A Korn-Poincaré-type inequality for special functions of bounded deformation

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Abstract

We present a Korn-Poincaré-type inequality in a planar setting which is in the spirit of the Poincaré inequality in SBV due to De Giorgi, Carriero, Leaci (see [14]). We show that for each function in SBD^2 one can find a modification which differs from the original displacement field only on a small set such that the distance of the modification from a suitable infinitesimal rigid motion can be controlled by an appropriate combination of the elastic and the surface energy. In particular, the result can be used to obtain compactness estimates for functions of bounded deformation.

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Contents

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1 Introduction

The propagation of crack has been studied in the realm of linearized elasticity since the seminal work of Griffith (see [21]) and led to a lot of realistic applications in engineering. Also from a mathematical point of view the theory is well developed (see [1, 3]) and adopted in many recent works in applied analysis (see e.g. [3, 4, 7, 8, 16, 22, 23]). A natural framework for the investigation of fracture models in a geometrically linear setting is given by the space of functions of bounded deformation, denoted by $BD(\Omega, \mathbb{R}^d)$, which consists of all functions $u \in L^1(\Omega, \mathbb{R}^d)$ whose symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d \times d}_{\rm sym}$-valued Radon measure. To study problems in fracture mechanics with variational methods Francfort and Marigo [17] have introduced energy functionals which are essentially of the form

$$\int_\Omega |e(u)|^2 \, dx + \mathcal{H}^{d-1}(J_u), \tag{1.1}$$

where $u \in SBD^2(\Omega, \mathbb{R}^d)$. (For the definition and properties of this space we refer to Section 2.1 below.) These so-called Griffith functionals comprise elastic bulk contributions for the unfractured regions of the body represented by the linear elastic strain $e(u) := \frac{1}{2}(\nabla u^T + \nabla u)$ and surface terms that assign energy contributions on the crack paths comparable to the size of the crack $\mathcal{H}^{d-1}(J_u)$.

A major difficulty of these problems is given by the fact that there is no control over the skew-symmetric part of the distributional derivative. Indeed, it would be desirable that uniform bounds on (1.1) induce estimates for the strain ∇u or the function u itself and that accordingly suitable compactness results can be derived. However, simple examples (see e.g. [1]) show that such properties cannot be inferred in general as the behavior of small pieces being almost or completely separated from the bulk part may not be controlled. On the one hand, this observation particularly implies that SBD is not contained in SBV. (For the definition and properties of SBV we refer to [2].) On the other hand, it leads to the natural question if certain estimates still hold (1) up to a small exceptional set or (2) after passing to a slightly modified displacement field. The goal of the work at hand is to show that indeed the distance of the function u from an infinitesimal rigid motion can be estimated by an appropriate combination of the energy terms given in (1.1).

The starting point of our analysis is the classical Korn-Poincaré inequality in BD (see [5, 24]) stating that there is a constant $C(\Omega)$ depending only on the domain $\Omega \subset \mathbb{R}^d$ such that

$$\|u - Pu\|_{L^{\frac{d}{d-1}}(\Omega)} \le C(\Omega)|Eu|(\Omega) \tag{1.2}$$

for all $u \in BD(\Omega, \mathbb{R}^d)$, where P is a linear projection onto the space of infinitesimal rigid motions and $|Eu|$ denotes the total variation of the symmetrized distributional derivative. This is a remarkable result in consideration of the fact that corresponding estimates also involving the absolutely continuous part of the derivative ∇u do not hold since Korn's inequality fails in BD (cf. [11]).

It first appears that the inequality is not adapted for problems of the form (1.1) as in $|Eu|(\Omega)$ the jump height is involved and in (1.1) we only have control over the size of the crack. However, the main strategy of our approach is to prove that one may indeed find bounds on the jump heights after a suitable modification of the jump set and the displacement field whose total energy (1.1) almost coincides with the original energy. In particular, (1.2) is then applicable and it can be shown that the distance of the modification from an infinitesimal rigid motion can already be controlled in terms of the elastic energy.

We goal of this work is to prove the following Korn-Poincaré-type inequality for SBD^2 functions. For $\mu > 0$ let $Q_\mu = (-\mu, \mu)^2$ and by $\mathrm{diam}(R)$ denote the diameter of a rectangle $R \subset Q_\mu$.

Theorem 1.1 *Let $1 \le p \le 2$. Let $\varepsilon > 0$ and $h_* > 0$ sufficiently small. Then there is a constant $C = C(h_*)$ and a universal constant $\bar{c} > 0$ such that for all $u \in SBD^2(Q_\mu, \mathbb{R}^2) \cap L^p(Q_\mu, \mathbb{R}^2)$ the following holds: We get pairwise disjoint, paraxial rectangles $R_1, \ldots, R_n$ with*

$$\sum\nolimits_{j=1}^n \mathrm{diam}(R_j) \le (1 + \bar{c}h_*)\big(\mathcal{H}^1(J_u) + \varepsilon^{-1}\|e(u)\|^2_{L^2(Q_\mu)}\big)$$

such that for $E := \bigcup_{j=1}^n R_j$ and the square $\tilde{Q} = (-\tilde{\mu}, \tilde{\mu})^2$ with $\tilde{\mu} = \max\{\mu - 2\sum_j \mathrm{diam}(R_j), 0\}$ we have $|E| \le \bar{c}(\sum_j \mathrm{diam}(R_j))^2$ and

$$\|u(x) - (A\,x + c)\|^2_{L^p(\tilde{Q}\setminus E)} \le C\mu^{4/p}\big(\|e(u)\|^2_{L^2(Q_\mu)} + \varepsilon\mathcal{H}^1(J_u)\big)$$

for some $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $c \in \mathbb{R}^2$.

As a direct consequence we obtain a modification $\tilde{u} \in SBD^2(Q_\mu, \mathbb{R}^2)$ with $\tilde{u} = u$ on $Q_\mu \setminus E$ and $\tilde{u}(x) = A\,x + c$ for $x \in E$ such that the estimate in Theorem

1.1 still holds. To the best of our knowledge our Korn-Poincaré inequality differs from other inequalities of this type available in the SBV-setting (see [6, 14]) as it is not based on a truncation of the function (which is forbidden in the SBD framework), but on a modification of the displacement field on an exceptional set $E = \bigcup_j R_j$. This set is associated to the parts of Q_μ being detached from the bulk part of Q_μ by J_u. In contrast to the recently established estimate in [9], Theorem 1.1 provides an exceptional set with a rather simple geometry. Most notably we have control over $\mathcal{H}^1(\partial E)$ which allows to apply compactness results for GSBD functions (see [12]).

Moreover, for $h_* \ll 1$ and $\varepsilon \gg \|e(u)\|^2_{L^2(Q_\mu)}(\mathcal{H}^1(J_u))^{-1}$ we obtain a fine estimate on the sum of the diameter of the rectangles which will be fundamental in the rigidity estimates in [18], where based on Theorem 1.1 we construct a modification whose crack length essentially coincides with the surface energy of the original displacement field. Moreover, this particular choice of ε shows that the distance from an infinitesimal rigid motions may indeed be controlled exclusively by the elastic part of the energy.

We comment that the original motivation for establishing Theorem 1.1 was the investigation of quantitative geometric rigidity results in SBD and the derivation of linearized Griffith energies from nonlinear models (see [18, 19]) generalizing the results in nonlinear elasticity theory (cf. [13, 20]) to the framework of brittle materials. Nevertheless, we believe that this inequality is of independent interest and may contribute to solve related problems in the future, especially concerning fracture models in the realm of linearized elasticity which are related to problems in SBV where Poincaré inequalities (see [14]) have proved to be useful. Moreover, we hope that a combination of the Korn-Poincaré inequality and the modification scheme presented in [18] may help to answer the question in which sense an analogous result to Theorem 1.1 can be derived for ∇u.

Similarly as other results in this context (see [6, 9, 14]), Theorem 1.1 provides an estimate only for functions whose jump set is small with respect to the size of the domain. In fact, if the jump set is large, the specimen may be separated into different parts of comparable size. Theorem 1.1 together with a localization argument shows that in the general case one can derive a piecewise Korn-Poincaré inequality, i.e. the specimen may be broken into different sets and on each connected component the distance of the displacement field from a certain infinitesimal rigid motion can be controlled. This problem is quite similar to the problem considered in [18] and we refer therein for further details.

The derivation of our main result is very involved as apart from the fact that the body may be disconnected by the jump set one has to face the problems that e.g. (1) the body might be still connected but only in a small region where the elastic energy is possibly large, (2) the crack geometry might become extremely complex including highly oscillating crack paths, (3) there might be infinite crack patterns occurring on different scales. The common difficulty of all these phenomena is the possible high irregularity of the jump set whence there are no uniform bounds for the constant in (1.2). Therefore, our proof is based on a modification algorithm which analyzes the problem iteratively on regions of mesoscopic size gradually becoming larger. In each step we have to carefully balance the elastic and surface contributions as well as the crack geometry and decide whether to establish an estimate for the function using (1.2) or to alter the jump set.

Although we in principle believe that the main strategy and a lot of techniques can be employed to treat the problem in arbitrary space dimension, we only prove the result in a planar setting in order to avoid even more technical difficulties concerning the topological structure of the crack geometries occurring in $d \geq 3$.

As the proof of Theorem 1.1 is very technical and long, we give a short overview and highlight the principal proof strategies for the convenience of the reader at the end of this introduction. The rest of the paper is then organized as follows. Section 2 is devoted to some preliminaries, where we carry out a careful analysis for the constant of the Korn-Poincaré inequality in BD (see (1.2)) and establish a trace theorem in SBD which allows to control the L^2-norm of the functions on the boundary. In Section 3 we introduce the notion of boundary components and provide a suitable modification scheme to alter the jump set. Section 4 is then devoted to a fine investigation of the elastic energy and the crack geometry in the neighborhood of a jump component. Section 6 contains the main technical estimates for the analysis of the jump height. In Section 5 we combine the previously established results and present an algorithm which iteratively modifies the jump set such that the estimates on the jump height may be applied.

Overview of the proof

Above we have already motivated that the fundamental idea in the derivation of Theorem 1.1 is the determination of the jump heights and the application of (1.2). More precisely, we have to show that $[u] \sim \sqrt{\varepsilon}$ in a suitable sense. By a density argument (see Theorem 2.3 below and cf. also [8]) we can assume that the jump set is contained in a finite number of rectangle boundaries. (We will call these sets boundary components or cracks in the following.) These boundaries will be altered during an iterative procedure. Clearly, we have to assure that in this process the length of the boundary components does not increase too much. To this end, it is convenient to measure the length of the jump set by a convex combination of the Hausdorff-measure $\mathcal{H}^1$ and the 'diameter' of a crack given by

$$|\Gamma|_\infty = \sqrt{|\pi_1\Gamma|^2 + |\pi_2\Gamma|^2}, \tag{1.3}$$

where Γ denotes the boundary component and π_1, π_2 the orthogonal projections onto the coordinate axes. One of the advantages of $|\cdot|_\infty$ in contrast to $\mathcal{H}^1$ is that due to the strict convexity of $|\cdot|_\infty$ it is often energetically favorable if different cracks are combined to one larger boundary component leading to a simplification of the jump set.

Nevertheless, during the modification process it cannot be avoided that additional cracks are added near original ones. To keep track of this amplification, the boundary components have to be assigned with a weight which indicates if (or: how much) this crack has already been 'used' to introduce another discontinuity set. Now a further difficulty arises from the the fact that during each iteration step of the modification these weights have to be carefully adjusted (see Section 3.2).

The overall aim of the modification is to assure that in a small neighborhood of a boundary component Γ the energy can be controlled. Indeed, it turns out that if the elastic energy exceeds $\varepsilon|\Gamma|_\infty$ or the size of the jump set in a neighborhood is much larger than $|\Gamma|_\infty$, then it is energetically favorable to replace the crack by a larger rectangle and to replace the function u in the interior of the rectangle by an infinitesimal rigid motion (see proof of Theorem 5.2a),b)). Moreover, the modification of the jump set occurs not only due to energetic but also due to geometrical reasons. Exploiting the properties of $|\cdot|_\infty$ we can find a finer characterization of the cracks in the neighborhood, e.g. one can show that there are at most two cracks whose size is comparable to $|\Gamma|_\infty$ (see Corollary 4.4). Furthermore, we can always find small stripes in the neighborhood which do not intersect the jump set (see Lemma 4.5).

Having these properties for the neighborhood of a rectangle Γ and assuming that for all smaller cracks Γ_l we have already established that $[u] \sim \sqrt{|\Gamma_l|_\infty \varepsilon}$, the main technical issue is to derive a trace estimate on the boundary of Γ. Then replacing the function u by an appropriate infinitesimal rigid motion in the interior of the rectangle we will indeed obtain $[u] \sim \sqrt{|\Gamma|_\infty \varepsilon}$ on the boundary of Γ. Consequently, the assertion can be proved using an algorithm which iteratively changes the jump set and determines the trace at boundary components once the required conditions in a neighborhood are fulfilled (see Theorem 5.2).

Obviously one expects that the crack opening of small cracks is generically small. In our framework this heuristically follows by a rescaling argument in (1.2) and the observation that after modification the energy in a neighborhood is bounded by $\sim |\Gamma|_\infty \varepsilon$. However, a rigorous investigation of the trace on the boundary of Γ is very subtle as due to the iterative application of the arguments the involved constants might become arbitrarily large. The proof will be carried out in several steps.

In the first step we assume that in the neighborhood N of Γ only small cracks Γ_l are present. This indeed induces that $|Eu|(N)$ is sufficiently small as on each Γ_l we have already shown $[u] \sim \sqrt{|\Gamma_l|_\infty \varepsilon}$. In general, the idea is to construct thin long paths in N which avoid cracks being to large. We then first measure the distance of the function from an infinitesimal rigid motion only on this path and may apply this result to estimate the distance in the whole set N afterwards (see Section 6.2). A major drawback of such a technique is that the constant in (1.2) crucially depends on the domain and explodes for sets getting arbitrarily thin. Consequently, in this context we have to carry out a careful quantitative analysis how the constant in (1.2) depends on the shape of the domain (see Section 2.2).

It turns out, however, that the paths in general cannot be selected in a way such that they only intersect sufficiently small cracks. Nevertheless, it can be shown that boundary components being too large for a direct application of the above ideas occupy only a comparably small region. In this region we then do not use the Korn-Poincaré inequality in (1.2), but circumvent the estimation of the surface energy by a slicing technique. Indeed, by the modification procedure alluded to above we always find small stripes in the neighborhood which do not intersect the jump set at all. The assertion then follows as this exceptional set can then taken arbitrarily small by an iterative application of the slicing method (see Section 6.3). We briefly note that such a technique is only employable as we treat a linear problem rendering the derivation of Theorem 1.1 easier than the geometrically nonlinear results in [18]. (Compare also [10], where the treatment of the linearized version is remarkably easier than the nonlinear problem due to the applicability of a slicing method.)

Finally, one has to face the problem that there are (at most) two other cracks Γ_1, Γ_2 intersecting N being larger than Γ. In particular, (1.2) cannot be directly applied since no estimate of the jump heights at Γ_1 and Γ_2 is available. However, the result can also be established in this case if the elastic and surface energy in the two areas close to Γ, Γ_1 and Γ, Γ_2 is sufficiently small (see Section 6.5). In fact, such a smallness assumption can always be inferred by a careful modification of the crack set (see proof of Theorem 5.2c) and Section 4.2). Finally, we remark that the result crucially depends on the application of a suitable L^2- trace theorem for SBD functions (see Lemma 2.3) which can be established in our framework because of the sufficiently regular jump set. Moreover, it is essential that there are at most two large cracks in a neighborhood. Already with three or four cracks the configurations might be significantly less rigid.

2 Preliminaries

In this preparatory section we recall the definition and basic properties of functions of bounded deformation, establish a trace theorem and analyze how the constant in the Korn-Poincaré-inequality (see (1.2)) depends on the shape of the domain.

2.1 Functions of bounded deformation

We collect the definitions and some fundamental properties of functions of bounded deformation. Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Recall that the space $BD(\Omega, \mathbb{R}^d)$ of *functions of bounded deformation* consists of functions $u \in L^1(\Omega, \mathbb{R}^d)$ whose symmetrized distributional derivative $Eu := \frac{(Du)^T + Du}{2}$ is a finite $R^{d\times d}_{\mathrm{sym}}$-valued Radon measure. In [1] it is shown that it can be decomposed as

$$Eu = e(u)\mathcal{L}^d + E^j u = e(u)\mathcal{L}^d + [u] \odot \xi_u \mathcal{H}^{d-1}|_{J_u} + \mathcal{E}^c(u), \tag{2.1}$$

where $e(u)$ is the absolutely continuous part of Eu with respect to the Lebesgue measure, $\mathcal{E}^c(u)$ denotes the 'Cantor part', J_u (the 'crack path') is an $\mathcal{H}^{d-1}$-rectifiable set in Ω, ξ_u is a normal of J_u and $[u] = u^+ - u^-$ (the 'crack opening') with $u^\pm$ being the one-sided limits of u at J_u. Here $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$.

The space $SBD(\Omega, \mathbb{R}^d)$ of *special functions of bounded deformation* consists of all $u \in BD(\Omega, \mathbb{R}^d)$ with $\mathcal{E}^c(u) = 0$. If in addition $e(u) \in L^2(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < \infty$, we write $u \in SBD^2(\Omega)$. For basic properties of this function space we refer to [1, 3].

We recall a Korn-Poincaré inequality and a trace theorem in BD (see [5, 24]).

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^d$ bounded, connected with Lipschitz boundary and let $P : L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d)$ be a linear projection onto the space of infinitesimal rigid motions. Then there is a constant $C > 0$, which is invariant under rescaling of the domain, such that for all $u \in BD(\Omega, \mathbb{R}^d)$*

$$\|u - Pu\|_{L^{\frac{d}{d-1}}(\Omega)} \leq C|Eu|(\Omega).$$

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^2$ bounded, connected with Lipschitz boundary. There exists a constant $C > 0$ such that the trace mapping $\gamma : BD(\Omega, \mathbb{R}^2) \to L^1(\partial\Omega, \mathbb{R}^2)$ is well defined and satisfies the estimate*

$$\|\gamma u\|_{L^1(\partial\Omega)} \leq C\big(\|u\|_{L^1(\Omega)} + |Eu|(\Omega)\big)$$

for each $u \in BD(\Omega, \mathbb{R}^2)$.

For sets which are related through bi-Lipschitzian homeomorphisms with Lipschitz constants of both the homeomorphism itself and its inverse uniformly bounded the constants in Theorem 2.1 can be chosen independently of these sets, see e.g. [20].

We now present a density result in $SBD^2(\Omega)$ being a variant of the theorem in [8] without the additional assumption that the functions lie in $L^2(\Omega)$. Although a generalization to arbitrary space dimension is possible, we state the result only for $d = 2$ for the sake of simplicity.

Theorem 2.3 *Let $1 \leq p < \infty$ and $\delta > 0$. For every $u \in SBD^2(\Omega) \cap L^p(\Omega)$ and for every set $\Omega' \subset\subset \Omega$ with Lipschitz boundary there are paraxial rectangles*

$R_1, \ldots, R_m$ *with* $|\bigcup_{j=1}^m R_j| \le \delta$ *and a function* $\tilde{u} \in H^1(\tilde{\Omega})$, *where* $\tilde{\Omega} = \Omega' \setminus \bigcup_{j=1}^m R_j$, *such that*

$$\begin{aligned} &(i) \quad \|u - \tilde{u}\|_{L^p(\tilde{\Omega})} \le \delta, \\ &(ii) \quad \|e(u) - e(\tilde{u})\|_{L^2(\tilde{\Omega})} \le \delta, \\ &(iii) \quad \sum\nolimits_{j=1}^m \mathrm{diam}(R_j) \le \mathcal{H}^1(J_u) + \delta. \end{aligned} \tag{2.2}$$

Proof. Let $u \in SBD^2(\Omega) \cap L^p(\Omega)$ be given and let $\nu > 0$ sufficiently small. Recall that J_u is rectifiable (see [2, Section 2.9]), i.e. there is a countable union of C^1 curves $(\Gamma_i)_{i\in\mathbb{N}}$ such that $\mathcal{H}^1(J_u \setminus \bigcup_i \Gamma_i) = 0$. Covering J_u with small balls and applying Besicovitch's covering theorem (see [15, Corollary 1, p. 35]) we find finitely many closed, pairwise disjoint balls $\overline{B_j} = \overline{B_{r_j}(x_j)}$, $j = 1, \ldots, n$ with $r_j \le \nu$ such that $\mathcal{H}^1(J_u \setminus \bigcup_{j=1}^n B_j) \le \nu$. Moreover, we get $\mathcal{H}^1(J_u \cap \overline{B_j}) \ge 2(1-\nu)r_j$ and for each B_j we find a C^1 curve Γ_{i_j} such that $\Gamma_{i_j} \cap \overline{B_j}$ is connected and $\mathcal{H}^1((\Gamma_{i_j} \triangle J_u) \cap \overline{B_j}) \le 2\nu r_j \le \frac{\nu}{1-\nu}\mathcal{H}^1(J_u \cap \overline{B_j})$. (Here $\triangle$ denotes the symmetric difference of two sets.) For a detailed proof we refer to [8, Theorem 2].

We choose paraxial rectangles Q_j with $\mathrm{diam}(Q_j) \le 2\sqrt{2}r_j$ such that $\mathcal{H}^1(\Gamma_{i_j} \cap (B_j \setminus Q_j)) = 0$ and $\mathrm{diam}(Q_j) \le \mathcal{H}^1(\Gamma_{i_j} \cap \overline{B_j})$. We then obtain

$$\begin{aligned} \sum\nolimits_j \mathrm{diam}(Q_j) &\le \sum\nolimits_j \mathcal{H}^1(\Gamma_{i_j} \cap \overline{B_j}) \\ &\le \Big(1 + \frac{\nu}{1-\nu}\Big) \sum\nolimits_j \mathcal{H}^1(J_u \cap \overline{B_j}) \le (1 + C\nu)\mathcal{H}^1(J_u). \end{aligned} \tag{2.3}$$

As $\mathcal{H}^1(J_u \setminus \bigcup_{j=1}^n B_j) \le \nu$, we then also get

$$\begin{aligned} \mathcal{H}^1(J_u \setminus \bigcup\nolimits_{j=1}^n Q_j) &\le \nu + \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n J_u \cap (B_j \setminus Q_j)\Big) \\ &\le \nu + \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n \Gamma_{i_j} \cap (B_j \setminus Q_j)\Big) + \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n (\Gamma_{i_j} \triangle J_u) \cap \overline{B_j}\Big) \\ &\le \nu + \frac{\nu}{1-\nu}\mathcal{H}^1(J_u) \le C\nu, \end{aligned} \tag{2.4}$$

where in the last step we have used $\mathcal{H}^1(\Gamma_{i_j} \cap (B_j \setminus Q_j)) = 0$. Choose paraxial rectangles $\tilde{Q}_j$ with $Q_j \subset\subset \tilde{Q}_j$ such that $\mathrm{diam}(\tilde{Q}_j) \le (1+\nu)\mathrm{diam}(Q_j)$ and

$$\mathcal{H}^1\Big(\bigcup\nolimits_j \partial\tilde{Q}_j \cap J_u\Big) = 0. \tag{2.5}$$

We define the sets $V = \Omega' \setminus \bigcup_j Q_j$ and $W = \Omega' \setminus \bigcup_j \tilde{Q}_j$ for $\Omega' \subset\subset \Omega$ with Lipschitz boundary. We observe that $W \subset V$ and $|\bigcup_j \tilde{Q}_j| \le C\nu\mathcal{H}^1(J_u)$ by the isoperimetric inequality. It is not restrictive to assume that corners of $Q_j, \tilde{Q}_j$ do not coincide and thus W, V are Lipschitz domains.

We now show that there is a function $\tilde{u} \in SBD^2(W)$ with $\|u - \tilde{u}\|_{L^p(W)} + \|e(u) - e(\tilde{u})\|_{L^2(W)} \leq C\nu$ such that $J_{\tilde{u}}$ is a finite union of closed segments satisfying $\mathcal{H}^1(J_{\hat{y}}) \leq C\nu$. To this end, we follow the proof in [8] and define

$$E(u, W) = \int_W V(e(u)) + \mathcal{H}^1(J_u \cap W)$$

as well as $E_c(u, W) = E(u, W) + c\mathcal{H}^1(J_u \cap W)$, where $V(A) := \frac{1}{2\pi} \int_{S^1} (\xi^T A \xi)^2 \, d\xi$ for $A \in \mathbb{R}^{2\times 2}$. (We drop the integration variable if no confusion arises.) As $u \in SBD^2(W) \cap L^p(W)$ with $E(u, W) < +\infty$ and W has Lipschitz boundary, by [8, Theorem 1] we find a sequence $u_n \in SBD^2(W) \cap L^p(W)$ with $\|u_n - u\|_{L^p(W)} \to 0$ such that $\overline{J_{u_n}}$ is a finite union of closed segments and

$$\begin{aligned} \limsup_{n\to\infty} E(u_n, W) &\leq E_c(u, W) \leq E(u, W) + C\nu \\ &\leq \int_W V(e(u)) + C\nu. \end{aligned} \tag{2.6}$$

In the second and third step we used (2.4). The proof is based on a discretization argument. Consequently, as a preparation an extension u' to some set $W' \supset\supset W$ with $E(u', W') \leq E(u, W) + \varepsilon$ for arbitrary $\varepsilon > 0$ had to be constructed (see [8, Lemma 3.2]). In our framework we can choose $u' = u$ due to $W \subset\subset V$ and (2.5). Observe that the assumption $u \in L^2(W)$ was only needed in [8, Lemma 3.2] for the construction of the extension and therefore may be dropped in the present context, where we obtain $u_n \to u$ in $L^p(W)$ instead of L^2-convergence. Moreover, strictly speaking, the theorem only states that J_{u_n} is essentially closed and contained in a finite union of closed segments. However, the proof shows that up to an infinitesimal perturbation of u_n (do not set $u_n = 0$ on a 'jump square', but $u_n = \tilde{c}$ for $\tilde{c} \approx 0$) the desired property can be achieved.

By [8, Lemma 5.1] we obtain weak convergence $e(u_n) \rightharpoonup e(u)$ in $L^2(W)$ up to a not relabeled subsequence. Together with the lower semicontinuity results $\int_W V(e(u)) \leq \liminf_{n\to\infty} \int_W V(e(u_n))$ and $\mathcal{H}^1(J_u) \leq \liminf_{n\to\infty} \mathcal{H}^1(J_{u_n})$ (see [8, Lemma 5.1]) we find by (2.6)

$$\int_W V(e(u)) \leq \limsup_{n\to\infty} \int_W V(e(u_n)) \leq \int_W V(e(u)) + C\nu.$$

Consequently, by weak convergence we obtain

$$\begin{aligned} \limsup_{n\to\infty} \|e(u_n) - e(u)\|^2_{L^2(W)} &\leq c \limsup_{n\to\infty} \int_W V(e(u_n) - e(u)) \\ &\leq c \limsup_{n\to\infty} \Big(\int_W V(e(u_n)) - \int_W V(e(u)) \Big) \\ &\leq C\nu. \end{aligned}$$

Then by (2.6) we also get $\limsup_{n\to\infty} \mathcal{H}^1(J_{u_n}) \le C\nu$. Now choose n large enough such that $\hat{u} := u_n$ satisfies $\|u - \hat{u}\|_{L^p(W)} + \|e(u) - e(\hat{u})\|_{L^2(W)} \le C\nu$ and $\mathcal{H}^1(J_{\hat{u}}) \le C\nu$ for some C sufficiently large. Choose a finite number of closed segments $(S_i)_i^{m'}$ such that $\overline{J_{\hat{y}}} \cap W \subset \bigcup_i S_i$ and $\mathcal{H}^1(\bigcup_i S_i) \le C\nu$. Then select paraxial rectangles T_i with $S_i \subset T_i$ and $\mathrm{diam}(T_i) \le (1+\nu)\mathcal{H}^1(S_i)$. Define $\tilde{\Omega} = W \setminus \bigcup_{i=1}^{m'} T_i$ and observe that similarly as before we get $|W \setminus \tilde{\Omega}| \le C\nu$. Now let $(R_j)_{j=1}^n$ be the union of the rectangles $(\tilde{Q}_j)_j$ and $(T_i)_i$. Choosing ν sufficiently small we obtain $|\Omega \setminus \tilde{\Omega}| \le |\Omega \setminus W| + |W \setminus \tilde{\Omega}| \le \delta$. Moreover, $\mathcal{H}^1(J_{\hat{u}}) \le C\nu$ together with (2.3) yields (2.2)(iii). Finally, define the function $\tilde{u} \in H^1(\tilde{\Omega})$ by $\tilde{u} = \hat{u}|_{\tilde{\Omega}}$ and observe that $\tilde{u}$ satisfies (2.2)(i),(ii). □

2.2 Korn-Poincaré: Dependence on the set shape

In general, the constant of the Korn-Poincaré inequality in Theorem 2.1 depends crucially on the set shape. This will be discussed in detail in this section. As a preparation we introduce some notation. For $s > 0$ we partition $\mathbb{R}^2$ up to a set of measure zero into squares $Q^s(p) = p + s(-1,1)^2$ for $p \in I^s := s(1,1) + 2s\mathbb{Z}^2$. Let

$$\mathcal{U}^s := \Big\{ U \subset \mathbb{R}^2 : U = \Big(\bigcup\nolimits_{p \in I} \overline{Q^s(p)} \Big)^\circ : \quad I \subset I^s \Big\}. \tag{2.7}$$

Here the superscript $\circ$ denotes the interior of a set. Consider $u \in SBD^2(U)$ with $U \in \mathcal{U}^s$. On a square $Q^s(p) \subset U$ and subsets $V \subset U$, $V \in \mathcal{U}^s$ we define for shorthand

$$\mathcal{E}(p) = \int_{Q^s(p)} |e(u)| + |D^j u|(Q^s(p)), \qquad \mathcal{E}(V) = \int_V |e(u)| + |D^j u|(V), \tag{2.8}$$

where $I^s(V) := \{p \in I^s : Q^s(p) \subset V\}$. Observe that $\mathcal{E}(V)$ differs from $|Eu|(V)$ as we consider the measure $D^j u$ instead of $E^j u$.

In order to quantify how the constant in Theorem 2.1 depends on the set shape we will estimate the variation from a square $Q^s(a)$ to a neighboring square $Q^s(b)$, $b = a + 2s\nu$ for $\nu = \pm \mathbf{e}_i$, $i = 1,2$ (cf. also [20]). For later purposes, we first consider more general rectangles and derive the difference of the deformation on adjacent squares as a special case. Let $b_1, b_2 \in \mathbb{R}^2$, and $B_i = b_i + (-l_i, l_i) \times (-m_i, m_i) \in \mathcal{U}^s$ for $i = 1,2$, where we assume without restriction that $l_1 \ge m_1 > 0$, $l_2 \ge m_2 > 0$. Suppose that there is a point $b_{12} \in \overline{B_1} \cap \overline{B_2}$.

For given $A_1, A_2, A_{12} \in \mathbb{R}^{2\times 2}_{\mathrm{skew}} = \{G \in \mathbb{R}^{2\times 2} : G^T = -G\}$ and $c_1, c_2, c_{12} \in \mathbb{R}^2$ we set $E_i := \|u - (A_i \,\cdot + c_i)\|^2_{L^2(B_i)}$ for $i = 1,2$ and suppose that

$$\|u - (A_{12} \,\cdot + c_{12})\|^2_{L^2(B_1 \cup B_2)} \le C(E_1 + E_2). \tag{2.9}$$

As above this implies

$$\|(A_i - A_{12}) \,\cdot + (c_i - c_{12})\|^2_{L^2(B_i)} \le C(E_1 + E_2) \quad \text{for } i = 1,2.$$

We let $B_i^- = b_i + (-l_i, 0) \times (-m_i, m_i)$, $B_i^+ = b_i + (0, l_i) \times (-m_i, m_i)$ and for shorthand we write $\hat{A}_i = A_i - A_{12}$, $\hat{c}_i = c_i - c_{12}$. We then derive

$$\begin{aligned} |B_i|\ l_i^2 |\hat{A}_i|^2 &\le 4\|\hat{A}_i\, l_i \mathbf{e}_1\|^2_{L^2(B_i^-)} = 4\|\hat{A}_i\,(\cdot + l_i\mathbf{e}_1) + \hat{c}_i - \hat{A}_i\ \cdot - \hat{c}_i\|^2_{L^2(B_i^-)} \\ &\le 8\|\hat{A}_i\ \cdot + \hat{c}_i\|^2_{L^2(B_i^+)} + 8\|\hat{A}_i\ \cdot + \hat{c}_i\|^2_{L^2(B_i^-)} \le C(E_1 + E_2) \end{aligned} \tag{2.10}$$

for $i = 1, 2$ and therefore

$$|B_1 \cup B_2|(l_1 + l_2)^2\ |A_1 - A_2|^2 \le C\kappa(E_1 + E_2), \tag{2.11}$$

where $\kappa = \frac{|B_1 \cup B_2|}{\min_j |B_j|}\left(\frac{l_1 + l_2}{\min_j l_j}\right)^2$. Observe that in the first equality we essentially used the skew symmetry. Since $|y - b_{12}| \le C(l_1 + l_2)$ for all $y \in B_1 \cup B_2$ we likewise compute

$$\begin{aligned} |B_1 \cup B_2||\hat{A}_i\, b_{12} + \hat{c}_i|^2 &\le C\frac{|B_1 \cup B_2|}{|B_i|}\|\hat{A}_i\ \cdot + \hat{c}_i\|^2_{L^2(B_i)} + C|B_1 \cup B_2|(l_1 + l_2)^2 |\hat{A}_i|^2 \\ &\le C\kappa(E_1 + E_2) \end{aligned}$$

for $i = 1, 2$. Employing the triangle inequality we then deduce

$$|B_1 \cup B_2||(A_2 - A_1)\, b_{12} + c_2 - c_1|^2 \le C\kappa(E_1 + E_2). \tag{2.12}$$

Consider $Z \subset B_1 \cup B_2$, $Z \in \mathcal{U}^s$. Similar arguments yield by (2.11)

$$\begin{aligned} \|(A_2 - A_1)\cdot + c_2 - c_1\|^2_{L^2(Z)} &\le C\|(A_2 - A_1) b_{12} + c_2 - c_1\|^2_{L^2(Z)} \\ &\quad + C|Z| \max\nolimits_j l_j^2 |A_1 - A_2|^2 \\ &\le C\frac{|Z|}{|B_1 \cup B_2|}\kappa(E_1 + E_2) \end{aligned} \tag{2.13}$$

and therefore by the triangle inequality

$$\begin{aligned} \|u - (A_1\ \cdot + c_1)\|^2_{L^2(B_2 \cap Z)} &\le C\|u - (A_2\ \cdot + c_2)\|^2_{L^2(B_2 \cap Z)} \\ &\quad + C\frac{|Z|}{|B_1 \cup B_2|}\kappa(E_1 + E_2). \end{aligned} \tag{2.14}$$

In particular, employing $Z = B_1 \cup B_2$ and recalling (2.9) we find

$$\|u - (A_1\ \cdot + c_1)\|^2_{L^2(B_1 \cup B_2)} \le C\kappa(E_1 + E_2). \tag{2.15}$$

Before we treat the case of two adjacent squares we observe that in the above estimates the constants may be refined in the case that $B_1 \subset B_2$ under additional assumptions on the energies. Let $\delta \ge Csl_2^{-1}$. Let $B_2 = (-l_2, l_2) \times (-s, s) \in \mathcal{U}^s$ and $B_1 \subset B_2$, $B_1 \in \mathcal{U}^s$ a general set such that $|B_2 \setminus B_1| \le \delta|B_2|$. In particular, this implies that the diameter of each connected component of $B_2 \setminus B_1$ is smaller than $C\delta l_2$. Moreover, we assume that for all $Z \subset B_2$, $Z \in \mathcal{U}^s$ one has

$$\|u - (A_i \ \cdot + c_i)\|^2_{L^2(B_i \cap Z)} \le |B_i \cap Z||B_2|^{-1} H_i \tag{2.16}$$

for $H_1, H_2 \ge 0$. Arguing similarly as in (2.11) we find $|A_1 - A_2|^2 \le C|B_2|^{-1} l_2^{-2}(H_1 + H_2)$. (Observe that the connectedness of B_1 is not necessary. Moreover, the estimate can also be derived if B_2 consists of several connected components.) We write $\tilde{A} = A_1 - A_2$ and $\tilde{c} = c_1 - c_2$ for shorthand. Let $b_0 \in B_1$ and $Q \subset B_1$ be the square containing b_0. Applying a scaled version of Young's inequality and using $s \le C\delta l_2$ we compute

$$\begin{aligned}
|Q||\tilde{A}\,b_0 + \tilde{c}|^2 = \|\tilde{A}\,b_0 + \tilde{c}\|^2_{L^2(Q)} &\le (1+\delta)\|\tilde{A} \ \cdot + \tilde{c}\|^2_{L^2(Q)} + (1 + \tfrac{1}{\delta})\|\tilde{A}\,(\cdot - b_0)\|^2_{L^2(Q)} \\
&\le (1+\delta)^2 \|u - (A_1 \ \cdot + c_1)\|^2_{L^2(Q)} + \tfrac{C}{\delta}\|u - (A_2 \ \cdot + c_2)\|^2_{L^2(Q)} \\
&\quad + \tfrac{C}{\delta}|Q|s^2|B_2|^{-1} l_2^{-2}(H_1 + H_2) \\
&\le (1 + C\delta)|B_2|^{-1}|Q|H_1 + \tfrac{C}{\delta}|B_2|^{-1}|Q|H_2 + C|B_2|^{-1}|Q|\delta(H_1 + H_2) \\
&\le (1 + C\delta)|B_2|^{-1}|Q|H_1 + \tfrac{C}{\delta}|B_2|^{-1}|Q|H_2.
\end{aligned}$$

Consider some connected $Z \subset B_2 \setminus B_1$, $Z \in \mathcal{U}^s$, and observe that we find some $b_0 \in B_1$ such that $|x - b_0| \le C\delta l_2$. Then repeating the above calculation, again employing Young's inequality, we derive

$$\begin{aligned}
\|\tilde{A} \ \cdot + \tilde{c}\|^2_{L^2(Z)} &\le (1+\delta)|Z||\tilde{A}\,b_0 + \tilde{c}|^2 + (1 + \tfrac{1}{\delta})|Z| \max_{x \in Z} |x - b_0|^2 |\tilde{A}|^2 \\
&\le (1 + C\delta)|B_2|^{-1}|Z|H_1 + \tfrac{C}{\delta}|B_2|^{-1}|Z|H_2.
\end{aligned}$$

Then Young's inequality yields

$$\begin{aligned}
\|u - (A_1 \ \cdot + c_1)\|^2_{L^2(Z)} &\le (1+\delta)\|\tilde{A} \ \cdot + \tilde{c}\|^2_{L^2(Z)} + (1 + \tfrac{1}{\delta})\|u - (A_2 \ \cdot + c_2)\|^2_{L^2(Z)} \\
&\le (1 + C\delta)|B_2|^{-1}|Z|H_1 + \tfrac{C}{\delta}|B_2|^{-1}|Z|H_2.
\end{aligned} \tag{2.17}$$

Finally, it is not hard to see that (2.17) holds for all $Z \subset B_2$, $Z \in \mathcal{U}^s$.

