IMPROVED CONVERGENCE THEOREMS FOR BUBBLE CLUSTERS. II. THE THREE-DIMENSIONAL CASE

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Abstract. Given a sequence $\{\mathcal{E}_k\}_k$ of almost-minimizing clusters in $\mathbb{R}^3$ which converges in L^1 to a limit cluster $\mathcal{E}$ we prove the existence of $C^{1,\alpha}$-diffeomorphisms f_k between $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ which converge in C^1 to the identity. Each of these boundaries is divided into $C^{1,\alpha}$-surfaces of regular points, $C^{1,\alpha}$-curves of points of type Y (where the boundary blows-up to three half-spaces meeting along a line at 120 degree) and isolated points of type T (where the boundary blows up to the two-dimensional cone over a one-dimensional regular tetrahedron). The diffeomorphisms f_k are compatible with this decomposition, in the sense that they bring regular points into regular points and singular points of a kind into singular points of the same kind. They are almost-normal, meaning that at fixed distance from the set of singular points each f_k is a normal deformation of ∂E, and at fixed distance from the points of type T, f_k is a normal deformation of the set of points of type Y. Finally, the tangential displacements are quantitatively controlled by the normal displacements. This improved convergence theorem is then used in the study of isoperimetric clusters in $\mathbb{R}^3$.

Contents

1. Introduction

1.1. **Overview.** This paper is the second part of [CLM14]. In [CLM14, Theorem 3.1], having in mind to address the convergence of stratified singular sets in geometric variational problems, we have detailed a procedure to construct structured diffeomorphisms between manifolds with boundary (in arbitrary dimension and codimension). This result was then used as the starting point to obtain an *improved convergence theorem* for planar almost-minimizing clusters, which in turn was used to the address a question posed by Almgren in [Alm76] concerning the classification of isoperimetric clusters. We discuss here the extension of these results to almost-minimizing clusters in $\mathbb{R}^3$. There are of course major difficulties in this extension, as the structure of singular sets is by far more complex in three-dimensions than in the planar case. Referring to the introduction of [CLM14] for detailed motivations, bibliographical references and further applications of improved convergence theorems, we directly pass to introduce the main results proved in this paper.

1.2. **Clusters.** A N-cluster $\mathcal{E}$ in $\mathbb{R}^n$ $(N, n \geq 2)$ is a family $\mathcal{E} = \{\mathcal{E}(h)\}_{h=1}^N$ of sets of locally finite perimeter in $\mathbb{R}^n$ such that $0 < |\mathcal{E}(h)|$ for $1 \leq h \leq N$ and $|\mathcal{E}(h) \cap \mathcal{E}(k)| = 0$ for $1 \leq h < k \leq N$. The set $\mathcal{E}(h)$ is the hth chamber of $\mathcal{E}$ and $\mathcal{E}(0) = \mathbb{R}^n \setminus \bigcup_{h=1}^N$ is the exterior chamber of $\mathcal{E}$. The volume $\mathrm{vol}\,(\mathcal{E})\in\mathbb{R}^N_+$ of $\mathcal{E}$ has hth entry given by $|\mathcal{E}(h)|$, and the perimeter of $\mathcal{E}$ relative to $F\subset\mathbb{R}^n$ is defined by

$$P(\mathcal{E};F)=\frac{1}{2}\sum_{h=0}^N P(\mathcal{E}(h);F)=\sum_{0\le h<k\le N}\mathcal{H}^{n-1}(F\cap\mathcal{E}(h,k))\,,\qquad P(\mathcal{E})=P(\mathcal{E};\mathbb{R}^n)\,,$$

where $\mathcal{E}(h,k)=\partial^*\mathcal{E}(h)\cap\partial^*\mathcal{E}(k)$ and ∂^*E denotes the reduced boundary of a set of locally finite perimeter E in $\mathbb{R}^n$. We shall always normalize (modulo Lebesgue null sets) the chambers $\mathcal{E}(h)$ so to have that $\mathrm{cl}\,(\partial^*\mathcal{E}(h))=\partial\mathcal{E}(h)$ for $h=0,...,N$, where cl stands for topological closure. In this way, setting

$$\partial\mathcal{E}=\bigcup_{h=1}^N\partial\mathcal{E}(h)\,,\qquad \partial^*\mathcal{E}=\bigcup_{h=1}^N\partial^*\mathcal{E}(h)=\bigcup_{0\le h<k\le N}\mathcal{E}(h,k)\,,\qquad \Sigma(\mathcal{E})=\partial\mathcal{E}\setminus\partial^*\mathcal{E}\,,$$

we have $P(\mathcal{E};F)=\mathcal{H}^{n-1}(F\cap\partial^*\mathcal{E})$ and $\mathrm{cl}\,(\partial^*\mathcal{E})=\partial\mathcal{E}$. An *isoperimetric cluster* is a N-cluster $\mathcal{E}$ in $\mathbb{R}^n$ such that

$$P(\mathcal{E})\le P(\mathcal{F})\qquad\text{whenever }\mathrm{vol}\,(\mathcal{E})=\mathrm{vol}\,(\mathcal{F}).$$

If $\mathcal{E}$ is an isoperimetric cluster, then $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in $\mathbb{R}^n$ (for some positive constants Λ and r_0 depending on $\mathcal{E}$ only) according to the following definition. Setting

$$\mathrm{d}_F(\mathcal{E},\mathcal{F})=\frac{1}{2}\sum_{h=0}^N|(\mathcal{E}(h)\Delta\mathcal{F}(h))\cap F|\,,\qquad \mathrm{d}(\mathcal{E},\mathcal{F})=\mathrm{d}_{\mathbb{R}^n}(\mathcal{E},\mathcal{F})\,,$$

for the L^1-distance between the N-clusters $\mathcal{E}$ and $\mathcal{F}$ in $F\subset\mathbb{R}^n$, one says that $\mathcal{E}$ is a (perimeter) *(Λ,r_0)-minimizing cluster in $\mathbb{R}^n$* if

$$P(\mathcal{E};B_{x,r})\le P(\mathcal{F};B_{x,r})+\Lambda\,\mathrm{d}(\mathcal{E},\mathcal{F})\,,\tag{1.1}$$

whenever $x\in\mathbb{R}^n$, $r<r_0$ and $\mathcal{E}(h)\Delta\mathcal{F}(h)\subset\subset B_{x,r}$ for every $h=1,...,N$. In this case, following [Alm76], $\partial^*\mathcal{E}$ is a $C^{1,\beta}$-hypersurface in $\mathbb{R}^n$ for every $\beta\in(0,1)$, $\mathcal{H}^{n-1}(\Sigma(\mathcal{E}))=0$, and $\mathcal{E}(h)$ is an open set for every $h=0,...,N$; see also [CLM14, Section 3]. If in addition $\mathcal{E}$ is an isoperimetric cluster, then $\partial\mathcal{E}$ is bounded and $\partial^*\mathcal{E}$ is a constant mean curvature (thus analytic) hypersurface.

