COUPLING RATE-INDEPENDENT AND RATE-DEPENDENT PROCESSES: EXISTENCE RESULTS

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Abstract. We address the analysis of an abstract system coupling a rate-independent process with a second order (in time) nonlinear evolution equation. We propose suitable weak solution concepts and obtain existence results by passing to the limit in carefully devised time-discretization schemes. Our arguments combine techniques from the theory of *gradient systems* with the toolbox for *rate-independent* evolution, thus reflecting the mixed character of the problem. Finally, we discuss applications to a class of rate-independent processes in visco-elastic solids with inertia, and to a recently proposed model for damage with plasticity.

Key words: rate-independent processes; gradient systems; inertia; energetic solutions; existence results; time discretization; generalized standard solids.

1. Introduction

The modeling of dissipative processes in mechanical systems via internal variables often leads to PDE systems which have the following structure

$$\mathrm{D}_u\mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } \mathbf{V}^* \quad \text{for a.a. } t\in(0,T), \tag{1.1a}$$

$$\partial\mathcal{R}(z'(t)) + \mathrm{D}_z\mathcal{E}(t,u(t),z(t)) = 0 \qquad \text{in } \mathbf{Z}^* \quad \text{for a.a. } t\in(0,T). \tag{1.1b}$$

Here, the (separable) Banach spaces $\mathbf{V}$ and $\mathbf{Z}$ are the state spaces for the (slow) variable u and the (fast) internal variable z, with z' its time derivative. A guiding example is the deformation of a body under the influence of dissipative processes such as, e.g., damage, plasticity, phase transformations, or delamination. Assuming small strains, the deformation of the body is described by its displacement field u, whereas, within the theory of generalized standard materials [HN75] (see also [Fré02]), the changes of its elastic behavior due to the evolving dissipative processes are modeled by an *internal variable* z. The evolution of the system results from a trade-off between the two competing mechanisms of energy conservation and energy dissipation, caused by time-dependent external loadings. Here and in what follows, the latter are included in the time-dependent energy functional $\mathcal{E}:[0,T]\times\mathbf{V}\times\mathbf{Z}\to(-\infty,+\infty]$, ($\mathrm{D}_u\mathcal{E}$ and $\mathrm{D}_z\mathcal{E}$ denoting its Gâteaux derivatives with respect to the variables u and z); dissipation in z is evaluated through a positive, lower semicontinuous, and convex dissipation potential $\mathcal{R}:\mathbf{Z}\to[0,+\infty]$, which we assume in addition to be positively 1-homogeneous. Thus system (1.1) is invariant under rescalings of the time variable, i.e. it is *rate-independent*. We refer to [Mie05] and [Mie11] for surveys on the analysis of such class of systems.

In this paper we want to analyze the case in which the variable u additionally evolves subject to *viscous* dissipation (for example, according to Kelvin-Voigt rheology in the frame of material modeling). On top of that we also allow for the presence of inertia. Thus, (1.1a) is replaced by the evolutionary equation

$$\varrho u''(t) + \partial\mathcal{V}(u'(t)) + \mathrm{D}_u\mathcal{E}(t,u(t),z(t)) \ni 0 \qquad \text{in } \mathbf{V}^* \quad \text{for a.a. } t\in(0,T), \tag{1.1c}$$

where $\varrho\geq 0$. The (lower semicontinuous, convex) dissipation potential $\mathcal{V}:\mathbf{V}\to[0,+\infty)$ has superlinear growth at infinity, namely $\lim_{\|v\|_{\mathbf{V}}\uparrow+\infty}\frac{\mathcal{V}(v)}{\|v\|_{\mathbf{V}}}=+\infty$. The kinetic energy leading to the inertial term $\varrho u''$ in (1.1c) is of

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the form

$$\mathcal{K}(u') = \frac{\varrho}{2}\|u'\|_{\mathbf{W}}^2 \qquad \text{with } \mathbf{W} \text{ a Hilbert space such that } \mathbf{V} \subset \mathbf{W} \text{ continuously and densely.} \tag{1.2}$$

Observe that, for $\varrho = 0$ system (1.1b, 1.1c) falls into the class of *(generalized) gradient systems*; in what follows, we will refer to it with the symbol $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$. In the case with inertia (i.e., if $\varrho > 0$), (1.1b, 1.1c) contains a *reversible* term and thus ceases to be a *pure* gradient system. We shall refer to it as an *evolutionary system* and denote it by $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$.

Despite its *mixed* rate-independent/rate-dependent character, (1.1b, 1.1c) retains all the difficulties attached to the analysis of purely rate-independent systems. In particular, due to the linear growth at infinity of the potential $\mathcal{R}$, for the variable z as a function of time we only have BV-regularity, which does not guarantee the existence of the pointwise derivative $z'(t)$ and thus calls for weak solvability concepts for (1.1b, 1.1c).

This issue was first addressed in [Rou09], focusing on a class of processes with a mixed rate-independent/rate-dependent character in generalized standard materials, modeled by a system of the type (1.1b,1.1c), with a *quadratic* dissipation potential $\mathcal{V}$. In the spirit of the concept of *energetic solutions* for rate-independent systems, cf. [Mie05], an *energetic-type* weak solvability concept was proposed and analyzed in [Rou09]. This has provided a sound mathematical basis for various subsequent extensions, in particular to systems coupling rate-independent, rate-dependent, and *thermal* processes in thermo-visco-elastic materials, cf. [Rou10]. It has also paved the way to the recent [RT14], where a broad class of models encompassing rate-independent and rate-dependent processes coupled with temperature evolution and phase separation has been tackled.

In this paper, though, we will confine ourselves to the isothermal setting. We aim to revisit the analysis carried out in [Rou09] from a more abstract and general perspective. This will allow us to extend the existence results in [Rou09] to a much broader class of dissipation potentials $\mathcal{V}$ and energy functionals $\mathcal{E}$. In particular, let us mention that the energies considered in [Rou09] need not be differentiable w.r.t. the variable z, as the energetic-type solution notion therein used does not involve the Gâteaux differential $\mathrm{D}_z\mathcal{E}$, but they are taken to be sufficiently smooth wr.t. u, and with suitable convexity properties. Here, instead, we focus on the case in which the functional $u \mapsto \mathcal{E}(t, u, z)$ is *as nonconvex and nonsmooth as possible.* Accordingly, we will replace the Gâteaux differential $\mathrm{D}_u\mathcal{E}$ in (1.1c) by the *Fréchet subdifferential* of $\mathcal{E}$ with respect to u, namely the multivalued operator $\partial_u^-\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \rightrightarrows \mathbf{V}^*$ defined at a point (t, u, z) in the domain of $\mathcal{E}$ by

$$\xi \in \partial_u^-\mathcal{E}(t, u, z) \quad \text{if and only if} \quad \liminf_{v \to u \text{ in } \mathbf{V}} \frac{\mathcal{E}(t, v, z) - \mathcal{E}(t, u, z) - \langle \xi, v - u\rangle_U}{\|v - u\|_{\mathbf{V}}} \geq 0. \tag{1.3}$$

Indeed, $\partial_u^-\mathcal{E}$ may be understood as a localization of the subdifferential of $\mathcal{E}$ (w.r.t. u) in the sense of convex analysis, and in fact it reduces to the latter object as soon as the functional $u \mapsto \mathcal{E}(t, u, z)$ is *convex*, whereas if $u \mapsto \mathcal{E}(t, u, z)$ is Gâteaux-differentiable, then $\partial_u^-\mathcal{E}(t, u, z)$ reduces to the singleton $\{\mathrm{D}_u\mathcal{E}(t, u, z)\}$. Therefore, we will study the subdifferential inclusion

$$\varrho u''(t) + \partial\mathcal{V}(u'(t)) + \partial_u^-\mathcal{E}(t, u(t), z(t)) \ni 0 \qquad \text{in } \mathbf{V}^* \quad \text{for a.a. } t \in (0, T), \tag{1.4a}$$

coupled with the (formally written)

$$\partial\mathcal{R}(z'(t)) + \mathrm{D}_z\mathcal{E}(t, u(t), z(t)) = 0 \qquad \text{in } \mathbf{Z}^* \quad \text{for a.a. } t \in (0, T). \tag{1.4b}$$

1.1. **Solution concepts for** (1.4)**.** As previously mentioned, already in the pure rate-independent context the lack of time regularity of z due to the rate-independent character of the flow rule (1.4b) makes it necessary to resort to a weak solvability concept. The one we are going to adopt in this coupled rate-independent/rate-dependent framework is the natural generalization of the energetic notion proposed for the specific class of systems in [Rou09], involving a *quadratic* dissipation $\mathcal{V}$.

In order to motivate our notion(s) of solution, let us momentarily continue to argue with the formally written system (1.4), observing that it can be rewritten as

$$\begin{aligned} &-(\varrho u''(t)+\xi(t)) \in \partial\mathcal{V}(u'(t)) \qquad \text{with } \xi(t) \in \partial_u^-\mathcal{E}(t, u(t), z(t)) \ni 0 \quad \text{in } \mathbf{V}^* \quad \text{for a.a. } t \in (0, T), \\ &-\mathrm{D}_z\mathcal{E}(t, u(t), z(t)) \in \partial\mathcal{R}(z'(t)) \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in } \mathbf{Z}^* \quad \text{for a.a. } t \in (0, T). \end{aligned} \tag{1.5}$$

Recall that, for a given convex and lower semicontinuous functional Φ one has $\zeta \in \partial\Phi(w)$ if and only if there holds $\Phi(w) + \Phi^*(\zeta) = \langle\zeta, w\rangle$, where the Fenchel-Moreau conjugate is defined by $\Phi^*(\zeta) := \sup_w\{\langle\zeta, w\rangle - \Phi(w)\}$.

Therefore, introducing the conjugates $\mathcal{V}^*$ and $\mathcal{R}^*$, (1.5) rewrites as

$$\begin{aligned} &\mathcal{V}(u'(t)) + \mathcal{V}^*(-(\varrho u''(t){+}\xi(t))) = -\left\langle (\varrho u''(t){+}\xi(t)), u'(t)\right\rangle_{\mathbf{V}} && \text{for a.a. } t \in (0,T),\\ &\mathcal{R}(z'(t)) + \mathcal{R}^*(-\mathrm{D}_z\mathcal{E}(t,u(t),z(t))) = -\left\langle \mathrm{D}_z\mathcal{E}(t,u(t),z(t)), z'(t)\right\rangle_{\mathbf{Z}} && \text{for a.a. } t \in (0,T). \end{aligned} \tag{1.6}$$

Now, taking into account the definition and properties of $\mathcal{V}^*$ and $\mathcal{R}^*$, it is not difficult to see that the two separate identities in (1.6) are equivalent to

$$\begin{aligned} \mathcal{V}(u'(t)) + \mathcal{V}^*(-(\varrho u''(t){+}\xi(t))) + \mathcal{R}(z'(t)) &= -\left\langle (\varrho u''(t){+}\xi(t)), u'(t)\right\rangle_{\mathbf{V}} - \left\langle \mathrm{D}_z\mathcal{E}(t,u(t),z(t)), z'(t)\right\rangle_{\mathbf{Z}}\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t),z(t)) + \partial_t\mathcal{E}(t,u(t),z(t)) - \mathcal{K}(u'(t)) \end{aligned} \tag{1.7a}$$

for a.a. $t \in (0,T)$, obtained by adding them up, using the chain rules for $\mathcal{E}$ and for the kinetic energy $\mathcal{K}$, and the fact that, by the 1-homogeneity of $\mathcal{R}$, its conjugate $\mathcal{R}^*$ is the indicator functional of the set $K^* = \partial\mathcal{R}(0) = \{\eta \in \mathbf{Z}^* : \langle \eta, \zeta\rangle_{\mathbf{Z}} \le \mathcal{R}(\zeta) \text{ for all } \zeta\}$. Therefore, one has in fact to add to (1.7a) the constraint

$$-\mathrm{D}_z\mathcal{E}(t,u(t),z(t)) \in \partial\mathcal{R}(0) \quad \text{in } \mathbf{Z}^* \quad \text{for a.a. } t \in (0,T)\,. \tag{1.7b}$$

We are now in the position to state our weak solvability notion for system (1.4): We call a pair $(u,z) : [0,T] \to \mathbf{V}\times\mathbf{Z}$ an *energetic solution* to the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, resp. evolutionary system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ for $\varrho > 0$, if it fulfills

(1) the subdifferential inclusion (1.4a);
(2) the *semistability condition*

$$\mathcal{E}(t,u(t),z(t)) \le \mathcal{E}(t,u(t),\tilde z) + \mathcal{R}(\tilde z{-}z(t)) \qquad \text{for all } \tilde z \in \mathbf{Z} \text{ for all } t \in [0,T]; \tag{1.8a}$$

(3) the *energy-dissipation* inequality

$$\begin{aligned} &\frac{\varrho}{2}\|u'(t)\|_{\mathbf{W}}^2 + \int_0^t \mathcal{V}(u'(s)) + \mathcal{V}^*(-\xi(s){-}\varrho u''(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(z,[0,t]) + \mathcal{E}(t,u(t),z(t))\\ &\qquad \le \frac{\varrho}{2}\|u'(0)\|_{\mathbf{W}}^2 + \mathcal{E}(0,u(0),z(0)) + \int_0^t \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s \qquad \text{for all } t \in [0,T], \end{aligned} \tag{1.8b}$$

with $\xi(s)$ a selection in $\partial_u^-\mathcal{E}(s,u(s),z(s))$ fulfilling (1.4a), i.e. $\varrho u''(s) + \omega(s) + \xi(s) = 0$ with $\omega(s) \in \partial\mathcal{V}(u'(s))$, for almost all $s \in (0,T)$.

First of all, let us remark that due to $\mathcal{V}(u'(t)) + \mathcal{V}^*(-\xi(s){-}\varrho u''(s)) = \langle \partial\mathcal{V}(u'(t)), u'(t)\rangle_{\mathbf{V}}$ by convexity, in the case of a quadratic dissipation potential $\mathcal{V}$ the energetic solutions in the sense of (1.8) coincide with the ones introduced in [Rou09, Def. 5.1]. Clearly, the energy-dissipation inequality (1.8b) is the integrated, inequality version of (1.7a), whereas the semistability (1.8a) can be understood as a weak form of condition (1.7b). This solution concept well reflects the *mixed* character of system (1.4). On the one hand, in the spirit of the energetic formulation of (purely) rate-independent systems, it features an energy (in-)equality and a (semi)stability condition in which the rate-dependent character of the variable u is tracked by the fact that only the variable z is allowed to vary, whereas u is kept fixed. On the other hand, (1.8b) is reminiscent of the energy identity (also often referred to as *De Giorgi principle*, cf. [Mie14])

$$\int_0^t \left(\mathcal{V}(u'(s)) + \mathcal{V}^*(-\mathrm{D}_u\mathscr{E}(s,u(s)))\right)\mathrm{d}s + \mathscr{E}(t,u(t)) = \mathscr{E}(0,u(0)) + \int_0^t \partial_t\mathscr{E}(s,u(s))\,\mathrm{d}s\,. \tag{1.9}$$

Its validity is *equivalent*, under suitable conditions on the energy functional $\mathscr{E}$ (not depending on the variable z), to the fact that u is a solution to the (*pure*) gradient system

$$\partial\mathcal{V}(u'(t)) + \mathrm{D}_u\mathscr{E}(t,u(t)) \ni 0 \qquad \text{in } \mathbf{V}^* \quad \text{for a.a. } t \in (0,T), \tag{1.10}$$

cf. [RMS08, MRS13b]. This principle has also been exploited for the Vlasov-Fokker-Planck equation in [DPZ13].

We will also work with a weaker solution notion, referred to as *weak energetic* solution, where the validity of the subdifferential inclusion (1.4a) is no longer claimed and thus only the semistability (1.8a) and the energy-dissipation inequality (1.8b) (with $\xi(s)$ a selection of $\partial_u^-\mathcal{E}(s,u(s),z(s))$ for a.e. $s \in (0,T)$) are required to hold, similarly as for energetic solutions to (purely) rate-independent system. Due to its intrinsically *variational* character, this concept will turn out to be particularly flexible for the *Evolutionary* Γ*-convergence analysis* for gradient and evolutionary systems in the forthcoming [RT15], in the same spirit as in [MRS08, Mie14].

The argument guaranteeing the passage from *weak energetic* to *energetic* solutions is drawn from the variational theory for gradient systems in [MRS13b]. Therein it was proved that, if the energy satisfies a suitable chain-rule inequality, then any curve fulfilling (1.9) a priori only in terms of an *inequality* "$\leq$", indeed complies with the energy *equality* (1.9), and it is thus a solution to (1.10). Similarly, in the forthcoming Proposition 3.2 we are going to show that any weak energetic solution along which the (integral) chain-rule inequality

$$\begin{aligned} &\int_0^t \langle \mathrm{D}_u \mathcal{E}(s,u(s),z(s)), u'(s)\rangle_{\mathbf{V}} \,\mathrm{d}s \\ &\leq \mathcal{E}(t,u(t),z(t)) - \mathcal{E}(0,u(0),z(0)) - \int_0^t \partial_t \mathcal{E}(s,u(s),z(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(z,[0,t]) \text{ for all } t\in[0,T], \end{aligned} \tag{1.11}$$

holds, is in fact an energetic solution (thus fulfilling the subdifferential inclusion (1.4a)), such that (1.8b) holds as an *energy-dissipation balance*. In Theorem 3.6 we will also provide some sufficient conditions on $\mathcal{E}$ guaranteeing the validity of (1.11) along weak energetic solutions. The proof of the latter result is in turn based on the combination of the semistability condition (1.8a) with Riemann sum techniques, with arguments typical of the theory of rate-independent systems, cf. [DMFT05, Mie05, MM05].

1.2. **Existence results.** To obtain existence results, we will follow an approach akin to the one for gradient flows and gradient systems in [Amb95, AGS08, RS06, RMS08, MRS13b], and for rate-independent evolution in [MT04, Mie05, Mie11, MRS08]: Namely, we will enucleate a series of *abstract conditions* on the energy $\mathcal{E}$ and on the dissipation potentials $\mathcal{V}$ and $\mathcal{R}$, guaranteeing the existence (via limit passage in carefully devised time-discretization schemes) of:

Theorem 1: energetic solutions to $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ for $\mathcal{V}$ *quadratic*;
Theorem 2: weak energetic solutions to $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ for *general* $\mathcal{V}$ with superlinear growth;
Theorem 3: weak energetic solutions to $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ for *general* $\mathcal{V}$ with superlinear growth;
Theorem 4: energetic solutions to $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ for $\mathcal{V}$ *quadratic*.

In accordance with the mixed rate-independent/rate-dependent character of the problem, our analysis for (weak) energetic solutions to the gradient (evolutionary) system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ ($(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, resp.) will combine tools for (pure) gradient systems with ideas from the theory of rate-independent processes. While postponing to Sections 2, 4, and 5 a detailed analysis of all of our hypotheses, let us briefly highlight here their most significant features, focusing on the case of the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$.

First of all, as standard for gradient flows, we require coercivity, i.e. that there exists $\tau_o>0$ such that for every $(u_o,z_o)\in\mathbf{V}\times\mathbf{Z}$

$$\text{the map } (u,z)\mapsto \mathcal{E}(t,u,z)+\tau_o\mathcal{V}\left(\frac{u-u_o}{\tau_o}\right)+\mathcal{R}(z-z_o) \text{ has sublevels bounded in } \mathbf{U}\times\mathbf{X}, \tag{1.12}$$

where the spaces $\mathbf{U}\subset\mathbf{V}$ and $\mathbf{X}\subset\mathbf{Z}$ with continuous embeddings. In fact, (1.12) is the minimal coercivity requirement to ensure that the *alternate* time-incremental minimization scheme

$$\begin{aligned} u_\tau^n &\in \operatorname*{Argmin}_{u\in\mathbf{V}}\left(\tau\mathcal{V}\left(\frac{u-u_\tau^{n-1}}{\tau}\right)+\mathcal{E}(t_n,u,z_\tau^{n-1})\right), \\ z_\tau^n &\in \operatorname*{Argmin}_{z\in\mathbf{Z}}\left(\tau\mathcal{R}\left(\frac{z-z_\tau^{n-1}}{\tau}\right)+\mathcal{E}(t_n,u_\tau^n,z)\right), \end{aligned} \tag{1.13}$$

yielding discrete solutions to (1.4) in the case of a *general* dissipation potential $\mathcal{V}$ with superlinear growth, does admit solutions. One then constructs approximate solutions to (1.4) by interpolating them suitably.

A crucial step in the existence proof is the derivation of a *discrete energy-dissipation* inequality satisfied by the approximate solutions: all a priori estimates stem from it, and one has to pass to the limit in this discrete inequality to obtain the time-continuous (1.8b). Again focusing on the case of gradient systems where $\varrho=0$, we will exploit two methods of proof to obtain the discrete version of (1.8b).

(1) For the case of a general dissipation potential, we will resort to specific techniques from the theory of *Minimizing Movements* for gradient flows employing the notion of *variational interpolant* of the discrete solutions due to E. De Giorgi, cf. [Amb95, AGS08].
(2) For $\mathcal{V}$ quadratic, we will make use of a suitable condition on the Fréchet subdifferential $\partial_u^-\mathcal{E}$, referred to as uniform Fréchet subdifferentiability, namely

$$\begin{aligned} &\exists\Lambda\geq 0\ \ \forall\, t\in[0,T],\ \forall\, u,\, v\in\mathbf{V},\ \forall\, z\in\mathbf{Z}\ \ \forall\,\xi\in\partial_u^-\mathcal{E}(t,u,z)\ : \\ &\qquad \mathcal{E}(t,v,z)-\mathcal{E}(t,u,z)\geq\ \langle\xi,v-u\rangle_{\mathbf{V}}-\Lambda\|v-u\|_{\mathbf{V}}^2 \end{aligned} \tag{1.14}$$

Essentially, (1.14) gives a *global* character to inequality (2.10) defining the Fréchet subdifferential. As we will see, it is guaranteed if, e.g., $u \mapsto \mathcal{E}(t,u,z)$ fulfills some λ-convexity property.

A key ingredient for passing to the limit in the *discrete energy-dissipation* inequality, and in the (discrete version) of the subdifferential inclusion (1.4a), will be a suitable closedness/continuity condition on $(\partial_u \mathcal{E}^-, \mathcal{E})$. It ensures that the graph of the Fréchet subdifferential $\partial_u \mathcal{E}$ is closed in the strong-weak topology of $\mathbf{V} \times \mathbf{V}^*$ along sequences with bounded energy, and in some qualified situations, leading to *enhanced* (weak) energetic solutions, also that the energy functional is itself continuous along such sequences. While in the case of general dissipation potentials we will have to impose this requirement additionally, in the analysis of the case with $\mathcal{V}$ quadratic we will see that closedness/continuity can be in fact derived as a consequence of the uniform Fréchet subdifferentiability condition, in combination with a further *recovery sequence condition* on $\mathcal{E}$.

Finally, in order to prove the semistability (1.8a) we will exploit a version of the mutual recovery sequence condition proposed in [MRS08] for the fully rate-independent setting. This condition allows us to pass to the limit in the discrete form of the semistability condition by requiring that, given a sequence $(u_n, z_n)_n \subset \mathbf{V} \times \mathbf{Z}$ semistable at times t_n and such that $(t_n, u_n, z_n) \to (t, u, z)$ in suitable topologies, for every $\tilde{z} \in \mathbf{Z}$ there exists a *recovery sequence* $(\tilde{z}_n)_n \subset \mathbf{Z}$ converging to $\tilde{z}$ and such that

$$\limsup_{n\to\infty} \left(\mathcal{R}(\tilde{z}_n - z_n) + \mathcal{E}(t_n, u_n, \tilde{z}_n) - \mathcal{E}(t_n, u_n, z_n) \right) \leq \mathcal{R}(\tilde{z} - z) + \mathcal{E}(t, u, \tilde{z}) - \mathcal{E}(t, u, z). \tag{1.15}$$

In this way, semistability (i.e. positivity of the r.h.s. in (1.15) for *any* $\tilde{z} \in \mathbf{Z}$) is preserved in the limit passage.

1.3. **Applications.** To illustrate the outcome of our existence analysis for system (1.4), we have focused on applications to

(1) the class of rate-independent processes in viscous solids tackled in [Rou09], revisiting and extending the existence results proved therein;
(2) a coupled damage-plasticity model recently proposed in [AMV14], whose analysis falls outside the scope of the results in [Rou09].

More precisely, in Section 6.1 we will consider systems pertaining to the class (1.4) and modeling the rate-independent evolution of an internal variable z, describing the elastic behavior of a body also subject to viscosity and inertia, at small strains. Thus, the displacement $\mathbf{u}$ of the body will play the role of the rate-dependent variable u. Following [Rou09], we will confine the discussion to a *quadratic* dissipation potential on the strain rate $\varepsilon(\mathbf{u}')$ and instead address quite a broad family of energy functionals of the form $\mathcal{E}(t,\mathbf{u},z) := \int_\Omega \varphi(\varepsilon(\mathbf{u}), z, \nabla z) - \mathbf{f}(t)\cdot\mathbf{u}\,\mathrm{d}x$, with $\mathbf{f}$ a given external force. In [Rou09], the author proposed several classes of conditions on the energy density φ ensuring the existence of an energetic solution to system (1.4). In Sec. 6.1 we will show that all the energies considered in [Rou09] fulfill the conditions of our abstract Thm. 1 (in the case where inertial effects are neglected), and Thm. 4, and thus we will recover the existence results from [Rou09]. This seems to indicate that our abstract approach has a *unifying* and ultimately *simplifying* character.

It also paves the way to extensions of the analysis developed in [Rou09], tackled in Sec. 6.2. In particular, therein we will deduce the existence of *weak energetic solutions* for rate-independent damage processes in viscous solids driven by *non-quadratic* dissipation potential acting on $\varepsilon(\mathbf{u}')$, and by energies encompassing a BV, instead of a Sobolev, regularizing gradient term in the damage variable.

Finally, in Sec. 6.3 we will show the *flexibility* of our approach by tackling a different type of mixed system, where both the rate-independent and the rate-dependent variables are internal variables. More specifically, we will consider a (purely) elastic body subject to damage, encompassing in the model the development of plasticity. While for the plastic tensor we will standardly assume a rate-independent evolution, we will take the dissipation potential acting on the damage rate to be quadratic. The displacement will be governed by the static balance of elastic energy. The resulting PDE system will be cast in the form (1.4) by introducing the *reduced energy* obtained by minimizing out the displacements. Then, with careful calculations we will check that Theorem 1 applies, yielding the existence of *energetic solutions*.

Plan of the paper. In Section 2 we set up the abstract functional analytic setting for (1.4) and fix the *basic conditions* on the energy $\mathcal{E}$ and the dissipation potentials $\mathcal{V}$ and $\mathcal{R}$ that will be adopted throughout the paper. Section 3 focuses on the notions of *energetic* and *weak energetic* solutions to (1.4), on their respective properties and mutual relations; finally, in Sec. 3.3 we show that, under uniform convexity of the energy the temporal regularity of the rate-independent variable z improves, thus extending to this mixed framework some of the results well known in the rate-independent context.

In Section 4 we present our existence results, Theorems 1 & 2, for the gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, in the case $\mathcal{V}$ quadratic and general $\mathcal{V}$ with superlinear growth, respectively. We thoroughly discuss all of our abstract conditions on $\mathcal{E}$, $\mathcal{V}$, $\mathcal{R}$, and finally illustrate them on some examples in Sec. 4.3. The existence Thms. 3 and 4 for the evolutionary system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, for $\mathcal{V}$ general and $\mathcal{V}$ quadratic, resp., are stated in Section 5, where we also develop some comparison with the existence and approximation results by E. Emmrich for abstract second order nonlinear evolution equations. Section 6 is devoted to the applications.

The proofs of Thms. 1–4 are given throughout Section 7, also relying on some tools from the theory of Young measures with values in Banach spaces, shortly recapped in Appendix A.

2. Setup

In the following, we will denote by $\|\cdot\|_X$ the norm of a Banach space X, and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. If X is a Hilbert space, its inner product shall be denoted by $(\cdot,\cdot)_X$. With the symbol $\mathbf{B}([0,T];X)$ we will denote the space of bounded, *everywhere defined* functions on $[0,T]$ with values in X.

Topological setup. The conditions on the map $u \mapsto \mathcal{E}(t,u,z)$ will involve two

$$\text{(separable) reflexive Banach spaces } \mathbf{U} \subset \mathbf{V} \text{ with a continuous and dense embedding.} \tag{2.1a}$$

Furthermore, in the case inertial effects are included in system (1.4), there will come into play

$$\text{a Hilbert space } \mathbf{W}, \text{ identified with its dual } \mathbf{W}^*, \text{ such that } \begin{cases} \mathbf{U} \Subset \mathbf{W} & \text{compactly,} \\ \mathbf{V} \subset \mathbf{W} & \text{continuously and densely.} \end{cases} \tag{2.1b}$$

Hence,

$$\mathbf{U} \subset \mathbf{V} \subset \mathbf{W} = \mathbf{W}^* \subset \mathbf{V}^* \subset \mathbf{U}^* \quad \text{with continuous and dense embeddings} \tag{2.2}$$

and

$$\langle w, u\rangle_{\mathbf{V}} = (w,u)_{\mathbf{W}} \qquad \text{for all } u \in \mathbf{V} \text{ and } w \in \mathbf{W}. \tag{2.3}$$

The assumptions on $z \mapsto \mathcal{E}(t,u,z)$ will feature two Banach spaces

$$\mathbf{X} \subset \mathbf{Z} \text{ continuously,} \quad \text{with } \mathbf{X} \text{ dual of a separable Banach space,} \tag{2.4a}$$

and $\mathbf{X}$ endowed with a topology $\sigma_{\mathbf{X}}$ such that

$$\begin{aligned} &\text{bounded sets in } \mathbf{X} \text{ are } \sigma_{\mathbf{X}}\text{-sequentially compact, and} \\ &z_n \overset{\sigma_{\mathbf{X}}}{\to} z \text{ in } \mathbf{X} \quad \Rightarrow \quad z_n \rightharpoonup z \text{ in } \mathbf{Z}. \end{aligned} \tag{2.4b}$$

Remark 2.1. Let us highlight that (2.4) encompasses the two following situations:

(1) the space $\mathbf{X}$ is reflexive: in this case, we take $\sigma_{\mathbf{X}}$ to be the weak topology on $\mathbf{X}$;
(2) the space $\mathbf{X}$ embeds *compactly* in $\mathbf{Z}$: in this case, $\sigma_{\mathbf{X}}$ is the strong topology on $\mathbf{Z}$.

Our basic conditions on the functional $\mathcal{E}$ mimic the conditions on energy $\mathcal{E} = \mathcal{E}_t(u)$ -independent of z-, proposed in [MRS13b].

Hypothesis 2.2 (Basic conditions on $\mathcal{E}$)**.** *The functional* $\mathcal{E} : [0,T] \times \mathbf{V} \times \mathbf{Z} \to (-\infty, +\infty]$ *has proper domain* $\mathrm{dom}(\mathcal{E}) = [0,T] \times \mathrm{D}_{\mathsf{u}} \times \mathrm{D}_{\mathsf{z}}$, *with* $\mathrm{D}_{\mathsf{u}} \subset \mathbf{U}$ *and* $\mathrm{D}_{\mathsf{z}} \subset \mathbf{X}$. *Moreover,*

$$\forall\, t \in [0,T] \quad \textit{the map } (u,z) \mapsto \mathcal{E}(t,u,z) \quad \textit{is weakly lower semicontinuous on } \mathbf{V} \times \mathbf{Z}, \tag{2.5a}$$

and $\mathcal{E}$ *is bounded from below, i.e.*

$$\exists\, C_0 > 0 \ \ \forall\, (t,u,z) \in [0,T] \times \mathrm{D}_{\mathsf{u}} \times \mathrm{D}_{\mathsf{z}} : \qquad \mathcal{E}(t,u,z) \geq C_0\,; \tag{2.5b}$$

in fact, since $\mathcal{E}$ is bounded from below, up to a shift we may always assume that it is bounded by a strictly positive constant. Furthermore, we require that

$$\begin{aligned} &\forall\,(u,z)\in \mathrm{D_u}\times\mathrm{D_z} \quad \text{the map } t\mapsto \mathcal{E}(t,u,z) \text{ is differentiable, with derivative } \partial_t\mathcal{E}(t,u,z) \text{ s.t.}\\ &\exists\, C_1,\, C_2>0 \ \ \forall\,(t,u,z)\in[0,T]\times \mathrm{D_u}\times\mathrm{D_z}: \qquad |\partial_t\mathcal{E}(t,u,z)|\le C_1\left(\mathcal{E}(t,u,z)+C_2\right) \text{ and fulfilling}\\ &\text{for all sequences } t_n\to t,\ u_n\rightharpoonup u \text{ in } \mathbf{V},\ z_n\rightharpoonup z \text{ in } \mathbf{Z} \ \text{ with } \sup_n \mathcal{E}(t_n,u_n,z_n)\le C\\ &\text{there holds } \limsup_{n\to\infty}\partial_t\mathcal{E}(t_n,u_n,z_n)\le \partial_t\mathcal{E}(t,u,z). \end{aligned} \tag{2.5c}$$

Let us set

$$\mathcal{G}(u,z) := \sup_{t\in[0,T]}\mathcal{E}(t,u,z) \qquad \text{for all } (u,z)\in \mathrm{D_u}\times\mathrm{D_z} \text{ and} \tag{2.6}$$

$$\mathcal{S}_E := \{(u,z)\in \mathrm{D_u}\times\mathrm{D_z}\,:\, \mathcal{G}(u,z)\le E\} \qquad \text{for } E>0. \tag{2.7}$$

As a consequence of the power estimate in (2.5c) and of Gronwall's inequality there holds

$$\forall\, s,t\in[0,T],\ \forall\,(u,z)\in\mathcal{S}_E:\ \mathcal{E}(t,u,z)+C_2\le \exp(C_1|t-s|)\left(\mathcal{E}(s,u,z)+C_2\right), \tag{2.8}$$

so that we in particular have the following estimate, which will play a crucial role in our analysis

$$\exists\, K_1,\, K_2>0 \ \ \forall\,(t,u,z)\in[0,T]\times \mathrm{D_u}\times\mathrm{D_z}: \qquad \mathcal{G}(u,z)\le K_1\mathcal{E}(t,u,z)+K_2\,. \tag{2.9}$$

Let us also recall that the Fréchet subdifferential $\partial_u^-\mathcal{E}:\mathbf{V}\rightrightarrows\mathbf{V}^*$ of $\mathcal{E}$ w.r.t. u is defined at $(t,u,z)\in[0,T]\times \mathrm{D_u}\times\mathrm{D_z}$ by

$$\xi\in\partial_u^-\mathcal{E}(t,u,z) \ \Leftrightarrow\ \mathcal{E}(t,v,z)-\mathcal{E}(t,u,z)\ge \langle\xi,v-u\rangle_{\mathbf{V}}+o(\|v-u\|_{\mathbf{V}}) \text{ as } v\to u \text{ in } \mathbf{V}. \tag{2.10}$$

It is not difficult to check that for every $(t,u,z)\in[0,T]\times \mathrm{D_u}\times\mathrm{D_z}$ the set $\partial_u^-\mathcal{E}(t,u,z)$ is closed and convex.

For the dissipation potentials $\mathcal{V}$ and $\mathcal{R}$ we require:

Hypothesis 2.3 (Basic conditions on $\mathcal{V}$ and $\mathcal{R}$)**.** *The functional $\mathcal{V}:\mathbf{V}\to[0,+\infty)$ is lower semicontinuous, convex, and fulfills*

$$\mathcal{V}(0)=0, \qquad \lim_{\|v\|_{\mathbf{V}}\uparrow+\infty}\frac{\mathcal{V}(v)}{\|v\|_{\mathbf{V}}}=+\infty. \tag{2.11a}$$

The functional $\mathcal{R}:\mathbf{Z}\to[0,+\infty]$, with domain $\mathrm{dom}(\mathcal{R})$, *is lower semicontinuous, convex, 1-positively homogeneous, i.e.*

$$\mathcal{R}(\lambda\zeta)=\lambda\mathcal{R}(\zeta) \qquad \text{for all } \zeta\in\mathbf{Z} \text{ and } \lambda\ge 0, \tag{2.11b}$$

and coercive

$$\exists\, C_R>0\ \ \forall\,\zeta\in\mathbf{Z} \qquad \mathcal{R}(\zeta)\ge C_R\|\zeta\|_{\mathbf{Z}}. \tag{2.11c}$$

Recall that the lower semicontinuity of a functional together with convexity results in its *weak* lower semicontinuity. Hereafter, we will denote by $\mathcal{V}^*:\mathbf{V}^*\to[0,+\infty)$ the convex conjugate of $\mathcal{V}$, defined by

$$\mathcal{V}^*(\xi):=\sup_{v\in\mathbf{V}}\left(\langle\xi,v\rangle_{\mathbf{V}}-\mathcal{V}(v)\right). \tag{2.12}$$

It follows from the fact that $\mathrm{dom}(\mathcal{V})=\mathbf{V}$ that $\mathcal{V}$ is continuous and that $\mathcal{V}^*$ as well has superlinear growth at infinity (cf. e.g. [ET74, Chap. 1, Cor. 2.5]), namely

$$\lim_{\|\xi\|_{\mathbf{V}^*}\uparrow+\infty}\frac{\mathcal{V}^*(\xi)}{\|\xi\|_{\mathbf{V}^*}}=+\infty. \tag{2.13}$$

We will denote by $\mathrm{Var}_{\mathcal{R}}$ the notion of total variation induced by $\mathcal{R}$. Given a curve $z:[0,T]\to\mathbf{Z}$ and a subinterval $[s,t]\subset[0,T]$, we set

$$\mathrm{Var}_{\mathcal{R}}(z;[s,t]):=\sup\left\{\sum_{j=1}^N\mathcal{R}(z(r_j)-z(r_{j-1}))\ :\quad s=r_0<r_1<\ldots<r_{N-1}<r_N=t\right\}. \tag{2.14}$$

Due to (2.11c), if $\mathrm{Var}_{\mathcal{R}}(z;[0,T])<+\infty$ then $z\in\mathrm{BV}([0,T];\mathbf{Z})$ and the $\mathcal{R}$-total variation estimates the total variation induced by $\|\cdot\|_{\mathbf{Z}}$, viz.

$$\mathrm{Var}_{\mathcal{R}}(z;[0,T])\ge\mathrm{Var}(z;[s,t]).$$

Remark 2.4 (State-dependent dissipation potentials)**.** Most of the forthcoming results, and in particular the existence Theorems 1–4, could be extended to the case that the dissipation potential $\mathcal{V}$ depends on the state variable $q = (u, z)$. More specifically, along the lines of [MRS13b] we could consider a family $(\mathcal{V}_q)_{q\in \mathrm{D_u}\times \mathrm{D_z}}$ of dissipation potentials such that

(1) for every $q \in \mathrm{D_u}\times\mathrm{D_z}$ the functional $\mathcal{V}_q : \mathbf{V} \to [0, +\infty)$ is convex and lower semicontinuous;
(2) the dissipation potentials $(\mathcal{V}_q)_{q\in \mathrm{D_u}\times \mathrm{D_z}}$ and $(\mathcal{V}^*_q)_{q\in \mathrm{D_u}\times \mathrm{D_z}}$ have superlinear growth at infinity, *uniformly* w.r.t. q in sublevels of the energy, i.e.

$$\forall\, S>0 : \quad \begin{cases} \lim_{\|v\|_{\mathbf{V}}\to+\infty} \inf_{\mathcal{G}(q)\leq S} \mathcal{V}_q(v) = +\infty, \\ \lim_{\|\xi\|_{\mathbf{V}^*}\to+\infty} \inf_{\mathcal{G}(q)\leq S} \mathcal{V}^*_q(\xi) = +\infty; \end{cases} \tag{2.15a}$$

(3) the dependence $q \mapsto \mathcal{V}_q$ is continuous, on sublevels of the energy, in the sense of Mosco-convergence (cf. e.g. [Att84]), i.e.

$$\forall\, S>0 : \quad q_n \rightharpoonup q \text{ in } \mathbf{V}\times\mathbf{Z}, \quad \mathcal{G}(q_n)\leq S, \quad v_n \rightharpoonup v \text{ in } \mathbf{V} \quad \Rightarrow \quad \liminf_{n\to\infty} \mathcal{V}_{q_n}(v_n) \geq \mathcal{V}_q(v),$$
$$\forall\, S>0 : \quad q_n \rightharpoonup q \text{ in } \mathbf{V}\times\mathbf{Z}, \quad \mathcal{G}(q_n)\leq S, \quad v \in \mathbf{V} \quad \Rightarrow \quad \begin{cases} \exists\, v_n \to v \text{ in } \mathbf{V}, \\ \lim_{n\to\infty} \mathcal{V}_{q_n}(v_n) = \mathcal{V}_q(v). \end{cases} \tag{2.15b}$$

The extension of our results to a family $(\mathcal{R}_q)$ of 1-positively homogeneous potentials seems to be more delicate, starting from the fact that it would not be completely clear how to define the total variation induced by $(\mathcal{R}_q)$. That is why, we choose to overlook this issue in the fully general setting, whereas in Sec. 6.3 we will briefly describe how one of our existence results, Thm. 1, could be adapted to encompass a very specific dependence of $\mathcal{R}$ on the *sole* variable u.

