\iota=1}^{j^*} T_\iota^{\frac{1}{2}}\Big)^2 + C\|\nabla y_* - \hat{R}_i\|^2_{L^2(W)} \le C\varepsilon^{1-\eta}$$

for $i = 1, \ldots, 4$.

It remains to give the exact definition of $W \in \mathcal{V}^{\hat{s}_0}_{143k}$ and to establish $\|W\|_* \le (1 + Ch_*)\mathcal{H}^1(J_y) + C\rho$. Recall $W_0 = \tilde{\Omega}_y$ and define $W_{j^*+1} := W^H$ for notational convenience. We now define W inductively.

Let $Y_0 = Y'_0 = Y''_0 = W_0$. Assume $Y_j \in \mathcal{V}^{\hat{s}_0}$ and $Y'_j \in \mathcal{V}^{\hat{s}_0}_{k_j}$, $Y''_j \in \mathcal{V}^{\hat{s}_0}$ are given with $|Y'_j \setminus Y_j| + |Y'_j \triangle Y''_j| = 0$, $|Y_j \setminus Y'_j| \le C_1 k_{j-1}$ and

$$\max\{\|Y'_j\|_*, \|Y''_j\|_*\} \le \|Y_j\|_* \le \|W_j\|_* + \sum\nolimits_{i=1}^{j-1} \beta^d_i,$$

where Y''_j has the property that all components not intersecting $\partial H^{\lambda_{j-1}}(W_j)$ coincide with components of Y'_j and the set $\big(X_t(H^{\lambda_{j-1}}(W_j))\big)_t$ of components of $H^{\lambda_{j-1}}(W_j)$ is a subset of the components of Y''_j. Moreover, suppose that $|Y'_j \setminus \bigcap_{i=0}^j W_i| = 0$ and $|\bigcap_{i=0}^j W_i \setminus Y'_j| \le \sum_{i=0}^{j-1} k_i$.

We now pass to step $j+1$. Let $X_1(W_{j+1}), \ldots, X_{n_{j+1}}(W_{j+1})$ be the components of W_{j+1} and define

$$Y_{j+1} = \big(Y_j'' \setminus \bigcup\nolimits_{t=1}^{n_{j+1}} X_t(W_{j+1})\big) \cup \bigcup\nolimits_{t=1}^{n_{j+1}} \partial X_t(W_{j+1}) \in \mathcal{V}^{\hat{s}_0}.$$

First observe that Y_{j+1} satisfies $|Y_{j+1} \setminus \bigcap_{i=0}^{j+1} W_i| = 0$ and $|\bigcap_{i=0}^{j+1} W_i \setminus Y_{j+1}| \le \sum_{i=0}^{j-1} k_i$. As $|W_{j+1} \setminus W_j^H| = 0$, we obtain $\bigcup_{t=1}^{n_{j+1}} \overline{X_t(W_{j+1})} \supset \bigcup_t \overline{X_t(W_j^H)}$ and then by the fact that $|W_j^H \setminus H^{\lambda_{j-1}}(W_j)| = 0$ we get $\bigcup_{t=1}^{n_{j+1}} \overline{X_t(W_{j+1})} \supset \bigcup_t \overline{X_t(H^{\lambda_{j-1}}(W_j))}$. As by hypothesis the components of $H^{\lambda_{j-1}}(W_j)$ are also components of Y_j'', we derive recalling $\beta_i^d = \|W_i\|_* - \|H^{\lambda_{j-1}}(W_j)\|_*$ and $\beta_0^d = 0$

$$\|Y_{j+1}\|_* \le \|Y_j''\|_* + \|W_{j+1}\|_* - \|H^{\lambda_{j-1}}(W_j)\|_* = \|W_{j+1}\|_* + \sum\nolimits_{i=1}^{j} \beta_i^d.$$

Observe that possibly $Y_{j+1} \notin \mathcal{V}^{\hat{s}_0}_{\mathrm{con}}$. However, by Lemma 4.2(ii) we find a set $Y'_{j+1} \in \mathcal{V}^{\hat{s}_0}$ with $|Y_{j+1} \setminus Y'_{j+1}| \le C_1 k_j$ and $\|Y'_{j+1}\|_* \le \|Y_{j+1}\|_*$. Here we essentially used the rectangular shape of the boundary components given by (5.56) and (5.6), respectively. Then it is elementary to see that $Y'_{j+1} \in \mathcal{V}^{\hat{s}_0}_{143k_j} \subset \mathcal{V}^{\hat{s}_0}_{k_{j+1}}$ and $|\bigcap_{i=0}^{j+1} W_i \setminus Y'_{j+1}| \le \sum_{i=0}^{j} k_i$. Moreover, if $j+1 \le j^*$, we let $Y''_{j+1} = (Y'_{j+1} \cap H^{\lambda_j}(W_{j+1})) \cup \partial H^{\lambda_j}(W_{j+1})$ and observe that Y''_{j+1} has the desired properties. In fact, $\|Y''_{j+1}\|_* \le \|Y_{j+1}\|_*$ follows as before. Components not intersecting $\partial H^{\lambda_j}(W_{j+1})$ are clearly components of Y'_{j+1}. Finally, by definition components of $H^{\lambda_j}(W_{j+1})$ are also components of Y''_{j+1}.

We finally define $W = Y'_{j^*+1} \cap \Omega^{3k} \in \mathcal{V}^{\hat{s}_0}_{143k}$. By (7.12) and (7.20)(i),(ii) we have

$$\beta_i^d \le \beta_{i-1} - \beta_i + C_1 t^i \beta_{i-1} + C\epsilon_{i-1}^{-1}\gamma_{i-1} \le \beta_{i-1} - \beta_i + C_1 t^i B + \rho t^{i-1}$$

for $i = 1, \ldots, j^*$. Recalling $\beta_0 = \|\tilde{\Omega}_y\|_*$, $\|W_*^H\|_* \le (1+C_1 t_{j^*})\beta_{j^*}$ and using (7.12), (7.27) as well as $t \le \rho$ we conclude

$$\begin{aligned}
\|W\|_* \le \|Y'_{j^*+1}\|_* &\le \|W^H\|_* + \sum\nolimits_{i=1}^{j^*} (\beta_{i-1} - \beta_i + C_1 t^i B + \rho t^{i-1}) \\
&\le \|W^H\|_* - \beta_{j^*} + \beta_0 + C_1\rho B + C_1\rho \le C_1\rho + \|\tilde{\Omega}_y\|_* + C_1\rho B + C_1\rho \\
&\le (1 + C_1\rho)\|\tilde{\Omega}_y\|_* + C_1\rho \le (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho,
\end{aligned}$$

as derided.

We now proceed as in the proof of Theorem 7.2 after equation (7.6) with the only difference that we take $\hat{s}_0$ instead of $s \sim \varepsilon^{\frac{\eta}{8}}$ in the application of Corollary 5.7. However, this does not change the analysis. This leads to a set $\Omega_y \in \mathcal{V}^{\hat{s}_0 \hat{m}}_{ck}$ with $\Omega_y \subset \Omega^{5k}$ and $|\Omega \setminus \Omega_y| \le C_1\rho$ for $k = \rho^{q-1}, m = 3\rho$ for which (7.2) can be established. $\square$

We now additionally treat the subatomistic regime by dropping the assumption $s \ge \kappa\varepsilon$.

Theorem 7.4 *Theorem 7.1 holds under the additional assumption that there is an $\tilde{\Omega}_y \subset \Omega^s$, $\tilde{\Omega}_y \in \mathcal{V}^s_\varepsilon$ for some $0 < s \ll \varepsilon$ such that $y \in H^1(\tilde{\Omega}_y)$, $\|\tilde{\Omega}_y\|_* \le (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho$ and $|\Omega \setminus \tilde{\Omega}_y| \le C_1\rho$ for a constant $C_1 = C_1(\Omega, M, \eta)$.*

Proof. Let again $\rho^{-1} \in \mathbb{N}$, $s_0 = \kappa\varepsilon$ and recall $\|\operatorname{dist}(\nabla y, SO(2))\|^2_{L^2(\Omega)} \le C\varepsilon$. As $\kappa \gg 1$ was chosen in dependence of T and $T = T(\rho, h_*)$ (see (7.15)), we can suppose $\kappa = \kappa(\rho, h_*)$. Applying Lemma 5.3 for s, $k = \rho^{-2}\kappa\varepsilon$, $m = \rho$ and $\epsilon = \rho^{-2}\kappa\varepsilon$, $U = \tilde{\Omega}_y \cap \Omega^k$ there is a set $W \subset \Omega^{3k}$ with $W \in \mathcal{V}^s_k$, $|\tilde{\Omega}_y \setminus W| \le C_1 k \le C_1\rho$ for ε small enough and

$$\|W\|_* \le \|\tilde{\Omega}_y\|_* + C\epsilon^{-1}\varepsilon \le \|\tilde{\Omega}_y\|_* + \rho.$$

The last inequality holds by choosing κ larger than C. Moreover, there are mappings $\hat{R}_i : \Omega^{3k} \to SO(2)$, $i = 1, \ldots, 4$, which are constant on $Q^k_i(q) \cap W$, $q \in I^k_i(\Omega^k)$, such that

$$\|\nabla y - \hat{R}_i\|^2_{L^2(W)} \le C\varepsilon + C\varepsilon\rho^{-2}\kappa\|\tilde{\Omega}_y\|_* \le C\rho^{-2}\kappa\varepsilon.$$