1.3. **Taylor's regularity theorem.** When $n=3$ much more can be said about $\Sigma(\mathcal{E})$ and the behavior of $\partial^*\mathcal{E}$ near $\Sigma(\mathcal{E})$ thanks to Taylor's theorem [Tay76]. In Theorem 1.1 below we formulate her result in our context. To this end, we denote by Y a reference closed cone in $\mathbb{R}^3$ defined by three half-planes meeting along their common boundary line (which contains the origin of $\mathbb{R}^3$) by forming 120 degrees angles. We denote by T a reference closed cone in $\mathbb{R}^3$ spanned by edges of a regular tetrahedron and with vertex at the barycenter of the tetrahedron – which is assumed to be the origin of $\mathbb{R}^3$. Both Y and T are two-dimensional cones in $\mathbb{R}^3$ (with vertex at the origin), and it turns out that, modulo isometries, they model (as tangent cones) all the possible singularities of (Λ,r_0)-minimizing clusters in $\mathbb{R}^3$. By exploiting [Tay76] one can indeed deduce the following result, where, given $M\subset\mathbb{R}^3$ and $x\in M$, we use the notation

$$\theta_M(x)=\lim_{r\to0^+}\frac{\mathcal{H}^2(M\cap B_{x,r})}{r^2}\qquad\text{(provided this limit exists)}\,.\tag{1.2}$$

Theorem 1.1. *There exists $\alpha\in(0,1)$ with the following property. If $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in $\mathbb{R}^3$, then $\theta_{\partial\mathcal{E}}(x)$ exists for every $x\in\partial\mathcal{E}$ and*

$$\begin{gathered}\partial^*\mathcal{E}=\{\theta_{\partial\mathcal{E}}=\pi\}\,,\qquad \Sigma(\mathcal{E})=\Sigma_Y(\mathcal{E})\cup\Sigma_T(\mathcal{E})\,,\\ \Sigma_Y(\mathcal{E})=\{\theta_{\partial\mathcal{E}}=\theta_Y(0)\}\,,\qquad \Sigma_T(\mathcal{E})=\{\theta_{\partial\mathcal{E}}=\theta_T(0)\}\,.\end{gathered}\tag{1.3}$$

Moreover, $\Sigma_T(\mathcal{E})$ is locally finite, there exists a locally finite family $\mathcal{S}(\mathcal{E})$ of closed connected topological surfaces with boundary in $\mathbb{R}^3$ such that

$$S^* = S \setminus \Sigma_T(\mathcal{E}) \text{ is a } C^{1,\alpha}\text{-surface with boundary in } \mathbb{R}^3 \text{ for every } S \in \mathcal{S}(\mathcal{E}),$$
$$\partial\mathcal{E} = \bigcup_{S\in\mathcal{S}(\mathcal{E})} S\,, \qquad \partial^*\mathcal{E} = \bigcup_{S\in\mathcal{S}(\mathcal{E})} \mathrm{int}\,(S^*)\,, \qquad \Sigma_Y(\mathcal{E}) = \bigcup_{S\in\mathcal{S}(\mathcal{E})} \mathrm{bd}\,(S^*)\,, \tag{1.4}$$

and there exists a locally finite family $\Gamma(\mathcal{E})$ of closed connected $C^{1,\alpha}$-curves with boundary in $\mathbb{R}^3$ such that

$$\Sigma_Y(\mathcal{E}) = \bigcup_{\gamma\in\Gamma(\mathcal{E})} \mathrm{int}\,(\gamma)\,, \qquad \Sigma_T(\mathcal{E}) = \bigcup_{\gamma\in\Gamma(\mathcal{E})} \mathrm{bd}\,(\gamma)\,. \tag{1.5}$$

Finally, for every $x \in \partial\mathcal{E}$ there exists a cone X in $\mathbb{R}^3$ (with vertex at the origin) such that, with hd_{B_R} *denoting the Hausdorff distance localized in the ball B_R (see* (2.1) *below), one has*

$$\lim_{r\to 0^+} \mathrm{hd}_{B_R}\Big(\frac{\partial\mathcal{E}-x}{r}, X\Big) = 0\,, \qquad \forall R > 0\,. \tag{1.6}$$

Here, if $x \in \partial^*\mathcal{E}$, then X is a plane, and if $x \in \Sigma(\mathcal{E})$, then $X = g(Y)$ or $X = g(T)$ for a linear isometry g of $\mathbb{R}^3$ depending on whether $x \in \Sigma_Y(\mathcal{E})$ or $x \in \Sigma_T(\mathcal{E})$. X is called the tangent cone to $\partial\mathcal{E}$ at x, and we set $X = T_x\partial\mathcal{E}$.

Remark 1.2 (Clusters of class $C^{2,1}$)**.** As a byproduct of (1.6) one sees that if $S \in \mathcal{S}(\mathcal{E})$ and $\nu_S \in C^{0,\alpha}(\mathrm{int}\,(S);\mathbb{S}^2)$ is such that $T_xS = \nu_S(x)^\perp$ for every $x \in \mathrm{int}\,(S)$, then ν_S can be extended by continuity to the whole S. If $\partial^*\mathcal{E}$ is a surface of class C^2, then $\nabla^S\nu_S$ is a continuous $\mathbb{R}^n\otimes\mathbb{R}^n$-field on $\mathrm{int}\,(S)$ (here we are using the convention adopted in [CLM14] that tangential gradients to manifolds are seen as linear maps on the whole ambient tangent space which take zero values on the orthogonal directions to the manifold). Correspondingly, we say that a (Λ, r_0)*-minimizing cluster $\mathcal{E}$ in $\mathbb{R}^3$ is of class $C^{2,1}$* if $\partial^*\mathcal{E}$ is of class $C^{2,1}$ and if, for every $S \in \mathcal{S}(\mathcal{E})$, $\nabla^S\nu_S$ can be extended by continuity to the whole S in such a way that for each $x, y \in S$ one has

$$\begin{aligned}
&\|\nabla^S\nu_S(y) - \nabla^S\nu_S(x)\| \le C\,|x-y|\,,\\
&|\nu_S(y) - \nu_S(x) - \nabla^S\nu_S(x)[x-y]| \le C\,|x-y|^2\,,\\
&|\nu_S(x)\cdot(y-x) - \nabla^S\nu_S(x)[x-y]\cdot(y-x)| \le C\,|x-y|^3\,,
\end{aligned}\tag{1.7}$$

for some constant C depending on $\mathcal{E}$ only, and where $\|\cdot\|$ denotes the operator norm on $\mathbb{R}^n\otimes\mathbb{R}^n$. We notice that by the higher regularity results of [KNS78] each isoperimetric cluster in $\mathbb{R}^3$ is of class $C^{2,1}$ (actually analytic). Moreover, (1.7) implies that each $\gamma \in \Gamma(\mathcal{E})$ is of class $C^{2,1}$.

1.4. **The improved convergence theorem and some applications.** If $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster in $\mathbb{R}^3$, then we say that $f \in C^{1,\alpha}(\partial\mathcal{E};\mathbb{R}^3)$ provided $f : \partial\mathcal{E} \to \mathbb{R}^3$ is continuous on $\partial\mathcal{E}$, $f \in C^{1,\alpha}(S^*)$ for every $S \in \mathcal{S}(\mathcal{E})$ and

$$\|f\|_{C^{1,\alpha}(\partial\mathcal{E})} := \sup_{S\in\mathcal{S}(\mathcal{E})} \|f\|_{C^{1,\alpha}(S^*)} < \infty\,.$$

If $\mathcal{E}$ and $\mathcal{F}$ are (Λ, r_0)-minimizing clusters in $\mathbb{R}^3$, then f is a $C^{1,\alpha}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$ provided f is an homeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$, $f \in C^{1,\alpha}(\partial\mathcal{E};\mathbb{R}^3)$, $f^{-1} \in C^{1,\alpha}(\partial\mathcal{F};\mathbb{R}^3)$ and

$$f(\Sigma_Y(\mathcal{E})) = \Sigma_Y(\mathcal{F})\,, \qquad f(\Sigma_T(\mathcal{E})) = \Sigma_T(\mathcal{F})\,.$$

Finally, if $\nu_\mathcal{E} : \partial^*\mathcal{E} \to S^2$ is any Borel vector field with $\nu_\mathcal{E}(x) \in \{\nu_{\mathcal{E}(h)}(x), \nu_{\mathcal{E}(k)}(x)\}$ for $x \in \mathcal{E}(h,k)$ and $f : \partial^*\mathcal{E} \to \mathbb{R}^3$, then we define the tangential component of f with respect to $\partial^*\mathcal{E}$, $\boldsymbol{\tau}_\mathcal{E} f : \partial^*\mathcal{E} \to \mathbb{R}^3$, as

$$\boldsymbol{\tau}_\mathcal{E} f(x) = f(x) - (f(x)\cdot\nu_\mathcal{E}(x))\,\nu_\mathcal{E}(x)\,, \qquad x \in \partial^*\mathcal{E}\,.$$

Our improved convergence theorem takes then the following form (here, $\alpha \in (0,1)$ is as in Theorem 1.1).