For the limit passage in (approximate versions) of the semistability condition, a key role will be played by the following recovery sequence condition, originally introduced in [MRS08], which is also termed *mutual* because it involves the energy $\mathcal{E}$ *and* the dissipation $\mathcal{R}$.

Hypothesis 2.5 (Mutual Recovery Sequence condition)**.** *Let $(t_n, u_n, z_n)_n \subset [0, T] \times \mathrm{D_u} \times \mathrm{D_z}$ fulfill for every $n \in \mathbb{N}$ the semistability condition* (1.8a)*, and suppose that $t_n \to t$, $(u_n, z_n) \rightharpoonup (u, z)$ in $\mathbf{V} \times \mathbf{Z}$ with $\sup_{n\in\mathbb{N}} \mathcal{G}(u_n, z_n) \leq C$.*

Then for every $\tilde z \in \mathbf{Z}$ there exists $\tilde z_n \rightharpoonup \tilde z$ in $\mathbf{Z}$ such that

$$\lim_{n\to\infty} \left(\mathcal{R}(\tilde z_n - z_n) + \mathcal{E}(t_n, u_n, \tilde z_n) - \mathcal{E}(t_n, u_n, z_n)\right) \leq \mathcal{R}(\tilde z - z) + \mathcal{E}(t, u, \tilde z) - \mathcal{E}(t, u, z). \tag{2.16}$$

Remark 2.6. In certain cases (cf. Remark 4.9), it will be sufficient to require a <u>weaker</u> variant of Hypothesis 2.5, in which we suppose that for every $\tilde z \in \mathbf{Z}$ it is possible to construct a recovery sequence $(\tilde z_n)_n$ fulfilling (2.16), whenever the sequence $(t_n, u_n, z_n)_n \subset [0, T] \times \mathrm{D_u} \times \mathrm{D_z}$ fulfills the conditions of Hypothesis 2.5 and, in addition, we have the *energy convergence*

$$\mathcal{E}(t_n, u_n, z_n) \to \mathcal{E}(t, u, z). \tag{2.17}$$

We close this section stating the main coercivity assumption on the map $\mathcal{E}(t, \cdot, \cdot)$. Observe that, in the in the spirit of the *variational approach* to gradient flows (cf. [AGS08, Chap. 2, Sec. 2.1]), we require a property on the sublevels of the functional $\mathcal{G}$ added with $\mathcal{V}$ and $\mathcal{R}$, in place of the sublevels of the sole $\mathcal{G}$. In fact, this is the minimal coercivity/compactness requirement on the energy to ensure the existence of solutions to the time-incremental minimization scheme(s) we shall use to discretize (1.4), as well as the compactness for the related family of approximate solutions to (1.4).

Hypothesis 2.7 (Coercivity)**.** *There exist $\tau_o > 0$ such that for all $u_o \in \mathbf{V}$ and $z_o \in \mathbf{Z}$*

$$\text{the map } (u, z) \mapsto \mathcal{G}(u, z) + \tau_o \mathcal{V}\left(\frac{u - u_o}{\tau_o}\right) + \mathcal{R}(z - z_o) \text{ has sublevels bounded in } \mathbf{U}\times\mathbf{X}. \tag{2.18}$$

Observe that (2.18) guarantees the separate coercivity properties

$$\forall\, \bar z \in \mathrm{D_z} \quad \text{the map } u \mapsto \mathcal{G}(u, \bar z) + \tau_o \mathcal{V}\left(\frac{u - u_o}{\tau_o}\right) \text{ has sublevels bounded in } \mathbf{U}, \tag{2.19a}$$
$$\forall\, \bar u \in \mathrm{D_u} \quad \text{the map } z \mapsto \mathcal{G}(\bar u, z) + \mathcal{R}(z - z_o) \text{ has sublevels bounded in } \mathbf{X}. \tag{2.19b}$$

As a consequence of (2.19b) and of (2.4), a sequence $(z_n)_n$ bounded in $\mathbf{Z}$ such that $\sup_{n\in\mathbb{N}} \mathcal{G}(\bar{u}, \bar{z}_n) \le C$ for some $C > 0$ and $\bar{u} \in \mathrm{D_u}$ admits a subsequence converging with respect to the $\sigma_{\mathbf{X}}$-topology. Instead, with (2.19a) we are requiring that $\mathcal{E}(t, \cdot, z)$ (added up with $\mathcal{V}$) has bounded sublevels in $\mathbf{U}$, which may or may not be compactly embedded in $\mathbf{V}$. In other words, we encompass in our analysis these two cases:

- the energy $\mathcal{E}(t, \cdot, z)$ and the dissipation potential $\mathcal{V}$ have sublevels bounded in the same space;
- the energy $\mathcal{E}(t, \cdot, z)$ has sublevels compact in the space $\mathbf{V}$ of the dissipation $\mathcal{V}$.

The following examples are prototypes of these two situations. The different choices for the space $\mathbf{V}$ reflect the different character of the equation ruling the evolution of the "viscous" variable u. To make this more transparent, we drop the dependence on the variable z.

Example 2.8 (Linear visco-elasticity)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We take $\mathbf{V} = H^1_0(\Omega;\mathbb{R}^d)$, the dissipation potential $\mathcal{V}(\mathbf{v}) := \int_\Omega \frac{1}{2}\varepsilon(\mathbf{v}) : \mathbb{D} : \varepsilon(\mathbf{v})\,\mathrm{d}x$ with $\mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}$ positive definite and symmetric ($\varepsilon(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^\top)\in\mathbb{R}^{d\times d}$ denoting the linearized strain tensor), and the energy functional $\mathcal{E} : [0,T]\times\mathbf{V} \to \mathbb{R}$ defined by*

$$\mathcal{E}(t, \mathbf{u}) := \int_\Omega \tfrac{1}{2}\varepsilon(\mathbf{u}):\mathbb{C}:\varepsilon(\mathbf{u})\,\mathrm{d}x - \int_\Omega \mathbf{f}(t)\cdot\mathbf{u}\,\mathrm{d}x,$$

with $\mathbb{C} \in \mathbb{R}^{d\times d\times d\times d}$ positive definite and symmetric, and $\mathbf{f} \in \mathrm{C}^1([0,T]; H^{-1}(\Omega;\mathbb{R}^d))$ a given external loading. In this case, $\mathcal{E}$ is coercive in the space $\mathbf{U} = \mathbf{V} = H^1_0(\Omega;\mathbb{R}^d)$. Choosing $\mathbf{W} = L^2(\Omega;\mathbb{R}^d)$, the viscous equation (1.4a) *yields classical linear visco-elasticity equation with inertia*

$$\varrho\ddot{\mathbf{u}} - \mathrm{div}(\mathbb{D}:\varepsilon(\dot{\mathbf{u}}) + \mathbb{C}:\varepsilon(\mathbf{u})) = \mathbf{f} \qquad \text{in } \Omega \times (0,T),$$

with homogeneous Dirichlet boundary conditions on $\mathbf{u}$.

Example 2.9 (L^2-gradient flow)**.** *We take $\mathbf{V} = L^2(\Omega)$, the dissipation potential*

$$\mathcal{V}(v) := \int_\Omega \tfrac{1}{2}|v|^2\,\mathrm{d}x \qquad \text{for every } v \in L^2(\Omega),$$

and the energy functional $\mathcal{E} : [0,T] \times \mathbf{V} \to (-\infty, +\infty]$ defined by

$$\mathcal{E}(t,u) := \begin{cases} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u)\right)\mathrm{d}x - \int_\Omega f(t)u\,\mathrm{d}x & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.20}$$

with $W \in \mathrm{C}^1(\mathbb{R})$ possibly nonconvex, and fulfilling the growth condition (cf. also Example 4.15) $c_1|u|^q - c_2 \le W(u) \le c_3|u|^q + c_4$ for positive constants $c_1, \ldots, c_4$, with the exponent q such that $2(q-1) \le 2^\star = \frac{2d}{d-2}$ (to fix ideas, if $d = 3$ we find the condition $q \le 4$, e.g. fulfilled by the double-well potential $W(u) = \frac{1}{4}(u^2-1)^2$). In this case, $\mathcal{E}$ is coercive in the space

$$\mathbf{U} = H^1(\Omega) \Subset \mathbf{V} = L^2(\Omega),$$

and, in the case $\varrho = 0$, (1.4a) *reduces to the* gradient flow

$$\dot{u} - \Delta u + W'(u) = f \qquad \text{in } \Omega \times (0,T).$$

3. Solution concepts and their properties

In Secs. 3.1 and 3.2 we give the two solvability notions for system (1.4) that will be studied throughout the paper and thoroughly examine their properties, as well as their relation. Finally, in Sec. 3.3 we show that, under suitable uniform convexity properties of the map $z \mapsto \mathcal{E}(t,u,z)$, for any (enhanced) *weak* energetic solution (u,z) the map $t \mapsto z(t)$ enjoys additional regularity properties.

3.1. Energetic solutions.

The following notion extends the one proposed in [Rou09, Def. 5.1] for the case of rate-independent processes in viscous solids, to the case of the abstract system (1.4), and to a *general* dissipation potential $\mathcal{V}$.

Definition 3.1 (Energetic solution)**.** *Let $\varrho \ge 0$. In the setting of Hypotheses 2.2 and 2.3, We call a pair $(u,z) : [0,T] \to \mathbf{V} \times \mathbf{Z}$ an* energetic solution *to the gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ (to the evolutionary system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ in the case $\varrho > 0$) if*

$$u \in L^\infty(0,T;\mathbf{U}) \cap W^{1,1}(0,T;\mathbf{V}), \qquad \varrho u' \in L^\infty(0,T;\mathbf{W}), \tag{3.1a}$$

$$z \in L^\infty(0,T;\mathbf{X}) \cap \mathrm{BV}([0,T];\mathbf{Z}) \tag{3.1b}$$

fulfill the

- *subdifferential inclusion for* u

$$\varrho u''(t)+\partial\mathcal{V}(u'(t))+\partial_u^-\mathcal{E}(t,u(t),z(t))\ni 0 \qquad \text{in } \mathbf{V}^* \quad \text{for a.a. } t\in(0,T), \tag{3.2}$$

viz. $\varrho u''(t)+\omega(t)+\xi(t)=0$, *with* $\xi(t)\in\partial_u^-\mathcal{E}(t,u(t),z(t))$ *and* $\omega(t)\in\partial\mathcal{V}(u'(t))$ *for almost all* $t\in(0,T)$*;*

- *semistability condition*

$$\mathcal{E}(t,u(t),z(t))\leq\mathcal{E}(t,u(t),\tilde{z})+\mathcal{R}(\tilde{z}-z(t)) \qquad \text{for all } \tilde{z}\in\mathbf{Z} \text{ for all } t\in[0,T]; \tag{3.3}$$

- *energy-dissipation inequality*

$$\begin{aligned}\frac{\varrho}{2}\|u'(t)\|_{\mathbf{W}}^2+\int_0^t\mathcal{V}(u'(s))+\mathcal{V}^*(-\xi(s)-\varrho u''(s))\,\mathrm{d}s+\mathrm{Var}_{\mathcal{R}}(z,[0,t])+\mathcal{E}(t,u(t),z(t))\\ \leq\frac{\varrho}{2}\|u'(0)\|_{\mathbf{W}}^2+\mathcal{E}(0,u(0),z(0))+\int_0^t\partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s \qquad \text{for all } t\in[0,T],\end{aligned} \tag{3.4}$$

with $\xi(s)$ *a selection in* $\partial_u^-\mathcal{E}(s,u(s),z(s))$ *fulfilling* (3.2) *for almost all* $s\in(0,T)$.

Finally, we call an energetic solution (u,z) enhanced *if it fulfills the energy-dissipation inequality* (3.4) *on any interval* (s,t)*, for all* $t\in(0,T]$ *and for almost all* $s\in(0,t)$.

Observe that, in the case $\mathcal{V}$ is a quadratic functional and independent of the state variable (cf. (4.1) later on), the energy-dissipation inequality (3.4) becomes

$$\begin{aligned}\frac{\varrho}{2}\|u'(t)\|_{\mathbf{W}}^2+\int_0^t 2\mathcal{V}(u'(s))\,\mathrm{d}s+\mathrm{Var}_{\mathcal{R}}(z,[0,t])+\mathcal{E}(t,u(t),z(t))\\ \leq\frac{\varrho}{2}\|u'(0)\|_{\mathbf{W}}^2+\mathcal{E}(0,u(0),z(0))+\int_0^t\partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s \qquad \text{for all } t\in[0,T].\end{aligned} \tag{3.5}$$

Our next result generalizes [Rou09, Prop. 5.2] (cf. also [Rou10, Prop. 3.2]) to the case of $\mathcal{E}(t,\cdot,z)$ possibly nonsmooth. It addresses the passage from energetic to pointwise solutions of system (1.4). To avoid overburdening the paper, we will state it in the case of gradient systems (i.e. for $\varrho=0$), and leave to the reader the straightforward generalization to the case with inertia. In Proposition 3.2, we have to assume that $\mathcal{E}(t,u,\cdot)$ is of class C^1, and that $\mathcal{E}$ fulfills a suitable *chain-rule inequality* featuring the Fréchet differential of $\mathcal{E}(t,u,\cdot)$ and the Fréchet subdifferential of $\mathcal{E}(t,\cdot,z)$. Hence, we show that, under the additional information that $z\in\mathrm{AC}([0,T];\mathbf{Z})$, then any energetic solution solves system (1.4) pointwise a.e. in $(0,T)$. Observe that the additional requirement that $\mathbf{Z}$ has the Radon-Nikodým property ensures that absolutely continuous curves with values in $\mathbf{Z}$ are differentiable in time almost everywhere on $(0,T)$. We also have to exclude the case that $\mathcal{R}$ takes the value $+\infty$, because the resulting constraint for the test functions in the semistability condition would not allow us to fully exploit it.

Proposition 3.2. *Let* $\mathbf{Z}$ *comply with the Radon-Nikodým property and* $\mathcal{R}$ *take values in* $[0,+\infty)$. *We suppose that* $\mathcal{E}(t,u,\cdot)\in\mathrm{C}^1(\mathbf{Z})$ *for every* $(t,u)\in[0,T]\times\mathrm{D_u}$, *and that the following chain-rule* inequality *holds:*

$$\begin{aligned}&\text{for every curve } (u,z)\in\mathrm{AC}([0,T];\mathbf{V}\times\mathbf{Z}) \text{ and every } \xi\in L^1(0,T;\mathbf{V}^*) \text{ with}\\ &\xi(t)\in\partial_u^-\mathcal{E}(t,u(t),z(t)) \text{ for a.a. } t\in(0,T) \text{ and such that}\\ &\sup_{t\in[0,T]}\mathcal{G}(u(t),z(t))<+\infty, \quad \int_0^T\mathcal{V}(u'(t))+\mathcal{V}^*(-\xi(t))\,\mathrm{d}t<+\infty,\\ &\text{then } t\mapsto\mathcal{E}(t,u(t),z(t)) \text{ is absolutely continuous on } [0,T] \text{ and}\\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t),z(t))\geq\partial_t\mathcal{E}(t,u(t),z(t))+\langle\xi(t),u'(t)\rangle_{\mathbf{V}}+\langle\mathrm{D}_z\mathcal{E}(t,u(t),z(t)),z'(t)\rangle_{\mathbf{Z}} \quad \text{for a.a. } t\in(0,T).\end{aligned} \tag{3.6}$$

Let (u,z) *be an energetic solution to the gradient system* $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *such that, in addition,* $z\in\mathrm{AC}([0,T];\mathbf{Z})$. *Then,* (u,z) *complies with the flow rule* (1.4b) *pointwise a.e. in* $(0,T)$.

Proof. Testing the semistability (3.3) with $\tilde{z}=z(t)+\epsilon\zeta$, where ζ is an arbitrary element in $\mathbf{Z}$ and $\epsilon>0$ is likewise arbitrary, and arguing in the very same way as in the proof of [Rou09, Prop. 5.2] leads to the inequality $\langle-\mathrm{D}_z\mathcal{E}(t,u(t),z(t)),\zeta\rangle_{\mathbf{V}}\leq\mathcal{R}(\zeta)$ for every $\zeta\in\mathbf{Z}$. In particular, we deduce

$$\langle-\mathrm{D}_z\mathcal{E}(t,u(t),z(t)),z'(t)\rangle_{\mathbf{Z}}\leq\mathcal{R}(z'(t)) \quad \text{for a.a. } t\in(0,T). \tag{3.7}$$

Now, we consider the energy-dissipation inequality (3.4), written for $\varrho = 0$, observing that, since $z \in \mathrm{AC}([0,T];\mathbf{Z})$, we have $\mathrm{Var}_{\mathcal{R}}(z,[0,t]) = \int_0^t \mathcal{R}(z'(s))\,\mathrm{d}s$. We continue (3.4) by applying the chain-rule inequality from (3.6). Hence, we obtain the following chain of inequalities for every $t \in [0,T]$

$$
\begin{aligned}
\int_0^t \mathcal{V}(u'(s)) + \mathcal{V}^*(-\xi(s))\,\mathrm{d}s + \int_0^t \mathcal{R}(z'(s))\,\mathrm{d}s &\le \mathcal{E}(0,u(0),z(0)) - \mathcal{E}(t,u(t),z(t)) + \int_0^t \partial_t \mathcal{E}(s,u(s),z(s))\,\mathrm{d}s \\
&\le \int_0^t \left(\langle -\xi(s), u'(s)\rangle_{\mathbf{V}} + \langle -\mathrm{D}_z\mathcal{E}(s,u(s),z(s)), z'(s)\rangle_{\mathbf{Z}} \right) \mathrm{d}s \\
&\le \int_0^t \mathcal{V}(u'(s)) + \mathcal{V}^*(-\xi(s))\,\mathrm{d}s + \int_0^t \mathcal{R}(z'(s))\,\mathrm{d}s,
\end{aligned}
\tag{3.8}
$$

where the last inequality follows from the definition (2.12) of $\mathcal{V}^*$, and from (3.7). Hence all inequalities hold as equalities and moreover, rearranging terms we conclude that $\int_0^t \left(\mathcal{V}(u'(s)) + \mathcal{V}^*(-\xi(s)) - \langle -\xi(s), u'(s)\rangle_{\mathbf{V}}\right) \mathrm{d}s = \int_0^t \left(\langle -\mathrm{D}_z\mathcal{E}(s,u(s),z(s)), z'(s)\rangle_{\mathbf{Z}} - \mathcal{R}(z'(s))\right) \mathrm{d}s$. Since the integrand in the l.h.s.-integral is positive by the definition (2.12) of $\mathcal{V}^*$ and the integrand on the r.h.s. is negative by (3.7), we necessarily conclude that both integrals (and in fact both integrands, pointwise) equal zero. Hence

$$
\langle -\mathrm{D}_z\mathcal{E}(t,u(t),z(t)), z'(t)\rangle_{\mathbf{Z}} = \mathcal{R}(z'(t)) \qquad \text{for a.a. } t \in (0,T). \tag{3.9}
$$

Combining this with (3.7) we get that $-\mathrm{D}_z\mathcal{E}(t,u(t),z(t)) \in \partial\mathcal{R}(z'(t))$ for almost all $t \in (0,T)$. This concludes the proof. □

Remark 3.3. A close perusal of the proof of Prop. 3.2 reveals that the validity of subdifferential inclusion (3.2) has never been used. In fact, the very same argument as in the above lines allows us to prove that, if a pair (u,z) fulfilling (3.1a)–(3.1b) with $z \in \mathrm{AC}([0,T];\mathbf{Z})$ complies with the semistability (3.3) and with the energy-dissipation inequality (3.4), and if the energy complies with the conditions from Prop. 3.2, then (u,z) also fulfills the flow rule (1.4b) and the subdifferential inclusion (3.2) pointwise a.e. in $(0,T)$. This is in accordance with the results from [MRS13b] for rate-dependent gradient systems.

3.2. **Weak energetic solutions.** Let us now introduce in the following definition a much weaker solvability notion, where the pointwise subdifferential inclusion for u is dropped. In fact, this may be hardly considered a solution concept for the gradient and evolutionary systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$. Still, it will be useful to bring it into play, especially in connection with the evolutionary Γ-convergence analysis of the latter systems which will be investigated in the forthcoming [RT15].

Definition 3.4 (Weak energetic solution)**.** *Let $\varrho \ge 0$. In the setting of Hypotheses 2.2 and 2.3, we say that a pair $(u,z) : [0,T] \to \mathbf{V}\times\mathbf{Z}$ as in* (3.1) *is a* weak energetic solution *to the gradient system* $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *(to the evolutionary system* $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *if $\varrho > 0$) if for all $t \in [0,T]$ it complies with the semistability condition* (3.3) *and with the energy-dissipation inequality* (3.4)*, with $\xi(s)$ a selection in $\partial_u^-\mathcal{E}(s,u(s),z(s))$ for a.a. $s \in (0,T)$.*

We call a weak energetic solution (u,z) enhanced *if it fulfills the energy-dissipation inequality* (3.4) *on any interval (s,t), for all $t \in (0,T]$ and for almost all $s \in (0,t)$.*

Although the above concept is extremely weak, slightly adapting the proof of Proposition 3.2 (cf. Remark 3.3) it can be shown that, if the energy complies with the conditions from the latter result, then any weak energetic solution such that in addition $z \in \mathrm{AC}([0,T];\mathbf{Z})$ is also an energetic solution, as well as a pointwise solution of system (3.2, 1.4b). In Proposition 3.5 and Theorem 3.6 ahead, we will provide a different set of conditions on the energy, still allowing us to pass from *weak* energetic to *energetic* solutions, but *without* the condition that z is absolutely continuous.

First of all, Prop. 3.5 guarantees that a weak energetic solution (u,z) is in fact an energetic solution, provided that (u,z) complies with a suitable "chain-rule inequality" in integral form, cf. (3.12) below. Then, in Thm. 3.6 we provide sufficient conditions on the energy $\mathcal{E}$, guaranteeing that any weak energetic solution to $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ (to $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ if $\varrho > 0$) complies with (3.12). Let us point out that, although the derivation of (3.12) is at the core of the proofs of [Rou09, Prop. 5.4] as well as [Rou10, Prop. 4.3], its role in deriving the subdifferential inclusion (3.2) from the energy-dissipation inequality (3.4) has never been highlighted in such a generality before.

Proposition 3.5. *Assume Hypotheses 2.2 and 2.3 and, in addition, that for every* $(t,z) \in [0,T] \times \mathrm{D}_{\mathbf{z}}$ *the map* $u \mapsto \mathcal{E}(t,u,z)$ *is Gâteaux-differentiable. In the case* $\varrho > 0$*, also suppose that*

$$\exists\, C_V, C_V' > 0 \ \ \forall\,(v,\xi) \in \mathbf{V} \times \mathbf{V}^* \qquad \mathcal{V}(v) + \mathcal{V}^*(\xi) \geq C_V |\langle \xi, v \rangle_{\mathbf{V}}| - C_V'. \tag{3.10}$$

Let (u,z) *be a weak energetic solution to the gradient system* $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ *(to the evolutionary system* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ *if* $\varrho > 0$*) such that, for* $\varrho > 0$*,* u *has the further regularity*

$$u \in W^{2,1}(0,T;\mathbf{V}^*). \tag{3.11}$$

If the map $t \mapsto \mathrm{D}_u \mathcal{E}(t,u(t),z(t))$ *is in* $L^\infty(0,T;\mathbf{V}^*)$ *and if* (u,z) *comply with the (integral) chain rule*

$$\begin{aligned} &\int_0^t \langle \mathrm{D}_u \mathcal{E}(s,u(s),z(s)), u'(s) \rangle_{\mathbf{V}} \,\mathrm{d}s \\ &\leq \mathcal{E}(t,u(t),z(t)) - \mathcal{E}(0,u(0),z(0)) - \int_0^t \partial_t \mathcal{E}(s,u(s),z(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(z,[0,t]) \text{ for all } t \in [0,T], \end{aligned} \tag{3.12}$$

then

(1) the energy-dissipation inequality (3.4) *turns into an identity, holding along any sub-interval* $[s,t]$*;*
(2) (u,z) *comply with the subdifferential inclusion* (3.2)*.*

Hence, (u,z) *is an energetic solution.*

Proof. Since $\int_0^t \mathcal{V}(u'(s)) + \mathcal{V}^*(-\mathrm{D}_u\mathcal{E}(s,u(s),z(s)) - \varrho u''(s))\,\mathrm{d}s \leq C$, from (3.10) we gather that the map $t \mapsto \langle -\mathrm{D}_u\mathcal{E}(t,u(t),z(t)) - \varrho u''(t)), u'(t)\rangle_{\mathbf{V}}$ is in $L^1(0,T)$. Since $t \mapsto \mathrm{D}_u\mathcal{E}(t,u(t),z(t))$ is in $L^\infty(0,T;\mathbf{V}^*)$ we have that $t \mapsto \langle -\mathrm{D}_u\mathcal{E}(t,u(t),z(t)), u'(t)\rangle_{\mathbf{V}}$ is in $L^1(0,T)$, hence we conclude in the case $\varrho > 0$ that

$$t \mapsto \langle u''(t)), u'(t)\rangle_{\mathbf{V}} \text{ is in } L^1(0,T). \tag{3.13}$$

Combining (3.12) with the energy-dissipation inequality (3.4) we conclude the chain of inequalities

$$\begin{aligned} &\int_0^t \mathcal{V}(u'(s)) + \mathcal{V}^*(-\mathrm{D}_u\mathcal{E}(s,u(s),z(s)) - \varrho u''(s))\,\mathrm{d}s \\ &\leq \frac{\varrho}{2}\|u'(0)\|_{\mathbf{W}}^2 - \frac{\varrho}{2}\|u'(t)\|_{\mathbf{W}}^2 - \int_0^t \langle \mathrm{D}_u\mathcal{E}(s,u(s),z(s)), u'(s)\rangle_{\mathbf{V}}\,\mathrm{d}s = \int_0^t \langle -\mathrm{D}_u\mathcal{E}(s,u(s),z(s)) - \varrho u''(s), u'(s)\rangle_{\mathbf{V}}\,\mathrm{d}s \end{aligned}$$

for all $t \in [0,T]$, where we have used that

$$\varrho\,\langle u''(t), u'(t)\rangle_{\mathbf{V}} = \frac{\varrho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u'\|_{\mathbf{W}}^2(t) \qquad \text{for a.a. } t \in (0,T)$$

since $(\mathbf{V}, \mathbf{W}, \mathbf{V}^*)$ is a Hilbert (or Gelfand) triple, cf. (2.3), as well as (3.13). Therefore, we find

$$\int_0^t (\mathcal{V}(u'(s)) + \mathcal{V}^*(-\mathrm{D}_u\mathcal{E}(s,u(s),z(s)) - \varrho u''(s)) - \langle -\mathrm{D}_u\mathcal{E}(s,u(s),z(s)) - \varrho u''(s), u'(s)\rangle_{\mathbf{V}})\,\mathrm{d}s \leq 0\,. \tag{3.14}$$

Now, it follows from the definition of the conjugate $\mathcal{V}^*$ that also the opposite of inequality (3.14) holds. Thus, we conclude that (3.14) holds as an equality a.e. in $(0,T)$. Hence,

$$-\varrho u''(s) - \mathrm{D}_u\mathcal{E}(s,u(s),z(s)) \in \partial\mathcal{V}(u'(s)) \qquad \text{for a.a. } s \in (0,T),$$

whence (3.2). Furthermore, from the above arguments it follows that (3.4) holds as an equality on every interval $[0,t] \subset [0,T]$. This concludes the proof. □

In the proof of Thm. 3.6 below, we derive the chain-rule inequality (3.12) from the semistability condition (3.3), mimicking the Riemann-sum procedure from the proof of [Rou09, Prop. 5.4], see also [Rou10, Prop. 4.3], which in turn is based on the argument first developed in [DMFT05].

Theorem 3.6. *Assume Hypotheses 2.2 and 2.3, that for every* $(t,z) \in [0,T] \times \mathrm{D}_{\mathbf{z}}$ *the map* $u \mapsto \mathcal{E}(t,u,z)$ *is Gâteaux-differentiable, and that*

$$\forall\, M > 0\ \exists\, S > 0\ \ \forall\, t \in [0,T],\ \forall\, u,\, u_1,\, u_2 \in \mathrm{D}_{\mathbf{u}},\ \bar{z} \in \mathrm{D}_{\mathbf{z}}\ :$$

$$\begin{cases} (u,\bar{z}) \in \mathcal{S}_M \ \Rightarrow\ \|\mathrm{D}_u\mathcal{E}(t,u,\bar{z})\|_{\mathbf{V}^*} \leq S, \\ (u_1,\bar{z}),\ (u_2,\bar{z}) \in \mathcal{S}_M \ \Rightarrow\ \|\mathrm{D}_u\mathcal{E}(t,u_1,\bar{z}) - \mathrm{D}_u\mathcal{E}(t,u_2,\bar{z})\|_{\mathbf{V}^*} \leq S\|u_1 - u_2\|_{\mathbf{V}}, \end{cases} \tag{3.15a}$$

and that $\partial_t\mathcal{E}$ satisfies analogous Lipschitz estimates, i.e.

$$\forall\, \widetilde{M}>0\ \exists\, \widetilde{S}>0\quad \forall\, t_1,t_2\in[0,T],\ \forall\, u_1,\, u_2\in \mathrm{D_u},\ \ \bar z\in \mathrm{D_z}\ :$$
$$(u_1,\bar z),\,(u_2,\bar z)\in \mathcal{S}_{\widetilde{M}}\ \Rightarrow\ \begin{cases}|\partial_t\mathcal{E}(t_1,u_1,\bar z)-\partial_t\mathcal{E}(t_2,u_1,\bar z)|\leq \widetilde{S}|t_1-t_2|,\\ |\partial_t\mathcal{E}(t_1,u_1,\bar z)-\partial_t\mathcal{E}(t_1,u_2,\bar z)|\leq \widetilde{S}\|u_1-u_2\|_{\mathbf{V}}.\end{cases} \tag{3.15b}$$

Let (u,z) be a weak energetic solution to the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ (to the evolutionary system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ if $\varrho>0$). Then, (u,z) complies with (3.12).

Proof. Preliminarily, we observe that along a weak energetic solution (u,z) there holds

$$\sup_{t\in[0,T]}\mathcal{G}(u(t),z(t))\leq C. \tag{3.16}$$

This can be checked by arguing in the very same way as in the proof of the forthcoming Proposition 7.2, namely starting from the energy-dissipation inequality (3.4) and exploiting estimate (2.9). Therefore, thanks to the first of (3.15a), we have that the map $t\mapsto \mathrm{D}_u\mathcal{E}(t,u(t),z(t))$ is in $L^\infty(0,T;\mathbf{V}^*)$.

Now, let $\eta>0$ be fixed, and let $(\mathsf{t}_i)_{i=1}^{N_\eta}$ be a partition of the interval $[0,t]$, with $0<\mathsf{t}_i-\mathsf{t}_{i-1}\leq\eta$ for all $i=1,\dots,N_\eta$. We test the semistability (3.3) at time $t=\mathsf{t}_{i-1}$ with $\tilde z=z(\mathsf{t}_i)$, obtaining

$$\mathcal{E}(\mathsf{t}_{i-1},u(\mathsf{t}_{i-1}),z(\mathsf{t}_{i-1}))\leq \mathcal{E}(\mathsf{t}_{i-1},u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))+\mathcal{R}(z(\mathsf{t}_i)-z(\mathsf{t}_{i-1})).$$

Using that

$$\begin{aligned}\mathcal{E}(\mathsf{t}_{i-1},u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))&=\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i))+\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))-\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i))\\&\quad+\mathcal{E}(\mathsf{t}_{i-1},u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))-\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))\end{aligned}$$

we deduce by the chain rule

$$\begin{aligned}\mathcal{E}(\mathsf{t}_{i-1},u(\mathsf{t}_{i-1}),z(\mathsf{t}_{i-1}))&\leq \mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i))+\mathcal{R}(z(\mathsf{t}_i)-z(\mathsf{t}_{i-1}))\\&\quad-\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\langle \mathrm{D}_u\mathcal{E}(\mathsf{t}_i,u(s),z(\mathsf{t}_i)),u'(s)\rangle_{\mathbf{V}}\,\mathrm{d}s-\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\partial_t\mathcal{E}(s,u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))\,\mathrm{d}s,\end{aligned}$$

which we add up over the index $i=1,\dots,N_\eta$. Taking into account that $\sum_{i=1}^{N_\eta}\mathcal{R}(z(\mathsf{t}_i)-z(\mathsf{t}_{i-1}))\leq \mathrm{Var}_{\mathcal{R}}(z;[0,t])$, we arrive at

$$\begin{aligned}\mathcal{E}(0,u(0),z(0))\leq \mathcal{E}(t,u(t),z(t))+\mathrm{Var}_{\mathcal{R}}(z;[0,t])&-\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\langle \mathrm{D}_u\mathcal{E}(\mathsf{t}_i,u(s),z(\mathsf{t}_i)),u'(s)\rangle_{\mathbf{V}}\,\mathrm{d}s\\&-\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\partial_t\mathcal{E}(s,u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))\,\mathrm{d}s.\end{aligned}$$

We now deal with the two latter terms. Observe that

$$\begin{aligned}&\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\langle \mathrm{D}_u\mathcal{E}(\mathsf{t}_i,u(s),z(\mathsf{t}_i)),u'(s)\rangle_{\mathbf{V}}\,\mathrm{d}s\\&=\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\langle \mathrm{D}_u\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i)),u'(\mathsf{t}_i)\rangle_{\mathbf{V}}\,\mathrm{d}s+\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\langle \mathrm{D}_u\mathcal{E}(\mathsf{t}_i,u(s),z(\mathsf{t}_i))-\mathrm{D}_u\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i)),u'(s)\rangle_{\mathbf{V}}\,\mathrm{d}s\\&+\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\langle \mathrm{D}_u\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i)),u'(s)-u'(\mathsf{t}_i)\rangle_{\mathbf{V}}\,\mathrm{d}s\doteq S_1^\eta+S_2^\eta+S_3^\eta.\end{aligned} \tag{3.17}$$

Now, in view of estimate (3.15a) (which is applicable thanks to (3.16)), we have that

$$|S_2^\eta|\leq C\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\|u(s)-u(\mathsf{t}_i)\|_{\mathbf{V}}\|u'(s)\|_{\mathbf{V}}\,\mathrm{d}s\leq C\|u'\|_{L^1(0,T;\mathbf{V})}\max_{i=1,\dots,N_\eta}\max_{s\in[\mathsf{t}_{i-1},\mathsf{t}_i]}\|u(s)-u(\mathsf{t}_i)\|_{\mathbf{V}}. \tag{3.18}$$

Since $u\in W^{1,1}(0,T;\mathbf{V})$, we have $\max_{i=1,\dots,N_\eta}\max_{s\in[\mathsf{t}_{i-1},\mathsf{t}_i]}\|u(s)-u(\mathsf{t}_i)\|_{\mathbf{V}}\to 0$ as $\eta\downarrow 0$, hence $S_2^\eta\to 0$. Moreover,

$$|S_3^\eta|\leq \|\mathrm{D}_u\mathcal{E}(\cdot,u(\cdot),z(\cdot))\|_{L^\infty(0,T;\mathbf{V}^*)}\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\|u'(s)-u'(\mathsf{t}_i)\|_{\mathbf{V}}\,\mathrm{d}s \tag{3.19}$$

where the latter estimate is again a consequence of (3.15a). We may choose the partition $(\mathsf{t}_i)_{i=1}^{N_\eta}$ in such a way that $\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\|u'(s)-u'(\mathsf{t}_i)\|_{\mathbf{V}}\,\mathrm{d}s\to 0$ as $\eta\downarrow 0$, hence also $S_3^\eta\to 0$, and such that $S_1^\eta\to\int_0^t\langle \mathrm{D}_u\mathcal{E}(s,u(s),z(s)),u'(s)\rangle_{\mathbf{V}}\,\mathrm{d}s$, to which, ultimately, the term on the left-hand side of (3.17) converges.

Analogously,

$$
\begin{aligned}
&\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\partial_t\mathcal{E}(s,u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))\,\mathrm{d}s\\
&=\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\partial_t\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i))\,\mathrm{d}s+\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\left(\partial_t\mathcal{E}(s,u(\mathsf{t}_i),z(\mathsf{t}_i))-\partial_t\mathcal{E}(\mathsf{t}_i,u(\mathsf{t}_i),z(\mathsf{t}_i))\right)\mathrm{d}s\\
&\quad+\sum_{i=1}^{N_\eta}\int_{\mathsf{t}_{i-1}}^{\mathsf{t}_i}\left(\partial_t\mathcal{E}(s,u(\mathsf{t}_{i-1}),z(\mathsf{t}_i))-\partial_t\mathcal{E}(s,u(\mathsf{t}_i),z(\mathsf{t}_i))\right)\mathrm{d}s\doteq S_4^\eta+S_5^\eta+S_6^\eta.
\end{aligned}
$$

With a suitable choice of the partition we have that $S_4^\eta\to\int_0^t\partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s$, while exploiting (3.15b) and arguing similarly as above we have that $S_5^\eta,\ S_6^\eta\to 0$. Hence, (3.12) ensues. □

3.3. **Improved temporal regularity of energetic solutions by uniform convexity of $\mathcal{E}$.** Under stronger convexity properties of the energy functional, it has been shown in [MT04, MR07, TM10] in the fully rate-independent setting that energetic solutions enjoy better temporal regularity. In particular, uniform convexity of the energy functional in the pair (u,z) implies temporal Hölder-, or even Lipschitz continuity, [TM10, Thm. 4.5]. In the coupled rate-independent/rate-dependent setting, however, the uniform convexity of the $\mathcal{E}(t,u(t),\cdot)$ in general only ensures the *continuity in time* of the semistable variable z, and in some special cases its Hölder continuity. To give a more precise outline, in what follows uniform convexity is only required with respect to the z-variable and the condition is formulated with respect to an additional Banach space $\mathbf{S}\supset\mathbf{X}$, which may or may not coincide with $\mathbf{Z}$ or $\mathbf{X}$, i.e.,

$$
\begin{aligned}
&\exists\,\alpha\geq 2, C_*>0,\ \forall\, z_0,z_1\in\mathbf{X},\ \forall\,\theta\in[0,1],\ z_\theta=\theta z_1+(1-\theta)z_0:\\
&\mathcal{E}(t,u(t),z_\theta)+\mathcal{R}(z_\theta-z_0)+C_*\theta(1-\theta)\|z_1-z_0\|_{\mathbf{S}}^\alpha\leq\theta\mathcal{E}(t,u(t),z_0)+(1-\theta)\mathcal{E}(t,u(t),z_1)+\mathcal{R}(z_1-z_0)\,.
\end{aligned}
\tag{3.20}
$$

The above convexity inequality (3.20) implies an improved semistability condition

$$
\mathcal{E}(t,u(t),z(t))+C_*\|z(z)-z(s)\|_{\mathbf{S}}^\alpha\leq\mathcal{E}(t,u(t),z(s))+\mathcal{R}(z(s)-z(t))\,,\tag{3.21}
$$

and via the energy-dissipation inequality (3.4), under further continuity assumptions on the partial derivatives $\partial_t\mathcal{E}$ and $\mathrm{D}_u\mathcal{E}$ one may therefrom obtain an estimate of the form $\|z(t)-z(s)\|_{\mathbf{S}}^\alpha\leq\omega(t-s)$ with ω a modulus of continuity also depending on u' and u''. To deduce this estimate, however, it is crucial that the energy-dissipation inequality (3.4) is satisfied on *every* subinterval $[s,t]\subset[0,T]$, which of course is a defining property of energetic solutions in the fully rate-independent setting, but a strengthening of the notion of energetic solutions in the coupled setting. Moreover, it has to be stressed, that ultimately concluding the continuity estimate for $\|z(t)-z(s)\|_{\mathbf{S}}^\alpha$ requires the Gâteaux differentiability of $\mathcal{E}(t,\cdot,z)$, and, in the case with inertia ($\rho>0$) additionally that $u\in W^{2,1}(0,T;\mathbf{V}^*)$.