Clearly, we also get $\|\nabla y - \hat{R}_i\|^4_{L^4(W)} \le C\rho^{-2}\kappa\varepsilon$ as $\|\nabla y\|_\infty \le M$. Then we apply Lemma 6.1 for $k = \rho^{-2}s_0$, $\nu = s_0$, $m = \rho$ and $\epsilon = \hat{c}\rho^{-3}\kappa\varepsilon$ to get sets $U \in \mathcal{V}^{s\hat{m}^2}_{71k}$ and $U^H \in \mathcal{V}^{\nu}_{72k}$ with $U, U^H \subset \Omega^{6k}$, $|U \setminus W| = 0$, $|U^H \setminus H^{\frac{\nu}{m}}(U)| = 0$ and

$$\|U\|_* \le (1 + C_1\rho)\|W\|_* + C\epsilon^{-1}\rho^{-2}\kappa\varepsilon \le \|\tilde{\Omega}_y\|_* + C_1\rho$$

as well as $|W \setminus U| \le C_1 k \le C_1\rho$ for ε small enough. Moreover, we find a function $\hat{y} \in H^1(U^H)$ such that by (6.2)

(*i*) $\|\operatorname{dist}(\nabla\hat{y}, SO(2))\|^2_{L^2(U^H)} \le CC^2_\rho(\rho^{-2}\kappa\varepsilon + \rho^{-3}\kappa\varepsilon\|W\|_*) \le CC^2_\rho\rho^{-3}\kappa\varepsilon$,

(*ii*) $\|\operatorname{dist}(\nabla\hat{y}, SO(2))\|^2_{L^\infty(U^H)} \le CC^6_\rho$,

(*iii*) $\|\nabla y - \nabla\hat{y}\|^2_{L^2(U')} \le CC^2_\rho\rho^{-3}\kappa\varepsilon, \quad \|\nabla y - \nabla\hat{y}\|^4_{L^4(U')} \le CC^8_\rho\rho^{-3}\kappa\varepsilon$,

where the second part of (iii) follows from (ii). Note that this also implies $\|\operatorname{dist}(\nabla\hat{y}, SO(2))\|^4_{L^4(U^H)} \le CC^8_\rho\rho^{-3}\kappa\varepsilon$. Setting $W_1 = U$, $W^H_1 = U^H$, $y_1 = \hat{y}$ we can now follow the proof of Theorem 7.3 beginning with (7.20) with the essential difference that we have to replace ε by $CC^8_\rho\rho^{-3}\kappa\varepsilon$. We then obtain the desired result for a possibly larger constant C_2 in (7.2). $\square$

7.3 Step 3: General case

We are now in a position to prove the general version of Theorem 7.1.

Proof of Theorem 7.1. Let $y \in SBV_M(\Omega) \cap L^2(\Omega)$ be given and let $\rho > 0$. It suffices to find a set $\tilde{\Omega} \in \mathcal{V}^s_\varepsilon$, $s > 0$, and a function $\tilde{y} \in H^1(\tilde{\Omega})$ with $\|\tilde{y}\|_{L^\infty(\tilde{\Omega})} + \|\nabla\tilde{y}\|_{L^\infty(\tilde{\Omega})} \le cM$ for a universal constant $c > 0$ such that

$$\begin{aligned} &|\Omega \setminus \tilde{\Omega}| \le C_1\rho, \quad \|\tilde{\Omega}\|_* \le (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho, \\ &\|y - \tilde{y}\|^2_{L^2(\tilde{\Omega})} + \|\nabla y - \nabla\tilde{y}\|^2_{L^2(\tilde{\Omega})} \le C_1\varepsilon\rho. \end{aligned} \tag{7.29}$$

Then the result follows from Theorem 7.4 applied on the function $\tilde{y}$. (Accordingly, replace M by cM in all estimates.) Note that we cannot just apply density results for SBV functions (see [12]) since in general such approximations do not preserve an L^∞ bound for the derivative. The problem may be bypassed by construction of a different approximation (see [7] and [22]) at the cost of a non exact approximation of the jump set which, however, suffices for our purposes.

Let $\mu = \varepsilon\rho$. Recall that J_y is rectifiable (see [2, Section 2.9]), i.e. there is a countable union of C^1 curves $(\Gamma_i)_{i\in\mathbb{N}}$ such that $\mathcal{H}^1(J_y \setminus \bigcup_i \Gamma_i) = 0$. Covering J_y with small balls and applying Besicovitch's covering theorem (see [18, Corollary 1, p. 35]) we find finitely many closed, pairwise disjoint balls $\overline{B_j} = \overline{B_{r_j}(x_j)}$, $j = 1, \dots, n$ with $r_j \le \mu$ such that $\mathcal{H}^1(J_y \setminus \bigcup_{j=1}^n B_j) \le \mu$. Moreover, we get $\mathcal{H}^1(J_y \cap \overline{B_j}) \ge 2(1-\mu)r_j$ and for each B_j we find a C^1 curve Γ_{i_j} such that $\Gamma_{i_j} \cap \overline{B_j}$ is connected and $\mathcal{H}^1((\Gamma_{i_j} \triangle J_y) \cap \overline{B_j}) \le 2\mu r_j \le \frac{\mu}{1-\mu}\mathcal{H}^1(J_y \cap \overline{B_j})$. For a detailed proof we refer to [7, Theorem 2].

We choose rectangles R_j with $|\partial R_j|_\infty \le 2\sqrt{2}r_j$ such that $\mathcal{H}^1(\Gamma_{i_j} \cap (B_j \setminus R_j)) = 0$ and $|\partial R_j|_\infty \le \mathcal{H}^1(\Gamma_{i_j} \cap \overline{B_j})$. We then obtain

$$\begin{aligned}\sum\nolimits_j |\partial R_j|_\infty &\le \sum\nolimits_j \mathcal{H}^1(\Gamma_{i_j} \cap \overline{B_j}) \\ &\le \Big(1 + \frac{\mu}{1-\mu}\Big) \sum\nolimits_j \mathcal{H}^1(J_y \cap \overline{B_j}) \le (1 + C_1\mu)\mathcal{H}^1(J_y)\end{aligned}$$

and likewise $\sum_j |\partial R_j|_{\mathcal{H}} \le C_1 \mathcal{H}^1(J_y)$. Choose rectangles Q_j with $R_j \subset\subset Q_j$ such that $|\partial Q_j|_* \le (1+\mu)|\partial R_j|_*$ and

$$\mathcal{H}^1\Big(\bigcup\nolimits_j \partial Q_j \cap J_y\Big) = 0. \tag{7.30}$$

As before it is not hard to see that $R_{j_1} \setminus R_{j_2}$ is connected for $1 \le j_1, j_2 \le n$. The rectangles $(Q_j)_j$ can be chosen in a way such that they fulfill the same property. Possibly replacing the rectangles by infinitesimally larger rectangles we can assume that there is some $s > 0$ such that $R_j, Q_j \in \mathcal{U}^s$ for $j = 1, \dots, n$. Then by Lemma 4.2(i) we find sets $W, V \in \mathcal{V}^s_\varepsilon$ with $|V \triangle (\Omega^\rho \setminus \bigcup_j R_j)| = 0$ and $|W \triangle (\Omega^\rho \setminus \bigcup_j Q_j)| = 0$. Note that $W^\circ \subset\subset V^\circ$ and $|\Omega \setminus W| \le C_1\rho$. It is not restrictive to assume that corners of R_j, Q_j do not coincide and thus W°, V° are Lipschitz domains. We get (recall Lemma 3.2)

$$\|W\|_* \le (1+\mu)\sum\nolimits_j |\partial R_j|_* \le (1 + C_1\rho + C_1 h_*)\mathcal{H}^1(J_y). \tag{7.31}$$

Moreover, as $\mathcal{H}^1(J_y \setminus \bigcup_{j=1}^n B_j) \le \mu$ we get

$$\begin{aligned}\mathcal{H}^1(J_y \setminus \bigcup\nolimits_{j=1}^n R_j) &\le \mu + \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n J_y \cap (B_j \setminus R_j)\Big) \\ &\le \mu + \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n \Gamma_{i_j} \cap (B_j \setminus R_j)\Big) + \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n (\Gamma_{i_j} \triangle J_y) \cap \overline{B_j}\Big) \\ &\le \mu + \frac{\mu}{1-\mu}\mathcal{H}^1(J_y) \le C_1\mu, \end{aligned} \tag{7.32}$$

where in the last step we have used $\mathcal{H}^1(\Gamma_{i_j} \cap (B_j \setminus R_j)) = 0$. We now show that there is a function $\hat{y} \in SBV(W^\circ)$ with $\|y - \hat{y}\|^2_{L^2(W)} + \|\nabla y - \nabla \hat{y}\|^2_{L^2(W)} \le C_1 \varepsilon \rho$ such that $\|\nabla \hat{y}\|_\infty \le cM$ and $J_{\hat{y}}$ is a finite union of closed segments satisfying $\mathcal{H}^1(J_{\hat{y}}) \le C_1 \mu \le C_1 \rho$. We apply a result by Chambolle obtained in [7] in an SBD-setting and rather cite the result as repeating the arguments. Therefore, we first obtain a control only over the symmetric part of the gradient. To derive the desired result we repeat the arguments for the function $v = (y^2, y^1)$ instead of $y = (y^1, y^2)$ to control also the skew part.