Now assume the special case that $B_1 = Q^s(a)$ and $B_2 = Q^s(b)$, $b = a + 2s\nu$ for $\nu = \pm \mathbf{e}_i$, $i = 1, 2$. Then by Theorem 2.1 we obtain $A(a), A(b), A(a,b) \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $c(a), c(b), c(a,b)$ such that

$$\begin{aligned}
&\|u - (A(p) \ \cdot + c(p))\|_{L^2(Q^s(p))} \le C\mathcal{E}(p) \quad \text{ for } p = a, b, \\
&\|u - (A(a,b) \ \cdot + c(a,b))\|_{L^2(Q^s(a,b))} \le C\mathcal{E}(a,b),
\end{aligned} \tag{2.18}$$

where for shorthand $\mathcal{E}(a,b) = \mathcal{E}(Q^s(a,b))$. As in this case $\kappa = 8$, (2.11) and (2.13) for $Z = Q^s(a,b)$ yield

$$\begin{aligned}
&s^2|A(a) - A(b)| \le C\mathcal{E}(a,b), \\
&\|(A(b) - A(a)) \ \cdot + c(b) - c(a)\|_{L^2(Q^s(a,b))} \le C\mathcal{E}(a,b).
\end{aligned}$$

More general, we consider a difference quotient with two arbitrary points $a, b \in I^s(U)$. We assume that there is a path $\xi = (\xi_1, \dots, \xi_m)$ such that

$$\begin{aligned} &\xi_1 = a, \quad \xi_m = b, \\ &\xi_j - \xi_{j-1} = \pm 2s\mathbf{e}_i \text{ for some } i = 1, 2, \quad \forall j = 2, \dots, m. \end{aligned}$$

Iterative application of the last estimate yields

$$\begin{aligned} &\|(A(b) - A(a)) \ \cdot + c(b) - c(a))\|_{L^2(Q^s(b))} \\ &\quad \le \sum\nolimits_{j=2}^m \|(A(\xi_j) - A(\xi_{j-1})) \ \cdot + c(\xi_j) - c(\xi_{j-1}))\|_{L^2(Q^s(b))} \\ &\quad \le \sum\nolimits_{j=2}^m \|(A(\xi_j) - A(\xi_{j-1})) \ \cdot + c(\xi_j) - c(\xi_{j-1}))\|_{L^2(Q^s(\xi_j))} \\ &\qquad + \sum\nolimits_{j=2}^m 2s|(A(\xi_j) - A(\xi_{j-1}))\,(b - \xi_j)| \\ &\quad \le C \sum\nolimits_{j=1}^m \mathcal{E}(\xi_j) + \sum\nolimits_{j=2}^m 2s|A(\xi_j) - A(\xi_{j-1})| m 2s \\ &\quad \le Cm \sum\nolimits_{j=2}^m \mathcal{E}(\xi_j, \xi_{j-1}) \end{aligned} \tag{2.19}$$

and therefore

$$\begin{aligned} &\|u - (A(a) \ \cdot + c(a))\|^2_{L^2(Q^s(b))} \le 2\|u - (A(b) \ \cdot + c(b))\|^2_{L^2(Q^s(b))} \\ &\quad + 2\|(A(b) - A(a)) \ \cdot + c(b) - c(a))\|^2_{L^2(Q^s(b))} \\ &\quad \le Cm^2 \Big(\sum\nolimits_{j=2}^m \mathcal{E}(\xi_j, \xi_{j-1}) \Big)^2 \le Cm^3 \sum\nolimits_{j=2}^m \big(\mathcal{E}(\xi_j, \xi_{j-1})\big)^2. \end{aligned} \tag{2.20}$$

In the last step we have used Hölder's inequality. We now apply the last estimate to derive a first weak Korn-Poincaré-type inequality.

Lemma 2.4 *Let $\mu, s > 0$ such that $l := \mu s^{-1} \in \mathbb{N}$. Then there is a constant $C > 0$ independent of μ, s such that for all connected sets $U \in \mathcal{U}^s$, $U \subset (-\mu, \mu)^2$, the following holds:*

(i) For all $u \in SBD^2(U)$ there is an $A \in \mathbb{R}^{2\times 2}_{\text{skew}}$ and $c \in \mathbb{R}^2$ such that

$$\int_U |u(x) - (A\,x + c)|^2 \, dx \le C(s^{-2}|U|)^3 (\mathcal{E}(U))^2.$$

(ii) More precisely, for all $V \subset U$, $V \in \mathcal{U}^s$ one has ($\fint_V$ stands for $\frac{1}{|V|}\int$)

$$|U| \fint_V |u(x) - (A\,x + c)|^2 \, dx \le C(s^{-2}|U|)^3 (\mathcal{E}(U))^2.$$

Proof. We first show (i). We fix $p_0 \in I^s(U)$ and consider an arbitrary $p \in I^s(U)$. As U is connected there is a path $\xi = (\xi_1 = p_0, \ldots, \xi_m = p)$ with $m \le |U|(2s)^{-2}$. We first apply (2.18) on each square and then by (2.20) we obtain

$$\begin{aligned}\|u - (A(p_0)\,\cdot + c(p_0))\|^2_{L^2(Q^s(p))} &\le C(s^{-2}|U|)^2 \Big(\sum\nolimits_{j=2}^m \mathcal{E}(\xi_{j-1}, \xi_j) \Big)^2 \\ &\le C(s^{-2}|U|)^2 (\mathcal{E}(U))^2.\end{aligned}$$

Then setting $A = A(p_0)$, $c = c(p_0)$ and summing over all $p \in I^s(U)$ we derive

$$\|u - (A(p_0)\,\cdot + c(p_0))\|^2_{L^2(U)} \le C\#I^s(U)(s^{-2}|U|)^2(\mathcal{E}(U))^2 \le C(s^{-2}|U|)^3(\mathcal{E}(U))^2.$$

Similarly, (ii) is a direct consequence of (i) since we may replace $\#I^s(U)$ by $\#I^s(V)$ if we only integrate over the set V. □

Remark 2.5 (i) The fact that we covered the sets U with squares is not essential. Recalling how we derived (2.11) we could equally well cover U with rectangles $R_i = t_i + (-a_i, a_i) \times (-b_i, b_i)$, where $c_1 a_i \le b_i \le c_2 a_i$ and $c_1 s \le b_i \le c_2 s$ for constants $0 < c_1 < c_2$. The constant in (2.18) only depends on c_1, c_2 as all the possible shapes are related to $(-s, s)^2$ through bi-Lipschitzian homeomorphisms with Lipschitz constants of both the homeomorphism itself and its inverse bounded (see Section 2.1).

(ii) In view of the proof of (i), in the choice of $A\,x + c$ we have the freedom to select any of the infinitesimal rigid motions which are given on each square $Q^s(p) \subset U$ by application of (2.18).

2.3 A trace theorem in SBD^2

By the trace theorem for BD functions (Theorem 2.2) one can control the L^1-norm of the function on the boundary. In our framework we may establish a trace theorem in L^2 for SBD^2 functions if the jump set is sufficiently regular: Let $Q_\mu = (-\mu, \mu)^2$ and recall the definition of $SBD^2(Q_\mu)$ in Section 2.1. We suppose that some $u \in SBD^2(Q_\mu)$ satisfies $J_u = \bigcup_j \Gamma_j \cap Q_\mu$, where $\Gamma_i = \partial R_i$ for rectangles $R_i = (a_1^i, a_2^i) \times (b_1^i, b_2^i) \subset \mathbb{R}^2$ (note that for the application we have in mind we do not require that the rectangles are subsets of Q_μ.). Clearly, as $u \in H^1(Q_\mu \setminus J_u)$ the trace is well defined in L^2. More precisely, we have the following statement.

Lemma 2.6 *Let $\mu > 0$. There is a constant $C > 0$ such that for all $u \in SBD^2(Q_\mu)$ with $J_u = \bigcup_{j=1}^n \Gamma_j \cap Q_\mu$, where $\Gamma_j = \partial R_j$, one has*

$$\begin{aligned}\int_{\partial Q_\mu} |u|^2 \, d\mathcal{H}^1 &\le C\mu \|e(u)\|^2_{L^2(Q_\mu)} + \tfrac{C}{\mu}\|u\|^2_{L^2(Q_\mu)} \\ &\quad + C \sum\nolimits_{j=1}^n \mathcal{H}^1(\Gamma_j) \; \sum\nolimits_{j=1}^n \Big((\mathcal{H}^1(\Gamma_j))^{-1} \int_{\Gamma_j \cap Q_\mu} |[u]|^2 \, d\mathcal{H}^1 \Big).\end{aligned} \tag{2.21}$$

Proof. Let $Q_\mu = (-\mu,\mu)^2$ and $u \in SBD^2(Q_\mu)$ with $J_u = \bigcup_{j=1}^n \Gamma_j \cap Q_\mu$. In what follows we drop the subscript μ for notational convenience. First by approximation of Sobolev functions on Lipschitz sets (see, e.g., [15, Section 4.2]) we may assume that $u|_{R_j}$ is smooth for $j = 0,\dots,n$, where $R_0 = Q \setminus \bigcup_{j=1}^n R_j$. We only consider the part $\partial' Q = (-\mu,\mu)\times\{\mu\}$ of the boundary. Let $\pi_x = \{x\}\times\mathbb{R}$ and compute for the second component u_2 by a slicing argument in $\mathbf{e}_2$-direction:

$$\begin{aligned}\int_{-\mu}^{\mu}\int_{-\mu}^{\mu} |u_2(x,\mu) - u_2(x,y)|^2\,dx\,dy &= \int_{-\mu}^{\mu}\int_{-\mu}^{\mu}\Big|\int_y^{\mu} D_2 u_2(x,t)\,dt\Big|^2\,dx\,dy\\ &\le C\int_{-\mu}^{\mu}\int_{-\mu}^{\mu}\Big(\mu\int_{-\mu}^{\mu}|\partial_2 u_2(x,t)|^2\,dt + \Big(\sum_{z\in J_u\cap\pi_x}|[u](z)|\Big)^2\Big)\,dx\,dy\\ &\le C\mu^2\|e(u)\|^2_{L^2(Q)} + C\mu\int_{-\mu}^{\mu}\Big(\sum_{z\in J_u\cap\pi_x}|[u](z)|\Big)^2\,dx.\end{aligned}$$

In the second step we have used Hölder's inequality. We now estimate the term on the right side. As Γ_j is a rectangle, except for two x-values there are exactly two points $t_j^1, t_j^2 \in \mathbb{R}$ such that $\Gamma_j \cap \pi_x = \{(x,t_j^1),(x,t_j^2)\}$ if $\Gamma_j\cap\pi_x \neq \emptyset$. We write $|\Gamma_j|_{\mathcal{H}} = \mathcal{H}^1(\Gamma_j)$, $|S|_{\mathcal{H}} = \sum_j \mathcal{H}^1(\Gamma_j)$ for shorthand. Letting $z_j^{k,x} = (x,t_j^k) \in \mathbb{R}^2$ and setting $|[u](z_j^{k,x})| = 0$ if $z_j^{k,x} \notin Q\cap\Gamma_j$, we then obtain by the discrete version of Jensen's inequality

$$\begin{aligned}\int_{-\mu}^{\mu}\Big(\sum_{z\in J_u\cap\pi_x}|[u](z)|\Big)^2\,dx &= 4\int_{-\mu}^{\mu}\Big(\sum\nolimits_j\sum\nolimits_{k=1,2}\frac{|\Gamma_j|_{\mathcal{H}}}{2|S|_{\mathcal{H}}}\,|[u](z_j^{k,x})|\frac{|S|_{\mathcal{H}}}{|\Gamma_j|_{\mathcal{H}}}\Big)^2\,dx\\ &\le 4\int_{-\mu}^{\mu}\sum\nolimits_j\sum\nolimits_{k=1,2}\frac{|\Gamma_j|_{\mathcal{H}}}{2|S|_{\mathcal{H}}}\,\Big(|[u](z_j^{k,x})|\frac{|S|_{\mathcal{H}}}{|\Gamma_j|_{\mathcal{H}}}\Big)^2\,dx\\ &\le 2|S|_{\mathcal{H}}\sum\nolimits_j\Big(|\Gamma_j|^{-1}_{\mathcal{H}}\int_{\Gamma_j\cap Q}|[u]|^2\,d\mathcal{H}^1\Big).\end{aligned}$$

Consequently, letting E be the right hand side of (2.21) we derive

$$\int_{\partial' Q}|u_2|^2\,d\mathcal{H}^1 \le \frac{C}{\mu}\Big(\int_{-\mu}^{\mu}\int_{-\mu}^{\mu}|u_2(x,\mu)-u_2(x,y)|^2\,dx\,dy + \|u\|^2_{L^2(Q)}\Big) \le CE.$$

The same argument with slicing in the directions $\xi_1 = \frac{1}{\sqrt{2}}(1,-1)$ and $\xi_1 = \frac{1}{\sqrt{2}}(-1,-1)$ yields

$$\int_{\partial_1' Q}|u\cdot\xi_1|^2\,d\mathcal{H}^1 \le CE, \qquad \int_{\partial_2' Q}|u\cdot\xi_2|^2\,d\mathcal{H}^1 \le CE,$$

where $\partial_1' Q = (-\mu,0)\times\{\mu\}$ and $\partial_2' Q = (0,\mu)\times\{\mu\}$. The claim now follows by combination of the previous estimates. ∎

3 Boundary components and modification of sets

In this section we first introduce the notion of *boundary components* for sets $\mathcal{U}^s$ and discuss some elementary properties. In particular, we define a suitable measure for the length of a component being a convex combination of the Hausdorff-measure $\mathcal{H}^1$ and the 'diameter'. Afterwards, we present a first modification procedure for sets and their associated boundary components.

3.1 Boundary components

Let $Q_\mu = (-\mu, \mu)^2$ and recall the definition of $\mathcal{U}^s$ in (2.7). We will concern ourselves with subsets $V \subset Q_\mu$ of the form

$$\mathcal{V}^s := \{V \subset Q_\mu : V = Q_\mu \setminus \bigcup_{i=1}^m X_i, \;\; X_i \in \mathcal{U}^s, \; X_i \text{ pairwise disjoint}\} \tag{3.1}$$

for $s > 0$. Note that each set in $V \in \mathcal{V}^s$ coincides with a set $U \in \mathcal{U}^s$ up to subtracting a set of zero Lebesgue measure, i.e. $U \subset V$, $\mathcal{L}^2(V \setminus U) = 0$. The essential difference of V and the corresponding U concerns the connected components of the complements $Q_\mu \setminus V$ and $Q_\mu \setminus U$. Observe that one may have $Q_\mu \setminus \bigcup_{i=1}^m X_i = Q_\mu \setminus \bigcup_{i=1}^{\hat{m}} \hat{X}_i$ with $(X_1, \ldots, X_m) \neq (\hat{X}_1, \ldots, \hat{X}_{\hat{m}})$, e.g. by combination of different sets (see Figure 1). In such a case we will regard $V_1 = Q_\mu \setminus \bigcup_{i=1}^m X_i$ and $V_2 = Q_\mu \setminus \bigcup_{i=1}^{\hat{m}} \hat{X}_i$ as different elements of $\mathcal{V}^s$. For this and the following sections we will always tacitly assume that all considered sets are elements of $\mathcal{V}^s$ for some small, fixed $s > 0$.

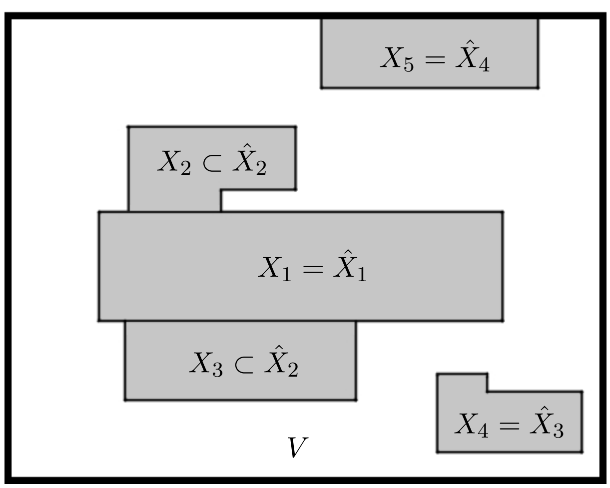

Figure 1: The square Q_μ with a subset V. The set V has two representations $V_1 = Q_\mu \setminus \bigcup_{i=1}^5 X_i$ and $V_2 = Q_\mu \setminus \bigcup_{i=1}^4 \hat{X}_i$, where $\hat{X}_2 = X_2 \cup X_3$, which are regarded as different elements of $\mathcal{V}^s$. The corresponding set $U \in \mathcal{U}^s$ arises from V by subtracting the black boundary lines $\bigcup_{i=1}^5 \partial X_i$.

Let $W \in \mathcal{V}^s$ and arrange the components $X_1, \ldots, X_m$ of the complement such that $\partial X_i \subset Q_\mu$ for $1 \le i \le n$ and $\partial X_i \cap \partial Q_\mu \neq \emptyset$ otherwise. Define $\Gamma_i(W) = \partial X_i$ for $i = 1, \ldots, n$. In the following we will often refer to these sets as *boundary components*. Note that $\bigcup_{i=1}^n \Gamma_i(W)$ might not cover $\partial W \cap Q_\mu$ completely if $n < m$. We frequently drop the subscript and write $\Gamma(W)$ or just Γ if no confusion arises. In addition to the Hausdorff-measure $|\Gamma|_{\mathcal{H}} = \mathcal{H}^1(\Gamma)$ (we will use both notations) we define the 'diameter' of a boundary component by $|\Gamma|_\infty := \sqrt{|\pi_1\Gamma|^2 + |\pi_2\Gamma|^2}$, where π_1, π_2 denote the orthogonal projections onto the coordinate axes.

Note that by definition of $\mathcal{V}^s$ (in contrast to the definition of $\mathcal{U}^s$) two components in $(\Gamma_i)_i$ might not be disjoint. Therefore, we choose an (arbitrary) order $(\Gamma_i)_{i=1}^n = (\Gamma_i(W))_{i=1}^n$ of the boundary components of W, introduce

$$\Theta_i = \Theta_i(W) = \Gamma_i \setminus \bigcup\nolimits_{j<i} \Gamma_j \tag{3.2}$$

for $i = 1, \ldots, n$ and observe that the boundary components $(\Theta_i)_i$ are pairwise disjoint. With a slight abuse of notation we define

$$|\Theta_i|_\infty = |\Gamma_i|_\infty.$$

Again we will often drop the subscript if we consider a fixed boundary component. We now introduce a convex combination of $|\cdot|_\infty$ and $|\cdot|_{\mathcal{H}}$. For an $h_* > 0$ to be specified below we set

$$|\Theta|_* = h_*|\Theta|_{\mathcal{H}} + (1 - h_*)|\Theta|_\infty. \tag{3.3}$$

For sets $W \in \mathcal{V}^s$ we then define

$$\|W\|_Z = \sum\nolimits_{j=1}^n |\Theta_j(W)|_Z$$

for $Z = \mathcal{H}, \infty, *$. Note that $\|W\|_\infty, \|W\|_{\mathcal{H}}$ and thus also $\|W\|_*$ are independent of the specific order which we have chosen in (3.2). Indeed, for $\|W\|_\infty$ this is clear as $|\Theta_i|_\infty = |\Gamma_i|_\infty$, for $\|W\|_{\mathcal{H}}$ it follows from the fact that $\|W\|_{\mathcal{H}} = \mathcal{H}^1(\bigcup_{i=1}^n \Gamma_i)$.

Before we introduce the modification procedure we collect some elementary properties of $|\cdot|_*$.

Lemma 3.1 *Let $W \subset Q_\mu$. Let $\Gamma = \Gamma(W)$ be a boundary component with $\Gamma = \partial X$ and let $\Theta \subset \Gamma$ be the corresponding set defined in (3.2). Moreover, let $V \in \mathcal{U}^s$ be a rectangle with $\overline{V} \cap \overline{X} \neq \emptyset$. Suppose that h_* is sufficiently small. Then*

(i) $|\Gamma|_* \ge |\partial R(\Gamma)|_*$ *if Γ is connected, where $R(\Gamma)$ denotes the smallest (closed) rectangle such that $\Gamma \subset R(\Gamma)$,*

(ii) $|\Theta|_* = |\Gamma|_* \Leftrightarrow |\Theta|_{\mathcal{H}} = |\Gamma|_{\mathcal{H}}$,

(iii) $|\partial(X \setminus \overline{V})|_\infty \le |\Theta|_\infty$ *and* $|\Theta \setminus \overline{V}|_{\mathcal{H}} \le |\Theta|_{\mathcal{H}}$,

(iv) $|\partial(V \cup X)|_* \le |\partial V|_* + |\Gamma|_*$,

(v) $|\partial(V \cup X)|_* \ge |\partial R(V \cup X)|_*$ *if* $\overline{V} \cup \overline{X}$ *is connected, where* $R(V \cup X)$ *denotes the smallest rectangle such that* $V \cup X \subset R(V \cup X)$.

Now assume that $\Gamma = \partial R$ *for a rectangle* $R \in \mathcal{U}^s$. *Then*

(vi) $\frac{1}{\sqrt{2}}|\Gamma|_{\mathcal{H}} \le 2|\Gamma|_\infty \le |\Gamma|_{\mathcal{H}}$,

(vii) $|\partial(V \cup R)|_* \le |\partial V|_* + \frac{1}{2}|\Gamma|_*$ *provided that* $\Gamma \setminus \overline{V}$ *is not connected and* $|\Gamma|_\infty \le c|\partial V|_\infty$ *for a constant* $c > 0$ *sufficiently small.*

Proof. If Γ is connected, we obtain $|\Gamma|_{\mathcal{H}} \ge |\partial R(\Gamma)|_{\mathcal{H}}$ and $|\Gamma|_\infty = |\partial R(\Gamma)|_\infty$. This yields (i) and likewise we obtain (v). Assertions (ii)-(iv) follow directly from the definition of $|\cdot|_*$, where in (ii) we particularly use $|\Theta|_\infty = |\Gamma|_\infty$. Claim (vi) is elementary. To see (vii) we assume without restriction $V = (-a, a) \times (0, b)$ and $\pi_1\Gamma = (-d, d)$ with $d > a$ as well as $\pi_2\Gamma \subset (0, b)$. An elementary calculation yields $|\partial(V \cup R)|_\infty = \sqrt{(2d)^2 + b^2} \le b + \frac{(2d)^2}{2b} \le b + \frac{2d}{4} \le |\partial V|_\infty + \frac{1}{4}|\Gamma|_\infty$. Here we used that $4d \le b$ for c small enough. As $|\partial(V \cup R)|_{\mathcal{H}} \le |\partial V|_{\mathcal{H}} + |\Gamma|_{\mathcal{H}}$ the claim now follows from (3.3) and (vi) if we choose h_* small enough. □

The properties stated here will be exploited frequently and we will not always refer to this lemma.

One method of the modification procedure below will be the 'combination' of different boundary components by adding additional sets to the original boundary (see case c) in the proof of Theorem 5.2). To keep track of the components we already 'used' to modify the boundary, it is convenient to introduce a weight $\omega_{\min} \le \omega(\Gamma_j) \le 1$ for all $\Gamma_j = \Gamma_j(W)$ with $\frac{1}{2} \le \omega_{\min} < 1$ to be specified below. We define $|\Theta_j|_{Z,\omega} = \omega(\Gamma_j)|\Theta_j|_Z$ and likewise a weighted version of $\|\cdot\|_Z$ by setting

$$\|W\|_{Z,\omega} := \sum\nolimits_j \omega(\Gamma_j)|\Theta_j|_Z \tag{3.4}$$

for $Z = \mathcal{H}, \infty, *$. For $Z = *$ we write for shorthand $|\cdot|_\omega = |\cdot|_{*,\omega}$ and $\|\cdot\|_\omega = \|\cdot\|_{*,\omega}$. We briefly note that in contrast to $\|\cdot\|_*$, the value of (3.4) depends on the order given in (3.2) and therefore we will always consider a specific order of the boundary components in the following.

3.2 Modification of sets

For $\lambda \ge 0$ and fixed small $\upsilon > 0$ let $\mathcal{W}^s_\lambda \subset \mathcal{V}^s$ be the subset consisting of the sets $W \in \mathcal{V}^s$ with a corresponding weight ω and an ordering of the boundary components $(\Gamma_i)_{i=1}^n$ such that the following properties are satisfied:

$$\begin{aligned}
&(i) && \Theta_i \subset \partial R_i,\ \Gamma_i \subset \overline{R_i} \ \text{ for a rectangle } R_i && \forall\, \Gamma_i:\ \omega(\Gamma_i) < 1,\\
&(ii) && |\partial R_i|_* \le \omega_{\min}^{-1}\,\omega(\Gamma_i)|\Theta_i|_* && \forall\, \Gamma_i:\ \omega(\Gamma_i) < 1,\\
&(iii) && R_i \setminus X_j \text{ is connected for all } j = 1,\dots,n && \forall\, \Gamma_i:\ \omega(\Gamma_i) < 1,\\
&(iv) && \omega(\Gamma_i) = 1 && \forall\, \Gamma_i:\ |\Gamma_i|_\infty \ge 19\upsilon\lambda,\\
&(v) && \Gamma_i = \Theta_i = \partial R_i \text{ for a rectangle } R_i && \forall\, \Gamma_i:\ \omega(\Gamma_i) = 1.
\end{aligned} \tag{3.5}$$

Observe that (iv),(v) imply that boundary components larger than $19\upsilon\lambda$ are always rectangular and pairwise disjoint. In particular, $\mathcal{W}_0^s$ consists of the sets where all boundary components are rectangular. By an elementary argumentation taking (3.3), (3.5)(i),(ii) into account and recalling $\omega_{\min} \ge \frac{1}{2}$, $h_* \ll 1$, we observe

$$|\Gamma_i|_\infty \le |\partial R_i|_\infty \le C|\Gamma_i|_\infty \quad \forall\, \Gamma_i:\ \omega(\Gamma_i) < 1, \tag{3.6}$$

i.e. the diameter of Γ_i and the corresponding rectangle R_i are comparable.

Consider a set $W = Q_\mu \setminus \bigcup_{i=1}^m X_i \in \mathcal{W}_\lambda^s$, $\lambda \ge 0$, and a rectangle $V \in \mathcal{U}^s$ with $|\partial V|_\infty \ge \lambda$ and $\overline{V} \subset Q_\mu$. We define the modification

$$\tilde{W} = Q_\mu \setminus \bigcup\nolimits_{i=0}^m \tilde{X}_i, \tag{3.7}$$

where $\tilde{X}_i = X_i \setminus \overline{V}$ for $i = 1,\dots,m$ and $\tilde{X}_0 = V$. We observe that $\tilde{W} = (W \setminus V) \cup \partial V$ (as a subset of $\mathbb{R}^2$). Therefore, for shorthand we will write $\tilde{W} = (W \setminus V) \cup \partial V$ to indicate the element of $\mathcal{V}^s$ which is given by (3.7).

We have the following boundary components of $\tilde{W}$: First let $\Gamma_0(\tilde{W}) = \partial V$ (it is convenient to start with index 0) and for $j \ge 1$ we have by construction $\Gamma_j(\tilde{W}) = \partial(X_j \setminus \overline{V})$. Observe that some boundary components may be empty and therefore reordering the indices we let $(\Gamma_j(\tilde{W}))_{j=1}^{\tilde{n}}$ for $\tilde{n} \le n$ be the nonempty boundary components. Clearly, for each $\Gamma_j(\tilde{W})$, $j \ge 1$, there is exactly one corresponding $\partial X_{i_j} = \Gamma_{i_j}(W)$ such that $\Gamma_j(\tilde{W}) = \partial(X_{i_j} \setminus \overline{V})$. (This mapping is injective.) We order the components of $\tilde{W}$ such that $1 \le j_1 < j_2$ if and only if $i_{j_1} < i_{j_2}$, i.e. we preserve the ordering of W.

We now define the corresponding subsets as in (3.2) and obtain $\Theta_0(\tilde{W}) = \partial V$ as well as $\Theta_j(\tilde{W}) = \Theta_{i_j}(W) \setminus \overline{V}$ for $j \ge 1$. Moreover, we choose the same corresponding rectangles as given for W by (3.5)(i), i.e. for $\Gamma_j(\tilde{W})$ with $\omega(\Gamma_j(\tilde{W})) < 1$ we define $R_j(\tilde{W}) = R_{i_j}(W)$.

From now on for notational convenience we may assume that $i_j = j$ for all $j \ge 1$. We obtain the following 'new' weights: Set $\omega(\Gamma_0(\tilde{W})) = 1$ and for $j \ge 1$

$$\omega(\Gamma_j(\tilde{W})) = \begin{cases} 1 & \text{if } \omega(\Gamma_j(W)) = 1,\\ \min\left\{\frac{|\Theta_j(W)|_*}{|\Theta_j(\tilde{W})|_*}\omega(\Gamma_j(W)), 1\right\} & \text{else.}\end{cases} \tag{3.8}$$

We note that

$$\omega(\Gamma_j(\tilde{W})) \geq \omega(\Gamma_j(W)) \quad \text{and} \quad \omega(\Gamma_j(\tilde{W}))|\Theta_j(\tilde{W})|_* \leq \omega(\Gamma_j(W))|\Theta_j(W)|_* \quad (3.9)$$

for all $j \geq 1$. To see this, it suffices to show $|\Theta_j(\tilde{W})|_* \leq |\Theta_j(W)|_*$. This follows from Lemma 3.1(iii) and the observation that by construction (recall in particular (3.2)) we have $\Gamma_j(\tilde{W}) = \partial(X_j \setminus \overline{V})$ and $\Theta_j(\tilde{W}) = \Theta_j(W) \setminus \overline{V}$.

Note that $\tilde{W}$ might not be an element of $\mathcal{W}^s_\lambda$. We now show, however, that $\tilde{W}$ can be modified to a set in $\mathcal{W}^s_\lambda$.

Lemma 3.2 *Let $\lambda \geq 0$ and $W \in \mathcal{W}^s_\lambda$. Let $\tilde{W} = (W \setminus V) \cup \partial V$ for a rectangle $V \in \mathcal{U}^s$ with $|\partial V|_\infty \geq \lambda$ and $\overline{V} \subset Q_\mu$. Then there is another rectangle $V' \in \mathcal{U}^s$ with $\overline{V} \subset \overline{V'} \subset Q_\mu$ such that $U := (W \setminus V') \cup \partial V' \in \mathcal{W}^s_\lambda$ and*

$$\|U\|_\omega \leq \|\tilde{W}\|_\omega, \quad (3.10)$$

where for both sets $\tilde{W}$, U we adjusted the weights as in (3.8).

Proof. Without restriction we can assume $\overline{V} \cap W \neq \emptyset$ as otherwise there is nothing to show. We first see that $\tilde{W}$ clearly satisfies (3.5)(i),(iv). (Recall that in (i) we take the same rectangles as for the boundary components of W.) To see (3.5)(ii) it suffices to note that for a given $\Theta_j(\tilde{W})$ with $\omega(\Gamma_j(\tilde{W})) < 1$, (3.8) implies $\omega(\Gamma_j(\tilde{W}))|\Theta_j(\tilde{W})|_* = \omega(\Gamma_j(W))|\Theta_j(W)|_*$. Possibly (3.5)(iii) or (3.5)(v) are violated, i.e. there are $\Gamma_{j_i}(\tilde{W})$, $i = 1, \ldots, k$, with $\omega(\Gamma_{j_i}(\tilde{W})) = 1$ such that $\Gamma_{j_i}(\tilde{W})$ is not rectangular or $\Gamma_{j_i}(\tilde{W}) \neq \Theta_{j_i}(\tilde{W})$ or there are sets $\Theta_{j_i}(\tilde{W})$, $i = k+1, \ldots, l$, such that for the corresponding rectangles R_{i_j} given by (3.5)(i) one has that $R_{j_i} \setminus X$ is disconnected for a suitable component X. Note that $\partial V \cap \Gamma_{j_i}(\tilde{W}) \neq \emptyset$ for $i = 1, \ldots, l$ as $W \in \mathcal{W}^s_\lambda$. So it remains to modify $\tilde{W}$ iteratively to obtain a set satisfying (3.5)(iii) and (3.5)(v).

Set $W_0 = \tilde{W}$ and $V_0 = V$. Assume $W_i = (W \setminus V_i) \cup \partial V_i \subset \tilde{W}$ has been constructed, where $V_i \in \mathcal{U}^s$ is a rectangle with $V \subset V_i$. Moreover, suppose that (3.10) holds replacing U by W_i and that W_i satisfies (3.5)(i),(ii),(iv) and

$$\Gamma(W_i) \cap \partial V_i \neq \emptyset \quad \text{for all } \Gamma(W_i) \in \mathcal{F}_i \quad (3.11)$$

for the boundary component $\partial V_i = \Gamma_0(W_i)$ with $\omega(\Gamma_0(W_i)) = 1$. Here $\mathcal{F}_i = \mathcal{F}^1_i \cup \mathcal{F}^2_i$, where $\mathcal{F}^1_i$ denotes the set of the not rectangular boundary components $\Gamma(W_i)$ with $\omega(\Gamma(W_i)) = 1$ and $\mathcal{F}^2_i$ denotes the set of boundary components for which the corresponding rectangle is disconnected. Observe that $|\partial V_i|_\infty \geq \lambda$ as $|\partial V|_\infty \geq \lambda$. If now $W_i \in \mathcal{W}^s_\lambda$ (i.e $\mathcal{F}^1_i = \mathcal{F}^2_i = \emptyset$), we stop and set $U = W_i$. Otherwise, we choose $\hat{\Gamma} \in \mathcal{F}_i$. If $\hat{\Gamma} \in \mathcal{F}^1_i$ we let $V_{i+1} \in \mathcal{U}^s$ be the smallest (closed) rectangle containing V_i and $\hat{\Gamma}$. By Lemma 3.1(v) we get $|\partial V_{i+1}|_* \leq |\partial(V_i \cup \hat{X})|_*$, where $\hat{X}$ is the component of $Q_\mu \setminus W_i$ corresponding to $\hat{\Gamma}$. Now by Lemma 3.1(iv) and (3.5)(v) we obtain

$$|\partial V_{i+1}|_* \leq |\partial(V_i \cup \hat{X})|_* \leq |\partial V_i|_* + |\hat{\Gamma}|_* = |\partial V_i|_\omega + |\hat{\Theta}|_\omega$$

for $\hat\Theta \subset \hat\Gamma$ as given by (3.2). If $\hat\Gamma \in \mathcal{F}^2_i \setminus \mathcal{F}^1_i$ we let $V_{i+1} \in \mathcal{U}^s$ be the smallest rectangle containing V_i and $\partial\hat R$, where $\hat R$ is the corresponding rectangle given by (3.5)(i). By (3.5)(ii) and $\omega_{\min} \geq \frac{1}{2}$ we derive $|\partial\hat R|_* \leq 2|\hat\Theta|_\omega$. Moreover, the fact that (3.5)(iii) holds for W, is violated for W_i and $W_i = (W\setminus V_i)\cup\partial V_i$ implies $\hat R\setminus V_i$ is disconnected. As $|\hat\Gamma|_\infty \leq 19\upsilon|\partial V_i|_\infty$ by (3.5)(iv) and thus $|\partial\hat R|_\infty \leq C\upsilon|\partial V_i|_\infty$ by (3.6), Lemma 3.1(v),(vii) then yields for υ sufficiently small

$$|\partial V_{i+1}|_* \leq |\partial(V_i\cup\hat R)|_* \leq |\partial V_i|_* + \tfrac{1}{2}|\partial\hat R|_* \leq |\partial V_i|_\omega + |\hat\Theta|_\omega.$$

Let $W_{i+1} = (W_i \setminus V_{i+1}) \cup \partial V_{i+1}$ (recall (3.7)) and adjust the weights of the boundary components of W_{i+1} as in (3.8). Recall that for all $\Gamma_j(W_{i+1})$ with $\Gamma_j(W_{i+1}) \neq \partial V_{i+1}$ we find a (unique) corresponding $\Gamma_j(W_i)$ with $\Gamma_j(W_i) \neq \partial V_i, \hat\Gamma$. By (3.9) we then derive

$$\begin{aligned}\|W_{i+1}\|_\omega &= |\partial V_{i+1}|_* + \sum\nolimits_{\Gamma_j(W_{i+1})\neq\partial V_{i+1}} \omega(\Gamma_j(W_{i+1}))|\Theta_j(W_{i+1})|_* \\ &\leq |\partial V_i|_\omega + |\hat\Theta|_\omega + \sum\nolimits_{\Gamma_j(W_i)\neq\partial V_i,\hat\Gamma} \omega(\Gamma_j(W_i))|\Theta_j(W_i)|_* = \|W_i\|_\omega.\end{aligned} \tag{3.12}$$

Consequently, (3.10) still holds and arguing as before we see that W_{i+1} satisfies (3.5)(i),(ii),(iv). Moreover, (3.5) (iii),(v) can only be violated if (3.11) holds with $\mathcal{F}_{i+1} \neq \emptyset$. We now continue with iteration step $i+1$ and observe that after a finite number of steps i^* we find a rectangle $V_{i^*} \supset V$ and a set $W_{i^*} = (W \setminus V_{i^*}) \cup \partial V_{i^*} \in \mathcal{W}^s_\lambda$ as in each step the number of boundary components decreases. Define $V' = V_{i^*}$ and $U = (W \setminus V') \cup \partial V'$.

Note that U and W_{i^*} coincide as sets in $\mathbb{R}^2$, but the weights have been obtained in a different way. Therefore, to see (3.10) it remains to show $\omega(\Gamma_j(W_{i^*})) = \omega(\Gamma_j(U))$ for all boundary components Γ_j. For $\Gamma_j = \partial V'$ it suffices to recall that $\omega(\partial V') = \omega(\partial V_i) = 1$ for all $1 \leq i \leq i^*$. If $\Gamma_j \cap \partial V' = \emptyset$ it follows from the fact that Γ_j has not been changed during the modification procedure. Otherwise, as $W_{i^*} \in \mathcal{W}^s_\lambda$ and thus boundary components of W_{i^*} with weight 1 are pairwise disjoint (see (3.5)(v)), we know that $\omega(\Gamma_j(W_{i^*})) < 1$. Let $\Theta_j \subset \Gamma_j$ as given in (3.2) and let $\Theta_j(\tilde W)$ be the component corresponding to Θ_j. Then by iterative application of (3.9) we get $\omega(\Gamma_j(\tilde W)) \leq \omega(\Gamma_j(W_{i^*})) < 1$ and thus using iteratively (3.8) we find

$$\omega(\Gamma_j(W_{i^*}))|\Theta_j|_* = \omega(\Gamma_j(W_{i^*-1}))|\Theta_j(W_{i^*-1})|_* = \ldots = \omega(\Gamma_j(\tilde W))|\Theta_j(\tilde W)|_*.$$

Consequently, again employing (3.8) we derive $\omega(\Gamma_j(W_{i^*})) = \frac{|\Theta_j(\tilde W)|_*}{|\Theta_j|_*}\omega(\Gamma_j(\tilde W)) = \omega(\Gamma_j(U))$, as desired. □

As a direct consequence of the above result, we get that sets in $\mathcal{V}^s$ can be modified such that the boundary components have rectangular form.

Corollary 3.3 *Let $W \in \mathcal{V}^s$ with connected boundary components. Then there is a subset $U \subset W$ such that $|W \setminus U| \leq c\|U\|_\infty^2$ for some $c > 0$ and all boundary components of U are rectangular and pairwise disjoint. Moreover, we have*

$$\|U\|_* \leq \|W\|_*.$$

In particular, if we introduce a weight ω corresponding to U by $\omega(\Gamma_j(U)) = 1$ for all j and define an (arbitrary) ordering of the boundary components we obtain $U \in \mathcal{W}_0^s$.