Theorem 3.7 (Continuity of weak energetic solutions by uniform convexity)**.** *Let $(u,z):\mathbf{V}\times\mathbf{Z}$ be a weak energetic solution for the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, resp. the evolutionary system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, which satisfies the energy-dissipation inequality* (3.4) *even on all subintervals $[s,t]\subset[0,T]$. Assume that $\mathcal{E}(t,\cdot,z)$ is Gâteaux-differentiable for every $(t,z)\in[0,T]\times\mathrm{D}_z$ and, if $\varrho>0$, that $u\in W^{2,1}(0,T;\mathbf{V}^*)$ (cf.* (3.11)*) and that $t\mapsto|\langle u''(t),u'(t)\rangle_{\mathbf{V}}|\in L^1(0,T)$. Moreover, assume that the energy functional $\mathcal{E}$ satisfies the following Hölder continuity conditions: There exists a Banach space $\mathbf{S}$, with $\mathbf{X}\subset\mathbf{S}$ continuously, and constants $c_*>0,\beta_u,\beta_z\in(0,1]$ such that for all $s,t\in[0,T]$, for all $(u_0,z_0),(u_0,z_1),(u_1,z_1)\in\mathcal{S}_E$:*

$$
\begin{aligned}
|\mathcal{E}(t,u_1,z_1)-\mathcal{E}(t,u_0,z_1)|&\leq c_*\|u_1-u_0\|_{\mathbf{V}}^{\beta_u}\,,&&(3.22)\\
|\partial_t\mathcal{E}(t,u_1,z_1)-\partial_t\mathcal{E}(t,u_0,z_0)|&\leq c_*\big(\|z_1-z_0\|_{\mathbf{S}}^{\beta_z}+\|u_1-u_0\|_{\mathbf{V}}^{\beta_u}\big)\,,&&(3.23)
\end{aligned}
$$

and, furthermore, that $\mathcal{E}(t,u(t),\cdot):\mathbf{Z}\to[0,+\infty]$ is uniformly convex as in (3.20)*.*

Then, $z\in\mathrm{C}^0([0,T];\mathbf{S})$, as it complies with the following estimate:

$$
C_*\|z(t)-z(s)\|_{\mathbf{S}}^\alpha\leq C|t-s|+\frac{\varrho}{2}\int_s^t|\langle u''(\zeta),u'(\zeta)\rangle_{\mathbf{V}}|\,\mathrm{d}\zeta+c_*\Big(\int_s^t\|u'(\zeta)\|_{\mathbf{V}}\,\mathrm{d}\zeta\Big)^{\beta_u}\,.\tag{3.24}
$$

If $\varrho = 0$ and if the dissipation $\mathcal{V}$ has p-growth for some $p > 1$, namely $\mathcal{V}(v) \geq C_1\|v\|_{\mathbf{V}}^p - C_2$ for some C_1, $C_2 > 0$, then $z \in \mathrm{C}^{0,h}([0,T];\mathbf{S})$ with the Hölder-exponent $h = \beta_u/(p'\alpha)$, where $p' = p/(p-1)$.

Proof. The proof carries over the steps of [TM10, Thm. 4.5] to the present coupled setting. For this, we first verify (3.21) by choosing $\theta \in (0,1)$, $z_0 = z(t)$ and $z_1 = z(s)$, $0 \leq s < t \leq T$ fixed, in the uniform convexity inequality (3.20). This yields

$$\begin{aligned} 0 \leq & \mathcal{E}(t,u(t),z_\theta) + \mathcal{R}(z_\theta - z(t)) - \mathcal{E}(t,u(t),z(t)) \\ \leq & \theta\big(\mathcal{E}(t,u(t),z(s) + \mathcal{R}(z(s)-z(t)) - \mathcal{E}(t,u(t),z(t) - C_*(1-\theta)\|z(s)-z(t)\|_{\mathbf{S}}^\alpha\big)\,. \end{aligned} \tag{3.25}$$

Dividing by $\theta > 0$ and letting $\theta \to 1$ results in the improved semistability inequality (3.21).

Now, estimate (3.24) is deduced from the energy-dissipation inequality (3.4) in combination with the continuity conditions (3.23) and (3.22)

$$\begin{aligned} & C_*\|z(t)-z(s)\|_{\mathbf{S}}^\alpha \leq \mathcal{E}(s,u(s),z(t)) + \mathcal{R}(z(t)-z(s)) - \mathcal{E}(s,u(s),z(s)) \\ & \leq \mathcal{E}(t,u(t),z(t)) + \mathrm{Var}_{\mathcal{R}}(z,[s,t]) - \mathcal{E}(s,u(s),z(s)) + \mathcal{E}(s,u(s),z(t)) - \mathcal{E}(t,u(t),z(t)) \\ & \leq \tfrac{\varrho}{2}\big(\|u'(s)\|_{\mathbf{W}}^2 - \|u'(t)\|_{\mathbf{W}}^2\big) - \int_s^t \mathcal{V}(u'(r)) + \mathcal{V}^*(-\mathrm{D}_u\mathcal{E}(r,u(r),z(r)) - \rho u''(r))\,\mathrm{d}r \\ & \qquad + \int_s^t \partial_t\mathcal{E}(r,u(r),z(r)) - \partial_t\mathcal{E}(r,u(s),z(t))\,\mathrm{d}r + \mathcal{E}(t,u(s),z(t)) - \mathcal{E}(t,u(t),z(t)) \\ & \leq \tfrac{\varrho}{2}\int_s^t |\langle u''(r),u'(r)\rangle_{\mathbf{V}}|\,\mathrm{d}r + c_*\int_s^t \|z(r)-z(t)\|_{\mathbf{S}}^{\beta_z} + \|u(r)-u(t)\|_{\mathbf{V}}^{\beta_u}\,\mathrm{d}r + c_*\left(\int_s^t \|u'(\zeta)\|_{\mathbf{V}}\,\mathrm{d}\zeta\right)^{\beta_u} \\ & \leq C|s-t| + \tfrac{\varrho}{2}\int_s^t |\langle u''(r),u'(r)\rangle_{\mathbf{V}}|\,\mathrm{d}r + c_*\left(\int_s^t \|u'(r)\|_{\mathbf{V}}\,\mathrm{d}r\right)^{\beta_u}\,. \end{aligned}$$

Here, we have used that $\|z(r)-z(t)\|_{\mathbf{S}} \leq 2C\sup_{t\in[0,T]}\|z(t)\|_{\mathbf{X}} \leq \tilde{C}$ for some $\tilde{C} > 0$, by the continuous embedding $\mathbf{X} \subset \mathbf{S}$ and the boundedness of energy-dissipation sublevels according to (2.18) in combination with (3.4). In a similar way, we have that $\|u(r)-u(t)\|_{\mathbf{V}}^{\beta_u} \leq \tilde{C}$. Moreover, we exploited that $0 < \int_s^t \mathcal{V}(u'(r)) + \mathcal{V}^*(-\mathrm{D}_u\mathcal{E}(r,u(r),z(r)) - \rho u''(r))\,\mathrm{d}r \leq C$ by the energy-dissipation inequality (3.4), so that the negative of this term can be dropped to find an estimate from above. Moreover, since $\langle u''(\cdot),u'(\cdot)\rangle_{\mathbf{V}} \in L^1(0,T)$, the first integral term on the right-hand side is absolutely continuous, whereas the second one is continuous due to $u \in L^1(0,T;\mathbf{V})$. This proves estimate (3.24) and yields $z \in \mathrm{C}^0([0,T];\mathrm{S})$.

To verify the Hölder continuity in the case that $\varrho = 0$ and $\mathcal{V}$ is of p-growth, we apply Hölder's inequality with exponent p to the third term on the r.h.s. of (3.24) and find that $C|s-t| + c_*\big(\int_s^t \|u'(\zeta)\|_{\mathbf{V}}\,\mathrm{d}\zeta\big)^{\beta_u} \leq |s-t|^{\beta_u/p'}\big(C|s-t|^{1-\beta_u/p'} + c_*\|u'\|_{L^p(0,T;\mathbf{V})}^{\beta_u}\big) \leq C|s-t|^{\beta_u/p'}$. □

4. Existence results for the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$

Throughout this section, we will focus on the case $\varrho = 0$.

In Section 4.1 we will first address the case in which the dissipation potential $\mathcal{V}$ is quadratic. In this context, our main assumption on the energy $\mathcal{E}$ will be a sort of *uniform Fréchet subdifferentiability* condition, drawn from [MRS13a]. Hence, in our first existence result, Theorem 1, we are going to show that the approximate solutions constructed by a carefully devised time-discretization scheme converge to an *energetic* solution of the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$.

In Section 4.2 we will extend our analysis to a reasonably broad class of convex dissipation potentials $\mathcal{V}$ with superlinear growth at infinity. For the limit passage from time-discrete to time-continuous, we will resort to an approach, and to ideas, from the *variational theory for gradient flows.* In this context, we will not require λ-convexity properties on the energy functional $\mathcal{E}$. However, we will need to *directly* impose a closedness condition on $\partial_u^-\mathcal{E}$ more stringent than the closedness property guaranteed by the uniform subdifferentiability condition. Hence, we will give our second existence result, Theorem 2, stating that the approximate solutions constructed by time discretization converge to a *weak* energetic solution to the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, which clearly turns into an *energetic* one under the additional condition (3.15) on $\mathcal{E}$.

In Section 4.3 we will compare the two sets of conditions on $\mathcal{E}$ in two examples. Furthermore, we will hint at possible extensions of Theorems 1 and 2.

4.1. **Case 1: $\mathcal{V}$ quadratic.** Throughout this section, we will confine the discussion to *quadratic* dissipation potentials for u, namely we will suppose that

$$\mathcal{V}(v) = \frac{1}{2}a(v,v) \qquad \text{with } a : \mathbf{V}\times\mathbf{V} \to \mathbb{R} \text{ a continuous and coercive bilinear form,} \tag{4.1}$$

thus inducing on $\mathbf{V}$ a Hilbert space structure. Hence, for every $v \in \mathbf{V}$

$$\partial\mathcal{V}(v) = \{Av\} \qquad \text{with } A : \mathbf{V} \to \mathbf{V}^* \text{ the linear operator associated with } a.$$

Still, throughout this section we will continue to use the notation $\partial\mathcal{V}$, also to make the comparison with the results of Sec. 4.2 more transparent. We now detail the time-discretization scheme which will give rise to the approximate solutions for the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$.

Time-discretization scheme. Let $\tau = T/N_\tau > 0$ be the time step, inducing a partition

$$\mathscr{P}_\tau = \{0 = t_0 < t_1 < \ldots < t_{N_\tau - 1} < t_{N_\tau} = T\}$$

of the interval $[0,T]$. We shall denote by $\bar{\mathsf{t}}_\tau$ and $\underline{\mathsf{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition $\mathscr{P}_\tau$, namely

$$\bar{\mathsf{t}}_\tau(0) = \underline{\mathsf{t}}_\tau(0) := 0, \quad \bar{\mathsf{t}}_\tau(t) := t_n \quad \text{for} \quad t \in (t_{n-1},t_n], \quad \underline{\mathsf{t}}_\tau(t) := t_{n-1} \quad \text{for} \quad t \in [t_{n-1},t_n). \tag{4.2}$$

Of course, for every $t \in [0,T]$ we have $\bar{\mathsf{t}}_\tau(t) \downarrow t$ and $\underline{\mathsf{t}}_\tau(t) \uparrow t$ as $\tau \downarrow 0$.

Starting from the Cauchy data $(u_0,z_0) \in \mathrm{D}_{\mathsf{u}} \times \mathrm{D}_{\mathsf{z}}$ (cf. (4.18) below), we construct discrete solutions $(u_\tau^n, z_\tau^n)_{n=1}^{N_\tau}$ by *alternate* time-incremental minimization. More precisely, we first find z_τ^n by minimizing with respect to z the sum of the energy and of the dissipation potential $\mathcal{V}$, with u_τ^{n-1} and z_τ^{n-1} given from the previous step. Then, we construct u_τ^n by minimizing, now with respect to u, the sum of the energy and of the dissipation, with z_τ^n now given.

Problem 4.1. *Let $(u_\tau^0, z_\tau^0) := (u_0,z_0) \in \mathrm{D}_{\mathsf{u}} \times \mathrm{D}_{\mathsf{z}}$, and for $n = 1,\ldots,N_\tau$, find*

$$z_\tau^n \in \operatorname*{Argmin}_{z\in\mathbf{Z}} \left(\tau\mathcal{R}\left(\frac{z - z_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, u_\tau^{n-1}, z)\right), \tag{4.3a}$$

$$u_\tau^n \in \operatorname*{Argmin}_{u\in\mathbf{V}} \left(\tau\mathcal{V}\left(\frac{u - u_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, u, z_\tau^n)\right). \tag{4.3b}$$

With the direct method in the Calculus of Variations, exploiting Hypotheses 2.2 and 2.7, it is not difficult to check that for all $0 < \tau < \tau_o$ Problem 4.1 admits a solution $(u_\tau^n, z_\tau^n)_{n=1}^{N_\tau}$. Then, we construct the approximate solutions as the left-continuous and right-continuous interpolants of $(u_\tau^n, z_\tau^n)_{n=1}^{N_\tau}$, defined by

$$\overline{u}_\tau(t) := u_\tau^n,\ \overline{z}_\tau(t) := z_\tau^n \quad \text{for } t \in (t_{n-1},t_n], \qquad \underline{u}_\tau(t) := u_\tau^{n-1},\ \underline{z}_\tau(t) := z_\tau^{n-1} \quad \text{for } t \in [t_{n-1},t_n). \tag{4.4}$$

We also consider he piecewise linear interpolants

$$u_\tau(t) := \frac{t - t_{n-1}}{\tau}u_\tau^n + \frac{t_n - t}{\tau}u_\tau^{n-1},\ z_\tau(t) := \frac{t - t_{n-1}}{\tau}z_\tau^n + \frac{t_n - t}{\tau}z_\tau^{n-1}, \quad \text{for } t \in [t_{n-1},t_n). \tag{4.5}$$

Uniform Fréchet subdifferentiability. For the derivation of all a priori estimates on the approximate solutions, it will be crucial to obtain a *discrete* form of the energy-dissipation inequality (3.4), cf. (7.30). Its proof will be based on a suitable condition on the Fréchet subdifferential $\partial_u^-\mathcal{E} : \mathbf{V} \rightrightarrows \mathbf{V}^*$, that can be interpreted as a *uniform Fréchet subdifferentiability* of the energy $u \mapsto \mathcal{E}(t,u,z)$ (in this connection, we refer to the discussions in [MRS13b, Sec. 2] and [MRS13a, Sec. 3] where analogous properties were introduced and exploited). We may understand it like this: While the inequality (2.10) defining the Fréchet subdifferential has a *local* character, with (4.6) below we make it *global*, at the price of having a negative term on the right-hand side. Observe that for our purposes it will be sufficient to require this property on energy sublevels.

Hypothesis 4.2 (Uniform Fréchet subdifferentiability)**.** *For all $E > 0$ there exists $\Lambda_E \geq 0$ such that*

$$\begin{aligned}\mathcal{E}(t,v,z) - \mathcal{E}(t,u,z) \geq\ &\langle \xi, v - u\rangle_{\mathbf{V}} - \Lambda_E \|v-u\|_{\mathbf{V}}^2 \\ &\textit{for all } t\in[0,T],\ (u,z),\ (v,z) \in \mathcal{S}_E \textit{ and for all } \xi \in \partial_u^-\mathcal{E}(t,u,z).\end{aligned} \tag{4.6}$$

Remark 4.3. Observe that, the validity of (4.6) implies on its own that $\xi \in \partial_u^- \mathcal{E}(t,u,z)$, cf. (2.10).

Notice that, if (4.6) holds with a remainder term of the type $\Lambda_E \|v-u\|_{\mathbf{V}} \|v-u\|_{\mathbf{Y}}$, and $\mathbf{Y}$ a space such that $\mathbf{V} \subset \mathbf{Y}$ with a continuous embedding, then (4.6) holds with the term $\widetilde{\Lambda_E}\|v-u\|_{\mathbf{V}}^2$ for some other $\widetilde{\Lambda_E} > 0$. This can be checked by observing that there exists $C_{\mathbf{V},\mathbf{Y}} > 0$ $\|v-u\|_{\mathbf{Y}} \le C_{\mathbf{V},\mathbf{Y}} \|v-u\|_{\mathbf{V}}$ for every $u,\, v \in \mathcal{S}_E$.

A sufficient condition for (4.6) is provided by the λ-convexity of the energy $u \mapsto \mathcal{E}(t,u,z)$, on energy sublevels, namely

Lemma 4.4. *Let $\mathcal{E} : [0,T] \times \mathbf{V} \times \mathbf{Z} \to (-\infty,+\infty]$ fulfill Hypothesis 2.2 and*

$$\begin{aligned} &\forall\, E>0 \quad \exists\, \Lambda_E \ge 0 \quad \forall\, t \in [0,T],\ (u,z),\ (v,z) \in \mathcal{S}_E,\ \forall\, \theta \in (0,1): \\ &\qquad \mathcal{E}(t,(1-\theta)u+\theta v, z) \le (1-\theta)\mathcal{E}(t,u,z) + \theta \mathcal{E}(t,v,z) + \Lambda_E \theta(1-\theta)\|v-u\|_{\mathbf{V}}^2 \,. \end{aligned} \tag{4.7}$$

Then, $\mathcal{E}$ complies with Hypothesis 4.2.

The *proof* follows the very same lines as the one in [MRS13a, Lemma 3.26], hence it is omitted.

Example 4.5. *Let $\phi : \mathbb{R}^{d\times d} \to \mathbb{R}$, $\phi(e) := \operatorname{tr} e|e|$ with $\operatorname{tr} e = \sum_{i=1}^d e_{ii}$ for any $e = (e_{ij})_{i,j=1}^d \in \mathbb{R}^{d\times d}$, $1 < d \in \mathbb{N}$. The function ϕ is not convex, as can be checked e.g. for $d=2$, and the diagonal matrices $e := \operatorname{diag}(1/2,1/2)$ and $\tilde e = \operatorname{diag}(-1,1)$. For $\theta = 1/2$, this choice gives $\frac12\sqrt{\frac58} = \phi(\theta e + (1-\theta)\tilde e) > \theta\phi(e) + (1-\theta)\phi(\tilde e) = \frac12\sqrt{\frac12}$. But ϕ is Λ-convex for any $\Lambda > \sqrt{d}$. To verify this statement, we calculate the derivatives*

$$\begin{aligned} \phi'(e)[\tilde e] &= \operatorname{tr}\tilde e|e| + \operatorname{tr} e\tfrac{e:\tilde e}{|e|}\,, \\ \phi''(e)[\tilde e,\hat e] &= \operatorname{tr}\tilde e\tfrac{e:\hat e}{|e|} + \operatorname{tr}\hat e\tfrac{e:\tilde e}{|e|} + \operatorname{tr} e\tfrac{\hat e:\tilde e}{|e|} - \operatorname{tr} e\tfrac{(e:\tilde e)(e:\hat e)}{|e|^3}\,, \\ \phi''(e)[\tilde e,\tilde e] &= 2\operatorname{tr}\tilde e\tfrac{e:\tilde e}{|e|} + + \operatorname{tr} e\tfrac{|\tilde e|^2}{|e|} - \tfrac{\operatorname{tr} e}{|e|}\big(\tfrac{e:\tilde e}{|e|}\big)^2\,. \end{aligned} \tag{4.8}$$

For any choice of $e, \tilde e \in \mathbb{R}^{d\times d}$ we aim to show that

$$\phi''(e)[\tilde e,\tilde e] \ge -2|\operatorname{tr}\tilde e||\tilde e|\,. \tag{4.9}$$

For this, we first assume that $\operatorname{tr} e, \operatorname{tr}\tilde e \ge 0$. Using that $\big(\frac{e:\tilde e}{|e|}\big)^2 \le |\tilde e|^2$ for the third term in (4.8), we find (4.9) as an estimate from below since the resulting term cancels out with the second term. With the same argument we verify (4.9) both if $\operatorname{tr} e \ge 0$ & $\operatorname{tr}\tilde e \le 0$ and if $\operatorname{tr} e \le 0$ & $\operatorname{tr}\tilde e \ge 0$. Finally, if $\operatorname{tr} e \le 0$ & $\operatorname{tr}\tilde e \le 0$, we equivalently apply $\big(\frac{e:\tilde e}{|e|}\big)^2 \le |\tilde e|^2$ to the second term in (4.8), so that the resulting term cancels out with the third term, and again obtain (4.9). Now, we can further estimate the right-hand side of (4.9) from below using that

$$|\operatorname{tr} e| \le \sum_{i=1}^d |e_{ii}| \le \sqrt{d}\Big(\sum_{i=1}^d (e_{ii}^2)\Big)^{1/2} \le \sqrt{d}\Big(\sum_{i,j=1}^d (e_{ij}^2)\Big)^{1/2} = \sqrt{d}|e|\,.$$

Thus, in total we have obtained that $\phi''(e)[\tilde e,\tilde e] \ge -2\sqrt{d}|\tilde e|^2$. In view of $\partial_e^2|e|[\tilde e,\tilde e] = 2|\tilde e|^2$, this proves that ϕ is Λ-convex for any $\Lambda \ge \sqrt{d}$. From this we conclude that the corresponding integral functional defined on a domain $\Omega \subset \mathbb{R}^d$,

$$\Phi : \mathbf{V} \to (-\infty,+\infty), \qquad \Phi(\mathbf{u}) = \int_\Omega \phi(\varepsilon(\mathbf{u}))\,\mathrm{d}x \quad \text{with } \phi(\varepsilon(\mathbf{u})) := |\operatorname{tr}\varepsilon(\mathbf{u})||\varepsilon(\mathbf{u})| \tag{4.10}$$

satisfies Λ-convexity (4.7) on $\mathbf{V} = H_0^1(\Omega;\mathbb{R}^d)$ for any $\Lambda > \sqrt{d}$. We also refer to Sec. 4.3 for further examples.

As mentioned in the above lines, the primary role of condition (4.6) is in the derivation of a *discrete energy-dissipation inequality.* Furthermore,

(1) In the case $\mathbf{U} \Subset \mathbf{V}$, in combination with Hypothesis 4.6 below, condition (4.6) will guarantee a closedness property of the graph of $\partial_u^- \mathcal{E}$ with respect to strong-weak topology, and a continuity property of $\mathcal{E}$ (cf. Lemma 4.7 ahead), that will play a key role in the passage to the time-continuous limit.
(2) If only the *continuous* embedding $\mathbf{U} \subset \mathbf{V}$ is assumed, it will be necessary to require, in addition to Hypothesis 4.2, a version of the closedness property of $\partial_u^- \mathcal{E}$ in Bochner spaces, cf. Hypothesis 4.8 below.

Case 1: A recovery sequence condition for $\mathcal{E}(t,\cdot,z)$. In the case $\mathbf{U} \Subset \mathbf{V}$, we impose the following

Hypothesis 4.6 (Recovery sequence condition for $\mathcal{E}(t,\cdot,z)$)**.** *For all sequences $(t_n)_n \subset [0,T]$, $(u_n)_n \subset \mathbf{V}$, and $(z_n)_n \subset \mathbf{Z}$ such that*

$$t_n \to t, \quad u_n \to u \text{ in } \mathbf{V}, \quad z_n \rightharpoonup z \text{ in } \mathbf{Z}, \quad \sup_n \mathcal{G}(u_n,z_n) \le C$$

and for every $\tilde{u} \in \mathrm{D_u}$ there exists $\tilde{u}_n \to \tilde{u}$ in $\mathbf{V}$ such that

$$\limsup_{n\to\infty} \left(\mathcal{E}(t_n,\tilde{u}_n,z_n) - \mathcal{E}(t_n,u_n,z_n)\right) \le \mathcal{E}(t,\tilde{u},z) - \mathcal{E}(t,u,z). \tag{4.11}$$

We postpone the proof of the following result to Section 7.2.

Lemma 4.7. *Suppose that $\mathbf{U} \Subset \mathbf{V}$ and let $\mathcal{E}$ fulfill Hypotheses 2.2, 4.2 and 4.6. Then, for all sequences $(t_n,u_n,z_n,\xi_n) \subset [0,T] \times \mathbf{V} \times \mathbf{Z} \times \mathbf{V}^*$ there holds*

$$\left.\begin{cases} t_n \to t, & \\ u_n \to u & \text{in } \mathbf{V}, \\ z_n \rightharpoonup z & \text{in } \mathbf{Z}, \\ \sup_n \mathcal{G}(u_n,z_n) \le C, & \\ \xi_n \rightharpoonup \xi & \text{in } \mathbf{V}^* \\ \xi_n \in \partial_u^- \mathcal{E}(t_n,u_n,z_n) & \end{cases}\right\} \Rightarrow \ \xi \in \partial_u^-\mathcal{E}(t,u,z) \ \text{ and } \ \mathcal{E}(t_n,u_n,z_n) \to \mathcal{E}(t,u,z). \tag{4.12}$$

Case 2: A "Bochner-space" version of closedness à la Minty of $\partial_u^-\mathcal{E}$. In the case $\mathbf{U} \subset \mathbf{V}$ continuously, it is no longer possible to deduce from Hypotheses 4.2 & 4.6 the closedness property (4.12). Therefore we have to require a version of it. Observe that (4.12) has a "pointwise-in-time" character, in that it only involves sequences converging in the spaces $\mathbf{V}$, $\mathbf{Z}$, and $\mathbf{V}^*$, and it is in fact suited to the application of Young measure arguments in the limit passage from time-discrete to time-continuous.

Instead, the forthcoming Hypothesis 4.8 considers sequences of time-dependent functions, converging in Lebesgue/Sobolev-Bochner spaces. It will be *directly* used in the limit passage in the discrete version of the equation for the rate-dependent variable u. Indeed, (4.14) below also features a lim sup condition, in the spirit of Minty's trick within the theory of maximal monotone operators (see e.g. [Att84]), that can be verified by arguing on the discrete u-equation, checking that all terms apart from the one to identify either pass to the limit, or can be handled by lower semicontinuity. For this, the quadratic character of $\mathcal{V}$ will play a crucial role.

Finally, in Hyp. 4.8 we will distinguish between a *weak*, and a *strong* form of the closedness property, depending on whether we also obtain convergence of the energies.

Hypothesis 4.8 (Closedness of $\partial_u^-\mathcal{E}$ à la Minty)**.** *For all sequences $\mathsf{t}_n : [0,T] \to [0,T]$, $(u_n)_n \subset L^\infty(0,T;\mathbf{U}) \cap H^1(0,T;\mathbf{V})$, $(z_n)_n \subset L^\infty(0,T;\mathbf{X}) \cap \mathrm{BV}([0,T];\mathbf{Z})$, and $(\xi_n)_n \subset L^2(0,T;\mathbf{V}^*)$ such that*

$$\exists\, C>0 \ \ \forall\, n \in \mathbb{N} \ \ \forall\, t \in [0,T]: \ \ \mathcal{G}(u_n(t),z_n(t)) \le C, \tag{4.13}$$

and moreover

$$\left.\begin{cases} \mathsf{t}_n \to \mathsf{t} \quad \text{pointwise a.e. in } (0,T), \\ u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;\mathbf{U}) \cap H^1(0,T;\mathbf{V}), \\ z_n \overset{*}{\rightharpoonup} z \text{ in } L^\infty(0,T;\mathbf{X}), \ z_n(t) \overset{\sigma_{\mathbf{X}}}{\to} z(t) \text{ in } \mathbf{X} \quad \text{for all } t \in [0,T], \\ \xi_n \rightharpoonup \xi \quad \text{in } L^2(0,T;\mathbf{V}^*), \\ \xi_n(t) \in \partial_u^-\mathcal{E}(\mathsf{t}_n(t),u_n(t),z_n(t)) \quad \text{for a.a. } t \in (0,T), \\ \limsup_{n\to\infty} \int_0^T \langle \xi_n, u_n\rangle_{\mathbf{V}}\,\mathrm{d}t \le \int_0^T \langle \xi, u\rangle_{\mathbf{V}}\,\mathrm{d}t, \end{cases}\right\} \tag{4.14}$$

$$\text{then there holds } \ \xi(t) \in \partial_u^-\mathcal{E}(\mathsf{t}(t),u(t),z(t)) \quad \text{for a.a. } t \in (0,T).$$

We will also consider a stronger form of (4.14), namely for all sequences

$$\begin{gathered} (\mathsf{t}_n)_n,\ (u_n)_n,\ (z_n)_n,\ (\xi_n)_n \text{ as in (4.14) there holds} \\ \xi(t) \in \partial_u^-\mathcal{E}(\mathsf{t}(t),u(t),z(t)), \text{ and } \mathcal{E}(\mathsf{t}_n(t),u_n(t),z_n(t)) \to \mathcal{E}(\mathsf{t}(t),u(t),z(t)) \quad \text{for a.a. } t \in (0,T). \end{gathered} \tag{4.15}$$

In Section 4.3 we will provide the example of an energy complying with Hypotheses 4.2, 4.6, and 4.8.

Statement of our first existence result. Under the basic conditions enucleated in Sec. 2, the uniform Fréchet subdifferentiability from Hyp. 4.2, and ($\mathbf{U} \Subset \mathbf{V}$ & Hyp. 4.6) or, in *alternative*, Hyp. 4.8, Theorem 1 below states that, up to a subsequence, any family of approximate solutions constructed via time discretization converge to an energetic solution (u, z) of the gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ for $\mathcal{V}$ quadratic. We also detail in which spaces the approximate solutions converge, distinguishing a first set of convergences which will hold also in the case of $\mathcal{V}$ with general superlinear growth (cf. Theorem 2 ahead), from convergence (4.17), related to the quadratic character of $\mathcal{V}$. Under suitable conditions we will also obtain the energy convergence (4.20), which will allow us to prove that the pair (u, z) is an *enhanced* energetic solution, cf. Def. 3.1.

Theorem 1 (Energetic solutions for $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, $\mathcal{V}$ quadratic)**.** *Assume Hypotheses 2.2, 2.3, 2.5, and 2.7. In addition, suppose that $\mathcal{V}$ is quadratic, cf.* (4.1)*, that $\partial_u^- \mathcal{E}$ complies with Hyp. 4.2, and that*

(1) either $\mathbf{U} \Subset \mathbf{V}$, Hyp. 4.2 holds, and in addition Hyp. 4.6 is valid;
(2) or, Hyp. 4.8 holds.

Let $u_0 \in \mathrm{D_u}$ and $z_0 \in \mathrm{D_z}$ be fixed and let $(\overline{u}_\tau, \underline{u}_\tau, u_\tau, \overline{z}_\tau, \underline{z}_\tau, z_\tau)_\tau$ be a family of approximate solutions constructed via the time-discretization scheme in Problem 4.1.

Then, for every sequence $\tau_k \downarrow 0$ as $k \to \infty$ there exist a (not relabeled) subsequence, and functions (u, z) as in (3.1) *such that, in addition, $u \in H^1(0, T; \mathbf{V})$ and the following convergences hold as $k \to \infty$*

$$u_{\tau_k} \rightharpoonup u \qquad \text{in } W^{1,1}(0, T; \mathbf{V}), \tag{4.16a}$$

$$\overline{u}_{\tau_k},\, u_{\tau_k},\, \tilde{u}_{\tau_k} \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(0, T; \mathbf{U}), \tag{4.16b}$$

$$\overline{u}_{\tau_k}(t),\, \underline{u}_{\tau_k}(t),\, u_{\tau_k}(t),\, \tilde{u}_{\tau_k}(t) \rightharpoonup u(t) \qquad \text{in } \mathbf{U} \text{ for all } t \in [0, T], \tag{4.16c}$$

$$\overline{z}_{\tau_k},\, z_{\tau_k} \overset{*}{\rightharpoonup} z \qquad \text{in } L^\infty(0, T; \mathbf{X}), \tag{4.16d}$$

$$\overline{z}_{\tau_k}(t),\, z_{\tau_k}(t) \overset{\sigma_{\mathbf{X}}}{\to} z(t) \qquad \text{in } \mathbf{X} \text{ for all } t \in [0, T], \tag{4.16e}$$

$$\big(\overline{z}_{\tau_k}(t) - \underline{z}_{\tau_k}(t)\big) \overset{\sigma_{\mathbf{X}}}{\to} 0 \qquad \text{in } \mathbf{X} \text{ for almost all } t \in (0, T), \tag{4.16f}$$

$$\overline{z}_{\tau_k}(t),\, z_{\tau_k}(t) \rightharpoonup z(t) \qquad \text{in } \mathbf{Z} \text{ for all } t \in [0, T], \tag{4.16g}$$

(while for $(\underline{z}_{\tau_k}(t))_k$ the analogues of (4.16e)–(4.16g) *hold for almost all $t \in (0, T)$), and, moreover,*

$$u_{\tau_k} \rightharpoonup u \quad \text{in } H^1(0, T; \mathbf{V}). \tag{4.17}$$

The pair (u, z) is an energetic *solution to the gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, satisfying the initial conditions*

$$u(0) = u_0, \qquad z(0) = z_0. \tag{4.18}$$

Furthermore,

(i) under the conditions in (1), we also have the additional convergences

$$\overline{u}_{\tau_k}(t),\, u_{\tau_k}(t),\, \underline{u}_{\tau_k}(t) \to u(t) \text{ in } \mathbf{V} \text{ for all } t \in [0, T], \tag{4.19}$$

$$\mathcal{E}(\overline{\mathsf{t}}_{\tau_k}(t), \overline{u}_{\tau_k}(t), \overline{z}_{\tau_k}(t)) \to \mathcal{E}(t, u(t), z(t)) \qquad \text{for a.a. } t \in (0, T), \tag{4.20}$$

and the pair (u, z) is an enhanced *energetic solution, complying with*

$$\begin{aligned} &\int_s^t \mathcal{V}(u'(r)) + \mathcal{V}^*(-\xi(r))\,\mathrm{d}r + \mathrm{Var}_{\mathcal{R}}(z, [s, t]) + \mathcal{E}(t, u(t), z(t)) \\ &\leq \mathcal{E}(s, u(s), z(s)) + \int_s^t \partial_t \mathcal{E}(r, u(r), z(r))\,\mathrm{d}r \qquad \text{for all } t \in (0, T] \text{ for } s = 0, \text{ and a.a. } s \in (0, t). \end{aligned} \tag{4.21}$$

(ii) If Hyp. 4.8 holds in the stronger form (4.15)*, then* (4.20) *and* (4.21) *are valid and (u, z) is enhanced.*

Remark 4.9. As it will be clear from the proof of Theorem 1, developed in Section 7.2, under the conditions guaranteeing the energy convergence (4.20), indeed it would be sufficient to adopt the *weaker* variant of Hypothesis 2.5 introduced in Remark 2.6.

4.2. **Case 2: $\mathcal{V}$ with general superlinear growth.** We now address the analysis of system (1.4) for a fairly broad class of dissipation potentials $\mathcal{V}$ that fulfill, in addition to condition (2.11a), the following property

$$\mathcal{V}^*(\xi_1) = \mathcal{V}^*(\xi_2) \qquad \text{for all } \xi_1, \xi_2 \in \partial\mathcal{V}(u) \quad \text{for all } u \in \mathbf{V}. \tag{4.22}$$

This condition will have a technical role in the derivation of a discrete energy-dissipation inequality, different from the one proved for $\mathcal{V}$ quadratic, for the approximate solutions arising from time discretization, cf. Lemma 7.1 ahead. The proof of such inequality, developed in [MRS13b], extends the *variational techniques* for gradient flows from [Amb95, AGS08] to the case of systems driven by general dissipation potentials with superlinear growth, complying with (4.22).

Time-discretization scheme. In the case of a general dissipation potential $\mathcal{V}$, we construct the discrete solutions by a scheme different from the one in Problem 4.1. In fact, first we solve a minimum problem for u, with u_τ^{n-1} and z_τ^{n-1} given from the previous step, and then we solve for z, with u_τ^n now given.

Problem 4.10. *Let $(u_\tau^0, z_\tau^0) := (u_0, z_0) \in \mathrm{D_u} \times \mathrm{D_z}$, and for $n = 1, \ldots, N_\tau$, find*

$$u_\tau^n \in \operatorname*{Argmin}_{u \in \mathbf{V}} \left(\tau \mathcal{V} \left(\frac{u - u_\tau^{n-1}}{\tau} \right) + \mathcal{E}(t_n, u, z_\tau^{n-1}) \right), \tag{4.23a}$$

$$z_\tau^n \in \operatorname*{Argmin}_{z \in \mathbf{Z}} \left(\tau \mathcal{R} \left(\frac{z - z_\tau^{n-1}}{\tau} \right) + \mathcal{E}(t_n, u_\tau^n, z) \right). \tag{4.23b}$$

With the direct method in the Calculus of Variations, exploiting Hypotheses 2.2 and 2.7, it is not difficult to check that for all $0 < \tau < \tau_o$ Problem 4.10 admits a solution $(u_\tau^n, z_\tau^n)_{n=1}^{N_\tau}$.

As in Section 4.1, we construct approximate solutions by taking the piecewise constant and linear interpolants of the elements $(u_\tau^n, z_\tau^n)_{n=1}^{N_\tau}$. Nonetheless, for the derivation of the approximate version of the energy-dissipation inequality (3.4) in the forthcoming Lemma 7.1, it will be crucial to resort to yet another notion of interpolant of $(u_\tau^n)_{n=1}^{N_\tau}$, which was first introduced by E. De Giorgi within the *Minimizing Movements* theory (see [Amb95, AGS08]). It is defined in the following way: the map $t \mapsto \tilde{u}_\tau(t)$ is Lebesgue measurable in $(0, T)$ and satisfies

$$\begin{cases} \tilde{u}_\tau(0) = u_0, \\ \tilde{u}_\tau(t) \in \mathrm{Argmin}_{u \in \mathbf{V}} \left\{ r \mathcal{V} \left(\frac{u - u_\tau^{n-1}}{r} \right) + \mathcal{E}(t, u, z_\tau^{n-1}) \right\} \quad \text{for } t = t_{n-1} + r \in (t_{n-1}, t_n]. \end{cases} \tag{4.24}$$

Hereafter, we will call $\tilde{u}_\tau$ *variational interpolant.* We refer to [RS06, Rmk. 4.4], [MRS13b, Sec. 4] for a thorough discussion on the existence of the above measurable selection. Observe that when $t = t_n$ the minimum problems (4.23a) and (4.24) coincide, hence we may assume $\overline{u}_\tau(t_n) = \underline{u}_\tau(t_n) = u_\tau(t_n) = \tilde{u}_\tau(t_n)$ for all $n = 1, \ldots, N_\tau$.

A closedness condition. In Section 7.1 we will see that the variational interpolant $\tilde{u}_\tau$ and the piecewise constant one $\overline{z}_\tau$ comply with the discrete energy-dissipation inequality (7.8). The latter will be the starting point in the derivation of all a priori estimates, and in it we will pass to the limit to conclude that any limit pair (u, z) of the approximate solutions complies with the energy-dissipation inequality (3.4). Let us stress that, the usage of the variational techniques from the theory of gradient flows enables us to obtain (7.8) without assuming the uniform Fréchet subdifferentiability condition from Hypothesis 4.2.

However, in order to pass to the limit in (7.8) as $\tau \downarrow 0$, we will exploit in a key way a closedness property of the graph of $\partial_u^- \mathcal{E}$ that has the same “pointwise-in-time” character as (4.12). Likewise, it will be exploited in combination with a Young measure argument. Nonetheless, as we are no longer requiring $\mathbf{U} \Subset \mathbf{V}$ like in Lemma 4.7, boundedness of the energies no longer turns the weak convergence in $\mathbf{V}$ into strong. Hence, the sequence $(u_n)_n$ featuring in (4.25a) below is supposed to be weakly converging in $\mathbf{V}$, only. As for Hyp. 4.8, we will also set apart the case in which, in addition, energy convergence is guaranteed.