We define

$$E(y, W^\circ) = \int_{W^\circ} V(e(\nabla y)) + \mathcal{H}^1(J_y \cap W^\circ)$$

and $E_c(y, W^\circ) = E(y, W^\circ) + c\mathcal{H}^1(J_y \cap W^\circ)$, where $V(A) := \frac{1}{2\pi} \int_{S^1} (\xi^T A \xi)^2 \, d\xi$ for $A \in \mathbb{R}^{2\times 2}$. As $y \in SBV_M(W^\circ) \cap L^2(W^\circ)$ with $E(y, W^\circ) < +\infty$ and W° has Lipschitz boundary, by [7, Theorem 1] we find a sequence $y_n \in SBD(W^\circ) \cap L^2(W^\circ)$ with $\|y_n - y\|_{L^2(W^\circ)} \to 0$ such that $\overline{J_{y_n}}$ is a finite union of closed segments and

$$\begin{aligned} \limsup_{n\to\infty} E(y_n, W^\circ) &\le E_c(y, W^\circ) \le E(y, W^\circ) + C_1 \mu \\ &\le \int_{W^\circ} V(e(\nabla y)) + C_1 \mu. \end{aligned} \tag{7.33}$$

In the second and third step we used (7.32). The proof is based on a discretization argument. Consequently, as a preparation an extension y' to some set $W' \supset\supset W^\circ$ with $E(y', W') \le E(y, W^\circ) + \delta$ for arbitrary $\delta > 0$ had to be constructed (see [7, Lemma 3.2]). In our framework we can choose $y' = y$ due to $W^\circ \subset\subset V^\circ$ and (7.30). Moreover, $\|y_n\|_\infty \le \|y\|_\infty$ holds. Although not stated explicitly in the theorem, the approximations satisfy $\|\nabla y_n\|_{L^\infty(W^\circ)} \le c\|\nabla y'\|_{L^\infty(W')} \le c\|\nabla y\|_{L^\infty(V)} \le cM$. (For a precise argument see the proof of [8, Theorem 3.1], where a similar construction is used.) Strictly speaking, the theorem only states that J_{y_n} is essentially closed and contained in a finite union of closed segments. However, the proof shows that up to an infinitesimal perturbation of y_n (do not set $y_n = 0$ on a 'jump square', but $y_n = \tilde{c}$ for $\tilde{c} \approx 0$) the desired property can be achieved.

By [7, Lemma 5.1] we obtain weak convergence $e(\nabla y_n) \rightharpoonup e(\nabla y)$ in $L^2(W^\circ)$ up to a not relabeled subsequence. Together with the lower semicontinuity results $\int_{W^\circ} V(e(\nabla y)) \le \liminf_{n\to\infty} \int_{W^\circ} V(e(\nabla y_n))$ and $\mathcal{H}^1(J_y) \le \liminf_{n\to\infty} \mathcal{H}^1(J_{y_n})$ (see [7, Lemma 5.1]) we find by (7.33)

$$\int_{W^\circ} V(e(\nabla y)) \le \limsup_{n\to\infty} \int_{W^\circ} V(e(\nabla y_n)) \le \int_{W^\circ} V(e(\nabla y)) + C_1 \mu.$$

Consequently, by weak convergence we obtain

$$\begin{aligned}\limsup_{n\to\infty}\|e(\nabla y_n)-e(\nabla y)\|^2_{L^2(W^\circ)} &\le c\limsup_{n\to\infty}\int_{W^\circ}V(e(\nabla y_n-\nabla y))\\ &\le c\limsup_{n\to\infty}\Big(\int_{W^\circ}V(e(\nabla y_n))-\int_{W^\circ}V(e(\nabla y))\Big)\\ &\le C_1\mu=C_1\varepsilon\rho.\end{aligned}$$

Then by (7.33) we also get $\limsup_{n\to\infty}\mathcal{H}^1(J_{y_n})\le C_1\mu\le C_1\rho$. We now repeat the argument for $v=(y^2,y^1)$ instead of y and observe that by construction the approximations can be chosen as $v_n=(y^2_n,y^1_n)$. We find that $y_n\in SBV(W^\circ)$ and $\limsup_{n\to\infty}\|\nabla y_n-\nabla y\|^2_{L^2(W^\circ)}\le C_1\varepsilon\rho$. Now choose n large enough such that $\hat{y}:=y_n$ satisfies

$$\|y-\hat{y}\|^2_{L^2(W^\circ)}+\|\nabla y-\nabla\hat{y}\|^2_{L^2(W^\circ)}\le C_1\varepsilon\rho,\quad \mathcal{H}^1(J_{\hat{y}})\le C_1\rho$$

for $C_1>0$ large enough. Choose a finite number of closed segments $(S_i)_i^m$ such that $\overline{J_{\hat{y}}}\cap W^\circ\subset\bigcup_i S_i$ and $\mathcal{H}^1(\bigcup_i S_i)\le C_1\rho$. For $s>0$ small choose $T_i\in\mathcal{U}^s$ as the smallest rectangle with $S_i\subset T_i$. Then by Lemma 4.2(i) we obtain a set $\tilde{\Omega}\in\mathcal{V}^s_\varepsilon$ with

$$\big|\tilde{\Omega}\triangle\big(W\setminus\bigcup_{j=1}^m T_m\big)\big|=0.$$

Observe that for s sufficiently small we obtain $\|\tilde{\Omega}\|_*\le\|W\|_*+C_1\rho$ and $|W\setminus\tilde{\Omega}|\le C_1\rho$. This together with (7.31) gives the two first parts of (7.29). Finally, define the function $\tilde{y}\in H^1(\tilde{\Omega})$ by $\tilde{y}=\hat{y}|_{\tilde{\Omega}}$ and observe that $\tilde{y}$ satisfies (7.29). □

8 Proof of the main SBD-rigidity result

This last section is devoted to the proof of the main SBD-rigidity result. We start with some preparations and then split up the proof into two steps concerning a suitable construction of the jump set and the definition of an extension. As before constants indicated by C_1 only depend on M,η,Ω and all constants do not depend on ρ and q unless stated otherwise.

Let $y\in SBV_M(\Omega)\cap L^2(\Omega)$ be given and let $\rho>0$, $\varrho=\rho^q$ for $q\in\mathbb{N}$ to be specified below. Set $k=\rho^{q-1}$ and $m=\rho$. Recall the definition $\Omega_\rho=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>C\rho\}$. We apply Theorem 7.1 and obtain a set $\Omega_y\subset\Omega_\rho$ with $\Omega_y\in\mathcal{V}^s_{ck}$ for s sufficiently small and $|\Omega\setminus\Omega_y|\le C_1\rho$ such that (7.2) holds for a modification $\tilde{y}\in H^1(\Omega_y)\cap SBV_{cM}(\Omega_y)$ with $\|y-\tilde{y}\|^2_{L^2(\Omega_y)}+\|\nabla y-\nabla\tilde{y}\|^2_{L^2(\Omega_y)}\le C_1\varepsilon\rho$. Recall from the proof of Theorem 7.2 and Corollary 5.7 that there is a set $\Omega^H_y\in\mathcal{V}^{3\varrho}_{ck}$ with $\Omega^\circ_y\subset\Omega^H_y$ and an extension $\hat{y}:\Omega^H_y\to\mathbb{R}^2$ of $\tilde{y}$ satisfying (5.58) and estimates of the form (5.57).

We first construct a modification of Ω^H_y and appropriate Jordan curves which separate the connected components. For a (closed) Jordan curve γ we denote by

$\mathrm{int}(\gamma)$ the interior of the curve. As connected components may be not simply connected we further introduce a generalization: We say a curve $\gamma = \gamma_0 \cup \bigcup_{j=1}^m \gamma_j$ is a *generalized Jordan curve* if it consists of pairwise disjoint Jordan curves $\gamma_0, \ldots, \gamma_m$ with $\gamma_j \in \mathrm{int}(\gamma_0)$ for $j = 1, \ldots, m$. We define the interior of γ by $\mathrm{int}(\gamma) = \mathrm{int}(\gamma_0) \setminus \bigcup_{j=1}^m \mathrm{int}(\gamma_j)$.