Proof. We follow the lines of the previous proof. Set $W_0 = W$ and assume $W_i \subset W$ has been constructed with $\|W_i\|_* \leq \|W\|_*$. If $W_i \in \mathcal{W}_0^s$ we stop, otherwise we find a component $\Gamma = \Gamma(W_i)$ which is not rectangular. Let $W_{i+1} = (W_i \setminus R(\Gamma)) \cup \partial R(\Gamma)$, where $R(\Gamma)$ is the smallest closed rectangle which contains Γ and all components Γ_j with $\Gamma_j \cap \Gamma \neq \emptyset$. Using Lemma 3.1(i) we clearly have $|\partial R(\Gamma)|_* \leq |\Gamma|_*$. As in the previous proof, in particular by (3.12), we then get $\|W_{i+1}\|_* \leq \|W_i\|_* \leq \|W\|_*$. We now continue with iteration step $i + 1$ and note that we find the desired set U after a finite number of iterations.

Let $(\Gamma_i)_i$ be the boundary components of U with corresponding sets $(X_i)_i$. It is elementary to see that $W \setminus U \subset \bigcup_i X_i$ and thus by the isoperimetric inequality we conclude $|W \setminus U| \leq \sum_i |X_i| \leq C \sum_i |\Gamma_i|^2 \leq C\|U\|_\infty^2$. ☐

4 Neighborhoods of boundary components

Consider $W \in \mathcal{W}_\lambda^s$, $\lambda \geq 0$. In this section we concern ourselves with neighborhoods of a boundary component $\Gamma = \Gamma(W)$ with $\omega(\Gamma) = 1$ and $|\Gamma|_\infty \geq \lambda$. This implies that Γ has rectangular shape by (3.5)(v). We begin with a rectangular neighborhood and show that essentially the neighborhood can contain at most two other 'large' boundary components. Afterwards we will introduce a dodecagonal neighborhood. The main condition which will allow us to investigate properties of the neighborhoods will be the following minimality condition for $\| \cdot \|_\omega$: We require

$$\|\tilde{W}\|_\omega \geq \|W\|_\omega \quad \text{for all rectangles } V \in \mathcal{U}^s \text{ with } \Gamma \subset \overline{V} \subset Q_\mu, \qquad (4.1)$$

where $\tilde{W} = (W \setminus V) \cup \partial V$ (recall (3.7)) and the weights are adjusted as in (3.8). In Section 5 we will see that (4.1) is one of the necessary conditions such that a trace estimate on Γ can be established (see Theorem 5.1). On the contrary, if (4.1) is violated for some $\tilde{W}$, we will show that it is convenient to replace W by $\tilde{W}$ (see case a) in the proof of Theorem 5.2).

Without restriction let $\Gamma = \partial X$ with $X = (-l_1, l_1) \times (-l_2, l_2)$ for $0 < l_2 \leq l_1$ and $l_1, l_2 \in s\mathbb{N}$.

4.1 Rectangular neighborhood

This section is devoted to the definition and properties of rectangular neighborhoods of Γ. As the technical proofs in this part are in principle not relevant to understand the proof of the main result in Section 5, they may be omitted on first reading. The essential points in this section are the definition of the neighborhood $N^t(\Gamma)$ (cf. Figure 2), the choice of the size of the neighborhoods (see (4.2) and (4.14)) and the properties that the length of ∂W in $N^t(\Gamma)$ can be controlled (see Lemma 4.1) as well as that there are at most two other 'large' boundary components (see Corollary 4.4 and Figure 3). Moreover, Lemma 4.5 shows that up to two small exceptional sets one can find a covering of the neighborhood (see Figure 4) such that on each element the projection $\|\cdot\|_\pi$ (see (4.6)) can be controlled which will be essential for a slicing argument in the proof of Theorem 5.1.

For $t \in s\mathbb{N}$ with $t \ll l_1$ we set

$$\begin{aligned} N^t(\Gamma) &:= (-t - l_1, l_1 + t) \times (-t - l_2, l_2 + t) \setminus \overline{X}, \\ N^t_{j,\pm}(\Gamma) &:= N^t(\Gamma) \cap \{\pm x_j \geq l_j\} \quad \text{for } j = 1, 2. \end{aligned}$$

(in the following we will use $\pm$ for shorthand if something holds for sets with index $+$ and $-$.) We drop Γ in the brackets if no confusion arises.

We cover $N^t_{2,\pm}$ up to a set of measure 0 with disjoint translates of a 'quasi square' $(0,\tilde{t}) \times (0,t)$, $\frac{\tilde{t}}{t} \approx 1$. If $l_2 \geq \frac{t}{2}$ we cover $N^t_{1,\pm} \setminus (N^t_{2,-} \cup N^t_{2,+})$ with translates of the rectangle $(0,t) \times (0,a)$ with $\frac{1}{2}t \leq a \leq t$. By $E^t_{\pm,\pm}$ we denote the four squares in the corners whose boundaries contain the points $(\pm l_1, \pm l_2)$, respectively. For $l_2 < \frac{t}{2}$ we cover each $N^t_{1,\pm}$ by itself, i.e. by a translate of the rectangle $(0,t) \times (0, 2t + 2l_2)$. For convenience we will often refer to these sets as 'squares' in the following. We number the squares by $Q^t_0, Q^t_1, \ldots, Q^t_n = Q^t_0$ such that $\overline{Q^t_j} \cap \overline{Q^t_{j+1}} \neq \emptyset$ for $j = 0, \ldots, n-1$ and let $J^t = \{Q^t_1, \ldots, Q^t_n\}$.

For shorthand we define $\bar{\tau} = v|\Gamma|_\infty$ for $0 < v \ll 1$ and we will assume that (possibly by passing to a smaller s)

$$\bar{\tau} = v|\Gamma|_\infty \in s\mathbb{N} \quad \text{and} \quad \bar{\tau} \gg s. \tag{4.2}$$

This assures that all the neighborhoods we consider below can be chosen as elements of $\mathcal{U}^s$. Let $(\Gamma_j)_j = (\Gamma_j(W))_j$ be the boundary components of W and $(\Theta_j)_j$ the corresponding subsets defined by (3.2). Let $(R_j)_j$ be the associated rectangles as given in (3.5)(i) and (3.5)(v), respectively. We will always add a subscript to avoid a mix up with Γ.

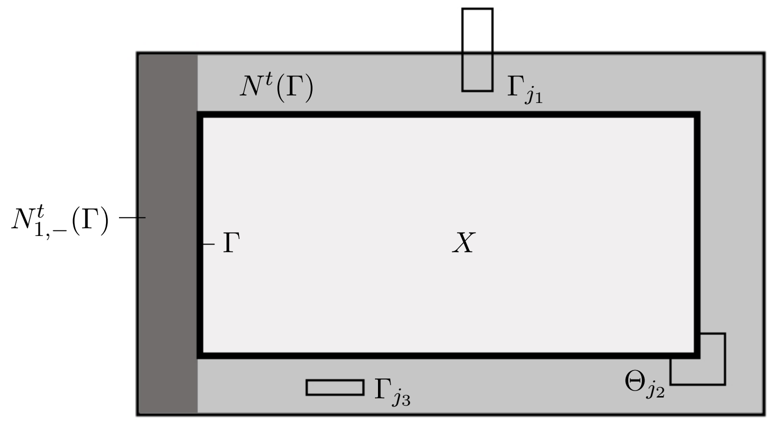

Figure 2: Neigborhood $N^t(\Gamma)$ with other small boundary components. The part $N^t_{1,-}(\Gamma)$ is colored in dark grey.

Lemma 4.1 *Let $\lambda \geq 0$ and $c > 0$. Let $W \in \mathcal{W}^s_\lambda$ and let Γ be a boundary component with $\omega(\Gamma) = 1$ and $|\Gamma|_\infty \geq \lambda$. Assume that (4.1) holds. Then there is a constant $C = C(c)$ such that*

(i) $|\partial W \cap N^t|_{\mathcal{H}} \leq C\frac{t}{h_*}$ *for all* $t \geq c\bar{\tau}$,

(ii) $|\Gamma_j \cap N^t|_{\mathcal{H}} \leq C\frac{t}{h_*}$ *for all* $t \in s\mathbb{N}$ *and all* Γ_j *with* $\omega(\Gamma_j) = 1$.

Proof. (i) Let $V = (-l_1-\hat{t}, l_1+\hat{t})\times(-l_2-\hat{t}, l_2+\hat{t}) \in \mathcal{U}^s$, where $\hat{t} = 2\max\{t, 19C\bar{\tau}\}$ with the constant C from (3.6). Define $\tilde{W} = (W\setminus V)\cup\partial V$ and adjust the weights as in (3.8). It is not hard to see that $|\partial V|_* \leq |\Gamma|_* + 8\hat{t}$.

Let $\mathcal{F}$ be the set of boundary components having nonempty intersection with N^t and let $\mathcal{G} \subset \mathcal{F}$ be the subset satisfying $\omega(\Gamma_j(W)) = 1$ for $\Gamma_j(W) \in \mathcal{G}$. By (3.5)(iv) and (3.6) we find $\Theta_j(W) \subset \partial R_j \subset V$ for $\Gamma_j(W) \in \mathcal{F}\setminus\mathcal{G}$. Recall that due to the choice of $\hat{t}$ for all $\Gamma_j(\tilde{W}) \in \mathcal{F}$ with $\Gamma_j(\tilde{W}) \neq \partial V$ we find a (unique) corresponding $\Gamma_j(W) \in \mathcal{G}$ with $\Gamma_j(W) \neq \partial V, \Gamma$. (Without restriction we take the same index.) For $\Gamma_j(W) \in \mathcal{G}$ it is elementary to see that $|\Theta_j(\tilde{W})|_* \leq |\Gamma_j(W)|_* - h_*|\Gamma_j(W)\cap V|_{\mathcal{H}}$. Consequently, using $\omega(\partial V) = \omega(\Gamma) = 1$ we derive

$$\begin{aligned}
\|\tilde{W}\|_\omega &= |\partial V|_* + \sum\nolimits_{\Gamma_j(\tilde{W})\neq\partial V} \omega(\Gamma_j(\tilde{W}))|\Theta_j(\tilde{W})|_* \\
&\leq |\Gamma|_\omega + 8\hat{t} + \sum\nolimits_{\Gamma_j(W)\in\mathcal{G}} (|\Gamma_j(W)|_* - h_*|\Gamma_j(W)\cap V|_{\mathcal{H}}) \\
&\quad + \sum\nolimits_{\Gamma_j(W)\notin\mathcal{F}} \omega(\Gamma_j(W))|\Theta_j(W)|_* \\
&\leq \|W\|_\omega + 8\hat{t} - \sum_{\Gamma_j(W)\in\mathcal{G}} h_*|\Gamma_j(W)\cap V|_{\mathcal{H}} - \sum_{\Gamma_j(W)\in\mathcal{F}\setminus\mathcal{G}} \omega(\Gamma_j(W))|\Theta_j(W)|_* \\
&\leq \|W\|_\omega + 8\hat{t} - \omega_{\min}h_*|\partial W\cap N^t|_{\mathcal{H}}.
\end{aligned}$$

For the components not being in $\mathcal{F}$ we proceeded as in (3.12). Since $\|W\|_\omega \leq \|\tilde{W}\|_\omega$ by condition (4.1) and $\omega_{\min} \geq \frac{1}{2}$, we find $|\partial W \cap N^t|_{\mathcal{H}} \leq C\frac{\hat{t}}{h_*} \leq C\frac{t}{h_*}$, where in the last step we used $t \geq c\bar{\tau}$.

(ii) We argue as in (i) with the difference that we set $\hat{t} = t$ and $\mathcal{F} = \mathcal{G} = \{\Gamma_j\}$. Then repeating the above calculation we obtain

$$\|\tilde{W}\|_\omega \leq \|W\|_\omega + 8\hat{t} - h_*|\Gamma_j \cap N^t|_{\mathcal{H}},$$

where for all other components we proceeded as in (3.12). We conclude by employing $\|W\|_\omega \leq \|\tilde{W}\|_\omega$. ☐

We now analyze the components intersecting N^t more precisely. In particular, we will show that at most two large boundary components lie in the neighborhood of Γ (see Corollary 4.4). The properties can be established by exploiting elementary geometric arguments and essential ideas of the procedure are exemplarily illustrated in Figure 3.

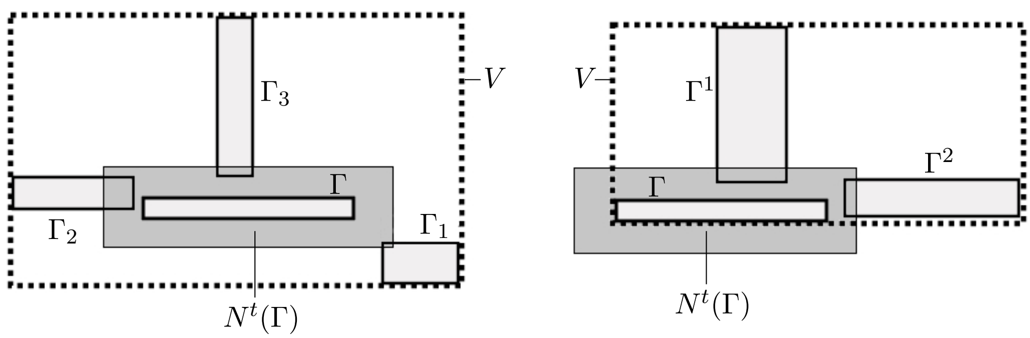

Figure 3: The left picture shows three boundary components $\Gamma_1, \Gamma_2, \Gamma_3$ intersecting $N^t(\Gamma)$. Below we argue that such a configuration violates (4.1) for $\tilde{W} = (W \setminus V) \cup \partial V$, where V is dotted rectangle. Indeed, one might have $|\partial V|_{\mathcal{H}} > |\Gamma|_{\mathcal{H}} + \sum_{k=1,2,3} |\Gamma_k|_{\mathcal{H}}$, but we can show that one always has $|\partial V|_\infty < |\Gamma|_\infty + \sum_{k=1,2,3} |\Gamma_k|_\infty$, whereby we obtain $|\partial V|_* < |\Gamma|_* + \sum_{k=1,2,3} |\Gamma_k|_*$ for h_* sufficiently small. Likewise, we can control the position of the at most two large components Γ^1, Γ^2 in $N^t(\Gamma)$: A configuration depicted on the right, where Γ^1, Γ^2 do not intersect opposite parts of $N^t(\Gamma)$, violates (4.1) for the dotted rectangle V.

We first introduce a coarser covering of N^t: Let $c\bar{\tau} \leq t \leq C\bar{\tau}$. Let $\mathcal{Y}^t$ be the union of connected sets Y having the form $Y = \left(\bigcup_{i=j}^k \overline{Q_i^t}\right)^\circ$ for $Q_i^t \in J^t$. Cover each set $N_{2,\pm}^t$ with seven sets $Y_{2,\pm}^j$ such that

$$|Y_{2,\pm}^j| \geq \bar{C}t|\Gamma|_\infty, \qquad \tfrac{1}{8}\bar{C}t|\Gamma|_\infty \leq |Y_{2,\pm}^j \cap Y_{2,\pm}^{j+1}| \leq \tfrac{1}{4}\bar{C}t|\Gamma|_\infty \tag{4.3}$$

for a constant $\bar{C} > 0$. If $l_2 \geq \frac{l_1}{2}$ we proceed likewise for $N_{1,\pm}^t$ passing possibly to a smaller constant $\bar{C}$. If $l_2 < \frac{l_1}{2}$ we cover $N_{1,\pm}^t$ by itself. Denote the covering by $\mathcal{C}^t = \mathcal{C}^t(\Gamma) = \{Y_1^t, \dots Y_m^t\}$ and order the sets in a way that $Y_i^t \cap Y_{i+1}^t \neq \emptyset$ for all $i = 1, \dots, m$, where by convention $Y_i^t = Y_{i \bmod m}^t$. In particular, (4.3) implies $Y_{i \bmod m}^t \cap Y_{j \bmod m}^t = \emptyset$ for $|i - j| \geq 2$.

This construction implies that for υ sufficiently small

$$R_j \cap Y_i^t \neq \emptyset \quad \Rightarrow \quad R_j \cap Y_{i+l}^t = \emptyset \quad \text{for } |l| \geq 3 \tag{4.4}$$

for all R_j and $i = 1, \dots, n$. To see this, we first observe that

$$|\partial R_j \cap N^t|_{\mathcal{H}} \leq Cth_*^{-1}. \tag{4.5}$$

Indeed, if $|\Gamma_j|_\infty < 19\bar{\tau}$, we obtain $|\partial R_j|_\infty \leq C\bar{\tau}$ by (3.6) and thus $|\partial R_j|_{\mathcal{H}} \leq 2\sqrt{2}|R_j|_\infty \leq Ct$. Otherwise, recalling $|\Gamma|_\infty \geq \lambda$, by (3.5)(iv),(v) we have $\partial R_j = \Gamma_j$ and thus employing Lemma 4.1(ii) we get $|\partial R_j \cap N^t|_{\mathcal{H}} \leq Cth_*^{-1}$.

If now $\operatorname{dist}(Y_i^t, Y_{i+l}^t) \geq \bar{C}|\Gamma|_\infty$ for some $|l| \geq 3$, (4.4) follows as $|\partial R_j \cap N^t|_{\mathcal{H}} \leq Cth_*^{-1} \ll \bar{C}|\Gamma|_\infty$ for υ small enough (depending on h_*).

On the other hand, suppose $\operatorname{dist}(Y_i^t, Y_{i+l}^t) \leq \bar{C}|\Gamma|_\infty$. This is only possible in the case $l_2 \leq \frac{l_1}{2}$ if (up to interchanging $+$ and $-$) $Y_i^t \subset N_{2,+}^t \setminus (N_{1,-}^t \cup N_{1,+}^t)$, $Y_{i+l}^t \subset N_{2,-}^t$ and $\operatorname{dist}(Y_i^t, N_{1,\pm}^t) \geq c|\Gamma|_\infty$ or $\operatorname{dist}(Y_{i+l}^t, N_{1,\pm}^t) \geq c|\Gamma|_\infty$. Now assume that (4.4) was wrong. Then by (4.3) and $|\partial R_j \cap N^t|_{\mathcal{H}} \leq Cth_*^{-1} \ll \bar{C}|\Gamma|_\infty$ this would imply $R_j \cap N_{2,\pm}^t \neq \emptyset$ and $R_j \cap (N_{1,-}^t \cup N_{1,+}^t) = \emptyset$. But then we would get that $R_j \setminus X$ is not connected which contradicts (3.5)(iii).

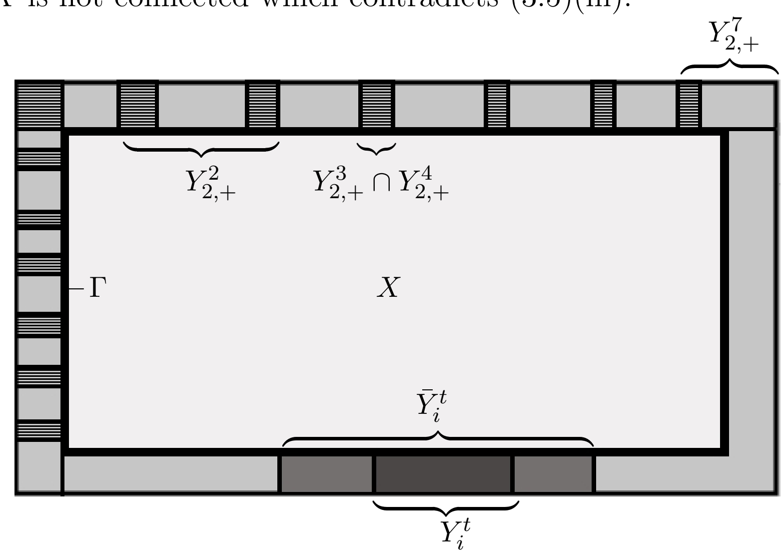

Figure 4: On the upper left side of the neigborhood $N^t(\Gamma)$ one can see elements of the partition $\mathcal{C}^t(\Gamma)$ (which are not necessarily of the same size). The sets where two elements overlap are striped. In the lower part an element Y_i^t and the corresponding enlarged set $\bar{Y}_i^t$ are highlighted.

For $Y \subset N^t$ we set $\mathcal{R}(Y) = \{R_j : R_j \cap Y \neq \emptyset\}$ and define

$$|\partial R_j|_\pi = \min\{|\partial R_j|_\infty,\ t - \max\nolimits_{i=1,2} \operatorname{dist}(\pi_i R_j, \pi_i \Gamma)\} \tag{4.6}$$

for all $R_j \in \mathcal{R}(N^t)$. It is obvious that $|\partial R_j|_\pi \leq |\partial R_j|_\infty$. For a set $Y \subset N^t$ we then define $\|Y\|_\pi = \sum_{R_j \in \mathcal{R}(Y)} |\partial R_j|_\pi$. The projection $\|\cdot\|_\pi$ is one essential object we will need to apply a slicing argument in the investigation of the jump heights in Section 6.

Remark 4.2 We have already introduced the (small) parameters $h_*, 1-\omega_{\min}, \upsilon$. In the following sections we will additionally consider q, r. The subsequent lemmas will hold if we choose the involved parameters sufficiently small. To avoid confusion about the relation of the different parameters, we state at this point that the parameters can be chosen in the order $h_*, q, 1-\omega_{\min}, r, \upsilon$. In what follows, we will not always repeat the relation of the parameters for convenience.

We now show that we can control $\|\cdot\|_\pi$ in a suitable way. For that purpose, for a set $Y_i^t \in \mathcal{C}^t$ we define

$$\bar{Y}_i^t = \overline{\bigcup_{|l|\leq 1} Y_{i+l}^t}.$$

Lemma 4.3 *Let $\lambda \geq 0$, $W \in \mathcal{W}_\lambda^s$. Let Γ be a boundary component with $\omega(\Gamma) = 1$ and $|\Gamma|_\infty \geq \lambda$. Assume that (4.1) holds. Let $c\bar{\tau} \leq t \leq C\bar{\tau}$. If we choose h_*, υ and $1-\omega_{\min}$ small enough, there are two sets $Y^1, Y^2 \in \mathcal{C}^t$ such that $\|Y^t\|_\pi \leq \frac{19}{20}t$ for all $Y^t \in \mathcal{C}^t$ with $Y^t \cap (\bar{Y}^1 \cup \bar{Y}^2) = \emptyset$.*
Additionally, if $\|Y^1\|_\pi, \|Y^2\|_\pi \geq \frac{19}{20}t$, then $\bar{Y}^1 \cup \bar{Y}^2$ intersects both $N_{1,+}^t$ and $N_{1,-}^t$ or both $N_{2,+}^t$ and $N_{2,-}^t$. If $l_2 \leq \frac{l_1}{2}$, then $\bar{Y}^1 \cup \bar{Y}^2$ intersects $N_{1,+}^t$ and $N_{1,-}^t$.

We briefly remark that by similar arguments the additional statement can also be proved without the extra assumption $\|Y^1\|_\pi, \|Y^2\|_\pi \geq \frac{19}{20}t$. We omit the proof of this fact here as we will not need it in the following.
Proof. For convenience we drop the superscript t in the following proof. We proceed in two steps:

In a) we first show that it is not possible that there are three sets $Y^1, Y^2, Y^3 \in \mathcal{C}$ such that $\bar{Y}^k \cap Y^l = \emptyset$ if $k \neq l$ and $\|Y^k\|_\pi > \frac{19}{20}t$ for $k, l = 1, 2, 3$. Provided that a) is proven we can then select the two desired sets Y^1, Y^2 as follows:

(1) If $\|Y\|_\pi \leq \frac{19}{20}t$ for all $Y \in \mathcal{C}$, we can choose arbitrary sets Y^1, Y^2 satisfying the additional condition. Otherwise, we can assume that there is some Y^* with $\|Y^*\|_\pi > \frac{19}{20}t$.

(2) If $\|Y\|_\pi \leq \frac{19}{20}t$ for all $Y \in \mathcal{C}^t$ with $Y \cap \bar{Y}^* = \emptyset$, we set $Y^1 = Y^*$ and choose Y^2 arbitrarily such that the additional condition holds.

(3) Otherwise, we set $Y^1 = Y^*$ and choose Y^2 with $\|Y^2\|_\pi > \frac{19}{20}t$ and $Y^2 \cap \bar{Y}^* = \emptyset$. Now a) indeed shows that $\|Y\|_\pi \leq \frac{19}{20}t$ for all $Y \in \mathcal{C}^t$ with $Y \cap (\bar{Y}^1 \cup \bar{Y}^2) = \emptyset$.

In step b) we concern ourselves with the additional assertions on the position of $\bar{Y}^1 \cup \bar{Y}^2$ in case (3).

a) Suppose that there are three sets $Y^1, Y^2, Y^3 \in \mathcal{C}$ such that $\bar{Y}^k \cap Y^l = \emptyset$ if $k \neq l$ and $\|Y^k\|_\pi > \frac{19}{20}t$ for $k, l = 1, 2, 3$. First note that the assumption implies that if e.g. $Y^1 = Y_i$, then $Y^2, Y^3 \notin \{Y_{i-2}, \ldots, Y_{i+2}\}$. Let V be the smallest rectangle containing Γ and the sets $\mathcal{R} := \bigcup_{k=1}^3 \mathcal{R}(Y^k)$. Define $\tilde{W} = (W \setminus V) \cup \partial V$ (recall (3.7)). Similarly as in (3.12) we intend to estimate $\|\tilde{W}\|_\omega$. To this end, we have to control the difference of $|\partial R_j|_*$ and $|\Theta_j|_\omega$ for $R_j \in \mathcal{R}$. By (3.5)(ii),(iv),(v) and (3.6) we have

$$|\Theta_j|_\omega = \omega(\Gamma_j)|\Theta_j|_* \geq \begin{cases} \omega_{\min}|\partial R_j|_* & |\partial R_j|_\infty \leq 19C\upsilon\lambda, \\ |\partial R_j|_* & \text{else,} \end{cases}$$

with the constant C from (3.6). For notational convenience we define $\|W\|_{\omega,\mathcal{R}} = \|W\|_\omega + \sum_{R_j \in \mathcal{R}}(|\partial R_j|_* - |\Theta_j|_\omega) = \sum_{R_j \notin \mathcal{R}} |\Theta_j|_\omega + \sum_{R_j \in \mathcal{R}} |\partial R_j|_*$. We get

$$\begin{aligned} \|W\|_{\omega,\mathcal{R}} &\leq \|W\|_\omega + \sum\nolimits_{R_j \in \mathcal{R},\, |\partial R_j|_\infty < 19C\upsilon\lambda} (\omega_{\min}^{-1} - 1)|\Theta_j|_\omega \\ &\leq \|W\|_\omega + (\omega_{\min}^{-1} - 1)\mathcal{H}^1(N^{t+C\bar{\tau}} \cap \partial W) \\ &\leq \|W\|_\omega + C\tfrac{t}{h_*}(\omega_{\min}^{-1} - 1). \end{aligned} \tag{4.7}$$

In the second step we used $|\cdot|_* \leq |\cdot|_{\mathcal{H}}$ and in the last step we applied Lemma 4.1(i). We will show below that

$$|\partial V|_\infty \leq |\Gamma|_\infty + \sum\nolimits_{R_j \in \mathcal{R}} |\partial R_j|_\infty - \tfrac{1}{50}t. \tag{4.8}$$

Moreover, it is not hard to see that $|\partial V|_{\mathcal{H}} \leq |\Gamma|_{\mathcal{H}} + \sum_{R_j \in \mathcal{R}} |\partial R_j|_{\mathcal{H}} + 8t$. Then recalling $\|W\|_{\omega,\mathcal{R}} = \sum_{R_j \notin \mathcal{R}} |\Theta_j|_\omega + \sum_{R_j \in \mathcal{R}} |\partial R_j|_*$ and arguing as in (3.12) we get $\|\tilde{W}\|_\omega - \|W\|_{\omega,\mathcal{R}} \leq -(1 - h_*)\frac{t}{50} + h_* 8t$. Consequently, for h_* small enough we get $\|\tilde{W}\|_\omega - \|W\|_{\omega,\mathcal{R}} < -\frac{t}{100}$ and thus by (4.7) we derive $\|\tilde{W}\|_\omega - \|W\|_\omega < 0$ for $1 - \omega_{\min}$ sufficiently small (with respect to h_*). This gives a contradiction to (4.1) and concludes the proof of a).

We now proceed to show (4.8). Assume $V = (-a_{1,-} - l_1, l_1 + a_{1,+}) \times (-a_{2,-} - l_2, l_2 + a_{2,+})$ and select (not necessarily pairwise different) $R_{k,\pm} \in \mathcal{R}$ such that $\pm(l_k + a_{k,\pm}) \in \pi_k \partial R_{k,\pm}$ for $k = 1, 2$. (If $a_{k,\pm} = 0$ then $R_{k,\pm} = \emptyset$.) We find by (4.6)

$$|\pi_k R_{k,\pm}| \geq a_{k,\pm} - \operatorname{dist}(\pi_k R_{k,\pm}, \pi_k \Gamma) \geq a_{k,\pm} - t + |\partial R_{k,\pm}|_\pi. \tag{4.9}$$

We suppose for the moment that $R_{k,+} \neq R_{k,-}$ for $k = 1, 2$. (In particular, this implies that three rectangles never coincide.) At the end of the proof we will briefly indicate how the following arguments can be adapted to the general case. We first assume that two rectangles coincide, e.g. $R = R_{1,-} = R_{2,-}$. By (4.9) and an elementary computation we obtain

$$\sqrt{a_{1,-}^2 + a_{2,-}^2} \leq |\partial R|_\infty + \sqrt{2}(t - |\partial R|_\pi) \leq |\partial R|_\infty + \sqrt{2}t - |\partial R|_\pi.$$

Otherwise, if e.g. $R_{1,-} \neq R_{2,-}$, again applying (4.9) we get

$$\begin{aligned}\sqrt{a_{1,-}^2 + a_{2,-}^2} &\leq \sqrt{(t + (|\partial R_{1,-}|_\infty - |\partial R_{1,-}|_\pi))^2 + (t + (|\partial R_{2,-}|_\infty - |\partial R_{2,-}|_\pi))^2} \\ &\leq \sqrt{2}t + |\partial R_{1,-}|_\infty - |\partial R_{1,-}|_\pi + |\partial R_{2,-}|_\infty - |\partial R_{2,-}|_\pi.\end{aligned}$$

Consequently, we obtain

$$F \leq 2\sqrt{2}t + \sum\nolimits_{k,\pm} (|\partial R_{k,\pm}|_\infty - |\partial R_{k,\pm}|_\pi), \tag{4.10}$$

where each rectangle is only counted once in the sum and

$$F = \sqrt{a_{1,-}^2 + a_{2,-}^2} + \sqrt{a_{1,+}^2 + a_{2,+}^2} \quad \text{or} \quad F = \sqrt{a_{1,-}^2 + a_{2,+}^2} + \sqrt{a_{1,+}^2 + a_{2,-}^2}.$$

Moreover, note that by assumption and (4.4) each boundary component R_j intersects at most one of the three sets Y^k. Therefore, as $|\cdot|_\infty \geq |\cdot|_\pi$ we obtain

$$\begin{aligned}|\partial V|_\infty &\leq |\Gamma|_\infty + F \leq |\Gamma|_\infty + 2\sqrt{2}t + \sum\nolimits_{R_j \in \mathcal{R}} (|\partial R_j|_\infty - |\partial R_j|_\pi) \\ &= |\Gamma|_\infty + \sum\nolimits_{R_j \in \mathcal{R}} |\partial R_j|_\infty + 2\sqrt{2}t - \sum\nolimits_{k=1}^{3} \|Y^k\|_\pi.\end{aligned} \tag{4.11}$$

As $\sum_k \|Y^k\|_\pi \geq 3 \cdot \frac{19}{20}t$ this gives (4.8).

b) Suppose that $Y^1, Y^2 \in \mathcal{C}$ with $\|Y^1\|_\pi, \|Y^2\|_\pi \geq \frac{19}{20}t$ have been chosen according to case (3) above. We show that $\bar{Y}^1 \cup \bar{Y}^2$ intersect both $N_{1,+}$ and $N_{1,-}$ or both $N_{2,+}$ and $N_{2,-}$ ($N_{1,\pm}$ if $l_2 \leq \frac{l_1}{2}$). Let V be the smallest rectangle containing Γ and the sets $\mathcal{R} := \bigcup_{k=1}^2 \mathcal{R}(Y^k)$. Set $\tilde{W} = (W \setminus V) \cup \partial V$. As before we define for convenience $\|W\|_{\omega,\mathcal{R}} = \|W\|_\omega + \sum_{R_j \in \mathcal{R}} (|\partial R_j|_* - |\Theta_j|_\omega)$ and note that (4.7) holds. Observe that by assumption and (4.4) each boundary component R_j intersects at most one of the two sets Y^k.

(i) First we assume $\bar{Y}_1 \cup \bar{Y}_2$ intersects at most two adjacent parts of the neighborhood, e.g. $(\bar{Y}_1 \cup \bar{Y}_2) \cap (N_{1,+}^t \cup N_{2,+}^t) = \emptyset$. This implies $V = (-a_1 - l_1, l_1) \times (-a_2 - l_2, l_2)$. Selecting (not necessarily different) R_1, R_2 such that $-l_k - a_k \in \pi_k \partial R_k$ for $k = 1, 2$ and proceeding as in (4.10) we obtain

$$\sqrt{a_1^2 + a_2^2} \leq \sqrt{2}t + \sum\nolimits_{k=1,2} (|R_k|_\infty - |R_k|_\pi)$$

and therefore

$$|\partial V|_\infty \leq |\Gamma|_\infty + \sum\nolimits_{R_j \in \mathcal{R}} |\partial R_j|_\infty + \sqrt{2}t - \sum\nolimits_{k=1,2} \|Y^k\|_\pi. \tag{4.12}$$

Moreover, we have $|\partial V|_\mathcal{H} \leq |\Gamma|_\mathcal{H} + 4t + \sum_{R_j \in \mathcal{R}} |\partial R_j|_\mathcal{H}$ which together with $\sum_k \|Y^k\|_\pi \geq 2 \cdot \frac{19}{20}t$ implies $\|\tilde{W}\|_\omega - \|W\|_{\omega,\mathcal{R}} \leq -\frac{t}{100}$ for h_* small enough. Recalling (4.7) we again obtain a contradiction to (4.1) for $1 - \omega_{\min}$ sufficiently small.

(ii) We finally show the additional statement that $\bar{Y}_1 \cup \bar{Y}_2$ intersects $N^t_{1,\pm}$ in the case $l_2 \le \frac{l_1}{2}$. Assume without restriction that $(\bar{Y}_1 \cup \bar{Y}_2) \cap N^t_{1,+}(\Gamma) = \emptyset$. Then we have $V = (-a_1 - l_1, l_1) \times (-a_- - l_2, l_2 + a_+)$ and select rectangles R_1, R_-, R_+ as before. (For the moment we assume $R_{k,+} \neq R_{k,-}$ for $k = 1, 2$.) Similarly as in a), we find

$$\sqrt{a_1^2 + a_-^2} \le \sqrt{2}t + \sum\nolimits_{k=1,-}(|\partial R_k|_\infty - |\partial R_k|_\pi), \quad a_+ \le t + |\partial R_+|_\infty - |\partial R_+|_\pi.$$

Then

$$|\partial V|_\infty = \max\nolimits_{c\in[0,1]} \big(\sqrt{1-c^2}|\pi_1\Gamma| + c|\pi_2\Gamma| + \sqrt{1-c^2}a_1 + c(a_+ + a_-)\big).$$

We define $f(c) = \frac{2}{\sqrt{5}}(\sqrt{1-c^2} + \frac{1}{2}c)$ for $c \ge \frac{1}{\sqrt{5}}$ and $f(c) = 1$ else. As $l_2 \le \frac{l_1}{2}$ an elementary argument yields $\sqrt{1-c^2}|\pi_1\Gamma| + c|\pi_2\Gamma| \le f(c)|\Gamma|_\infty$. Using $|\Gamma|_\infty \ge (C\upsilon)^{-1}t$ we then obtain

$$\begin{aligned}|\partial V|_\infty &\le \max\nolimits_{c\in[0,1]} \big(f(c)|\Gamma|_\infty + \sqrt{2}t + ct + \sum\nolimits_{k=1,\pm}(|\partial R_k|_\infty - |\partial R_k|_\pi)\big) \\ &\le |\Gamma|_\infty + t\max\nolimits_{c\in[0,1]} r(c) + \sum\nolimits_{R_j\in\mathcal{R}} |\partial R_j|_\infty - \sum\nolimits_{k=1,2} \|Y^k\|_\pi,\end{aligned} \tag{4.13}$$

where $r(c) = (f(c)-1)(C\upsilon)^{-1} + \sqrt{2} + c$. A computation yields $r(c) \le \frac{19}{10} - \frac{1}{100}$ for $c \le \sqrt{\frac{9}{40}}$ as $f \le 1$. Otherwise we have $\max_{[\sqrt{\frac{9}{40}},1]}(f(c)-1) < 0$ and thus for υ sufficiently small we also obtain $r(c) \le \frac{19}{10} - \frac{1}{100}$ for $c \in [\sqrt{\frac{9}{40}}, 1]$. Moreover, we have $|\partial V|_{\mathcal{H}} \le |\Gamma|_{\mathcal{H}} + 6t + \sum_{R_j\in\mathcal{R}} |\partial R_j|_{\mathcal{H}}$ which together with $\sum_k \|Y^k\|_\pi \ge 2\cdot\frac{19}{20}t$ implies $\|\tilde{W}\|_\omega - \|W\|_\omega < 0$ for $h_*, 1-\omega_{\min}$ small enough. This again yields the desired contradiction.

To finish the proof we briefly indicate how to proceed if e.g. $R_{2,-} = R_{2,+}$. This may happen in the cases a) and b)ii) above if $l_2 \ll l_1$. In this case we reduce the problem to the above treated situation by applying a translation argument: We replace R_j by $R'_j := R_j - a_{2,+}\mathbf{e}_2$ for all $R_j \in \mathcal{R}$ as well as V by $V' := V - a_{2,+}\mathbf{e}_2$. Then we may set $R_{2,+} = \emptyset$ and can repeat the arguments above to derive (4.11) and (4.13), respectively, for V' and $R'_{k,\pm}$. But then (4.11) and (4.13) also hold for the original sets V and $R_j \in \mathcal{R}$ as $|V'|_* = |V|$ and $|R'_{k,\pm}|_* = |R_{k,\pm}|_*$. Consequently, we may then proceed as before and can employ (4.7) to derive a contradiction to (4.1). □

As a corollary we obtain that at most two large boundary components lie in the neighborhood of Γ.

Corollary 4.4 *Let $\lambda \ge 0$, $W \in \mathcal{W}^s_\lambda$. Let Γ be a boundary component with $\omega(\Gamma) = 1$ and $|\Gamma|_\infty \ge \lambda$. Assume that (4.1) holds. Let $\bar{\tau} \le \bar{t} \le C\bar{\tau}$. Then for h_*, υ and $1-\omega_{\min}$ small enough there are at most two boundary components Γ_1 and*

Γ_2 *with* $|\Gamma_i|_\infty \geq 19\bar{t}$ *having nonempty intersection with* $N^{\bar{t}}$.
If Γ_1, Γ_2 *exist,* $\Gamma_1 \cup \Gamma_2$ *intersects both* $N^{\bar{t}}_{1,+}$ *and* $N^{\bar{t}}_{1,-}$ *or both* $N^{\bar{t}}_{2,+}$ *and* $N^{\bar{t}}_{2,-}$. *Additionally, if* $l_2 \leq \frac{l_1}{2}$ *then* $\Gamma_1 \cup \Gamma_2$ *intersects both* $N^{\bar{t}}_{1,+}$, $N^{\bar{t}}_{1,-}$ *and* $|\pi_1\Gamma_k| \geq \frac{1}{2}|\pi_2\Gamma_k|$ *for* $k = 1, 2$.