Hypothesis 4.11 (Closedness of $\partial_u^- \mathcal{E}$, Continuity of $\mathcal{E}$)**.** *For all sequences $(t_n, u_n, z_n)_n \subset [0, T] \times \mathrm{D_u} \times \mathrm{D_z}$ and $(\xi_n)_n \subset \mathbf{V}^*$*

$$\left. \begin{cases} t_n \to t, \ u_n \rightharpoonup u \text{ in } \mathbf{V}, \ z_n \rightharpoonup z \text{ in } \mathbf{Z}, \ \sup_n \mathcal{G}(u_n, z_n) \leq C, \\ \xi_n \rightharpoonup \xi \text{ in } \mathbf{V}^*, \\ \xi_n \in \partial_u^- \mathcal{E}(t_n, u_n, z_n) \end{cases} \right\} \Rightarrow \ \xi \in \partial_u^- \mathcal{E}(t, u, z). \tag{4.25a}$$

We will also consider the following enhanced closedness/continuity condition:

$$\begin{gathered} \text{for all sequences } (t_n, u_n, z_n)_n \text{ and } (\xi_n)_n \text{ in the conditions of (4.25a) there holds} \\ \xi \in \partial_u^- \mathcal{E}(t, u, z) \text{ and } \lim_{n \to \infty} \mathcal{E}(t_n, u_n, z_n) = \mathcal{E}(t, u, z). \end{gathered} \tag{4.25b}$$

Observe that, since $\mathcal{E}$ is lower semicontinuous, it is sufficient to have $\limsup_{n\to\infty}\mathcal{E}(t_n,u_n,z_n)\leq\mathcal{E}(t,u,z)$ for (4.25b) to hold.

Our second existence result. The proof of Theorem 2 will be developed in Sec. 7.1.

Theorem 2 (Weak energetic solutions for $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, $\mathcal{V}$ general)**.** *Assume Hypotheses 2.2, 2.3, 2.5, and 2.7. In addition, suppose that $\mathcal{V}$ complies with* (4.22)*, and that $\partial_u^-\mathcal{E}$ complies with* (4.25a) *from Hyp. 4.11. Let $(\overline{u}_\tau,\underline{u}_\tau,u_\tau,\tilde{u}_\tau,\overline{z}_\tau,\underline{z}_\tau,z_\tau)_\tau$ be a family of approximate solutions constructed via the time-discretization scheme in Problem 4.10, starting from data $u_0\in\mathrm{D}_{\mathsf{u}}$ and $z_0\in\mathrm{D}_{\mathsf{z}}$.*

Then, for every sequence $\tau_k\downarrow 0$ as $k\to\infty$ there exist a (not relabeled) subsequence, and functions (u,z) as in (3.1) *such that convergences* (4.16) *hold and the pair (u,z) is a* weak *energetic solution to the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, fulfilling the initial conditions $u(0)=u_0$ and $z(0)=z_0$.*

Furthermore, if the closedness condition holds in the strongest form (4.25b)*, then (u,z) fulfills* (4.21) *and is thus an* enhanced *weak energetic solution.*

As observed in Remark 4.9, if the closedness condition holds in the form (4.25b), then it is sufficient to adopt the *weaker* variant of Hypothesis 2.5 introduced in Remark 2.6: this will be clear from the proof.

Finally, the following result is a consequence of Prop. 3.5 and of Theorem 3.6.

Corollary 4.12. *Under the assumptions of Theorem 2, if in addition $\mathcal{E}$ complies with* (3.15)*, then (u,z) from Thm. 2 is an* energetic *solution to the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$.*

4.3. **Examples.** In what follows, we illustrate Hypotheses 4.2, 4.6, 4.8, and 4.11 on a prototypical class of energy functionals. Since the latter conditions essentially concern the dependence of $\mathcal{E}$ on the variable u, to make the discussion more transparent we will consider a very simple (yet significant) dependence on the variable z, postponing to Section 6 more complex examples.

Admissible energy functionals. Following the discussion in [MRS13a, Sec. 5], we consider energies $\mathcal{E}:[0,T]\times L^2(\Omega)\times L^1(\Omega)\to(-\infty,+\infty]$ of the form

$$\mathcal{E}(t,u,z):=\begin{cases}\int_\Omega g(z)\beta(\nabla u)+W(u)\,\mathrm{d}x-\langle f(t),u\rangle_{W^{1,p}(\Omega)} & \text{if } u\in\mathrm{D}\\ +\infty & \text{if } u\in L^2(\Omega)\setminus\mathrm{D}\end{cases}\tag{4.26}$$

for all $t\in[0,T]$ and $z\in L^1(\Omega)$, where $\Omega\subset\mathbb{R}^d$ is a bounded Lipschitz domain, the nonlinear functions g, β, and W fulfill

$$g\in\mathrm{BC}(\mathbb{R})\text{ and }\exists\,c_0>0\;\forall\,z\in\mathbb{R}\,:\;g(z)\geq c_0;\tag{4.27a}$$

$$\beta:\mathbb{R}^d\to[0,+\infty)\text{ is differentiable, convex and}\tag{4.27b}$$

$$\exists\,p>1,\;\exists\,c_1,\,c_2,\,c_3>0\;\forall\,A\in\mathbb{R}^d\,:\quad c_1|A|^p-c_2\leq\beta(A)\leq c_3(|A|^p+1);\tag{4.27c}$$

$$W:\mathbb{R}\to(-\infty,+\infty]\text{ is bounded from below and }\lambda\text{-convex for some }\lambda\in\mathbb{R};\tag{4.27d}$$

(in (4.27a) BC stands for *bounded and continuous*), while $f\in\mathrm{C}^1([0,T];W^{1,p}(\Omega)^*)$. To fix ideas we consider zero Dirichlet boundary conditions and thus set

$$\mathrm{D}:=\{u\in W_0^{1,p}(\Omega)\,:\;W(u)\in L^1(\Omega)\}=\mathrm{D}_{\mathsf{u}}.\tag{4.28}$$

It can be easily verified that $\partial_t\mathcal{E}(t,u,z)=-\langle f(t),u\rangle_{W^{1,p}(\Omega)}$ complies with (2.5c). Taking into account that W is bounded from below and exploiting the Poincaré inequality it is also immediate to check that $\mathcal{E}$ is itself bounded from below, and the sublevels of the functional $\mathcal{G}(u):=\sup_{t\in[0,T]}\mathcal{E}(t,u)$ are bounded in $\mathbf{U}:=W^{1,p}(\Omega)$. Thus, the choice of the ambient space $\mathbf{V}$ determines restrictions on p, namely

$$\text{if }\mathbf{V}=L^2(\Omega),\text{ we have }\mathbf{U}\Subset\mathbf{V}\text{ for all }p>2d/(2+d);\tag{4.29a}$$

$$\text{if }\mathbf{V}=H^1(\Omega),\text{ we have }\mathbf{U}\subset\mathbf{V}\text{ for all }p\geq 2.\tag{4.29b}$$

We now discuss the uniform Fréchet subdifferentiability condition in Hypothesis 4.2. In fact, the energy in Example 4.13 below also complies with the Fréchet subdifferentiability condition in the forthcoming Hyp. 5.2.

Example 4.13. *Since W is λ-convex (we may suppose $\lambda < 0$ without loss of generality), we have for every u, $v \in \mathrm{D}$ and $\theta \in [0,1]$*

$$\mathcal{E}(t,(1-\theta)u+\theta v,z) \leq (1-\theta)\mathcal{E}(t,u,z)+\theta\mathcal{E}(t,v,z)-\frac{\lambda}{2}\theta(1-\theta)\|v-u\|^2_{L^2(\Omega)}. \tag{4.30}$$

Hence, (4.7) *(and, a fortiori,* (4.6)*) holds.*

Note that in the special case $\beta(\nabla u) = \frac{1}{2}|\nabla u|^2$, arguing as in [MRS13a, Ex. 5.1]*, we can improve* (4.30) *by the 1-convexity of β. Hence, in place of $-\frac{\lambda}{2}\theta(1-\theta)\|v-u\|^2_{L^2(\Omega)}$, on the r.h.s. of* (4.30) *we find*

$$-\frac{1}{2}\theta(1-\theta)(\|\nabla(v-u)\|^2_{L^2(\Omega)}+\lambda\|v-u\|^2_{L^2(\Omega)}) \leq \frac{1}{2}\theta(1-\theta)((-1-\rho\lambda)\|\nabla(v-u)\|^2_{L^2(\Omega)}-\lambda C_\rho\|v-u\|^2_{L^1(\Omega)})$$

thanks to the Gagliardo-Nirenberg inequality $\|w\|_{L^2(\Omega)} \leq C_{\mathrm{GN}}\|w\|^{2/(d+2)}_{L^1(\Omega)}\|\nabla w\|^{d/(d+2)}_{L^2(\Omega)} + \|w\|_{L^1(\Omega)}$ and to Young's inequality, with $\rho > 0$ arbitrary and $C_\rho > 0$ accordingly determined. Therefore, choosing $\rho \leq 1/|\lambda|$, we conclude (4.7) *(and, a fortiori,* (4.6)*) in a stronger form (cf. Remark 4.3), with the remainder term involving the square norm of $\mathbf{Y} := L^1(\Omega)$, and $\Lambda_E = \frac{1}{2}C_\rho|\lambda|$.*

Let us now turn to Hypothesis 4.6, which is assumed if $\mathbf{U} \Subset \mathbf{V}$. If $\mathbf{V} = L^2(\Omega)$ and $\mathbf{Z} = L^1(\Omega)$, for every $\tilde{u} \in \mathrm{D}_{\mathsf{u}} \subset W^{1,p}(\Omega)$, it is sufficient to take the recovery sequence $\tilde{u}_n \equiv \tilde{u}$. Then, for $t_n \to t$ and $z_n \to z$ in $L^1(\Omega)$ we have $g(z_n) \to g(z)$ in $L^q(\Omega)$ for all $q \in [1,\infty)$, and by dominated convergence $\int_\Omega g(z_n)\beta(\nabla\tilde{u})\,\mathrm{d}x \to \int_\Omega g(z)\beta(\nabla\tilde{u})\,\mathrm{d}x$. By the continuity of f, we have $\mathcal{E}(t_n,\tilde{u},z_n) \to \mathcal{E}(t,\tilde{u},z)$ and (4.11) ensues.

In the following example, for simplicity we particularize the discussion of Hypothesis 4.8 to the cases $\beta(A) = \frac{1}{p}|A|^p$ and $W(u) = (u^2-1)^2/4$, but it could be extended to a uniformly convex β with p-growth, cf. condition (6.4) in Sec. 6.1 ahead, and to a λ-convex potential W complying with suitable growth conditions. Since Hyp. 4.8 comes into play in the case the compact embedding $\mathbf{U} \Subset \mathbf{V}$ is no longer required, we will focus on the case $\mathbf{V} = H^1(\Omega)$ and $\mathbf{U} \subset \mathbf{V}$ continuously, but not compactly.

Example 4.14. *We take $\mathbf{V} = H^1(\Omega)$ and, for $p \geq 2$ (cf.* (4.29b)*)*

$$\mathcal{E}(t,u,z) = \int_\Omega \left(\frac{g(z)}{p}|\nabla u|^p + \tfrac{1}{4}(u^2-1)^2\right)\mathrm{d}x - \langle f(t),u\rangle_{W^{1,p}(\Omega)} \quad \text{if } u \in \mathrm{D}, \text{ cf. (4.28)},$$

and $+\infty$ otherwise. We omit other contributions to $\mathcal{E}$ that would only depend on the variable z as they do not affect the Fréchet subdifferential of $\mathcal{E}$ w.r.t. u; nonetheless, in order to discuss Hyp. 4.8 let us implicitly assume that such contributions contain a regularizing gradient term and that, therefore,

$$\mathbf{X} = W^{1,q}(\Omega) \quad \text{for some } q > 1, \quad \text{while} \quad \mathbf{Z} = L^1(\Omega). \tag{4.31}$$

It can be checked that, for $(t,u,z) \in [0,T]\times\mathrm{D}\times L^1(\Omega)$

$$\begin{aligned}\partial_u^-\mathcal{E}(t,u,z) \neq \emptyset \quad &\Leftrightarrow \quad -\mathrm{div}(g(z)|\nabla u|^{p-2}\nabla u)+u^3-u-f(t) \in H^1(\Omega)^* \\ \text{and in that case} \quad &\partial_u^-\mathcal{E}(t,u,z) = \{-\mathrm{div}(g(z)|\nabla u|^{p-2}\nabla u)+u^3-u-f(t)\}.\end{aligned}$$

Observe that, for a sequence $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;H^1(\Omega))$, whence $u_n \to u$ strongly in $\mathrm{C}^0([0,T];W^{1-\epsilon,p}(\Omega))$ for all $\epsilon \in (0,1)$, there holds

$$u_n^3 - u_n \to u^3 - u \qquad \text{in } L^2((0,T)\times\Omega). \tag{4.32}$$

Furthermore, let $(\mathsf{t}_n)_n$ a sequence in the conditions of (4.14)*. From $f \in \mathrm{C}^1([0,T];W^{1,p}(\Omega)^*)$ it follows that*

$$f(\mathsf{t}_n) \to f(\mathsf{t}) \quad \text{in } L^\infty(0,T;W^{1,p}(\Omega)^*). \tag{4.33}$$

Finally, for a sequence $(z_n)_n$ converging to z as indicated in (4.14)*, by the compact embedding $W^{1,q}(\Omega) \Subset L^q(\Omega)$ we have $z_n(t) \to z(t)$ in $L^q(\Omega)$ for all $t \in [0,T]$ and hence, since $g \in \mathrm{BC}(\mathbb{R})$, we have $g(z_n) \to g(z)$ in $L^\sigma((0,T)\times\Omega)$ for all $1 \leq \sigma < \infty$.*

Now, let us consider the sequence $\xi_n = -\mathrm{div}(g(z_n)|\nabla u_n|^{p-2}\nabla u_n) + u_n^3 - u_n - f(\mathsf{t}_n)$, weakly converging to some ξ in $L^2(0,T;\mathbf{V}^*)$. In view of (4.32) *and* (4.33)*, $-\mathrm{div}(g(z_n)|\nabla u_n|^{p-2}\nabla u_n)$ weakly converges in $L^2(0,T;W^{1,p}(\Omega)^*)$ to $\zeta := \xi - W'(u) + f(\mathsf{t})$. We now aim to show that $\zeta = -\mathrm{div}(g(z)|\nabla u|^{p-2}\nabla u)$. For this, we use the condition $\limsup_{n\to\infty}\int_0^T \langle\xi_n,u_n\rangle_{\mathbf{V}}\,\mathrm{d}t \leq \int_0^T \langle\xi,u\rangle_{\mathbf{V}}\,\mathrm{d}t$ which, in view of* (4.32) *and of* (4.33)*, in particular implies that*

$$\limsup_{n\to\infty}\int_0^T \langle -\mathrm{div}(g(z_n)|\nabla u_n|^{p-2}\nabla u_n),u_n\rangle_{W^{1,p}(\Omega)}\,\mathrm{d}t \leq \int_0^T \langle\zeta,u\rangle_{W^{1,p}(\Omega)}\,\mathrm{d}t. \tag{4.34}$$

Taking into account that the functionals $\mathcal{F}_n(u) := \int_0^T \int_\Omega \frac{g(z_n)}{p} |\nabla u|^p \, \mathrm{d}x \, \mathrm{d}t$ Mosco*-converge (cf.* [Att84]*) in* $L^p(0, TW^{1,p}(\Omega)$ *to* $\mathcal{F}(u) := \int_0^T \int_\Omega \frac{g(z)}{p} |\nabla u|^p \, \mathrm{d}x \, \mathrm{d}t$*, thanks to the* $\limsup$ *inequality in* (4.34) *it is sufficient to conclude that* $\zeta = -\mathrm{div}(g(z)|\nabla u|^{p-2}\nabla u)$ *almost everywhere in* $(0,T)$*. Hence,* $\xi = -\mathrm{div}(g(z)|\nabla u|^{p-2}\nabla u) + u^3 - u - f(\mathsf{t})$ *a.e. in* $(0,T)$*, which concludes the proof of* (4.14)*.*

Furthermore, from (4.34) *we also infer that* $\int_0^T \int_\Omega \frac{g(z_n)}{p} |\nabla u_n|^p \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \frac{g(z)}{p} |\nabla u|^p \, \mathrm{d}x \, \mathrm{d}t$*, whence the convergence of the energies in* (4.15)*.*

Finally, we illustrate the closedness Hypothesis 4.11 with the functional $\mathcal{E}$ from Example 4.14, but with a *quadratic* leading term $\int_\Omega \frac{g(z)}{2} |\nabla u|^2 \, \mathrm{d}x$.

Example 4.15. *We take* $\mathbf{X}$ *and* $\mathbf{Z}$ *as in* (4.31)*,* $\mathbf{V} = L^2(\Omega)$*, and* $\mathcal{E} : [0,T] \times L^2(\Omega) \times L^1(\Omega) \to (-\infty, +\infty]$*,*

$$\mathcal{E}(t,u,z) = \int_\Omega \left(\frac{g(z)}{2} |\nabla u|^2 + \tfrac{1}{4}(u^2-1)^2 \right) \mathrm{d}x - \ \langle f(t), u \rangle_{W^{1,2}(\Omega)} \quad \text{if } u \in \mathrm{D}, \text{ cf. (4.28)},$$

and $+\infty$ *otherwise. It is not difficult to check that the Fréchet subdifferential of* $u \mapsto \mathcal{E}(t,u,z)$ *w.r.t. the* $L^2(\Omega)$*-topology has domain*

$$\begin{aligned} \mathrm{dom}(\partial_u^- \mathcal{E}) = \{(t,u,z) \in [0,T] \times H^1(\Omega) \times L^1(\Omega) \ : \ -\mathrm{div}(g(z)\nabla u) + u^3 - u - f(t) \in L^2(\Omega)\} \text{ and} \\ \partial_u^- \mathcal{E}(t,u,z) = \{-\mathrm{div}(g(z)\nabla u) + u^3 - u - f(t)\} \text{ for all } (t,u,z) \in \mathrm{dom}(\partial_u^- \mathcal{E}). \end{aligned} \tag{4.35}$$

Let $u_n \rightharpoonup u$ *in* $L^2(\Omega)$ *with* $\sup_n \mathcal{G}(u_n, z_n) \le C$*. Since* $\mathcal{E}(t,u,z) \ge c \int_\Omega |\nabla u|^2 \, \mathrm{d}x - C$ *for some* c*,* $C > 0$*, we conclude* $u_n \rightharpoonup u$ *in* $H^1(\Omega)$*, hence* $u_n \to u$ *in* $L^{2^\star - \epsilon}(\Omega)$ *for all* $\epsilon \in (0, 2^\star - 1]$*. Thus,* $u_n^3 - u_n \rightharpoonup u^3 - u$ *in* $L^2(\Omega)$*. By continuity of* f*,* $t_n \to t$ *implies* $f(t_n) \to f(t)$ *in* $L^2(\Omega)$*. From* $\xi_n \rightharpoonup \xi$ *in* $L^2(\Omega)$ *we infer that* $(-\mathrm{div}(g(z_n)\nabla u_n))_n$ *is bounded in* $L^2(\Omega)$*, hence it weakly converges to some* ζ *in* $L^2(\Omega)$*. With the same arguments as in Example 4.14 we find* $\zeta = -\mathrm{div}(g(z)\nabla u)$*, hence* $\xi = -\Delta u + W'(u) - f(t)$*, and* $\int_\Omega \frac{g(z_n)}{2} |\nabla u_n|^2 \, \mathrm{d}x \to \int_\Omega \frac{g(z)}{2} |\nabla u|^2 \, \mathrm{d}x$*, whence* $u_n \to u$ *in* $H^1(\Omega)$ *strongly. In fact the stronger property* (4.25b) *ensues.*

Observe that in the above example, the *linear* character of the higher order term $-\Delta u$ contributing to the Fréchet subdifferential $\partial_u^- \mathcal{E}$ has played a major role.

5. Existence results for the evolutionary system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$

In this section we address the case with inertia, i.e. we allow for $\varrho \neq 0$, and investigate the weak solvability of the evolutionary system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, supplemented with the initial conditions

$$u(0) = u_0, \quad u'(0) = v_0, \quad z(0) = z_0 \tag{5.1}$$

with $(u_0, z_0) \in \mathrm{D}_\mathsf{u} \times \mathrm{D}_\mathsf{z}$ and $v_0 \in \mathbf{W}$. Combining the assumptions and techniques from Sections 4.1 and 4.2, we will obtain the existence of weak energetic solutions for a general convex $\mathcal{V}$ with superlinear growth, cf. Thm. 3, and of energetic solutions in the case of a quadratic dissipation potential $\mathcal{V}$, cf. Thm. 4.

Time discretization. We construct our discrete solutions via the following scheme

Problem 5.1. *Let* $(u_\tau^0, z_\tau^0) := (u_0, z_0) \in \mathrm{D}_\mathsf{u} \times \mathrm{D}_\mathsf{z}$ *and set*

$$u_\tau^{-1} := u_0 - \tau v_0, \tag{5.2}$$

and for $n = 1, \ldots, N_\tau$*, find*

$$z_\tau^n \in \operatorname*{Argmin}_{z \in \mathbf{Z}} \left(\tau \mathcal{R}\left(\frac{z - z_\tau^{n-1}}{\tau} \right) + \mathcal{E}(t_n, u_\tau^{n-1}, z) \right), \tag{5.3a}$$

$$u_\tau^n \in \operatorname*{Argmin}_{u \in \mathbf{V}} \left(\frac{\varrho}{\tau^2} \| u - 2u_\tau^{n-1} + u_\tau^{n-2} \|_{\mathbf{W}}^2 + \tau \mathcal{V}\left(\frac{u - u_\tau^{n-1}}{\tau} \right) + \mathcal{E}(t_n, u, z_\tau^n) \right). \tag{5.3b}$$

Observe that Euler-Lagrange equation for the minimum problem (5.3b) now reads (taking into account that $\mathbf{W} = \mathbf{W}^* \subset \mathbf{V}^*$, cf. the proof of Prop. 7.6 for more details)

$$\varrho \frac{u_\tau^n - 2u_\tau^{n-1} + u_\tau^{n-2}}{\tau^2} + \partial \mathcal{V}\left(\frac{u - u_\tau^{n-1}}{\tau} \right) + \partial_u^- \mathcal{E}(t_n, u_\tau^n, z_\tau^n) \ni 0 \qquad \text{in } \mathbf{V}^*. \tag{5.4}$$

We will denote by v_τ the piecewise linear interpolant of the values $((u_\tau^n - u_\tau^{n-1})/\tau)_{n=1}^{N_\tau}$, namely

$$v_\tau(t) = \frac{t-t_{n-1}}{\tau}\frac{u_\tau^n - u_\tau^{n-1}}{\tau} + \frac{t_n - t}{\tau}\frac{u_\tau^{n-1} - u_\tau^{n-2}}{\tau} \qquad \text{for } t \in (t_{n-1}, t_n]. \tag{5.5}$$

Therefore, $v_\tau'(t) = \frac{u_\tau^n - u_\tau^{n-1}}{\tau} - \frac{u_\tau^{n-1} - u_\tau^{n-2}}{\tau} = \frac{u_\tau^n - 2u_\tau^{n-1} + u_\tau^{n-2}}{\tau}$ for all $t \in (t_{n-1}, t_n]$, and (5.4) rephrases as

$$\varrho v_\tau'(t) + \partial \mathcal{V}(u_\tau'(t)) + \overline{\xi}_\tau(t) \ni 0 \qquad \text{with } \overline{\xi}_\tau(t) \in \partial_u^- \mathcal{E}(\overline{\mathfrak{t}}_\tau(t), \overline{u}_\tau(t), \overline{z}_\tau(t)) \qquad \text{for a.a. } t \in (0,T). \tag{5.6}$$

We now introduce and motivate the two enhanced conditions on the Fréchet subdifferential of $\mathcal{E}$ that we will have to impose in our existence results for $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$.

Enhanced Fréchet subdifferentiability and subgradient estimate. In order to recover the discrete version of the energy-dissipation inequality (3.4), we shall have to resort to a slightly stronger version of the uniform Fréchet subdifferentiability property from Hypothesis 4.2, in which we require estimate (4.6) to hold with a constant independent of the energy sublevel, and where we replace $\|\cdot\|_{\mathbf{V}}^2$ by $\|\cdot\|_{\mathbf{W}} \mathcal{V}(\cdot)^{1/2}$. Observe that, since $\mathbf{V} \subset \mathbf{W}$ continuously, we are in fact strengthening (4.6), cf. Remark 4.3.

Hypothesis 5.2 (Enhanced Fréchet subdifferentiability)**.** *There exists a constant* $\Lambda > 0$ *such that*

$$\begin{aligned} &\forall\, t \in [0,T], \quad \forall\, (u,z),\, (v,z) \in \mathrm{D_u} \times \mathrm{D_z} \quad \forall\, \xi \in \partial_u^- \mathcal{E}(t,u,z)\ : \\ &\mathcal{E}(t,v,z) - \mathcal{E}(t,u,z) \geq \langle \xi, v-u \rangle_{\mathbf{V}} - \Lambda \|v-u\|_{\mathbf{W}} \mathcal{V}(v-u)^{1/2} \end{aligned} \tag{5.7}$$

In addition, observe that in (5.6) the inertial term makes it necessary to estimate the term $\overline{\xi}_\tau \in \partial_u^- \mathcal{E}(\cdot, \overline{u}_\tau, \overline{z}_\tau)$ independently, since no comparison estimate is possible. In this direction, the following condition requires that the $\mathbf{V}^*$-norm of the elements in the Fréchet subdifferential $\partial_u^- \mathcal{E}$ to be estimated by the energy itself. As we will see later on, this condition e.g. holds for the energy functional in a model for adhesive contact in visco-elastic materials. This will also determine the spatial regularity for u'', with u the energetic solution to the evolutionary system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ arising in the time-continuous limit of the time-discrete scheme (5.3).

Hypothesis 5.3 (Subgradient estimate)**.** *There exist constants* C_3, C_4, $C_5 > 0$ *and* $\sigma \in [1, \infty)$ *such that*

$$\|\xi\|_{\mathbf{V}^*}^\sigma \leq C_3 \mathcal{E}(t,u,z) + C_4 \|u\|_{\mathbf{V}} + C_5 \qquad \textit{for all } \xi \in \partial_u^- \mathcal{E}(t,u,z) \textit{ and all } (t,u,z) \in [0,T] \times \mathrm{D_u} \times \mathrm{D_z}. \tag{5.8}$$

We postpone to the end of this section a discussion of Hypothesis 5.3 in the context of two examples.

An existence result for general $\mathcal{V}$. We now give our first result for the evolutionary system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, stating the existence of *weak* energetic solutions. In addition to the basic conditions from Section 2 and to Hypotheses 5.2 & 5.3, we will also need to enforce a suitable closedness condition of $\partial_u^- \mathcal{E}$ in order to pass to the time-continuous limit and identify the elements in the Fréchet subdifferential $\partial_u^- \mathcal{E}$. This will be done by

- either directly requiring Hypothesis 4.11;
- or by imposing $\mathbf{U} \Subset \mathbf{V}$. Then, the desired closedness property for $\partial_u^- \mathcal{E}$ will derive from combining Hypotheses 4.6 and the uniform Fréchet subdifferentiability 5.2. Indeed, the proof of Lemma 4.7 goes through in this setting as well, guaranteeing the closedness (4.12).

Theorem 3 (Weak energetic solutions to $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, $\mathcal{V}$ general)**.** *Assume Hypotheses 2.2, 2.3, 2.5, 2.7 and, in addition, Hypotheses 5.2 and 5.3. Moreover, assume*

- *either Hypothesis 4.11,*
- *or,* $\mathbf{U} \Subset \mathbf{V}$ *with Hypothesis 4.6.*

Then, for every sequence $\tau_k \downarrow 0$ *as* $k \to \infty$ *there exist a (not relabeled) subsequence, and functions* (u,z) *as in* (3.1) *such that, in addition,* $u \in W^{2,1}(0,T;\mathbf{V}*)$*, and convergences* (4.16) *hold. Moreover,*

$$u_{\tau_k}' \overset{*}{\rightharpoonup} u' \qquad \textit{in } L^\infty(0,T;\mathbf{W}), \tag{5.9}$$

$$v_{\tau_k}' \rightharpoonup u'' \qquad \textit{in } L^1(0,T;\mathbf{V}^*), \tag{5.10}$$

and the pair (u,z) *is a* weak *energetic solution to the evolutionary system* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$*, fulfilling the Cauchy conditions* (5.1).

Furthermore, if Hypothesis 4.11 holds with the additional continuity condition (4.25b)*, or under Hypothesis 4.6 in the case* $\mathbf{U} \Subset \mathbf{V}$*, then* (u, z) *comply with the energy-dissipation inequality*

$$\begin{aligned}&\frac{\varrho}{2}\|u'(t)\|_{\mathbf{W}}^2 + \int_s^t \mathcal{V}(u'(r)) + \mathcal{V}^*(-\xi(r) - \varrho u''(r))\,\mathrm{d}r + \mathrm{Var}_{\mathcal{R}}(z, [s,t]) + \mathcal{E}(t, u(t), z(t)) \\ &\leq \frac{\varrho}{2}\|u'(s)\|_{\mathbf{W}}^2 + \mathcal{E}(s, u(s), z(s)) + \int_s^t \partial_t \mathcal{E}(r, u(r), z(r))\,\mathrm{d}r \qquad \text{for all } t \in (0,T] \text{ for } s = 0, \text{ and a.a. } s \in (0,t),\end{aligned} \tag{5.11}$$

with $\xi(t)$ *a selection in* $\partial_u^- \mathcal{E}(t, u(t), z(t))$*, and thus* (u, z) *is an enhanced weak energetic solution.*

We then have the analogue of Corollary 4.12.

Corollary 5.4. *Under the assumptions of Theorem 3, if in addition* $\mathcal{E}$ *complies with the smoothness properties* (3.15)*, then* (u, z) *from Thm. 3 is an energetic solution.*

An existence result for $\mathcal{V}$ quadratic. In the case of a *quadratic* dissipation potential $\mathcal{V}$, we will be able to show the existence of *energetic* solutions to $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ in the special setting that

$$\mathbf{V} \Subset \mathbf{W}. \tag{5.12}$$

This will guarantee sufficient compactness information for (the sequence approximating) u' to allow for the limit passage in the (discrete) viscous equation. Clearly, in view of the compact embedding $\mathbf{U} \Subset \mathbf{W}$ from (2.1b), (5.12) is for example fulfilled in the case in which $\mathbf{V} = \mathbf{U}$, cf. Example 2.8. Therefore, to get the closedness of $\partial_u^- \mathcal{E}$ we will resort to the closedness condition à la Minty from Hypothesis 4.8.

Theorem 4 (Energetic solutions for $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, $\mathcal{V}$ quadratic)**.** *Assume Hypotheses 2.2, 2.3, 2.5, 2.7, and let* $\mathcal{V}$ *be quadratic, cf.* (4.1)*. In addition, assume Hypotheses 4.8, 5.2, 5.3, and* (5.12)*.*

Then, for every sequence $\tau_k \downarrow 0$ *as* $k \to \infty$ *there exist a (not relabeled) subsequence, and functions* (u, z) *as in* (3.1) *such that, in addition,* $u \in H^1(0,T;\mathbf{V}) \cap H^2(0,T;\mathbf{V}^*)$ *and convergences* (4.16) *hold, as well as* (5.9)*. Moreover,*

$$v'_{\tau_k} \rightharpoonup u'' \quad \text{in } L^2(0,T;\mathbf{V}^*), \qquad u'_{\tau_k} \to u' \quad \text{in } L^p(0,T;\mathbf{W}) \quad \text{for all } 1 \leq p < \infty \tag{5.13}$$

and the pair (u, z) *is an* energetic *solution to the evolutionary system* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$*, fulfilling* (5.1)*.*

Finally, if Hyp. 4.8 holds in the stronger form (4.15)*, then the energy-dissipation inequality holds in the form* (5.11)*, with* $\xi(t)$ *a selection in* $\partial_u^- \mathcal{E}(t, u(t), z(t))$ *also fulfilling*

$$\varrho u''(t) + \partial\mathcal{V}(u'(t)) + \xi(t) = 0 \qquad \text{for a.a. } t \in (0,T), \tag{5.14}$$

and (u, z) *is thus an* enhanced *energetic solution.*

The *proofs* of Theorems 3 and 4 will be developed in Sec. 7.3. As in the previous sections, we will see that, in both cases, under the conditions yielding (5.11), it is sufficient to resort to the weaker variant of the mutual recovery sequence condition from Hypothesis 2.5, cf. Rmk. 2.6.

Remark 5.5. In Remark 7.7 we will check that for $\mathcal{V}$ quadratic it would indeed be possible to work under a *weaker* version of the Fréchet subdifferentiability from Hypothesis 5.2, namely

$$\begin{aligned}&\forall\, t \in [0,T],\ \forall\,(u,z),\ (v,z) \in \mathrm{D}_{\mathsf{u}} \times \mathrm{D}_{\mathsf{z}} \quad \forall\, \xi \in \partial_u^- \mathcal{E}(t,u,z) : \\ &\mathcal{E}(t,v,z) - \mathcal{E}(t,u,z) \geq \langle \xi, v-u \rangle_{\mathbf{V}} - \Lambda\|v-u\|_{\mathbf{V}}^2,\end{aligned} \tag{5.15}$$

i.e. the former (4.6), but with a constant *independent* of the energy sublevel.

Remark 5.6. In the case of $\mathcal{V}$ quadratic, it would be possible to weaken the subgradient estimate in Hyp. 5.3 by replacing $\|\xi\|_{\mathbf{V}^*}$ with $\|\xi\|_{\mathbf{U}^*}$, with ξ an element in the Fréchet subdifferential $\partial_u^{-,\mathbf{U}} \mathcal{E}(t,u,z)$ of $\mathcal{E}$ in the $\mathbf{U} - \mathbf{U}^*$ duality. Accordingly, taking into account that for every $v \in \mathbf{V}$, $\partial\mathcal{V}(v) \subset \mathbf{V}^* \subset \mathbf{U}^*$, (3.2) could be formulated in a weaker way as a subdifferential inclusion in $\mathbf{U}^*$, and one would have $u \in H^2(0,T;\mathbf{U}^*)$.

The proof of this extension follows the same lines as the argument for Thm. 4, with suitable modifications such as the closedness argument to be formulated in the $\mathbf{U} - \mathbf{U}^*$ duality.

Remark 5.7 (Comparison with the results by E. Emmrich & coworkers)**.** In the papers [ET11, EŠ11, EŠ13], E. Emmrich and coauthors have obtained a series of deep results on the existence of solutions (whose basics were already established in the seminal paper [LS65]) and on the full discretization (in time *and* space) for this class of second order nonlinear evolution equations

$$u'' + Au' + Bu = f \qquad \text{in } V^*, \text{ a.e. in } (0,T). \tag{5.16}$$

Here, $V = V_A \cap V_B$ and V_A, V_B are the separable reflexive Banach spaces on which the two operators A and B are defined. The authors neither suppose $V_A \subset V_B$ (continuously), nor do they require $V_B \subset V_A$, thus allowing for a wide class of applications, although (5.16) is not coupled to the rate-independent evolution of an additional internal variable z.

In [ET11, EŠ11] the following assumptions are made on the *time-dependent* operators A and B: It is required that $A : V_A \to V_A^*$ has a main part d-monotone, hemicontinuous, (cf. e.g. [Rou05, Chap. 2, Definitions 2.1, 2.3]) and satisfying a certain growth condition, that is perturbed by a non-monotone but locally Hölder continuous operator, with suitable growth. The main part of the operator $B : V_B \to V_B^*$ is linear, symmetric, and strongly positive, and it is also perturbed by a locally Hölder continuous operator, with suitable growth. The analysis in [ET11, EŠ11] can thus be compared to ours in the (particular) case in which the energy functional as a function of u is quadratic up to a lower order perturbation. Observe that, while we restrict to the subdifferential of a possibly *nonsmooth, but convex* dissipation potential $\mathcal{V}$, the operator A in (5.16) could be the differential of a convex smooth potential with a nonconvex, but lower order, perturbation.

Recently, in [EŠ13], the authors instead consider the case in which B is the Gâteaux differential of an energy bounded from below, with superlinear growth at infinity in the sense of (2.11a). It is assumed that B maps bounded sets of V_B into bounded sets of V_B^* and that it is demicontinuous. However, in this case the analysis is restricted to the case in which A is the linear operator associated with a bilinear form inducing an inner product on V_A and a norm $\|\cdot\|_A$ equivalent to $\|\cdot\|_{V_A}$. This corresponds to our choice of a *quadratic* dissipation potential $\mathcal{V}$. It has to be noted that in [EŠ13] the operators A and B are required to satisfy a condition of *Andrews-Ball* type, namely that there exists $\lambda \geq 0$ such that the operator $(B + \lambda A) : V \to V^*$ is monotone:

$$\exists\, \lambda \geq 0 \qquad \forall\, v, w \in V \qquad \langle Bv - Bw, v - w\rangle_V \geq -\lambda \|v - w\|_A^2. \tag{5.17}$$

This exactly corresponds to the uniform Fréchet subdifferentiability condition (5.15).

Examples for the subgradient estimate in Hypothesis 5.3. It is not difficult to verify that Hypothesis 5.3 is, e.g., satisfied by the energy in Example (4.15), with the space $\mathcal{V} = H^1(\Omega)$; observe that, in the latter space the closedness from Hyp. 4.11 is also fulfilled. The following example has a structure similar to Ex. 4.15, but it features a *nonsmooth* double-well term W, with a kink at $|u| = 2$. Again, to make calculations more transparent we neglect the z-dependence.

Example 5.8. *Let us consider* $W_2 : \mathbb{R} \to \mathbb{R}$

$$W_2(u) := \begin{cases} a(1-u^2)^2 & \text{if } |u| \leq 2, \\ b(1-u^2)^2 + 9(a-b) & \text{if } |u| > 2, \end{cases} \qquad \text{with constants } 0 < a < b\,. \tag{5.18}$$

It is a Λ*-convex function for any* $\Lambda \geq b$*, i.e., for all* $u, \tilde{u} \in \mathbb{R}^d$*, for all* $\theta \in [0,1]$

$$W_2(\theta u + (1-\theta)\tilde{u}) \leq \theta W_2(u) + (1-\theta) W_2(\tilde{u}) + \frac{\Lambda}{2}\theta(1-\theta)|u - \tilde{u}|^2\,. \tag{5.19}$$

Its Fréchet subdifferential is given by

$$\partial^- W_2(u) = \begin{cases} \{-4au(1-u^2)\} & \text{if } |u| < 2, \\ \big\{ -4\alpha u(1-u^2),\, \alpha \in [a,b]\big\} & \text{if } |u| = 2, \\ \{-4bu(1-u^2)\} & \text{if } |u| > 2. \end{cases}$$

and every $\xi \in \partial^- W_2(u)$ *satisfies*

$$|\xi|^{4/3} \leq (4\alpha(|u| + |u|^3))^{4/3} \leq 48\alpha|u|^4 \leq \tfrac{48^{4/3}\alpha^{1/3}}{\min\{a,1\}}(W_2(u) + 9(b-a) + 2bu^2)\,,$$

which leads to subdifferential estimate (5.8) *for* $\mathbf{V} = L^4(\Omega)$ *defined on a bounded Lipschitz domain* $\Omega \subset \mathbb{R}^d$*. Let* $\mathbf{W} = L^2(\Omega)$ *and let* $\mathcal{W}_2 : \mathbf{U} \to [0, +\infty)$*,* $\mathcal{W}_2(u) := \int_\Omega W_2(u)\,\mathrm{d}x$*. Hence, by convexity inequality* (5.19) *and the continuous embedding* $\|w\|_{\mathbf{W}} \leq C_{\mathbf{UW}}\|w\|_{\mathbf{U}}$ *we obtain the enhanced* λ*-convexity*

$$\mathcal{W}_2(\theta u + (1-\theta)\tilde{u}) \leq \theta \mathcal{W}_2(u) + (1-\theta)\mathcal{W}_2(\tilde{u}) + \frac{\Lambda}{2} C_{\mathbf{UW}} \theta(1-\theta)\|u - \tilde{u}\|_{\mathbf{W}} \|u - \tilde{u}\|_{\mathbf{V}}^{1/2}\,, \tag{5.20}$$

which, in turn, implies the enhanced Fréchet subdifferentiability (5.7) *upon choosing* $\mathcal{V}(u) := \frac{1}{4}\int_\Omega |u|^4\,\mathrm{d}x$.