Lemma 8.1 *Let $\rho > 0$, $M > 0$ and $q \in \mathbb{N}$. There is a constant $C_1 = C_1(M) > 0$ such that for all $\Omega_y^H \in \mathcal{V}_{ck}^{3\varrho}$ as given above we find $\hat{\Omega} \subset \Omega_\rho$ with $\mathcal{H}^1(\partial\hat{\Omega}) \le C_1$, $|\Omega_y^H \setminus \hat{\Omega}| \le C_1\rho$ and a set $S \subset \Omega_\rho \setminus \hat{\Omega}$ such that*

(i) $\mathcal{H}^1(S) \le \|\Omega_y^H\|_* + C_1\rho$,

(ii) for all $\hat{P}_i$ there is a generalized Jordan curve γ in $S \cup \partial\Omega_\rho$ such that $\mathrm{int}(\gamma) \cap \hat{\Omega} = \hat{P}_i$, *where $(\hat{P}_i)_i$ denote the connected components of $\hat{\Omega}$,*

(iii) $\mathrm{int}(\gamma) \cap \hat{\Omega} \neq \emptyset$ *for all Jordan curves γ in $S \cup \partial\Omega_\rho$,*

(iv) $\mathrm{dist}(x, S) \le C_1\rho^{q-2}$ *for all $x \in \Omega_\rho \setminus \hat{\Omega}$,*

(v) $(S \cup \partial\Omega_\rho) \cap X_t(\hat{\Omega})$ *is connected for all components $X_t(\hat{\Omega})$ of $\Omega_\rho \setminus \hat{\Omega}$.*

Proof. In contrast to the previous sections, where it was essential to avoid the combination of different cracks, we now combine boundary components: Choose a set $\hat{\Omega}_y^H \in \mathcal{V}^{3\varrho}$ satisfying $\hat{\Omega}_y^H \subset \Omega_y^H$, $|\Omega_y^H \setminus \hat{\Omega}_y^H| = 0$ and $|\Gamma_j(\hat{\Omega}_y^H) \cap \Gamma_l(\hat{\Omega}_y^H)|_{\mathcal{H}} = 0$ for $j \neq l$. Clearly, by (7.7) and (5.36) we have $\mathcal{H}^1(\hat{\Omega}_y^H) \le \mathcal{H}^1(\Omega_y^H) \le C_1$.

Letting $Y_1, \ldots, Y_m$ be the connected components of $\hat{\Omega}_y^H$ satisfying $|\partial Y_j|_\infty \le \rho^{q-2}$ for $j = 1, \ldots, m$ we define $\tilde{\Omega} = \hat{\Omega}_y^H \setminus \bigcup_{j=1}^m Y_j$. As $|\partial Y_j|_\infty \le \rho^{q-2}$ for $j = 1, \ldots, m$, the isoperimetric inequality implies $|\bigcup_{j=1}^m Y_j| \le C_1\rho^{q-2}\|\hat{\Omega}_y^H\|_{\mathcal{H}} \le C_1\rho$ and thus $|\Omega_y^H \setminus \tilde{\Omega}| \le C_1\rho$.

Let $Z \subset \Omega_\rho \setminus \tilde{\Omega}$ be the largest set in $\mathcal{U}^{\rho^{q-2}}$ such that $\mathrm{dist}_\infty(x, \partial\tilde{\Omega} \setminus \partial\Omega_\rho) \ge \rho^{q-2}$ for all $x \in Z$ and define $\hat{\Omega} = \tilde{\Omega} \cup \overline{Z}$. (Observe that Z is typically not connected.) It is not hard to see that

$$\mathrm{dist}(x, \partial\hat{\Omega} \setminus \partial\Omega_\rho) \le C_1\rho^{q-2} \quad \text{for all } \ x \in \Omega_\rho \setminus \hat{\Omega}. \tag{8.1}$$

Moreover, we get $|\Omega_y^H \setminus \hat{\Omega}| \le C_1\rho$ and $\mathcal{H}^1(\hat{\Omega}) \le C_1$. In fact, for each connected component Z^i of $\overline{Z}$ we find boundary components $(X_j^i = X_j^i(\Omega_y^H))_j$ and $(Y_j^i)_j$ such that $\partial Z^i \subset \bigcup_j \overline{X_j^i} \cup \bigcup_j \overline{Y_j^i}$ and thus by $|\partial X_j^i|_\infty \le c\rho^{q-1}$, $|\partial Y_j^i|_\infty \le \rho^{q-2}$ we obtain $|\partial Z^i|_{\mathcal{H}} \le C_1(\sum_j |\partial X_j^i|_{\mathcal{H}} + \sum_j |\partial Y_j^i|_{\mathcal{H}})$. We recall $\mathcal{H}^1(\Omega_y^H) \le C_1$ and observe that for different components Z^{i_1}, Z^{i_2} one has $(\bigcup_j \overline{X_j^{i_1}} \cup \bigcup_j \overline{Y_j^{i_1}}) \cap (\bigcup_j \overline{X_j^{i_2}} \cup \bigcup_j \overline{Y_j^{i_2}}) = \emptyset$.

Let $\hat{P}_1, \ldots, \hat{P}_n$ be the connected components of $\hat{\Omega}$ and define $\mathcal{F}(\hat{P}_i) = \{X_j = X_j(\Omega_y^H) : \overline{X_j} \cap \overline{\hat{P}_i} \neq \emptyset\}$. (Here it is essential that we take the components of Ω_y^H.) By $Z_j \in \mathcal{U}^{3\varrho}$ we denote the smallest rectangle containing X_j.

(I) As a preparation we consider the special case that there is only one connected component $\hat{P}_1$. Moreover, we first suppose that $\Omega_\rho \setminus \hat{\Omega}$ consists of one connected component only. We can choose a set S in $\bigcup_{Z_j \in \mathcal{F}(\hat{P}_1)} \overline{Z_j}$ consisting of segments such that $S \cup (\partial\Omega_\rho \setminus \hat{\Omega})$ is connected,

$$\mathcal{H}^1(S) \le (1 + C_1\rho) \sum\nolimits_{X_j \in \mathcal{F}(\hat{P}_1)} |\Gamma_j|_\infty \le (1 + C_1\rho) \sum\nolimits_{X_j \in \mathcal{F}(\hat{P}_1)} |\Gamma_j|_* \tag{8.2}$$

and $\mathrm{dist}(x, S) \le C_1\rho^{q-2}$ for all $x \in \partial\hat{P}_1 \setminus \partial\Omega_\rho$ for a sufficiently large constant. Indeed, a set with the desired properties can be constructed in the following way. By the definition of $|\cdot|_\infty$ we first see that we can choose a piecewise affine Jordan curve γ in $\bigcup_{X_j \in \mathcal{F}(\hat{P}_1)} \overline{Z_j} \cup \partial\Omega_\rho$ such that $\hat{P}_1 \subset \mathrm{int}(\gamma)$ and $S_0 := \gamma \cap \Omega_\rho^\circ$ satisfies $\mathcal{H}^1(S_0) \le \sum_{X_j \in \mathcal{G}(S_0)} |\Gamma_j|_\infty$, where $\mathcal{G}(S_0) = \{X_j : X_j \cap S_0 \neq \emptyset\}$. (If $\gamma \cap \Omega_\rho^\circ = \emptyset$, we let $S_0 = \{p_0\}$ for some point $p_0 \in \Omega_\rho \setminus \hat{\Omega}$.) Assume a connected set S_l consisting of segments has been constructed such that

$$\mathcal{H}^1(S_l) \le \sum\nolimits_{X_j \in \mathcal{G}(S_l)} |\Gamma_j|_\infty + C_1 l \rho^{q-1}. \tag{8.3}$$

If $\mathrm{dist}(x, S_l) \le C_1\rho^{q-2}$ for all $x \in \partial\hat{P}_1 \setminus \partial\Omega_\rho$, we stop. Otherwise, there is some $y \in \partial\hat{P}_1 \setminus \partial\Omega_\rho$ such that $\mathrm{dist}(y, S_l) > C_1\rho^{q-2}$. By the definition of $|\cdot|_\infty$ it is elementary to see that we can find a piecewise affine, continuous curve T_{l+1} with $T_{l+1} \cap S_l \neq \emptyset$, $y \in T_{l+1}$, $\#(\mathcal{G}(T_{l+1}) \cap \mathcal{G}(S_l)) = 1$ such that $\mathcal{H}^1(T_{l+1}) \le \sum_{X_j \in \mathcal{G}(T_{l+1})} |\Gamma_j|_\infty$. Then using that $|\Gamma(\Omega_y^H)|_\infty \le 2\sqrt{2} \cdot ck \le C_1\rho^{q-1}$ and $\#(\mathcal{G}(T_{l+1}) \cap \mathcal{G}(S_l)) = 1$ we find that (8.3) is satisfied for $S_{l+1} := S_l \cup T_{l+1}$.

After a finite number of iterations $n \in \mathbb{N}$ we find that $\mathrm{dist}(y, S_n) \le C_1\rho^{q-2}$ for all $y \in \partial\hat{P}_1 \setminus \partial\Omega_\rho$ and set $S_* = S_n$. Indeed, this follows from the fact that in each iteration $\mathcal{G}(S_l)$ increases and the assertion clearly holds if S_l intersects all boundary components since $\max_j |\Gamma_j(\Omega_y^H)|_\infty \le C_1\rho^{q-1}$. As $\mathcal{H}^1(T_l) > C_1\rho^{q-2}$, it is not hard to see that $n \le C_1\rho^{2-q} \sum_{X_j \in \mathcal{F}(\hat{P}_1)} |\Gamma_j|_\infty$ and thus (8.2) holds replacing S by S_*.

Observe that possibly $S_* \cup (\partial\Omega_\rho \setminus \hat{\Omega})$ is not connected. Therefore, we choose some point y in each connected component of $\partial\Omega_\rho \setminus \hat{\Omega}$ (which may be several if Ω_ρ is not simply connected) and repeat the construction below (8.3) for each y. We obtain a set S such that (8.2) still holds and $S \cup (\partial\Omega_\rho \setminus \hat{\Omega})$ is connected.

If $\Omega_\rho \setminus \hat{\Omega}$ consists of several connected components $X_t(\hat{\Omega})$, we repeat the arguments on each component separately possibly starting with $S_0 = \{p_0\}$ for some $p_0 \in X_t(\hat{\Omega})$.