We remark that the additional statement $|\pi_1\Gamma_k| \geq \frac{1}{2}|\pi_2\Gamma_k|$ also holds if only one Γ_k exists.
Proof. Let $\bar{\tau} \leq \bar{t} \leq C\bar{\tau}$ be given and assume that there are three components Γ_k, $k = 1, 2, 3$, intersecting $N^{\bar{t}}$ with $|\Gamma_k|_\infty \geq 19\bar{t}$. By (3.5)(iv),(v), (4.2) and $\bar{t} \geq \bar{\tau}$ we see that Γ_k are rectangular with $\omega(\Gamma_k) = 1$ for $k = 1, 2, 3$. Set $t = 20\bar{t}$ and recalling (4.6) we observe that $|\Gamma_k|_\pi \geq 19\bar{t} = \frac{19}{20}t$. We now may follow the lines of the proof of Lemma 4.3 with the essential difference that we replace the set of rectangles $\mathcal{R} = \bigcup_{k=1}^3 \mathcal{R}(Y^k)$ (see beginning of step a)) by $\mathcal{R} = \{\Gamma_1\} \cup \{\Gamma_2\} \cup \{\Gamma_3\}$ and in (4.11) we replace $\sum_{k=1}^3 \|Y^k\|_\pi$ by $\sum_{k=1}^3 |\Gamma_k|_\pi$. Noting that $\sum_{k=1}^3 |\Gamma_k|_\pi \geq 3 \cdot \frac{19}{20}t$ we again obtain a contradiction to (4.1) and thus there are at most two large components Γ_k, $k = 1, 2$, in $N^{\bar{\tau}}$. Likewise, we can proceed to determine the possible position of the two sets.

It remains to show that $|\pi_1\Gamma_k| < \frac{1}{2}|\pi_2\Gamma_k|$ leads to a contradiction if $l_2 \leq \frac{l_1}{2}$. Let V be the smallest rectangle containing Γ, Γ_k and derive similarly as in (4.13)

$$\begin{aligned}|\partial V|_\infty &\leq \max\nolimits_{c\in[0,1]} \left(\sqrt{1-c^2}|\pi_1\Gamma| + c|\pi_2\Gamma| + \sqrt{1-c^2}(\bar{\tau} + |\pi_1\Gamma_k|) + c(\bar{\tau} + |\pi_2\Gamma_k|)\right) \\ &\leq \max\nolimits_{c\in[0,1]} \left(f(c)|\Gamma|_\infty + \sqrt{2}\bar{\tau} + \sqrt{1-c^2}|\pi_1\Gamma_k| + c|\pi_2\Gamma_k|)\right),\end{aligned}$$

where we used $\sqrt{1-c^2}|\pi_1\Gamma| + c|\pi_2\Gamma| \leq f(c)|\Gamma|_\infty$ due to the fact that $|\pi_2\Gamma| \leq \frac{1}{2}|\pi_2\Gamma|$. Likewise, we use the assumption $|\pi_1\Gamma_k| < \frac{1}{2}|\pi_2\Gamma_k|$ to find $\sqrt{1-c^2}|\pi_1\Gamma_k| + c|\pi_2\Gamma_k| \leq f(\sqrt{1-c^2})|\Gamma_k|_\infty$ and thus obtain

$$|\partial V|_\infty \leq |\Gamma|_\infty + |\Gamma_k|_\infty + \sqrt{2}\bar{\tau} + \max_{c\in[0,1]} \left((f(c)-1)\upsilon^{-1}\bar{\tau} + (f(\sqrt{1-c^2})-1)19\bar{t}\right),$$

where we used $|\Gamma|_\infty \geq \upsilon^{-1}\bar{\tau}$ and $|\Gamma|_\infty \geq 19\bar{t}$. Again separating the cases $c \geq \sqrt{\frac{9}{40}}$, where $\max_{[\sqrt{\frac{9}{40}},1]}(f(c)-1) < 0$, and $c \leq \sqrt{\frac{9}{40}}$, where $(f(\sqrt{1-c^2})-1)19\bar{\tau} \leq -3\bar{t}$, we obtain for υ small enough $|\partial V|_\infty \leq |\Gamma|_\infty + |\Gamma_k|_\infty - \bar{\tau}$. As $|\partial V|_{\mathcal{H}} \leq |\Gamma|_{\mathcal{H}} + 4\bar{t} + |\Gamma_k|_{\mathcal{H}}$ we derive $\|\tilde{W}\|_\omega - \|W\|_\omega < 0$ for h_* small enough, where $\tilde{W} = (W \setminus V) \cup \partial V$. This gives a contradiction to (4.1) and finishes the proof. □

We now use Corollary 4.4 for $\bar{t} = \bar{\tau}$ to find (at most) two Γ_i, $i = 1, 2$, with $|\Gamma_1|_\infty, |\Gamma_2|_\infty \geq 19\bar{\tau}$ intersecting $N^{\bar{\tau}}$. We can choose

$$\tfrac{1}{800}\bar{\tau} \leq \tau \leq \tfrac{1}{2}\bar{\tau} \tag{4.14}$$

in such a way that the neighborhood $N^\tau = N^\tau(\Gamma)$ satisfies $\Gamma_i \cap N^{\tau/20} \neq \emptyset$ or $\Gamma_i \cap N^\tau = \emptyset$ for $i = 1, 2$: If $\Gamma_1, \Gamma_2 \cap N^{\bar{\tau}/800} \neq \emptyset$ choose $\tau = \frac{\bar{\tau}}{2}$, if $\Gamma_1, \Gamma_2 \cap N^{\bar{\tau}/800} = \emptyset$ choose $\tau = \frac{\bar{\tau}}{800}$, otherwise choose either $\tau = \frac{\bar{\tau}}{2}$ or $\tau = \frac{1}{40}\bar{\tau}$. For shorthand we set

$N = N^\tau(\Gamma)$, $N_{j,\pm} = N^\tau(\Gamma)_{j,\pm}$, $J = J^\tau = \{Q_0, \dots, Q_n\}$, $\mathcal{Y} = \mathcal{Y}^\tau$ and $\mathcal{C} = \mathcal{C}^\tau = \{Y_1, \dots, Y_m\}$ (recall the constructions before (4.2) and Lemma 4.3).

We are now in a position to formulate the main lemma of this section.

Lemma 4.5 *Let $\lambda \geq 0$, $W \in \mathcal{W}^s_\lambda$. Let Γ be a boundary component with $\omega(\Gamma) = 1$ and $|\Gamma|_\infty \geq \lambda$. Assume that* (4.1) *holds. Choosing h_*, υ and $1-\omega_{\min}$ small enough we obtain sets $K_1, K_2 \in \mathcal{Y}$ with $|K_j| \leq C\frac{\tau^2}{h_*}$, $j = 1, 2$, and* $\mathrm{dist}(K_1, K_2) \geq c|\Gamma|_\infty$ *for some $c > 0$ small enough such that*

(i) The covering $\{\hat{Y}_1, \dots, \hat{Y}_k\}$ of $N \setminus (K_1 \cup K_2)$ consisting of the connected components of $\{Y \setminus (K_1 \cup K_2) : Y \in \mathcal{C}\}$, satisfies $\|\hat{Y}_i\|_\pi \leq \frac{19}{20}\tau$ for all $i = 1, \dots, k$.

(ii) $\Gamma_i \cap N \subset K_1 \cup K_2$ for all components Γ_i with $|\Gamma_i|_\infty \geq 19\bar{\tau}$.

Proof. By Lemma 4.3 we obtain that there are two sets $Y^1, Y^2 \in \mathcal{C}$ with $\bar{Y}^1 \cap \bar{Y}^2 = \emptyset$ such that $\|Y\|_\pi \leq \frac{19}{20}\tau$ for $Y \in \mathcal{C}$ with $Y \cap (\bar{Y}^1 \cup \bar{Y}^2) = \emptyset$. We only construct the set K_1. Choose $Y_i = Y^1$ and set $S_l = Y_{i+l} \in \mathcal{C}$ for $|l| \leq 3$. In particular, we have $S_l \cap S_0 \neq \emptyset$ for $l = -1, 1$ and $\|S_l\|_\pi \leq \frac{19}{20}\tau$ for $l = -3, 3$. Set $S = \bigcup_{l=-2}^{2} S_l$.

Arguing as in (4.12) or (4.13) for $t = \tau$, respectively, depending on whether S is contained in at most two adjacent parts of the neighborhood or S intersects three parts of the neighborhood (possible for $l_2 \leq \frac{l_1}{2}$), we derive $|\partial V|_\infty \leq |\Gamma|_\infty + \sum_{R_j \in \mathcal{R}(S)} |\partial R_j|_\infty + (\frac{19}{10} - \frac{1}{100})\tau - \|S\|_\pi$, where V is the smallest rectangle containing Γ and $\mathcal{R}(S)$. (Note that in the above calculation we possibly have to repeat the translation argument indicated at the end of the proof of Lemma 4.5.) Thus, arguing as in the proof of Lemma 4.3, in particular taking (4.7) and condition (4.1) into account, we find

$$0 \leq \|\tilde{W}\|_\omega - \|W\|_\omega \leq (1 - h_*)\tfrac{19}{10}\tau - (1 - h_*)\|S\|_\pi \tag{4.15}$$

for h_*, $1 - \omega_{\min}$ small enough, where $\tilde{W} = (W \setminus V) \cup \partial V$. We now construct the set K_1 and the corresponding (at most) two connected components T_1, T_2 of $S \setminus K_1$ by distinction of the two following cases:

a) If there is some R_j with $|\partial R_j|_\pi \geq \frac{19}{20}\tau$ we choose $K_1 \in \mathcal{Y}$ as the smallest set such that $R_j \cap N \subset K_1$. Then the (at most) two connected components T_1, T_2 of $S \setminus K_1$ satisfy $\|T_i\|_\pi \leq \frac{19}{20}\tau$ by (4.15). Using (4.5) we derive that $|K_1| \leq C\frac{\tau^2}{h_*}$, as desired.

b) Otherwise, we choose K_1 as follows. Assume $S = (\bigcup_{i=1}^{n'} \overline{Q_i})^\circ$ for $Q_i \in J$ and let $k \in \{0, \dots, n'\}$ be the index (if existent) such that $\|(\bigcup_{i=1}^{k} \overline{Q_i})^\circ\|_\pi \leq \frac{19}{20}\tau$ and $\|(\bigcup_{i=1}^{k+1} \overline{Q_i})^\circ\|_\pi > \frac{19}{20}\tau$. Now define $T_1 = (\bigcup_{i=1}^{k} \overline{Q_i})^\circ$ and choose $K_1 = (\bigcup_{i=k+1}^{l} \overline{Q_i})^\circ$ for l large enough such that $|K_1| \geq \bar{c}\frac{\tau^2}{h_*}$. Finally, let $T_2 = S \setminus (\overline{T_1 \cup K_1})$ and observe that for $\bar{c}$ large enough also $\|T_2\|_\pi \leq \frac{19}{20}\tau$ by (4.15) and (4.5) since each rectangle can intersect at most one of the sets T_1, T_2.

Let S_l^1, S_l^2 be the connected components of $S_l \backslash K_1$ for $l = -2, -1, 0, 1, 2$. Both cases a),b) above imply $\|S_l^i \setminus K_1\|_\pi \le \max_{k=1,2} \|T_k\|_\pi \le \frac{19}{20}\tau$ for $l = -2, \ldots, 2$, $i = 1, 2$, which gives assertion (i). Assertion (ii) follows from the construction of the set K_1 and definition (4.14). Indeed, if $\Gamma_i \cap N \neq \emptyset$, then $\Gamma_i \cap N^{\tau/20}$ and thus recalling (4.6) we find $|\Gamma_i|_\pi \ge \frac{19}{20}\tau$ and then $\Gamma_i \cap N \subset \bar{Y}^1 \cup \bar{Y}^2$. Finally, $\mathrm{dist}(K_1, K_2) \ge c|\Gamma|_\infty$ follows directly from the fact that in the case $\|Y^1\|_\pi, \|Y^2\|_\pi \ge \frac{19}{20}\tau$ the set $\bar{Y}^1 \cup \bar{Y}^2$ intersects both $N_{1,+}$ and $N_{1,-}$ or both $N_{2,+}$ and $N_{2,-}$ ($N_{1,\pm}$ if $l_2 \le \frac{l_1}{2}$). □

4.2 Dodecagonal neighborhood

We now introduce neighborhoods of Γ which in general have dodecagonal shape and differ from $N^t(\Gamma)$ near the corners of Γ. These neighborhoods will be essential in the modification algorithm below (see Section 5.2) as we have to treat the modification near the corners of a boundary component with special care. For $t > 0$ we define

$$\hat{M}^t(\Gamma) = \bigcup_{i=1,2} \{x \in N^t(\Gamma) : \ |x_i + l_i| \ge q h_*^{-1} t, |x_i - l_i| \ge q h_*^{-1} t\} \qquad (4.16)$$

for $q \gg 1$ to be specified below. Moreover, for $\tilde{l} = l_1 + \min\{t, q^{-1} h_* l_2\}$ let

$$M^t(\Gamma) := \mathrm{co}(\hat{M}^t(\Gamma) \cup \Gamma \cup (\tilde{l}, 0) \cup (-\tilde{l}, 0)) \cap N^t(\Gamma), \qquad (4.17)$$

where $\mathrm{co}(\cdot)$ denotes the convex hull of a set. Observe that $M^t(\Gamma) \supset \hat{M}^t(\Gamma)$ and that $M^t(\Gamma)$, $\hat{M}^t(\Gamma)$ differ by some triangles. Moreover, the shape of $M^t(\Gamma)$ is dodecagonal for $l_2 > q h_*^{-1} t$ and decagonal otherwise, cf. Figure 7. For shorthand we write $M = M^\tau(\Gamma)$ and $\hat{M} = \hat{M}^\tau(\Gamma)$ for a choice of τ satisfying (4.14). For later reference we also define

$$M_k^t(\Gamma) = M^t(\Gamma) \cap (N_{k,+}^t(\Gamma) \cup N_{k,-}^t(\Gamma)) \quad \text{ for } k = 1, 2. \qquad (4.18)$$

Recall the definition in (4.2). Let $K_1, K_2 \in \mathcal{Y}$ be the sets constructed in Lemma 4.5. Let $\Gamma_m = \Gamma_m(W)$ be another boundary component satisfying $\Gamma_m \cap K \neq \emptyset$ for some $K \in \{K_1, K_2\}$ and $|\Gamma_m|_\infty \ge \frac{q^2 \bar{\tau}}{h_*}$ with q given in (4.16). For q large enough we have $|\Gamma_m|_\infty \ge 19\bar{\tau}$ and thus $\omega(\Gamma_m) = 1$ by (3.5)(iv). Moreover, (4.14) implies that Γ_m is one of the (at most) two rectangular boundary components given by Corollary 4.4. By the choice in (4.14), K is constructed in case a) of the proof of Lemma 4.5 and therefore it is not hard to see that K is contained in one of the sets $N_{j,\pm}$, $j = 1, 2$. Let $X_m \in \mathcal{U}^s$ be the corresponding component of $Q_\mu \setminus W$. We now treat two different cases depending on whether K is near a corner of Γ or not:

(I) Assume $K \cap \hat{M} \neq \emptyset$. As K is contained in one of the sets $N_{j,\pm}$, $j = 1, 2$ we assume e.g. $K \subset N_{1,-}$. As $|K| \le C\frac{\tau^2}{h_*}$ by Lemma 4.5, we find $|\pi_2 \Gamma_m| \le C\frac{\tau}{h_*}$ and thus $|\pi_1\Gamma_m| \gg |\pi_2\Gamma_m|$. Consequently, for q sufficiently large we have $|\pi_1\Gamma_m| \gg \bar{\tau}$ which implies

$$\Gamma_m \cap \{-l_1 - 21\bar{\tau}\} \times \mathbb{R} \neq \emptyset. \tag{4.19}$$

Let $Q_1, Q_2 \in J$ be the neighboring squares of K, i.e. $Q_i \cap K = \emptyset$ and $\partial K \cap \partial Q_i \neq \emptyset$ for $i = 1, 2$. Let $\Psi = (\overline{Q_1 \cup K \cup Q_2} \setminus X_m)^\circ$ and observe that $\Psi \subset N_{1,-}$ as $K \cap \hat{M} \neq \emptyset$. By (4.19) the set $\Psi = \Psi_1 \cup \Psi_2 \cup \Psi_3$ decomposes into three rectangles, where (up to translation and sets of measure zero) $\Psi_1 = (0, \tau) \times (0, \tau + a_1)$, $\Psi_2 = (0, \psi) \times (0, \hat{\psi})$ and $\Psi_3 = (0, \tau) \times (0, \tau + a_3)$ for $-\frac{1}{2}\tau \leq a_1, a_3 \leq \tau$. (Recall the construction of K in the proof of Lemma 4.5 a).) Furthermore, let

$$\Phi = \{x \in Q_\mu : \operatorname{dist}(x, \Psi) \leq 20\bar{\tau}\}.$$

Before we go on with case (II) we state two observations. We say that two sets are C-Lipschitz equivalent if they are related through a bi-Lipschitzian homeomorphism with Lipschitz constants of both the homeomorphism itself and its inverse bounded by C.

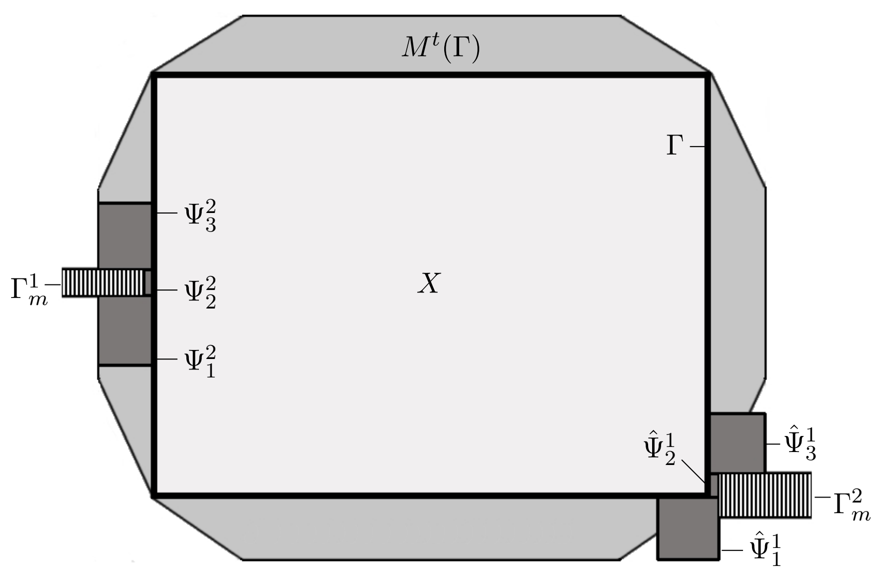

Figure 5: Neigborhood $M^t(\Gamma)$ with two other boundary components Γ_m^1, Γ_m^2 (the interiors X_m^1, X_m^2 are striped) and corresponding neighborhoods $\hat{\Psi}^1$ and Ψ^2.

Lemma 4.6 *Let Γ, Γ_m with $\omega(\Gamma_m) = \omega(\Gamma) = 1$ and $|\Gamma|_\infty \geq \lambda$, $|\Gamma_m|_\infty \geq q^2 \frac{\bar{\tau}}{h_*}$ be given. In the situation of (I) the following holds:*

(i) Let $V \in \mathcal{U}^s$ be the smallest rectangle containing X and X_m. Then $\Phi \subset V$.

(ii) $\hat{\psi} \leq C \frac{\psi}{h_*}$. In particular, there is a suitable set $\Psi_2 \subset \Psi_2^* \subset \Psi$ such that each set Ψ_1, Ψ_2^*, Ψ_3 is $C(h_*)$-Lipschitz equivalent to a square.

Proof. (i) As $K \cap \hat{M} \neq \emptyset$ we get that $\Phi \subset N^* := N_{1,-}^{21\bar{\tau}} \setminus (N_{2,+}^{21\bar{\tau}} \cup N_{2,-}^{21\bar{\tau}})$ if we again choose q large enough. By (4.19) we have $\partial N^* \cap \Gamma_m \neq \emptyset$. Therefore, the smallest rectangle V containing Γ and Γ_m satisfies $N^* \subset \overline{V}$ which gives the assertion.

(ii) By Lemma 4.1(ii) we obtain $\hat{\psi} \le |\Gamma_m \cap N^{2\psi}|_{\mathcal{H}} \le C\frac{\psi}{h_*}$. If also $\hat{\psi} \ge \frac{h_*}{C}\psi$ we set $\Psi_2^* = \Psi_2$, otherwise we choose some $\Psi_2^* \supset \Psi_2$ with $|\pi_2 \Psi_2^*| = \psi$. ◻

(II) Assume now $K \cap \hat{M} = \emptyset$. (i) We first treat the case $l_2 \gg \frac{\tau}{h_*}$ and similarly as in (I) suppose without restriction that $K \subset N_{1,-}$. Again let Q_1, Q_2 be the neighboring squares of K and set $\hat{\Psi} = (\overline{Q_1 \cup K \cup Q_2} \setminus X_m)^\circ$. If $Q_j \subset N_{1,-}$ for $j = 1, 2$ the set $\hat{\Psi}$ decomposes as before in (I).

Otherwise, we may assume that e.g. $Q_1 \subset N_{2,-} \setminus N_{1,-}$. Observe that then $K_1, Q_2 \subset N_{1,-}$ as $|K_1| \le C\frac{\tau^2}{h_*}$ and $l_2 \gg \frac{\tau}{h_*}$. As indicated in Figure 5, the set $\hat{\Psi}$ contains three rectangles $\hat{\Psi}_1, \hat{\Psi}_2, \hat{\Psi}_3$, where (up to translation and sets of measure zero) $\hat{\Psi}_1 = (0, \tau + \psi) \times (0, \tau)$, $\hat{\Psi}_2 = (0, \psi) \times (0, \hat{\psi})$ and $\hat{\Psi}_3 = (0, \tau) \times (0, \tau + a_3)$ for $0 \le a_3 \le \tau$. Note that $\hat{\psi} = 0$ is possible and that an argumentation as in Lemma 4.6 yields $\hat{\psi} \le C\frac{\psi}{h_*}$. Now let

$$\Psi_j = \hat{\Psi}_j \setminus (M^{21\bar{\tau}}(\Gamma) \cup M^{21\bar{\tau}_m}(\Gamma_m)),\ j = 1, 2, 3, \qquad \Psi = \Big(\bigcup_{j=1}^{3} \overline{\Psi_j}\Big)^\circ,$$

where $\bar{\tau}_m = \upsilon|\Gamma_m|_\infty$. Furthermore, let $\Phi = \{x \in Q_\mu : \mathrm{dist}(x, \Psi) \le 20\bar{\tau}\}$.

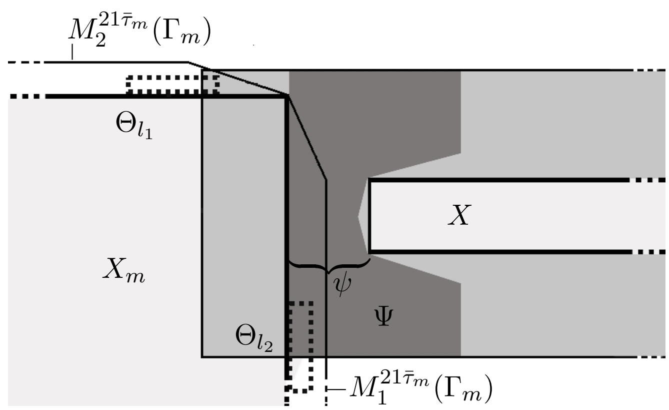

Figure 6: Sketch of Ψ (grey) in the case (II)(ii), where only parts of the boundary components Γ,Γ_m are depicted. In particular $M^{21\bar{\tau}_m}(\Gamma_m) \cap \Psi \neq \emptyset$ and $M^{21\bar{\tau}}(\Gamma) \cap \Psi = \emptyset$. Also note that $M^{21\bar{\tau}_m}(\Gamma_m)$ is dodecagonal, whereas $M^{21\bar{\tau}}(\Gamma)$ is decagonal. Moreover, for later reference (see proof of Lemma 5.5) we have also drawn two boundary components $\Theta_{l_1}, \Theta_{l_2} \subset M^{21\bar{\tau}_m}(\Gamma_m)$ in dashed lines.

(ii) We finally treat the case that l_2 is small with respect to l_1 (i.e. $l_2 \le C\frac{\tau}{h_*}$) which particularly implies that $M^{21\bar{\tau}}(\Gamma)$ is decagonal. Suppose without restriction that $K \subset N_{1,-}$. If $K \cap N_{2,+} = \emptyset$ or $K \cap N_{2,-} = \emptyset$ we may proceed as before in (II)(i). Otherwise, the set $\hat{\Psi} \supset \hat{\Psi}_1 \cup \hat{\Psi}_2 \cup \hat{\Psi}_3$ contains three rectangles, where (up to translation and sets of measure zero) $\hat{\Psi}_1 = (0, \tau+\psi) \times (0, \tau)$, $\hat{\Psi}_2 = (0, \psi) \times (0, 2l_2)$ and $\hat{\Psi}_3 = (0, \tau+\psi) \times (0, \tau)$ (cf. Figure 6). The same argumentation as in Lemma 4.6(ii) yields $2l_2 \le C\frac{\psi}{h_*}$. We let $\Psi_j = \hat{\Psi}_j \setminus M^{21\bar{\tau}}(\Gamma)$ for $j = 1, 2, 3$. Observe that in contrast to case (II)(i) we only subtract the set $M^{21\bar{\tau}}(\Gamma)$. We now have the following properties.

Lemma 4.7 *Let Γ, Γ_m with $\omega(\Gamma_m) = \omega(\Gamma) = 1$ and $|\Gamma|_\infty \ge \lambda$, $|\Gamma_m|_\infty \ge q^2 \frac{\bar{\tau}}{h_*}$ be given. In the situation of (II) the following holds:*

(i) Let $V \in \mathcal{U}^s$ be the smallest rectangle containing X and X_m. Then we have $\Phi \cap \{x : x_1 \ge -l_1 - \psi\} \cap M^{21\bar{\tau}_m}(\Gamma_m) \subset V$.

(ii) In the cases (II)(i),(ii) we have $\hat{\psi} \le C\frac{\psi}{h_*}$ and $2l_2 \le C\frac{\psi}{h_*}$, respectively. Moreover, there is a suitable set $\Psi_2 \subset \Psi_2^* \subset \Psi$ such that each set Ψ_1, Ψ_2^*, Ψ_3 is $C(h_*)$-Lipschitz equivalent to a square.

Proof. (i) It suffices to note that $\{x : x_2 \ge -l_1 - \psi\} \cap M^{21\bar{\tau}_m}(\Gamma_m) \subset [-l_1 - \psi, \infty) \times \pi_2\Gamma_m$ and $\pi_1\Phi \subset (-\infty, l_1]$ (cf. Figure 6).

(ii) The bounds on $\hat{\psi}$ and l_2 were already discussed above. As in the proof of Lemma 4.6(ii) we can choose $\hat{\Psi}_2^* \supset \hat{\Psi}_2$ such that $\hat{\Psi}_2^*$ is $C(h_*)$-Lipschitz equivalent to a square. Let $\Psi_2^* = \hat{\Psi}_2^* \setminus (M^{21\bar{\tau}}(\Gamma) \cup M^{21\bar{\tau}_m}(\Gamma_m))$ or $\Psi_2^* = \hat{\Psi}_2^* \setminus M^{21\bar{\tau}}(\Gamma)$, respectively, depending on the cases (II)(i) and (II)(ii). For q sufficiently large in (4.16) it is elementary to see that Ψ_1, Ψ_2^*, Ψ_3 are $C(h_*)$-Lipschitz equivalent to a square. □

5 Proof of the Korn-Poincaré-inequality

This section is devoted to the main proof of Theorem 1.1. We concern ourselves with functions $u \in H^1(W)$ on $W \in \mathcal{V}^s$, $W \subset Q_\mu$ (recall (2.7), (3.1)). In the following we will again omit to write $\mathcal{V}^s$. For shorthand we set $\alpha(U) = \|e(u)\|^2_{L^2(U)}$ for $U \subset W$.

As a further preparation, we define $H(W) \supset W \in \mathcal{V}^s$ as the 'variant of W without holes'. Arrange the components $X_1, \ldots, X_m$ such that $\partial X_i \subset Q_\mu$ for $1 \le i \le n$ and $\partial X_i \cap \partial Q_\mu \neq \emptyset$ otherwise. We set

$$H(W) = W \cup \bigcup\nolimits_{j=1}^n X_j. \tag{5.1}$$

The main idea will be to analyze the trace of u at the boundary components. Therefore, we will have to change the set W iteratively. We first introduce further conditions for the neighborhood of a boundary component which allows us to apply a trace estimate. Then we present the main modification algorithm. Afterwards, the proof of Theorem 1.1 will be straightforward by employing Theorem 2.1 and Theorem 2.3.

5.1 Conditions for boundary components and trace estimate

Recall definition (3.5) and assume that in an iteration step $W_i \in \mathcal{W}^s_\lambda$ for $\lambda \geq 0$ with the corresponding weight ω and a specific ordering of the boundary components $(\Gamma(W_i)_j)_{j=1}^n$ is given. Consider $\Gamma = \Gamma(W_i)$ with $|\Gamma|_\infty \geq \lambda$ and recall that $\Gamma = \Theta$ is rectangular by (3.5)(v). Let $\hat{N} = N^{2\hat{\tau}}(\Gamma)$, where

$$\hat{\tau} = q^2 \bar{\tau} h_*^{-1} = q^2 \upsilon h_*^{-1} |\Gamma|_\infty \ll |\Gamma|_\infty \tag{5.2}$$

with q from (4.16) and $\bar{\tau}$ as defined in (4.2). Recall that $\hat{\tau}$ is the least length of boundary components considered in Section 4.2. The latter inequality holds if we choose υ sufficiently small with respect to q. For $\varepsilon > 0$ and for $D = D(h_*)$ sufficiently large we require

$$\alpha(\hat{N} \cap W_i) + \varepsilon |\partial W_i \cap \hat{N}|_{\mathcal{H}} \leq D \varepsilon \hat{\tau}. \tag{5.3}$$

Moreover, let Ψ^j and ψ^j, $j = 1, 2$, be defined as in Section 4.2 (I),(II) corresponding to the sets K_j, $j = 1, 2$, provided by Lemma 4.5. We introduce the condition

$$\alpha(\Psi^j \cap W_i) + \varepsilon |\partial W_i \cap \Psi^j|_{\mathcal{H}} \leq D(1 - \omega_{\min})^{-1} \varepsilon \psi^j \tag{5.4}$$

for $j = 1, 2$, where $D = D(h_*)$ as in (5.3). For $\eta \geq 0$ we let $\mathcal{T}_\eta(W_i)$ be the set of $\Gamma_l(W_i)$ satisfying $|\Gamma_l(W_i)|_\infty \leq \eta$ and

$$N^{2\hat{\tau}_l}(\partial R_l) \subset H(W_i),$$

where $\hat{\tau}_l = q^2 \upsilon |\Gamma_l|_\infty h_*^{-1}$ (cf. (5.2)) and R_l is the corresponding rectangle given in (3.5)(i) or (3.5)(v), respectively. Moreover, recalling again the definition of $\Theta_l(W_i) \subset \Gamma_l(W_i) = \partial X_l$ in (3.2) we define

$$S_\lambda(W_i) = \bigcup\nolimits_{\Gamma_l(W_i) \in \mathcal{S}_\lambda(W_i)} \Theta_l(W_i), \tag{5.5}$$

where $\mathcal{S}_\lambda(W_i) = \{\Gamma_l : |\Gamma_l|_\infty > \lambda\} \cup \{\Gamma_l : \omega(\Gamma_l) = 1,\ N^{2\hat{\tau}_l}(\partial R_l) \not\subset H(W_i)\}$. (Note that by (3.5)(iv) all components of $\mathcal{S}_\lambda(W_i)$ have weight 1.)

We assume that for all $\Gamma_l(W_i) \in \mathcal{T}_{\hat{\tau}}(W_i)$ there are $A_l \in \mathbb{R}^{2\times 2}_{\text{skew}}$, $c_l \in \mathbb{R}^2$ such that for the extension $\bar{u} \in SBD(W_{\hat{\tau}})$, $W_{\hat{\tau}} := W_i \cup \bigcup_{\Gamma_l(W_i) \in \mathcal{T}_{\hat{\tau}}(W_i)} X_l$, defined by

$$\bar{u}(x) = \begin{cases} A_l\, x + c_l & x \in X_l \ \text{ for } \ \Gamma_l(W_i) \in \mathcal{T}_{\hat{\tau}}(W_i) \\ u(x) & \text{else,} \end{cases} \tag{5.6}$$

we have the trace estimate

$$\begin{aligned} \int_{\Theta_l(W_i)} |\bar{u}(x) - (A_l\, x + c_l)|^2 \, d\mathcal{H}^1(x) &= \int_{\Theta_l(W_i)} |[\bar{u}](x)|^2 \, d\mathcal{H}^1(x) \\ &\leq C_* \frac{\varepsilon^4}{\upsilon} |\Theta_l(W_i)|_*^2 \end{aligned} \tag{5.7}$$

for some $C_* = C_*(h_*) > 0$ sufficiently large. (The left hand side has to be understood as the trace of $\bar{u}|_{W_{\hat{\tau}} \setminus X_l}$ on $\Theta_l(W_i)$.)

We now state that under suitable conditions also $\Gamma = \Gamma(W_i)$ with $|\Gamma|_\infty \geq \lambda$ satisfies an estimate similar to (5.7).

Theorem 5.1 *Let $\upsilon, h_*, \varepsilon, \omega_{\min} > 0$ and $\lambda > 0$. Then there is a constant $\hat{C} = \hat{C}(h_*) > 0$ such that for υ sufficiently small (depending on h_* and $\omega_{\min}$) the following holds: For all $W_i \in \mathcal{W}^s_\lambda$, for all $u \in H^1(W_i)$ and boundary components $\Gamma = \Gamma(W_i)$ with $|\Gamma|_\infty \geq \lambda$ such that (4.1), (5.3), (5.4), $N^{2\hat{\tau}}(\Gamma) \subset H(W_i)$ hold and (5.7) is satisfied for $\mathcal{T}_{\hat{\tau}}(W_i)$ one has (in the sense of traces)*

$$\int_\Gamma |\bar{u}(x) - (A\,x - c)|^2 \, d\mathcal{H}^1(x) \leq \Big(\hat{C} + \frac{C_*}{2}\Big) \frac{\varepsilon}{\upsilon^4} |\Gamma|_*^2 \tag{5.8}$$

for suitable $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $c \in \mathbb{R}^2$.

As the proof of this assertion is very technical and involves several steps we postpone it to Section 6.

5.2 Modification algorithm

We now show that we may modify the set W iteratively such that successively we find a component Γ which satisfies the conditions (4.1), (5.3), (5.4) and (5.7) such that Theorem 5.1 can be applied. Define $\mathcal{W}^s = \bigcup_{\lambda \geq 0} \mathcal{W}^s_\lambda$.

Theorem 5.2 *Let $\varepsilon > 0$ and $h_* \geq \sigma > 0$ sufficiently small. Let $C_1 = C_1(\sigma, h_*) \geq 1$ large, $0 < C_2 = C_2(\sigma, h_*) < 1$ small enough and let $c > 0$ be a universal constant. For all $W \in \mathcal{V}^s$ with connected boundary components and $u \in H^1(W)$ there is a set $U \in \mathcal{W}^{C_2 s}$ with $|U \setminus W| = 0$ and an extension $\bar{u}$ defined by*

$$\bar{u}(x) = \begin{cases} A_l\, x + c_l & x \in X_l \quad \text{for all } \Gamma_l(U) \text{ with } N^{2\hat{\tau}_l}(\partial R_l) \subset H(U), \\ u(x) & \text{else,} \end{cases} \tag{5.9}$$

such that for all $\Gamma_l(U)$ with $N^{2\hat{\tau}_l}(\partial R_l) \subset H(U)$

$$\int_{\Theta_l(U)} |[\bar{u}](x))|^2 \, d\mathcal{H}^1(x) \leq C_1 \varepsilon |\Theta_l(U)|_*^2. \tag{5.10}$$

Moreover, one has $|W \setminus U| \leq c\|U\|_\infty^2$ and

$$\varepsilon\|U\|_* + \alpha(U) \leq (1+\sigma)(\varepsilon\|W\|_* + \alpha(W)). \tag{5.11}$$

Remark 5.3 (i) In the proofs of Theorem 5.2 and Theorem 5.1 we will see that the constants $C_i = C_i(\sigma, h_*)$ have polynomial growth in σ: We find $z \in \mathbb{N}$ large enough such that $C_1(\sigma, h_*) \leq C(h_*)\sigma^{-z}$ and $C_2(\sigma, h_*) \geq C(h_*)\sigma^z$.

(ii) Although we only state that the extensions are elements of SBD, it is clear that they also lie in SBV due to the regularity of the jump set and Korn's inequality.

The proof relies on an iterative modification procedure. First choose

$$C_2 = C\upsilon \tag{5.12}$$

for C small enough and consider $W \in \mathcal{V}^s$ as an element of $\mathcal{V}^{C_2 s}$ such that (4.2) is satisfied for all boundary components $\Gamma_l(W)$. From now on we will always tacitly assume that all involved sets lie in $\mathcal{V}^{C_2 s}$ and write $\mathcal{W}_\lambda$ instead of $\mathcal{W}_\lambda^{C_2 s}$. In the proof below we will show that C_2 is in fact a constant only depending on h_* and σ.

We set $W_0 = \hat{W}$, where $\hat{W}$ is the modification constructed in Corollary 3.3. Choosing an ordering of the boundary components and setting $\omega(\Gamma_j(W_0)) = 1$ for all j we obtain $W_0 \in \mathcal{W}_0$. Moreover, we let $\lambda_0 = 0$, $B_0^0 = \emptyset$. Assume that $\lambda_0 \le \ldots \le \lambda_i$ and that $W_i \subset \ldots \subset W_0$, $W_j \in \mathcal{W}_{\lambda_j}$, are given (the inclusion holds up to sets of negligible $\mathcal{L}^2$-measure) as well as $\{B_k^j : k = 0, \ldots, j\}$ for $j = 0, \ldots, i$. In each iteration step the sets B_k^j, $k = 0, \ldots, j$, will describe the set where we already 'used' the 'energy lying in the set' to modify W.