Observe that the corresponding energy augmented by a Dirichlet integral, i.e. $\mathcal{E}(t,u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + W_2(u)\,\mathrm{d}x$ *complies, in the context of the space* $\mathbf{V} = H^1_0(\Omega)$, *with Hypotheses 5.2 and 5.3 and, in addition, with Hypotheses 2.2, 2.7, and with the closedness à la Minty from Hyp. 4.8.*

We conclude with the example of an energy functional modeling rate-independent *adhesive contact* between two visco-elastic bodies with inertia.

Example 5.9 (Adhesive contact between visco-elastic materials with inertia)**.** *We consider two visco-elastic bodies* Ω_+ *and* Ω_-, *bonded along a prescribed contact surface* Γ_C, *during a timespan* $[0,T]$. *The adhesiveness of the bonding is modeled by the* delamination variable $z : (0,T)\times\Gamma_\mathrm{C} \to [0,1]$, *describing the fraction of the effective molecular links between* Ω_+ *and* Ω_-, *so that* $z(t,x) = 1$ *(*$z(t,x) = 0$*, respectively), means fully intact (broken) bonding at time* $t \in (0,T)$ *and at the material point* $x \in \Gamma_\mathrm{C}$. *We postulate* a rate-independent *evolution for* z. *The other state variable is the displacement field* $\mathbf{u} : (0,T)\times\Omega \to \mathbb{R}^d$ *(with* $\Omega := \Omega_-\cup\Omega_+$*), subject to viscosity and inertia. We denote by* $[\![\mathbf{u}]\!]$ *its jump across* Γ_C.

Following [RR11], *we model this phenomenon by the energy*

$$\begin{aligned}\mathcal{E}(t,\mathbf{u},z) := &\int_{\Omega\setminus\Gamma_\mathrm{C}} \tfrac{1}{2}\varepsilon(\mathbf{u}) : \mathbb{C} : \varepsilon(\mathbf{u})\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} \kappa z \left|[\![\mathbf{u}]\!]\right|^2 \mathrm{d}\mathscr{H}^{d-1}(x) \\ &+ \int_{\Gamma_\mathrm{C}} \left(J([\![u]\!]) + I_{[0,1]}(z) - a_0 z\right) \mathrm{d}\mathscr{H}^{d-1}(x) - \langle \mathbf{f}(t), \mathbf{u}\rangle_{H^1(\Omega;\mathbb{R}^d)},\end{aligned} \tag{5.21}$$

where $\mathscr{H}^{d-1}$ *is the* $(d-1)$*-dimensional Hausdorff measure,* $\mathbb{C} \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ *the symmetric, positive definite elasticity tensor,* κ *a fixed positive constant modulating the adhesive contact contribution,* $J : \mathbb{R}^d \to [0,+\infty]$ *a convex functional,* $a_0 \geq 0$ *the specific energy stored by delamination, and* $\mathbf{f} \in W^{1,1}(0,T;H^1(\Omega\setminus\Gamma_\mathrm{C};\mathbb{R}^d)^*)$ *a given external loading incorporating volume and surface forces acting on the Neumann part* Γ_N *of the boundary* $\partial\Omega$ *(whereas on the Dirichlet part* Γ_D *null displacement is imposed). The dissipation potentials are*

$$\mathcal{V}(\mathbf{v}) := \int_{\Omega\setminus\Gamma_\mathrm{C}} \tfrac{1}{2}\varepsilon(\mathbf{v}) : \mathbb{D} : \varepsilon(\mathbf{v})\,\mathrm{d}x, \qquad \mathcal{R}(\zeta) := \int_{\Gamma_\mathrm{C}} \mathrm{R}(\zeta)\,\mathrm{d}\mathscr{H}^{d-1}(x)$$

where $\mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ *is the symmetric, positive definite viscosity tensor. The density of the* 1*-positively homogeneous potential* $\mathcal{R}$ *is given by* $\mathrm{R}(\zeta) = -a_1\zeta + I_{(-\infty,0]}(\zeta)$, *with* $a_1 \geq 0$ *the specific energy dissipated by delamination. Inertial effects are encompassed in the model through the kinetic energy* $\mathcal{K}(\mathbf{w}) := \int_\Omega \frac{\varrho}{2}|\mathbf{w}|^2\,\mathrm{d}x$.

The existence of energetic solutions to the evolutionary system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, *where the ambient spaces are, naturally,* $\mathbf{V} = H^1_{\Gamma_\mathrm{D}}(\Omega\setminus\Gamma_\mathrm{D};\mathbb{R}^d)$, $\mathbf{W} = L^2(\Omega;\mathbb{R}^d)$, *and* $\mathbf{Z} = L^1(\Gamma_\mathrm{C})$, *stems from the analysis in* [RR11], *also encompassing the coupling with thermal processes. It is indeed possible to show that the evolutionary system* $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *in fact complies with all of the assumptions of Theorem 4. Here we will just confine the discussion to Hypothesis 5.3, highlighting a significant point in the analysis of contact systems with inertia.*

The Fréchet subdifferential $\partial^-_{\mathbf{u}}\mathcal{E}$ *is given by* $\partial^-_{\mathbf{u}}\mathcal{E}(t,\mathbf{u},z) = \{\mathrm{D}_u\mathcal{E}(t,\mathbf{u},z)\}$, *with*

$$\langle \mathrm{D}_{\mathbf{u}}\mathcal{E}(t,\mathbf{u},z),\mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^d)} := \int_{\Omega\setminus\Gamma_\mathrm{C}} \varepsilon(\mathbf{u}) : \mathbb{C} : \varepsilon(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} \left(\kappa z [\![\mathbf{u}]\!]\,[\![\mathbf{v}]\!] + \zeta [\![\mathbf{v}]\!]\right) \mathrm{d}\mathscr{H}^{d-1}(x) - \langle \mathbf{f}(t),\mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^d)}$$

for all $\mathbf{v} \in H^1(\Omega;\mathbb{R}^d)$, *where* $\zeta \in L^2(\Gamma_\mathrm{C})$ *fulfills* $\zeta(x) \in \partial J([\![\mathbf{u}(x)]\!])$ *for a.a.* $x \in \Gamma_\mathrm{C}$. *Now, while the* $H^1(\Omega;\mathbb{R}^d)^*$*-norm of all the other contributions to* $\mathrm{D}_{\mathbf{u}}\mathcal{E}(t,\mathbf{u},z)$ *is controlled by* $\mathcal{E}$ *in the sense of* (5.8), *for the validity of Hyp. 5.3 the convex analysis subdifferential* $\partial J : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ *cannot be an unbounded operator. Therefore, the choice* $J([\![\mathbf{u}]\!]) := I_{[0,+\infty)}([\![\mathbf{u}]\!]\cdot\nu)$ *(*ν *denoting the unit normal to* Γ_C*, oriented from* Ω_+ *to* Ω_-*), is not admissible. Observe that such a choice would lead to a flow rule for the delamination variable encompassing the* Signorini contact *conditions, and in fact the existence of solutions to the adhesive contact system with Signorini conditions and inertia has been an open problem in the literature on contact problems for a long time, cf.* [RR11, Rmk. 5.3]. *Instead, one can for instance take for* J *the Yosida approximation* $J_\epsilon(\mathbf{u}) := -\frac{1}{\epsilon}([\![\mathbf{u}]\!]\cdot\nu)^-$, *with* $\epsilon > 0$, $(\cdot)^-$ *denoting the negative part. This corresponds to the so-called* normal compliance conditions.

6. Applications

In this section we collect some examples of mechanical systems with a mixed rate-dependent/rate-independent character, to which our existence Theorems 1–4 apply.

6.1. **Energetic solutions to a class of rate-independent processes in visco-elastic solids.** Here we address a class of rate-independent systems in a visco-elastic body $\Omega \subset \mathbb{R}^d$, also subject to inertia, in the frame of the theory of *generalized standard solids*, which was first analyzed in [Rou09]. The variable governed by viscosity is the displacement $\mathbf{u}$ (in a small-strain regime, $\varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$ denoting the small strain tensor), while the internal variable z, in general taking values in $\mathbb{R}^m$, describes some rate-independent process such as, e.g., plasticity with hardening, damage, or phase transformations in a solid.

Energetics. The energy has the form

$$\mathcal{E}(t, \mathbf{u}, z) := \int_\Omega \varphi(\varepsilon(\mathbf{u}), z, \nabla z)\,\mathrm{d}x - \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x \tag{6.1}$$

where $\mathbf{f}$ is a given external force and the energy density $\varphi : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to [0, +\infty)$ is generally assumed smooth, although in [Rou09, Rmk. 3.3] it is hinted, without further details, that an extension to nonsmooth energies could be possible.

The "viscous" dissipation potential $\mathcal{V}$ is quadratic and has the form

$$\mathcal{V}(\mathbf{v}) := \int_\Omega \tfrac{1}{2}\varepsilon(\mathbf{v}) : \mathbb{D} : \varepsilon(\mathbf{v}) \quad \text{with } \mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}} \text{ positive definite,} \tag{6.2}$$

while the "rate-independent" potential $\mathcal{R}$ is

$$\begin{aligned} \mathcal{R}(\zeta) = \int_\Omega \mathrm{R}(\zeta)\,\mathrm{d}x \quad &\text{with } \mathrm{R} : \mathbb{R}^m \to [0, +\infty] \text{ convex, positively homogeneous of degree 1, and} \\ &\exists\, c > 0 \ \ \forall\, \zeta \in \mathbb{R}^m : \quad \mathrm{R}(\zeta) \geq c|\zeta|, \end{aligned} \tag{6.3}$$

whereas the kinetic energy is $\mathcal{K}(\mathbf{v}) = \frac{\varrho}{2}\int_\Omega |\mathbf{v}|^2\,\mathrm{d}x$ with $\varrho \geq 0$. Therefore, our own Hypothesis 2.3 is satisfied with the spaces $\mathbf{V} = H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d)$, with Γ_{D} a portion of the boundary $\partial\Omega$ on which zero Dirichlet boundary conditions are imposed, and $\mathbf{Z} = L^1(\Omega; \mathbb{R}^m)$, while $\mathbf{W} = L^2(\Omega; \mathbb{R}^d)$.

Properties of the energy. Concerning the energy density φ, Roubíček provides classes of sufficient conditions ensuring the existence of *energetic* solutions. In fact, in [Rou09] other weak solvability notions are proposed and related existence results are obtained. Nonetheless, in the next lines we will only focus on the energetic concept and highlight the relations between Roubíček's assumptions on φ for the existence result [Rou09, Prop. 5.3], and the abstract conditions for our own existence Thms. 1 and 4 (for $\mathcal{V}$ quadratic).

First of all, it is required (cf. [Rou09, (12a)]) that $\varphi(e, z, Z) \geq \tilde{c}(|e|^p + |Z|^q)$ for some $\tilde{c} > 0$ and $p,\, q > 1$. This sets the spaces $\mathbf{U}$ and $\mathbf{X}$ in which the coercivity Hypothesis 2.7 is verified, namely

$$\mathbf{U} = W^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d) \cap H^1_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^d) \quad \text{and} \quad \mathbf{X} = W^{1,q}(\Omega; \mathbb{R}^m).$$

It is not difficult to check that Hyp. 2.2 is also fulfilled.

The other basic condition (cf. [Rou09, (16)]) is that there exists $\ell \geq 0$ such that the mapping $(e, z, Z) \mapsto \varphi(e, z, Z) + \ell|e|^2$ is convex, which guarantees the validity of our uniform Fréchet differentiability Hyps. 4.2 & 5.2.

In [Rou09, Prop. 5.3] Roubíček also proposes three sets of conditions (cf. (66a)/(66b)/(66c) therein) that he alternatively uses to verify the mutual recovery sequence condition from Hyp. 2.5. More in detail,

(a) either it is assumed ([Rou09, (66a)]) that R is continuous (and in particular takes values in $[0, +\infty)$), in combination with the condition that $\partial_e\varphi$ is *p-strongly monotone*, i.e.

$$\begin{aligned} &\exists\, \alpha > 0 \ \ \forall\, (z, Z) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}\ \forall\, e,\, \tilde{e} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \\ &\alpha(|\tilde{e}|^{p-2}\tilde{e} - |e|^{p-2}e) : (\tilde{e} - e) \leq (\partial_e\varphi(\tilde{e}, z, Z) - \partial_e\varphi(e, z, Z)). \end{aligned} \tag{6.4}$$

Observe that (6.4) implies our Hyp. 4.8, whereas the combination of (6.4) and continuity of R allows for the choice of the constant recovery sequence $\tilde{z}_n = \tilde{z}$ in Hyp. 2.5;

(b) or, (cf. [Rou09, (66b)]), in addition to (6.4) suitable growth properties of $(z, Z) \mapsto \varphi(z, Z)$ and of $(z, Z) \mapsto \partial_{(z,Z)}\varphi(z, Z)$ are required. This again ensures Hyp. 4.8, while the recovery sequence $(\tilde{z}_n)_n$ for Hyp. 2.5 is constructed via the so-called *binomial trick*;

(c) or it is supposed, cf. [Rou09, (66c)]), that $\varphi(e, z, Z) = \varphi_1(e, z, Z) + \varphi_2(z, Z)$ with $(e, Z) \mapsto \varphi_1(e, z, Z)$ affine and fulfilling suitable growth conditions, and φ_2 quadratic. This guarantees Hyps. 2.5 and 4.8.

Existence of energetic solutions. All in all, we have verified that the class of material models in [Rou09] complies, in the case $\varrho = 0$, with all the conditions in Theorem 1. Thus our own result applies, yielding the existence of *energetic* solutions.

In the case $\varrho > 0$, ROUBÍČEK additionally assumes a growth condition for $\partial_e \varphi$ (cf. [Rou09, (21)]) that is akin to our own Hypothesis 5.3, although slightly weaker. Indeed, it leads to a weaker formulation of the momentum equation than ours (5.14) (cf. Thm. 4), in which the inertial term is integrated by parts in time.

6.2. **Rate-independent damage in visco-elastic solids.** In this section we highlight some possible extensions, which are included in our analytical results but not yet covered by the results by ROUBÍČEK, at the example of partial damage in visco-elastic solids at small strains, located in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, observed over the finite timespan $[0,T]$. Here, the state variables under consideration are the displacement field $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}^d$, with $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ the small strain tensor, and the damage variable $z : [0,T] \times \Omega \to [0,1]$. In particular, $z(t,x) = 1$ corresponds to the case that the material is fully intact at the point $(t,x) \in [0,T] \times \Omega$, while $z(t,x) = 0$ refers to maximal damage at (t,x). As common in the modeling of rate-independent damage without healing, the 1-homogeneous dissipation potential is

$$\mathcal{R}(\zeta) := \int_\Omega \mathrm{R}(\zeta)\,\mathrm{d}x\,, \quad \mathrm{R}(\zeta) := \begin{cases} |\zeta| & \text{if } \zeta \le 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{6.5}$$

The energy functional $\mathcal{E} : [0,T] \times \mathbf{V} \times \mathbf{Z} \to (-\infty, +\infty]$ has the following structure

$$\mathcal{E}(t,u,z) := \begin{cases} \int_\Omega g(z) W(\varepsilon(\mathbf{u}) + \varepsilon(\mathbf{u}_\mathrm{D}(t))) - \mathbf{f}(t)\cdot\mathbf{u}\,\mathrm{d}x + \mathcal{G}(z) & \text{if } (\mathbf{u},z) \in \mathbf{U} \times \mathbf{X}, \\ +\infty & \text{otherwise.} \end{cases} \tag{6.6}$$

Here, $\mathbf{f} \in \mathrm{C}^1([0,T]; \mathbf{V}^*)$ is a given external loading and $\mathbf{u}_\mathrm{D} \in \mathrm{C}^1([0,T]; \mathbf{U})$ the extension of a Dirichlet datum, prescribed on the Dirichlet boundary $\Gamma_\mathrm{D} = \partial\Omega$, into the domain Ω; note that we have set $\Gamma_\mathrm{D} = \partial\Omega$, only for the shortness of the exposition. The energy contributions $g(\cdot)W(\cdot)$ and $\mathcal{G}(\cdot)$, encompassing a gradient term in z, will have to be chosen such that the topological setup (2.1) and (2.4) is satisfied. Here we want to point out that the latter condition includes the case of (the non-reflexive space) $\mathbf{X} = \mathrm{BV}(\Omega)$, which allows in particular in the example of damage that

$$\mathcal{G} : \mathbf{X} \to [0,+\infty], \quad \mathcal{G}(z) := |\mathrm{D}z|(\Omega) + \int_\Omega I_{[0,1]}(z) \tag{6.7}$$

is composed of the total variation $|\mathrm{D}z|(\Omega)$ of z in Ω and the indicator function of the interval $[0,1]$ with $I_{[0,1]}(z) = 0$ if $z \in [0,1]$ and $I_{[0,1]}(z) = +\infty$ otherwise. This is our choice of $\mathcal{G}$ throughout this section. In particular, by [AFP05, Rem. 3.5, Thm. 3.23], it is ensured that $\mathcal{G}(z_n) \le C$ for a sequence $(z_n)_n$ implies that there exists z with $\mathcal{G}(z) \le C$ and a subsequence such that $z_n \to z$ in $L^1(\Omega)$, and that $\mathcal{G}$ is lower semicontinuous with respect to strong convergence in $L^1(\Omega)$. Moreover, to ensure that the energy of the system decreases with increasing damage, which is a typical feature of material damage, the function g is chosen as follows:

$$g \in \mathrm{C}^0([0,1]) \text{ monotonically increasing with } \quad 0 < g_0 < g(0) < g(1) < g_1 < +\infty\,; \tag{6.8}$$

the fact that $0 < g_0 < g(0)$ features partial damage as this constraint ensures the coercivity of $\mathcal{E}(t,\cdot,0)$ given that W is chosen suitably. More precisely, for the density W we assume

$$\text{Convexity:} \quad W \in \mathrm{C}^0(\mathbb{R}^{d\times d}, \mathbb{R}^+) \text{ is convex,} \tag{6.9a}$$

$$\text{Coercivity:} \quad \exists\, C > 0, p \in (1,\infty),\ \forall e \in \mathbb{R}^{d\times d} : \ W(e) \ge C|e|^p\,, \tag{6.9b}$$

$$\text{Stress control:} \quad \exists\, C > 0, p' = p/(p-1),\ \forall e \in \mathbb{R}^{d\times d} : \ |\partial_e^{\sup} W(e)|^{p'} \le C(W(e)+1)\,, \tag{6.9c}$$

where $\partial_e W : \mathbb{R}^{d\times d} \rightrightarrows \mathbb{R}^{d\times d}$ is the convex analysis subdifferential, and $|\partial_e^{\sup} W(e)| := \sup\{|\xi| : \xi \in \partial_e W(e)\}$. As for the viscous dissipation $\mathcal{V} : \mathbf{V} \to [0,+\infty)$ we stay general for the moment and only assume that

$$\mathbf{U} \subset \mathbf{V} \text{ continuously, and } \mathcal{V} \text{ complies with Hyp. 2.3.} \tag{6.10}$$

In addition, we are going to consider the case of inertia with a kinetic energy given by

$$\mathcal{K}(\mathbf{u}') := \frac{\varrho}{2}\|\mathbf{u}'\|^2_{\mathbf{W}}, \text{ where } \mathbf{W} := L^2(\Omega, \mathbb{R}^d)\,. \tag{6.11}$$

We are going to use this example to emphasize how the growth assumptions on W affect the choice of $\mathcal{V}$, and which of the existence theorems is applicable in the different cases.

In view of (6.5), (6.7), and (6.9) we have

$$\mathbf{U} := \{v \in W^{1,p}(\Omega, \mathbb{R}^d),\ v = 0 \text{ on } \Gamma_{\rm D}\}\,, \quad \mathbf{X} := \mathrm{BV}(\Omega) \cap L^\infty(\Omega)\,, \quad \mathbf{Z} := L^1(\Omega)\,. \tag{6.12}$$

We now summarize the basic properties of $\mathcal{E}$ and of the corresponding system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$.

Proposition 6.1 (Properties of $\mathcal{E}$ and $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$)**.** *Let the evolutionary system* $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$ *be given by* (6.5)–(6.12)*. Then, the following statements hold:*

(1) The energy functional $\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \to (-\infty, +\infty]$ *has bounded sublevels in* $\mathbf{U} \times \mathbf{X}$*, i.e. also* (2.18) *from Hyp. 2.7 holds true.*

(2) $\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \to (-\infty, +\infty]$ *is bounded from below and lower semicontinuous w.r.t. weak convergence in* $\mathbf{V}$ *for* $\mathbf{u}$ *and strong convergence in* $L^q(\Omega)$*, for any* $q \in [1, \infty)$*, for* z*, whence* (2.5) *in Hyp.* (2.2)*.*

(3) The mutual recovery sequence condition from Hyp. 2.5 holds true.

Proof. **Ad** (1)**:** The boundedness of the energy sublevels ensues on the one hand directly from the definition of $\mathcal{G}$ and on the other hand from coercivity estimate (6.9b) together with Korn's inequality and the bounds on the given data, since

$$\begin{aligned}\mathcal{E}(t, \mathbf{u}, z) &\geq \int_\Omega g_0 C|\varepsilon(\mathbf{u}{+}\mathbf{u}_{\rm D})|^p \,\mathrm{d}x - \sup_t \|\mathbf{f}(t)\|_{\mathbf{V}^*}\|\mathbf{u}\|_{\mathbf{V}} \\ &\geq g_0 C 2^{1-p} C_{\rm K}\|\mathbf{u}\|^p_{\mathbf{U}} - aC\|\mathbf{u}_{\rm D}(t)\|^p_{\mathbf{U}} - C_{\mathbf{VU}} \sup_t \|\mathbf{f}(t)\|_{\mathbf{V}^*}\|\mathbf{u}\|_{\mathbf{U}} \\ &\geq g_0 C 2^{-p} C_{\rm K}\|\mathbf{u}\|^p_{\mathbf{U}} - \tilde{C} = c\|\mathbf{u}\|^p_{\mathbf{U}} - \tilde{C}\end{aligned} \tag{6.13}$$

In the last line we have used Young's inequality with exponent p and the factor $\epsilon = (g_0 C 2^{-p} C_{\rm K} p)^{1/p}$.

Ad (2)**:** The boundedness from below is a direct consequence from the above deduced estimate together with the fact that $\mathcal{G}(z) \geq 0$ for all $z \in \mathbf{Z}$. The lower semicontinuity of $\mathcal{E}(t, \cdot, \cdot)$ w.r.t. weak convergence in $\mathbf{V}$ and strong convergence in $L^q(\Omega)$ is ensured by [AFP05, Rem. 3.5, Thm. 3.23], assumption (6.9a), and lower semicontinuity results such as e.g. [FL07, Thm. 7.5].

Ad (3)**:** For the construction of the mutual recovery sequence we refer to [Tho13]. For given $\hat{z} \in \mathbf{X}$ with $\hat{z} \leq z$ a.e. in Ω this construction is such that $0 \leq \hat{z}_k \leq z_k$ a.e. in Ω, $\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$, $\limsup_{k\to\infty} \big(\mathcal{G}(\hat{z}_k) - \mathcal{G}(z_k)\big) \leq \mathcal{G}(\hat{z}) - \mathcal{G}(z)$. Hence, also $\mathcal{R}(\hat{z}_k - z_k) \to \mathcal{R}(\hat{z} - z)$. In order to find that $\limsup_{k\to\infty} \int_\Omega (g(\hat{z}_k) - g(z_k)) W(\varepsilon(\mathbf{u}_k + \mathbf{u}_{\rm D}(t))\,\mathrm{d}x \leq \int_\Omega (g(\hat{z}) - g(z)) W(\varepsilon(\mathbf{u}{+}\mathbf{u}_{\rm D}(t))\,\mathrm{d}x$ we observe that the expression is non-positive for all $k \in \mathbb{N}$. We now define the functional $\tilde{\mathcal{E}} : \mathbf{Z} \times \mathbf{Z} \times \mathbf{V} \to [0, +\infty]$, $\tilde{\mathcal{E}}(z, \hat{z}, \mathbf{u}) := \int_\Omega (g(z) - g(\hat{z}) W(\varepsilon(\mathbf{u}{+}\mathbf{u}_{\rm D})) + I_{[0,+\infty)}(z - \hat{z})\,\mathrm{d}x$ and observe that $\tilde{\mathcal{E}}$ is lower semicontinuous with respect to strong convergence in $L^q(\Omega)$ and weak convergence in $\mathbf{V}$. Hence, we obtain the desired estimate by lower semicontinuity. □

In order to establish the whole bunch of assumptions of the abstract existence results, it remains to discuss the conditions on $\partial_u^- \mathcal{E}(t, \cdot, z)$ and $\partial_t \mathcal{E}(t, \mathbf{u}, z)$. For this, we note that $u \mapsto \mathcal{E}(t, u, z)$ is convex, hence $\partial_u^- \mathcal{E}(t, u, z)$ coincides with the convex analysis subdifferential $\partial_u \mathcal{E}(t, u, z)$ and will thus be denoted by $\partial_u \mathcal{E}(t, u, z)$ hereafter. By the sum rule, any element $\xi \in \partial_u \mathcal{E}(t, u, z) \subset \mathbf{V}^*$ is represented by

$$\begin{aligned}&\xi = -\operatorname{div} g(z)\tilde{\xi} - \mathbf{f}(t) \quad \text{with } \tilde{\xi} \in \partial_e W(\varepsilon(\mathbf{u}{+}\mathbf{u}_{\rm D}(t))) \quad \text{for all } (t, u, z) \in \mathrm{dom}(\partial_u \mathcal{E}), \text{ where} \\ &\mathrm{dom}(\partial_u \mathcal{E}) = \{(t, u, z) \in [0, T] \times \mathbf{V} \times \mathbf{Z},\ -\operatorname{div} g(z)\tilde{\xi} - \mathbf{f}(t) \in \mathbf{V}^* \ \&\ \tilde{\xi} \in \partial_e W(\varepsilon(\mathbf{u}{+}\mathbf{u}_{\rm D}(t)))\}\,.\end{aligned} \tag{6.14}$$

Therefore, in view of the definition of the convex analysis subdifferential, we have that the Fréchet subdifferentiability conditions are satisfied, cf. , Hyp. 4.2 for $\varrho = 0$ and Hyp. 5.2 for $\varrho > 0$. Moreover, due to (6.14) and the regularity of $\mathbf{u}_{\rm D}$ we find that the power of the energy is given by

$$\begin{aligned}\partial_t \mathcal{E}(t, \mathbf{u}, z) = \langle -\operatorname{div} g(z)\tilde{\xi}, \mathbf{u}_{\rm D}'(t)\rangle_{\mathbf{V}} - \langle \mathbf{f}'(t), \mathbf{u}\rangle_{\mathbf{V}} &= \int_\Omega g(z)\tilde{\xi} : \varepsilon(\mathbf{u}_{\rm D}'(t))\,\mathrm{d}x - \langle \mathbf{f}'(t), \mathbf{u}\rangle_{\mathbf{V}} \\ &\text{for all } \tilde{\xi} \in \partial_e W(\varepsilon(\mathbf{u}{+}\mathbf{u}_{\rm D}(t)))\,,\end{aligned} \tag{6.15}$$

where the above equivalence is also due to $\Gamma_{\rm D} = \partial\Omega$. The integral $\int_\Omega g(z)\tilde{\xi} : \varepsilon(\mathbf{u}_{\rm D}'(t))\,\mathrm{d}x$ is well defined for all $\tilde{\xi} \in \partial_e W(\varepsilon(\mathbf{u}{+}\mathbf{u}_{\rm D}(t)))$, thanks to the stress control condition (6.9c) and the regularity of $\mathbf{u}_{\rm D}$. It thus remains to verify the closedness of the subdifferential $\partial_u \mathcal{E}(t, \cdot, z)$, and this will also help us deduce the properties of $\partial_t \mathcal{E}(t, \mathbf{u}, z)$ stated in condition (2.5c). Moreover, if $\varrho > 0$, also the subgradient estimate proposed in Hyp. 5.3 has to be checked. To do so, we are going to distinguish between the case that $\mathcal{V}$ has general superlinear growth and the case that $\mathcal{V}$ is quadratic. The latter case e.g. covers Kelvin-Voigt rheology, where $\mathcal{V}$ is given by

(6.2), and hence $\mathbf{V} = H^1_0(\Omega, \mathbb{R}^d)$ so that, for $\mathbf{U} \subset \mathbf{V}$ continuously, $p \geq 2$ is necessary. One may also consider a weak damping of the form $\mathcal{V}(\mathbf{u}') := \frac{1}{2}\int_\Omega c|\mathbf{u}'|^2\,\mathrm{d}x$, i.e. here $\mathbf{V} = L^2(\Omega, \mathbb{R}^d)$ and $\mathbf{U} \Subset \mathbf{V}$ for any $p > 2d/(d+2)$. In the case that $\mathcal{V}$ is of general superlinear growth we have in mind that e.g. $\mathcal{V}(\mathbf{u}') := \|\varepsilon(\mathbf{u}')\|^q_{L^r(\Omega,\mathbb{R}^{d\times d})}$ with $1 < r \leq p$ and $q > 1$, such that $\mathbf{V} = W^{1,r}(\Omega, \mathbb{R}^d)$ and $\mathbf{U} \subset \mathbf{V}$ continuously. But also a weak damping can be handled of the form $\mathcal{V}(\mathbf{u}') := \|\mathbf{u}'\|^q_{L^r(\Omega,\mathbb{R}^d)}$ with $p > rd/(d+r)$ and $r, q > 1$, such that $\mathbf{V} = L^r(\Omega, \mathbb{R}^d)$ and $\mathbf{U} \Subset \mathbf{V}$. Moreover we will treat gradient systems ($\varrho = 0$) and evolutionary systems ($\varrho > 0$) separately.

The case $\varrho = 0$ and $\mathcal{V}$ quadratic. In this setting (which already falls outside the scope of the results in [Rou09]), as outlined in (4.1), the viscous dissipation potential $\mathcal{V}(v) := \frac{1}{2}a(v, v)$ is defined via a continuous and coercive bilinear form $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$. Now, the assumptions of Theorem 1 have to be checked. In view of Prop. 6.1, condition (2.5c) on $\partial_t\mathcal{E}(t, \mathbf{u}, z)$ and the closedness of $\partial_u\mathcal{E}$ have to be established. According to Section 4.1, this can be done by verifying the recovery sequence Hypothesis 4.6 if $\mathbf{U} \Subset \mathbf{V}$, whereas, if only $\mathbf{U} \subset \mathbf{V}$ continuously, the closedness is to be verified via the closedness-argument à la Minty from Hyp. 4.8.

Proposition 6.2 (Properties of $\partial_u\mathcal{E}(t, \cdot, z)$ and $\partial_t\mathcal{E}(t, \mathbf{u}, z)$)**.** *Let the assumptions of Prop. 6.1 hold true and assume that $\mathcal{V} : \mathbf{V} \to [0, +\infty)$ is quadratic.*

(1) Let $\mathbf{V} \Subset \mathbf{U}$. Then $\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \to (-\infty, +\infty]$ satisfies the recovery sequence Hypothesis 4.6.
(2) Let $\mathbf{U} \subset \mathbf{V}$ continuously. Then $\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \to (-\infty, +\infty]$ satisfies the closedness à la Minty from Hypothesis 4.8.
(3) For any $(t, \mathbf{u}, z) \in [0, T] \times \mathbf{V} \times \mathbf{Z}$ with $\mathcal{E}(t, u, z) \leq C$ the power of the energy $\partial_t\mathcal{E}$ fulfills condition (2.5c)*.*

Proof. **Ad** (1)**:** For any given $\tilde{\mathbf{u}} \in \mathbf{V}$, the constant sequence $\tilde{\mathbf{u}}_k = \tilde{\mathbf{u}}$ serves as a recovery sequence in Hyp. 4.6.

Ad (2)**:** Consider a sequence $(\mathsf{t}_k, \mathbf{u}_k, z_k)_k \subset [0, T] \times \mathbf{V} \times \mathbf{Z}$ complying with the assumptions in (4.14), i.e. $\mathsf{t}_k \to \mathsf{t}$ pointwise a.e. in $(0, T)$, $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $L^\infty(0, T; \mathbf{U}) \cap H^1(0, T; \mathbf{V})$, $z_k \overset{*}{\rightharpoonup} z$ in $L^\infty(0, T; L^q(\Omega))$ and $z_k(t) \to z(t)$ in $L^q(\Omega)$ for any $1 \leq q < \infty$ and all $t \in [0, T]$, $\xi_k \rightharpoonup \xi$ in $L^2(0, T; \mathbf{V}^*)$ with $\xi_k \in \partial_u\mathcal{E}(\mathsf{t}_k(t), \mathbf{u}_k(t), z_k(t))$ for a.e. $t \in (0, T)$, and $\limsup_k \int_0^T \langle \xi_k, \mathbf{u}_k\rangle_\mathbf{V} \leq \langle \xi, \mathbf{u}\rangle_\mathbf{V}$. We have to verify that also $\xi(t) \in \partial_u\mathcal{E}(\mathsf{t}(t), \mathbf{u}(t), z(t))$ for a.e. $t \in (0, T)$. For this, we may repeat the very same arguments as in Example 4.14.

Ad (3)**:** We first establish the power control. For this, we note that $|\partial_t\mathcal{E}(t, \mathbf{u}, z)| \leq |\langle -\operatorname{div} g(z)\tilde{\xi}, \mathbf{u}'_\mathrm{D}(t)\rangle_\mathbf{V}| + |\langle \mathbf{f}'(t), \mathbf{u}\rangle_\mathbf{V}|$ and we handle each of the two terms separately. For the term involving $\mathbf{f}$ we see that

$$|\langle \mathbf{f}'(t), \mathbf{u}\rangle_\mathbf{V}| \leq \|\mathbf{f}'(t)\|_{\mathbf{V}^*}\|\mathbf{u}\|_\mathbf{V} \leq C_\mathbf{f} c^{-1}(\mathcal{E}(t, \mathbf{u}, z) + \tilde{C}) \tag{6.16}$$

due to the regularity of $\mathbf{f}$ and the fact that the energy sublevels are uniformly bounded by Prop. 6.1(1), more precisely, we used (6.13). For the term involving $\mathbf{u}_\mathrm{D}$ we find

$$\begin{aligned}
|\langle -\operatorname{div} g(z)\tilde{\xi}, \mathbf{u}'_\mathrm{D}(t)\rangle_\mathbf{V}| &\leq \int_\Omega |g(z)\partial_e W(\varepsilon(\mathbf{u}) + \varepsilon(\mathbf{u}_\mathrm{D}(t))) : \varepsilon(\mathbf{u}'_\mathrm{D}(t))|\,\mathrm{d}x \\
&\overset{(1)}{\leq} \|g(z)\partial_e W(\varepsilon(\mathbf{u}) + \varepsilon(\mathbf{u}_\mathrm{D}(t)))\|_{L^{p'}(\Omega,\mathbb{R}^{d\times d})}\|\varepsilon(\mathbf{u}'_\mathrm{D}(t))\|_{L^p(\Omega,\mathbb{R}^{d\times d})} \\
&\overset{(2)}{\leq} C_\mathrm{D} g_1^{1/p}\|g(z)W(\varepsilon(\mathbf{u}) + \varepsilon(\mathbf{u}_\mathrm{D}(t)))\|^{1/p'}_{L^1(\Omega,\mathbb{R}^{d\times d})} \\
&\leq C_\mathrm{D} g_1^{1/p}\big(\|g(z)W(\varepsilon(\mathbf{u}) + \varepsilon(\mathbf{u}_\mathrm{D}(t)))\|_{L^1(\Omega,\mathbb{R}^{d\times d})} + 1\big)^{1/p'} - \langle \mathbf{f}(t), \mathbf{u}\rangle_\mathbf{V} + \sup_{t\in[0,T]}\|\mathbf{f}(t)\|_{\mathbf{V}^*}\|\mathbf{u}\|_\mathbf{V} \\
&\leq C_\mathrm{D} g_1^{1/p}\mathcal{E}(t, \mathbf{u}, z) + C_\mathrm{D} g_1^{1/p} + C_\mathbf{f} c^{-1}(\mathcal{E}(t, \mathbf{u}, z) + \tilde{C})\,.
\end{aligned} \tag{6.17}$$

For (1), we have applied Hölder's inequality, while for (2) we have used that $g_0 < g(z) < g_1$, hence $g(z)^{p'} < g_1^{p'-1}g(z)$, and $(p'-1)/p' = 1/p$. Moreover, we have introduced $C_\mathrm{D} > \sup_{t\in[0,T]}\|e(\mathbf{u}'_\mathrm{D}(t))\|_{L^p(\Omega,\mathbb{R}^{d\times d})}$. In order to re-establish $\mathcal{E}(t, \mathbf{u}, z)$, we have eliminated the power $1/p' < 1$ by adding 1 under the root, we have added the work of the external loadings and compensated its possible non-positivity by the corresponding continuity estimate. Finally, as for (6.16), we have used $C_\mathbf{f}$ as the uniform bound on $\mathbf{f}$ and the bound on $\mathbf{u}$ by (6.13).

Secondly, we convince ourselves that $\limsup_k \partial_t\mathcal{E}(t_k, \mathbf{u}_k, z_k) \leq \partial_t\mathcal{E}(t, \mathbf{u}, z)$ if $\mathcal{E}(t_k, \mathbf{u}_k, z_k) \leq C$ for all $k \in \mathbb{N}$, $t_k \to t$, $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $\mathbf{V}$, and $z_k \to z$ in $L^1(\Omega)$. In fact, choose a sequence $\tilde{\xi}_k \in \partial_e W(\varepsilon(\mathbf{u}_k + \mathbf{u}_\mathrm{D}(t_k)))$ such that $\partial_t\mathcal{E}(t_k, \mathbf{u}_k, z_k) = \int_\Omega g(z_k)\tilde{\xi}_k : \varepsilon(\mathbf{u}'_\mathrm{D}(t_k))\,\mathrm{d}x - \langle \mathbf{f}'(t_k), \mathbf{u_k}\rangle_\mathbf{V}$, cf. (6.15). It follows from (6.9c) and from $\mathcal{E}(t_k, \mathbf{u}_k, z_k) \leq C$ that, up to a subsequence $\tilde{\xi}_k \rightharpoonup \xi$ in $L^{p'}(\Omega; \mathbb{R}^{d\times d})$ and, by the strong-weak closedness of $\partial_e W$, we conclude that $\tilde{\xi} \in \partial_e W(\varepsilon(\mathbf{u} + \mathbf{u}_\mathrm{D}(t)))$, where we have also used that $\mathbf{u}_\mathrm{D}(t_k) \to \mathbf{u}_\mathrm{D}(t)$ in $\mathbf{U}$. Taking into account the available convergences and the boundedness properties of g, with the dominated convergence theorem one shows that $\lim_{k\to\infty}\int_\Omega g(z_k)\tilde{\xi}_k:\varepsilon(\mathbf{u}'_{\mathrm{D}}(t_k))\,\mathrm{d}x = \int_\Omega g(z)\tilde{\xi}:\varepsilon(\mathbf{u}'_{\mathrm{D}}(t))\,\mathrm{d}x$. By the regularity properties of $\mathbf{f}$ we also find that $\mathbf{f}(t_k)\to\mathbf{f}(t)$ in $\mathbf{V}^*$. Thus, in view of the representation (6.15) of $\partial_t\mathcal{E}$ we even obtain by weak-strong convergence arguments that $\partial_t\mathcal{E}(t_k,\mathbf{u}_k,z_k)\to\partial_t\mathcal{E}(t,\mathbf{u},z)$. □

Now, Theorem 1 guarantees the existence of an energetic solution.