We see that (i),(v) are satisfied, (ii) holds with γ and (iii) follows from the fact that in the construction of the sets T_l above we do not obtain additional 'loops'. Moreover, (iv) follows from the fact that each $x \in \Omega_\rho \setminus \hat{\Omega}$ satisfies $\mathrm{dist}(x, \partial\hat{P}_1 \setminus \partial\Omega_\rho) \le C_1\rho^{q-2}$ by (8.1).

(II) We now consider an arbitrary number of connected components. Choose Jordan curves γ^i in $\bigcup_{X_j \in \mathcal{F}(\hat{P}_i)} \overline{Z_j} \cup \partial\Omega_\rho$ such that $\hat{P}_i \subset \mathrm{int}(\gamma^i) \cap \hat{\Omega}$ and $\mathcal{H}^1(\gamma^i \cap \Omega_\rho^\circ) \le$

$\sum_{X_j \in \mathcal{G}(\gamma^i)} |\Gamma_j|_\infty$. We first assume that $\hat{P}_i = \mathrm{int}(\gamma^i) \cap \hat{\Omega}$, i.e. $\mathrm{int}(\gamma^i)$ does not contain other components of $\hat{\Omega}$, and treat the general case in (III). As the sets $(\mathcal{F}(\hat{P}_i))_{i=1}^n$ might be not disjoint, we have to combine the different curves in a suitable way. Define $G_i = \bigcup_{X_j \in \mathcal{G}(\gamma^i)} \overline{Z_j}$. It is not restrictive to assume that $\bigcup_{1 \le i \le n} G_i$ is connected as otherwise we apply the following arguments on each component separately. For $B \subset \mathbb{R}^2$ we define

$$\mathrm{Int}(B) = \{x \in \mathbb{R}^2 : \exists \text{ Jordan curve } \gamma^i \text{ in } B : x \in \mathrm{int}(\gamma^i)\}.$$

It is not hard to see that we can order the sets $(\hat{P}_i)_i$ in a way such that for all $1 \le l \le n$ we have $\bigcup_{1 \le i \le l} G_i$ is connected and $\mathrm{Int}(\bigcup_{1 \le i \le l} G_i) \cap \hat{P}_j = \emptyset$ for all $j > l$. In fact, to see the second property, assume the first l sets $G_1, \ldots, G_l$ have already been chosen. Select some other component $\hat{P}_k$, $k > l$, with corresponding G_k. If the desired property is satisfied, we reorder and set $G_{l+1} = G_k$, otherwise we find some $\hat{P}_{k'}$, $k' > l, k' \neq k$, with corresponding $G_{k'}$ such that $\hat{P}_{k'} \subset \mathrm{Int}(\bigcup_{1 \le i \le l} G_i \cup G_k)$. Possibly repeating this procedure we finally find a set G_{l+1} such that $\mathrm{Int}(\bigcup_{1 \le i \le l+1} G_i) \cap \hat{P}_j = \emptyset$ for all $j > l+1$.

We now proceed iteratively. Set $S_0 = \emptyset$ and assume a connected set S_l has been constructed with

$$
\begin{aligned}
&(a) \quad \mathcal{H}^1(S_l \cap \Omega_\rho) \le (1 + C_1\rho) \sum\nolimits_{X_j \in \bigcup_{1 \le i \le l} \mathcal{G}(\mathrm{int}(\gamma^i))} |\Gamma_j|_* + C_1(l-1)\rho^{q-1},\\
&(b) \quad \text{for all } 1 \le i \le l \text{ there is a Jordan curve } \gamma \text{ in } S_l \text{ such that } \mathrm{int}(\gamma) \cap \hat{\Omega} = \hat{P}_i,\\
&(c) \quad \mathrm{dist}(x, S_l) \le C_1\rho^{q-2} \text{ for all } x \in \bigcup\nolimits_{i=1}^{l} \partial\hat{P}_i \setminus \partial\Omega_\rho.
\end{aligned}
\tag{8.4}
$$

Let T_{l+1} be the (unique) connected component of $\gamma^{l+1} \setminus \bigcup_{1 \le i \le l} G_i$ such that $\hat{P}_{l+1} \subset \mathrm{Int}(\bigcup_{1 \le i \le l} G_i \cup T_{l+1})$. Now choose two segments T^j_{l+1}, $j = 1, 2$, with $\mathcal{H}^1(T^j_{l+1}) \le C_1\rho^{q-1}$, $T^j_{l+1} \cap S_l \neq \emptyset$, $T^j_{l+1} \cap T_{l+1} \neq \emptyset$ such that $\hat{S}_{l+1} := S_l \cup T_{l+1} \cup \bigcup_{j=1,2} T^j_{l+1}$ satisfies $\hat{P}_{l+1} \subset \mathrm{Int}(\hat{S}_{l+1})$ and

$$\mathcal{H}^1(\hat{S}_{l+1} \cap \Omega_\rho) \le (1 + C_1\rho) \sum\nolimits_{X_j \in \bigcup_{1 \le i \le l} \mathcal{G}(\mathrm{int}(\gamma^i)) \cup \mathcal{G}(T_{l+1})} |\Gamma_j|_* + C_1 l \rho^{q-1}.$$

By the order of the sets $(\hat{P}_i)_i$ it is not hard to see that there is a Jordan curve γ in $\hat{S}_{l+1}$ with $\mathrm{int}(\gamma) \cap \hat{\Omega} = \hat{P}_{l+1}$. Observe that $\mathrm{dist}(x, \gamma^{l+1}) \le C_1\rho^{q-2}$ for all $x \in \partial\hat{P}_{l+1} \setminus \partial\Omega_\rho$ might not hold. Therefore, following the lines of (I) we choose a (possibly not connected) set $R_{l+1} \subset \mathrm{int}(\gamma^{l+1})$ such that such that $S_{l+1} := \hat{S}_{l+1} \cup R_{l+1}$ is connected in each component of $\Omega_\rho \setminus \hat{\Omega}$, $\mathrm{dist}(x, S_{l+1}) \le C_1\rho^{q-2}$ for all $x \in \partial\hat{P}_{l+1} \setminus \partial\Omega_\rho$ and

$$\mathcal{H}^1(R_{l+1}) \le (1 + C_1\rho) \sum\nolimits_{X_j \in \mathcal{G}(\mathrm{int}(\gamma^{l+1})) \setminus \mathcal{G}(\hat{S}_{l+1})} |\Gamma_j|_*.$$

Now it is not hard to see that (a)-(c) are satisfied for S_{l+1}.

After the last iteration step we define $S_* = S_n \cap \Omega_\rho$. Observe that by construction (see before (8.1)) each $\hat{P}_i$ satisfies $|\partial \hat{P}_i|_\infty \geq \rho^{q-2}$. Thus $n \leq C_1 \rho^{2-q}$ and then we obtain $\mathcal{H}^1(S_*) \leq \|\Omega_y^H\|_* + C_1\rho$ since $n\rho^{q-1} \leq C_1\rho$. Similarly as before, $S_* \cup \partial\Omega_\rho$ might not be connected in the components of $\Omega_\rho \setminus \hat{\Omega}$. Consequently, we proceed as in (I) (see construction below (8.3)) to find a set $S \supset S_*$ such that (i) still holds and $S \cup \partial\Omega_\rho$ is connected in the components of $\Omega_\rho \setminus \hat{\Omega}$. This gives (v). Moreover, (b) implies (ii) and similarly as in (I) also (iii) holds. (Here we do not have to consider generalized Jordan curves.) Finally, to see (iv) we use (c) and the fact that each $x \in \Omega_\rho \setminus \hat{\Omega}$ satisfies $\mathrm{dist}(x, \partial\hat{\Omega} \setminus \partial\Omega_\rho) \leq C_1\rho^{q-2}$ by (8.1).

(III) We now finally treat the case that the components $(\hat{P}_i)_{i=0}^n$ may also contain other components of $\hat{\Omega}$. To simplify the exposition we assume that there is exactly one component, say $\hat{P}_0$, such that $\hat{P}_0 \neq \mathrm{int}(\gamma^0) \cap \hat{\Omega}$. The general case follows by inductive application of the following arguments.

We proceed as in (II) (assuming we had $\hat{P}_0 = \mathrm{int}(\gamma^0) \cap \hat{\Omega}$) and construct a set S' particularly satisfying (i),(iii),(v). We have to verify (ii) for $\hat{P}_0$ and find a set $S \supset S'$ such that (iv) is satisfied and (i),(iii),(v) still hold. By $(\hat{P}_{i_j})_j$ we denote the components with $\hat{P}_{i_j} \subset \mathrm{int}(\gamma_0)$. As (ii) holds for these components we find pairwise disjoint Jordan curves $\gamma_1, \ldots, \gamma_m$ with $\bigcup_j \hat{P}_{i_j} \subset \bigcup_{j=1}^m \mathrm{int}(\gamma_j) \subset \mathrm{int}(\gamma_0)$. Consequently, defining the generalized Jordan curve $\gamma = \bigcup_{j=0}^m \gamma_j$ we find $\hat{P}_0 = \mathrm{int}(\gamma) \cap \hat{\Omega}$ which gives (ii).