Theorem 1.3. *Given $\Lambda \geq 0$, $r_0 > 0$ and a bounded (Λ, r_0)-minimizing cluster $\mathcal{E}$ in $\mathbb{R}^3$ of class $C^{2,1}$, then there exist positive constants μ_0 and C_0 (depending on Λ and $\mathcal{E}$) with the following property. If $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ, r_0)-minimizing clusters in $\mathbb{R}^3$ such that $\mathrm{d}(\mathcal{E}_k, \mathcal{E}) \to 0$ as $k \to \infty$, then for every $\mu < \mu_0$ there exist $k(\mu) \in \mathbb{N}$ and a sequence of maps $\{f_k\}_{k\geq k(\mu)}$ such that each f_k is a $C^{1,\alpha}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ with*

$$\begin{aligned} \|f_k\|_{C^{1,\alpha}(\partial\mathcal{E})} &\leq C_0\,, \\ \lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1(\partial\mathcal{E})} &= 0\,, \\ \|\boldsymbol{\tau}_{\mathcal{E}}(f_k - \mathrm{Id})\|_{C^1(\partial^*\mathcal{E})} &\leq \frac{C_0}{\mu}\,\|f_k - \mathrm{Id}\|_{C^1(\Sigma_Y(\mathcal{E}))}\,, \\ \boldsymbol{\tau}_{\mathcal{E}}(f_k - \mathrm{Id}) &= 0\,, \qquad \text{on } \partial\mathcal{E} \setminus I_\mu(\Sigma(\mathcal{E}))\,. \end{aligned} \tag{1.8}$$

Remark 1.4. The last property in (1.8) says that f_k is almost-normal on $\partial\mathcal{E}$, meaning that it is a normal deformation of $\partial\mathcal{E}$ at a fixed distance from $\Sigma(\mathcal{E})$. Actually more is true, as it will become apparent from the proof of Theorem 1.1: the diffeomorphisms f_k is also almost normal on $\Sigma_Y(\mathcal{E})$. More precisely, for each $\gamma \in \Gamma(\mathcal{E})$, denoting by $\pi^\gamma_x v$ the projection of $v \in \mathbb{R}^3$ on $T_x\gamma$, and setting $(\pi^\gamma h)(x) = \pi^\gamma_x(h(x))$ for $h : \gamma \to \mathbb{R}^3$, then

$$\pi^\gamma(f_k - \mathrm{Id}) = 0 \quad \text{on } \gamma \setminus I_\mu(\Sigma_T(\mathcal{E}))\,, \qquad \|\pi^\gamma(f_k - \mathrm{Id})\|_{C^1(\gamma)} \leq \frac{C_0}{\mu}\,\|f_k - \mathrm{Id}\|_{C^0(\gamma\cap\Sigma_T(\mathcal{E}))}\,, \tag{1.9}$$

see in particular Lemma 4.5 and Lemma 4.6 below. Notice that the penultimate condition in (1.8) and the second condition in (1.9) express a quantitative control on the tangential displacements in terms of the corresponding normal displacements.

There are of course many different applications of Theorem 1.3 that one may wish to explore. One direction is definitely the discussion of global stability inequalities. In the case of the planar counterpart of Theorem 1.3, namely [CLM14, Theorem 1.5], this kind of analysis has been performed on planar double-bubbles [CLM12] and hexagonal honeycombs [CM14]. Another interesting direction is discussing the relation between strict stability (positive second variation) and local minimality. Leaving for future investigations these kind of questions, we discuss here two more immediate consequences of Theorem 1.3, whose planar analogs have been presented in [CLM14, Theorem 1.9, Theorem 1.10].

The first result is an application to the classification problem for isoperimetric clusters [Alm76, VI.1(6)]. We introduce an equivalence relation $\approx$ on the family of clusters in $\mathbb{R}^3$ that are (Λ, r_0)-minimizing cluster for some choice of $\Lambda \geq 0$ and $r_0 > 0$, by setting $\mathcal{E} \approx \mathcal{F}$ if and only if there exists a $C^{1,\alpha}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$.

Theorem 1.5. *For every $m_0 \in \mathbb{R}^N_+$ there exists $\delta > 0$ such that if Ω is the family of the isoperimetric clusters in $\mathbb{R}^3$ with $|\mathrm{vol}\,(\mathcal{E}) - m_0| < \delta$, then $\Omega/_{\approx}$ is a finite set.*

One can also qualitatively describe global minimizers of the cluster perimeter in the presence of a sufficiently small potential energy term.

Theorem 1.6. *Let $m_0 \in \mathbb{R}^N_+$ be such that there exists a unique (modulo isometries) isoperimetric cluster $\mathcal{E}_0$ in $\mathbb{R}^3$ with $\mathrm{vol}\,(\mathcal{E}_0) = m_0$, and let $g : \mathbb{R}^3 \to [0,\infty)$ be a continuous function with $g(x) \to \infty$ as $|x| \to \infty$. Then there exists $\delta_0 > 0$ (depending on $\mathcal{E}_0$ and g only) such that for every $\delta < \delta_0$ and $|m - m_0| < \delta_0$ there exists a minimizer $\mathcal{E}$ in*

$$\inf\Big\{P(\mathcal{E}) + \delta\sum_{h=1}^N \int_{\mathcal{E}(h)} g(x)\,dx : \mathrm{vol}\,(\mathcal{E}) = m\Big\}\,, \tag{1.10}$$

and necessarily it must be $\mathcal{E} \approx \mathcal{E}_0$.

Theorem 1.5 and Theorem 1.6 are deduced from Theorem 1.3 in exactly the same way as [CLM14, Theorem 1.9 and Theorem 1.10] are obtained from [CLM14, Theorem 1.5]. The only significant difference with the planar case is that in $\mathbb{R}^3$ obtaining compactness from perimeter bounds is a subtler issue. Considering that this kind of question has been discussed at length in the companion paper [CLM12], see in particular Appendix A therein, and taking into account the already considerable length of the present two-part paper, we shall omit a detailed presentation of the proofs of Theorem 1.5 and Theorem 1.6.

1.5. **Organization of the paper.** In section 2 we recall the results of Taylor [Tay76] and, more recently of David [Dav09, Dav10], which provide us with the local description of singular sets needed in order to begin our analysis. In particular, we prove Theorem 1.1. In section 3 we show the stratified Hausdorff convergence of singular sets, while in section 4 we prove the converge of the decomposition of $\partial\mathcal{E}_k$ into curves and surfaces introduced in Theorem 1.1 to the corresponding decomposition of $\partial\mathcal{E}$. In section 5 we finally deduce Theorem 1.3, while in Appendix A we present a technical result bridging between our "distributional" context based on the theory of sets of finite perimeter and the theory of $(\mathbf{M},\xi,\delta)$-minimal sets by Almgren used in Taylor's and David's papers.

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2. Structure of (Λ,r_0)-minimizing clusters in $\mathbb{R}^3$

The goal of this section is the proof of Theorem 1.1. We first recall the results of Taylor [Tay76] and David [Dav09, Dav10] in section 2.2. The main result proved here is then Theorem 2.1, section 2.3, which enables one to use exploit Taylor's regularity theory to boundaries of (Λ,r_0)-minimizing clusters. Finally, in section 2.4, we prove Theorem 1.1.