Theorem 6.3. *Let the assumptions of Prop. 6.2 hold true. Then the gradient system* $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *admits an energetic solution.*

The case $\varrho = 0$ and $\mathcal{V}$ of general superlinear growth. In this frame, the closedness of $\partial_u\mathcal{E}(t,\cdot,z)$ is to be verified via Hypothesis 4.25. This hypothesis requires the identification of the limit (as an element of the limit subdifferential) via *weak convergence both* of the sequence $(\mathbf{u}_k)_k$ and the elements $(\xi_k)_k$ of their corresponding subdifferentials *without any additional identification qualification* (such as the existence of a recovery sequence for the energy functional, cf. Hyp. 4.6, or the closedness à la Minty, cf. Hyp. 4.8). In view of the form (6.6) of the energy functional, such identification can be carried out if W is of quadratic nature. In other words, in case of a viscous dissipation potential of general superlinear growth, the energy functional has to be of quadratic growth in the respective variable, i.e., here, we may consider

$$W(e) := \tfrac{1}{2}e:\mathbb{C}:e \quad \text{with } \mathbb{C}\in\mathbb{R}^{d\times d\times d\times d} \text{ symmetric and positively definite.} \tag{6.18}$$

Hence, $\mathbf{U} = H^1_0(\Omega,\mathbb{R}^d)$ and the condition $\mathbf{U}\subset\mathbf{V}$ restricts $\mathbf{V}$ to be either a Sobolev space $W^{1,q}_0(\Omega,\mathbb{R}^d)$ with $1<q\le 2$ or a Lebesgue space $L^q(\Omega,\mathbb{R}^d)$ with $1<q\le 2d/(d-2)$ for $d>2$ and $1<q<\infty$ for $d=2$. By the representation formula (6.14), we have with the ansatz (6.18) that, if $\mathbf{u}_k\rightharpoonup\mathbf{u}$ in $\mathbf{V}$, $z_k\to z$ in $\mathbf{Z}$, $\xi_k\rightharpoonup\xi$ in $\mathbf{V}^*$ with $\xi_k\in\partial_u\mathcal{E}(t,\mathbf{u}_k,z_k)$ for all $k\in\mathbb{N}$, then also $\xi\in\partial_u\mathcal{E}(t,\mathbf{u},z)$. With this argument it can be observed that also $\limsup_k \partial_t\mathcal{E}(t_k,\mathbf{u}_k,z_k)\le\mathcal{E}(t,\mathbf{u},z)$, which is required in (2.5c). Therefore we now summarize

Proposition 6.4 (Properties of $\partial_u\mathcal{E}(t,\cdot,z)$ and $\partial_t\mathcal{E}(t,u,z)$)**.** *Let the assumptions of Prop. 6.1 hold true and let $\mathcal{E}(t,\cdot,z)$ be quadratic as in* (6.18) *as well as $\mathbf{U}=H^1_0(\Omega,\mathbb{R}^d)\subset\mathbf{V}$. Then, the system* $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *satisfies the closedness Hypothesis 4.25 as well as condition* (2.5c).

Now, we may invoke Theorem 2, which ensures the existence of a weak energetic solution.

Theorem 6.5. *Let the assumptions of Prop. 6.4 be satisfied. Then the gradient system* $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *admits a weak energetic solution.*

The case $\varrho > 0$ and $\mathcal{V}$ of general superlinear growth. As above, the closedness of $\partial_u\mathcal{E}(t,\cdot,z)$ has to be verified via Hypothesis 4.25 and hence, also here the energy functional has to be quadratic with W of the form (6.18) and $\mathbf{U}=H^1_0(\Omega,\mathbb{R}^d)$. In addition, if $\varrho>0$, also the subgradient estimate due to Hyp. 5.3 has to be shown. More precisely, for $\xi\in\partial_u\mathcal{E}(t,\mathbf{u},z)$ it has to be established that $|\langle\xi,v\rangle_{\mathbf{V}}|\le C\,(\mathcal{E}(t,\mathbf{u},z)+\|\mathbf{u}\|_{\mathbf{V}}+1)$ for all $v\in\mathbf{V}$. However, taking a closer look at the representation formula (6.14) we see that the stress control (6.9c) can only imply the subgradient estimate in terms of the energy if the divergence can be swapped to the test function $v\in\mathbf{V}$ by partial integration. But this requires that $\varepsilon(v)\in L^p(\Omega,\mathbb{R}^{d\times d})$ with $p\ge 2$ for any $v\in\mathbf{V}$, hence $\mathbf{V}\subset W^{1,p}_0(\Omega;\mathbb{R}^d)$ with $p\ge 2$. On other hand, $\mathbf{U}\subset\mathbf{V}$ enforces $p\le 2$. All in all, this leads to $\mathbf{V}=\mathbf{U}=H^1_0(\Omega,\mathbb{R}^d)$.

Proposition 6.6 (Properties of $\partial_u\mathcal{E}(t,\cdot,z)$ and $\partial_t\mathcal{E}(t,\mathbf{u},z)$)**.** *Let the assumptions of Prop. 6.4 be satisfied and let $\mathbf{V}=\mathbf{U}=H^1_0(\Omega,\mathbb{R}^d)$. Then, the evolutionary system* $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *satisfies the closedness Hypothesis 4.25 as well as condition* (2.5c) *and $\partial_u\mathcal{E}(t,\cdot,z)$ complies with the subgradient estimate* (5.8) *from Hyp. 5.3.*

Due to these findings the abstract existence Theorem 3 yields the existence of a weak energetic solution.

Theorem 6.7. *Let the assumptions of Prop. 6.6 hold true. Then the evolutionary system* $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *admits a weak energetic solution.*

The case $\varrho > 0$, $\mathcal{V}$ quadratic, and $\mathbf{V}\Subset\mathbf{W}$. Here, the topological setup $\mathbf{U}\subset\mathbf{V}\Subset\mathbf{W}$ together with (6.12) determines that $\mathbf{V}:=W^{1,r}_0(\Omega;\mathbb{R}^d)$ with $r\in[r^*,p]$ for $r^*=2d/(d+2)$ if $d>2$, $r^*>1$ if $d=2$. However, as already seen above, the subgradient estimate due to Hyp. 5.3 enforces that $\mathbf{V}=\mathbf{U}$; using the stress control (6.9c) it can be verified with calculations similar to the ones in the proof of Prop. 6.2(3). Moreover, for this setting, the closedness of $\partial_u\mathcal{E}(t,\cdot,z)$ is again to be verified via the closedness-argument à la Minty, as already done in Prop. 6.2(2). Therefrom then also follows the $\limsup$-qualification of $\partial_t\mathcal{E}$, cf. condition (2.5c), again arguing along the lines of Prop. 6.2(3).

Proposition 6.8 (Properties of $\partial_u \mathcal{E}(t,\cdot,z)$ and $\partial_t \mathcal{E}(t,\mathbf{u},z)$)**.** *Let the assumptions of Prop. 6.2 be satisfied and let $\mathcal{V}$ be quadratic and $\mathbf{V} = \mathbf{U} \Subset \mathbf{W}$. Then, the evolutionary system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ satisfies the closedness Hypothesis 4.8, condition* (2.5c) *and $\partial_u \mathcal{E}(t,\cdot,z)$ complies with the subgradient estimate* (5.8) *from Hyp. 5.3.*

This enables us to deduce the existence of energetic solutions via Theorem 4.

Theorem 6.9. *Let the assumptions of Prop. 6.8 be satisfied. Then the evolutionary system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ admits an energetic solution.*

6.3. **Rate-dependent damage and rate-independent plasticity with hardening at small strains.** We now focus on a coupled elasto-plastic damage model (at small strains) that was proposed in [AMV14] and first mathematically analyzed in [Cri14, CL15]. While [Cri14] addresses the existence of energetic solutions (*quasistatic evolutions*) for the fully rate-independent evolution of plasticity and damage, [CL15] studies the system regularized by viscosity in the damage variable, and develops the asymptotic analysis for vanishing viscosity.

In what follows, we aim to revisit the mixed rate-dependent (for the damage variable) / rate-independent (for the plasticity tensor) system studied in [CL15]. There, the authors have proved the existence of a weak notion of solution featuring a formulation of the damage flow rule in terms of a Kuhn-Tucker inequality, in place of a standard subdifferential inclusion, coupled with an energy-dissipation balance. In fact, such a formulation is akin to the weak solvability concept proposed and analyzed in [HK11, HK13] for (rate-dependent) damage coupled with phase separation.

Here we are going to address a variant of this damage/plasticity system, featuring hardening and a gradient regularization for the plastic tensor, and where unidirectionality of damage is neglected. To this regularized system it is possible to apply our abstract approach, leading to the existence of *energetic solutions*, which are in fact stronger than the solutions obtained in [CL15]. In particular, we will solve the damage flow rule pointwise a.e. in the space-time domain, cf. (6.34) below.

Energy. We consider damage in a body located in a sufficiently smooth (cf. (6.28) below) bounded domain $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$. The "viscous" and "rate-independent" variables are, respectively

$$u := \alpha \in [0,1] \text{ damage parameter}; \qquad z := p \in \mathbb{R}^{d\times d}_{\mathrm{dev}} := \{\pi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \,:\, \mathrm{tr}(\pi) = 0\} \text{ plastic tensor.}$$

The overall energy is given by the sum of three contributions

$$\mathcal{E}(t,\alpha,p) := \mathcal{E}_1(\alpha) + \mathcal{E}_2(p) + \mathcal{E}_3(t,\alpha,p). \tag{6.19}$$

We take

$$\mathcal{E}_1 : H^s(\Omega) \to (-\infty,+\infty]\,, \quad \mathcal{E}_1(\alpha) := a_s(\alpha,\alpha) + \int_\Omega I_{[0,1]}(\alpha) + \gamma(\alpha)\,\mathrm{d}x\,, \tag{6.20}$$

where, along the lines of [KRZ13], and as also done in [CL15], we choose for the gradient regularization of the damage variable the bilinear form

$$a_s(\alpha_1,\alpha_2) = \int_\Omega\int_\Omega \frac{\big(\nabla\alpha_1(x)-\nabla\alpha_1(y)\big)\cdot\big(\nabla\alpha_2(x)-\nabla\alpha_2(y)\big)}{|x-y|^{d+2(s-1)}}\,\mathrm{d}x\,\mathrm{d}y \quad \text{with } s \in \left(\frac{d}{2},2\right) \tag{6.21}$$

associated with the s-Laplacian operator $A_s : H^s(\Omega) \to H^s(\Omega)$ (with $H^s(\Omega)$ we denote the space $W^{s,2}(\Omega)$). In what follows, we will exploit in a crucial way that, since $s > \frac{d}{2}$, there holds

$$H^s(\Omega) \Subset \mathrm{C}^{0,\delta}(\overline{\Omega}) \quad \text{for some } \delta \in (0,1]. \tag{6.22}$$

The other contributions to the energy $\mathcal{E}_1$ are the indicator function of the interval $[0,1]$, enforcing the constraint $z \in [0,1]$, and a smooth, but possibly nonconvex, function $\gamma \in \mathrm{C}^1([0,1])$. Unlike in [CL15], where the case of perfect plasticity was considered, we encompass hardening and gradient plasticity in our model through

$$\mathcal{E}_2 : H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}) \to (0,+\infty)\,, \quad \mathcal{E}_2(p) := \int_\Omega \tfrac12 p{:}\mathbb{H}{:}p + |\nabla p|^2\,\mathrm{d}x \tag{6.23a}$$

with $\mathbb{H}$ a symmetric, positive definite operator (of Prager/Ziegler type) on $\mathbb{R}^{d\times d}_{\mathrm{dev}}$ such that

$$\exists\,\mathfrak{h}>0 \ \ \forall\,p \in \mathbb{R}^{d\times d}_{\mathrm{dev}} : \qquad p{:}\mathbb{H}{:}p \geq \mathfrak{h}|p^2|\,. \tag{6.23b}$$

Finally, we define $\mathcal{E}_3 : [0,T] \times \mathrm{C}^0(\overline{\Omega};[0,1]) \times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ by minimizing out the displacement from the elastic energy, namely

$$
\begin{aligned}
&\mathcal{E}_3(t,\alpha,p) := \min_{\mathbf{u}\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)} \mathcal{I}(t,\alpha,\mathbf{u},p) \quad \text{with} \\
&\mathcal{I}(t,\alpha,\mathbf{u},p) := \int_\Omega \tfrac{1}{2} g(\alpha)(\varepsilon(\mathbf{u}+\mathbf{u}_{\mathrm{D}})-p):\mathbb{C}:(\varepsilon(\mathbf{u}+\mathbf{u}_{\mathrm{D}})-p)\,\mathrm{d}x - \langle \mathbf{f}(t),\mathbf{u}\rangle \ .
\end{aligned}
\tag{6.24}
$$

Here, following [KRZ13, Sec. 2.4], we assume that

$$
\begin{aligned}
&\mathbb{C} \in \mathrm{C}^{0,\delta}(\overline{\Omega};\mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}},\mathbb{R}^{d\times d}_{\mathrm{sym}})) \quad \text{for } \delta\in(0,1] \text{ as in (6.22)},\\
&\exists\, c_0>0 \quad \text{for all } \xi\in\mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ and almost all } x\in\Omega: \quad \mathbb{C}(x)\xi:\xi \geq c_0|\xi|^2,
\end{aligned}
\tag{6.25}
$$

and that the constitutive function g fulfills

$$
g\in \mathrm{C}^1([0,1]), \quad \text{and } \exists\, c_1>0\ \forall\, z\in[0,1]: \quad g(z)\geq c_1. \tag{6.26}
$$

Along the footsteps of [KRZ13], we assume for the Dirichlet datum $\mathbf{u}_{\mathrm{D}}$ and the external load $\mathbf{f}$ that

$$
\mathbf{u}_{\mathrm{D}} \in \mathrm{C}^{1,1}([0,T];W^{1,4}(\Omega;\mathbb{R}^d)), \qquad \mathbf{f}\in \mathrm{C}^{1,1}([0,T];W^{-1,4}(\Omega;\mathbb{R}^d)). \tag{6.27}
$$

Let us mention in advance that (6.27) will have a crucial role in the derivation of *higher integrability* estimates for the (unique) minimizer of the energy $\mathcal{I}(t,\alpha,\cdot,p)$, cf. Lemma 6.10 below, along with the condition that

$$
\Omega \text{ has } \mathrm{C}^1\text{-boundary } \partial\Omega, \text{ and Dirichlet boundary } \Gamma_{\mathrm{D}} = \partial\Omega. \tag{6.28}
$$

Dissipation. For the damage variable, we consider the quadratic dissipation potential

$$
\mathcal{V}(\alpha') := \int_\Omega \frac{1}{2}|\alpha'|^2\,\mathrm{d}x \qquad \forall\,\alpha'\in \mathbf{V} := L^2(\Omega). \tag{6.29}
$$

Clearly, $\mathcal{V}$ complies with Hypothesis 2.3. Observe that $\mathcal{V}$ does not encompass the unidirectionality constraint $\alpha'\leq 0$, since, as mentioned in the introduction, in the analysis developed in this paper we cannot allow for $\mathcal{V}$ taking the value $+\infty$.

For the plastic tensor, along the footsteps of [AMV14], we include in the 1-homogeneous dissipation potential a dependence on the damage variable, namely we define

$$
\begin{aligned}
&\mathcal{R}: \mathrm{C}^0(\overline{\Omega};[0,1])\times \mathbf{Z} \to [0,+\infty), \quad \text{with } \mathbf{Z} := L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}), \quad \text{by } \mathcal{R}(\alpha,\pi') := \int_\Omega H(\alpha,\pi')\,\mathrm{d}x \quad \text{with}\\
&H:[0,1]\times\mathbb{R}^{d\times d}_{\mathrm{dev}} \to [0,+\infty) \text{ continuous }, H(\alpha,\cdot) \text{ convex and 1-positively homogeneous } \forall\,\alpha\in[0,1],\\
&\exists\, r,\, R>0 \ \ \forall\,(\alpha,\pi')\in[0,1]\times\mathbb{R}^{d\times d}_{\mathrm{dev}}: \qquad r|\pi|\leq H(\alpha,\pi')\leq R|\pi'|\,.
\end{aligned}
\tag{6.30a}
$$

It follows from (6.30a) that for every $\alpha\in[0,1]$ the dissipation density $H(\alpha,\cdot)$ is the support function of a closed convex $K(\alpha)\subset\mathbb{R}^{d\times d}_{\mathrm{dev}}$, i.e.

$$
H(\alpha,\pi') = \sup_{\sigma\in K(\alpha)} \sigma:\pi'\,.
$$

In the lines of [Cri14, Rmk. 2.1], we further specialize to the case in which

$$
K(\alpha) = h(\alpha)K \text{ with } \begin{cases} h:[0,1]\to[0,+\infty) \text{ continuous, and } \exists\, c_2>0\ \forall\,\alpha\in[0,1]: \ h(\alpha)\geq c_2>0,\\ K\subset\mathbb{R}^{d\times d}_{\mathrm{dev}} \text{ closed, convex, with } B_r(0)\subset K\subset B_R(0) \text{ for some } 0<r<R. \end{cases} \tag{6.30b}
$$

Hence, we have the following explicit formula for H

$$
H(\alpha,\pi') = h(\alpha)\sup_{\sigma\in K}\sigma:\pi' \doteq h(\alpha)|\pi'|_K, \tag{6.31}
$$

where we have used the norm $|\cdot|_K$ induced by K, equivalent to the Euclidean norm thanks to (6.30b). It is immediate to verify that for every $\alpha\in\mathrm{C}^0(\overline{\Omega};[0,1])$ the dissipation potential $\mathcal{R}(\alpha,\cdot)$ complies with Hyp. 2.3, with a constant uniform w.r.t. α.

The special form (6.31) of H naturally leads to the following definition for the total variation functional induced by $\mathcal{R}$: we set on an interval $[s,t]\subset[0,T]$

$$
\mathrm{Var}_{\mathcal{R}}(\alpha,p;[s,t]) := \iint_{[s,t]\times\overline{\Omega}} h(\alpha)|p'|_K(\mathrm{d}x\,\mathrm{d}r) \quad \text{for every } \alpha\in\mathrm{C}^0([0,T];\mathrm{C}^0(\overline{\Omega};[0,1])) \text{ and } p\in\mathrm{BV}([0,T];\mathbf{Z}). \tag{6.32}
$$

where the Radon measure $p' \in \mathrm{M}([0,T]\times\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ is the distributional derivative of p, and $|p'|_K$ its total variation with respect to the norm $|\cdot|_K$.

The notion of energetic solution and the time-discrete scheme. We now revisit the notion of *energetic* solution to the system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ given by (6.19), (6.29), and (6.30), adapting Def. 3.1, for $\varrho=0$, to the present situation with a *state-dependent* dissipation potential $\mathcal{R}$. Hence, the semistability condition reads

$$\mathcal{E}(t,\alpha(t),p(t)) \leq \mathcal{E}(t,\alpha(t),\tilde p) + \mathcal{R}(\alpha(t),\tilde p - p(t)) \qquad \forall\,\tilde p\in L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}) \quad \text{for all } t\in[0,T], \tag{6.33}$$

whereas the energy-dissipation inequality has the same form as (3.4) but features the α-dependent total variation $\mathrm{Var}_{\mathcal{R}}(\alpha,p;[0,t])$. Finally, let us highlight that the subdifferential inclusion for α reads

$$\begin{aligned}&\alpha'(t) + A_s\alpha(t) + \partial I_{[0,1]}(\alpha(t)) + \gamma'(\alpha(t))\\ &\quad + \frac12 g'(\alpha)(\varepsilon(\mathbf{u}_{\min}(t)+\mathbf{u}_{\mathrm{D}}(t)) - p(t)) : \mathbb{C} : (\varepsilon(\mathbf{u}_{\min}(t)+\mathbf{u}_{\mathrm{D}}(t)) - p(t)) \ni 0 \qquad \text{in } L^2(\Omega) \text{ for a.a. } t\in(0,T),\end{aligned} \tag{6.34}$$

where we have used the explicit formula (6.48) below for the Fréchet subdifferential of $\alpha\mapsto\mathcal{E}(t,\alpha,p)$ (w.r.t. the $L^2(\Omega)$-topology), with $\mathbf{u}_{\min}(t)\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$ the (unique) minimizer of $\mathcal{I}(t,\alpha(t),\cdot,p(t))$.

In order to prove the existence of energetic solutions to $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$, we pass to the limit in the time-discretization scheme from Problem 4.1, where the first minimum problem (4.3a) now reads

$$p^n_\tau \in \operatorname*{Argmin}_{p\in\mathbf{Z}} \left(\tau\mathcal{R}\left(\alpha^{n-1}_\tau, \frac{p-p^{n-1}_\tau}{\tau}\right) + \mathcal{E}(t_n,\alpha^{n-1}_\tau,p)\right), \tag{6.35}$$

whereas (4.3b) stays the same. Hence, for the limit passage in the semistability condition it will be necessary to verify the following mutual recovery sequence condition, tailored to the fact that the map $\alpha\mapsto\mathcal{G}(\alpha,p)$ has sublevels bounded in $H^s(\Omega)$, cf. Proposition 6.13 below, and that we will be able to prove for every $t\in[0,T]$ that $(\overline{\alpha}_\tau(t)-\underline{\alpha}_\tau(t))\rightharpoonup 0$ in $H^s(\Omega)$, cf. (4.16c) :

Let $(t_n,\bar\alpha_n,\alpha_n,p_n)_n\subset[0,T]\times H^s(\Omega)\times H^s(\Omega)\times H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ fulfill for every $n\in\mathbb{N}$ the semistability condition

$$\mathcal{E}(t_n,\alpha_n,p_n)\leq\mathcal{E}(t_n,\alpha_n,\tilde p)+\mathcal{R}(\bar\alpha_n,\tilde p-p_n) \qquad \forall\,\tilde p\in L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}),$$

and suppose that $t_n\to t$, $(\alpha_n,p_n)\rightharpoonup(\alpha,p)$ in $H^s(\Omega)\times H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ with $\sup_{n\in\mathbb{N}}\mathcal{G}(\alpha_n,p_n)\leq C$, and that $(\alpha_n-\bar\alpha_n)\rightharpoonup 0$ in $H^s(\Omega)$. Then for every $\tilde p\in L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ there exists $\tilde p_n\rightharpoonup\tilde p$ in $L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that

$$\lim_{n\to\infty}\left(\mathcal{R}(\bar\alpha_n,\tilde p_n-p_n)+\mathcal{E}(t_n,\alpha_n,\tilde p_n)-\mathcal{E}(t_n,\alpha_n,p_n)\right)\leq\mathcal{R}(\alpha,\tilde p-p)+\mathcal{E}(t,\alpha,\tilde p)-\mathcal{E}(t,\alpha,p). \tag{6.36}$$

Properties of the reduced energy $\mathcal{E}_3$. Prior to verifying the hypotheses of Thm. 1, we start with the following preliminary results, providing basic properties of the reduced energy $\mathcal{E}_3$ from (6.24). In Lemmas 6.10–6.12 (whose proofs are indeed drawn from [KRZ13]), we will distinguish between the properties valid for $p\in L^2(\Omega;\mathbb{R}^{d\times d})$, and those requiring $p\in L^4(\Omega;\mathbb{R}^{d\times d})$.

Lemma 6.10. *Assume* (6.25)–(6.28)*. Then, for every $(t,\alpha,p)\in[0,T]\times\mathrm{C}^0(\overline{\Omega};[0,1])\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ there exists a unique minimizer $\mathbf{u}_{\min}=\mathbf{u}_{\min}(t,\alpha,p)\in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$ and*

$$\begin{aligned}&\forall\,\epsilon\in(0,c_0c_1)\quad\exists\,d_1^\epsilon>0\quad\forall\,(t,\alpha,p)\in[0,T]\times\mathrm{C}^0(\overline{\Omega};[0,1])\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}):\\ &\mathcal{E}_3(t,\alpha,p)\geq\frac12(c_0c_1-\epsilon)\|\mathbf{u}_{\min}\|^2_{H^1(\Omega;\mathbb{R}^d)}+\frac12\left(c_0c_1-\frac1\epsilon\right)\|p\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}\\ &\qquad\qquad - d_1^\epsilon(\|\mathbf{u}_{\mathrm{D}}\|^2_{L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^d))}+\|\mathbf{f}\|^2_{L^\infty(0,T;W^{-1,4}(\Omega;\mathbb{R}^d))})\end{aligned} \tag{6.37}$$

Moreover, if $p\in L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$, then $\mathbf{u}_{\min}\in W^{1,4}(\Omega;\mathbb{R}^d)$ and

for every $M>0$ there exist $d_2(M)$, $d_3(M)>0$ such that

$$\begin{aligned}&\forall\,(t,\alpha,p)\in[0,T]\times\mathrm{C}^0(\overline{\Omega};[0,1])\times L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}),\ \|\alpha\|_{\mathrm{C}^0(\overline{\Omega})}+\|p\|_{L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}\leq M:\\ &\quad\|\mathbf{u}_{\min}(t,\alpha,p)\|_{W^{1,4}(\Omega;\mathbb{R}^d)}\leq d_2(M)\left(\|\mathbf{u}_{\mathrm{D}}\|_{L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^d))}+\|\mathbf{f}\|_{L^\infty(0,T;W^{-1,4}(\Omega;\mathbb{R}^d))}\right);\end{aligned} \tag{6.38}$$

$$\begin{aligned}&\forall\,(t_1,\alpha_1,p),\ (t_2,\alpha_2,p)\in[0,T]\times\mathrm{C}^0(\overline{\Omega};[0,1])\times L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}),\ \|\alpha_1\|_{\mathrm{C}^0(\overline{\Omega})}+\|\alpha_2\|_{\mathrm{C}^0(\overline{\Omega})}+\|p\|_{L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}\leq M:\\ &\|\mathbf{u}_{\min}(t_1,\alpha_1,p)-\mathbf{u}_{\min}(t_2,\alpha_2,p)\|_{W^{1,4}(\Omega;\mathbb{R}^d)}\\ &\quad\leq d_3(M)(|t_1-t_2|+\|\alpha_1-\alpha_2\|_{\mathrm{C}^0(\overline{\Omega})})\left(\|\mathbf{u}_{\mathrm{D}}\|_{L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^d))}+\|\mathbf{f}\|_{L^\infty(0,T;W^{-1,4}(\Omega;\mathbb{R}^d))}\right).\end{aligned} \tag{6.39}$$

Sketch of the proof. Estimate (6.37) can be easily checked by elementary calculations, also making use of Korn's inequality (cf. also the calculations for [KRZ13, Lemma 2.4]. The existence of a minimizer standardly follows from the coercivity and lower semicontinuity properties of the functional $\mathcal{I}(t,\alpha,\cdot,p)$, via the *direct method*; its uniqueness is due to the uniform convexity of $\mathcal{I}(t,\alpha,\cdot,p)$. Estimates (6.38) and (6.39) rely on higher integrability results for weak solutions to elliptic equations in smooth domains, cf. e.g. [Giu03, Sec. 10.4]: we refer to the proofs of [KRZ13, Lemmas 2.4–2.5, Prop. 2.11] for all details. □

Let us highlight that (6.39) is a continuous dependence estimate for $\mathbf{u}_{\min}$ with *fixed* p. It will play a crucial role for the study of the continuity properties of $\mathrm{D}_\alpha\mathcal{E}_3(\cdot,\cdot,p)$ in our next result.

Lemma 6.11. *Assume* (6.25)–(6.28). *Then:*

(1) For every $(\alpha,p)\in \mathrm{C}^0(\overline{\Omega};[0,1])\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ *the map* $t\mapsto \mathcal{E}_3(\alpha,p)$ *is in* $\mathrm{C}^1([0,T];\mathbb{R})$ *with*

$$\partial_t\mathcal{E}_3(t,\alpha,p)=\int_\Omega g(\alpha)\mathbb{C}(\varepsilon(\mathbf{u}_{\min}(t,\alpha,p)+\mathrm{u}_{\mathrm{D}}(t))-p)\colon\varepsilon(\mathrm{u}'_{\mathrm{D}}(t)\,\mathrm{d}x-\ \langle \mathbf{f}'(t),\mathbf{u}_{\min}(t,\alpha,p)\rangle_{H^1(\Omega;\mathbb{R}^d)},\tag{6.40}$$

and there exists a constant $d_4>0$ *such that for all* $(t,\alpha,p)\in[0,T]\times\mathrm{C}^0(\overline{\Omega});[0,1])\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$

$$|\partial_t\mathcal{E}_3(t,\alpha,p)|\le d_4\big(\|\mathrm{u}_{\mathrm{D}}\|^2_{\mathrm{C}^1([0,T];W^{1,4}(\Omega;\mathbb{R}^d))}+\|\mathbf{f}\|^2_{\mathrm{C}^1([0,T];W^{-1,4}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d))}+\|p\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}\big).\tag{6.41}$$

Moreover, there exist $d_5>0$ *depending on* $\|\mathrm{u}_{\mathrm{D}}\|_{\mathrm{C}^{1,1}([0,T];W^{1,4}(\Omega;\mathbb{R}^d))}$, $\|\mathbf{f}\|_{\mathrm{C}^{1,1}([0,T];W^{-1,4}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d))}$, *and such that for all* $(t_i,\alpha_i,p_i)\in[0,T]\times\mathrm{C}^0(\overline{\Omega};[0,1])\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$, $i=1,2$,

$$|\partial_t\mathcal{E}_3(t_1,\alpha_1,p_1)-\partial_t\mathcal{E}_3(t_2,\alpha_2,p_2)|\le d_5(|t_1-t_2|+\|\alpha_1-\alpha_2\|_{\mathrm{C}^0(\overline{\Omega})}+\|p_1-p_2\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})})\,.\tag{6.42}$$

(2) For every $t\in[0,T]$ *and* $p\in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ *the functional* $\alpha\mapsto\mathcal{E}_3(t,\alpha,p)$ *is Gâteaux-differentiable on* $\mathrm{C}^0(\overline{\Omega};[0,1])$ *with Gâteaux derivative*

$$\mathrm{D}_\alpha\mathcal{E}_3(t,\alpha,p)[\beta]=\int_\Omega \tfrac12 g'(\alpha)(\varepsilon(\mathbf{u}_{\min}(t,\alpha,p)+\mathbf{u}_{\mathrm{D}})-p)\colon\mathbb{C}\colon(\varepsilon(\mathbf{u}_{\min}(t,\alpha,p)+\mathbf{u}_{\mathrm{D}})-p)\beta\,\mathrm{d}x\qquad \text{for all }\beta\in\mathrm{C}^0(\overline{\Omega}).\tag{6.43}$$

If $p\in L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$, *then* $\mathrm{D}_\alpha\mathcal{E}_3(t,\alpha,p)$ *can be identified with an element in* $L^2(\Omega)$ *and, moreover, there exists a constant* $d_6>0$, *depending on* $\|\mathrm{u}_{\mathrm{D}}\|_{\mathrm{C}^{1,1}([0,T];W^{1,4}(\Omega;\mathbb{R}^d))}$, $\|\mathbf{f}\|_{\mathrm{C}^{1,1}([0,T];W^{-1,4}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d))}$, *and* $p\in L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$, *such that for every* $t_1,\,t_2\in[0,T]$ *and* $\alpha_1,\,\alpha_2\in\mathrm{C}^0(\overline{\Omega})$

$$\|\mathrm{D}_\alpha\mathcal{E}_3(t_1,\alpha_1,p)-\mathrm{D}_\alpha\mathcal{E}_3(t_2,\alpha_2,p)\|_{L^2(\Omega)}\le d_6\left(|t_1-t_2|+\|\alpha_1-\alpha_2\|_{\mathrm{C}^0(\overline{\Omega})}\right)\,.\tag{6.44}$$

Sketch of the proof. We refer to [KRZ13, Lemma 2.6] for the proof of (6.40)–(6.42). Formula (6.43) can be obtained straightforwardly and, in view of (6.38), one sees that $\mathrm{D}_\alpha\mathcal{E}_3(t,\alpha,p)\in L^2(\Omega)$ as soon as $p\in L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. The proof of (6.44) can be obtained by suitably adapting the calculations for [KRZ13, Lemma 2.8]. □

Finally, we examine the continuity properties of $\mathcal{E}_3$.

Lemma 6.12. *Assume* (6.25)–(6.28). *Then,*

(1) for every $p\in L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ *the functional* $(t,\alpha)\mapsto\mathcal{E}_3(t,\alpha,p)$ *is in* $\mathrm{C}^{1,1}([0,T];\mathrm{C}^0(\overline{\Omega};[0,1]))$;
(2) for every sequence $(t_n,\alpha_n,p_n)\subset[0,T]\times\mathrm{C}^0(\overline{\Omega};[0,1])\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$

$$\left.\begin{cases} t_n\to t,\\ \alpha_n\to\alpha & \text{in } \mathrm{C}^0(\overline{\Omega}),\\ p_n\to p & \text{in } L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})\end{cases}\right\}\ \Rightarrow\ \mathcal{E}_3(t_n,\alpha_n,p_n)\to\mathcal{E}_3(t,\alpha,p)\,.\tag{6.45}$$

Sketch of the proof. The Fréchet differentiability, with Lipschitz continuous differential, of $\mathcal{E}_3(\cdot,\cdot,p)$ is a consequence of Lemma 6.11. To show (6.45), we first observe that, given a sequence $(t_n,\alpha_n,p_n)_n$ as in (6.45), the sequence $\mathbf{u}_n:=\mathbf{u}_{\min}(t_n,\alpha_n,p_n)$, bounded in $H^1(\Omega;\mathbb{R}^d)$ by (6.37), up to a (not relabeled) subsequence weakly converges to some limit $\mathbf{u}$ in $H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$. With standard lower semicontinuity arguments we find

$$\liminf_{n\to\infty}\mathcal{E}_3(t_n,\alpha_n,p_n)=\liminf_{n\to\infty}\mathcal{I}(t_n,\alpha_n,\mathbf{u}_n,p_n)\ge\mathcal{I}(t,\alpha,\mathbf{u},p)\ge\mathcal{E}_3(t,\alpha,p)\,.$$

We now show that $\mathbf{u}$ is the unique minimizer of $\mathcal{J}(t,\alpha,\cdot,p)$ by observing that

$$\mathcal{J}(t,\alpha,\mathbf{u},p) \leq \limsup_{n\to\infty} \mathcal{J}(t_n,\alpha_n,\mathbf{u}_n,p_n) \overset{(1)}{\leq} \limsup_{n\to\infty} \mathcal{J}(t_n,\alpha_n,\mathbf{v},p_n) \overset{(2)}{=} \mathcal{J}(t,\alpha,\mathbf{v},p) \quad \forall\, \mathbf{v} \in H^1_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)\,, \tag{6.46}$$

where (1) is due to $\mathbf{u}_n \in \mathrm{Argmin}\,\mathcal{J}(t_n,\alpha_n,\cdot,p_n)$, and (2) is guaranteed by the convergence properties of $(t_n,\alpha_n,p_n)_n$. Choosing $\mathbf{v}=\mathbf{u}$ in (6.46), we find that $\lim_{n\to\infty}\mathcal{J}(t_n,\alpha_n,\mathbf{u}_n,p_n) = \mathcal{J}(t,\alpha,\mathbf{u},p)$, whence (6.45). □

Properties of the energy $\mathcal{E}$. We are now in the position to check that the energy functional $\mathcal{E}$ complies with the conditions of Thm. 1.

Proposition 6.13. *Assume* (6.25)–(6.28). *Let $\mathcal{E}$ be defined by* (6.19), (6.20), (6.23), *and* (6.24) *and, in addition, suppose that the constants $\mathfrak{h}$, c_0, and c_1 from* (6.23b), (6.25), *and* (6.26), *respectively, fulfill*

$$c_0c_1 > \frac{1}{\mathfrak{h}+c_0c_1}\,. \tag{6.47}$$

Then,

(1) $\mathcal{E}$ complies with Hypotheses 2.2 and 2.7, with the spaces $\mathbf{U}=H^s(\Omega)$ and $\mathbf{X}=H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$.
(2) The mutual recovery sequence condition (6.36) *is satisfied.*
(3) The Fréchet subdifferential $\partial^-_\alpha\mathcal{E}(t,\alpha,p): L^2(\Omega)\rightrightarrows L^2(\Omega)$ is given at any point $(t,\alpha,p)\in\mathrm{dom}(\partial^-_\alpha\mathcal{E})$ by

$$\partial^-_\alpha\mathcal{E}(t,\alpha,p) = A_s\alpha + \partial I_{[0,1]}(\alpha) + \gamma'(\alpha) + \tfrac12 g'(\alpha)(\varepsilon(\mathbf{u}_{\min}(t,\alpha,p)+\mathbf{u}_{\mathrm{D}})-p):\mathbb{C}:(\varepsilon(\mathbf{u}_{\min}(t,\alpha,p)+\mathbf{u}_{\mathrm{D}})-p), \tag{6.48}$$

and the uniform Fréchet subdifferentiability condition from Hyp. 4.2 is satisfied.
(4) $\mathcal{E}$ complies with Hyp. 4.6.

Proof. **Ad (1):** Taking into account that the function γ is bounded from below on $[0,1]$, the positive definiteness of the hardening tensor $\mathbb{H}$, and the previously proved (6.37), we find that $\mathcal{E}$ is bounded from below. Indeed, from the hardening contribution one gains the positive term $\frac{\mathfrak{h}}{2}\|p\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}$, and it is sufficient to choose the constant $\epsilon\in(0,c_0c_1)$ in (6.37) such that $\mathfrak{h}+c_0c_1-\frac{1}{\epsilon}>0$. This is possible thanks to the compatibility condition (6.47). In this way, we also also straightforwardly verify the coercivity property (2.18).

To check (2.5a) in Hyp. 2.7, let us consider a sequence $(\alpha_n,p_n)\rightharpoonup(\alpha,p)$ in $L^2(\Omega)\times L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that $\liminf_{n\to\infty}\mathcal{E}(t,\alpha_n,p_n)<+\infty$: up to a subsequence we may suppose that $\sup_n\mathcal{E}(t,\alpha_n,p_n)<+\infty$. Hence, by (2.18), we find that $(\alpha_n,p_n)\rightharpoonup(\alpha,p)$ in $H^s(\Omega)\times H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})\Subset \mathrm{C}^0(\overline{\Omega})\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. Thus, by (6.45) we conclude that $\mathcal{E}_3(t,\alpha_n,p_n)\to\mathcal{E}_3(t,\alpha,p)$. The lower semicontinuity inequality for $\mathcal{E}_1$ and $\mathcal{E}_2$ is standard.

Clearly, for every $(t,\alpha,p)\in[0,T]\times\mathrm{C}^0(\overline{\Omega};[0,1])\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ we have $\partial_t\mathcal{E}(t,\alpha,p)=\partial_t\mathcal{E}_3(t,\alpha,p)$. Hence, (2.5c) is a consequence of (6.41), which gives a control of the power in terms of the energy, and of (6.42), yielding the upper semicontinuity of the power. This concludes the verification of Hypothesis 2.2.