Let $(Y_j)_j$ be the components of $\Omega_\rho \setminus \hat{\Omega}$ which are completely contained in $\mathrm{int}(\gamma_0)$. We observe that (iv) may be violated for $x \in Y^* := \bigcup_j Y_j \setminus \bigcup_{j=1}^m \mathrm{int}(\gamma_j)$. We now proceed similarly as in (I) to obtain a set $R \subset Y^*$ such that $S := S' \cup R$ is connected in the connected components of $\Omega_\rho \setminus \hat{\Omega}$ and $\mathrm{dist}(x, S) \leq C_1\rho^{q-2}$ for all $x \in \partial\hat{P}_1 \cap Y^*$. This implies (iii),(v) and together with (8.1) also (iv). Arguing similarly as in (II) we find that (i) is still satisfied since the sum in (8.4)(a) does not run over the components contained in Y^*. □

We finally can give the proof of Theorem 2.1 by constructing an extension $\hat{y}$ of the function $\tilde{y}$. We briefly note that the function $\hat{y}$ has to be defined as an extension of the approximation and not of the original deformation y as only in this case we obtain the correct surface energy due to the higher regularity of the jump set of $\tilde{y}$ and the available trace estimates. Recall the definition of $E_\varepsilon^\rho(y, U)$ in (2.3), in particular $f_\varepsilon^\rho(x) = \min\{\frac{x}{\sqrt{\varepsilon\rho}}, 1\}$.

Proof of Theorem 2.1. Let $\Omega_y \subset \Omega_\rho$ with $\Omega_y \in \mathcal{V}^s$ and $\Omega_y^H \in \mathcal{V}^{3\varrho}$ with $\Omega_y^\circ \subset \Omega_y^H$ be given. Recall that $|\Omega \setminus \Omega_y| \leq C_1\rho$. Let $\tilde{y} \in H^1(\Omega_y)$ be the approximation of $y \in SBV_M(\Omega)$ with $\|y - \tilde{y}\|^2_{L^2(\Omega_y)} + \|\nabla y - \nabla\tilde{y}\|^2_{L^2(\Omega_y)} \leq C_1\varepsilon\rho$ and let $\hat{y} \in SBV_{cM}(\Omega_y^H) \cap L^2(\Omega_y^H)$ be the extension of $\tilde{y}$ given by Corollary 5.7. Let $\hat{\Omega}$ be the set constructed in Lemma 8.1. We first consider the jumps of $\hat{y}$ in $(\Omega_y^H \cap \hat{\Omega})^\circ$. By

(5.64) and Hölder's inequality we find

$$\Big(\int_{J_{\hat y}\cap(\Omega_y^H)^\circ}|[\hat y]|\,d\mathcal{H}^1\Big)^2 \le \Big(\sum\nolimits_{Q_t\subset\Omega_y^H}\int_{J_{\hat y}\cap\overline{Q_t}}|[\hat y]|\,d\mathcal{H}^1\Big)^2$$
$$\le \sum\nolimits_{Q_t}|J_{\hat y}\cap\overline{Q_t}|_{\mathcal{H}}\cdot\sum\nolimits_{Q_t}|J_{\hat y}\cap\overline{Q_t}|_{\mathcal{H}}^{-1}\Big(\int_{J_{\hat y}\cap\overline{Q_t}}|[\hat y]|\,d\mathcal{H}^1\Big)^2$$
$$\le C\mathcal{H}^1(J_{\hat y})\cdot\sum\nolimits_{Q_t}CC_\rho^2\varrho^2(\gamma(N_t)+\delta_4(N_t)+\epsilon|\partial W\cap N_t|_{\mathcal{H}}),$$

where W as defined in (7.4), $N_t := N(Q_t) = \{x\in W : \mathrm{dist}(x,Q_t)\le C\rho^{q-1}\}$ and $\gamma(N_t) = \|\nabla\,\mathrm{dist}(\nabla\hat y, SO(2))\|^2_{L^2(W)}$, $\delta_4(N_t) = \sum_{i=1}^4\|\nabla\hat y - \hat R_i\|^4_{L^4(W)}$ (recall (7.6)). As each $x\in\Omega$ is contained in at most $\sim\rho^{-2}$ different N_t we find by (7.3), (7.4), (7.6), (5.58) and the fact that $\epsilon = \hat c\rho^{-1}\varepsilon$

$$\Big(\int_{J_{\hat y}\cap(\Omega_y^H\cap\hat\Omega)^\circ}|[\hat y]|\,d\mathcal{H}^1\Big)^2 \le C\rho^{-2}\cdot CC_\rho^2\varrho^2\epsilon \le C\varrho^2\rho^{-3}C_\rho^2\varepsilon.$$

(Note that in the general case the set W and the rigid motions $\hat R_i$ were defined differently (see e.g. (7.28)), but here and in the following we prefer to refer to the proof of Theorem 7.2 for the sake of simplicity.) By Remark 3.9(i) we get for $q = q(h_*)$ sufficiently large

$$\int_{J_{\hat y}\cap(\Omega_y^H\cap\hat\Omega)^\circ}|[\hat y]|\,d\mathcal{H}^1 \le CC_\rho\rho^{q-\frac32}\sqrt{\varepsilon} = C\rho^{q-(\frac32+z)}\sqrt{\varepsilon}\le\rho^2\sqrt{\varepsilon}.$$

Recalling that $f^\rho_\varepsilon(x)\le\rho^{-1}\frac{x}{\sqrt{\varepsilon}}$ for $x\ge 0$ we get

$$\int_{J_{\hat y}\cap(\Omega_y^H\cap\hat\Omega)^\circ} f^\rho_\varepsilon(|[\hat y]|)\,d\mathcal{H}^1 \le \varepsilon^{-1/2}\rho^{-1}\int_{J_{\hat y}\cap(\Omega_y^H\cap\hat\Omega)^\circ}|[\hat y]|\,d\mathcal{H}^1\le\rho. \tag{8.5}$$

We now concern ourselves with the components of $\partial\hat\Omega$. Let Y_t be a connected component of $\Omega_\rho\setminus(\hat\Omega\cup S)$, where S is the set constructed in Lemma 8.1. Set $S_t = S\cap\overline{Y_t}$ and $\Gamma_t = \overline{Y_t}\cap\partial\hat\Omega$. We observe that by Lemma 8.1(ii),(iii) Γ_t is a Jordan curve if $\overline{Y_t}\cap\partial\Omega_\rho = \emptyset$.

Define $J = I^\varrho(\hat\Omega)$ and for Γ_t we choose $J(\Gamma_t)\subset J$ such that $\overline{Q^\varrho(p)}\cap\Gamma_t\neq\emptyset$ for all $p\in J(\Gamma_t)$. We set $M(\Gamma_t) = \bigcup_{p\in J(\Gamma_t)}\overline{Q^\varrho(p)}$. For later purpose, for components with $|\Gamma_t|_\infty > 2\rho^{q-2}$ we introduce a finer partition of $M(\Gamma_t)$: Define $J(\Gamma_t) = I_1\dot\cup\ldots\dot\cup I_n$ and the connected sets $B_i = \bigcup_{p\in I_i}\overline{Q^\varrho(p)}$ such that $\rho^{-2}\le\#I_i\le C\rho^{-2}$, $i = 1,\ldots,n$, for a constant $C\gg 1$. For $|\Gamma_t|_\infty\le 2\rho^{q-2}$ we let $I_1 = J(\Gamma_t)$. It is elementary to see that $n\le\max\{C|\Gamma_t|_{\mathcal{H}}\rho^{2-q},1\}\le C|\Gamma_t|_{\mathcal{H}}\rho^{-q}$, where we used $|\Gamma_t|_{\mathcal{H}}\ge C\rho^q$.

Consider $\bar R_j:\Omega_y^H\to SO(2)$ and $\bar c_j:\Omega_y^H\to\mathbb{R}^2$, $j = 1,\ldots,4$, as given in (7.9). Recall the definition $\tilde\Omega = \hat\Omega\setminus\overline{Z}\subset\Omega_y^H$ before (8.1). We extend the function $\hat y$

to $\hat{\Omega}$ by setting $\hat{y} = \mathbf{id}$ on $\hat{\Omega} \setminus \tilde{\Omega}$ and likewise let $\bar{R}_j = \mathbf{Id}$, $\bar{c}_j = 0$ on $\hat{\Omega} \setminus \tilde{\Omega}$. (If $\overline{Z} \cap \Omega_y^H \neq \emptyset$, we redefine the function on this set.) Applying Corollary 5.7 on each $Q_j^{3\varrho}(p) \subset \hat{\Omega}$ with $Q_j^{3\varrho}(p) \cap M(\Gamma_t) \neq \emptyset$, we get

$$\begin{aligned} \|\hat{y} - (\bar{R}_j \ \cdot + \bar{c}_j)\|^2_{L^2(B_i)} &\leq C\varrho^2 C_\rho^2 \cdot \rho^{-2}\rho^{q-1}\epsilon \cdot \#I_i = C\rho^{3q-6}C_\rho^2\varepsilon, \\ \|\hat{y} - (\bar{R}_j \ \cdot + \bar{c}_j)\|^2_{L^1(\partial B_i)} &\leq C\rho^{3q-6}C_\rho^2\varepsilon, \end{aligned} \tag{8.6}$$

for $j = 1, \ldots, 4$ and $i = 1, \ldots, n$. Here we used $k = \rho^{q-1}$, $\epsilon = \hat{c}\varepsilon\rho^{-1}$ and the fact that each $N(Q_j^{3\varrho}(p))$ contains $\sim m^{-2} = \rho^{-2}$ different $Q^{3\varrho}(p) \subset \Omega_y^H$. The triangle inequality then yields

$$\|(\bar{R}_{j_1} \ \cdot + \bar{c}_{j_1}) - (\bar{R}_{j_2} \ \cdot + \bar{c}_{j_2})\|^2_{L^2(B_i)} \leq C\rho^{3q-6}C_\rho^2\varepsilon$$

for $1 \leq j_1, j_2 \leq 4$ and $i = 1, \ldots, n$. The strategy will be to cover Y_t with n different rigid motions. We argue as in (7.10)f. and (3.18) to get $\hat{R}_i \in SO(2)$, $\hat{c}_i \in \mathbb{R}^2$ such that

$$\|\hat{y} - (\hat{R}_i \ \cdot + \hat{c}_i)\|^2_{L^2(B_i)} \leq C(\#I_i)^4\rho^{3q-6}C_\rho^2\varepsilon \leq C\rho^{3q-14}C_\rho^2\varepsilon.$$