2.1. **Sets and manifolds.** We set $B(x,r)=B_{x,r}$ for the ball of center $x\in\mathbb{R}^n$ and radius $r>0$, and set $B_r=B_{0,r}=B(0,r)$, $B=B_1$, $\mathbb{S}^{n-1}=\partial B$. Given $S\subset\mathbb{R}^n$, $\mathring{S}$, ∂S, $\mathrm{cl}\,(S)$ are the interior, the boundary and the closure of S, while $I_\varepsilon(S)=\{x\in\mathbb{R}^n:\mathrm{dist}(x,S)<\varepsilon\}$ is the ε-neighborhood of S, $\varepsilon>0$. Given $S,T\subset\mathbb{R}^n$ we define the Hausdorff distance between S and T localized in $K\subset\mathbb{R}^n$ as

$$\mathrm{hd}_K(S,T)=\max\Big\{\sup\{\mathrm{dist}(y,S):y\in T\cap K\},\sup\{\mathrm{dist}(y,T):y\in S\cap K\}\Big\}\,,\qquad(2.1)$$

and set $\mathrm{hd}_{x,r}(S,T)=\mathrm{hd}_{B_{x,r}}(S,T)$ and $\mathrm{hd}(S,T)=\mathrm{hd}_{\mathbb{R}^n}(S,T)$. If S is a k-dimensional (embedded) C^1-manifold in $\mathbb{R}^n$, then we set dist_S for the geodesic distance on S and denote by $N_\varepsilon(S)$ the normal ε-neighborhood to S. If S is a C^1-manifold with boundary in $\mathbb{R}^n$, then $\mathrm{int}\,(S)$ and $\mathrm{bd}\,(S)$ denote, respectively, the interior and the boundary points of S. If S is a topological manifold with boundary in $\mathbb{R}^n$, then we use $\mathrm{bd}_{\,\tau}(S)$ for the boundary points of S, and we set

$$[S]_\rho=S\setminus I_\rho(\mathrm{bd}_{\,\tau}(S))\,,\qquad\forall\rho>0\,.\qquad(2.2)$$

The terms curve, surface and hypersurface are used in place of 1-dimensional manifold, 2-dimensional manifold and $(n-1)$-dimensional manifold in $\mathbb{R}^n$. If S is a k-dimensional C^1-manifold in $\mathbb{R}^n$, $x\in S$, and $f:S\to\mathbb{R}^m$, then we set

$$\nabla^S f(x)[v]=\begin{cases}\lim_{t\to0}\frac{f(\gamma(t))-f(x)}{t} & \text{if } v\in T_xS,\ \gamma\in C^1((-\varepsilon,\varepsilon);S),\ \gamma(0)=x,\ \gamma'(0)=v\,,\\ 0 & \text{if } v\in(T_xS)^\perp\,.\end{cases}$$

and we let $\|f\|_{C^1(S)} = \sup_{x\in S}|f(x)| + \|\nabla^S f(x)\|$, where $\|L\| = \sup\{|L[v]| : |v| = 1\}$ for every linear map $L : \mathbb{R}^n \to \mathbb{R}^m$. For $\alpha \in (0,1]$ and S of class $C^{1,\alpha}$, we set

$$\begin{aligned} [\nabla^S f]_{C^{0,\alpha}(S)} &= \sup_{x,y\in S,\, x\neq y} \frac{\|\nabla^S f(x) - \nabla^S f(y)\|}{|x-y|^\alpha}\,, \\ \|f\|_{C^{1,\alpha}(S)} &= \sup_{x\in S} |f(x)| + \|\nabla^S f(x)\| + [\nabla^S f]_{C^{0,\alpha}(S)}\,. \end{aligned}$$

Finally, given an orientable k-dimensional $C^{1,\alpha}$-manifold S in $\mathbb{R}^n$ which admits a global normal frame of class $C^{1,\alpha}$ (i.e., such that for every $x \in S$ there exists an orthonormal basis $\{\nu_S^{(i)}(x)\}_{i=1}^{n-k}$ of $(T_xS)^\perp$ with the property $\nu_S^{(i)} \in C^{1,\alpha}(S)$ for each i), we write

$$\|S\|_{C^{1,\alpha}} \le L\,,$$

if

$$\begin{cases} |\nu_S^{(i)}(x) - \nu_S^{(i)}(y)| \le L\,|x-y|^\alpha\,, \\ |\nu_S^{(i)}(x)\cdot(y-x)| \le L|y-x|^{1+\alpha}\,, \end{cases} \qquad \forall x,y\in S\,, i = 1,...,n-k\,. \tag{2.3}$$

2.2. **$(\mathbf{M},\xi,\delta)$-minimal sets and Taylor's theorem.** Let $\delta > 0$ and let $\xi : (0,\infty) \to [0,\infty)$ be an increasing function such that $\xi(0^+) = 0$. Consider an open set $A \subset \mathbb{R}^n$ and a bounded set M which is relatively closed in A. We assume that, for some $1 \le k \le n-1$, one has $\mathcal{H}^k(M) < \infty$ and $\mathcal{H}^k(M \cap B_{x,r}) > 0$ for every $r > 0$ and $x \in M$: in this way, $\mathcal{H}^k \llcorner M$ is a finite Radon measure on A with $M = A \cap \mathrm{spt}(\mathcal{H}^k \llcorner M)$. Under these assumptions, one says that M is a (k-dimensional) $(\mathbf{M},\xi,\delta)$-minimal set in A if

$$\mathcal{H}^k(W\cap M) \le (1+\xi(r))\,\mathcal{H}^k(f(W\cap M))\,,$$

whenever $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map with $W \cup f(W) \subset\subset A$ and $\mathrm{diam}(W \cup f(W)) = r < \delta$, where $W = \{f \ne \mathrm{Id}\}$.

Let $\mathrm{reg}(M)$ denote the set of points at which M admits an approximate tangent plane, and set $\sigma(M) = M \setminus \mathrm{reg}(M)$. As a consequence of [Alm76, III.3(7)], if M is a $(\mathbf{M},\xi,\delta)$-minimal set in A for $\xi(r) = C\,r^\gamma$, $\gamma \in (0,1)$, then $\mathrm{reg}(M)$ is a k-dimensional $C^{1,\beta}$-manifold in A for every $\beta < \gamma/2$, $\sigma(M)$ is closed, and $\mathcal{H}^k(\sigma(M)) = 0$.

In the case $k = 2$, $n = 3$, Taylor [Tay76] has improved this regularity result to a sharp degree. Let Y and T be the reference cones introduced in section 1. Taylor shows that if M is a two-dimensional $(\mathbf{M},\xi,\delta)$-minimal set in $A \subset \mathbb{R}^3$ (for $\xi(r) = C\,r^\gamma$, $\gamma \in (0,1)$), then $\theta_M(x)$ exists for every $x \in M$ (see (1.2)) and

$$\mathrm{reg}(M) = \{\theta_M = \pi\}\,, \qquad \sigma(M) = \sigma_Y(M) \cup \sigma_T(M)\,, \tag{2.4}$$

where

$$\sigma_Y(M) = \big\{\theta_M = \theta_Y(0)\big\}\,, \qquad \sigma_T(M) = \big\{\theta_M = \theta_T(0)\big\}\,. \tag{2.5}$$

Moreover, there exists $\alpha \in (0,\gamma)$ such that for every $x \in \sigma(M)$ there exist $r_x > 0$, an open set $U \subset \mathbb{R}^3$ with $0 \in U$, and a $C^{1,\alpha}$-diffeomorphism Φ between U and $B_{x,r_x} \subset\subset A$ with $\Phi(0) = x$ such that,

$$\begin{aligned} &\text{if } x \in \sigma_Y(M), \text{ then } \Phi(Y\cap U) = M \cap B_{x,r_x} \text{ and } \Phi(\sigma_Y(Y)\cap U) = \sigma_Y(M)\cap B_{x,r_x};\\ &\text{if } x \in \sigma_T(M), \text{ then } \Phi(T\cap U) = M \cap B_{x,r_x} \text{ and } \Phi(\sigma_Y(T)\cap U) = \sigma_Y(M)\cap B_{x,r_x}. \end{aligned} \tag{2.6}$$