Ad (2): Let $(t_n,\alpha_n,p_n)_n\subset[0,T]\times L^2(\Omega)\times L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ be a sequence in the conditions of Hyp. 2.5. Thanks to Hyp. 2.7 and by the compact embeddings $H^s(\Omega)\Subset\mathrm{C}^0(\overline{\Omega})$ and $H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})\Subset L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ we have $(\alpha_n,p_n)\to(\alpha,p)$ in $\mathrm{C}^0(\overline{\Omega})\times L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. Hence, for (6.36) we may choose the constant recovery sequence $\tilde p_n:=\tilde p$, taking into account that

$$\begin{cases}\mathcal{R}(\bar\alpha_n,\tilde p-p_n)\to\mathcal{R}(\alpha,\tilde p-p) & \text{as } \bar\alpha_n\to\alpha \text{ in } \mathrm{C}^0(\overline{\Omega}) \text{ and } p_n\to p \text{ in } L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}),\\ \mathcal{E}(t_n,\alpha_n,\tilde p)\to\mathcal{E}(t,\alpha,p) & \text{by Lemma 6.12.}\end{cases}$$

Ad (3): The representation formula for $\partial^-_\alpha\mathcal{E}$ ensues from the sum rule for the Fréchet subdifferential, taking into account that $\mathcal{E}(t,\cdot,p)$ is given by the sum of either Gâteaux differentiable or convex terms. Next, we observe that for every $t\in[0,T]$, α_1, $\alpha_2\in\mathrm{C}^0(\overline{\Omega})$, and $p\in L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ and for every $\xi_i\in\partial^-_\alpha\mathcal{E}(t,\alpha_i,p)$, $i=1,2$:

$$\begin{aligned}\int_\Omega(\xi_1-\xi_2)\,(\alpha_1-\alpha_2)\,\mathrm{d}x &\geq a_s(\alpha_1-\alpha_2,\alpha_1-\alpha_2)+\int_\Omega(\gamma'(\alpha_1)-\gamma'(\alpha_2))\,(\alpha_1-\alpha_2)\,\mathrm{d}x\\ &\quad+\int_\Omega(\mathrm{D}_\alpha\mathcal{E}_3(t,\alpha_1,p)-\mathrm{D}_\alpha\mathcal{E}_3(t,\alpha_2,p))\,(\alpha_1-\alpha_2)\,\mathrm{d}x\\ &\geq\|\alpha_1-\alpha_2\|^2_{H^s(\Omega)}-\|\alpha_1-\alpha_2\|^2_{L^2(\Omega)}-C\|\alpha_1-\alpha_2\|^2_{L^2(\Omega)}-d_6\|\alpha_1-\alpha_2\|_{\mathrm{C}^0(\Omega)}\|\alpha_1-\alpha_2\|_{L^2(\Omega)},\end{aligned}$$

where the first inequality is due to the monotonicity of $\partial I_{[0,1]}$, and the second one follows from adding and subtracting $\|\alpha_1-\alpha_2\|^2_{L^2(\Omega)}$, since $\sqrt{\|\alpha\|^2_{L^2(\Omega)}+a_s(\alpha,\alpha)}$ gives the $H^s(\Omega)$-norm, from using that γ is Lipschitz continuous on $[0,1]$, and from estimate (6.44). Using now that

$$\forall \epsilon>0 \ \ \exists C_\epsilon>0 \ \ \forall \alpha \in H^s(\Omega) \ : \ \|\alpha\|_{\mathrm{C}^0(\overline{\Omega})} \leq \epsilon \|\alpha\|_{H^s(\Omega)} + C_\epsilon \|\alpha\|_{L^2(\Omega)}$$

in view of the embeddings $H^s(\Omega) \Subset \mathrm{C}^0(\overline{\Omega}) \subset L^2(\Omega)$, we conclude that

$$\begin{aligned} &\exists\, d_7,\, d_8>0 \ \forall\,(t,p) \in [0,T]\times L^4(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}) \ \forall\, \alpha_1,\alpha_2 \in H^s(\Omega) \ \forall\, \xi_i \in \partial_\alpha^- \mathcal{E}(t,\alpha_i,p), \ i=1,2 \ : \\ &\qquad \int_\Omega (\xi_1-\xi_2)\,(\alpha_1-\alpha_2)\,\mathrm{d}x \geq d_7 \|\alpha_1-\alpha_2\|_{H^s(\Omega)}^2 - d_8 \|\alpha_1-\alpha_2\|_{L^2(\Omega)}^2 . \end{aligned} \tag{6.49}$$

It is standard to deduce from (6.49) that $\mathcal{E}$ complies with the λ-convexity (4.7), whence Hyp. 4.2.

Ad (4): Hypothesis 4.6 can be verified with the choice of the constant recovery sequence $\tilde\alpha_n := \tilde\alpha$, taking into account that, for a sequence $(t_n,\alpha_n,p_n)_n$ as in Hyp. 4.6,

$$\begin{aligned} \limsup_{n\to\infty} \left(\mathcal{E}(t_n,\tilde\alpha,p_n) - \mathcal{E}(t_n,\alpha_n,p_n)\right) &= \limsup_{n\to\infty} \left(\mathcal{E}_1(\tilde\alpha) + \mathcal{E}_3(t_n,\tilde\alpha,p_n) - \mathcal{E}_1(\alpha_n) - \mathcal{E}_3(t_n,\alpha,p_n)\right) \\ &\leq \mathcal{E}_1(\tilde\alpha) + \mathcal{E}_3(t,\tilde\alpha,p) - \mathcal{E}_1(\alpha) - \mathcal{E}_3(t,\alpha,p) = \mathcal{E}(t,\tilde\alpha,p) - \mathcal{E}(t,\alpha,p) \end{aligned}$$

thanks to (6.45) and the lower semicontinuity properties of $\mathcal{E}_1$ and $\mathcal{E}_3$. □

Hence, from Theorem 1 we conclude

Theorem 6.14. *Assume* (6.25)–(6.28), *and* (6.47). *Then, for every* $(\alpha_0,p_0) \in H^s(\Omega) \cap H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ *there exists an enhanced energetic solution* (α,p), *with*

$$\alpha \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)) \quad \textit{and} \quad p \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})) \cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))$$

to the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *given by* (6.19), (6.29), *and* (6.30) *such that* $\alpha(0)=\alpha_0$ *and* $p(0)=p_0$.

7. Proofs of Theorems 1 – 4

In the proof of our existence results for the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ we will reverse the order in Section 4 and indeed start with the case of a general dissipation potential $\mathcal{V}$ in Sec. 7.1. In this general frame, on the one hand, we will employ *variational* techniques tailored to the proof of the discrete energy-dissipation inequality. On the other hand, in the limit passage to the time-continuous limit we will resort to Young measure arguments that will play a major role also in the particular case $\mathcal{V}$ quadratic, cf. Sec. 7.2. Finally, in Sec. 7.3 we will address the proofs of Theorems 3 ($\mathcal{V}$ general) and 4 ($\mathcal{V}$ quadratic).

7.1. **Proof of Theorem 2.** For the time-discrete analysis of the rate-dependent differential inclusion (3.2), we will adapt the arguments from the proof of [MRS13b, Thm. 4.4], see also [RS06]. More precisely, we will start by providing some "stationary estimates" for the solutions of the time-incremental minimum problem (4.23a), yielding the discrete $(u_\tau^n)_{n=1}^{N_\tau}$, at each fixed time-step. In particular, in Lemma 7.1 below, we will give the crucial time-discrete energy-dissipation inequality (7.5). We shall combine it with the estimates associated with the minimum problem (4.23b), yielding the discrete $(z_\tau^n)_{n=1}^{N_\tau}$, to derive the a priori estimates on the approximate solutions in Proposition 7.2. Also relying on Young measure compactness arguments, in Prop. 7.3 we derive a series of convergences (along suitable subsequences) of the discrete solutions to a pair (u,z), which we will then show to be a *weak energetic solution* to the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$.

Preliminarily, we fix some properties of the minimization problem

$$\mathcal{I}_r(\bar t,\bar u,\bar z) := \inf_{u\in \mathrm{D}_{\mathsf{u}}} \left(r\mathcal{V}\left(\frac{u-\bar u}{r}\right) + \mathcal{E}(\bar t + r, u, \bar z)\right), \quad \text{for given } (\bar t,\bar u,\bar z) \in [0,T]\times \mathrm{D}_{\mathsf{u}} \times \mathrm{D}_{\mathsf{z}} \text{ and } 0<r<T-\bar t, \tag{7.1}$$

in Lemma 7.1 below, which we recall here from [MRS13b] for the reader's convenience. Observe that it is in the proof of this Lemma, in particular in the derivation of the energy-dissipation inequality (7.5), that condition (4.22) on $\mathcal{V}$ comes into play, cf. the proof of [MRS13b, Lemma 6.1].

Lemma 7.1. [MRS13b, Lemma 6.1] *Under Hypotheses 2.2, 2.3, 2.7, condition* (4.22) *on* $\mathcal{V}$, *and Hyp. 4.11, for every* $(\bar t,\bar u,\bar z) \in [0,T]\times \mathrm{D}_{\mathsf{u}} \times \mathrm{D}_{\mathsf{z}}$ *and* $0<r<\min\{\tau_o, T-\bar t\}$ *the set*

$$\mathrm{M}_r(\bar t,\bar u,\bar z) := \operatorname*{Argmin}_{u\in \mathrm{D}_{\mathsf{u}}} \left(r\mathcal{V}\left(\frac{u-\bar u}{r}\right) + \mathcal{E}(\bar t + r, u, \bar z)\right) \neq \emptyset,$$

and there exists a measurable selection $r \in (0, T - \bar{t}) \mapsto u_r \in \mathrm{M}_r(\bar{t}, \bar{u}, \bar{z})$ *fulfilling for every* $r \in (0, T - \bar{t})$ *the Euler-Lagrange equation*

$$\partial \mathcal{V}\left(\frac{u_r - u}{r}\right) + \partial_u^- \mathcal{E}(\bar{t} + r, u_r, \bar{z}) \ni 0 \qquad \text{in } \mathbf{V}^*, \tag{7.2}$$

as well as

$$\exists\, C > 0 \ \ \forall\, r \in (0, T - \bar{t}) : \qquad \mathcal{G}(u_r, \bar{z}) \leq \mathcal{G}(\bar{u}, \bar{z}), \tag{7.3}$$

$$\lim_{r \downarrow 0} \|u_r - \bar{u}\|_{\mathbf{V}} = 0, \qquad \lim_{r \downarrow 0} \mathcal{I}_r(\bar{t}, \bar{u}, \bar{z}) = \mathcal{E}(\bar{t}, \bar{u}, \bar{z}), \tag{7.4}$$

and complying with the energy-dissipation inequality

$$r_0 \mathcal{V}\left(\frac{u_{r_0} - \bar{u}}{r_0}\right) + \int_0^{r_0} \mathcal{V}^*(-\xi_r)\,\mathrm{d}r + \mathcal{E}(\bar{t} + r_0, u_{r_0}, \bar{z}) \leq \mathcal{E}(\bar{t}, \bar{u}, \bar{z}) + \int_0^{r_0} \partial_t \mathcal{E}(\bar{t} + r, u_r, \bar{z})\,\mathrm{d}r \tag{7.5}$$

for any $r_0 \in (0, T - \bar{t})$, *with* ξ_r *any selection in* $\partial_u^- \mathcal{E}(\bar{t} + r, u_r, \bar{z}) \cap \left(-\partial \mathcal{V}\left(\frac{u_r - u}{r}\right)\right)$.

While referring to [MRS13b] for the proof, let us only comment here on the Euler-Lagrange equation: it is an immediate consequence of [MRS13b, Prop. 4.2], which can be applied thanks to Hypothesis 4.11.

Exploiting (7.5), we will now derive the discrete version (7.8) of the energy-dissipation inequality (3.4), whence all of the a priori estimates on the approximate solutions arising from the time-discrete scheme in Problem 4.10. The Euler-Lagrange equation (7.2) rewrites for the variational interpolant $\tilde{u}_\tau$ from (4.24) as

$$\partial \mathcal{V}\left(\frac{\tilde{u}_\tau(t) - \underline{u}_\tau(t)}{t - \underline{\mathsf{t}}_\tau(t)}\right) + \partial_u^- \mathcal{E}(t, \tilde{u}_\tau(t), \underline{z}_\tau(t)) \ni 0 \qquad \text{in } \mathbf{V}^* \text{ for a.a. } t \in (0, T).$$

In what follows, we will denote by $\tilde{\xi}_\tau : (0, T) \to \mathbf{V}^*$ a measurable map fulfilling

$$\tilde{\xi}_\tau(t) \in \partial_u^- \mathcal{E}(t, \tilde{u}_\tau(t), \underline{z}_\tau(t)), \quad -\tilde{\xi}_\tau(t) \in \partial \mathcal{V}\left(\frac{\tilde{u}_\tau(t) - \underline{u}_\tau(t)}{t - \underline{\mathsf{t}}_\tau(t)}\right) \qquad \text{for a.a. } t \in (0, T). \tag{7.6}$$

For later use, we also point out that the minimum problem (4.23a) yields

$$\overline{\xi}_\tau(t) \in \partial_u^- \mathcal{E}(t, \overline{u}_\tau(t), \underline{z}_\tau(t)), \quad -\overline{\xi}_\tau(t) \in \partial \mathcal{V}(u_\tau'(t)) \qquad \text{for a.a. } t \in (0, T), \tag{7.7}$$

with $\overline{\xi}_\tau : (0, T) \to \mathbf{V}^*$ the piecewise constant interpolant of the elements $(\xi_\tau^n)_{n=1}^{N_\tau}$ fulfilling the Euler-Lagrange equation for (4.23a).

Proposition 7.2. *Assume Hypotheses 2.2, 2.3, 2.7, and condition* (4.22) *on* $\mathcal{V}$.

Then, the interpolants $(\overline{u}_\tau, \underline{u}_\tau, u_\tau, \tilde{u}_\tau, \overline{z}_\tau, \underline{z}_\tau, z_\tau)_\tau$ *and* $(\tilde{\xi}_\tau)_\tau$ *of the discrete solutions to Problem 4.10 comply with the discrete energy-dissipation inequality*

$$\begin{aligned} &\int_{\overline{\mathsf{t}}_\tau(s)}^{\overline{\mathsf{t}}_\tau(t)} \mathcal{V}(u_\tau'(r)) + \mathcal{V}^*(-\tilde{\xi}_\tau(r))\,\mathrm{d}r + \int_{\overline{\mathsf{t}}_\tau(s)}^{\overline{\mathsf{t}}_\tau(t)} \mathcal{R}(z_\tau'(r))\,\mathrm{d}r + \mathcal{E}(\overline{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \overline{z}_\tau(t)) \\ &\leq \mathcal{E}(\overline{\mathsf{t}}_\tau(s), \overline{u}_\tau(s), \overline{z}_\tau(s)) + \int_{\overline{\mathsf{t}}_\tau(s)}^{\overline{\mathsf{t}}_\tau(t)} \partial_t \mathcal{E}(r, \tilde{u}_\tau(r), \underline{z}_\tau(r))\,\mathrm{d}r, \end{aligned} \tag{7.8}$$

and the discrete semistability condition

$$\mathcal{E}(\overline{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \overline{z}_\tau(t)) \leq \mathcal{E}(\overline{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \tilde{z}) + \mathcal{R}(\tilde{z} - \overline{z}_\tau(t)) \qquad \text{for all } \tilde{z} \in \mathbf{Z} \text{ and for all } t \in (0, T]. \tag{7.9}$$

Moreover, there exists $C > 0$ *such that for every* $0 < \tau < \tau_o$ *the following estimates hold*

$$\begin{aligned} &\sup_{t \in (0,T)} \left(\mathcal{G}(\overline{u}_\tau(t), \overline{z}_\tau(t)) + \tau_o \mathcal{V}\left(\frac{\overline{u}_\tau(t) - u_0}{\tau_o}\right) + \mathcal{R}(\overline{z}_\tau(t) - z_0)\right) \leq C, \\ &\sup_{t \in (0,T)} \left(\mathcal{G}(\tilde{u}_\tau(t), \underline{z}_\tau(t)) + \tau_o \mathcal{V}\left(\frac{\tilde{u}_\tau(t) - u_0}{\tau_o}\right) + \mathcal{R}(\underline{z}_\tau(t) - z_0)\right) \leq C, \\ &\textit{and analogously for } \sup_{t \in (0,T)} \mathcal{G}(\overline{u}_\tau(t), \underline{z}_\tau(t)) + \tau_o \mathcal{V}((\overline{u}_\tau(t) - u_0)/\tau_o); \end{aligned} \tag{7.10a}$$

$$\sup_{t \in (0,T)} |\partial_t \mathcal{E}(t, \tilde{u}_\tau(t), \underline{z}_\tau(t))| \leq C; \tag{7.10b}$$

$$\int_0^T \mathcal{V}(u_\tau'(r))\,\mathrm{d}r \leq C, \qquad \int_0^T \mathcal{V}^*(-\tilde{\xi}_\tau(r))\,\mathrm{d}r \leq C, \qquad \int_0^T \mathcal{R}(z_\tau'(r))\,\mathrm{d}r \leq C; \tag{7.10c}$$

the families $(u'_\tau)_\tau \subset L^1(0,T;\mathbf{V})$ and $(\tilde{\xi}_\tau)_\tau \subset L^1(0,T;\mathbf{V}^*)$ are uniformly integrable, and

$$\sup_{t\in(0,T)} \|\overline{u}_\tau(t) - \underline{u}_\tau(t)\|_{\mathbf{V}} + \sup_{t\in(0,T)} \|u_\tau(t) - \underline{u}_\tau(t)\|_{\mathbf{V}} + \sup_{t\in(0,T)} \|\tilde{u}_\tau(t) - \underline{u}_\tau(t)\|_{\mathbf{V}} = o(1) \text{ as } \tau \downarrow 0. \tag{7.10d}$$

Proof. The discrete semistability (7.9) is a direct consequence of the minimum problem (4.23b) and of the triangle inequality satisfied by $\mathcal{R}$. The proof of the remaining items in the statement follows the lines of [MRS13b, Prop. 6.3]. Therefore we shall just outline its main steps, referring to [MRS13b] for all details.

In order to derive (7.8), let us first of all fix two nodes $t_{n-1}, t_n \in \mathscr{P}_\tau$ and write (7.5) with $\bar{t} = t_{n-1}$, $\bar{u} = u_\tau^{n-1}$, $\bar{z} = z_\tau^{n-1}$, $r_0 = t - t_{n-1}$, $u_{r_0} = \tilde{u}_\tau(t)$, $u_r = \tilde{u}_\tau(r)$, and $\xi_r = \tilde{\xi}_\tau(r)$, for $r \in (t_{n-1}, t)$. Thus we obtain with a change of variables in the second integral on the left-hand side

$$\begin{aligned}(t - t_{n-1})\mathcal{V}\left(\frac{\tilde{u}_\tau(t) - \underline{u}_\tau(t)}{t - t_{n-1}}\right) + \int_{t_{n-1}}^t \mathcal{V}^*(-\tilde{\xi}_\tau(r))\,\mathrm{d}r + \mathcal{E}(t, \tilde{u}_\tau(t), \underline{z}_\tau(t))\\ \leq \mathcal{E}(t_{n-1}, \underline{u}_\tau(t), \underline{z}_\tau(t)) + \int_{t_{n-1}}^t \partial_t \mathcal{E}(r, \tilde{u}_\tau(r), \underline{z}_\tau(r))\,\mathrm{d}r,\end{aligned} \tag{7.11}$$

whence, for $t = t_n$,

$$\int_{t_{n-1}}^{t_n} \mathcal{V}\left({u'}_\tau(r)\right) + \mathcal{V}^*(-\tilde{\xi}_\tau(r))\,\mathrm{d}r + \mathcal{E}(t_n, u_\tau^n, z_\tau^{n-1}) \leq \mathcal{E}(t_{n-1}, u_\tau^{n-1}, z_\tau^{n-1}) + \int_{t_{n-1}}^{t_n} \partial_t \mathcal{E}(r, \tilde{u}_\tau(r), \underline{z}_\tau(r))\,\mathrm{d}r. \tag{7.12}$$

On the other hand, we deduce from (4.23b) that

$$\tau\mathcal{R}\left(\frac{z_\tau^n - z_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, u_\tau^n, z_\tau^n) \leq \mathcal{E}(t_n, u_\tau^n, z_\tau^{n-1}).$$

Adding this to (7.12), due to the cancelation of the term $\mathcal{E}(t_n, u_\tau^n, z_\tau^{n-1})$ we obtain

$$\begin{aligned}\int_{t_{n-1}}^{t_n} \mathcal{V}\left({u'}_\tau(r)\right) + \mathcal{V}^*(-\tilde{\xi}_\tau(r))\,\mathrm{d}r + \tau\mathcal{R}\left(\frac{z_\tau^n - z_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n, u_\tau^n, z_\tau^n)\\ \leq \mathcal{E}(t_{n-1}, u_\tau^{n-1}, z_\tau^{n-1}) + \int_{t_{n-1}}^{t_n} \partial_t \mathcal{E}(r, \tilde{u}_\tau(r), \underline{z}_\tau(r))\,\mathrm{d}r\end{aligned}$$

which gives (7.8) upon summing up over the index n.

Using the power control condition from (2.5c) and arguing in the very same way as in the proof of [MRS13b, Prop. 6.3], one infers that

$$\sup_{t\in(0,T)} \left(\mathcal{G}(\overline{u}_\tau(t), \overline{z}_\tau(t)) + \int_0^{\bar{\mathsf{t}}_\tau(t)} \mathcal{V}(u'_\tau(s))\,\mathrm{d}s\right) \leq C\,. \tag{7.13}$$

In order to conclude the first of (7.10), we proceed in the following way: For $t_0 \in (0,T)$ fixed, $1 \leq n_0 \leq N_\tau$ fulfill $\bar{\mathsf{t}}_\tau(t_0) = n_0\tau$; suppose that $n_0\tau \leq \tau_o$, with $\tau_o > 0$ from Hyp. 2.7. Then,

$$\int_0^{\bar{\mathsf{t}}_\tau(t_0)} \mathcal{V}(u'_\tau(s))\,\mathrm{d}s = n_0\tau \sum_{n=1}^{n_0} \frac{1}{n_0}\mathcal{V}\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right) \overset{(1)}{\geq} \bar{\mathsf{t}}_\tau(t_0)\mathcal{V}\left(\sum_{n=1}^{n_0} \frac{u_\tau^n - u_\tau^{n-1}}{\bar{\mathsf{t}}_\tau(t_0)}\right) \overset{(2)}{\geq} \tau_o\mathcal{V}\left(\frac{\overline{u}_\tau(t_0) - u_0}{\tau_o}\right),$$

where (1) follows from the convexity of $\mathcal{V}$, and (2) from the fact that for every $v \in \mathbf{V}$ the map $\tau \mapsto \mathcal{V}\left(\frac{v}{\tau}\right)$ is nonincreasing. Then, from (7.13) we conclude that $\tau_o\mathcal{V}\left(\frac{\overline{u}_\tau(t_0) - u_0}{\tau_o}\right) \leq C$ for $t_0 \in (0,T)$ sufficiently small such that $\bar{\mathsf{t}}_\tau(t_0) \leq \tau_o$. Then, the estimate

$$\sup_{t\in(0,T)} \tau_o\mathcal{V}\left(\frac{\overline{u}_\tau(t) - u_0}{\tau_o}\right) \leq C \tag{7.14}$$

can be concluded with a standard argument, dividing $[0,T]$ into subintervals of length less or equal than τ_o. The calculations for the estimate of the $\mathcal{R}$-dissipation are simpler by 1-homogeneity, and we thus conclude the first of (7.10a). The second of (7.10a) analogously ensues from (7.11). Let us only finally comment on the bound for $\sup_{t\in[0,T]} \mathcal{G}(\overline{u}_\tau(t), \underline{z}_\tau(t))$ in (7.10a): From (4.23a) we deduce for every $t \in [0,T]$

$$\mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \underline{z}_\tau(t)) \leq \mathcal{E}(\bar{\mathsf{t}}_\tau(t), \underline{u}_\tau(t), \underline{z}_\tau(t)) = \mathcal{E}(\underline{\mathsf{t}}_\tau(t), \underline{u}_\tau(t), \underline{z}_\tau(t)) + \int_{\underline{\mathsf{t}}_\tau(t)}^{\bar{\mathsf{t}}_\tau(t)} \partial_t \mathcal{E}(s, \underline{u}_\tau(t), \underline{z}_\tau(t))\,\mathrm{d}s \leq C,$$

where the last estimate ensues from the bound for $\sup_{t\in[0,T]} \mathcal{G}(\overline{u}_\tau(t), \overline{z}_\tau(t))$, and from (2.5c). □

The next result shall also involve the Young measure limit of the family $(\tilde{\xi}_\tau)_\tau$. We refer to the Appendix for a self-contained exposition of the Young measure compactness result underlying the proof of Proposition 7.3.

Proposition 7.3. *Assume Hypotheses 2.2, 2.3, 2.7, and the closedness* (4.25a) *from Hypothesis 4.11.*

Then, for every vanishing sequence $(\tau_k)_k$ of time-steps there exist a (not relabeled) subsequence, a pair (u,z) as in (3.1)*, a function $\mathscr{E} \in \mathrm{BV}([0,T])$, and a time-dependent Young measure $\boldsymbol{\mu} = (\mu_t)_{t\in(0,T)} \in \mathscr{Y}(0,T;\mathbf{V}^*)$ such that convergences* (4.16) *hold as $k \to \infty$ and, in addition*

$$\begin{cases} \mathscr{E}(t) = \lim_{k\to\infty} \mathcal{E}(\bar{\mathsf{t}}_{\tau_k}(t), \overline{u}_{\tau_k}(t), \overline{z}_{\tau_k}(t)) & \text{for all } t \in [0,T], \\ \mathscr{E}(t) \le \liminf_{k\to\infty} \mathcal{E}(\bar{\mathsf{t}}_{\tau_k}(t), \overline{u}_{\tau_k}(t), \underline{z}_{\tau_k}(t)) & \text{for almost all } t \in (0,T), \\ \mathscr{E}(t) \ge \mathcal{E}(t, u(t), z(t)) & \text{for all } t \in [0,T], \end{cases} \tag{7.15}$$

while $\boldsymbol{\mu}$ is the limit Young measure of $(\tilde{\xi}_{\tau_k})_k$, whence

$$\tilde{\xi}_{\tau_k} \rightharpoonup \xi \quad \text{in } L^1(0,T;\mathbf{V}^*) \qquad \text{with } \xi(t) = \int_{\mathbf{V}^*} \zeta \,\mathrm{d}\mu_t(\zeta) \quad \text{for a.a. } t \in (0,T). \tag{7.16}$$

Moreover, the following energy-dissipation inequality holds for all $0 \le s \le t \le T$

$$\begin{aligned} &\int_s^t \mathcal{V}(u'(r))\,\mathrm{d}r + \int_s^t \int_{\mathbf{V}^*} \mathcal{V}^*(-\zeta)\,\mathrm{d}\mu_r(\zeta)\,\mathrm{d}r + \mathrm{Var}_{\mathcal{R}}(z;[s,t]) + \mathscr{E}(t) \\ &\le \mathscr{E}(s) + \int_s^t \partial_t \mathcal{E}(r, u(r), z(r))\,\mathrm{d}r. \end{aligned} \tag{7.17}$$

Proof. Convergences (4.16a)–(4.16c) ensue from estimates (7.10) via standard compactness arguments, cf. [Sim87], also taking into account the coercivity from Hyp. 2.7. The pointwise $\sigma_{\mathbf{X}}$-convergence for $(\overline{z}_{\tau_k})_k$ (which then implies (4.16g) thanks to (2.4b)), is a consequence of the bounds in $L^\infty(0,T;\mathbf{X}) \cap \mathrm{BV}([0,T];\mathbf{Z})$ via a Helly-type argument (cf. e.g. [MT04, Thm. 6.1]), which also gives

$$\mathrm{Var}_{\mathcal{R}}(z;[s,t]) \le \liminf_{k\to\infty} \int_s^t \mathcal{R}(z'_{\tau_k}(r))\,\mathrm{d}r,$$

whence $z \in \mathrm{BV}([0,T];\mathbf{Z})$. Finally, convergence (4.16f) ensues from a standard argument, cf. e.g. the proof of [RTP15, Thm. 4.1]. With (4.16f) at hand, it is then easy to deduce convergences (4.16d) and (4.16e) for $(z_{\tau_k})_k$ as well. We refer to the proof of [MRS13b, Prop. 6.4] for all details on the first and third inequalities in (7.15) and on (7.16). The limit passage in (7.8), leading to (7.17), also follows from the arguments for [MRS13b, Prop. 6.4], combined with the observation that

$$\liminf_{k\to\infty} \int_{\bar{\mathsf{t}}_{\tau_k}(s)}^{\bar{\mathsf{t}}_{\tau_k}(t)} \mathcal{R}(z'_{\tau_k}(r))\,\mathrm{d}r \ge \mathrm{Var}_{\mathcal{R}}(z;[s,t])$$

for every $0 \le s \le t \le T$, thanks to (4.16g). Finally, let us comment on the second of (7.15). It ensues from the minimum problem (4.23b), yielding

$$\mathcal{E}(\bar{\mathsf{t}}_{\tau_k}(t), \overline{u}_{\tau_k}(t), \overline{z}_{\tau_k}(t)) \le \mathcal{E}(\bar{\mathsf{t}}_{\tau_k}(t), \overline{u}_{\tau_k}(t), \underline{z}_{\tau_k}(t)) \qquad \text{for all } t \in [0,T].$$

For later use, observe also that (2.5c) yields

$$\limsup_{\tau_k \downarrow 0} \partial_t \mathcal{E}(t, \tilde{u}_{\tau_k}(t), \underline{z}_{\tau_k}(t)) \le \partial_t \mathcal{E}(t, u(t), z(t)) \qquad \text{for a.a. } t \in (0,T). \tag{7.18}$$

□

We are now in the position to conclude the **proof of Theorem 2**. In order to obtain the energy-dissipation inequality (3.4) we need to gain further insight into the Young measure energy-dissipation inequality (7.17), and in particular into the properties of the limit $\boldsymbol{\mu} = (\mu_t)_{t\in(0,T)}$ of $(\tilde{\xi}_{\tau_k})$. It follows from Theorem A.2 ahead that for almost all $t \in (0,T)$ the measure μ_t is concentrated on the limit points of $(\tilde{\xi}_{\tau_k}(t))_k$ with respect to the weak topology of $\mathbf{V}^*$. Now, due to the first of (7.6), $\tilde{\xi}_{\tau_k}(t) \in \partial_u^- \mathcal{E}(t, \tilde{u}_{\tau_k}(t), \underline{z}_{\tau_k}(t))$ for almost all $t \in (0,T)$. Combining the pointwise convergences (4.16c) and (4.16g) for $(\tilde{u}_{\tau_k}(t))_k$ and $(\underline{z}_{\tau_k}(t))_k$ with the closedness condition (4.25a), we ultimately conclude that

$$\text{the support of } \mu_t \text{ is a subset of } \partial_u^- \mathcal{E}(t, u(t), z(t)) \text{ for almost all } t \in (0,T). \tag{7.19}$$

Since the latter set is closed and convex, we conclude that

$$\xi(t) = \int_{\mathbf{V}^*} \zeta \, \mathrm{d}\mu_t(\zeta) \in \partial_u^- \mathcal{E}(t, u(t), z(t)) \qquad \text{for a.a. } t \in (0,T)\,, \tag{7.20}$$

Then, by Jensen's inequality we have

$$\int_s^t \int_{\mathbf{V}^*} \mathcal{V}^*(-\zeta) \, \mathrm{d}\mu_r(\zeta) \, \mathrm{d}r \geq \int_s^t \mathcal{V}^*(-\xi(r)) \, \mathrm{d}r \qquad \text{for every } 0 \leq s \leq t \leq T. \tag{7.21}$$

Hence, in view of (7.21) and (7.18) we are in the position to deduce from (7.17), written on the interval $(0,t)$, the energy-dissipation inequality (3.4).

Furthermore, observe that, when it holds, the enhanced closedness/continuity condition (4.25b) guarantees

$$\lim_{k\to\infty} \mathcal{E}(\bar{\mathsf{t}}_{\tau_k}(t), \overline{u}_{\tau_k}(t), \overline{z}_{\tau_k}(t)) = \mathscr{E}(t) = \mathcal{E}(t, u(t), z(t)) \qquad \text{for almost all } t \in (0,T). \tag{7.22}$$

To check this, from estimate (7.10c) and Fatou's Lemma we deduce that $\liminf_{k\to\infty} \|u_{\tau_k}'(t)\|_{\mathbf{V}} < +\infty$ for almost all $t \in (0,T)$. Since $\partial\mathcal{V} : \mathbf{V} \rightrightarrows \mathbf{V}^*$ is a bounded operator, we then have

$$\liminf_{k\to\infty} \|\overline{\xi}_{\tau_k}(t)\|_{\mathbf{V}^*} < +\infty \qquad \text{for a.a. } t \in (0,T), \tag{7.23}$$

where $t \mapsto \overline{\xi}_\tau(t)$ is the selection in $(-\partial\mathcal{V}(u_\tau'(t))) \cap \partial^- \mathcal{E}(t, \overline{u}_\tau(t), \underline{z}_\tau(t))$ fulfilling the Euler Lagrange equation (7.7). We are thus in the position to apply condition (4.25b) (also in view of (7.10a)), thus concluding that

$$\liminf_{k\to\infty} \mathcal{E}(\bar{\mathsf{t}}_{\tau_k}(t), \overline{u}_{\tau_k}(t), \underline{z}_{\tau_k}(t)) = \mathcal{E}(t, u(t), z(t)) \qquad \text{for a.a. } t \in (0,T). \tag{7.24}$$

But then, by the second and the third of (7.15), we infer $\mathcal{E}(t,u(t),z(t)) \leq \mathscr{E}(t) \leq \mathcal{E}(t,u(t),z(t))$ for almost all $t \in (0,T)$, whence (7.22). Therefore, from (7.17) we gather the enhanced energy-dissipation inequality (4.21).

Finally, with a standard procedure in the analysis of rate-independent processes we pass to the limit in the discrete semistability (7.9) by resorting to the mutual recovery sequence condition (2.16). More detailed, for all $t \in (0,T)$ we apply Hypothesis 2.5 to the sequences $(\overline{u}_{\tau_k}(t), \overline{z}_{\tau_k}(t))_k$, converging to $(u(t), z(t))$ as in (4.16c) and (4.16g), and thus for every $\tilde{z} \in \mathbf{Z}$ we construct a sequence $\tilde{z}_k \rightharpoonup z$ in $\mathbf{Z}$ such that (2.16) holds. Exploiting (2.16), we thus conclude that the semistability inequality (3.3) holds with $\tilde{z}$. Observe that, under the enhanced condition (4.25b), we have the additional energy convergence (7.22), hence we may employ the weaker variant of (2.16) in Remark 2.6. This concludes the proof. ■

7.2. **Proof of Theorem 1.** Preliminarily, observe that from (4.3b) we conclude that

$$0 \in \partial^- \left(\tau \mathcal{V}\left(\frac{\cdot - u_\tau^{n-1}}{\tau} \right) + \mathcal{E}(t_n, \cdot, z_\tau^n) \right)(u_\tau^n) = \partial\mathcal{V}\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau} \right) + \partial_u^- \mathcal{E}(t_n, u_\tau^n, z_\tau^n), \tag{7.25}$$

where the last equality follows from the sum rule for the Fréchet subdifferential, since $\mathcal{V}$ is Fréchet differentiable. Hence we obtain the Euler-Lagrange equation

$$\partial\mathcal{V}\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau} \right) + \xi_\tau^n \ni 0, \qquad \xi_\tau^n \in \partial_u^- \mathcal{E}(t_n, u_\tau^n, z_\tau^n) \quad \text{for all } n = 1, \ldots, N_\tau. \tag{7.26}$$

We denote by $\overline{\xi}_\tau$ the piecewise constant interpolant of the elements $(\xi_\tau^n)_{n=1}^{N_\tau} \subset \mathbf{V}^*$, so that (7.26) rewrites as

$$\partial\mathcal{V}\left(u_\tau'(t)\right) + \overline{\xi}_\tau(t) \ni 0, \qquad \overline{\xi}_\tau(t) \in \partial_u^- \mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \overline{z}_\tau(t)) \qquad \text{for a.a. } t \in (0,T). \tag{7.27}$$

We start with the counterpart to Proposition 7.2. Observe that the result below holds for *any* dissipation potential $\mathcal{V}$, not necessarily quadratic. Instead, the *uniform Fréchet subdifferentiability* from Hypothesis 4.2 plays here a key role.

Proposition 7.4. *Assume Hypotheses 2.2, 2.3, 2.7, and 4.2. Let* $(\overline{u}_\tau, \underline{u}_\tau, u_\tau, \overline{z}_\tau, \underline{z}_\tau, z_\tau)_\tau$ *and* $(\overline{\xi}_\tau)_\tau$ *be the approximate solutions constructed from the time-discretization scheme in Problem 4.1.*

Then, there exists $C>0$ such that for every $0<\tau<\tau_o$ the following estimates hold

$$\sup_{t\in(0,T)}\left(\mathcal{G}(\overline{u}_\tau(t),\overline{z}_\tau(t))+\tau_o\mathcal{V}\left(\frac{\overline{u}_\tau(t)-u_0}{\tau_o}\right)+\mathcal{R}(\overline{z}_\tau(t)-z_0)\right)\leq C,\qquad \sup_{t\in(0,T)}|\partial_t\mathcal{E}(t,\underline{u}_\tau(t),\underline{z}_\tau(t))|\leq C, \tag{7.28a}$$

$$\int_0^T\mathcal{V}\left(u_\tau'(r)\right)\mathrm{d}r\leq C,\qquad \int_0^T\mathcal{R}(z_\tau'(r))\,\mathrm{d}r\leq C, \tag{7.28b}$$

$$\int_0^T\mathcal{V}^*(-\overline{\xi}_\tau(r))\,\mathrm{d}r\leq C, \tag{7.28c}$$

and (7.10d) *holds.*

Furthermore, the interpolants comply with the discrete semistability condition

$$\mathcal{E}(\bar{\mathsf{t}}_\tau(t),\underline{u}_\tau(t),\overline{z}_\tau(t))\leq\mathcal{E}(\bar{\mathsf{t}}_\tau(t),\underline{u}_\tau(t),\tilde{z})+\mathcal{R}(\tilde{z}-\overline{z}_\tau(t))\qquad \textit{for all } \tilde{z}\in\mathbf{Z} \textit{ and for all } t\in(0,T], \tag{7.29}$$

and with the discrete energy-dissipation inequality

$$\begin{aligned}&\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\mathcal{V}\left(u_\tau'(r)\right)+\mathcal{V}^*(-\overline{\xi}_\tau(r))\,\mathrm{d}r+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\mathcal{R}(z_\tau'(r))\,\mathrm{d}r+\mathcal{E}(\bar{\mathsf{t}}_\tau(t),\overline{u}_\tau(t),\overline{z}_\tau(t))\\ &\leq\mathcal{E}(\bar{\mathsf{t}}_\tau(s),\overline{u}_\tau(s),\overline{z}_\tau(s))+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\partial_t\mathcal{E}(r,\underline{u}_\tau(r),\underline{z}_\tau(r))\,\mathrm{d}r+\Lambda_C\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\|\overline{u}_\tau(r)-\underline{u}_\tau(r)\|_{\mathbf{V}}\|u_\tau'(r)\|_{\mathbf{V}}\,\mathrm{d}r,\end{aligned} \tag{7.30}$$

with $\Lambda_C>0$ depending on C from (7.28a) *via* (4.6).

Observe that, for $\mathcal{V}$ quadratic, estimates (7.28b) and (7.28c) in particular yield

$$\|u_\tau'(r)\|_{L^2(0,T;\mathbf{V})}\leq C,\qquad \|\overline{\xi}_\tau(r)\|_{L^2(0,T;\mathbf{V}^*)}\leq C. \tag{7.31}$$

Proof. While the discrete semistability (7.29) is again a direct consequence of the minimum problem (4.3a), the discrete energy-dissipation inequality (7.30) follows from a combination of the time-discrete scheme (4.3) with the uniform Fréchet subdifferentiability condition (4.6) on energy sublevels. Hence, it can be derived only after obtaining the energy estimate (7.28a). Therefore, we need to derive an intermediate energy-dissipation inequality, i.e. (7.34) below.

Let us fix $n\in\{1,\ldots,N_\tau\}$. From the minimum problem (4.3a) we gather

$$\tau\mathcal{R}\left(\frac{z_\tau^n-z_\tau^{n-1}}{\tau}\right)+\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^n)\leq\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^{n-1}), \tag{7.32}$$

while (4.3b) yields

$$\tau\mathcal{V}\left(\frac{u_\tau^n-u_\tau^{n-1}}{\tau}\right)+\mathcal{E}(t_n,u_\tau^n,z_\tau^n)\leq\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^n). \tag{7.33}$$

Adding (7.32) and (7.33) and summing over the index n leads to the intermediate inequality for all $0\leq s\leq t\leq T$

$$\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\left(\mathcal{V}\left(u_\tau'(r)\right)+\mathcal{R}(z_\tau'(r))\right)\mathrm{d}r+\mathcal{E}(\bar{\mathsf{t}}_\tau(t),\overline{u}_\tau(t),\overline{z}_\tau(t))\leq\mathcal{E}(\bar{\mathsf{t}}_\tau(s),\overline{u}_\tau(s),\overline{z}_\tau(s))+\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\partial_t\mathcal{E}(r,\underline{u}_\tau(r),\underline{z}_\tau(r))\,\mathrm{d}r \tag{7.34}$$

whereby estimates (7.28a) and (7.28b) ensue from the very same arguments as for the proof of Proposition 7.2. Moreover, we recover the stability estimates (7.10d) as a consequence of the dissipation bound in (7.28b).