Here we used Hölder's inequality (cf. (3.18)). A similar argument shows that we even find

$$\sum\nolimits_{j=-1,0,1} \|\hat{y} - (\hat{R}_{i+j} \ \cdot + \hat{c}_{i+j})\|^2_{L^2(B_i)} \leq C\rho^{3q-14}C_\rho^2\varepsilon \tag{8.7}$$

for $i = 1, \ldots, n$, where (in the case that Γ_t is a Jordan curve) we set $\hat{R}_{n+1} = \hat{R}_1$, $\hat{c}_{n+1} = \hat{c}_1$ and $\hat{R}_0 = \hat{R}_n$, $\hat{c}_0 = \hat{c}_n$. Without restriction recalling Remark 3.15(iii) we can assume that $\hat{R}_i \in \mathrm{im}_{\bar{R}_4}(M(\Gamma_t)) \subset SO(2)$, where $\mathrm{im}_{\bar{R}_4}$ denotes the image of the function $\bar{R}_4$. For shorthand let $\bar{R} = \bar{R}_4$ and $\bar{c} = \bar{c}_4$. By (8.6) and (8.7) we get

$$\sum\nolimits_{j=-1,0,1} \|(\hat{R}_{i+j} \ \cdot + \hat{c}_{i+j}) - (\bar{R} \ \cdot + \bar{c})\|^2_{L^2(B_i)} \leq C\rho^{3q-14}C_\rho^2\varepsilon. \tag{8.8}$$

Using Hölder's inequality and passing to the trace on each $Q^{3\varrho}(p)$ we obtain for all $i = 1, \ldots, n$

$$\begin{aligned} &\sum\nolimits_{j=-1,0,1} \|(\hat{R}_{i+j} \cdot + \hat{c}_{i+j}) - (\bar{R} \ \cdot + \bar{c})\|^2_{L^1(B_i \cap \Gamma_t)} \\ &\qquad \leq C\sum\nolimits_{j=-1,0,1} |B_i \cap \Gamma_t|_{\mathcal{H}} \|(\hat{R}_{i+j} \ \cdot + \hat{c}_{i+j}) - (\bar{R} \ \cdot + \bar{c})\|^2_{L^2(B_i \cap \Gamma_t)} \\ &\qquad \leq C\varrho\rho^{-2} \cdot \varrho^{-1}\rho^{3q-14}C_\rho^2\varepsilon \leq C\rho^{3q-16}C_\rho^2\varepsilon. \end{aligned}$$

Together with (8.6) this implies

$$\sum\nolimits_{j=-1,0,1} \|\hat{y} - (\hat{R}_{i+j} \ \cdot + \hat{c}_{i+j})\|^2_{L^1(B_i \cap \Gamma_t)} \leq C\rho^{3q-16}C_\rho^2\varepsilon.$$

This and the fact that $n \le C|\Gamma_t|_{\mathcal{H}}\rho^{-q}$ yield

$$H_1 := \sum_i \sum_{j=-1,0,1} \|\hat{y} - (\hat{R}_{i+j} \ \cdot + \hat{c}_{i+j})\|_{L^1(B_i \cap \Gamma_t)} \le C|\Gamma_t|_{\mathcal{H}} \rho^{\frac{q}{2}-8} C_\rho \sqrt{\varepsilon}. \quad (8.9)$$

For the difference of the rigid motions we get by the triangle inequality and (8.7)

$$\sum_{j_1,j_2=-1,0,1} \|(\hat{R}_{i+j_1} \ \cdot + \hat{c}_{i+j_1}) - (\hat{R}_{i+j_2} \ \cdot + \hat{c}_{i+j_2})\|^2_{L^2(B_i)} \le C\rho^{3q-14} C_\rho^2 \varepsilon.$$

Let $\tilde{B}_i = \{x \in \Omega : \mathrm{dist}(x, B_i) \le \bar{C}\rho^{q-2}\}$. Arguing similarly as in (3.17) it is not hard to see that

$$\begin{aligned} \sum_{j_1,j_2=-1,0,1} &\|(\hat{R}_{i+j_1} \ \cdot + \hat{c}_{i+j_1}) - (\hat{R}_{i+j_2} \ \cdot + \hat{c}_{i+j_2})\|^2_{L^2(\tilde{B}_i)} \\ &\le C(\rho^{-2})^2 \cdot \rho^{-4} \cdot \rho^{3q-14} C_\rho^2 \varepsilon \le C\rho^{3q-22} C_\rho^2 \varepsilon \end{aligned} \quad (8.10)$$

as $\frac{|\tilde{B}_i|}{|B_i|} \le C\rho^{-4}$ and $\frac{|\partial \tilde{B}_i|_\infty}{|\partial B_i|_\infty} \le C\rho^{-2}$. Define $\tilde{I}_i = I^\varrho(\tilde{B}_i)$. Again using Hölder's inequality, passing from the traces to a bulk integral and recalling $n \le C|\Gamma_t|_{\mathcal{H}}\rho^{-q}$, $\#\tilde{I}_i \le C\rho^{-4}$ we derive (let $\cdot = (\hat{R}_{i+j_1} \ \cdot + \hat{c}_{i+j_1}) - (\hat{R}_{i+j_2} \ \cdot + \hat{c}_{i+j_2})$ for shorthand)

$$\begin{aligned} H_2 &:= \sum_i \sum_{p \in \tilde{I}_i} \sum_{j_1,j_2=-1,0,1} \| \cdot \|_{L^1(\partial Q^\varrho(p))} \\ &\le C \sum_i \sum_{p \in \tilde{I}_i} \sum_{j_1,j_2=-1,0,1} \varrho^{1/2} \| \cdot \|_{L^2(\partial Q^\varrho(p))} \\ &\le C \sum_i (\#\tilde{I}_i)^{\frac{1}{2}} \Big(\sum_{p \in \tilde{I}_i} \sum_{j_1,j_2=-1,0,1} \varrho \| \cdot \|^2_{L^2(\partial Q^\varrho(p))} \Big)^{1/2} \\ &\le C \sum_i \rho^{-2} \Big(\sum_{j_1,j_2=-1,0,1} \| \cdot \|^2_{L^2(\tilde{B}_i)} \Big)^{1/2} \le C|\Gamma_t|_{\mathcal{H}} \rho^{\frac{q}{2}-13} C_\rho \sqrt{\varepsilon}. \end{aligned} \quad (8.11)$$

By $(T_j)_j$ we denote the connected components of $Q^\varrho(p) \setminus (\hat{\Omega} \cup S)$ for all $Q^\varrho(p)$ with $Q^\varrho(p) \cap Y_t \neq \emptyset$. We now choose suitable rigid motions: Observe that $\mathrm{dist}(\Gamma_t \cup \partial\Omega_\rho, x) \le C_1 \rho^{q-2}$ for all $x \in Y_t$ by Lemma 8.1(iv) and the fact that Y_t is a connected component of $\Omega_\rho \setminus (\hat{\Omega} \cup S)$. Therefore, for every T_j with $\mathrm{dist}(T_j, \partial\Omega_\rho) \gg \rho^{q-2}$ we find some (possibly non unique) B_{i_j} with $\mathrm{dist}(T_j, B_{i_j}) \le C\rho^{q-2}$. In particular, we get $T_j \subset \tilde{B}_{i_j}$ choosing $\bar{C}$ in the definition of $\tilde{B}_i$ large enough. We define

$$\hat{y}(x) = \hat{R}_{i_j} \, x + \hat{c}_{i_j} \quad \text{for } x \in T_j \cap Y_t \cap \Omega_{2\rho} \quad (8.12)$$

for all j and note that we have found an extension $\hat{y}$ to $Y_t \cap \Omega_{2\rho}$. (If $Y_t \cap \Omega^H_y \neq \emptyset$, we redefine the function on this set.) Taking Lemma 8.1(v) into account the choice of B_{i_j} can be done in a way that for neighboring sets T_1, T_2 with $\overline{T}_1 \cap \overline{T}_2 \neq \emptyset$ one has $i_1 - i_2 \in \{-1, 0, 1\}$ and that $\mathcal{H}^1(J_{\hat{y}} \cap Y_t) \le C_1 \mathcal{H}^1(\Gamma_t)$. Now by (8.9) and (8.11) it is not hard to see that

$$\int_{(J_{\hat{y}} \cap \overline{Y_t}) \setminus S} |[\hat{y}]| \, d\mathcal{H}^1 \le CH_1 + CH_2 \le C|\Gamma_t|_{\mathcal{H}} \rho^{\frac{q}{2}-13} C_\rho \sqrt{\varepsilon}.$$