Note that $\sigma_Y(Y)$ is the boundary line shared by the three half-planes defining Y, while $\sigma_Y(T)$ is the union of four open half-lines sharing 0 as the common origin of their closures. In [Dav09, Dav10], David addresses the regularity of two-dimensional $(\mathbf{M},\xi,\delta)$-minimal set in $\mathbb{R}^n$ with $n \ge 3$ under a certain admissibility assumption on their possible tangent cones. This assumption is always satisfied when $n = 3$. In particular, he recovers Taylor's result, and actually proves some estimates that shall be useful in the sequel. For the sake of clarity we now give a precise statement of the result we shall use. In doing so, it is convenient to say that a closed set $X \subset \mathbb{R}^3$ is a *minimal cone* if either X is a plane through the origin, $X = \rho(Y)$, or $X = \rho(T)$ for a linear isometry ρ of $\mathbb{R}^3$. (In particular, X is a cone with respect to 0.)

Theorem A. *There exist positive constants $\alpha\,, \varepsilon_0 < 1$ and $C_0 \ge 1$ with following property.*

Let M be a closed set in $\mathbb{R}^3$ such that $\mathcal{H}^2 \llcorner M$ is a Radon measure and $\mathcal{H}^2(M \cap B_{x,r}) > 0$ for every $x \in M$ and $r > 0$, and assume that for some $L \ge 0$ and $\rho_0 > 0$ one has

$$\mathcal{H}^2(M \cap W) \le \mathcal{H}^2(f(M \cap W)) + L\, r^3\,, \tag{2.7}$$

whenever $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz map with $\mathrm{diam}(W \cup f(W)) = r < \rho_0$, $W = \{f \neq \mathrm{Id}\}$.
(a) There exists λ depending on L and ρ_0 such that

$$r \in (0, \rho_0) \mapsto \frac{\mathcal{H}^2(M \cap B_{x,r})}{r^2} + \lambda\, r$$

is increasing for every $x \in M$; moreover, for every $x \in M$ there exist $r_x \in (0, \rho_0/2)$ and a minimal cone X' with $\theta_{X'}(0) = \theta_M(x)$ such that

$$\varepsilon = \frac{\mathrm{hd}_{x,r_x}(M, x + X')}{r_x} + \Big(\frac{\mathcal{H}^2(M \cap B_{x,r_x})}{r_x^2} + \lambda\, r_x - \theta_M(x)\Big) \le \varepsilon_0\,. \tag{2.8}$$

(b) If $x \in M$, $r_x \in (0, \rho_0/2)$, and X' is a minimal cone with $\theta_{X'}(0) \le \theta_M(x)$ such that (2.8) holds, then there exists a minimal cone X such that $\theta_X(0) = \theta_M(x)$ and

$$\mathrm{hd}_{0,1}(X, X') \le C_0\, \varepsilon\,,$$
$$\frac{\mathrm{hd}_{x,r}(M, x + X)}{r} \le C_0\, \big(\frac{r}{r_x}\big)^{\alpha}\, \varepsilon\,, \qquad \forall r < \frac{r_x}{C_0}\,.$$

Moreover, for every $r \le r_x/C_0$ there exists a $C^{1,\alpha}$-diffeomorphism Φ between $B_{0,2r}$ and $\Phi(B_{0,2r})$ such that

$$\begin{aligned}
&\Phi(0) = x,\ B_{x,r} \subset \Phi(B_{0,2\,r}),\ \text{and}\ B_{0,r/C_0} \subset \Phi^{-1}(B_{x,r})\,,\\
&\Phi(X \cap B_{0,2r}) \cap B_{x,r} = M \cap B_{x,r}\,,\\
&\Phi(\sigma_Y(X) \cap B_{0,2r}) \cap B_{x,r} = \sigma_Y(M) \cap B_{x,r}\,,\\
&\|\Phi\|_{C^{1,\alpha}(B_{0,2r})} + \|\Phi^{-1}\|_{C^{1,\alpha}(\Phi(B_{0,2r}))} \le C_0\,.
\end{aligned} \tag{2.9}$$

Proof. As explained in [Dav10, Definition 1.10, Equation (1.13)] the results from [Dav10] apply to sets satisfying the almost-minimality condition (2.7). Assertion (a) then follows from [Dav10, Equation (3.13), Proposition 3.14], and assertion (b) is deduced by [Dav10, Theorem 12.8, Corollary 12.25]. □

2.3. (Λ, r_0)**-minimizing clusters as** $(\mathbf{M}, \xi, \delta)$**-minimal sets.** The theory of section 2.2 can be applied to the boundaries of (Λ, r_0)-minimizing clusters.

Theorem 2.1. *If $\mathcal{E}$ is a perimeter (Λ, r_0)-minimizing N-cluster in $\mathbb{R}^n$, then there exists positive constants L and ρ_0 (depending on Λ, r_0, n, N and $\max_{1 \le h \le N} |\mathcal{E}(h)|$ only) such that*

$$\mathcal{H}^{n-1}(W \cap \partial\mathcal{E}) \le \mathcal{H}^{n-1}(f(W \cap \partial\mathcal{E})) + L\, r^n\,, \tag{2.10}$$

whenever $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map and $\mathrm{diam}(W \cup f(W)) = r < \rho_0$, *where* $W = \{f \neq \mathrm{Id}\}$. *In particular, if $n = 3$, then $M = \partial\mathcal{E}$ satisfies the assumptions of Theorem A.*

The proof of Theorem 2.1 is discussed in Appendix A.

2.4. **Proof of Theorem 1.1.** *Step one*: Let ε_0, α and C_0 be as in Theorem A, and let $\mathcal{E}$ be a (Λ, r_0)-minimizing cluster in $\mathbb{R}^3$. By Theorem 2.1 we can apply Theorem A to $M = \partial\mathcal{E}$. In particular, by (2.4) and (2.5), $\theta_{\partial\mathcal{E}}(x)$ is defined for every $x \in \partial\mathcal{E}$, and thus we get

$$\partial\mathcal{E} = \{\theta_{\partial\mathcal{E}} = \pi\} \cup \{\theta_{\partial\mathcal{E}} = \theta_Y(0)\} \cup \{\theta_{\partial\mathcal{E}} = \theta_T(0)\}\,.$$

We set $\Sigma_Y(\mathcal{E}) = \{\theta_{\partial\mathcal{E}} = \theta_Y(0)\} = \sigma_Y(\partial\mathcal{E})$ and $\Sigma_T(\mathcal{E}) = \{\theta_{\partial\mathcal{E}} = \theta_T(0)\} = \sigma_T(\partial\mathcal{E})$. Again by Theorem A, for every $x \in \partial\mathcal{E}$ there exist a minimal cone X_x in $\mathbb{R}^3$ and $r_x > 0$ such that $\theta_{\partial\mathcal{E}}(x) = \theta_{X_x}(0)$,

$$\frac{\mathrm{hd}_{x,r}(\partial\mathcal{E}, x + X_x)}{r} \le C_0 \left(\frac{r}{r_x}\right)^\alpha, \qquad \forall r < \frac{r_x}{C_0}\,, \tag{2.11}$$

and there exists a $C^{1,\alpha}$-diffeomorphism Φ_x between $U_x = B_{0,2s_x}$ ($s_x = r_x/2C_0$) and $A_x = \Phi_x(B_{0,2s_x})$ such that $\Phi_x(0) = x$, $B_{x,s_x} \subset A_x$ and

$$\begin{aligned} \Phi_x(X_x \cap U_x) \cap B_{x,s_x} &= \partial\mathcal{E} \cap B_{x,s_x}\,,\\ \Phi_x(\sigma_Y(X_x) \cap U_x) \cap B_{x,s_x} &= \Sigma_Y(\mathcal{E}) \cap B_{x,s_x}\,,\\ \|\Phi_x\|_{C^{1,\alpha}(U_x)} + \|\Phi_x^{-1}\|_{C^{1,\alpha}(A_x)} &\le C_0\,. \end{aligned} \tag{2.12}$$