Suppose that in an iteration step i the following conditions are satisfied:

$$\varepsilon \|W_i\|_\omega + \alpha(W_i) \le \varepsilon \|W\|_* + \alpha(W) + h_*(1 - \omega_{\min}) \sum\nolimits_{j=0}^{i} \alpha(B_j^i) \tag{5.13}$$

as well as

$$\begin{aligned}
&(i) \ \text{Each } x \in Q_\mu \text{ lies in at most two different } B_{j_1}^i, B_{j_2}^i \text{ and} \\
&\quad\ \text{each } x \in W_i \text{ lies in at most one } B_j^i,\ j, j_1, j_2 \in \{0, \ldots, i\}, \\
&(ii) \ \text{Either } \Theta_l(W_i) \subset B_j^i \text{ for some } 0 \le j \le i \text{ or} \\
&\quad\ \Gamma_l(W_i) \in \mathcal{G}_i := \big\{\Gamma_l : \Gamma_l \cap \bigcup\nolimits_{j=0}^{i} B_j^i = \emptyset,\ \omega(\Gamma_l) = 1\big\} \ \text{ for all } \Gamma_l(W_i), \\
&(iii) \ \text{Each } B_j^i \text{ with } B_j^i \cap W_i \neq \emptyset, \ \text{ satisfies } B_j^i \cap W_i \subset M_k^{\eta_l^i}(\Gamma_l(W_i)), \\
&\qquad\qquad \text{for some } \Gamma_l(W_i) \in \mathcal{G}_i \text{ and } k \in \{1, 2\},\ j = 0, \ldots, i.
\end{aligned} \tag{5.14}$$

Here $\eta_l^i := 21\upsilon \min\{|\Gamma_l(W_i)|_\infty, \lambda_i\} = \min\{21\bar{\tau}_l, 21\upsilon\lambda_i\}$ and the neighborhood M_k was defined in (4.18). Moreover, recalling (5.5) we suppose

$$\begin{aligned}
\alpha(N^{\hat{\tau}_l}(\Gamma_l(W_i)) \cap W_i) + \varepsilon |N^{\hat{\tau}_l}(\Gamma_l(W_i)) \cap (\partial W_i \setminus S_{\lambda_i}(W_i))|_{\mathcal{H}} \le D\varepsilon\hat{\tau}_l \\
\text{for all } \Gamma_l(W_i) \in \mathcal{G}_i \cap \mathcal{T}_{\lambda_i}(W_i)
\end{aligned} \tag{5.15}$$

where D is defined as in (5.3). Furthermore, recalling (5.6) we assume that there is an extension $\bar{u}_i \in SBD^2(W_{\lambda_i})$ such that all boundary components $\Gamma_l(W_i) \in \mathcal{T}_{\lambda_i}(W_i)$ satisfy

$$\int_{\Theta_l(W_i)} |\bar{u}_i(x) - (A_l\, x - c_l)|^2 \, d\mathcal{H}^1 \le \hat{C} \sum_{n=0}^{i} \Big(\frac{2}{3}\Big)^n \ \frac{\omega(\Gamma_l(W_i))^2}{\hat{\omega}_i(\Gamma_l(W_i))^2} \frac{\varepsilon}{\upsilon^4} |\Theta_l(W_i)|_*^2 \tag{5.16}$$

for $A_l \in \mathbb{R}^{2\times 2}_{\rm skew}, c_l \in \mathbb{R}^2$, where $\hat{\omega}_i(\Gamma_l(W_i)) := 1 - \frac{1-\omega_{\min}}{2}\#\{j = 0,\ldots,i : \Theta_l(W_i) \subset B^i_j\}$ and $\hat{C}$ is the constant from (5.8). In particular, this implies that (5.7) is satisfied if we replace C_* by $\hat{C}\omega_{\min}^{-2}\sum_{n=0}^i (2/3)^n$ as $\hat{\omega}_i(\Gamma_l(W_i)) \geq \omega_{\min}$ (see (5.14)(i), (ii)). Finally, we assume

$$\begin{aligned} &(i)\ \ \omega(\Gamma_l(W_i)) \geq \hat{\omega}_i(\Gamma_l(W_i)), && \forall\, \Gamma_l,\\ &(ii)\ \ |\partial R_l(W_i)|_* \leq \omega(\Gamma_l(W_i))(\hat{\omega}_i(\Gamma_l(W_i)))^{-1}|\Theta_l(W_i)|_*, && \forall\, \Gamma_l:\ \omega(\Gamma_l) < 1. \end{aligned} \tag{5.17}$$

The second condition is a refinement of (3.5)(ii).

Recall that $\partial W_i \cap Q_\mu \subset W_i$ by definition (see (3.1)). This particularly implies that each $\Theta_l(W_i)$ is contained in at most one set B^i_j (see (5.14)(i),(ii)). Before we give the proof of Theorem 5.2 we first observe that the above stated properties are preserved under modification.

Lemma 5.4 *Let $\varepsilon > 0$ and $\lambda \geq 0$. Let $W_i \in \mathcal{W}_\lambda$, $\bar{u}_i$ and $\{B^i_j : j = 0,\ldots,i\}$ be given such that* (5.13)-(5.17) *hold for W_i, $\bar{u}_i$ and λ (replace λ_i by λ). For a rectangle $\overline{V} \subset Q_\mu$ with $|\partial V|_\infty > \lambda$, let $\tilde{W}_i = (W_i \setminus V) \cup \partial V$ and assume that (recall* (3.7), (3.8)*)*

$$\varepsilon\|\tilde{W}_i\|_\omega + \alpha(\tilde{W}_i) \leq \varepsilon\|W_i\|_\omega + \alpha(W_i).$$

Let $W_{i+1} \in \mathcal{W}_\lambda$ be the set given by Lemma 3.2 and define $B^{i+1}_j = B^i_j \setminus S_\lambda(W_{i+1})$ for $j = 0,\ldots,i$ (recall (5.5)*) and $B^{i+1}_{i+1} = \emptyset$ as well as $\bar{u}_{i+1} = \bar{u}_i$. Then* (5.13)-(5.17) *hold for W_{i+1}, $\bar{u}_{i+1}$ and λ (replace λ_{i+1} by λ).*

Proof. Let $\tilde{W} = (W_i \setminus V) \cup \partial V$ for some rectangle V with $|V|_\infty > \lambda$ and choose $V' \supset V$ such that $W_{i+1} := (W_i \setminus V') \cup \partial V' \in \mathcal{W}_\lambda$ as in Lemma 3.2. Then by assumption and (3.10) we have $\varepsilon\|W_{i+1}\|_\omega + \alpha(W_{i+1}) \leq \varepsilon\|W_i\|_\omega + \alpha(W_i)$, where we adjust the weights as described in (3.8). As $|B^i_j \setminus B^{i+1}_j| = 0$ for all $j = 0,\ldots,i$, (5.13) is trivially satisfied.

Clearly, (5.14)(i) still holds as $B^{i+1}_j \subset B^i_j$ for all $j = 0,\ldots,i$. Moreover, $\Gamma_l(W_{i+1}) \cap \bigcup_{j=0}^{i+1} B^{i+1}_j = \emptyset$ for all $\Gamma_l(W_{i+1}) \in \mathcal{S}_\lambda(W_{i+1})$ by definition. Consequently, $\mathcal{S}_\lambda(W_{i+1}) \subset \mathcal{G}_{i+1}$ and to confirm (5.14)(ii) it suffices to consider the components not lying in $\mathcal{S}_\lambda(W_{i+1})$. Let $\Gamma_l(W_{i+1}) \notin \mathcal{S}_\lambda(W_{i+1})$. Using that $\partial V' \in \mathcal{S}_\lambda(W_{i+1})$ and arguing similarly as in Section 3.2 (see remark after (3.7)) we find a (unique) corresponding $\Gamma_l(W_i)$ (for notational convenience we use the same index) such that $\Theta_l(W_{i+1}) = \Theta_l(W_i) \setminus \overline{V'}$. If $\Gamma_l(W_i) \in \mathcal{G}_i$ we immediately get $\Gamma_l(W_{i+1}) \in \mathcal{G}_{i+1}$ by (3.9) and the fact that the sets $(B^i_j)^i_{j=1}$ do not become larger. Consequently, we can assume that $\Theta_l(W_i) \subset B^i_j$ for some j and it now remains to show $\Theta_l(W_{i+1}) \subset B^{i+1}_j$. To see this, it suffices to observe $\Theta_l(W_{i+1}) = \Theta_l(W_i) \setminus \overline{V'} \subset \Theta_l(W_i)$ and $\Theta_l(W_{i+1}) \cap S_\lambda(W_{i+1}) = \emptyset$, where the latter holds as the sets $(\Theta_j(W_{i+1}))_j$ are pairwise disjoint (see (3.2)).

Due to the modification procedure (see the construction of V' in the proof of Lemma 3.2), for all $\Gamma_l(W_i) \in \mathcal{G}_i$ we find a $\Gamma_j(W_{i+1}) \in \mathcal{G}_{i+1}$ such that $\Gamma_l(W_i) \subset$

$\overline{X_j(W_{i+1})}$, where $\partial X_j(W_{i+1}) = \Gamma_j(W_{i+1})$). In fact, one can choose either $\Gamma_l(W_i)$ itself or $\partial V'$. (Note that both are elements of $\mathcal{G}_{i+1}$.) Therefore, $M_k^{\eta_l^i}(\Gamma_l(W_i)) \subset M_k^{\eta_j^{i+1}}(\Gamma_j(W_{i+1}))$ for $k = 1, 2$ and thus condition (5.14)(iii) holds.

To confirm (5.17)(i) we first observe that $\hat{\omega}_{i+1}(\partial V') = \omega(\partial V') = 1$. Moreover, we see that $\hat{\omega}_{i+1}(\Gamma_l(W_{i+1})) = \hat{\omega}_i(\Gamma_l(W_i))$ for all $\Gamma_l(W_{i+1}) \neq \partial V'$, where $\Gamma_l(W_i)$ is the unique corresponding component. In fact, $\hat{\omega}_{i+1}(\Gamma_l(W_{i+1})) \geq \hat{\omega}_i(\Gamma_l(W_i))$ follows immediately from the definition of $(B_j^{i+1})_{j=1}^{i+1}$ and the reverse inequality follows from the observation that $\Theta_l(W_i) \subset B_j^i$ implies $\Theta_l(W_{i+1}) \subset B_j^{i+1}$ (see above). Now this together with the fact that $\omega(\Gamma_l(W_{i+1})) \geq \omega(\Gamma_l(W_i))$ (see (3.9)) yields (5.17)(i).

We now show that (5.16) remains true. Similarly as before we find for all $\Gamma_l(W_{i+1}) \in \mathcal{T}_\lambda(W_{i+1})$ a (unique) corresponding $\Gamma_l(W_i)$. If $\Gamma_l(W_{i+1}) \cap \partial V' = \emptyset$, then $\Theta_l(W_{i+1}) = \Theta_l(W_i)$ and there is nothing to show since $\Gamma_l(W_i) \in \mathcal{T}_\lambda(W_i)$ due to the fact that $|\Gamma_l(W_i)|_\infty = |\Gamma_l(W_{i+1})|_\infty \leq \lambda$ and $\omega(\Gamma_l(W_i))$, $\hat{\omega}_i(\Gamma_l(W_i))$ remain unchanged. Otherwise, $\omega(\Gamma_l(W_{i+1})) < 1$ by (3.5)(v) and thus

$$\omega(\Gamma_l(W_{i+1}))|\Theta_l(W_{i+1}))|_* = \omega(\Gamma_l(W_i))|\Theta_l(W_i)|_* \tag{5.18}$$

by (3.8), which together with the fact that $\bar{u}_{i+1} = \bar{u}_i$ and $\hat{\omega}_{i+1}(\Gamma_l(W_{i+1})) = \hat{\omega}_i(\Gamma_l(W_i))$ implies (5.16). To see that $|\Gamma_l(W_i)|_\infty \leq \lambda$ also holds in this case (and thus $\Gamma_l(W_i) \in \mathcal{T}_\lambda(W_i)$) we note that $|\Gamma_l(W_{i+1})|_\infty \leq 19\upsilon\lambda$ by (3.5)(iv) and therefore (5.18) together with (3.5)(i) and (3.6) implies $|\Gamma_l(W_i)|_\infty \leq \lambda$ for υ small enough.

Observe that by the same argument as in (5.18) property (5.17)(ii) is satisfied. (Recall that in the modification procedure we never change the rectangles ∂R_l.)

Finally, (5.15) holds. Indeed, for a given $\Gamma_l(W_{i+1}) \in \mathcal{T}_\lambda(W_{i+1}) \cap \mathcal{G}_{i+1}$ we deduce $\Gamma_l(W_{i+1}) \cap \partial V' = \emptyset$ by (3.5)(v) and thus $\Gamma_l(W_{i+1}) = \Gamma_l(W_i)$, where $\Gamma_l(W_i)$ is the corresponding component of W_i. The assertion now follows from the i-th iteration step of (5.15). In fact, for the left part it suffices to recall $W_{i+1} \subset W_i$. For the right part we note $S_\lambda(W_{i+1}) = \partial V' \cup (S_\lambda(W_i) \cap \partial W_{i+1}) \supset S_\lambda(W_i) \cap \partial W_{i+1}$ (again recall that we did not change the rectangles ∂R_l) and $\partial W_{i+1} \setminus \partial W_i \subset \partial V' \subset S_\lambda(W_{i+1})$ which then yields $\partial W_{i+1} \setminus S_\lambda(W_{i+1}) \subset \partial W_i \setminus S_\lambda(W_i)$ by an elementary computation. □

We are now in a position to prove Theorem 5.2.

Proof of Theorem 5.2. Using Corollary 3.3 we first see that (5.13)-(5.17) hold for $W_0 = \hat{W}$ and $\lambda_0 = 0$, $B_0^0 = \emptyset$. Assume that $W_i \in \mathcal{W}_{\lambda_i}$, λ_i, $\{B_j^i : j = 0, \dots, i\}$ and $\bar{u}_i$ have already been constructed and that (5.13)-(5.17) hold.

If now all $\Gamma_l(W_i)$ with $N^{2\hat{\tau}_l}(\partial R_l(W_i)) \subset H(W_i)$ satisfy $|\Gamma_l(W_i)|_\infty \leq \lambda_i$ we stop and set $U = W_i$. We observe that in this case (5.10) holds for $C_1 = \hat{C}\sum_{n=0}^\infty (2/3)^n \omega_{\min}^{-2}\upsilon^{-4}$ by (5.16). Otherwise, there is some smallest $\Gamma = \Gamma(W_i)$ with respect to $|\cdot|_\infty$ satisfying $|\Gamma|_\infty > \lambda_i$ and $N^{2\hat{\tau}}(\Gamma) \subset H(W_i)$. To simplify the exposition, we will suppose that the choice of Γ is unique. At the end of the proof we briefly indicate the necessary changes if there are several components of the same size.

Choose $\omega_{\min} \geq \sqrt{\frac{3}{4}}$. We observe that $\mathcal{T}_{\hat{\tau}} \subset \mathcal{T}_{\lambda_i}$ for $\hat{\tau}$ as defined in (5.2). Indeed, for $\hat{\tau} \leq \lambda_i$ it is obvious and for $\hat{\tau} > \lambda_i$ it follows from the choice of Γ with respect to $|\cdot|_\infty$. Thus, by (5.16) we get that (5.7) is satisfied replacing C_* by $\frac{4}{3}\hat{C}\sum_{n=0}^{i}(2/3)^n$. If Γ additionally fulfills (4.1), (5.3) and (5.4), we may apply Theorem 5.1. Therefore, recalling $\omega(\Gamma) = 1$ we get that for suitable $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c \in \mathbb{R}^2$

$$\int_\Gamma |\bar{u}_i(x) - (Ax+c)|^2\,dx \leq \Big(\hat{C} + \frac{1}{2}\cdot\frac{4}{3}\hat{C}\sum\nolimits_{n=0}^{i}\Big(\frac{2}{3}\Big)^n\Big)\frac{\varepsilon}{\upsilon^4}|\Gamma|_*^2$$
$$\leq \hat{C}\sum\nolimits_{n=0}^{i+1}\Big(\frac{2}{3}\Big)^n\frac{\omega(\Gamma)^2}{\hat{\omega}(\Gamma)^2}\frac{\varepsilon}{\upsilon^4}|\Gamma|_*^2.$$

Thus, (5.16) holds, as desired. We define $\bar{u}_{i+1}(x) = A\,x + c$ for $x \in X$ and $\bar{u}_{i+1} = \bar{u}_i$ else, where $\partial X = \Gamma$. Moreover, we set $W_{i+1} = W_i$, $\lambda_{i+1} = |\Gamma|_\infty$, $B_j^{i+1} = B_j^i$ for $j = 0,\ldots,i$ and $B_{i+1}^{i+1} = \emptyset$. Clearly, (5.13)-(5.17) still hold due to choice of Γ with respect to $|\cdot|_\infty$. In particular, for (5.16) we note that $\mathcal{T}_{\lambda_{i+1}}(W_{i+1}) = \mathcal{T}_{\lambda_i}(W_i)\cup\{\Gamma\}$. Likewise, (5.15) is fulfilled by (5.3) and the fact that $S_{\lambda_{i+1}}(W_{i+1}) = S_{\lambda_i}(W_i)$ (recall (5.5)). Moreover, (5.14)(iii) still holds as $\lambda_{i+1} \geq \lambda_i$. As also (3.5) is satisfied for λ_{i+1}, we get $W_{i+1} \in \mathcal{W}_{\lambda_{i+1}}$. We continue with the next iteration step.

Otherwise (a) (4.1), (b) (5.3) or (c) (5.4) is violated.

In case (a) we find some $V \supsetneq \Gamma$ such that setting $\tilde{W}_i = (W_i \setminus V)\cup\partial V$, we get $\|\tilde{W}_i\|_\omega \leq \|W_i\|_\omega$ and therefore

$$\varepsilon\|\tilde{W}_i\|_\omega + \alpha(\tilde{W}_i) \leq \varepsilon\|W_i\|_\omega + \alpha(W_i).$$

Here we adjusted the weights as in (3.8). Let $\lambda_{i+1} = |\Gamma|_\infty$. It is not hard to see that $W_i \in \mathcal{W}_{\lambda_{i+1}}$ satisfies (5.13)-(5.17) also for λ_{i+1}. In fact, (5.16) follows from the choice of Γ with respect to $|\cdot|_\infty$ and the fact that $|\partial V|_\infty > \lambda_{i+1}$. For the other properties we may argue as before. Now Lemma 5.4 yields a set $W_{i+1} \in \mathcal{W}_{\lambda_{i+1}}$ with $W_{i+1} \subset \tilde{W}_i$ as well as $(B_j^{i+1})_{j=1}^{i+1}$ and $\bar{u}_{i+1} = \bar{u}_i$ such that (5.13)-(5.17) hold for W_{i+1}, $\bar{u}_{i+1}$ and λ_{i+1}. We now continue with the next iteration step.

In case b) set $\tilde{W}_i = (W_i\setminus V)\cup\partial V$, where V is the smallest rectangle containing $N^{4\hat{\tau}}(\Gamma)$. Observe that $|\partial V|_* \leq |\Gamma|_* + C\hat{\tau}$. Choosing $D = D(h_*) \geq \frac{2C}{h_*} \geq \frac{C}{h_*\omega_{\min}}$ and arguing as in the proof of Lemma 4.1 we derive

$$\varepsilon\|\tilde{W}_i\|_\omega + \alpha(\tilde{W}_i) \leq \varepsilon\|W_i\|_\omega + \alpha(W_i) + C\varepsilon\hat{\tau} - \alpha(V\cap W_i)$$
$$- \varepsilon h_*\omega_{\min}|\partial W_i \cap \hat{N}|_{\mathcal{H}} \leq \varepsilon\|W_i\|_\omega + \alpha(W_i).$$

As usual we adjusted the weights as in (3.8). We now may proceed as in case (a) and then continue with the next iteration step.

Finally, consider case (c). Let $\Psi_i = \Psi^j$ and $\psi_i = \psi^j$, where Ψ^j is a set such that (5.4) is violated. As derived above in Section 4.2, we find a boundary component $\Gamma_m = \Gamma_m(W_i)$ with $|\Gamma_m|_\infty \geq \hat{\tau}$, $\omega(\Gamma_m) = 1$. Moreover, there is a rectangle $T \subset Q_\mu$ with $|\partial T|_{\mathcal{H}} \leq 4\psi_i$ and $\overline{T} \cap \Gamma \neq \emptyset$, $\overline{T} \cap \Gamma_m \neq \emptyset$ (cf. Figure 8).

Let $\mathcal{A}_i \subset (\Gamma_l(W_i))_l \backslash \{\Gamma, \Gamma_m\}$ be the boundary components with $\Theta_l(W_i) \cap \Psi_i \neq \emptyset$ or, if $\Gamma_l(W_i) = \Theta_l(W_i) \in \mathcal{G}_i$, with $M^{\eta_l^i}(\Gamma_l(W_i)) \cap \Psi_i \neq \emptyset$. We now define an additional set B^i_{i+1}, where we will 'use the energy' to modify W_i. Let

$$B^i_{i+1} = \Big((\Psi_i \cap W_i) \cup \bigcup\nolimits_{\Gamma_l(W_i) \in \mathcal{A}_i} \Theta_l(W_i) \Big) \setminus \bigcup\nolimits_{B^i_j \in \mathcal{B}_i} B^i_j,$$

where $\mathcal{B}_i := \{B^i_j : B^i_j \cap W_i \subset M^{\eta_l^i}(\Gamma_l(W_i)) \text{ for some } \Gamma_l(W_i) \in \mathcal{A}_i \cap \mathcal{G}_i\}$. In the definition of B^i_{i+1} it is essential to subtract the set on the right hand side such that we will be able to assure (5.14)(i).

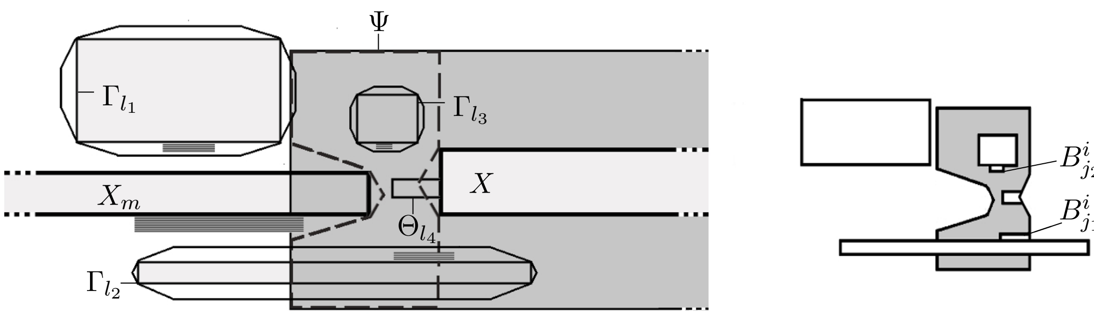

Figure 7: On the left side Ψ (the set surrounded by the dashed grey line) and parts of Γ, Γ_m are sketched. Observe that $M^{21\bar{\tau}_m}(\Gamma_m) \cap \Psi = M^{21\bar{\tau}}(\Gamma) \cap \Psi = \emptyset$. Moreover, the picture includes several boundary components with corresponding dodecagonal or decagonal neighborhoods as well as four striped sets $B^i_{j_1}, \ldots, B^i_{j_4}$. On the right hand side the resulting B^i_{i+1} is drawn, where ∂B^i_{i+1} is black and the interior $(B^i_{i+1})^\circ$ is grey. Observe that in general only parts of ∂B^i_{i+1} are contained in B^i_{i+1}.

Note that by (5.14) we have for all $\Gamma_l(W_i) \in \mathcal{A}_i$ either $\Theta_l(W_i) \subset B^i_{i+1}$ or $\Theta_l(W_i) \cap B^i_{i+1} = \emptyset$ depending on whether $\Theta_l(W_i) \cap \bigcup_{B^i_j \in \mathcal{B}_i} B^i_j = \emptyset$ or $\Theta_l(W_i) \subset B^i_j \in \mathcal{B}_i$. Moreover, the components $\Gamma_l(W_i) \notin \mathcal{A}_i$ clearly satisfy $\Theta_l(W_i) \cap B^i_{i+1} = \emptyset$. Denote by $\tilde{\mathcal{A}}_i \subset \mathcal{A}_i$ the boundary components completely contained in B^i_{i+1} and observe that $\mathcal{G}_i \cap \mathcal{A}_i \subset \tilde{\mathcal{A}}_i$.

By (5.21) below we obtain $|\Gamma_l(W_i)|_\infty < 19\bar{\tau}$ for all $\Gamma_l(W_i) \in \mathcal{A}_i$ which by (5.2) for υ sufficiently small implies $N^{2\hat{\tau}_l}(\partial R_l) \subset N^{2\hat{\tau}}(\Gamma) \subset H(W_i)$. Moreover, as $|\Gamma_l(W_i)|_\infty < |\Gamma|_\infty$, by the choice of Γ with respect to $|\cdot|_\infty$ we obtain $|\Gamma_l(W_i)|_\infty \leq \lambda_i$ for all $\Gamma_l(W_i) \in \mathcal{A}_i$ and thus $\mathcal{A}_i \subset \mathcal{T}_{\lambda_i}(W_i)$. As $\mathcal{T}_{\lambda_i}(W_i) \cap \mathcal{S}_{\lambda_i}(W_i) = \emptyset$, this also yields $\Psi_i \cap S_{\lambda_i}(W_i) = \emptyset$. This together with (5.15) shows that $\gamma(M^{21\bar{\tau}_l}(\Gamma_l(W_i))) \leq$

$D\varepsilon\hat{\tau}_l$ for all $\Gamma_l(W_i) \in \mathcal{A}_i \cap \mathcal{G}_i$, where

$$\gamma(A) := \alpha(A \cap W_i \cap \Psi_i) + \varepsilon|A \cap (\partial W_i \cap \Psi_i)|_{\mathcal{H}}$$

for $A \subset \mathbb{R}^2$. Observe that $\eta_l^i = 21\bar{\tau}_l$ as $|\Gamma_l(W_i)|_\infty \le \lambda_i$ for all $\Gamma_l(W_i) \in \mathcal{A}_i$. Using the definition of $\mathcal{B}_i$, (5.2), (5.15) and recalling $D = D(h_*)$ we find for υ small enough (with respect to h_*)

$$\begin{aligned}\sum\nolimits_{B_j^i \in \mathcal{B}_i} \gamma(B_j^i) &\le \sum\nolimits_{\Gamma_l(W_i) \in \mathcal{A}_i \cap \mathcal{G}_i} \gamma(M^{21\bar{\tau}_l}(\Gamma_l(W_i))) \\ &\le \sum\nolimits_{\Gamma_l(W_i) \in \tilde{\mathcal{A}}_i} \varepsilon|\Gamma_l(W_i)|_\infty \le \varepsilon|B_{i+1}^i \cap \partial W_i|_{\mathcal{H}}.\end{aligned}$$

In the first step we used that $B_{j_1}^i \cap B_{j_2}^i \cap W_i = \emptyset$ for $j_1 \neq j_2$ by (5.14)(i) and $\partial W_i \cap Q_\mu \subset W_i$. The last step follows from the definition of $\tilde{\mathcal{A}}_i$. Recall that The fact that (5.4) is violated and the definition of B_{i+1}^i then imply

$$\begin{aligned}D(1-\omega_{\min})^{-1}\varepsilon\psi_i &< \alpha(\Psi_i \cap W_i) + \varepsilon|\partial W_i \cap \Psi_i|_{\mathcal{H}} \\ &\le \alpha(\Psi_i \cap W_i) + \varepsilon|\partial W_i \cap \Psi_i|_{\mathcal{H}} - \sum\nolimits_{B_j^i \in \mathcal{B}_i} \gamma(B_j^i) \\ &\quad + \varepsilon|\partial W_i \cap B_{i+1}^i|_{\mathcal{H}} \\ &\le \alpha(B_{i+1}^i) + 2\varepsilon|\partial W_i \cap B_{i+1}^i|_{\mathcal{H}}.\end{aligned} \tag{5.19}$$

We adjust the weights for components in $\tilde{\mathcal{A}}_i$: Let $W_i^* = W_i$ and $\omega(\Gamma_l(W_i^*)) = \omega(\Gamma_l(W_i)) - \frac{1-\omega_{\min}}{2}$ for $\Gamma_l(W_i^*) = \Gamma_l(W_i) \in \tilde{\mathcal{A}}_i$ and $\omega(\Gamma_l(W_i^*)) = \omega(\Gamma_l(W_i))$ otherwise. (The set as a subset of $\mathbb{R}^2$ is left unchanged, we have only changed the weights of the boundary components.) This implies

$$\|W_i^*\|_\omega \le \|W_i\|_\omega - \tfrac{1}{2}h_*(1-\omega_{\min})|\partial W_i \cap B_{i+1}^i|_{\mathcal{H}}. \tag{5.20}$$

We briefly note that (5.16), (5.17) are still satisfied for W_i^* if we replace $\hat{\omega}_i$ by $\hat{\omega}_{i+1}^*$, where $\hat{\omega}_{i+1}^*(\Gamma_l(W_i^*)) := 1 - \frac{1-\omega_{\min}}{2}\#\{j = 0, \ldots, i+1 : \Theta_l(W_i) \subset B_j^i\}$. Indeed, as $\omega_i(\Gamma_l(W_i)) \ge \hat{\omega}_i(\Gamma_l(W_i))$ by (5.17) we find

$$\frac{\omega_i(\Gamma_l(W_i^*))}{\hat{\omega}_{i+1}^*(\Gamma_l(W_i^*))} = \frac{\omega_i(\Gamma_l(W_i)) - (1-\omega_{\min})/2}{\hat{\omega}_i(\Gamma_l(W_i)) - (1-\omega_{\min})/2} \ge \frac{\omega_i(\Gamma_l(W_i))}{\hat{\omega}_i(\Gamma_l(W_i))} \ge 1$$

for $\Gamma_l(W_i) \in \tilde{\mathcal{A}}_i$. (Observe that $\hat{\omega}_{i+1}^*$ may slightly differ from the desired $\hat{\omega}_{i+1}$ as given in (5.16). Below we will see, however, that the properties are still satisfied for $\hat{\omega}_{i+1}$.)

We set $\tilde{W}_i = (W_i^* \setminus V) \cup \partial V$, where $\overline{V}$ is the smallest rectangle containing Γ, Γ_m and T. As usual we define $\omega(\partial V) = 1$ and adjust the other weights as in (3.8). We then derive by (5.19) and (5.20)

$$\begin{aligned}\|\tilde{W}_i\|_\omega &\le \|W_i^*\|_\omega + |\partial T|_{\mathcal{H}} \le \|W_i\|_\omega - \tfrac{1}{2}h_*(1-\omega_{\min})|\partial W_i \cap B_{i+1}^i|_{\mathcal{H}} + |\partial T|_{\mathcal{H}} \\ &\le \|W_i\|_\omega + h_*(1-\omega_{\min})\tfrac{1}{4\varepsilon}\alpha(B_{i+1}^i) - \tfrac{1}{4}h_*D\psi_i + |\partial T|_{\mathcal{H}},\end{aligned}$$

where for the other boundary components not involved we proceeded as in (3.12). Recall $|\partial T|_{\mathcal{H}} \le 4\psi_i$. Now choosing $D \ge \frac{16}{h_*}$ we conclude

$$\varepsilon\|\tilde{W}_i\|_\omega + \alpha(\tilde{W}_i) \le \varepsilon\|W_i\|_\omega + \alpha(W_i) + h_*(1-\omega_{\min})\alpha(B^i_{i+1}).$$

Define $\lambda_{i+1} = |\Gamma|_\infty$ and $\bar{u}_{i+1} = \bar{u}_i$. Observe that $W_i^* \in \mathcal{W}_{\lambda_i}$. In fact, (3.5)(iv) follows from the definition of the weights and (5.21) below. Moreover, (3.5)(ii) is a consequence of (5.17)(ii). As before, following the proof of Lemma 5.4, we find a set $W_{i+1} = (W_i^* \setminus V') \cup \partial V' \in \mathcal{W}_{\lambda_{i+1}}$ for a rectangle $V' \supset V$ with $W_{i+1} \subset \tilde{W}_i$ and $B^{i+1}_j = B^i_j \setminus S_{\lambda_{i+1}}(W_{i+1})$ for $j = 0, \dots, i+1$ such that (5.13), (5.15) hold and (5.16), (5.17) are satisfied for $\hat{\omega}_{i+1}$. Observe that (5.14) does not follow from Lemma 5.4 as $B^{i+1}_{i+1} \neq \emptyset$. We postpone the proof of (5.14) to Lemma 5.5 below. We now continue with the next iteration step.

In each iteration step either the number of components satisfying (5.16) increases or the volume of W_i decreases by at least $(2C_2 s)^2$. Consequently, after a finite number of steps, denoted by i^*, we find a set $U = W_{i^*} \in \mathcal{W}^s_{\lambda_U}$, $\lambda_U \ge 0$, satisfying (5.16) for all boundary components $\Gamma_l(U)$ with $N^{2\hat{\tau}_l}(\partial R_l(U)) \subset H(U)$. Let $\bar{u} = \bar{u}_{i^*}$. Then (5.10) holds for all such boundary components as $\sum_n (2/3)^n < \infty$, $\hat{\omega}(\Gamma_l(U)) \ge \omega_{\min}$ for all $\Gamma_l(U)$ by (5.14)(i) and since υ can be chosen in dependence of h_*. Similarly, by (5.14)(i) we find $\sum_{j=0}^{i^*} \alpha(B^{i^*}_j) \le 2\alpha(W)$ and by (5.17) we get $\omega(\Gamma_l(U)) \ge \omega_{\min}$ for all $\Gamma_l(U)$. Setting $\sigma = 2(1-\omega_{\min})$, by (5.13) and $h_* \le 1$ we conclude

$$\varepsilon\|U\|_* + \alpha(U) \le (1 - \tfrac{1}{2}\sigma)^{-1}\varepsilon\|W\|_* + (1+\sigma)\alpha(W) \le (1+\sigma)(\varepsilon\|W\|_* + \alpha(W)).$$

As υ is chosen in dependence of h_* and σ, the constant C_2 in (5.12) depends only on h_* and σ. Finally, the property $|W \setminus U| \le c\|U\|^2_\infty$ relies on the isoperimetric inequality and can be derived as in Corollary 3.3.

It remains to indicate the necessary changes if in some iteration step i the choice of Γ is not unique. If there are several components $\Gamma_1, \dots, \Gamma_m$ with $\lambda_{i+1} := |\Gamma_j|_\infty > \lambda_i$ for $j = 1, \dots, m$ we choose an order such that $\Gamma_1, \dots, \Gamma_{m'}$, $m' \le m$, are the components satisfying (4.1), (5.3), (5.4). We now apply Theorem 5.1 successively on each Γ_j, $j = 1, \dots, m'$, and replace $\mathcal{T}_{\lambda_{i+1}}(W_{i+1})$ in (5.15), (5.16) by $\mathcal{T}^j_{\lambda_{i+1}}(W_{i+1}) := \mathcal{T}_{\lambda_i}(W_i) \cup \bigcup_{k=1}^j \{\Gamma_k\}$. For each Γ_j, $j = m'+1, \dots, m$, we proceed as in one of the cases a) - c) and let $\mathcal{T}^j_{\lambda_{i+1}}(W_{i+1}) := \mathcal{T}_{\lambda_i}(W_i) \cup \bigcup_{k=1}^{m'} \{\Gamma_k\}$ in (5.15), (5.16). $\square$

It remains to show (5.14) in case c).

Lemma 5.5 *If in the i-th iteration step of the above modification procedure case c) is applied, then* (5.14) *holds for* W_{i+1}.

Proof. We first show that

$$|\Gamma_l(W_i)|_\infty < 19\bar{\tau} \quad \text{and} \quad \mathrm{dist}(\Gamma_l(W_i), \Psi_i) \le \bar{\tau} \qquad \text{for all } \Gamma_l(W_i) \in \mathcal{A}_i. \tag{5.21}$$

For sets $\Gamma_l = \Gamma_l(W_i)$ intersecting $\Psi_i \subset N^{\bar{\tau}}(\Gamma)$ this is clear by construction of Ψ_i and Corollary 4.4. (We can assume that property (4.1) holds and Corollary 4.4 is applicable as otherwise we would have applied case (a).) Now assume $\Gamma_l \cap N^{\bar{\tau}}(\Gamma) = \emptyset$ but $M^{\eta^i_l}(\Gamma_l) \cap \Psi_i \neq \emptyset$ for $\Gamma_l \in \mathcal{G}_i$, which implies $\mathrm{dist}(\Gamma_l, \Psi_i) < \eta^l_i \leq 21\upsilon\lambda_i \leq 21\bar{\tau}$. In particular, this yields $\Gamma_l \cap N^{22\bar{\tau}}(\Gamma) \neq \emptyset$. Recall that $\Gamma_m \cap \overline{\Psi}_1 \neq \emptyset$ and $|\Gamma_m|_\infty \geq \hat{\tau} \geq 19 \cdot 22\bar{\tau}$ for q large enough. Therefore, applying Corollary 4.4 for $\bar{t} = 22\bar{\tau}$ we derive that that $|\Gamma_l|_\infty \leq 19 \cdot 22\bar{\tau}$. Repeating the above arguments we obtain $\mathrm{dist}(\Gamma_l, \Psi_i) \leq 21\eta^l_i = 21\bar{\tau}_l \leq \upsilon \cdot 21 \cdot 19 \cdot 22\bar{\tau} < \frac{\bar{\tau}}{2}$ for υ small enough due to the choice of Γ with respect to $|\cdot|_\infty$. This gives the second part of (5.21). Moreover, we have $\Gamma_l \cap N^{\bar{\tau}}(\Gamma) \neq \emptyset$ as $M^{21\bar{\tau}_l}(\Gamma_l) \cap N^{\tau}(\Gamma) \neq \emptyset$ and $\tau \leq \frac{\bar{\tau}}{2}$ by (4.14). This, however, gives a contradiction to the assumption and thus $\Gamma_l \cap N^{\bar{\tau}}(\Gamma) \neq \emptyset$. Then the first part of (5.21) follows again from Corollary 4.4.

We now show that (5.14) holds for W_{i+1}. Note that by Lemma 5.4 we have $W_{i+1} = (W^*_i \setminus V') \cup \partial V'$ for a rectangle V' which contains Γ, Γ_m and T. Moreover, recall that $W_i = W^*_i$ only differ by the definition of the weights.

First of all, to see (5.14)(ii) it suffices to show that either $\Theta_l(W^*_i) \subset B^i_j$ for some $0 \leq j \leq i+1$ or $\Theta_l(W^*_i) \cap \bigcup_{j=0}^{i+1} B^i_j = \emptyset$ and $\omega(\Gamma_l(W^*_i)) = 1$. In fact, we can then follow the argumentation in the proof of Lemma 5.4 to obtain the desired property also for the sets $B^{i+1}_j = B^i_j \setminus S_{\lambda_{i+1}}(W_{i+1})$, $j = 0, \ldots, i+1$.

Recall that $\Theta_l(W^*_i) \subset B^i_{i+1}$ or $\Theta_l(W^*_i) \cap B^i_{i+1} = \emptyset$ for all $(\Gamma_l(W^*_i))_l$. Thus, if $\Theta_l(W^*_i) \not\subset B^i_j$ for some $0 \leq j \leq i+1$ we find $\Theta_l(W^*_i) \cap \bigcup_{j=0}^{i+1} B^i_j = \emptyset$ by (5.14)(ii) for iteration step i. This particularly implies $\Gamma_l(W^*_i) \notin \tilde{\mathcal{A}}_i$ as $\Theta_l(W^*_i) \cap B^i_{i+1} = \emptyset$. Again by (5.14)(ii) and the construction of the weights in (5.20) we then get $\omega(\Gamma_l(W^*_i)) = 1$, as desired.