Repeating the arguments for all components Y_t we obtain a configuration $\hat{y} \in SBV_{cM}(\Omega_\rho)$ with $\hat{y} = \tilde{y}$ on $\Omega_y^* := \Omega_y \cap \tilde{\Omega}$, where by Lemma 8.1 we have $|\Omega \setminus \Omega_y^*| \leq C_1\rho$. (With a slight abuse of notation we replace Ω_y^* by Ω_y in the assertion of Theorem 2.1.) Summing over all Y_t and recalling that $\mathcal{H}^1(\partial\hat{\Omega}) \leq C_1$ by Lemma 8.1 we get

$$\sum\nolimits_t \int_{(J_{\hat{y}} \cap \overline{Y_t}) \setminus S} f_\varepsilon^\rho(|[\hat{y}]|)\, d\mathcal{H}^1 \leq C\rho^{\frac{q}{2}-13} C_\rho \leq \rho$$

for $q = q(h_*)$ sufficiently large. Together with (8.5), Lemma 8.1(i) and (7.2)(i) this implies

$$\int_{J_{\hat{y}}} f_\varepsilon^\rho(|[\hat{y}]|)\, d\mathcal{H}^1 \leq \int_{J_{\hat{y}} \setminus S} f_\varepsilon^\rho(|[\hat{y}]|)\, d\mathcal{H}^1 + \mathcal{H}^1(S) \leq (1 + C_1 h_*)\mathcal{H}^1(J_y) + C_1\rho.$$

Choosing $h_* = \rho$ we finally get

$$\int_{J_{\hat{y}}} f_\varepsilon^\rho(|[\hat{y}]|)\, d\mathcal{H}^1 \leq \mathcal{H}^1(J_y) + C_1\rho. \tag{8.13}$$

We observe $\nabla\hat{y} \in SO(2)$ on $\Omega_\rho \setminus \Omega_y$ (see construction in Corollary 5.7, (8.12) and recall $\hat{y} = \mathbf{id}$ in $\hat{\Omega} \setminus \tilde{\Omega}$). As $\|\tilde{y} - y\|^2_{L^2(\Omega_y)} + \|\nabla\tilde{y} - \nabla y\|^2_{L^2(\Omega_y)} \leq C_1\varepsilon\rho$ we obtain $E_\varepsilon^\rho(\hat{y}, \Omega_\rho) \leq E_\varepsilon(y) + C_1\rho$ which gives (2.4). Here we used $\|\nabla\tilde{y}\|_\infty + \|\nabla y\|_\infty \leq cM$ and the regularity of the stored energy density W.

Let $(P_j)_j$ be the connected components of $\Omega_\rho \setminus S$. By Lemma 8.1(ii),(iii) it is not hard to see that for every index j there is a (unique) connected component $\hat{P}_j$ of $\hat{\Omega}$ such that $\hat{P}_j \subset P_j$. Then there is either a connected component P_j^H of Ω_y^H such that $\hat{P}_j = P_j^H$ (see proof of Theorem 7.2) or $\hat{y} = \mathbf{id}$ on $\hat{P}_j$ (see construction before (8.6)). We now define (2.5) by $u(x) = \hat{y}(x) - (R_j\, x + c_j)$ for $x \in P_j$, where $R_j\, x + c_j$ is either the rigid motion on P_j^H given in Theorem 7.1 or $R_j = \mathbf{Id}$, $c_j = 0$, respectively. For later purpose, we note that for (8.13) we can also write

$$\sum\nolimits_j \tfrac{1}{2}P(P_j, \Omega_\rho) + \int_{J_{\hat{y}} \setminus \partial P} f_\varepsilon^\rho(|[\hat{y}]|)\, d\mathcal{H}^1 \leq \mathcal{H}^1(J_y) + C_1\rho, \tag{8.14}$$

where $\partial P = \bigcup_j \partial P_j$ and $P(P_j, \Omega)$ denotes the perimeter of P_j in Ω_ρ.

It remains to confirm (2.6). First, (i) follows by $\mathcal{H}^1(J_{\hat{y}} \cap (\Omega_y^H)^\circ) \leq C_1$ (see (5.58) and (7.7)), $\mathcal{H}^1(\partial\hat{\Omega}) \leq C_1$ (see Lemma 8.1) and the fact that the $\mathcal{H}^1$-measure of the jump set added in the construction of $\hat{y}$ (see (8.12)) is controlled by $\mathcal{H}^1(\partial\hat{\Omega})$ and $\mathcal{H}^1(S)$. In view of (7.2)(ii)-(iv) (see also (7.11)) the properties (ii)-(iv) already hold on the set $\hat{\Omega}$ for a sufficiently large constant $C(\rho, q) = C(\rho)$. (Recall $q = q(h_*)$ and the definition $h_* = \rho$. See also Remark 5.1.)

Recall that $\Omega_\rho \setminus \hat{\Omega} \subset \bigcup_t \overline{Y_t}$. Repeating the arguments leading to (7.11) we find by (8.8), (8.10) and (8.12)

$$\sum\nolimits_j \|\hat{y} - (R_j\, \cdot + c_j)\|^2_{L^2(P_j \setminus \hat{\Omega})} \leq C(\rho)\varepsilon.$$

This gives (ii). Moreover, as on each $Q^\varrho(p) \subset P_j \setminus \hat{\Omega}$ we have $\nabla \hat{y} = R$ for some $R \in \mathrm{im}_{\bar{R}_4}(\hat{\Omega})$ (see construction before (8.8)) we get

$$\begin{aligned}\|\nabla \hat{y} - R_j\|^p_{L^p(P_j \setminus \hat{\Omega})} &\le C(\rho)\|\bar{R}_4 - R_j\|^p_{L^p(P_j \cap \hat{\Omega})} \\ &\le C(\rho)\Big(\|\nabla \hat{y} - \bar{R}_4\|^p_{L^p(P_j \cap \hat{\Omega})} + \|\nabla \hat{y} - R_j\|^p_{L^p(P_j \cap \hat{\Omega})}\Big)\end{aligned}$$

for $p = 2, 4$. By (7.9) and (7.11) this yields

$$\sum\nolimits_j \|\nabla \hat{y} - R_j\|^4_{L^4(P_j \setminus \hat{\Omega})} \le C(\rho)\varepsilon, \quad \sum\nolimits_j \|\nabla \hat{y} - R_j\|^2_{L^2(P_j \setminus \hat{\Omega})} \le C(\rho)\varepsilon^{1-\eta}.$$

This together with (5.11) gives (iii),(iv). □

Having completed the main rigidity result, we can now prove the linearized version. We may essentially follow the proof of Theorem 2.1 with some minor changes. The proof, however, is considerably simpler as a lot of estimates and lemmas can be skipped.

Proof of Theorem 2.3. We only give a short sketch of the proof. Define $y = \mathbf{id} + u$. As the approximation argument presented in the proof of Theorem 7.1 also holds in the SBD-setting, it again suffices to prove the result under the assumption that there is some $\tilde{\Omega}_u \in \mathcal{V}^s_\varepsilon$ such that $u|_{\tilde{\Omega}_u} \in H^1(\tilde{\Omega}_u)$. We skip Section 5.1 and always set $\hat{R}_i = \mathbf{Id}$ for $i = 1, \ldots, 4$. Similarly as in Lemma 5.6 we find sets Ω_u, $\Omega_u^H \in \mathcal{V}^{3\varrho}_{9k}$ for $k = \rho^{q-1}$, $\varrho = \rho^q$, as well as mappings $\bar{A}_j : \Omega_u^H \to \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $\bar{c}_j : \Omega_u^H \to \mathbb{R}^2$, which are constant on $Q^{3\varrho}_j(p)$, $p \in I^{3\varrho}_j(\Omega^{3k})$, such that

$$\begin{aligned}&(i) \quad \|u - (\bar{A}_j \ \cdot + \bar{c}_j)\|^2_{L^2(\Omega_u)} \le CC^2_\rho \varrho^2(\alpha + \epsilon\|W\|_*), \\ &(iii) \quad \|(\bar{A}_{j_1} \ \cdot + \bar{c}_{j_1}) - (\bar{A}_{j_2} \ \cdot + \bar{c}_{j_2})\|^2_{L^2(\Omega_u^H)} \le CC^2_\rho \varrho^2(\alpha + \epsilon\|W\|_*)\end{aligned}$$

for $j_1, j_2 = 1, \ldots, 4$, $j = 1, \ldots, 4$, where $\alpha = \|e(\nabla u)\|^2_{L^2(\tilde{\Omega}_u)}$ and $\epsilon = \hat{c}\rho^{-1}\varepsilon$. This can be established following the lines of the proof of Lemma 5.6 with the difference that in (5.46) we do not replace $\mathbf{Id} + A$ by a different rigid motion $\bar{R}$, but proceed with $\mathbf{Id} + A$. Analogously, we find an extension Ω_u^H as constructed in Corollary 5.7 and then we obtain the result up to a small set following the lines of Theorem 7.2. Finally, the jump set and the extension to Ω_ρ may be constructed as in Section 8. □

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