We claim that $\partial^*\mathcal{E} = \{\theta_{\partial\mathcal{E}} = \pi\}$. Indeed, $\theta_{\partial\mathcal{E}} = \pi$ on $\partial^*\mathcal{E}$ by De Giorgi's structure theorem for sets of finite perimeter. At the same time, if $\theta_{\partial\mathcal{E}}(x) = \pi$ for some $x \in \partial\mathcal{E}$, then X_x is a plane and thus, by (2.12), $B_{x,s_x} \setminus \partial\mathcal{E}$ has two distinct connected components. Hence there exists $0 \le h < k \le N$ such that $B_{x,s_x} \cap \mathcal{E}(j) \neq \emptyset$ if and only if $j = h, k$, so that $\mathcal{E}(h)$ is an open set with boundary of class $C^{1,\alpha}$ in B_{x,s_x}. In particular, $B_{x,s_x} \cap \partial\mathcal{E}(h) = B_{x,s_x} \cap \partial^*\mathcal{E}(h)$ and thus $x \in \partial^*\mathcal{E}$. We have thus proved

$$\partial^*\mathcal{E} = \{\theta_{\partial\mathcal{E}} = \pi\}\,, \qquad \text{and so} \qquad \Sigma(\mathcal{E}) = \Sigma_Y(\mathcal{E}) \cup \Sigma_T(\mathcal{E})\,, \tag{2.13}$$

that is, (1.3) holds.

Step two: Let now $x \in \Sigma_T(\mathcal{E})$. By $\sigma_T(X) = \{0\}$, $\Phi_x(0) = x$, $\Phi_x(\sigma_Y(X) \cap U_x) \cap B_{x,s_x} = \Sigma_Y(\mathcal{E}) \cap B_{x,s_x}$ and (2.13) we conclude that $\Sigma_T(\mathcal{E}) \cap B_{x,s_x} = \{x\}$. In particular, $\Sigma_T(\mathcal{E})$ is locally finite. By an analogous argument we check that $\Sigma_Y(\mathcal{E})$ is a $C^{1,\alpha}$-curve in $\mathbb{R}^3$, relatively open in $\Sigma(\mathcal{E})$, while (as we already know even when $n \ge 4$) $\partial^*\mathcal{E}$ is a $C^{1,1/2}$-surface in $\mathbb{R}^3$, relatively open in $\partial\mathcal{E}$. Let $\{M_i\}_{i\in I}$ and $\{\sigma_j\}_{j\in J}$ denote the connected components of $\partial^*\mathcal{E}$ and $\Sigma_Y(\mathcal{E})$ respectively, so that

$$\partial^*\mathcal{E} = \bigcup_{i\in I} M_i\,, \qquad \Sigma_Y(\mathcal{E}) = \bigcup_{j\in J} \sigma_j\,. \tag{2.14}$$

By (2.12), $\{M_i\}_{i\in I}$ and $\{\sigma_j\}_{j\in J}$ are locally finite, and each M_i is a connected $C^{1,\beta}$-surface in $\mathbb{R}^3$ for every $\beta \in (0,1)$, while each σ_j is a connected $C^{1,\alpha}$-curve in $\mathbb{R}^3$. In the following steps we check that (1.4) and (1.5) hold with

$$\mathcal{S}(\mathcal{E}) = \{S_i = \mathrm{cl}\,(M_i)\}_{i\in I}\,, \qquad \Gamma(\mathcal{E}) = \{\gamma_j = \mathrm{cl}\,(\sigma_j)\}_{j\in J}\,.$$

Step three: We first check that for each $j \in J$ there exist $0 \le k_1^j < k_2^j < k_3^j \le N$ such that $\sigma_j \cap \partial\mathcal{E}(h) \neq \emptyset$ if and only if $h \in \{k_1^j, k_2^j, k_3^j\}$. This follows immediately by (2.12), by the connectedness of σ_j and by means of a covering argument.

Step four: We prove that (1.5) holds with $\gamma_j = \mathrm{cl}\,(\sigma_j)$. We first check that γ_j is a connected $C^{1,\alpha}$-curve with boundary in $\mathbb{R}^3$. This is trivial if $\sigma_j = \gamma_j$, so let $\gamma_j \setminus \sigma_j \neq \emptyset$. Since $\sigma_j \subset \Sigma_Y(\mathcal{E}) \subset \Sigma(\mathcal{E})$ and $\Sigma(\mathcal{E})$ is closed we have $\gamma_j \setminus \sigma_j \subset \Sigma(\mathcal{E})$. At the same time, by (2.12) and by connectedness of σ_j, we have $\Sigma_Y(\mathcal{E}) \cap \gamma_j = \Sigma_Y(\mathcal{E}) \cap \sigma_j$, so that $\gamma_j \setminus \sigma_j \subset \Sigma_T(\mathcal{E})$. Let $x \in \gamma_j \setminus \sigma_j$, then by (2.12) and by $x \in \Sigma_T(\mathcal{E})$, $\Sigma_Y(\mathcal{E}) \cap B_{x,s_x}$ consists of four distinct $C^{1,\alpha}$-diffeomorphic images ρ_1, ρ_2, ρ_3 and ρ_4 of $(0,1)$. Without loss of generality we may assume that $\rho_1 \subset \sigma_j \cap B_{x,s_x}$. By showing that $\rho_1 = \sigma_j \cap B_{x,s_x}$ and by invoking again (2.12) we see that γ_j is $C^{1,\alpha}$-diffeomorphic to $[0,1)$ in a neighborhood of x, as required. To this end, it is enough to check that $\rho_m \cap \sigma_j \cap B_{x,s_x} = \emptyset$

for $m=2,3,4$. Indeed, by (2.12), for each $m=1,2,3,4$ there exist $0\le h_1^m<h_2^m<h_3^m\le N$ such that

$$\rho_m=\Sigma_Y(\mathcal{E})\cap B_{x,s_x}\cap\partial\mathcal{E}(h_1^m)\cap\partial\mathcal{E}(h_2^m)\cap\partial\mathcal{E}(h_3^m)\,, \tag{2.15}$$

and such that for every $1\le m<m'\le 4$ it holds

$$\#\Big(\{h_1^m,h_2^m,h_3^m\}\cap\{h_1^{m'},h_2^{m'},h_3^{m'}\}\Big)=2\,. \tag{2.16}$$

By step three, (2.15) and $\rho_1\subset\sigma_j\cap B_{x,s_x}$ it must be

$$\{k_1^j,k_2^j,k_3^j\}=\{h_1^1,h_2^1,h_3^1\}\,.$$

Hence, if $\rho_m\subset\sigma_j\cap B_{x,s_x}$ for some $m=2,3,4$, then

$$\{k_1^j,k_2^j,k_3^j\}=\{h_1^m,h_2^m,h_3^m\}\,,$$

thus leading to a contradiction with (2.16). This proves that γ_j is a connected $C^{1,\alpha}$-curve with boundary in $\mathbb{R}^3$ with

$$\mathrm{int}\,(\gamma_j)=\sigma_j\subset\Sigma_Y(\mathcal{E})\,,\qquad \mathrm{bd}\,(\gamma_j)\subset\Sigma_T(\mathcal{E})\,.$$

By (2.14) we find

$$\Sigma_Y(\mathcal{E})=\bigcup_{j\in J}\mathrm{int}\,(\gamma_j)\,,\qquad \bigcup_{j\in J}\mathrm{bd}\,(\gamma_j)\subset\Sigma_T(\mathcal{E})\,.$$

Finally, if $x\in\Sigma_T(\mathcal{E})$, then, by (2.12), $x\in\mathrm{cl}\,(\Sigma_Y(\mathcal{E}))$, and thus $x\in\gamma_j=\mathrm{cl}\,(\sigma_j)$ for some $j\in J$, and (1.5) holds.