We concern ourselves with (5.14)(i). First, the assertion is clear for $x \in W_{i+1} \setminus W_i$ as $W_{i+1} \setminus W_i \subset \partial V'$ and $\partial V' \in S_{\lambda_{i+1}}(W_{i+1})$. For $x \notin W_{i+1} \setminus W_i$ it is enough to show the property for $(B^i_j)_{j=1}^{i+1}$ since $B^{i+1}_j \subset B^i_j$ for $j = 0, \ldots, i+1$. As $\bigcup_{l=1}^n \Theta_l(W_i) \subset W_i$ and thus $B^i_{i+1} \subset W_i$, it is elementary to see that it suffices to confirm $B^i_{i+1} \cap \bigcup_{j=0}^i B^i_j \subset W_i \setminus W_{i+1}$. Recall that $\Gamma, \Gamma_m \notin \mathcal{A}_i$. By the definition of B^i_{i+1} we have

$$B^i_{i+1} \cap \bigcup\nolimits_{j=0}^{i} B^i_j \subset B^i_{i+1} \cap \big(f(M^{21\bar{\tau}}(\Gamma)) \cup f(M^{21\bar{\tau}_m}(\Gamma_m))\big),$$

where $f(A) = A$ if $A \cap \Psi_i \neq \emptyset$ and $f(A) = \emptyset$ else for $A \subset \mathbb{R}^2$. (The possible different cases can be seen in Figure 5, 6, 7.) To see this, let $\mathcal{A}^* \subset \{\Gamma, \Gamma_m\}$ such that the boundary component is contained in $\mathcal{A}^*$ if the corresponding neighborhood intersects Ψ_i. Observe that if $B^i_j \cap B^i_{i+1} \neq \emptyset$, then by (5.14)(iii) (for W_i) we get $B^i_j \cap B^i_{i+1} \subset B^i_j \cap W_i \subset M^{\eta^i_l}(\Gamma_l)$ for some $\Gamma_l \in \mathcal{G}_i \setminus \tilde{\mathcal{A}}_i = \mathcal{G}_i \setminus \mathcal{A}_i$. (The last equality follows from $\mathcal{G}_i \cap \mathcal{A}_i \subset \tilde{\mathcal{A}}_i$.) On the other hand, by (5.14)(ii),(iii) we derive that each $\Gamma_l(W_i) \in \mathcal{A}_i$ with $\Theta_l(W_i) \subset B^i_j$ satisfies $\Gamma_l(W_i) \notin \mathcal{G}_i$, $\Theta_l(W_i) \cap \Psi_i \neq \emptyset$ and thus $\Theta_l(W_i) \cap M^{\eta^i_l}(\Gamma_l) = \emptyset$ for all $\Gamma_l \in \mathcal{G}_i \setminus (\mathcal{A}_i \cup \mathcal{A}^*)$. Likewise, we get

$\Psi_i \cap M^{\eta_l^i}(\Gamma_l) = \emptyset$ for all $\Gamma_l \in \mathcal{G}_i \setminus (\mathcal{A}_i \cup \mathcal{A}^*)$ and thus $(B^i_j \cap B^i_{i+1}) \cap M^{\eta_l^i}(\Gamma_l) = \emptyset$ for all $\Gamma_l \in \mathcal{G}_i \setminus (\mathcal{A}_i \cup \mathcal{A}^*)$. This implies $B^i_j \cap B^i_{i+1} \subset M^{\eta_l^i}(\Gamma_l)$ for some $\Gamma_l \in \mathcal{A}^*$.

Setting $\Phi := \{x \in Q_\mu : \mathrm{dist}(x, \Psi_i) \le 20\bar{\tau}\}$ and recalling (5.21) we then find

$$B^i_{i+1} \cap \bigcup\nolimits_{j=0}^{i} B^i_j \subset \Phi \cap \big(f(M^{21\bar{\tau}}(\Gamma)) \cup f(M^{21\bar{\tau}_m}(\Gamma_m))\big).$$

We now differ the cases (I) and (II) as considered in Lemma 4.6, 4.7. In case (I) we get $\Phi \cap W_{i+1} = \emptyset$ as by Lemma 4.6(i) the rectangle V' satisfies $\Phi \subset V'$. In (II)(i) the assertion follows as $M^{21\bar{\tau}}(\Gamma), M^{21\bar{\tau}_m}(\Gamma_m) \cap \Psi_i = \emptyset$. Finally, in (II)(ii) it suffices to derive

$$B^i_{i+1} \cap \bigcup\nolimits_{j=0}^{i} B^i_j \subset \Phi \cap \{x : x_1 \ge -l_1 - \psi\} \cap M^{21\bar{\tau}_m}(\Gamma_m), \tag{5.22}$$

where without restriction we treat the case $\Gamma_m \cap N(\Gamma) \subset N_{1,-}(\Gamma)$. Then Lemma 4.7(i) gives $\Phi \cap \{x : x_1 \ge -l_1 - \psi\} \cap M^{21\bar{\tau}_m}(\Gamma_m) \subset V'$ which finishes the proof of (5.14)(i). To see (5.22), first note that $f(M^{21\bar{\tau}}(\Gamma)) = \emptyset$ and $\Psi_i \subset \{x : x_1 \ge -l_1 - \psi\}$. Consequently, recalling (5.14)(ii),(iii) if the assertion was wrong, there would be some $\Gamma_l(W_i) \in \mathcal{A}_i \setminus \mathcal{G}_i$ which satisfies $\Theta_l(W_i) \subset M^{21\bar{\tau}_m}(\Gamma_m)$ and $\Theta_l(W_i) \cap \{x : x_1 < -l_1 - \psi\} \neq \emptyset$. Again by (5.14)(iii) we then get $\Theta_l(W_i) \subset M_2^{21\bar{\tau}_m}(\Gamma_m)$ (see Figure 6) and therefore $\Theta_l(W_i) \cap \Psi_i = \emptyset$. This implies $\Gamma_l(W_i) \notin \mathcal{A}_i$ and yields a contradiction.

Finally we show (5.14)(iii). It suffices to consider B^{i+1}_{i+1} as for the other sets the property follows from Lemma 5.4. Without restriction we set $V' = (-v_1, v_1) \times (-v_2, v_2)$. We first observe that $\Phi \setminus \overline{V'} \subset N^{21\bar{\lambda}}(\partial V')$, where $\bar{\lambda} = \upsilon\lambda_{i+1}$. This is a consequence of the definition of Φ and the fact that $\bar{\lambda} = \bar{\tau}$. We may assume that $|\pi_1 \Gamma_m| \ge \frac{1}{2}|\pi_2\Gamma_m|$. In fact, if $l_2 \le \frac{l_1}{2}$ this follows from Corollary 4.4, if $l_2 \ge \frac{l_1}{2}$ then l_1, l_2 are comparable and the assumption holds possibly after a rotation of the components by $\frac{\pi}{2}$. Recalling $|\Gamma_m|_\infty \ge \hat{\tau}$ we thus obtain $|\pi_1\Gamma_m| \ge \frac{1}{\sqrt{5}}\hat{\tau} = \frac{q^2\bar{\tau}}{\sqrt{5}h_*}$. Recall that $|\pi_1\Gamma_m \cap \pi_1\Gamma| \le C\frac{\bar{\tau}}{h_*}$ by Lemma 4.1(ii).

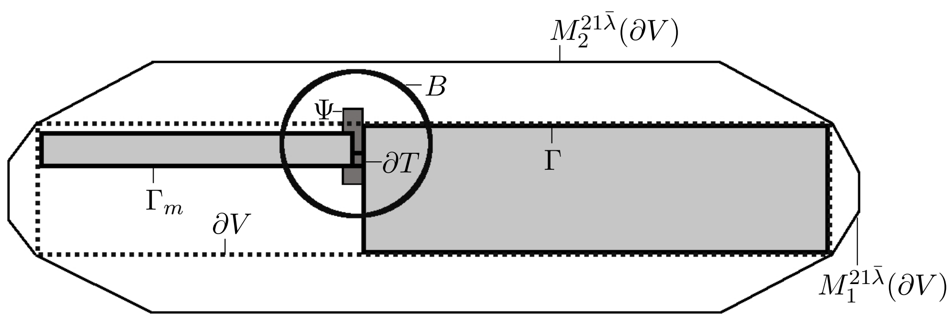

Figure 8: Sketch of the components Γ, Γ_m, ∂T and in dashed lines the corresponding rectangle V (which in this example coincides with V'). The ball B is chosen large enough such that $\Phi \subset B$. (The proportions were adapted for illustration purposes.)

We now find for all $x \in \Phi \setminus V'$ with $x_1, x_2 \geq 0$

$$\frac{v_1 - x_1}{x_2 - v_2} \geq \frac{\min\{|\pi_1\Gamma|, |\pi_1\Gamma_m|\} - C\bar{\tau}h_*^{-1}}{21\bar{\tau}} \geq \frac{q^2\bar{\tau}(\sqrt{5}h_*)^{-1} - C\bar{\tau}h_*^{-1}}{21\bar{\tau}} \geq qh_*^{-1}$$

for q sufficiently large and may proceed likewise for $\pm x_1, \pm x_2 \geq 0$. Thus, upon recalling (4.16) and (4.17), we obtain $B^i_{i+1} \cap W_{i+1} \subset \Phi \setminus \overline{V'} \subset M_2^{21\bar{\lambda}}(\partial V')$. As $\upsilon|\partial V'|_\infty \geq \bar{\lambda}$, $\omega(\partial V') = 1$ and $\partial V' \cap \bigcup_{j=0}^{i+1} B_j^{i+1} = 0$ (since $\partial V' \subset S_{\lambda_{i+1}}(W_{i+1})$) we finally obtain (5.14)(iii). □

Remark 5.6 (i) During the modification process in Theorem 5.2 the components $X_{n+1}(W), \dots, X_m(W)$ at the boundary of Q_μ might be changed and the corresponding components of U are given by $X_j(U) = X_j(W) \setminus \overline{H(U)}$ for $j = n+1, \dots, m$. In particular, we observe $|\partial X_j(U)|_* \leq |\partial X_j(W)|_*$ arguing as in Lemma 3.1.

(ii) In general, the components of the set U might not be connected as they can be separated by other components during the modification process. However, by application of Corollary 3.3 we obtain a set $U' \subset U$ with $\|U'\|_* \leq \|U\|_*$ and $|U \setminus U'| \leq C\|U'\|^2_\infty \leq C\mu\|U'\|_\infty$ such that all components of U' are pairwise disjoint and rectangular and thus particularly connected. Moreover, recalling the modification process (cf. Section 8.2) we find that for each $\Gamma(U)$ the corresponding rectangle $R(U)$ given by (3.5) is contained in a component of U'.

5.3 Proof of the main theorem

We now are in a position to prove our Korn-Poincaré-type inequality. We split the proof into three steps and begin with a corollary of Theorem 5.2. In what follows, we will frequently employ (5.10) and in doing so we apply the inequalities

$$|\Theta|_{\mathcal{H}} \leq C|\Theta|_* \leq C|\Gamma|_\infty \leq C|\Gamma|_{\mathcal{H}}, \qquad |\Theta|_* \leq |\partial R|_* \leq |\partial R|_{\mathcal{H}} \tag{5.23}$$

for a boundary component $\Theta \subset \Gamma$ and the corresponding rectangle R given by (3.5). The properties follow from (3.5)(i), (3.6) and Lemma 3.1(vi)). Moreover, we observe that for $W \in \mathcal{V}^s$ and a subset $A \subset Q_\mu$ one has $\sum_{\Gamma_l(W)} |\Gamma_l(W) \cap A|_{\mathcal{H}} \leq 2|\partial W \cap A|$. Recall the definition of $\mathcal{W}^{C_2 s}$ in Theorem 5.2 as well as (2.8) and (2.1).

Corollary 5.7 *Let $\varepsilon, \mu, h_* > 0$. Let $U \subset Q_\mu = (-\mu, \mu)^2$, $U \in \mathcal{W}^{C_2 s}$ and $u \in H^1(U)$. Assume there is a square $\tilde{Q} = (-\tilde{\mu}, \tilde{\mu})^2 \subset Q$ such that (5.10) is satisfied for all components $\Theta_l(U)$ having nonempty intersection with $\tilde{Q}$, where $\bar{u}$ is the extension of u defined in (5.6). Then there is a universal constant C such that*

$$|E\bar{u}|(\tilde{Q})^2 \leq (\mathcal{E}(\tilde{Q}))^2 \leq C\tilde{\mu}^2\|e(u)\|^2_{L^2(U\cap\tilde{Q})} + CC_1\mu\varepsilon|\partial U \cap \tilde{Q}|_{\mathcal{H}}|\partial U \cap Q|_{\mathcal{H}},$$

where C_1 is the constant in Theorem 5.2.

Proof. Recall $\alpha(V) = \|e(\bar{u})\|^2_{L^2(V)}$ for $V \subset U$. Note that by Hölder's inequality we have

$$|E\bar{u}|(V)^2 \le (\mathcal{E}(V))^2 \le C|V|\alpha(V) + C\Big(\int_{\overline{V}\cap J_{\bar{u}}} |[u]|\, d\mathcal{H}^1\Big)^2. \tag{5.24}$$

for $V \subset U$. Moreover, observe that $J_{\bar{u}} \cap \tilde{Q} = \bigcup_l \Theta_l(U) \cap \tilde{Q}$. We now derive by (5.10) and (5.23)

$$\begin{aligned}(\mathcal{E}(\tilde{Q}))^2 &\le C\tilde{\mu}^2\alpha(\tilde{Q}) + C\big(\textstyle\sum_{\Theta_l(U)\cap\tilde{Q}\neq\emptyset} |\Theta_l(U)\cap\tilde{Q}|^{\frac{1}{2}}_{\mathcal{H}}\|[\bar{u}]\|_{L^2(\Theta_l(U))}\big)^2\\ &\le C\tilde{\mu}^2\alpha(\tilde{Q}) + CC_1\varepsilon|\partial U\cap\tilde{Q}|_{\mathcal{H}}\textstyle\sum_l |\Gamma_l(U)|^2_\infty\\ &\le C\tilde{\mu}^2\alpha(\tilde{Q}\cap U) + CC_1\mu\varepsilon|\partial U\cap\tilde{Q}|_{\mathcal{H}}|\partial U\cap Q|_{\mathcal{H}}.\end{aligned}$$

In the second and third step we employed Hölder's inequality. In the last step we used $\alpha(\tilde{Q}\setminus U)$ by (5.9) as well as $|\Gamma_l(U)|_\infty \le 2\mu$ and $\sum_l |\Gamma_l(U)|_\infty \le C|\partial U\cap Q|_{\mathcal{H}}$. □

We now formulate and prove the main theorem of this paper first in terms of sets $W \in \mathcal{V}^s$. The assertion follows combining Theorem 5.2, Corollary 5.7 and Theorem 2.1.

Theorem 5.8 *Let $\varepsilon, \mu > 0$ and $h_* > 0$ sufficiently small. There is a constant $C = C(h_*)$ and a universal constant $\bar{c} > 0$ such that for all sets $W \subset Q_\mu = (-\mu,\mu)^2$, $W \in \mathcal{V}^s$, with connected boundary components, and all $u \in H^1(W)$ the following holds: There is a set $U \in \mathcal{W}^{C_2 s}$ with $|U\setminus W| = 0$, $|W\setminus U| \le \bar{c}|\partial U\cap Q_\mu|^2_\infty$ and*

$$\varepsilon\|U\|_* + \|e(u)\|^2_{L^2(U)} \le (1+h_*)\big(\varepsilon\|W\|_* + \|e(u)\|^2_{L^2(W)}\big)$$

such that for the square $\tilde{Q} = (-\tilde{\mu},\tilde{\mu})^2$ with $\tilde{\mu} = \max\{\mu - 2|\partial U\cap Q_\mu|_{\mathcal{H}}, 0\}$ we have

$$\|u(x) - (A\,x + c)\|^2_{L^2(U\cap\tilde{Q})} \le C\mu^2\|e(u)\|^2_{L^2(U\cap\tilde{Q})} + C\mu\varepsilon|\partial U\cap Q|^2_{\mathcal{H}}$$

for some $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $c \in \mathbb{R}^2$.

Proof. Choose $\sigma = \sigma(h_*) \le h_*$ and apply Theorem 5.2 to get a set $U \in \mathcal{W}^{C_2 s}$ with $|U\setminus W| = 0$ satisfying (5.11). We can assume that $\tilde{\mu} = \mu - 2|\partial U\cap Q_\mu|_{\mathcal{H}} > 0$ as otherwise there is nothing to show. By definition of $\tilde{Q}$ and (5.2) it is not hard to see that every boundary component $\Gamma_l(U)$ with $\Theta_l(U)\cap\tilde{Q}\neq\emptyset$ fulfills $N^{2\hat{\tau}_l}(\partial R_l(U)) \subset H(U)$ and therefore (5.10) holds. The claim now follows from Theorem 2.1 and Corollary 5.7. □

We can now finally give the proof of Theorem 1.1.
Proof of Theorem 1.1. Let $u \in SBD^2(Q_\mu)\cap L^p(Q_\mu)$ for $1 \le p \le 2$ be given. By Theorem 2.3 we find a set $W = \tilde{Q}\setminus\bigcup_{j=1}^n Q_j$ with rectangles $(Q_j)_j$ and

a modification $\tilde{u} \in H^1(W)$ with $\|\tilde{u} - u\|_{L^p(W)} + \|e(\tilde{u}) - e(u)\|_{L^2(W)} \le \delta$ for $\delta > 0$ arbitrarily small. Possibly replacing the rectangles by infinitesimally larger rectangles we can assume that there is some $s > 0$ such that $Q_j \in \mathcal{U}^s$ for $j = 1, \ldots, n$ and thus $W \in \mathcal{V}^s$ with connected boundary components and

$$\|W\|_* \le (1 + \bar{c}h_*) \sum\nolimits_{j=1}^n \mathrm{diam}(Q_j) \le (1 + \bar{c}h_*)(\mathcal{H}^1(J_u) + \delta) \le (1 + \bar{c}h_*)\mathcal{H}^1(J_u)$$

for δ sufficiently small. We now apply Theorem 5.8 on $\tilde{u}$ and W. Up to a modification by applying Corollary 3.3 we can assume that the components of U are pairwise disjoint rectangles $R_1, \ldots, R_n$ with $\sum_j |R_j|_* \le \|U\|_*$ which yields $\sum_j |R_j|_\infty \le (1 + \bar{c}h_*)\|U\|_* \le (1 + \bar{c}h_*)\big(\mathcal{H}^1(J_u) + \varepsilon^{-1}\|e(\tilde{u})\|^2_{L^2(W)}\big)$. Clearly, we have $|Q_\mu \setminus U| \le \bar{c}(\sum_j |R_j|_\infty)^2$. Finally, as we may assume $\|U\|_* \le C\mu$ (otherwise $\tilde{Q} = \emptyset$) we conclude by Hölder's inequality

$$\begin{aligned}\|u(x) - (A\,x + c)\|^2_{L^p(U \cap \tilde{Q})} &\le C\delta^2 + C\mu^{4/p-2}\|\tilde{u}(x) - (A\,x + c)\|^2_{L^2(U \cap \tilde{Q})} \\ &\le C\delta^2 + C\mu^{4/p}\|e(\tilde{u})\|^2_{L^2(U\cap\tilde{Q})} + C\mu^{4/p-1}\varepsilon\|U\|^2_* \\ &\le C(1 + \mu^{4/p})\delta^2 + C\mu^{4/p}\big(\|e(u)\|^2_{L^2(Q_\mu)} + \varepsilon\mathcal{H}^1(J_u)\big).\end{aligned}$$

As δ was arbitrary, we obtain the desired estimate. □

6 Trace estimates for boundary components

This section is entirely devoted to the proof of Theorem 5.1. We start with some preliminary estimates including an approximation of u by a piecewise infinitesimal rigid motion. Here we also discuss the passage from an estimate in the neighborhood to a trace estimate. Afterwards the proof is performed in several steps. We will first assume that in a neighborhood of Γ only small boundary components are present (Step 1). Then we suppose that we have a bound on the projection $\|\cdot\|_\pi$ (recall definition (4.6)) which will allow us to apply a slicing method in the regions of the domain where too large boundary components exist (Step 2). In this context, we have to be particularly careful at the corners of Γ (Step 3). Finally, we present the general proof taking into account the possible existence of sets Ψ^1, Ψ^2 discussed in Section 4.2 (Step 4). At this point, the trace theorem we derived in Section 2.3 will play an essential role.

6.1 Preliminary estimates

Assume $h_*, q, \omega_{\min} > 0$ have been chosen in the previous section (in this order, see Remark 4.2). The parameter $\upsilon > 0$ considered before is not assumed to be already chosen, but will be specified below. Moreover, let $r > 0$ such that $r(1 - \omega_{\min})^3 \ge \upsilon$. This implies $\upsilon = \upsilon(h_*, q, \omega_{\min}, r)$. Moreover, we will show

$r = r(h_*, q)$ and recalling that $\sigma = 2(1-\omega_{\min})$ (see proof of Theorem 5.2) as well as using $q = q(h_*)$ we will find $\upsilon \sim C(h_*)\sigma^3$ (cf. Remark 5.3).

Let $\varepsilon > 0$, $\lambda \geq 0$ and let $u \in H^1(W)$ for $W = W_i \in \mathcal{W}^s_\lambda$. (We drop the subscript i in the following.) Recall $\alpha(U) = \|e(u)\|^2_{L^2(U)}$ for $U \subset W$. Let $\Gamma = \Gamma(W)$ with $|\Gamma|_\infty \geq \lambda$ and the corresponding neighborhoods $N(\Gamma) = N^\tau(\Gamma)$ and $\hat{N}(\Gamma) = N^{2\hat{\tau}}(\Gamma)$ be given. In addition, we define $\tilde{N}(\Gamma) = N(\Gamma) \setminus (X^1 \cup X^2)$, where $\partial X^1 = \Gamma^1$, $\partial X^2 = \Gamma^2$ are the boundary components satisfying $|\Gamma^i|_\infty \geq \hat{\tau} = q^2 h_*^{-1}\upsilon|\Gamma|_\infty$ and $\Gamma_i \cap N(\Gamma) \neq \emptyset$, see Corollary 4.4 (note that $X_1, X_2 = \emptyset$ is possible). As before, for shorthand we will write N, $\hat{N}$ and $\tilde{N}$ if no confusion arises. By Remark 2.5(i) it is not restrictive to assume that $J = J(\Gamma)$ as defined before equation (4.2) consists of (almost) squares. Suppose that (4.1), (5.3), (5.4) and (5.7) for $\mathcal{T}_{\hat{\tau}}(W)$ hold. Assume that $N^{2\hat{\tau}} \subset H(W)$.

Note that the inclusion $W \cap N \subset \tilde{N}$ may be strict due to boundary components Γ_l with $|\Gamma_l|_\infty < \hat{\tau}$ having nonempty intersection with N. Observe that by (3.5)(iv),(v) and (3.6) for q large we have $|\partial R_l|_\infty < \hat{\tau}$ for these Γ_l, where R_l is the corresponding rectangle given by (3.5)(i),(v). Then for υ sufficiently small we get

$$N^{2\hat{\tau}_l}(\partial R_l) \subset N^{2\hat{\tau}}(\Gamma) \subset H(W) \tag{6.1}$$

and thus $\Gamma_l \in \mathcal{T}_{\hat{\tau}}(W)$. Consequently, we can extend u as an SBD function from $N \cap W$ to $\tilde{N}$ as defined in (5.6). For convenience we denote this extension still by u. Clearly, one has $\alpha(\tilde{N}) = \alpha(W \cap N)$.

We now begin with some preliminary estimates. Recall definition (2.8) as well as (5.24). First assume $\tilde{N} = N$. We apply Theorem 2.1 on each $Q \in J$ recalling that the constant is invariant under rescaling of the domain: This yields functions $\bar{A} : N \to \mathbb{R}^{2\times 2}_{\text{skew}}$, $\bar{c} : N \to \mathbb{R}^2$ being constant on each $Q \in J$ such that by (5.3) and (5.7) we obtain

$$\begin{aligned}
\int_N |u(x) - (\bar{A}\,x + \bar{c})|^2\,dx &\leq C \sum\nolimits_{Q\in J} (\mathcal{E}(Q))^2 \leq C(\mathcal{E}(N))^2 \\
&\leq C\upsilon|\Gamma|^2_\infty \alpha(N) + C\big(\sum\nolimits_l |\Theta_l \cap N|^{1/2}_{\mathcal{H}} \|[u]\|_{L^2(\Theta_l)}\big)^2 \\
&\leq C\upsilon^2|\Gamma|^3_\infty\varepsilon + CC_*\upsilon^{-4}\varepsilon|\partial W \cap N|_{\mathcal{H}}\big(\sum\nolimits_l |\Theta_l|_*\big)^2 \\
&\leq C\upsilon^2|\Gamma|^3_\infty\varepsilon + CC_*\upsilon^{-4}\varepsilon|\partial W \cap \hat{N}|^3_{\mathcal{H}} \leq C(1+C_*)\upsilon^{-1}|\Gamma|^3_\infty\varepsilon
\end{aligned} \tag{6.2}$$

for some $C = C(h_*, q)$. In the third step we employed Hölder's inequality and $|N| \leq C\upsilon|\Gamma|^2_\infty$. In the penultimate step we used $\sum_l |\Theta_l|_* \leq C\sum_l |\Gamma_l|_{\mathcal{H}} \leq C|\partial W \cap \hat{N}|_{\mathcal{H}}$ by (5.23) and (6.1). The constants used in this section may as usual vary from line to line but are always independent of the parameters r, $\omega_{\min}$ and υ.

In the general case, recall the definition of Ψ^1 and Ψ^2 in Section 4.2. By Lemma 4.6(ii) and Lemma 4.7(ii) it is not restrictive to assume that Ψ^i_1, $\Psi^{i,*}_2$, Ψ^i_3 are squares for $i = 1, 2$. Similarly as in the previous estimate we obtain functions

$\bar{A} : N \to \mathbb{R}^{2\times 2}_{\rm skew}$, $\bar{c} : N \to \mathbb{R}^2$ being constant on each $Q \in J$ with $Q \cap (\Psi^1 \cup \Psi^2) = \emptyset$ and Ψ^i_j, $i = 1, 2$, $j = 1, 2, 3$, such that

$$\begin{aligned} &(i) \quad \int_{\tilde{N}\setminus(\Psi^1\cup\Psi^2)} |u(x) - (\bar{A}\,x + \bar{c})|^2\,dx \le C(1 + C_*)\upsilon^{-1}|\Gamma|^3_\infty \varepsilon, \\ &(ii) \quad \int_{\Psi^i_j} |u(x) - (\bar{A}\,x + \bar{c})|^2\,dx \le C(1 + C_*)\upsilon^{-3}|\Gamma|^2_\infty \psi^i \varepsilon, \end{aligned} \tag{6.3}$$

To see (ii), we apply Theorem 2.1 on the sets Ψ^i_1, $\Psi^{i,*}_2$, Ψ^i_3 and follow the lines of the previous estimate to obtain that the left hand side is bounded by $C\upsilon|\Gamma|^2_\infty \alpha(\Psi^i_j) + CC_*\upsilon^{-4}\varepsilon|\partial W \cap \Psi^i_j|_{\mathcal{H}}|\partial W \cap \hat{N}|^2_{\mathcal{H}}$. We then use $\alpha(\Psi^i_j) \le D\varepsilon(1-\omega_{\min})^{-1}\psi^i \le D\varepsilon\upsilon^{-1}\psi^i$ and $|\partial W \cap \Psi^i|_{\mathcal{H}} \le C\upsilon^{-1}\psi^i$ by (5.3), (5.4).

The goal will be to replace the functions $\bar{A}$, $\bar{c}$ in (6.3) by constants $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c \in \mathbb{R}^2$ such that

$$\begin{aligned} &(i) \quad \int_{\tilde{N}\setminus(\Psi^1_2\cup\Psi^2_2)} |u(x) - (A\,x + c)|^2\,dx \le C(1 + rC_*)\upsilon^{-3}|\Gamma|^3_\infty \varepsilon, \\ &(ii) \quad \int_{\Psi^i_2} |u(x) - (A\,x + c)|^2\,dx \le C(1 + rC_*)\upsilon^{-4}|\Gamma|^2_\infty \psi^i \varepsilon, \end{aligned} \tag{6.4}$$

for $i = 1, 2$ and for $r(1-\omega_{\min})^3 \ge \upsilon$. Then the trace theorem applied on each square (if J or Ψ^j, $j = 1, 2$, consist also of rectangles, they can be covered by possibly overlapping squares) implies the assertion:

To satisfy the assumptions of Lemma 2.6, the jump set has to be the union of rectangle boundaries. Therefore, we extend $J_u \cap \tilde{N}$ to $\tilde{J}_u = \bigcup_l \partial R_l \cap \tilde{N}$ by $[u](x) = 0$ for $x \in \tilde{J}_u \setminus J_u$, where R_l are the corresponding rectangles given in (3.5)(i). We observe that by (3.5)(ii) and (5.23) we get

$$\sum\nolimits_l |\partial R_l|_{\mathcal{H}} \le C \sum\nolimits_l |\Theta_l|_*, \quad \sum\nolimits_l |\partial R_l|^{-1}_{\mathcal{H}} |\Theta_l|^2_* \le C \sum\nolimits_l |\Theta_l|_*. \tag{6.5}$$

Note that by every boundary component Γ_l is contained in at most $C(h_*)$ different squares (see Lemma 4.1). Then by Lemma 2.6, either for $\mu \sim \upsilon|\Gamma|_\infty$ or $\mu \sim \psi^i$, (6.4) and (5.7) we obtain for υ small enough

$$\begin{aligned} &\int_\Gamma |u(x) - (A\,x + c)|^2\,d\mathcal{H}^1(x) \\ &\le C\upsilon|\Gamma|_\infty \alpha(\tilde{N}) + CC_* \sum\nolimits_l |\partial R_l|_{\mathcal{H}} \sum\nolimits_l |\partial R_l|^{-1}_{\mathcal{H}} \varepsilon\upsilon^{-4}|\Theta_l|^2_* \\ &\quad + C(\upsilon|\Gamma|_\infty)^{-1}\|u - (A\,\cdot + c)\|^2_{L^2(\tilde{N}\setminus(\Psi^1_2\cup\Psi^2_2))} + \sum\nolimits_{i=1}^2 C(\psi^i)^{-1}\|u - (A\,\cdot + c)\|^2_{L^2(\Psi^i_2)} \\ &\le C(1 + rC_*)\upsilon^{-2}|\Gamma|^2_\infty \varepsilon + C(1 + rC_*)\upsilon^{-4}|\Gamma|^2_\infty \varepsilon \le C(1 + rC_*)\varepsilon\upsilon^{-4}|\Gamma|^2_*, \end{aligned}$$

where for the first two terms we proceeded similarly as in (6.2), also taking (6.5) into account. Finally, choosing $\hat{C} = C = C(h_*, q)$ and $r = r(h_*, q)$ small enough (i.e. also υ small enough) such that $r\hat{C} \le \frac{1}{2}$ we get (5.8), as desired. Consequently, it suffices to establish (6.4).

6.2 Step 1: Small boundary components

We first treat the case that only small components Γ_l lie in N. For $1 > r \geq \upsilon > 0$ define $T = \lfloor \log_r(\upsilon) \rfloor$ and let

$$\mathcal{S}_t(Q) = \{\Gamma_l : \Gamma_l \cap Q \neq \emptyset,\ \upsilon^4 r^{-2t}|\Gamma|_\infty < |\Gamma_l|_\infty \leq \upsilon^4 r^{-2t-2}|\Gamma|_\infty\}$$

for all $t \in \mathbb{N}$ and $\mathcal{S}_0(Q) = \{\Gamma_l : \Gamma_l \cap Q \neq \emptyset, |\Gamma_l|_\infty \leq \upsilon^4 r^{-2}|\Gamma|_\infty\}$.

Lemma 6.1 *Theorem 5.1 holds under the additional assumption that there is some* $\frac{T}{4}+2 \leq t \leq \frac{T}{2}-1$ *such that* $\bigcup_{s>t} \mathcal{S}_s(Q) = \emptyset$ *for all* $Q \in J$ *and* $\sum_Q \#\mathcal{S}_t(Q) \leq \upsilon^{-3} r^{2t+3}$.

Proof. We first observe that the assumption implies $\tilde{N} = N$. Let $\frac{T}{4}+2 \leq t \leq \frac{T}{2}-1$ with the above properties be given and write $\hat{\upsilon} = \upsilon^2 r^{-\frac{2t+1}{2}}$ for shorthand. For later we note that

$$\upsilon^{\frac{7}{4}} r^{-\frac{3}{2}} \leq \hat{\upsilon} \leq \upsilon^{\frac{3}{2}} \sqrt{r}. \tag{6.6}$$

We cover N with squares $\hat{Q}(\xi) = \hat{Q}^{\hat{\upsilon}|\Gamma|_\infty}(\xi)$ of length $2\hat{\upsilon}|\Gamma|_\infty$ and midpoint ξ. (If the sets in $J = J(\Gamma)$ constructed in Section 4.1 are not perfect squares, the sets $\hat{Q}(\xi)$ shall be chosen appropriately. The difference in the possible shapes, however, does not affect the following estimates by Remark 2.5(i).) We will now consider a rectangular path, i.e. a path $\xi = (\xi_0, \ldots \xi_n = \xi_0)$ of square midpoints intersecting all $Q \in J$ such that there are indices i_1, i_2, i_3 with $\xi_j - \xi_{j-1} = \pm 2\hat{\upsilon}|\Gamma|_\infty \mathbf{e}_1$ for all $0 \leq j \leq i_1$, $i_2 \leq j \leq i_3$ and $\xi_j - \xi_{j-1} = \pm 2\hat{\upsilon}|\Gamma|_\infty \mathbf{e}_2$ else. Observe that the number of squares in a path satisfies $n \leq C\hat{\upsilon}^{-1}$ and that we can find $\sim \upsilon\hat{\upsilon}^{-1}$ disjoint rectangular paths in N. Consequently, by assumption and (5.3) we can find at least one rectangular path $P := \bigcup_j \hat{Q}(\xi_j)$ such that

$$\alpha(P) \leq C\hat{\upsilon}\varepsilon|\Gamma|_\infty, \quad \sum_{\Gamma_l \in \hat{\mathcal{S}}(P)} |\Gamma_l|_\infty \leq C\hat{\upsilon}|\Gamma|_\infty, \quad \#\hat{\mathcal{S}}^t(P) \leq C\frac{\hat{\upsilon}}{\upsilon}\upsilon^{-3} r^{2t+3} \tag{6.7}$$

for some sufficiently large constant $C = C(h_*, q)$, where $\hat{\mathcal{S}}(P) = \{\Gamma_l : \Gamma_l \cap P \neq \emptyset\}$ and $\hat{\mathcal{S}}^t(P) = \hat{\mathcal{S}}(P) \cap \bigcup_Q \mathcal{S}_t(Q)$. Here we used that each $\Gamma_l \in \bigcup_{Q\in J}\bigcup_{s\leq t} \mathcal{S}_s(Q)$ intersects at most four adjacent squares $\hat{Q}$ because $|\partial R_l|_\infty \leq C\hat{\upsilon}^2 r^{-1}|\Gamma|_\infty \ll \hat{\upsilon}|\Gamma|_\infty$ by (3.6), (6.6) and $\Gamma_l \subset \overline{R_l}$ (see (3.5)(i)). Observe that the above path can be chosen in the way that also $|P \cap Q| \geq C\frac{\hat{\upsilon}}{\upsilon}|Q|$ for $Q = E_{\pm,\pm}$, where $E_{\pm,\pm}$ denote the squares in the corners of N, i.e. $(\pm l_1, \pm l_2) \cap \overline{E_{\pm,\pm}} \neq \emptyset$. This implies $|P \cap Q| \geq C\frac{\hat{\upsilon}}{\upsilon}|Q|$ for all $Q \in J$. It is convenient to write the above estimate in the form

$$\upsilon^4 r^{-2t-1\pm 1} = \hat{\upsilon}^2 r^{\pm 1}, \quad \#\hat{\mathcal{S}}^t(P)\upsilon^4 r^{-2t-2} \leq C\hat{\upsilon}r. \tag{6.8}$$

We now apply Lemma 2.4(i) with $s = \hat{\upsilon}|\Gamma|_\infty$ and $|V| = |P| \sim \hat{\upsilon}|\Gamma|^2_\infty$. Recall that we get $\|[u]\|_{L^1(\Theta_l)} \le \sqrt{|\Theta_l|_{\mathcal{H}}}\|[u]\|_{L^2(\Theta_l)} \le C\sqrt{C_*\varepsilon\upsilon^{-4}}|\Theta_l|^{3/2}_*$ by (5.7), (5.23) and Hölder's inequality. Arguing similarly as in (6.2) we find $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c \in \mathbb{R}^2$ such that by (6.1), (6.7), (6.8) and (5.23)

$$\begin{aligned}\int_P |u(x) - (A\,x + c)|^2\,dx &\le C\hat{\upsilon}^{-3}(\mathcal{E}(P))^2 \\ &\le C\hat{\upsilon}^{-3}\hat{\upsilon}|\Gamma|^2_\infty\alpha(P) + CC_*\hat{\upsilon}^{-3}\frac{\varepsilon}{\upsilon^4}\big(\textstyle\sum_{\Gamma_l\in\hat{\mathcal{S}}(P)}(|\Theta_l|_*)^{3/2}\big)^2 \\ &\le C\hat{\upsilon}^{-1}|\Gamma|^3_\infty\varepsilon + CC_*\hat{\upsilon}^{-1}r^{-1}|\Gamma|_\infty\frac{\varepsilon}{\upsilon^4}\big(\textstyle\sum_{\Gamma_l\in\hat{\mathcal{S}}^t(P)}|\Gamma_l|_\infty\big)^2 \\ &\quad + CC_*\hat{\upsilon}^{-1}r|\Gamma|_\infty\frac{\varepsilon}{\upsilon^4}\big(\textstyle\sum_{\Gamma_l\in\hat{\mathcal{S}}(P)\setminus\hat{\mathcal{S}}^t(P)}|\Gamma_l|_\infty\big)^2 \\ &\le C\hat{\upsilon}^{-1}|\Gamma|^3_\infty\varepsilon + CC_*\hat{\upsilon}r\frac{\varepsilon}{\upsilon^4}|\Gamma|^3_\infty + CC_*\hat{\upsilon}r\frac{\varepsilon}{\upsilon^4}|\Gamma|^3_\infty.\end{aligned} \tag{6.9}$$

Observing that $\hat{\upsilon}^{-1} \le \hat{\upsilon}\upsilon^{-4}$ by definition of $\hat{\upsilon}$, we derive

$$\int_P |u(x) - (A\,x + c)|^2\,dx \le C(1 + C_*r)\,\hat{\upsilon}\frac{\varepsilon}{\upsilon^4}|\Gamma|^3_\infty =: F. \tag{6.10}$$

We now pass from an estimate on P to an estimate on N. For later purpose in Section 6.3, we consider general subsets $V \subset N$ consisting of squares in J. Then repeating (6.9) we obtain by Lemma 2.4(ii) $\|u(x) - (A\,x + c)\|^2_{L^2(P\cap V)} \le |P \cap V||P|^{-1}CF$.