We now obtain (7.30) by rephrasing the discrete Euler-Lagrange equation (7.26) as

$$\mathcal{V}\left(\frac{u_\tau^n-u_\tau^{n-1}}{\tau}\right)+\mathcal{V}^*(-\xi_\tau^n)=\left\langle-\xi_\tau^n,\frac{u_\tau^n-u_\tau^{n-1}}{\tau}\right\rangle_{\mathbf{V}}. \tag{7.35}$$

On the other hand, from $\xi_\tau^n\in\partial_u^-\mathcal{E}(t_n,u_\tau^n,z_\tau^n)$ and (4.6) we obtain

$$-\langle\xi_\tau^n,u_\tau^n-u_\tau^{n-1}\rangle_{\mathbf{V}}\leq\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^n)-\mathcal{E}(t_n,u_\tau^n,z_\tau^n)+\Lambda_C\|u_\tau^n-u_\tau^{n-1}\|_{\mathbf{V}}^2,$$

with $\Lambda_C>0$ depending on C, taking into account that $(u_\tau^n,z_\tau^n)\in\mathcal{S}_C$ by (7.28a), and that (u_τ^{n-1},z_τ^n) as well as belongs to $\mathcal{S}_C$ since

$$\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^n)\overset{(7.32)}{\leq}\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^{n-1})=\mathcal{E}(t_{n-1},u_\tau^{n-1},z_\tau^{n-1})+\int_{t_{n-1}}^{t_n}\partial_t\mathcal{E}(s,u_\tau^{n-1},z_\tau^{n-1})\,\mathrm{d}s\overset{(7.28a)}{\leq}\tilde{C}. \tag{7.36}$$

Combining (7.35) and (7.36) we arrive at

$$\tau\mathcal{V}\left(\frac{u_\tau^n-u_\tau^{n-1}}{\tau}\right)+\tau\mathcal{V}^*(-\xi_\tau^n)\leq\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^n)-\mathcal{E}(t_n,u_\tau^n,z_\tau^n)+\Lambda_C\|u_\tau^n-u_\tau^{n-1}\|_{\mathbf{V}}^2,$$

which we add to (7.32), thus obtaining, with the cancelation of two terms and using that $\mathcal{E}(t_n,u_\tau^{n-1},z_\tau^n)=\mathcal{E}(t_{n-1},u_\tau^{n-1},z_\tau^n)+\int_{t_{n-1}}^{t_n}\partial_t\mathcal{E}(s,u_\tau^{n-1},z_\tau^{n-1})\,\mathrm{d}s$,

$$\begin{aligned}&\tau\mathcal{V}\left(\frac{u_\tau^n-u_\tau^{n-1}}{\tau}\right)+\tau\mathcal{V}^*(-\xi_\tau^n)+\tau\mathcal{R}\left(\frac{z_\tau^n-z_\tau^{n-1}}{\tau}\right)+\mathcal{E}(t_n,u_\tau^n,z_\tau^n)\\&\leq\mathcal{E}(t_{n-1},u_\tau^{n-1},z_\tau^{n-1})+\int_{t_{n-1}}^{t_n}\partial_t\mathcal{E}(s,u_\tau^{n-1},z_\tau^{n-1})\,\mathrm{d}s+\Lambda_C\|u_\tau^n-u_\tau^{n-1}\|_{\mathbf{V}}^2\,.\end{aligned}\tag{7.37}$$

Adding up over the index n leads to (7.30).

Observe that the last term on the right-hand side of (7.30) can be estimated as

$$\Lambda_C\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)}\|\overline{u}_\tau(r)-\underline{u}_\tau(r)\|_{\mathbf{V}}\|u'_\tau(r)\|_{\mathbf{V}}\,\mathrm{d}r\leq C\tau\int_0^T\|u'_\tau(r)\|_{\mathbf{V}}^2\,\mathrm{d}r\leq C\tau,\tag{7.38}$$

where for the last estimate we have relied on the previously obtained (7.28b). Thus, from (7.30) we deduce (7.28c). □

The proof of the forthcoming Proposition 7.5 relies on the same Young measure argument as Prop. 7.3, hence we omit the details. Let us only mention that, in order to conclude the energy-dissipation inequality (7.40) below we pass to the limit in (7.30), relying on the bound on (7.38) to conclude that the last term on the right-hand side of (7.30) tends to zero. We also use (7.21) from the proof of Theorem 2.

Proposition 7.5. *Assume Hypotheses 2.2, 2.3, 2.7, and 4.2. Let $\mathcal{V}$ be quadratic, cf.* (4.1).

Then, for every vanishing sequence $(\tau_k)_k$ of time-steps there exist a (not relabeled) subsequence, a pair (u,z) as in (3.1) *and a function $\mathscr{E}\in\mathrm{BV}([0,T])$, such that convergences* (4.16) *hold as $k\to\infty$ as well as convergence* (4.17) *and the energy convergence* (7.15). *There also exists a Young measure $\boldsymbol{\mu}=(\mu_t)_{t\in(0,T)}$, limit of $(\overline{\xi}_{\tau_k})_k$, such that*

$$\overline{\xi}_{\tau_k}\rightharpoonup\xi\quad\text{ in }L^2(0,T;\mathbf{V}^*)\qquad\text{with }\xi(t)=\int_{\mathbf{V}^*}\zeta\,\mathrm{d}\mu_t(\zeta)\quad\text{for a.a. }\ t\in(0,T).\tag{7.39}$$

Moreover, the following energy-dissipation inequality holds for all $0\leq s\leq t\leq T$

$$\int_s^t\mathcal{V}\left(u'(r)\right)\mathrm{d}r+\int_s^t\mathcal{V}^*(-\xi(r))\,\mathrm{d}r+\mathrm{Var}_{\mathcal{R}}(z;[s,t])+\mathscr{E}(t)\leq\mathscr{E}(s)+\int_s^t\partial_t\mathcal{E}(t,r,u(r))\,\mathrm{d}r.\tag{7.40}$$

We are now in the position to develop the **proof of Theorem 1**. The energy-dissipation inequality (3.4) on $(0,t)$ follows from (7.40), taking into account that $\mathscr{E}(t)\geq\mathcal{E}(t,u(t),z(t))$ for all $t\in(0,T]$, and that $\mathscr{E}(0)=\mathcal{E}(0,u_0,z_0)$ by (7.15). It remains to prove the subdifferential inclusion (3.2). For this, we distinguish two cases:

Case (1)**: Hyp. 4.6 holds:** It follows from $\mathbf{U}\Subset\mathbf{V}$ that

$$\overline{u}_{\tau_k},\,\underline{u}_{\tau_k},\,u_{\tau_k}\to u\quad\text{ in }L^\infty(0,T;\mathbf{V}).\tag{7.41}$$

Hence, we have convergence (4.19) and, in view of Lemma 4.7, the energy convergence (4.20) holds. Moreover, Lemma 4.7 guarantees that for almost all $t\in(0,T)$ the set of the limit points of the sequence $(\overline{\xi}_{\tau_k}(t))_k$, hence the support of the measure μ_t, is contained in $\partial_u^-\mathcal{E}(t,u(t),z(t))$. Then, with the same argument as in the proof of Theorem 2, cf. (7.20), we find that $\xi(t)\in\partial_u^-\mathcal{E}(t,u(t),z(t))$ for a.a. $t\in(0,T)$. On the other hand, since $\mathcal{V}$ is quadratic, passing to the limit as $k\to\infty$ in (7.27) we conclude that $-\xi(t)\in\partial\mathcal{V}(u'(t))=\{Au'(t)\}$ for almost all $t\in(0,T)$, and (3.2) ensues.

Case (2)**: Hyp. 4.8 holds:** Passing to the limit in (7.27), exploiting the fact that $\mathcal{V}$ is quadratic, we get $-\xi(t)\in\partial\mathcal{V}(u'(t))=\{Au'(t)\}$ for almost all $t\in(0,T)$. We now use Hypothesis 4.8 to show that

$$\xi(t)\in\partial_u^-\mathcal{E}(t,u(t),z(t))\qquad\text{for a.a. }\ t\in(0,T).\tag{7.42}$$

With this aim, we test (7.27) by $\overline{u}_{\tau_k}$, integrate in time, and find for every $t \in [0,T]$ that

$$\begin{aligned}\limsup_{k\to\infty} \int_0^{\overline{\mathsf{t}}_{\tau_k}(t)} \langle \overline{\xi}_{\tau_k}, \overline{u}_{\tau_k}\rangle_{\mathbf{V}}\,\mathrm{d}s &= -\liminf_{k\to\infty} \int_0^{\overline{\mathsf{t}}_{\tau_k}(t)} \langle A u'_{\tau_k}, \overline{u}_{\tau_k}\rangle_{\mathbf{V}}\,\mathrm{d}s \\ &\leq -\liminf_{k\to\infty} \tfrac12 a(\overline{u}_{\tau_k}(t), \overline{u}_{\tau_k}(t)) + \tfrac12 a(u_0,u_0) \\ &\leq -\tfrac12 a(u(t),u(t)) + \tfrac12 a(u_0,u_0) = -\int_0^t \langle Au', u\rangle_{\mathbf{V}}\,\mathrm{d}s = \int_0^t \langle \xi, u\rangle_{\mathbf{V}}\,\mathrm{d}s.\end{aligned} \tag{7.43}$$

Then, in view of the closedness property (4.14) we conclude (7.42). If (4.15) holds in addition, we infer the energy convergence (4.20).

Clearly, under the energy convergence (4.20) it is possible to conclude that (u,z) fulfills the energy-dissipation inequality (3.4) for all $t \in (0,T]$ and almost all $s \in (0,t)$. For the limit passage in the discrete semistability (7.29) we resort to Hypothesis 2.5.

Observe that, under Hypothesis 4.6 or under the enhanced (4.15), we also have $\mathcal{E}(\underline{\mathsf{t}}_{\tau_k}(t), \underline{u}_{\tau_k}(t), \underline{z}_{\tau_k}(t)) \to \mathcal{E}(t,u(t),z(t))$ as $k\to\infty$ for almost all $t \in (0,T)$. Then, arguing as in (7.36) we infer that

$$\limsup_{k\to\infty} \mathcal{E}(\overline{\mathsf{t}}_{\tau_k}(t), \underline{u}_{\tau_k}(t), \overline{z}_{\tau_k}(t)) \leq \mathcal{E}(t,u(t),z(t)),$$

whence $\mathcal{E}(\overline{\mathsf{t}}_{\tau_k}(t), \underline{u}_{\tau_k}(t), \overline{z}_{\tau_k}(t)) \to \mathcal{E}(t,u(t),z(t))$ for almost all $t \in (0,T)$. Therefore, it is sufficient to exploit the weaker variant of Hypothesis 2.5 in Remark 2.6. This concludes the proof. ∎

We close this Section with the proof of **Lemma 4.7**: to prove that $\xi \in \partial_u^- \mathcal{E}(t,u,z)$, we will show that

$$\mathcal{E}(t,v,z) - \mathcal{E}(t,u,z) \geq \langle \xi, v-u\rangle_{\mathbf{V}} - \Lambda\|v-u\|_{\mathbf{V}}^2 \qquad \text{for every } v \in \mathrm{D}_{\mathsf{u}} \tag{7.44}$$

With this aim, we use Hypothesis 4.6 and for every fixed $v \in \mathrm{D}_{\mathsf{u}}$ we consider a recovery sequence $v_n \to v$ in $\mathbf{V}$ such that (4.11) holds. By Hyp. 4.2, there exists $\Lambda = \Lambda_C > 0$ such that for every $n \in \mathbb{N}$

$$\mathcal{E}(t_n,v_n,z_n) - \mathcal{E}(t_n,u_n,z_n) \geq \langle \xi_n, v_n-u_n\rangle_{\mathbf{V}} - \Lambda\|v_n-u_n\|_{\mathbf{V}}^2. \tag{7.45}$$

We pass to the limit in (7.45) combining the convergences in (4.12) with (4.11), and conclude (7.44).

To prove the energy convergence, it is sufficient to check $\limsup_n \mathcal{E}(t_n,u_n,z_n) \leq \mathcal{E}(t,u,z)$. This can be deduced from choosing $v=u$ in (4.6), whence

$$\mathcal{E}(t,u,z) \geq \limsup_{n\to\infty} \mathcal{E}(t_n,u_n,z_n) + \lim_{n\to\infty}\big(\langle \xi_n, u-u_n\rangle_{\mathbf{V}} - \Lambda\|u-u_n\|_{\mathbf{V}}^2\big) = \limsup_{n\to\infty} \mathcal{E}(t_n,u_n,z_n).$$

∎

7.3. **Proof of Theorems 3 and 4.** The following result collects all the a priori estimates on the approximate solutions constructed from the time-discretization scheme in Problem 5.1. In Proposition 7.6 below we will derive the a priori estimates under the uniform subdifferentiability Hypothesis 5.2, whereas in Remark 7.7 we will hint at the slightly different properties for which it is possible to deduce them under the *weaker* (5.15) in the case $\mathcal{V}$ quadratic.

Proposition 7.6. *Assume Hypotheses 2.2, 2.3, 2.7, as well as Hypotheses 5.2 and 5.3. Let $(\overline{u}_\tau, \underline{u}_\tau, u_\tau, \overline{z}_\tau, \underline{z}_\tau, z_\tau)_\tau$ be the approximate solutions constructed from the time-discretization scheme in Problem 5.1, with $(v_\tau)_\tau$ from* (5.5) *and $(\overline{\xi}_\tau)_\tau$ complying with* (5.6).

Then, the interpolants satisfy the discrete semistability condition (7.29) *and the discrete energy-dissipation inequality*

$$\begin{aligned}&\frac{\varrho}{2}\|u'_\tau(t)\|_{\mathbf{W}}^2 + \int_{\overline{\mathsf{t}}_\tau(s)}^{\overline{\mathsf{t}}_\tau(t)} \mathcal{V}(u'_\tau(r)) + \mathcal{V}^*(-\overline{\xi}_\tau(r) - \varrho v'_\tau(r))\,\mathrm{d}r + \int_{\overline{\mathsf{t}}_\tau(s)}^{\overline{\mathsf{t}}_\tau(t)} \mathcal{R}(z'_\tau(r))\,\mathrm{d}r + \mathcal{E}(\overline{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \overline{z}_\tau(t)) \\ &\leq \frac{\varrho}{2}\|u'_\tau(s)\|_{\mathbf{W}}^2 + \mathcal{E}(\overline{\mathsf{t}}_\tau(s), \overline{u}_\tau(s), \overline{z}_\tau(s)) + \int_{\overline{\mathsf{t}}_\tau(s)}^{\overline{\mathsf{t}}_\tau(t)} \partial_t \mathcal{E}(r, \underline{u}_\tau(r), \underline{z}_\tau(r))\,\mathrm{d}r + \Lambda\tau^{1/2}\int_{\overline{\mathsf{t}}_\tau(s)}^{\overline{\mathsf{t}}_\tau(t)} \|u'_\tau(r)\|_{\mathbf{W}} \mathcal{V}(u'_\tau(r))^{1/2}\,\mathrm{d}r,\end{aligned} \tag{7.46}$$

with $\Lambda>0$ as in (5.7). *Furthermore, there exists $C>0$ such that for every $0<\tau<\tau_o$ there hold the stability property* (7.10d), *the energy-dissipation estimates* (7.28a)–(7.28b), *and*

$$\int_0^T \mathcal{V}^*(-\overline{\xi}_\tau(r) - \varrho v_\tau'(r))\,\mathrm{d}r \le C, \tag{7.47a}$$

$$\|u_\tau'\|_{L^\infty(0,T;\mathbf{W})} \le C \tag{7.47b}$$

Hence,

$$\|\overline{\xi}_\tau\|_{L^\infty(0,T;\mathbf{V}^*)} \le C, \tag{7.47c}$$

$$(v_\tau')_\tau \text{ is uniformly integrable in } L^1(0,T;\mathbf{V}^*). \tag{7.47d}$$

Proof. Preliminarily, we observe that the Euler-Lagrange equation (5.4) follows from the same argument as in (7.25) in the case $\mathcal{V}$ quadratic, taking into account the sum rule

$$\begin{aligned}
&\partial^-\left(\frac{\varrho}{\tau^2}\|\cdot - 2u_\tau^{n-1} + u_\tau^{n-2}\|_{\mathbf{W}}^2 + \tau\mathcal{V}\left(\frac{\cdot - u_\tau^{n-1}}{\tau}\right) + \mathcal{E}(t_n,\cdot,z_\tau^n)\right)(u_\tau^n)\\
&= \varrho\frac{u_\tau^n - 2u_\tau^{n-1} + u_\tau^{n-2}}{\tau^2} + \partial\mathcal{V}\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right) + \partial_u^-\mathcal{E}(t_n,u_\tau^n,z_\tau^n)
\end{aligned}$$

due to the Fréchet differentiability of $\mathcal{V}$ and of $\|\cdot\|_{\mathbf{W}}^2$.

For $\mathcal{V}$ general, we apply [MRS13b, Prop. 4.2] to $\mathcal{V}$ and to $\mathcal{E}_n := \mathcal{E}(t_n,\cdot,z_\tau^n) + \frac{\varrho}{\tau^2}\|\cdot - 2u_\tau^{n-1} + u_\tau^{n-2}\|_{\mathbf{W}}^2$, and then observe that the Fréchet subdifferential of $\mathcal{E}_n$ at u_τ^n decomposes into the sum of the Fréchet subdifferential of $\mathcal{E}$ and the singleton $\{\frac{\varrho}{\tau^2}(u_\tau^n - 2u_\tau^{n-1} + u_\tau^{n-2})\}$.

We now start with the proof of (7.46). We rephrase the Euler-Lagrange equation (5.4) as (cf. (7.35))

$$\begin{aligned}
&\mathcal{V}\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right) + \mathcal{V}^*\left(-\xi_\tau^n - \varrho\frac{u_\tau^n - 2u_\tau^{n-1} + u_\tau^{n-2}}{\tau^2}\right)\\
&= \langle -\xi_\tau^n - \varrho\frac{u_\tau^n - 2u_\tau^{n-1} + u_\tau^{n-2}}{\tau^2}, \frac{u_\tau^n - u_\tau^{n-1}}{\tau}\rangle_{\mathbf{V}}\\
&\le -\langle \xi_\tau^n, \frac{u_\tau^n - u_\tau^{n-1}}{\tau}\rangle_{\mathbf{V}} + \frac{\varrho}{2\tau}\frac{\|u_\tau^{n-1} - u_\tau^{n-2}\|_{\mathbf{W}}^2}{\tau^2} - \frac{\varrho}{2\tau}\frac{\|u_\tau^n - u_\tau^{n-1}\|_{\mathbf{W}}^2}{\tau^2}.
\end{aligned} \tag{7.48}$$

On the other hand, from $\xi_\tau^n \in \partial_u^-\mathcal{E}(t_n,u_\tau^n,z_\tau^n)$ and the Fréchet subdifferentiability condition (5.7) we obtain

$$\langle \xi_\tau^n, u_\tau^n - u_\tau^{n-1}\rangle_{\mathbf{V}} \le \mathcal{E}(t_n,u_\tau^n,z_\tau^n) - \mathcal{E}(t_n,u_\tau^{n-1},z_\tau^n) + \Lambda\|u_\tau^n - u_\tau^{n-1}\|_{\mathbf{W}}\mathcal{V}(u_\tau^n - u_\tau^{n-1})^{1/2}. \tag{7.49}$$

Combining (7.48) and (7.49) with estimate (7.32) which derives from the minimum problem (5.3a), we get

$$\begin{aligned}
&\frac{\varrho}{2}\frac{\|u_\tau^n - u_\tau^{n-1}\|_{\mathbf{W}}^2}{\tau^2} + \tau\mathcal{V}\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right) + \tau\mathcal{V}^*\left(-\xi_\tau^n - \varrho\frac{u_\tau^n - 2u_\tau^{n-1} + u_\tau^{n-2}}{\tau^2}\right) + \mathcal{E}(t_n,u_\tau^n,z_\tau^n) + \mathcal{R}(z_\tau^n - z_\tau^{n-1})\\
&\le \frac{\varrho}{2}\frac{\|u_\tau^{n-1} - u_\tau^{n-2}\|_{\mathbf{W}}^2}{\tau^2} + \mathcal{E}(t_{n-1},u_\tau^{n-1},z_\tau^{n-1}) + \int_{t_{n-1}}^{t_n}\partial_t\mathcal{E}(s,u_\tau^{n-1},z_\tau^{n-1})\,\mathrm{d}s + \Lambda\|u_\tau^n - u_\tau^{n-1}\|_{\mathbf{W}}\mathcal{V}(u_\tau^n - u_\tau^{n-1})^{1/2}.
\end{aligned}$$

Adding up over the index n leads to (7.46).

We are now in the position to deduce estimates (7.28a)–(7.28b) and (7.47). Indeed, we estimate the last term on the right-hand side of (7.46) by

$$\begin{aligned}
\Lambda\tau\left\|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right\|_{\mathbf{W}}\mathcal{V}(u_\tau^n - u_\tau^{n-1})^{1/2} &\overset{(1)}{\le} \Lambda\tau\left\|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right\|_{\mathbf{W}}\tau^{1/2}\mathcal{V}\left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right)^{1/2}\\
&= \Lambda\tau^{1/2}\int_{\underline{\mathsf{t}}_\tau(t)}^{\overline{\mathsf{t}}_\tau(t)}\|u_\tau'(r)\|_{\mathbf{W}}\mathcal{V}({u'}_\tau(r))^{1/2}\,\mathrm{d}r\\
&\overset{(2)}{\le} \frac{1}{2}\int_{\underline{\mathsf{t}}_\tau(t)}^{\overline{\mathsf{t}}_\tau(t)}\mathcal{V}({u'}_\tau(r))\,\mathrm{d}r + C\tau\int_{\underline{\mathsf{t}}_\tau(t)}^{\overline{\mathsf{t}}_\tau(t)}\|u_\tau'(r)\|_{\mathbf{W}}^2\,\mathrm{d}r,
\end{aligned}$$

where (1) follows from the fact that $\mathcal{V}(u_\tau^n - u_\tau^{n-1}) \leq \tau\mathcal{V}(\frac{u_\tau^n - u_\tau^{n-1}}{\tau})$ by the convexity of $\mathcal{V}$ and the fact that $\mathcal{V}(0) = 0$, while (2) is due to Young's inequality. Therefore, from (7.46), we gather

$$
\begin{aligned}
&\frac{\varrho}{2}\|u'_\tau(t)\|_{\mathbf{W}}^2 + \int_0^{\bar{\mathsf{t}}_\tau(t)} \frac{1}{2}\mathcal{V}\left(u'_\tau(r)\right) + \mathcal{V}^*(-\bar{\xi}_\tau(r) - \varrho v'_\tau(r))\,\mathrm{d}r + \int_0^{\bar{\mathsf{t}}_\tau(t)} \mathcal{R}(z'_\tau(r))\,\mathrm{d}r + \mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \overline{z}_\tau(t)) \\
&\leq \frac{\varrho}{2}\|v_0\|_{\mathbf{W}}^2 + \mathcal{E}(0, u_0, z_0) + \int_0^{\bar{\mathsf{t}}_\tau(t)} \partial_t\mathcal{E}(r, \underline{u}_\tau(r), \underline{z}_\tau(r))\,\mathrm{d}r + C\tau\int_0^{\bar{\mathsf{t}}_\tau(t)} \|u'_\tau(r)\|_{\mathbf{W}}^2\,\mathrm{d}r
\end{aligned}
$$

and we conclude by estimating the power term on the right-hand side via (2.5c), as in the proof of Proposition 7.2, and by applying the Gronwall Lemma. Therefore, estimates (7.28a)–(7.28b) and (7.47a)–(7.47b) ensue.

Finally, (7.47c) is a consequence of the bound for the energy (7.28a) combined with Hypothesis 5.3. Then, (7.47d) follows from (7.47a), taking into account that $\mathcal{V}^*$ has superlinear growth. □

Remark 7.7. First of all, let us mention that for $\mathcal{V}$ quadratic (cf. (4.1)), and under (5.15), the time-incremental minimum problem for u in (5.3b) has a unique solution u_τ^n. To see this, let u^1, u^2 two solutions to (5.3b). We subtract the Euler-Lagrange equation for u^2 from the one for u^1, and test the relation thus obtained by $u^1 - u^2$. We thus conclude

$$
\frac{\varrho}{\tau^2}\|u^1 - u^2\|_{\mathbf{W}}^2 + \frac{1}{\tau}a(u^1 - u^2, u^1 - u^2) + \langle \xi^1 - \xi^2, u^1 - u^2\rangle_{\mathbf{V}} = 0,
$$

with $\xi^i \in \partial_u^- \mathcal{E}(t_n, u^i, z_\tau^n)$ fulfilling the Euler-Lagrange equation for $i = 1, 2$. Now, from (5.15) we gather that

$$
\langle \xi^1 - \xi^2, u^1 - u^2\rangle_{\mathbf{V}} \geq -\Lambda\|u^1 - u^2\|_{\mathbf{V}}^2.
$$

Combining this with the above relation, using that $a(\cdot,\cdot)$ is coercive with respect to the norm of $\mathbf{V}$, and choosing $\tau > 0$ sufficiently small, we ultimately deduce that $\|u^1 - u^2\|_{\mathbf{W}}^2 + c\|u^1 - u^2\|_{\mathbf{V}}^2 \leq 0$, whence $u^1 = u^2$.

Moreover, observe that the very same arguments as in the proof of Prop. 7.6 yield that, in addition to the discrete semistability (7.29), the approximate solutions fulfill the discrete energy-dissipation inequality

$$
\begin{aligned}
&\frac{\varrho}{2}\|u'_\tau(t)\|_{\mathbf{W}}^2 + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \mathcal{V}\left(u'_\tau(r)\right) + \mathcal{V}^*(-\bar{\xi}_\tau(r) - \varrho v'_\tau(r))\,\mathrm{d}r + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \mathcal{R}(z'_\tau(r))\,\mathrm{d}r + \mathcal{E}(\bar{\mathsf{t}}_\tau(t), \overline{u}_\tau(t), \overline{z}_\tau(t)) \\
&\leq \frac{\varrho}{2}\|u'_\tau(s)\|_{\mathbf{W}}^2 + \mathcal{E}(\bar{\mathsf{t}}_\tau(s), \overline{u}_\tau(s), \overline{z}_\tau(s)) + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \partial_t\mathcal{E}(r, \underline{u}_\tau(r), \underline{z}_\tau(r))\,\mathrm{d}r + \Lambda\tau^{1/2}\int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \|u'_\tau(r)\|_{\mathbf{V}}^2\,\mathrm{d}r,
\end{aligned}
\tag{7.50}
$$

whereby estimates (7.47) ensue. In particular, since $\mathcal{V}$ is quadratic, due to the quadratic growth of $\mathcal{V}^*$ we have

$$
\|v'_\tau\|_{L^2(0,T;\mathbf{V}^*)} \leq C. \tag{7.51}
$$

We are now in the position to develop the **proof of Theorem 3**. Let $(\tau_k)_k$ be a vanishing sequence. With standard weak and strong compactness arguments (cf., e.g., [Sim87]), taking into account estimates (7.28) and Hyp. 2.7, we find a quadruple (u, z, ξ, v) with (u, z) as in (3.1), $\xi \in L^\infty(0,T;\mathbf{V}^*)$, and $v \in L^1(0,T;\mathbf{V}^*)$ such that convergences (4.16) hold, and, in addition,

$$
\begin{aligned}
&u_{\tau_k} \overset{*}{\rightharpoonup} u && \text{in } W^{1,\infty}(0,T;\mathbf{W}), && (7.52)\\
&u_{\tau_k} \to u && \text{in } \mathrm{C}^0([0,T];\mathbf{W}), && (7.53)\\
&\overline{u}_{\tau_k}(t) \to u(t) && \text{in } \mathbf{W} \quad \text{for every } t \in [0,T], && (7.54)\\
&\overline{\xi}_{\tau_k} \overset{*}{\rightharpoonup} \xi && \text{in } L^\infty(0,T;\mathbf{V}^*), && (7.55)\\
&v_{\tau_k} \rightharpoonup v && \text{in } W^{1,1}(0,T;\mathbf{V}^*). && (7.56)
\end{aligned}
$$

We easily find that $v' = u''$, whence $u \in W^{2,1}(0,T;\mathbf{V}^*)$.

Next, we prove that

$$
\xi(t) \in \partial_u^- \mathcal{E}(t, u(t), z(t)) \qquad \text{for a.a. } t \in (0,T). \tag{7.57}
$$

We distinguish two cases:

(1) Under the weak closedness condition from Hypothesis 4.11, we employ the very same Young measure argument as in the proof of Theorem 2, cf. (7.20), and conclude (7.57).

(2) In the case $\mathbf{U} \Subset \mathbf{V}$ with Hypothesis 4.6, to prove (7.57) we again resort to the Young measure argument from the proof of Theorem 1: In order to conclude that the set of the weak limit points of $(\xi_{\tau_k}(t))_k$ w.r.t. the topology of $\mathbf{V}^*$ is contained in $\partial_u^- \mathcal{E}(t, u(t), z(t))$, we use the closedness property (4.12) from Lemma 4.7. A close perusal of its proof shows that it extends to the case in which the *uniform Fréchet subdifferentiability* holds in the form of Hypothesis 4.3. In fact, the remainder term in (7.45) now reads $-\Lambda \mathcal{V}(v_n - u_n)^{1/2} \|v_n - u_n\|_{\mathbf{W}}$, and it goes to zero as $n \to \infty$ since $\mathbf{V} \subset \mathbf{W}$ continuously, and $\mathcal{V}$ is continuous on $\mathbf{V}$. Likewise, it also extends to the case when the Fréchet subdifferentiability Hypothesis 4.2 is replaced by its weaker variant (5.15).

We now address the limit passage in the discrete energy-dissipation inequality (7.50): It follows from (7.55)–(7.56) that

$$\liminf_{k\to\infty} \int_{\bar{\mathsf{t}}_{\tau_k}(s)}^{\bar{\mathsf{t}}_{\tau_k}(t)} \mathcal{V}^*(-\overline{\xi}_{\tau_k}(r) - \varrho v'_{\tau_k}(r))\, \mathrm{d}r \geq \int_s^t \mathcal{V}^*(-\xi(r) - \varrho u''(r))\, \mathrm{d}r$$

whereas for the remainder term on the right-hand side of (7.50) we have

$$\Lambda \tau^{1/2} \int_{\bar{\mathsf{t}}_{\tau_k}(s)}^{\bar{\mathsf{t}}_{\tau_k}(t)} \|u'_{\tau_k}(r)\|_{\mathbf{W}} \mathcal{V}(u'_{\tau_k}(r))^{1/2}\, \mathrm{d}r \leq \Lambda \tau^{1/2} \left(\int_{\bar{\mathsf{t}}_{\tau_k}(s)}^{\bar{\mathsf{t}}_{\tau_k}(t)} \|u'_{\tau_k}(r)\|_{\mathbf{W}}^2\, \mathrm{d}r \right) \left(\int_{\bar{\mathsf{t}}_{\tau_k}(s)}^{\bar{\mathsf{t}}_{\tau_k}(t)} \mathcal{V}(u'_{\tau_k}(r))\, \mathrm{d}r \right) \to 0 \text{ as } k \to \infty$$

in view of the previously obtained estimates (7.28b) and (7.47b). Therefore, we conclude the energy-dissipation inequality (3.4) on every interval $(0, t)$, and the enhanced (5.11) under the stronger requirement (4.25b) from Hyp. 4.11, or under Hyp. 4.6.

Arguing as in the proof of Theorem 2, we use Hypothesis 2.5 (possibly in its weaker form from Remark 2.6 under (4.25b), or Hyp. 4.6), to pass to the limit in the discrete semistability (7.29) and conclude that (u, z) fulfill the semistability (3.3).

Thus, we conclude that (u, z) is a weak energetic solution to the evolutionary system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$. ■

Finally, we present the proof of **Theorem 4**, just dwelling on its differences from the proof of Thm. 3. Let $(\tau_k)_k$ be a vanishing sequence. The very same compactness arguments as in the proof of Thm. 3 yield that there exists a quadruple (u, z, ξ, v) with (u, z) as in (3.1), such that convergences (4.16) and (7.52)–(7.55) hold. Moreover,

$$v_{\tau_k} \rightharpoonup v = u' \qquad \text{in } H^1(0, T; \mathbf{V}^*), \tag{7.58}$$

and, in view of the estimates for $(u'_{\tau_k})_k$ in $L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; \mathbf{V}^*) \cap H^1(0, T; \mathbf{V}^*)$ and of the compact embedding $\mathbf{V} \Subset \mathbf{W}$, we may also suppose, up to a further extraction, that

$$\begin{aligned} &u'_{\tau_k} \to u' \quad \text{in } L^p(0, T; \mathbf{W}) \quad \text{for all } 1 \leq p < \infty, \\ &u'_{\tau_k}(t) \rightharpoonup u'(t) \quad \text{in } \mathbf{W} \quad \text{for all } t \in [0, T]. \end{aligned} \tag{7.59}$$

Exploiting convergences (7.52)–(7.55) as well as (7.58), and the fact that $\partial\mathcal{V}$ is a linear and bounded operator A, we first pass to the limit in (5.6) and conclude that (u, z) fulfill

$$\varrho u''(t) + A u'(t) + \xi(t) = 0 \qquad \text{for a.a. } t \in (0, T), \tag{7.60}$$

with $\xi \in L^\infty(0, T; \mathbf{V}^*)$ from (7.55). In order to show that $\xi(t) \in \partial_u^- \mathcal{E}(t, u(t), z(t))$ for almost all $t \in (0, T)$, we exploit Hypothesis 4.8. Hence, we test the discrete momentum equation (5.6) by $\overline{u}_{\tau_k}$ and integrate in time, thus obtaining

$$\int_0^{\bar{\mathsf{t}}_{\tau_k}(t)} \langle \overline{\xi}_{\tau_k}(s), \overline{u}_{\tau_k}(s) \rangle_{\mathbf{V}}\, \mathrm{d}s = -\varrho \int_0^{\bar{\mathsf{t}}_{\tau_k}(t)} (v'_{\tau_k}(s), \overline{u}_{\tau_k}(s))_{\mathbf{W}}\, \mathrm{d}s - \int_0^{\bar{\mathsf{t}}_{\tau_k}(t)} \langle A u'_{\tau_k}(s), \overline{u}_{\tau_k}(s) \rangle\, \mathrm{d}s \doteq I_1 + I_2.$$

Now, the discrete integration-by-part formula yields

$$\begin{aligned} I_1 &= \varrho \int_0^{\bar{\mathsf{t}}_{\tau_k}(t)} (u'_{\tau_k}(s), u'_{\tau_k}(s - \tau_k))_{\mathbf{W}}\, \mathrm{d}s - \varrho(\overline{u}_{\tau_k}(t), u'_{\tau_k}(t))_{\mathbf{W}} + \varrho(u_0, \dot{u}^0_\tau)_{\mathbf{W}} \\ &\overset{k\to\infty}{\to} \varrho \int_0^t (u'(s), u'(s))_{\mathbf{W}}\, \mathrm{d}s - \varrho(u'(t), u(t))_{\mathbf{W}} + \varrho(u_0, v_0)_{\mathbf{W}} = -\varrho \int_0^t (u''(s), u(s))_{\mathbf{W}}\, \mathrm{d}s, \end{aligned}$$

with $\dot{u}^0_\tau := \frac{u_0 - u_\tau^{-1}}{\tau}$ and u_τ^{-1} from (5.2), where the limit passage ensues from convergences (7.59). With the very same calculations as for (7.43), we find that $\limsup_{k\to\infty} I_2 \leq -\int_0^t \langle A u', u \rangle_{\mathbf{V}}\, \mathrm{d}s$. Taking into account (7.60), we then conclude that $\limsup_{k\to\infty} \int_0^T \langle \overline{\xi}_{\tau_k}(t), \overline{u}_{\tau_k}(t)\rangle_{\mathbf{V}}\, \mathrm{d}t \leq \int_0^T \langle \xi(t), u(t)\rangle_{\mathbf{V}}\, \mathrm{d}t$, whence the identification of $\xi(t)$ as an element in $\partial_u^- \mathcal{E}(t, u(t), z(t))$ by Hypothesis 4.8. This concludes the proof. ∎

Appendix A. Young measure tools

For the reader's convenience, we collect here the definition of parameterized (or Young) measure with values in infinite-dimensional spaces, and the Young measure compactness result from [MRS13b] (cf. also [RS06]).

Notation. Given an interval $I \subset \mathbb{R}$, we denote by $\mathscr{L}_I$ the σ-algebra of the Lebesgue measurable subsets of I and, given a reflexive Banach space B, by $\mathscr{B}(B)$ its Borel σ-algebra. We use the symbol $\otimes$ for product σ-algebrae. We recall that a $\mathscr{L}_I \otimes \mathscr{B}(B)$-measurable function $h : I \times B \to (-\infty, +\infty]$ is a *normal integrand* if for a.a. $t \in (0, T)$ the map $x \mapsto h_t(x) = h(t, x)$ is lower semicontinuous on B.

We consider the space B endowed with the *weak* topology, and say that a $\mathscr{L}_{(0,T)} \otimes \mathscr{B}(B)$–measurable functional $\mathcal{H} : (0, T) \times B \to (-\infty, +\infty]$ is a *weakly-normal integrand* if for a.a. $t \in (0, T)$ the map

$$w \mapsto h(t, w) \text{ is sequentially lower semicontinuous on } B \text{ w.r.t. the weak topology.} \tag{A.1}$$

We denote by $\mathscr{M}(0, T; \mathcal{V})$ the set of all $\mathscr{L}_{(0,T)}$-measurable functions $y : (0, T) \to B$. A sequence $(w_n) \subset \mathscr{M}(0, T; B)$ is said to be *weakly-tight* if there exists a weakly-normal integrand $\mathcal{H} : (0, T) \times B \to (-\infty, +\infty]$ such that the map

$$w \mapsto \mathcal{H}_t(w) \text{ has compact sublevels w.r.t. the weak topology of } B \text{, and}$$

$$\sup_n \int_0^T \mathcal{H}(t, w_n(t))\, \mathrm{d}t < \infty.$$

Definition A.1 ((Time-dependent) Young measures). *A* Young measure *in the space B is a family $\boldsymbol{\mu} := \{\mu_t\}_{t\in(0,T)}$ of Borel probability measures on B such that the map on $(0, T)$*

$$t \mapsto \mu_t(A) \quad \textit{is} \quad \mathscr{L}_{(0,T)}\textit{-measurable} \quad \textit{for all } A \in \mathscr{B}(B). \tag{A.2}$$

We denote by $\mathscr{Y}(0, T; B)$ the set of all Young measures in B.

Theorem A.2. [MRS13b, Theorems A.2, A.3] *Let $\{\mathcal{H}_n\}$, $\mathcal{H} : (0, T) \times B \to (-\infty, +\infty]$ be weakly-normal integrands such that for all $w \in B$ and for a.a. $t \in (0, T)$*

$$\mathcal{H}(t, w) \leq \inf \left\{ \liminf_{n\to\infty} \mathcal{H}_n(t, w_n) \,:\, w_n \rightharpoonup w \textit{ in } B \right\}. \tag{A.3}$$

Let $(w_n) \subset \mathscr{M}(0, T; B)$ be a weakly-tight sequence. Then, there exist a subsequence (w_{n_k}) and a Young measure $\boldsymbol{\mu} = \{\mu_t\}_{t\in(0,T)}$ such that for a.a. $t \in (0, T)$

$$\mu_t \textit{ is concentrated on the set } L(t) := \bigcap_{p=1}^{\infty} \overline{\{w_{n_k}(t) \,:\, k \geq p\}}^{\mathrm{w}} \tag{A.4}$$

of the limit points of the sequence $(w_{n_k}(t))$ with respect to the weak topology of B and, if the sequence $t \mapsto \mathcal{H}_{n_k}^-(t, w_{n_k}(t))$ is uniformly integrable, there holds

$$\liminf_{k\to\infty} \int_0^T \mathcal{H}_{n_k}(t, w_{n_k}(t))\, \mathrm{d}t \geq \int_0^T \int_{\mathcal{V}} \mathcal{H}(t, w)\, \mathrm{d}\mu_t(w)\, \mathrm{d}t\,. \tag{A.5}$$

If in addition $(w_n) \subset L^p(0, T; \mathcal{V})$ is bounded and, in the case $p = 1$, also uniformly integrable, then, up to extracting a further subsequence from (w_{n_k}) we have that, setting

$$\mathrm{w}(t) := \int_B w\, \mathrm{d}\mu_t(w) \qquad \textit{for a.a. } t \in (0, T)\,,$$

there holds

$$w_{n_k} \rightharpoonup \mathrm{w} \quad \textit{in } L^p(0, T; B), \tag{A.6}$$

with $\rightharpoonup$ replaced by $\overset{*}{\rightharpoonup}$ if $p = \infty$.

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