Step five: We prove (1.4). By (2.14) and $\mathrm{cl}\,(\partial^*\mathcal{E})=\partial\mathcal{E}$ we see that $\partial\mathcal{E}=\bigcup_{i\in I}S_i$. We now claim that $S_i^*=S_i\setminus\Sigma_T(\mathcal{E})$ is a $C^{1,\alpha}$-surface with boundary in $\mathbb{R}^3$ with

$$\mathrm{int}\,(S_i^*)=M_i\,,\qquad \mathrm{bd}\,(S_i^*)\subset\Sigma_Y(\mathcal{E})\,. \tag{2.17}$$

Since $S_i^*\cap M_i=M_i$ we have that S_i^* is locally $C^{1,\beta}$-diffeomorphic to a disk at every $x\in S_i^*\cap M_i$ for every $\beta\in(0,1)$. If $x\in S_i^*\setminus M_i$, then $x\in\Sigma_Y(\mathcal{E})$. By (2.12) and by arguing as in step three and step four one checks that S_i^* is locally $C^{1,\alpha}$-diffeomorphic to a half-disk at every $x\in S_i^*\setminus M_i$. This proves (2.17), thus (1.4) up to the inclusion $\Sigma_Y(\mathcal{E})\subset\bigcup_{i\in I}\mathrm{bd}\,(S_i^*)$, which follows from (2.12) and the fact that $\partial^*\mathcal{E}=\bigcup_{i\in I}M_i$. The fact that S_i is a connected topological surface with boundary similarly follows from (2.12). Finally (1.6) follows by (1.3) and (2.11).

3. Hausdorff convergence of singular sets and tangent cones

The goal of this section is showing the convergence of singular sets and tangent cones for clusters in $\mathbb{R}^3$. Precisely, given $\Lambda\ge 0$ and $r_0>0$ we assume that

$$\begin{aligned}&\mathcal{E}\text{ is a }(\Lambda,r_0)\text{-minimizing cluster in }\mathbb{R}^3\text{ with }\partial^*\mathcal{E}\text{ of class }C^{2,1}\,,\\ &\{\mathcal{E}_k\}_{k\in\mathbb{N}}\text{ is a sequence of }(\Lambda,r_0)\text{-minimizing clusters in }\mathbb{R}^3\,,\\ &\mathrm{d}_{B_R}(\mathcal{E}_k,\mathcal{E})\to 0\text{ as }k\to\infty\text{ for every }R>0\,.\end{aligned} \tag{3.1}$$

Our starting point is the following result from [CLM14] (which holds *verbatim* for arbitrary n). Here and in the following, in analogy to (2.2) but with a slight abuse of notation, we set

$$[\partial\mathcal{E}]_\rho=\partial^*\mathcal{E}\setminus I_\rho(\Sigma(\mathcal{E}))\,,\qquad \forall\rho>0\,.$$

Theorem 3.1. *If* (3.1) *holds, then $\mathcal{E}$ is a (Λ,r_0)-minimizing cluster in $\mathbb{R}^3$, $\mathcal{H}^2\llcorner\partial^*\mathcal{E}_k\xrightarrow{*}\mathcal{H}^2\llcorner\partial\mathcal{E}$ as $k\to\infty$ as Radon measures, and there exist positive constants ρ_0 (depending on $\mathcal{E}$) and C (depending on Λ and $\mathcal{E}$) such that:*

(i) *for every $R>0$ one has $\mathrm{hd}_{B_R}(\partial\mathcal{E}_k,\partial\mathcal{E})\to 0$ as $k\to\infty$, and, actually,*

$$\lim_{k\to\infty}\mathrm{hd}_{B_R}\Big(\partial\mathcal{E}_k(i)\cap\partial\mathcal{E}_k(j),\partial\mathcal{E}(i)\cap\partial\mathcal{E}(j)\Big)=0\,,\qquad\forall 0\le i<j\le N\,; \tag{3.2}$$

(ii) *for every* $R > R' > 0$ *and* $\varepsilon > 0$ *there exist* $k_0 \in \mathbb{N}$ *such that*

$$\Sigma(\mathcal{E}_k) \cap B_{R'} \subset I_\varepsilon(\Sigma(\mathcal{E}) \cap B_R)\,, \qquad \forall k \geq k_0\,; \tag{3.3}$$

(iii) *for every* $R > R' > 0$ *and* $\rho < \rho_0$ *there exist* $k_0 \in \mathbb{N}$, $\varepsilon \in (0,\rho)$, $R'' \in (R',R)$ *and* $\{\psi_k\}_{k\geq k_0} \subset C^{1,\beta}([\partial\mathcal{E}]_\rho)$ *for every* $\beta \in (0,1)$ *such that*

$$(B_{R'} \cap \partial\mathcal{E}_k) \setminus I_{2\rho}(\Sigma(\mathcal{E}) \cap B_R) \subset (\mathrm{Id} + \psi_k \nu_{\mathcal{E}})([\partial\mathcal{E}]_\rho \cap B_{R''}) \subset B_R \cap \partial^*\mathcal{E}_k\,, \tag{3.4}$$

$$N_\varepsilon(B_{R'} \cap [\partial\mathcal{E}]_\rho) \cap \partial\mathcal{E}_k = (\mathrm{Id} + \psi_k\,\nu_{\mathcal{E}})(B_{R'} \cap [\partial\mathcal{E}]_\rho)\,, \tag{3.5}$$

for every $k \geq k_0$, *with*

$$\begin{aligned} &\lim_{k\to\infty} \|\psi_k\|_{C^1(B_{R''}\cap[\partial\mathcal{E}]_\rho)} = 0\,,\\ &\sup_{k\geq k_0} \|\psi_k\|_{C^{1,\beta}(B_{R''}\cap[\partial\mathcal{E}]_\rho)} \leq C(\beta,\Lambda,\mathcal{E},R',R) \quad \forall \beta \in (0,1)\,. \end{aligned} \tag{3.6}$$

Proof. This follows from [CLM14, Theorem 4.9 and Theorem 4.12]. □

We are now ready to prove the main result of this section. The constants α, ε_0 and C_0 will be the ones introduced in Theorem A.