Since $|P \cap Q| \ge C\frac{\hat{\upsilon}}{\upsilon}|Q|$ for all $Q \in J$ we find $\frac{|V|}{|N|} \ge C\frac{|V\cap P|}{|P|} \ge C\upsilon$. Therefore, by (6.2) (recall that $\tilde{N} = N$) and $\upsilon^2 \le \hat{\upsilon}r$ (see (6.6)) we also have

$$\int_V |u(x) - (\bar{A}\,x + \bar{c})|^2\,dx \le |V \cap P||P|^{-1}CF \le C|V||N|^{-1}F.$$

We apply (2.15) on each $Q \subset V$ with $B_1 = P \cap Q$, $B_2 = Q$ noting that $\bar{A}, \bar{c}$ are constant on each square. (Although $P\cap Q$ is not a rectangle if $Q = E_{\pm,\pm}$, we can still argue as in (2.15) since $P\cap Q$ consists of two rectangles.) As $|P\cap Q| \ge C\frac{\hat{\upsilon}}{\upsilon}|Q|$ for all $Q \in J$ we obtain

$$\|u(x){-}(A\,x{+}c)\|^2_{L^2(Q)} \le C\frac{\upsilon}{\hat{\upsilon}}\Big(\int_Q |u(x){-}(\bar{A}\,x{+}\bar{c})|^2\,dx + \int_{P\cap Q}|u(x){-}(A\,x{+}c)|^2\,dx\Big)$$

and thus summing over all $Q \subset V$ we derive

$$|N|\fint_V |u(x) - (A\,x + c)|^2\,dx \le C\hat{\upsilon}^{-1}\upsilon F = C(1 + C_*r)\,\frac{\varepsilon}{\upsilon^3}|\Gamma|^3_\infty. \tag{6.11}$$

Consequently, setting $V = N$, (6.4)(i) is established, as desired. □

6.3 Step 2: Subset with small projection of components

The next step will be the case that $\|\cdot\|_\pi$ is not too large. For that purpose, recall (4.6) and the definition of $\mathcal{Y}$ (see before (4.3)). Consider some $U \in \mathcal{Y}$ with $|U| \geq Cv|\Gamma|^2_\infty$. Moreover, by $\mathcal{Y}'$ we denote the set of subsets of N consisting of squares in J. (In contrast to $\mathcal{Y}$ the connectedness of the sets is not required.) In this section we show that for all $Z \subset U$, $Z \in \mathcal{Y}'$, one has

$$|U| ⨍_Z |u(x) - (A_U\, x + c_U)|^2\, dx \leq C(1 + rC_*)v^{-3}|\Gamma|^3_\infty \varepsilon \tag{6.12}$$

for $A_U \in \mathbb{R}^{2\times 2}_{\text{skew}}$, $c_U \in \mathbb{R}^2$. Recall that $E_{\pm,\pm}$ denote the squares at the corners of Γ (see construction before (4.2)).

Lemma 6.2 *Let $r \geq v > 0$. Let $U \in \mathcal{Y}$ with $|U| \geq Cv|\Gamma|^2_\infty$ and $U \cap E_{\pm,\pm} = \emptyset$ be given and assume that $\|U\|_\pi \leq \frac{19}{20}\tau$. Then there is a subset $U' \subset U$, $U' \in \mathcal{Y}'$, with $|U \setminus U'| \leq Cr|U|$ such that* (6.12) *holds for all $Z \subset U'$, $Z \in \mathcal{Y}'$.*

Proof. Let $U \in \mathcal{Y}$ be given with $\|U\|_\pi \leq \frac{19}{20}\tau$ and assume without restriction $U \subset N_{2,+} \setminus (N_{1,-} \cup N_{1,+})$. By the choice of τ in (4.14) we obtain that all Γ_l having nonempty intersection with U satisfy $|\Gamma_l|_\infty < 19\bar{\tau}$. In particular, this implies $U \cap \tilde{N} = U$. Let $(\partial R_l)_l$ be the rectangles corresponding to $(\Gamma_l)_l$ as given by (3.5)(i),(v). We first prove that there is a $\frac{T}{4} + 2 \leq t \leq \frac{T}{2} - 1$ such that $\sum_{Q \subset U} \#\mathcal{S}_t(Q) \leq v^{-3}r^{2t+3}$ as in the assumption of Lemma 6.1. If the claim were false, we would have (assume without restriction that $T \in 4\mathbb{N}$)

$$\begin{aligned} |\partial W \cap N^{(1+19C)\bar{\tau}}|_{\mathcal{H}} &\geq C \sum\nolimits_{t=\frac{T}{4}+2}^{\frac{T}{2}-1} \sum\nolimits_{Q\subset U} \#\mathcal{S}_t(Q) v^4 r^{-2t}|\Gamma|_\infty \geq CTvr^3|\Gamma|_\infty \\ &\geq C\log_r(v) r^3 v|\Gamma|_\infty \gg v|\Gamma|_\infty \end{aligned} \tag{6.13}$$

for v small enough (with respect to $r = r(h_*, q)$) giving a contradiction to (5.3). In the first step we used that $|\partial R_l|_\infty \leq 19C\bar{\tau}$ by (3.6) which implies $\Gamma_l \subset N^{(1+19C)\bar{\tau}}(\Gamma)$ and assures that Γ_l intersects only a uniformly bounded number of different squares $Q \subset U$ (independently of r, v). As before, we define $\hat{v} = v^2 r^{-\frac{2t+1}{2}}$ for shorthand.

As in the previous proof we will select a path in U with certain properties. Recalling (4.6) it is not hard to see that $|\pi_2(R_l \cap U)| \leq |\partial R_l|_\pi$. As by assumption $\|U\|_\pi \leq \frac{19}{20}\tau$, we find a set $S \subset (l_2, l_2 + \tau)$ being the union of intervals $2k's + (-s, s), k' \in \mathbb{Z}$, with $|S| \geq \frac{\tau}{20}$ such that the stripe $\hat{U} = U \cap (\mathbb{R} \times S)$ satisfies $\partial W \cap \hat{U} = \emptyset$. We cover U by k horizontal paths $\mathcal{P} = (P_i)_i$, $i = 1, \ldots, k$ consisting of $\hat{Q}(\xi) = \hat{Q}^{\hat{v}|\Gamma|_\infty}(\xi)$, i.e. $k = \lceil (2\hat{v}|\Gamma|_\infty)^{-1}\tau \rceil$ as $|\pi_2 U| = \tau$. We can find a subset $\hat{\mathcal{P}}_1 \subset \mathcal{P}$ with $\#\hat{\mathcal{P}}_1 \geq c_1 k$ for c_1 small enough such that

$$\Gamma_l \cap P_i = \emptyset \quad \text{for all } |\Gamma_l|_\infty \geq \bar{C}^2 \hat{v}|\Gamma|_\infty \;\text{ and }\; P_i \in \hat{\mathcal{P}}_1 \tag{6.14}$$

and $|S \cap \pi_2 \bigcup_{P_i \in \hat{\mathcal{P}}_1} P_i| \geq \frac{\tau}{21}$, if $\bar{C}$ is chosen sufficiently large. Indeed, for $\{\Gamma_l : |\pi_2 \Gamma_l| \geq \bar{C}\hat{\upsilon}|\Gamma|_\infty\}$ this follows by an elementary argument. On the other hand, by (3.6) we see that each component in $\mathcal{G} := \{\Gamma_l : |\Gamma_l|_\infty \geq \bar{C}^2\hat{\upsilon}|\Gamma|_\infty, |\pi_2\Gamma_l| \leq \bar{C}\hat{\upsilon}|\Gamma|_\infty\}$ intersects at most $\sim \frac{\bar{C}\hat{\upsilon}|\Gamma|_\infty}{\hat{\upsilon}|\Gamma|_\infty} = \bar{C}$ different $P_i \in \mathcal{P}$ and thus using (5.3) $\mathcal{G}$ intersects at most $\bar{C}\frac{C\tau}{\bar{C}^2\hat{\upsilon}|\Gamma|_\infty} \sim \frac{C\tau}{\hat{\upsilon}|\Gamma|_\infty \bar{C}} \ll \frac{\tau}{|\Gamma|_\infty \hat{\upsilon}}$ different $P_i \in \mathcal{P}$.

Moreover, it is not hard to see that there is a subset $\hat{\mathcal{P}}_2 \subset \hat{\mathcal{P}}_1$ with $\#\hat{\mathcal{P}}_2 \geq c_2 k$ for c_2 sufficiently small such that

$$|\pi_2(P_i \cap \hat{U})| \geq \tfrac{1}{22} 2\hat{\upsilon}|\Gamma|_\infty \quad \text{for all} \quad P_i \in \hat{\mathcal{P}}_2. \tag{6.15}$$

Recall that we have already found a $\frac{T}{4}+2 \leq t \leq \frac{T}{2}-1$ such that $\sum_{Q\subset U} \#\mathcal{S}_t(Q) \leq \upsilon^{-3} r^{2t+3}$. Using (6.14) we can now choose a path $P = \overline{\bigcup_j \hat{Q}(\xi_j)} \in \hat{\mathcal{P}}_2$ such that (6.7) is satisfied possibly passing to a larger constant $C > 0$ depending on $\bar{C}$. (The essential difference to the argument developed in (6.7) is the fact that every boundary component may intersect not only four squares but a number depending on $\bar{C}$.) Observe that $n \sim \hat{\upsilon}^{-1}$, where n denotes the number of squares in the path P. Recall $\hat{\mathcal{S}}(P) = \{\Gamma_l : \Gamma_l \cap P \neq \emptyset\}$ and let

$$\hat{\mathcal{S}}^t_>(P) = \{\Gamma_l : \Gamma_l \cap P \neq \emptyset, \Gamma_l \in \bigcup_{Q\in J} \bigcup_{s>t} \mathcal{S}_s(Q)\}. \tag{6.16}$$

Moreover, define $\mathcal{K} = \{\hat{Q} = \hat{Q}(\xi_j) : \hat{Q} \cap \Gamma_l = \emptyset \text{ for all } \Gamma_l \in \hat{\mathcal{S}}^t_>(P)\}$. By (6.7) it is elementary to see that $\#\hat{\mathcal{S}}^t_>(P) \leq C\hat{\upsilon}|\Gamma|_\infty(\upsilon^4 r^{-2t-2}|\Gamma|_\infty)^{-1} = C\hat{\upsilon}^{-1}r$. Consequently, as by (6.14) every $\Gamma_l \in \hat{\mathcal{S}}^t_>(P)$ intersects only a uniformly bounded number of adjacent sets, we find

$$n - \#\mathcal{K} \leq C\#\hat{\mathcal{S}}^t_>(P) \leq C\hat{\upsilon}^{-1}r. \tag{6.17}$$

Consider two squares $\hat{Q}(a), \hat{Q}(b) \in \mathcal{K}$ and the path $(\xi_0 = a, \xi_1, \ldots, \xi_m = b)$. Define $D = \overline{\bigcup_{j=0}^m \hat{Q}(\xi_j)}$. Without restriction we assume $\hat{Q}_0 := \hat{Q}(a) = \mu(-1,1)^2$ and $\hat{Q}_m = \hat{Q}(b) = \mu((2m,0)+(-1,1)^2)$, where for shorthand we write $\mu = \hat{\upsilon}|\Gamma|_\infty$. We will now derive an estimate of the form (2.19). First of all, Theorem 2.1 (see also (2.18)), Theorem 2.2 and a rescaling argument show

$$\|u - (A^i \cdot + c^i)\|_{L^1(\partial\hat{Q}_i)} \leq C\mathcal{E}(\hat{Q}_i) \tag{6.18}$$

for $A^i \in \mathbb{R}^{2\times 2}_{\text{skew}}$, $c^i \in \mathbb{R}^2$, $i = 0, m$, and a constant independent of μ. For shorthand let $\mathcal{E} = \mathcal{E}(\hat{Q}_0) + \mathcal{E}(\hat{Q}_m)$ and define $\hat{\alpha}(D) = \int_D |e(u)|$. We claim that

$$\begin{aligned} &\mu^2|a^0 - a^m| + \mu|c^0_1 - c^m_1| \leq C\mathcal{E} + C\hat{\alpha}(D), \\ &\mu|c^0_2 - c^m_2| \leq Cm\mathcal{E} + Cm\hat{\alpha}(D), \end{aligned} \tag{6.19}$$

where c^i_j denotes the j-th component of c^i, $i = 0, m$, and a^0, a^m are defined such that $A^i = \begin{pmatrix} 0 & a^i \\ -a^i & 0 \end{pmatrix}$. By (6.15) we find two (measurable) sets $B_1, B_2 \subset \mu(-1,1)$

such that $|B_j| \geq \frac{\mu}{44}$, $\mathrm{dist}(B_1, B_2) \geq \frac{\mu}{22}$ and $E := \mu(-1, 2m+1) \times B_1 \cup B_2 \subset W$. We apply a slicing argument in the first coordinate direction and obtain

$$\begin{aligned} \int_{B_1 \cup B_2} & |u_1(\mu(2m-1), y) - u_1(\mu, y)| \, dy \\ & \leq \int_{B_1 \cup B_2} \Big| \int_{\mu}^{\mu(2m-1)} \partial_1 u_1(t, y) \, dt \Big| \, dy \leq C\hat{\alpha}(E). \end{aligned} \tag{6.20}$$

This together with (6.18) and the triangle inequality yields

$$\|(a^0 - a^m) \, \cdot + (c_1^0 - c_1^m)\|_{L^1(B_1 \cup B_2)} \leq C\mathcal{E} + C\hat{\alpha}(E).$$

Choose $f : B_1 \to \mathbb{R}$ such that $\mathbf{id} + f : B_1 \to B_2$ is piecewise constant and bijective. Thanks to $|B_1| \geq \frac{\mu}{44}$ and $f(y) \geq \frac{\mu}{22}$ for $y \in B_1$ we derive

$$\begin{aligned} \mu^2 |a^0 - a^m| \leq C\|(a^0 - a^m) \, f(\cdot)\|_{L^1(B_1)} &\leq C\|(a^0 - a^m) \, \cdot + (c_1^0 - c_1^m)\|_{L^1(B_1)} \\ + C\|(a^0 - a^m) \, (\cdot + f(\cdot)) + (c_1^0 - c_1^m)\|_{L^1(B_1)} &\leq C\mathcal{E} + C\hat{\alpha}(E) \end{aligned}$$

and likewise $\mu|c_1^0 - c_1^m| \leq C\mathcal{E} + C\hat{\alpha}(E)$. This gives the first bound in (6.19) since $E \subset D$. Analogously, we slice in $\zeta = \mu(2m-2, c)$ direction for $0 < c < 1$. By (6.7) we find $|\pi_2(\partial W \cap P_i)| \leq C\mu$. Consequently, choosing c small enough and recalling (6.15), we find a set $B_3 \subset \mu(-1, 1-c)$ with $|B_3| \geq \frac{1}{24} 2\hat{v}|\Gamma|_\infty = \frac{\mu}{12}$ such that $\{\mu\} \times B_3 + [0,1]\zeta \subset W$. Letting $\bar{\zeta} = \frac{\zeta}{|\zeta|}$ we get

$$\int_{B_3} |u(\mu, y) \cdot \bar{\zeta} - u((\mu, y) + \zeta) \cdot \bar{\zeta}| \, dy \leq C\hat{\alpha}(E)$$

similarly to (6.20) and thus, using (6.18) and the fact that $A^m \, \bar{\zeta} \cdot \bar{\zeta} = 0$, we derive

$$\int_{B_3} |(A^0 - A^m) \, (\mu, y)^T \cdot \bar{\zeta} + (c^0 - c^m) \cdot \bar{\zeta}| \, dy \leq C\mathcal{E} + C\hat{\alpha}(E).$$

This together with first part of (6.19) then leads to

$$\mu|c_2^0 - c_2^m + (a^m - a^0)\mu| \frac{c}{|\zeta|} \leq C\hat{\alpha}(E) + C\mathcal{E}$$

and implies the second part of (6.19) as $E \subset D$. Summarizing, (6.19) yields

$$\|c^0 - c^m + (A^0 - A^m) \, \cdot \|_{L^2(\hat{Q}(a))} \leq Cm(\mathcal{E}(\hat{Q}_0) + \mathcal{E}(\hat{Q}_m)) + Cm\hat{\alpha}(D), \tag{6.21}$$

which is an estimate of the form (2.19) with the difference that $\mathcal{E}_R$ is replaced by the elastic part of the energy $\hat{\alpha}$ in squares not contained in $\mathcal{K}$. We briefly note that in (6.21) we can replace $\hat{Q}(a)$ by $\hat{Q}(b)$ due to (6.19).

Define $\tilde{P} = \overline{\bigcup_{\hat{Q}\in\mathcal{K}} \hat{Q}}$. Recall that the essential point for the derivation of Lemma 2.4 was an estimate of the form (2.19), (2.20). Consequently, arguing similarly as in Lemma 2.4(i) for $s = \hat{\upsilon}|\Gamma|_\infty$ and $|V| = |\tilde{P}| \sim \hat{\upsilon}|\Gamma|^2_\infty$ and derive

$$\|u - (A\ \cdot + c)\|^2_{L^2(\tilde{P})} \le C\hat{\upsilon}^{-3}((\mathcal{E}(\tilde{P}))^2 + (\hat{\alpha}(P))^2) \tag{6.22}$$

for suitable $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $c \in \mathbb{R}^2$. Recall the definition of $\mathcal{K}$ (cf. (6.16)) and note that $\Gamma_l \cap \tilde{P} = \emptyset$ for all $\Gamma_l \in \hat{\mathcal{S}}^t_>$. Proceeding as in (6.9) and (6.10), in particular using (6.7) and (6.8), it is not hard to see that

$$\|u - (A\ \cdot + c)\|^2_{L^2(\tilde{P})} \le C(1 + C_* r)\, \hat{\upsilon}\frac{\varepsilon}{\upsilon^4}|\Gamma|^3_\infty. \tag{6.23}$$

Note that the difference to the estimate in the proof of Lemma 6.1 is that due to the above slicing argument it suffices to consider the elastic part of the energy in the connected components of $P \setminus \tilde{P}$. Now let $J' \subset J$ be the set of squares such that $|Q \cap \tilde{P}| \ge C\upsilon\hat{\upsilon}|\Gamma|^2_\infty \ge C\frac{\hat{\upsilon}}{\upsilon}|Q|$ for all $Q \in J'$. Setting $U' = \big(\bigcup_{Q\in J'} \overline{Q}\big)^\circ \in \mathcal{Y}'$ it is not hard to see that $|U \setminus U'| \le Cr|U|$ for r small enough as $|P \setminus \tilde{P}| \le Cr|P|$ by (6.17). Let $Z \subset U'$, $Z \in \mathcal{Y}'$. As before in Lemma 6.1, applying Lemma 2.4(ii) instead of Lemma 2.4(i), (6.23) yields

$$\|u - (A\ \cdot + c)\|^2_{L^2(\tilde{P}\cap Z)} \le C|\tilde{P} \cap Z||\tilde{P}|^{-1}(1 + C_* r)\, \hat{\upsilon}\frac{\varepsilon}{\upsilon^4}|\Gamma|^3_\infty.$$

Then applying (2.15), (6.2) and arguing as in (6.11) we derive

$$|U'|\fint_Z |u(x) - (A\,x + c)|^2\,dx \le C(1 + C_* r)\frac{\varepsilon}{\upsilon^3}|\Gamma|^3_\infty.$$

As $|U \setminus U'| \le Cr|U|$, this gives (6.12), as desired. □

The next step will be to replace U' by U in Lemma 6.2. To this end, we will apply the above arguments iteratively.

Lemma 6.3 *Let $r \ge \upsilon > 0$. Let $U \in \mathcal{Y}$ with $|U| \ge C\upsilon|\Gamma|^2_\infty$ and $U \cap E_{\pm,\pm} = \emptyset$ be given and assume that $\|U\|_\pi \le \frac{19}{20}\tau$. Then (6.12) holds.*

Proof. Define $U_1 = U'$ and $J_1 = J'$ as given in Lemma 6.2. Assume that $U_i = \big(\bigcup_{Q\in J_i} \overline{Q}\big)^\circ$, $J_i \subset J$, with $U_1 \subset \ldots \subset U_i$ is given such that for $\bar{C} > 0$ sufficiently large

$$|U \setminus U_i| \le \bar{C}r^i|U| \tag{6.24}$$

and for all $Z \subset U_i$, $Z \in \mathcal{Y}'$, one has

$$\fint_Z |u - (A\,x + c)|^2\,dx \le |Z||U|^{-1}C\prod_{j=0}^{i-1}\big(1 + \bar{C}r^{\frac{j}{8}}\big)\ G, \tag{6.25}$$

where $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c \in \mathbb{R}^2$ as given by Lemma 6.2 and $G := (1 + C_* r)\frac{\varepsilon}{\upsilon^3}|\Gamma|^3_\infty$.

Observe that (6.24), (6.25) hold for $i = 1$ by Lemma 6.2. We now pass from i to $i + 1$ and suppose $i \le T + 2$. First, it is not restrictive to assume that $|U \setminus U_i| \ge \bar{C} r^{i+1}|U|$ for $\bar{C} > 0$ as above since otherwise we may set $U_{i+1} = U_i$. We cover $U \setminus U_i$ with pairwise disjoint, connected sets $N_i^1, \dots, N_i^m \in \mathcal{Y}$, such that

$$\tfrac{1}{2} r^{\frac{i}{8}}|N_i^k| \le |N_i^k \setminus U_i| \le 2r^{\frac{i}{8}}|N_i^k| \tag{6.26}$$

for all $k = 1, \dots, m$. This can be done in the following way: Let $V_0 = U = (\bigcup_{j=1}^n \overline{Q_j})^\circ$. First, to construct $\tilde{N}_i^1$ let l_1, l_2 be the smallest and largest index, respectively, such that $Q_l \subset U \setminus U_i$ and choose $l = l_1$ if $l_1 < n - l_2$ and $l = l_2$ otherwise. Then add neighbors $Q_{l-1}, Q_{l+1} \subset U$, $Q_{l-2}, Q_{l+2} \subset U$, $\dots$ until $|\tilde{N}_i^1 \setminus U_i| \le 2r^{\frac{i}{8}}|\tilde{N}_i^1|$ holds. (I.e. the right inequality in (6.26) is satisfied.) This is possible due to the fact that $|V_0 \setminus U_i| \le \bar{C} r^i |V_0| \le \frac{1}{2} r^{\frac{i}{8}}|V_0|$ by (6.24) for r sufficiently small. Then note that also $r^{\frac{i}{8}}|\tilde{N}_i^1| \le |\tilde{N}_i^1 \setminus U_i|$ holds, in particular the left inequality in (6.26) is fulfilled. We now define V_1 as the connected component of $V_0 \setminus \tilde{N}_i^1$ which is not completely contained in U_i. (If both are contained in U_i we have finished.) We repeat the procedure on sets V_j to define $\tilde{N}_i^j$, $1 \le j \le k$, satisfying (6.26), where k is the smallest index such that $|V_k \setminus U_i| > \frac{1}{2} r^{\frac{i}{8}}|V_k|$. We now define $N_i^j = \tilde{N}_i^j$ for $j < k$ and $N_i^k := \tilde{N}_i^k \cup V_k$.

It remains to show that also N_i^k satisfies (6.26). Recall $|V_{k-1} \setminus U_i| \le \frac{1}{2} r^{\frac{i}{8}}|V_{k-1}|$. As due to the choice of l and the fact that $r^{\frac{i}{8}}|\tilde{N}_i^{k-1}| \le |\tilde{N}_i^{k-1} \setminus U_i|$ we have $|V_k| \ge \frac{1}{2}|V_{k-1}| - |\tilde{N}_i^{k-1}|$ and $|V_k \setminus U_i| = |V_{k-1} \setminus U_i| - |\tilde{N}_i^{k-1} \setminus U_i| \le \frac{1}{2} r^{\frac{i}{8}}|V_{k-1}| - r^{\frac{i}{8}}|\tilde{N}_i^{k-1}|$, we find $|V_k \setminus U_i| \le r^{\frac{i}{8}}|V_k|$. This together with (6.26) for $\tilde{N}_i^k$ implies the desired property for N_i^k.

Let $N_i = \bigcup_{k=1}^m N_i^k$. Similarly as in (6.13) we find some $\frac{T}{4} + 2 \le t \le \frac{T}{2} - 1$ such that for $t_i = t + \frac{9}{8} \cdot \frac{i}{2}$ we have $\sum_{Q \subset N_i} \#\mathcal{S}_{t_i}(Q) \le \upsilon^{-3} r^{2t_i+3}$. Again set $\hat{\upsilon} = \upsilon^2 r^{-\frac{2t+1}{2}}$. Arguing as in (6.15) we can find a horizontal path P_i consisting of $\hat{Q}(\xi_j) = \hat{Q}^{\hat{\upsilon}|\Gamma|_\infty}(\xi_j)$, $j = 1, \dots, n_i$, and lying in N_i such that (6.7), (6.14) and (6.15) are satisfied replacing t by t_i. By (6.24) and (6.26) we obtain

$$\bar{C} r^{i+1}\hat{\upsilon}^{-1} \le n_i \le C\bar{C} r^{i - \frac{i}{8}}\hat{\upsilon}^{-1}. \tag{6.27}$$

Clearly, in general the path P_i is not connected. Define $\hat{\mathcal{S}}^{t_i}_>(P_i)$ and $\mathcal{K}_i$ similarly as in (6.16). By (6.7) it is elementary to see that

$$\#\hat{\mathcal{S}}^{t_i}_>(P_i) \le C\hat{\upsilon}|\Gamma|_\infty(\upsilon^4 r^{-2t_i-2}|\Gamma|_\infty)^{-1} \le C\hat{\upsilon}^{-1} r^{\frac{9}{8}i} r = C\hat{\upsilon}^{-1} r^{\frac{9}{8}i+1}.$$

Therefore, letting $\tilde{P}_i = \overline{\bigcup_{\hat{Q} \in \mathcal{K}_i} \hat{Q}} \subset P_i$ we find by (6.14) (cf. (6.17))

$$n_i - \#\mathcal{K}_i \le C\hat{\upsilon}^{-1} r^{\frac{9}{8}i+1} \quad \text{and} \quad |P_i| - |\tilde{P}_i| \le C r^{\frac{9}{8}i+1}\hat{\upsilon}|\Gamma|^2_\infty. \tag{6.28}$$

We now repeat the slicing arguments above on each N_i^k and obtain expressions similar to (6.21). As before, applying Lemma 2.4(i) we get (cf. (6.22))

$$\|u - (A^k \cdot + c^k)\|^2_{L^2(\tilde{P}_i \cap N_i^k)} \le C(n^k)^3(\mathcal{E}(\tilde{P}_i \cap N_i^k) + \hat{\alpha}(P_i \cap N_i^k))^2$$

for suitable $A^k \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c^k \in \mathbb{R}^2$, where as before $\hat{\alpha}(D) = \int_D |e(u)|$ for $D \subset N$. Here n^k denotes the number of squares forming the path $P_i \cap N_i^k$, particularly $n_i = \sum_{k=1}^m n^k$. We observe that by (6.7) the estimate in (6.8) can now be replaced by

$$\upsilon^4 r^{-2t_i - 1 \pm 1} = \hat{\upsilon}^2 r^{-\frac{9}{8}i \pm 1}, \qquad \#\hat{\mathcal{S}}^{t_i}(P)\upsilon^4 r^{-2t_i-2} \le C\hat{\upsilon} r.$$

Consequently, recalling $n_i \le C\bar{C} r^{\frac{7}{8}i}\hat{\upsilon}^{-1}$ by (6.27), $t_i = t + \frac{9}{8}\cdot\frac{i}{2}$ and following the arguments in (6.9), (6.22) and (6.23) we obtain

$$\begin{aligned}
&\sum\nolimits_k (n^k)^{-1}\|u - (A^k \cdot + c^k)\|^2_{L^2(N_i^k \cap \tilde{P}_i)} \\
&\quad \le C\sum\nolimits_k (n^k)^2(\mathcal{E}(N_i^k \cap \tilde{P}_i) + \hat{\alpha}(N_i^k \cap P_i))^2 \le Cn_i^2(\mathcal{E}(\tilde{P}_i) + \hat{\alpha}(P_i))^2 \\
&\quad \le C\hat{\upsilon} r^{\frac{7}{4}i}\hat{\upsilon}^{-3}(\mathcal{E}(\tilde{P}_i) + \hat{\alpha}(P_i))^2 \le C\hat{\upsilon} r^{\frac{7}{4}i}\, r^{-\frac{9}{8}i}F \le C\hat{\upsilon} r^{\frac{i}{2}}F,
\end{aligned}$$

where F was defined in (6.10). Observe that in the calculation the additional $r^{-\frac{9}{8}i}$ in front of F occurs as in (6.8) $\upsilon^4 r^{-2t-1\pm 1}$ was replaced by $\upsilon^4 r^{-2t-1\pm 1} r^{-\frac{9}{8}i}$. Moreover, the above estimate can be repeated applying Lemma 2.4(ii) instead of Lemma 2.4(i): For $Z \subset N_i$, $Z \in \mathcal{Y}'$, we obtain

$$\sum\nolimits_k (n^k)^{-1}|N_i^k \cap \tilde{P}_i| \fint_{N_i^k \cap \tilde{P}_i \cap Z} |u(x) - (A^k\, x + c^k)|^2\, dx \le C\hat{\upsilon} r^{\frac{i}{2}}F.$$

Define $J_i^k \subset J$ such that $|Q \cap (\tilde{P}_i \cap N_i^k)| \ge C\upsilon\hat{\upsilon}|\Gamma|^2_\infty \ge C\frac{\hat{\upsilon}}{\upsilon}|Q|$ for $Q \in J_i^k$ and set $\hat{N}_i^k = \bigcup_{Q \in J_i^k} Q$. Assume $\hat{N}_i^k \cap Z \neq \emptyset$ which implies $|\hat{N}_i^k \cap Z| \ge \upsilon^2|\Gamma|^2_\infty$. Observe $\hat{\upsilon} \ge \upsilon^{\frac{7}{4}} r^{-\frac{3}{2}} \ge \upsilon^2 r^{-\frac{i}{4}-1}$ by (6.6) and the fact that $i \le T+2$. As $|N_i^k|(n^k)^{-1} \le C\hat{\upsilon}\upsilon|\Gamma|^2_\infty$, we find by (6.2)

$$\begin{aligned}
&\sum\nolimits_k (n^k)^{-1}|\hat{N}_i^k| \fint_{\hat{N}_i^k \cap Z} |u(x) - (\bar{A}\, x + \bar{c})|^2\, dx \\
&\quad \le C\hat{\upsilon}\upsilon^{-1}\int_U |u(x) - (\bar{A}\, x + \bar{c})|^2\, dx \le C\hat{\upsilon}\upsilon^{-1}\ \upsilon^3(\hat{\upsilon} r)^{-1}F \le C\hat{\upsilon} r^{\frac{i}{4}}F.
\end{aligned}$$

Again arguing as in (6.11), in particular applying (2.15), we derive

$$\sum\nolimits_k (n^k)^{-1}|\hat{N}_i^k| \fint_{Z \cap \hat{N}_i^k} |u(x) - (A^k\, x + c^k)|^2\, dx \le C\hat{\upsilon} r^{\frac{i}{4}}\ \hat{\upsilon}^{-1}\upsilon F = Cr^{\frac{i}{4}}\upsilon F. \quad (6.29)$$

We set $U_i^k = \hat{N}_i^k$ if $|N_i^k \setminus \hat{N}_i^k| \le r^{\frac{i}{8}}|N_i^k|$ and $U_i^k = \emptyset$ else for all $k = 1, \dots, m$. We now estimate the difference between A, c given in (6.25) and A^k, c^k for $k =$

$1,\dots,m$. Consider U_i^k such that $U_i^k = \hat{N}_i^k$. Then $|U_i^k| \geq (1 - r^{\frac{i}{8}})|N_i^k|$ and thus by (6.26) we have

$$|U_i^k \cap U_i| \geq |N_i^k \cap U_i| - |N_i^k \setminus \hat{N}_i^k| \geq (1 - Cr^{\frac{i}{8}})|N_i^k| \geq (1 - Cr^{\frac{i}{8}})|U_i^k|$$

for r sufficiently small and some $C > 0$. Consequently, we are in the position to apply (2.17) for $B_2 = U_i^k$, $B_1 = U_i^k \cap U_i$ and $s = \frac{\tau}{2}$, $\delta = Cr^{\frac{i}{8}}$, where we observe $\delta \geq Cs|\pi_1(U_i^k)|^{-1}$ by (6.26). (Recall the remark in Section 2.2 that B_2 does not have to be connected.) Set $\bar{C}_i = C\prod_{j=0}^{i-1}\big(1 + \bar{C}r^{\frac{j}{8}}\big)$ (cf. (6.25)). Using (6.25) and (6.29), in particular recalling that the sets $(\hat{N}_i^k)_k$ are pairwise disjoint, we find for $Z \subset U$, $Z \in \mathcal{Y}'$

$$\begin{aligned}
&\|u - (A\ \cdot + c)\|^2_{L^2(U_i^k \cap U_i \cap Z)} \leq |U_i^k \cap U_i \cap Z||U_i^k|^{-1}H_1^k,\\
&\|u - (A^k\ \cdot + c^k)\|^2_{L^2(U_i^k \cap Z)} \leq |U_i^k \cap Z||U_i^k|^{-1}H_2^k,
\end{aligned}$$

where $H_1^k = |U_i^k||U|^{-1}\bar{C}_iG$ and $H_2^k = Cn^kr^{\frac{i}{4}}\upsilon F$. Therefore, (2.17) yields

$$\begin{aligned}
\|u - (A\ \cdot + c)\|^2_{L^2(U_i^k \cap Z)} &\leq |U_i^k \cap Z||U_i^k|^{-1}(1 + Cr^{\frac{i}{8}})|U_i^k||U|^{-1}\bar{C}_iG\\
&\quad + |U_i^k \cap Z||U_i^k|^{-1}Cr^{-\frac{i}{8}}n^kr^{\frac{i}{4}}\upsilon F.
\end{aligned} \tag{6.30}$$

For shorthand we write $U^* = \big(\bigcup_{k=1}^m \overline{U_i^k}\big)^\circ$ and define $U_{i+1} = (\overline{U_i} \cup \overline{U^*})^\circ$. We recall $N_i = \bigcup_{k=1}^m N_i^k$ as constructed in (6.26). We claim

$$|N_i \setminus U^*| \leq Cr^{i+1}|U| \tag{6.31}$$

and postpone the proof of this assertion to the end of the proof. Then (6.27) for $\bar{C}$ sufficiently large implies $|U^*| \geq |N_i| - Cr^{i+1}|U| \geq cn_i\hat{\upsilon}|U|$. As for $U_i^k \neq \emptyset$ we have $|N_i^k| \leq (1 - r^{\frac{i}{8}})^{-1}|U_i^k|$, it is not hard to see that $\frac{|U^*|}{|U_i^k|} \leq C\frac{n_i}{n^k}$ and thus $n^k|U_i^k|^{-1} \leq C|U^*|^{-1}n_i \leq C|U|^{-1}\hat{\upsilon}^{-1}$. Let $V \subset U_{i+1}$, $V \in \mathcal{Y}'$. Now by (6.30), the fact that the sets U_i^k are pairwise disjoint and $F \leq C\hat{\upsilon}\upsilon^{-1}G$ we derive

$$\begin{aligned}
\sum\nolimits_k \|u - (A\ \cdot + c)\|^2_{L^2(U_i^k \cap V)} &\leq |U^* \cap V||U|^{-1}\big(\bar{C}_i(1 + Cr^{\frac{i}{8}})G + Cr^{\frac{i}{8}}G\big)\\
&\leq |U^* \cap V||U|^{-1}\bar{C}_{i+1}G,
\end{aligned}$$

The last estimate follows for $\bar{C}$ sufficiently large. By (6.25) we now conclude for $V \subset U_{i+1}$

$$\begin{aligned}
\|u - (A\ \cdot + c)\|^2_{L^2(V)} &= \|u - (A\ \cdot + c)\|^2_{L^2(V \setminus U^*)} + \sum\nolimits_k \|u - (A\ \cdot + c)\|^2_{L^2(U_i^k \cap V)}\\
&\leq |V \setminus U^*||U|^{-1}\bar{C}_iG + |U^* \cap V||U|^{-1}\bar{C}_{i+1}G\\
&\leq |V||U|^{-1}\bar{C}_{i+1}G.
\end{aligned}$$

This yields (6.25). To see (6.24) for $i+1$, we apply (6.31) to obtain $|U \setminus U_{i+1}| \leq |(U \setminus U_i) \setminus N_i| + |N_i \setminus U^*| \leq 0 + \bar{C}r^{i+1}|U|$. Here we used that $U \setminus U_i \subset N_i$.

Finally, we choose $i_* \le T+2$ large enough such that $|U \setminus U_{i_*}| \le \bar{C} r \upsilon |U| \ll (\upsilon |\Gamma|_\infty)^2$ for r sufficiently small which implies $U_{i_*} = U$. Consequently, thanks to (6.25), (6.12) holds.

It remains to show (6.31). First, by (6.28) and the construction of $\hat{N}_i^k$ we have $|\bigcup_k N_i^k \setminus \bigcup_k \hat{N}_i^k| = \sum_k |N_i^k \setminus \hat{N}_i^k| \le C r^{\frac{9}{8}i+1} \upsilon |\Gamma|_\infty^2 \le C r^{\frac{9}{8}i+1} |U|$. Therefore, it suffices to prove

$$\sum\nolimits_k |\hat{N}_i^k \setminus U_i^k| \le r^{-\frac{i}{8}} \sum\nolimits_k |N_i^k \setminus \hat{N}_i^k| \tag{6.32}$$

as then we conclude $|N_i \setminus U^*| \le \sum_k |N_i^k \setminus \hat{N}_i^k| + \sum_k |\hat{N}_i^k \setminus U_i^k| \le C r^{i+1} |U|$.

To see (6.32) we observe that if $\hat{N}_i^k \neq U^k$, then $|\hat{N}_i^k| \le |N_i^k| < r^{-\frac{i}{8}} |N_i^k \setminus \hat{N}_i^k|$. Consequently, we calculate $\sum_k |\hat{N}_i^k \setminus U_i^k| = \sum_{k: U_i^k = \emptyset} |\hat{N}_i^k| \le r^{-\frac{i}{8}} \sum_{k: U_i^k = \emptyset} |N_i^k \setminus \hat{N}_i^k| \le r^{-\frac{i}{8}} \sum_k |N_i^k \setminus \hat{N}_i^k|$, as desired. □

Remark 6.4 We briefly note that the previous proof shows that the assertion of Lemma 6.3 holds for $U \in \mathcal{Y}$ with $U \cap E_{\pm,\pm} = \emptyset$ of arbitrary size. In fact, we can choose $0 \le i_0 \le T+2$ such that $C r^{i_0+1} \upsilon |\Gamma|_\infty^2 < |U| \le C r^{i_0} \upsilon |\Gamma|_\infty^2$ and begin the induction in (6.24), (6.25) not for $i = 0$, but for $i = i_0$. For the first step $i = i_0$ we do not apply Lemma 6.2, but follow the lines of the proof of Lemma 6.3 for one single set $N_{i_0}^1 = U$.

We now drop the assumption that $U \in \mathcal{Y}$ does not intersect a corner of Γ.

Corollary 6.5 *Let $r \ge \upsilon > 0$. Let $U \in \mathcal{Y}$ be given and assume that $\|U\|_\pi \le \frac{19}{20}\tau$. Then* (6.12) *holds.*

Proof. Assume without restriction $E_{+,+} \subset U$ and define $U' = U \setminus E_{+,+}$. Using Lemma 6.3 and Remark 6.4 we find

$$|U'| \fint_Z |u(x) - (A\,x + c)|^2 \, dx \le CG$$

for $Z \subset U'$, $Z \in \mathcal{Y}'$, where $G := (1 + C_* r) \frac{\varepsilon}{\upsilon^3} |\Gamma|_\infty^3$. Let $Q \in J$, $Q \subset U'$ such that $\partial Q \cap E_{+,+} \neq \emptyset$. Setting $Z = Q$ in the above inequality and arguing as in (6.2) we find $\hat{A} \in \mathbb{R}^{2\times 2}_{\rm skew}$, $\hat{c} \in \mathbb{R}^2$ such that

$$\int_Q |u(x) - (A\,x + c)|^2 \, dx \le C\upsilon G, \quad \int_{Q \cup E_{+,+}} |u(x) - (\hat{A}\,x + \hat{c})|^2 \, dx \le C\upsilon^2 r^{-1} G.$$

Applying (2.15) on $B_1 = Q$ and $B_2 = Q \cup E_{+,+}$ we find $\|u - (A\,\cdot + c)|^2_{L^2(Q \cup E_{+,+})} \le C\upsilon G$ as $\upsilon \le r$. Now it is not hard to see that (6.12) is satisfied. □

6.4 Step 3: Neighborhood with small projection of components

Recall the covering $\mathcal{C}$ of the neighborhood N introduced in (4.3). We now treat the case that $\|U\|_\pi$ is small for all $U \in \mathcal{C}$. It is essential that adjacent elements of the covering overlap sufficiently. Therefore, we introduce another covering $\hat{\mathcal{C}} \subset \mathcal{C}$ as follows. First assume $l_2 \geq \frac{l_1}{2}$. If some $U \in \mathcal{C}$ intersects only one of the four sets $N_{j,\pm}$, $j = 1, 2$, we let $U \in \hat{\mathcal{C}}$. Then eight sets $U^1_{\pm,\pm}, U^2_{\pm,\pm}$ remain where $U^i_{\pm,\pm} \cap E_{\pm,\pm} \neq \emptyset$ and $U^i_{\pm,\pm} \subset N_{i,-} \cup N_{i,+}$ for $i = 1, 2$. As before $E_{\pm,\pm}$ denote the sets at the corners of Γ. Add the four sets $U^1_{\pm,\pm} \cup U^2_{\pm,\pm}$ to $\hat{\mathcal{C}}$. If $l_2 < \frac{l_1}{2}$ we proceed likewise with the only difference that instead of $U^1_{\pm,\pm} \cup U^2_{\pm,\pm}$ we add the two sets $U^2_{k,+} \cup U^1_{k,+} \cup U^1_{k,-} \cup U^2_{k,-}$, $k = +, -$, to $\hat{\mathcal{C}}$. (Note that by definition of $\mathcal{C}$ we have $U^1_{\pm,+} = U^1_{\pm,-} = N_{1,\pm}$ in this case.)

Lemma 6.6 *Theorem 5.1 holds under the additional assumption that $\|U\|_\pi \leq \frac{19}{20}\tau$ for all $U \in \mathcal{C}$.*

Proof. It suffices to show that for all $U \in \hat{\mathcal{C}}$ there are $A_U \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c_U \in \mathbb{R}^2$ such that

$$|U| \fint_Z |u(x) - (A_U\, x + c_U)|^2\, dx \leq C(1 + rC_*)\upsilon^{-3}|\Gamma|^3_\infty \varepsilon \qquad (6.33)$$

holds for all $Z \subset U$, $Z \in \mathcal{Y}'$. Indeed, the desired result then follows from the construction of the covering $\hat{\mathcal{C}}$ and the arguments developed in Section 2.2: Write $\hat{\mathcal{C}} = \{U_1, \ldots, U_n\}$ with $U_{i-1} \cap U_i \neq \emptyset$ for all $i = 1, \ldots, n$, where $U_0 = U_n$. Now let

$$\mathcal{D} = \{U_i \setminus \overline{U_{i-1} \cup U_{i+1}} : i = 0, \ldots, n-1\} \cup \{U_i \cap U_{i+1} : i = 0, \ldots, n-1\},$$

where $U_{-1} = U_{n-1}$. Note that the elements in $\mathcal{D}$ are pairwise disjoint. We write $\mathcal{D} = \{V_1, \ldots, V_m\}$ such that $\partial V_{i-1} \cap \partial V_i \neq \emptyset$ for $i = 1, \ldots, m$, where $V_0 = V_m$. By (4.3) and the definition of the 'combined sets' in $\hat{\mathcal{C}}$, we find $|V_i| \sim \upsilon|\Gamma|^2_\infty$. Clearly, (6.33) also holds for all $V_i \in \mathcal{D}$ for corresponding infinitesimal rigid motions as each set is contained in an element of $\hat{\mathcal{C}}$. We can now estimate the difference of the infinitesimal rigid motions of $B_1 = V_{i-1}$ and $B_2 = V_i$, $i = 1, \ldots, m$, proceeding as in (2.11), (2.13) and (2.15). Here it is essential to observe that assumption (2.9) is satisfied as $B_1 \cup B_2 \subset U$ for some $U \in \hat{\mathcal{C}}$ and so (6.33) may be applied. We now obtain (6.4) following the argument in (2.19), (2.20) replacing the squares $(Q(\xi_j))_j$ by the elements of $\mathcal{D}$ and noting that $\#\mathcal{D}$ is uniformly bounded independently of υ.

More general, taking (6.33) and (2.14) into account, we have even shown that

$$|N| \fint_V |u(x) - (A\, x + c)|^2\, dx \leq C(1 + rC_*)\upsilon^{-3}|\Gamma|^3_\infty \varepsilon \qquad (6.34)$$

for all $V \subset N$, $V \in \mathcal{Y}'$.

It remains to establish (6.33) for $U \in \hat{\mathcal{C}}$. By assumption and Lemma 6.3 the assertion is clear if $U \cap E_{\pm,\pm} = \emptyset$ as then particularly $U \in \mathcal{C}$. Therefore, we first let $l_2 \geq \frac{l_1}{2}$ and assume that e.g. $U \cap E_{+,+} \neq \emptyset$. The necessary changes for the case $l_2 \leq \frac{l_1}{2}$ are indicated at the end of the proof.

As in (6.13) we find $\frac{T}{4} + 2 \leq t \leq \frac{T}{2} - 1$ such that $\sum_{Q \subset U} \# \mathcal{S}_t(Q) \leq \upsilon^{-3} r^{2t+3}$. Again let $\hat{\upsilon} = \upsilon^2 r^{-\frac{2t+1}{2}}$. As before, the main strategy will be to construct a suitable path in U. Let $(\Gamma_l)_l$ be the boundary components such that the corresponding rectangles $(\partial R_l)_l$ given by (3.5)(i) and (3.5)(v), respectively, satisfy $\partial R_l \cap U \neq \emptyset$ and $|\partial R_l|_\pi \neq |\partial R_l|_\infty$. Let $V_l \subset \overline{N}$ be the smallest rectangle containing $R_l \cap N$ and $(l_1 + \tau, l_2 + \tau)$. We partition $(V_l)_l$ into $\mathcal{V}_1$ and $\mathcal{V}_2$ depending on whether $|\pi_1 V_l| \leq |\pi_2 V_l|$ or $|\pi_1 V_l| > |\pi_2 V_l|$. Recalling (4.6) it is not hard to see that $|\pi_j V_l| = |R_l|_\pi$ for $V_l \in \mathcal{V}_j$ for $j = 1, 2$. Let $a_j = \inf\{s \in \mathbb{R} : s \in \pi_j V_l \text{ for a } V_l \in \mathcal{V}_j\}$ and define the stripes

$$A_1 = (-\infty, a_1) \times (-\infty, a_2) \cap N_{1,+} \cap U, \quad A_2 = (-\infty, a_1) \times (-\infty, a_2) \cap N_{2,+} \cap U.$$

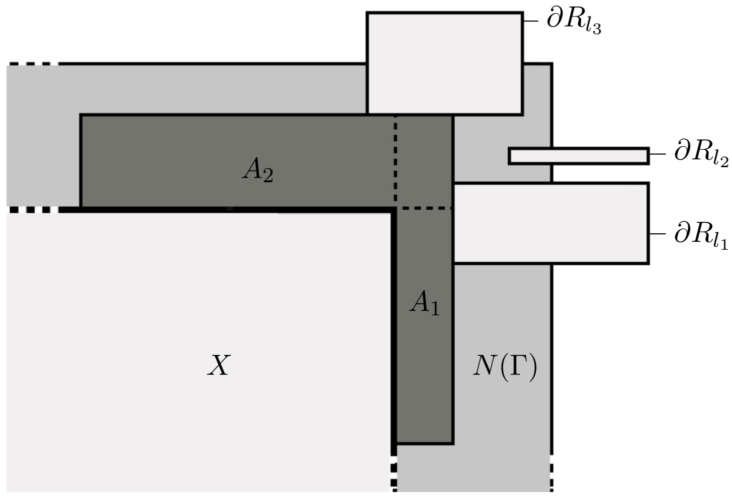

Figure 9: Sketch of a part of $N(\Gamma)$ containing $U^1_{+,+} \cup U^2_{+,+}$. The sets A_1, A_2 are highlighted in grey.

As by assumption $\|U\|_\pi \leq \frac{19}{20}\tau$ for all $U \in \mathcal{C}$, we find sets $S_j \subset (l_j, a_j)$ with $|S_j| \geq \frac{\tau}{20}$ such that the stripes $\hat{A}_1 = A_1 \cap (S_1 \times \mathbb{R}) \in \mathcal{U}^s$ and $\hat{A}_2 = A_2 \cap (\mathbb{R} \times S_2) \in \mathcal{U}^s$ satisfy $\partial W \cap \hat{A}_j = \emptyset$ for $j = 1, 2$. Moreover, observe that $|a_j - l_j| \geq \frac{\tau}{20}$ for $j = 1, 2$. We cover A_1 by vertical paths $\mathcal{P}_1 = (P^1_i)_i$, $i = 1, \ldots, k_1$, and A_2 by horizontal paths $\mathcal{P}_2 = (P^2_i)_i$, $i = 1, \ldots, k_2$, consisting of squares $\hat{Q}^{\hat{\upsilon}|\Gamma|_\infty}(\xi) = \hat{Q}(\xi)$, i.e. $k_j = \lceil (2\hat{\upsilon}|\Gamma|_\infty)^{-1}(a_j - l_j) \rceil$. As in (6.15) it is not hard to see that there are subsets $\hat{\mathcal{P}}_j \subset \mathcal{P}_j$ with $\#\hat{\mathcal{P}}_j \geq ck_j \geq c\upsilon\hat{\upsilon}^{-1}$ for $c > 0$ sufficiently small such that (6.14) and (6.15) hold for all $P^j_i \in \hat{\mathcal{P}}_j$, $j = 1, 2$. We can now choose

$P^j \in \hat{\mathcal{P}}_j$, $j = 1, 2$, such that (6.7) is satisfied possibly passing to a larger constant. Moreover, this can be done in a way that $Q^* := P^1 \cap P^2$ satisfies

$$\begin{aligned} &\sum\nolimits_{\Gamma_k \cap Q^* \neq \emptyset} |\Gamma_k|_\infty \le \tilde{C} \upsilon^{-1} \hat{\upsilon}^2 |\Gamma|_\infty, \\ &Q^* \cap \Gamma_k = \emptyset \ \text{ for all } \Gamma_k : |\Gamma_k| \ge \tilde{C} \hat{\upsilon}^2 \upsilon^{-1} |\Gamma|_\infty, \end{aligned} \tag{6.35}$$

for $\tilde{C} > 0$ sufficiently large. To see the latter, note that we have $\sim \tau^2 (\hat{\upsilon} |\Gamma|_\infty)^{-2} = \upsilon^2 \hat{\upsilon}^{-2}$ possibilities to combine paths in $\hat{\mathcal{P}}_1$, $\hat{\mathcal{P}}_2$ such that (6.7) hold. Moreover, we also have $\sim \tau^2 (\hat{\upsilon} |\Gamma|_\infty)^{-2} = \upsilon^2 \hat{\upsilon}^{-2}$ possibilities to combine paths in $\hat{\mathcal{P}}_1$, $\hat{\mathcal{P}}_2$ such that Q^* additionally has empty intersection with all Γ_k satisfying $|\Gamma_k|_\infty \ge \tilde{C} \hat{\upsilon}^2 \upsilon^{-1} |\Gamma|_\infty$. This follows from (6.14) and the fact that by (5.3) we derive

$$\#\{\hat{Q} : \exists \Gamma_k : \tilde{C} \hat{\upsilon}^2 \upsilon^{-1} |\Gamma|_\infty \le |\Gamma_k|_\infty \le C \hat{\upsilon} |\Gamma|_\infty, \Gamma_k \cap \hat{Q} \neq \emptyset\} \le C \tilde{C}^{-1} \upsilon^2 \hat{\upsilon}^{-2}.$$

Since all other components Γ_k intersect at most four adjacent squares $\hat{Q}$, using again (5.3) we can select Q^* such that also $\sum_{\Gamma_k \cap Q^* \neq \emptyset} |\Gamma_k|_\infty \le C \upsilon |\Gamma|_\infty \upsilon^{-2} \hat{\upsilon}^2$ holds.

Let $P = \hat{P}^1 \cup Q^* \cup \hat{P}^2$, where $\hat{P}^j$, $j = 1, 2$, is the connected component of $P^j \setminus Q^*$ not completely contained in $E_{+,+}$. Denote the midpoints of the squares in P by $(\xi_1, \ldots, \xi_n)$. Recall the definition of $\hat{\mathcal{S}}^t_>$ in (6.16) and let

$$\mathcal{K} = \{\hat{Q} = \hat{Q}(\xi_j) : \hat{Q} \cap \Gamma_l = \emptyset \ \text{ for all } \ \Gamma_l \in \hat{\mathcal{S}}^t_>(P)\} \cup \{Q^*\}.$$

Consider two sets $\hat{Q}(a), \hat{Q}(b) \in \mathcal{K}$ and the path $(\xi_0 = a, \ldots, \xi_m = b)$. We can repeat the slicing method of the previous proofs and end up with an estimate of the form (cf. (6.21))

$$\|c_a - c_b + (A_a - A_b) \cdot\|_{L^2(\hat{Q}(a))} \le Cm \Big(\mathcal{E}(\hat{Q}(a)) + \mathcal{E}(\hat{Q}(b)) + \mathcal{E}(\hat{Q}^*) + \hat{\alpha}(D) \Big)$$

for suitable $A_a, A_b \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c_a, c_b \in \mathbb{R}^2$, where $D = \bigcup_{j=0}^m \hat{Q}(\xi_j)$. In fact, if $\hat{Q}(a), \hat{Q}(b) \subset \hat{P}^j$ for some $j = 1, 2$, this follows immediately. Otherwise, we apply the arguments leading to (6.19) on each pair $\hat{Q}(a), Q^*$ and $Q^*, \hat{Q}(b)$ and employ the triangle inequality.

Defining $\tilde{P} = \overline{\bigcup_{\hat{Q} \in \mathcal{K}} \hat{Q}}$ and arguing as in (6.22), we then obtain

$$\|u - (A \cdot + c)\|^2_{L^2(\tilde{P})} \le C \hat{\upsilon}^{-3} ((\mathcal{E}(\tilde{P}))^2 + (\alpha(P))^2) \le C(1 + C_* r) \frac{\varepsilon \hat{\upsilon}}{\upsilon^4} |\Gamma|^3_\infty + C \frac{1}{\hat{\upsilon}^3} (\mathcal{E}(\hat{Q}^*))^2$$

for some $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c \in \mathbb{R}^2$. In the last step we proceeded as in (6.23) (see also (6.9)), observing that the paths $\tilde{P}$ defined here and in the proof of Lemma 6.2 differ essentially by the square Q^*. By (6.6) we get $\hat{\upsilon}^3 \upsilon^{-3} \le \hat{\upsilon} r$ and using (6.35) as well as (5.7) we derive (cf. (6.9))

$$\begin{aligned} \hat{\upsilon}^{-3} (\mathcal{E}(\hat{Q}^*))^2 &\le C \hat{\upsilon}^{-1} |\Gamma|^3_\infty \varepsilon + C \hat{\upsilon}^{-3} C_* \frac{\varepsilon}{\upsilon^4} \Big(\sum\nolimits_{\Gamma_l \cap Q^* \neq \emptyset} (|\Theta_l|_*)^{3/2} \Big)^2 \\ &\le C \hat{\upsilon}^{-1} |\Gamma|^3_\infty \varepsilon + C |\Gamma|_\infty \hat{\upsilon}^{-1} \upsilon^{-1} C_* \frac{\varepsilon}{\upsilon^4} \Big(\sum\nolimits_{\Gamma_l \cap Q^* \neq \emptyset} |\Gamma_l|_\infty \Big)^2 \\ &\le C(1 + C_* r)\, \hat{\upsilon} \frac{\varepsilon}{\upsilon^4} |\Gamma|^3_\infty. \end{aligned} \tag{6.36}$$

Consequently, we have re-derived (6.23). Proceeding as in Lemma 6.2 we obtain a set $U' \subset U$ with $|U \setminus U'| \le Cr|U|$ such that

$$|U| \fint_Z |u(x) - (A\,x + c)|^2\,dx \le C(1 + C_* r)\frac{\varepsilon}{v^3}|\Gamma|_\infty^3$$

for $Z \subset U'$, $Z \in \mathcal{Y}'$. On the other hand, by Corollary 6.5 we find $A_j \in \mathbb{R}^{2\times 2}_{\rm skew}$, $c_j \in \mathbb{R}^2$ for $j = 1, 2$ such that

$$|U \cap N_{j,+}| \fint_{Z \cap N_{j,+}} |u(x) - (A_j\,x + c_j)|^2\,dx \le C(1 + C_* r)\frac{\varepsilon}{v^3}|\Gamma|_\infty^3 \qquad (6.37)$$

for all $Z \subset U$, $Z \in \mathcal{Y}'$. Now (6.34) follows by applying (2.14) on $B_1 = U' \cap N_{j,+}$ and $B_2 = U \cap N_{j,+}$.

Finally, the essentially difference in the treatment of the case $l_2 \le \frac{l_1}{2}$ is that in the construction of the path P one has to choose two sets Q_1^*,Q_2^* where the path changes its direction. Following the above arguments it is not hard to see that these sets can be selected so that the required conditions are satisfied. Note that in the derivation of (6.37) we then exploit that Corollary 6.5 also holds for sets which are much smaller than $v|\Gamma|_\infty^2$. □

6.5 Step 4: General case

We are eventually in a position to give the proof of Theorem 5.1. We briefly remark that the following proof crucially depends on the trace theorem established in Lemma 2.3 and the fact that there are at most two large cracks in the neighborhood of Γ.

Proof of Theorem 5.1. In the general situation we possibly have $N \neq \tilde{N} = N \setminus (X_1 \cup X_2)$. Let $\hat{\mathcal{C}}$ be the covering considered in Lemma 6.6. Let K_1, K_2 with $\mathrm{dist}(K_1, K_2) \ge c|\Gamma|_\infty$ be the sets given by Lemma 4.5 and let $\tilde{\mathcal{C}}$ be the covering of $N \setminus (K_1 \cup K_2)$ consisting of the connected components of the sets $U \setminus (K_1 \cup K_2)$, $U \in \hat{\mathcal{C}}$. To simplify the exposition we prefer to present first a special case where K_1, K_2 have the form $K_- := K_1 = (-\tau - l_1, -l_1) \times (-\tau, \tau)$ and $K_+ := K_2 = (l_1, l_1 + \tau) \times (-\tau, \tau)$. Moreover, we suppose that the sets $\Psi^\pm$ associated to boundary components larger than $\hat{\tau}$ – if they exist at all – have the form $\Psi^\pm = \Psi_1^\pm \cup \Psi_2^\pm \cup \Psi_3^\pm$, where $\Psi_1^\pm = (\pm l_1, \pm(l_1 + \tau)) \times (\psi^\pm, 2\tau)$, $\Psi_2^\pm = (\pm l_1, \pm(l_1 + \psi^\pm)) \times (-\psi^\pm, \psi^\pm)$ and $\Psi_3^\pm = (\pm l_1, \pm(l_1 + \tau)) \times (-2\tau, -\psi^\pm)$. Here $\psi^\pm$ denote the corresponding values to $\Psi^\pm$ (see Section 4.2).

If Ψ^- or Ψ^+ do not exist, we set $\Psi_2^- = K_1$, $\Psi_2^+ = K_2$, respectively, and let $\Psi_j^\pm$, $j = 1, 3$, be the adjacent squares. In addition, we then define $\psi^\pm = \tau$. We will treat both cases simultaneously in the following.

This special case already covers the fundamental ideas of the proof as the arguments essentially rely on the property that $\mathrm{dist}(K_1, K_2) \ge c|\Gamma|_\infty$ and the fact that the shapes of all sets are comparable (through homeomorphisms with constants depending on h_*) to squares. We will indicate the necessary adaptions for the general case at the end of the proof.

Let $N'_\pm = N \cap \{\pm x_2 \geq 0\} \setminus (K_1 \cup K_2)$. By Lemma 4.5 the assumptions of Lemma 6.6 are satisfied on each set N'_+ and N'_-. Consequently, there are $A_\pm \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c_\pm \in \mathbb{R}^2$ such that for all $V_\pm \subset N'_\pm$, $V_\pm \in \mathcal{Y}$, one has

$$|N'_\pm| \fint_{V_\pm} |u(x) - (A_\pm\, x - c_\pm)|^2\, dx \leq C(1 + C_* r)\frac{\varepsilon}{\upsilon^3}|\Gamma|^3_\infty =: G \tag{6.38}$$

by (6.34). We let $\Xi^+_0 = (l_1, l_1 + \psi^+) \times \{0\}$, $\Xi^-_0 = (-l_1 - \psi^-, -l_1) \times \{0\}$ and without restriction (possibly after a small translation in $\mathbf{e}_2$-direction) we can assume $\mathcal{H}^1(\Xi^\pm_0 \cap \partial W) = 0$. The goal is to show

$$\int_{\Xi^\pm_0} |(A_+ - A_-)\, x + (c_+ - c_-)|^2\, d\mathcal{H}^1(x) \leq C\frac{\psi^\pm}{\upsilon|\Gamma|^2_\infty}G. \tag{6.39}$$

We prove this only for Ξ^-_0. As a preparation let $\tilde\Psi^-_1 = (-\frac{\tau}{2} - l_1, -l_1) \times (\psi^-, \frac{3}{2}\tau)$, $\tilde\Psi^-_3 = (-\frac{\tau}{2} - l_1, -l_1) \times (-\frac{3}{2}\tau, -\psi^-)$ and $\tilde\Psi^- = \big(\overline{\tilde\Psi^-_1} \cup \overline{\Psi^-_2} \cup \overline{\tilde\Psi^-_3}\big)^\circ$. We observe

$$\sum\nolimits_{\Gamma_l \cap \tilde\Psi^- \neq \emptyset} |\Gamma_l|_{\mathcal{H}} \leq C(1 - \omega_{\min})^{-1}\psi^- \tag{6.40}$$

for $C = C(h_*, q)$. If $\psi^- \geq c(1 - \omega_{\min})\tau$ this follows from (5.3) and the fact that $\Gamma_l \subset \hat N = N^{2\hat\tau}(\Gamma)$ for all Γ_l with $\Gamma_l \cap \tilde\Psi^- \neq \emptyset$ (recall $|\Gamma_l|_\infty \leq \hat\tau$ by the construction in Section 4.2). If $\psi^- \leq c(1 - \omega_{\min})\tau$, by (5.4) we obtain $|\Psi^- \cap \partial W|_{\mathcal{H}} \leq D(1 - \omega_{\min})^{-1}\psi^- \ll \tau$ taking $c > 0$ sufficiently small. Thus, we can assume that $\Gamma_l \subset \Psi^-$ if $\Gamma_l \cap \tilde\Psi^- \neq \emptyset$. This implies $\sum_{\Gamma_l \cap \tilde\Psi^-_i \neq \emptyset} |\Gamma_l|_{\mathcal{H}} \leq C|\Psi^- \cap \partial W|_{\mathcal{H}}$ and gives the assertion.

Recall that $\upsilon \leq r(1 - \omega_{\min})^3$ (see beginning of Section 6.1). Applying Theorem 2.1 we obtain by (5.4), (5.7), (6.40) and the fact that $\psi^- \leq \upsilon|\Gamma|_\infty$ (cf. also (6.2) for a similar estimate)

$$\begin{aligned}
&\int_{\tilde\Psi^-_i} |u(x) - (A_i\, x + c_i)|^2\, dx \\
&\quad\leq C|\tilde\Psi^-_i|\alpha(\tilde\Psi^-_i) + CC_*\varepsilon\upsilon^{-4}|\partial W \cap \tilde\Psi^-_i|\big(\sum\nolimits_{\Gamma_l \cap \tilde\Psi^-_i \neq \emptyset} |\Gamma_l|_\infty\big)^2 \\
&\quad\leq C\upsilon^2|\Gamma|^2_\infty(1 - \omega_{\min})^{-1}\varepsilon\psi^- + CC_*(1 - \omega_{\min})^{-3}\varepsilon\upsilon^{-2}|\Gamma|^2_\infty\psi^- \\
&\quad\leq C(1 + C_* r)\upsilon^{-3}|\Gamma|^2_\infty\psi^-\varepsilon
\end{aligned} \tag{6.41}$$

for $A_i \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $c_i \in \mathbb{R}^2$, $i = 1, 3$. Likewise using particularly (6.40) and $|\Psi^-_2| \leq C(\psi^-)^2$ we get

$$\int_{\Psi^-_2} |u(x) - (A_2\, x + c_2)|^2\, dx \leq C(1 + C_* r)\upsilon^{-4}|\Gamma|_\infty(\psi^-)^2\varepsilon \tag{6.42}$$

for $A_2 \in \mathbb{R}^{2\times 2}_{\text{skew}}$ and $c_2 \in \mathbb{R}^2$. By (6.38) for $V_\pm = \Psi_1^- \setminus K_1, \Psi_3^- \setminus K_1$ we see that

$$\|u - (A_+\, x + c_+)\|^2_{L^2(\Psi_1^- \setminus K_1)} + \|u - (A_-\, x + c_-)\|^2_{L^2(\Psi_3^- \setminus K_1)} \le C\upsilon G.$$

Applying (2.11) and (2.12) on $B_1 = \Psi_i^- \setminus K_1$, $B_2 = \tilde{\Psi}_i^-$ $i = 1, 3$, we then derive by (6.41) employing $\psi^- \le \upsilon|\Gamma|_\infty$

$$\begin{aligned} &\tau^2 |A_+ - A_1|^2 + |c_+ - c_1 + (A_+ - A_1)\, b_+|^2 \le C(\upsilon|\Gamma|^2_\infty)^{-1} G, \\ &\tau^2 |A_- - A_3|^2 + |c_- - c_3 + (A_- - A_3)\, b_-|^2 \le C(\upsilon|\Gamma|^2_\infty)^{-1} G, \end{aligned} \tag{6.43}$$

where $b_- = (-l_1, -\tau)^T$ and $b_+ = (-l_1, \tau)^T$. Furthermore, Lemma 2.6, (5.4), (5.7), (6.5), (6.40) and (6.41) yield

$$\begin{aligned} &\int_{\partial \tilde{\Psi}_i^-} |u(x) - (A_i\, x + c_i)|^2 \, d\mathcal{H}^1(x) \\ &\qquad \le C(\upsilon|\Gamma|_\infty)^{-1} \|u - (A_i\, \cdot + c_i)\|^2_{L^2(\tilde{\Psi}_i^-)} + C\upsilon|\Gamma|_\infty \alpha(\tilde{\Psi}_i^-) \\ &\qquad\quad + CC_* \varepsilon \upsilon^{-4} \sum\nolimits_{\Gamma_l \cap \tilde{\Psi}_i^- \neq \emptyset} |\Theta_l|_* \sum\nolimits_{\Gamma_l \cap \tilde{\Psi}_i^- \neq \emptyset} |\Theta_l|_* \\ &\qquad \le C(1 + rC_*)\upsilon^{-4} |\Gamma|_\infty \psi^- \varepsilon \le C\psi^- (\upsilon|\Gamma|^2_\infty)^{-1} G \end{aligned} \tag{6.44}$$

for $i = 1, 3$, where we tacitly assumed that all boundary components are rectangular (cf. discussion after (6.4)). In the penultimate step we again used $\psi^- \le \upsilon|\Gamma|_\infty$ and $\upsilon \le r(1 - \omega_{\min})^3$. Likewise, we get

$$\int_{\partial \Psi_2^- \cup \Xi_0^-} |u(x) - (A_2\, x + c_2)|^2 \, d\mathcal{H}^1(x) \le C\psi^- (\upsilon|\Gamma|^2_\infty)^{-1} G, \tag{6.45}$$

where we replaced $(\upsilon|\Gamma|_\infty)^{-1}$ by $(\psi^-)^{-1}$ and used (6.42) instead of (6.41). Observe that (6.45) is well defined in the sense of traces since $\mathcal{H}^1(\Xi_0^- \cap \partial W) = 0$. Define $\Xi_\pm^- = (-\frac{\psi^-}{2} - l_1, -l_1) \times \{\pm \psi^-\}$ and note that $\Xi_+^- \subset \partial \Psi_2^- \cap \partial \tilde{\Psi}_1^-$ and $\Xi_-^- \subset \partial \Psi_2^- \cap \partial \tilde{\Psi}_3^-$. Again up to a small translation in $\mathbf{e}_2$-direction we may suppose $\mathcal{H}^1(\Xi_\pm^- \cap \partial W) = 0$. Combining the estimates (6.43), (6.44) for $i = 1, 3$ we obtain

$$\int_{\Xi_+^-} |u(x) - (A_+\, x + c_+)|^2 \, d\mathcal{H}^1 + \int_{\Xi_-^-} |u(x) - (A_-\, x + c_-)|^2 \, d\mathcal{H}^1 \le C\psi^- (\upsilon|\Gamma|^2_\infty)^{-1} G.$$

Using once more the techniques provided in Section 2.2 we may estimate the difference of $A_\pm$, A_2 and $c_\pm$, c_2 on the boundaries $\Xi_\pm^-$ (replace the sets B_1, B_2 in (2.11), (2.12) by the surfaces $\Xi_\pm^-$) and obtain by (6.45) an expression similar to (6.43):

$$(\psi^-)^2 |A_\pm - A_2|^2 + |c_\pm - c_2 + (A_\pm - A_2)\, b_2|^2 \le C(\upsilon|\Gamma|^2_\infty)^{-1} G, \tag{6.46}$$

where $b_2 = (-l_1, 0)^T$. Together with (6.45) this leads to

$$\int_{\Xi_0^-} |u(x) - (A_\pm\, x + c_\pm)|^2 \, d\mathcal{H}^1(x) \le C\psi^- (\upsilon|\Gamma|^2_\infty)^{-1} G$$

and then by the triangle inequality we derive

$$\int_{\Xi_0^-} |(A_+ - A_-)\,x + (c_+ - c_-)|^2 \, d\mathcal{H}^1(x) \le C\psi^-(\upsilon|\Gamma|_\infty^2)^{-1}G.$$

This gives the desired estimate (6.39). From (6.39) applied on both sets, Ξ_0^- and Ξ_0^+, we deduce

$$-C(l_1\upsilon)^2|(A_+ - A_-)\,\mathbf{e}_1|^2 + |-(A_+ - A_-)\,l_1\mathbf{e}_1 + (c_+ - c_-)|^2 \le C(\upsilon|\Gamma|_\infty^2)^{-1}G$$

and

$$-C(l_1\upsilon)^2|(A_+ - A_-)\,\mathbf{e}_1|^2 + |(A_+ - A_-)\,l_1\mathbf{e}_1 + (c_+ - c_-)|^2 \le C(\upsilon|\Gamma|_\infty^2)^{-1}G.$$

Combining these two estimates we find for υ sufficiently small $l_1^2|A_+ - A_-|^2 = 2l_1^2|(A_+ - A_-)\mathbf{e}_1|^2 \le C(\upsilon|\Gamma|_\infty^2)^{-1}G$ and then also $|c_+ - c_-|^2 \le C(\upsilon|\Gamma|_\infty^2)^{-1}G$. (This is the step where we fundamentally use $\mathrm{dist}(K_1, K_2) \ge c|\Gamma|_\infty$.) We choose $A = A_-$ and $c = c_-$. Recalling the definition of G and $|N| \le \upsilon|\Gamma|_\infty^2$ we obtain by (6.38) for $V_\pm = N'_\pm$

$$\int_{N'_+ \cup N'_-} |u(x) - (A\,x - c)|^2 \, dx \le C(1 + C_* r)\frac{\varepsilon}{\upsilon^3}|\Gamma|_\infty^3,$$

which together with the estimates (6.41) and (6.43) gives (6.4)(i). Finally, (6.46) yields $(\psi^-)^2|A - A_2|^2 \le C(\upsilon|\Gamma|_\infty^2)^{-1}G$ and $|c - c_2 + (A - A_2)\,(-l_1, 0)^T|^2 \le C(\upsilon|\Gamma|_\infty^2)^{-1}G$. Then by (6.42) and the fact that $|\Psi_2^-| \le C(\psi^-)^2 \le C\psi^-\upsilon|\Gamma|_\infty$ we conclude

$$\int_{\Psi_2^-} |u(x) - (A\,x + c)|^2 \, dx \le C(1 + rC_*)\upsilon^{-3}|\Gamma|_\infty^2\psi^-\varepsilon$$

giving (6.4)(ii). The estimate for Ψ_2^+ follows analogously.

It remains to briefly indicate the necessary adaptions for the general case. The main differences are (i) the shape of the sets $\Psi_i^\pm, i = 1, 2, 3$ and (ii) the position of the sets K_1, K_2. For (i) we observe that $\Psi_i^\pm, i = 1, 3$, are $C(h_*)$-Lipschitz equivalent to a square by Lemma 4.6(ii) and Lemma 4.7(ii) whereby (6.41) can still be derived (cf. Remark 2.5(i)). (Note that the sets are even related by affine mappings.) Likewise, an estimate of the form (6.42) can be derived for sets $(\Psi_2^\pm)^* \supset \Psi_2^\pm$ which have been constructed in Section 4.2. Moreover, although not stated explicitly in Section 2.3, the trace estimate used in (6.44), (6.45) can also be applied for sets being an affine transformation of a square. The rest of the arguments concerning the difference of infinitesimal rigid motions (see (6.43), (6.46)) remains unchanged. For (ii) we observe that in the derivation of $|A_+ - A_-|^2 \le Cl_1^{-2}(\upsilon|\Gamma|_\infty^2)^{-1}G$ we fundamentally used that $\mathrm{dist}(K_1, K_2) \sim l_1$, but the exact position of the sets K_1, K_2 was not essential. □

We briefly explain Remark 5.3. At the beginning of Section 6.1 we have already observed that $\upsilon \sim C(h_*)\sigma^3$. Now the property for C_2 follows immediately (see (5.12)). For C_1 we use (5.16) and the fact that $\hat{C} = \hat{C}(h_*)$ (see end of Section 6.1).

We close this section with an estimate for the skew symmetric matrices involved in the above results which will be needed in the derivation of the nonlinear rigidity estimates in [18].

Lemma 6.7 *Let be given the situation of Theorem 5.2 for a function $u \in H^1(W)$ and define $y = \bar{R}\,(\mathbf{id}+u)$, where $\mathbf{id}$ denotes the identity function and $\bar{R} \in SO(2)$. Let $V \subset Q_\mu$ be a rectangle and let $\mathcal{F}(V)$ be the boundary components $(\Gamma_l)_l = (\Gamma_l(U))_l$ satisfying $N^{\hat{\tau}_l}(\partial R_l) \subset V$ and (5.10). Then there is a $C_3 = C_3(\sigma, h_*)$ such that*

$$\sum\nolimits_{\Gamma_l \in \mathcal{F}(V)} |X_l|^2_\infty |A_l|^p \le C_3\big(\|\nabla y - \bar{R}\|^p_{L^p(V\cap W)} + (\varepsilon s^{-1})^{\frac{p}{2}-1}\varepsilon|\partial U \cap V|_{\mathcal{H}}\big)$$

for $p = 2, 4$, where $X_l \subset Q_\mu$, $A_l \in \mathbb{R}^{2\times 2}_{\rm skew}$ is given in (5.9).

Remark 6.8 Similarly as in Remark 5.3 we note that the constant $C_3 = C_3(\sigma, h_*)$ has polynomial growth in σ, i.e. $C_3(\sigma, h_*) \le C(h_*)\sigma^{-z}$ for some $z \in \mathbb{N}$.

Proof. Let $p = 2, 4$. Consider a component $\Gamma = \Gamma_l(U)$ with corresponding rectangle R and X with $\partial X = \Gamma$. It suffices to show $|R|^2_\infty |A|^p \le C_3\big(\|\nabla y - \bar{R}\|^p_{L^p(\tilde{N})} + (\varepsilon s^{-1})^{\frac{p}{2}-1}\varepsilon|\Gamma|_{\mathcal{H}}\big)$ for this component, where $\tilde{N} = N^{\hat{\tau}}(\partial R) \setminus \bigcup_{\Gamma_l \in \mathcal{I}(\Gamma)} N^{\hat{\tau}_l}(\partial R_l)$ and $\mathcal{I}(\Gamma) = \{\Gamma_l : |\Gamma_l|_\infty \le |\Gamma|_\infty\}$. Then the assertion follows by summation over all components and the fact that $|X|_\infty \le |R|_\infty$.

As Γ satisfies (5.10), we observe that we applied Theorem 5.1 on ∂R in some iteration step, in particular (6.4) is satisfied. Choose $U \in \mathcal{C}$ with $U \subset N^{2,+}$ as considered in Lemma 6.3. By assumption we find a set $S \subset (l_2, l_2 + \tau)$ with $|S| \ge \frac{1}{2}\frac{\tau}{20}$ such that for $T = (\mathbb{R} \times S) \cap U$ we have $T \cap \bigcup_{\Gamma_l \in \mathcal{I}(\Gamma)} N^{\hat{\tau}_l}(\partial R_l) = \emptyset$ by (5.2). It is not restrictive to assume that S is connected as otherwise we follow the subsequent arguments for every connected component of S. Recall $|\Gamma|_\infty \le |\partial R|_\infty \le C|\Gamma|_\infty$ by (3.6). The Poincaré inequality and a rescaling argument imply

$$\int_T |u(x) - \hat{c}|^2\,dx \le C|T|^{1-\frac{2}{p}}|\Gamma|^2_\infty \|\nabla y - \bar{R}\|^2_{L^p(T)}$$

for a constant $\hat{c} \in \mathbb{R}^2$ and $p = 2, 4$. This together with (6.4) yields

$$\int_T |A\,x + c - \hat{c}|^2\,dx \le C(\upsilon|\Gamma|^2_\infty)^{1-\frac{2}{p}}|\Gamma|^2_\infty\|\nabla y - \bar{R}\|^2_{L^p(T)} + C\hat{C}\sum_{n=0}^\infty \Big(\frac{2}{3}\Big)^n \upsilon^{-3}|\Gamma|^3_\infty\varepsilon.$$

For the constant in the latter part see below (5.16). Arguing as in (2.11) we find

$$|T||\Gamma|^2_\infty|A|^2 \le C\int_T |A\,x + c - \hat{c}|^2\,dx.$$

Thus, by $|T| \geq Cv|\Gamma|^2_\infty$ and an elementary calculation we derive

$$|R|^2_\infty |A|^4 \leq C|\Gamma|^2_\infty |A|^4 \leq Cv^{-1}\|\nabla y - \bar{R}\|^4_{L^4(T)} + Cv^{-8}\varepsilon^2.$$

As $|\partial R|_* \geq s$, we obtain $|\Gamma|_{\mathcal{H}} \geq C|\partial R|_* \geq Cs$ by (3.5)(ii). Choose C_3 large enough and recall $T \subset N^{\bar{\tau}}(\Gamma) \setminus \bigcup_{\Gamma_l \in \mathcal{I}(\Gamma)} N^{\bar{\tau}_l}(\partial R_l) \subset V$ as well as the fact that $v \geq C(h_*)\sigma^3$. This yields

$$|R|^2_\infty |A|^4 \leq C|\Gamma|^2_\infty |A|^4 \leq C_3\|\nabla y - \bar{R}\|^4_{L^4(\tilde{N})} + C_3\varepsilon s^{-1}\varepsilon|\Gamma|_{\mathcal{H}}.$$

giving the claim for $p = 4$. Likewise, for $p = 2$ we deduce

$$\begin{aligned}|R|^2_\infty |A|^2 \leq C|\Gamma|^2_\infty |A|^2 &\leq Cv^{-1}\|\nabla y - \bar{R}\|^2_{L^2(T)} + Cv^{-4}\varepsilon|\Gamma|_\infty \\ &\leq C_3\|\nabla y - \bar{R}\|^2_{L^2(\tilde{N})} + C_3\varepsilon|\Gamma|_{\mathcal{H}}.\end{aligned}$$

□

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