Theorem 3.2. *If* (3.1) *holds, then*

$$\begin{aligned} &\lim_{k\to\infty} \mathrm{hd}_{B_R}(\Sigma(\mathcal{E}_k),\Sigma(\mathcal{E})) = \lim_{k\to\infty} \mathrm{hd}_{B_R}(\Sigma_Y(\mathcal{E}_k),\Sigma_Y(\mathcal{E})) = 0\,,\\ &\lim_{k\to\infty} \mathrm{hd}_{B_R}(\Sigma_T(\mathcal{E}_k),\Sigma_T(\mathcal{E})) = 0\,, \end{aligned} \tag{3.7}$$

for every $R > 0$. *Moreover, if* $x \in \Sigma(\mathcal{E})$, $x_k \in \Sigma(\mathcal{E}_k)$, $x_k \to x$ *as* $k \to \infty$, *and* $\theta_{\partial\mathcal{E}_k}(x_k) = \theta_{\partial\mathcal{E}}(x)$ *for every* $k \in \mathbb{N}$, *then*

$$\lim_{k\to\infty} \mathrm{hd}_{0,1}(T_x\partial\mathcal{E}, T_{x_k}\partial\mathcal{E}_k) = 0\,, \tag{3.8}$$

and there exists $s_x > 0$, *and for every* $r < s_x$ *there exist* $k_{x,r} \in \mathbb{N}$ *and* $C^{1,\alpha}$*-diffeomorphisms* Φ_r *and* $\Phi_{k,r}$ *defined on* $B_{0,2r}$ *such that*

$$\begin{aligned} &\Phi_r(0) = x,\ B_{x,r} \subset \Phi_r(B_{0,2r}),\ \text{and } B_{0,r/C_0} \subset (\Phi_r)^{-1}(B_{x,r})\,,\\ &\Phi_r(B_{0,2r} \cap T_x\partial\mathcal{E}) \cap B_{x,r} = B_{x,r} \cap \partial\mathcal{E}\,,\\ &\Phi_r(B_{0,2r} \cap \sigma_Y(T_x\partial\mathcal{E})) \cap B_{x,r} = B_{x,r} \cap \Sigma_Y(\mathcal{E})\,,\\ &\|\Phi_r\|_{C^{1,\alpha}(B_{0,2r})} + \|\Phi_r^{-1}\|_{C^{1,\alpha}(\Phi_r(B_{0,2r}))} \leq C_0\,; \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\Phi_{k,r}(0) = x,\ B_{x_k,r} \subset \Phi_{k,r}(B_{0,2r}),\ \text{and } B_{0,r/C_0} \subset (\Phi_{k,r})^{-1}(B_{x_k,r})\,,\\ &\Phi_{k,r}(B_{0,2r} \cap T_{x_k}\partial\mathcal{E}_k) \cap B_{x_k,r} = B_{x_k,r} \cap \partial\mathcal{E}_k\,,\\ &\Phi_{k,r}(B_{0,2r} \cap \sigma_Y(T_{x_k}\partial\mathcal{E}_k)) \cap B_{x_k,r} = B_{x_k,r} \cap \Sigma_Y(\mathcal{E}_k)\,,\\ &\|\Phi_{k,r}\|_{C^{1,\alpha}(B_{0,2r})} + \|\Phi_{k,r}^{-1}\|_{C^{1,\alpha}(\Phi_{k,r}(B_{0,2r}))} \leq C_0\,. \end{aligned} \tag{3.10}$$

Proof of Theorem 3.2. Step one: We prove (3.8), (3.9) and (3.10). Up to a translation, we can assume that $x_k = x$ for every k. We first prove that for every $\eta \in (0,\varepsilon_0)$ we can find $k_x \in \mathbb{N}$ such that $\mathrm{hd}_{0,1}(T_x\partial\mathcal{E}, T_x\partial\mathcal{E}_k) < \eta$ if $k \geq k_x$. We start by noticing that by Theorem 2.1 we can find L and $\rho_0 > 0$ such that (2.7) holds with $M = \partial\mathcal{E}_k$, and then that, by Theorem A-(i), we can find $\lambda > 0$ such that, for each $k \in \mathbb{N}$,

$$r \mapsto \frac{\mathcal{H}^2(\partial\mathcal{E}_k \cap B_{x,r})}{r^2} + \lambda\, r\,, \tag{3.11}$$

is increasing on $(0,\rho_0)$. We now claim that there exists $s_x \in (0,\rho_0/2)$ and $k_x \in \mathbb{N}$ such that

$$\frac{\mathrm{hd}_{x,s_x}(\partial\mathcal{E}_k, x + T_x(\partial\mathcal{E}))}{s_x} + \Big(\frac{\mathcal{H}^2(\partial\mathcal{E}_k \cap B_{x,s_x})}{s_x^2} + \lambda\, s_x - \theta_{\partial\mathcal{E}_k}(x)\Big) \leq \min\big\{\frac{\eta}{C_0}, \varepsilon_0\big\}\,. \tag{3.12}$$

Since $X' = T_x\partial\mathcal{E}$ satisfies $\theta_{X'}(0) = \theta_{\partial\mathcal{E}}(x) = \theta_{\partial\mathcal{E}_k}(x)$, by Theorem A-(ii) and (3.12) we will deduce the existence of minimal cones X_k such that if $k \ge k_x$, then

$$\mathrm{hd}_{0,1}(X_k, X') \le \eta\,, \qquad \frac{\mathrm{hd}_{x,r}(\partial\mathcal{E}_k, x + X_k)}{r} \le \Big(\frac{r}{s_x}\Big)^{\alpha}\,, \qquad \forall r < \frac{s_x}{C_0}\,.$$

The second inequality will then imply (in the limit $r \to 0^+$) that $X_k = T_x(\partial\mathcal{E}_k)$, so that the first inequality will give us $\mathrm{hd}_{0,1}(T_x\partial\mathcal{E}, T_x\partial\mathcal{E}_k) < \eta$, as required. We now check (3.12). For a.e. $r > 0$ one has $P(\mathcal{E}_k; B_{x,r}) \to P(\mathcal{E}; B_{x,r})$, so that (3.11) gives us

$$\limsup_{k\to\infty} \frac{\mathcal{H}^2(B_{x,r} \cap \partial\mathcal{E}_k)}{r^2} \le \frac{\mathcal{H}^2(\mathrm{cl}\,(B_{x,r}) \cap \partial\mathcal{E})}{r^2}\,, \qquad \forall r > 0\,. \tag{3.13}$$

Since $r^{-2}\mathcal{H}^2(\mathrm{cl}\,(B_{x,r}) \cap \partial\mathcal{E}) \to \theta_{\partial\mathcal{E}}(x) = \theta_{\partial\mathcal{E}_k}(x)$ as $r \to 0^+$, by combining the definition of tangent cone to $\partial\mathcal{E}$ at x with (3.13) we can find $s_x \in (0, \rho_0/2)$ such that

$$\frac{\mathrm{hd}_{x,r}(\partial\mathcal{E}, x + T_x(\partial\mathcal{E}))}{r} + \Big(\frac{\mathcal{H}^2(\partial\mathcal{E} \cap B_{x,r})}{r^2} + \lambda\, r - \theta_{\partial\mathcal{E}}(x)\Big) \le \frac{1}{2}\,\Big\{\frac{\eta}{C_0}, \varepsilon_0\Big\}\,, \tag{3.14}$$

for every $r \in (0, s_x]$. Moreover, by Theorem 3.1-(i), for every $r \le s_x$ we can find $k_{x,r} \in \mathbb{N}$ such that

$$\frac{\mathrm{hd}_{x,r}(\partial\mathcal{E}_k, \partial\mathcal{E})}{r} \le \frac{1}{2}\,\Big\{\frac{\eta}{C_0}, \varepsilon_0\Big\}\,, \qquad \forall k \ge k_{x,r}\,. \tag{3.15}$$

If we take $r = s_x$ and $k_x = k_{x,s_x}$ then (3.15) reduces to (3.12), and thus proves (3.8). More generally, by combining (3.14) and (3.15) one is able to apply Theorem 2.1 to prove (3.9) and (3.10).

Step two: We prove the first line of (3.7). By Theorem 3.1-(ii) and since $\mathrm{cl}\,(\Sigma_Y(\mathcal{F})) = \Sigma(\mathcal{F})$ for every (Λ, r_0)-minimizing cluster $\mathcal{F}$ in $\mathbb{R}^3$, it is enough to show that $\mathrm{hd}_{B_R}(\Sigma_Y(\mathcal{E}_k), \Sigma_Y(\mathcal{E})) \to 0$ (for every $R > 0$) as $k \to \infty$. Arguing by contradiction and thanks to [CLM14, Lemma 4.14], we find a sequence $\delta_j \to 0$ as $j \to \infty$ and $(\delta_j, \delta_j^{-1})$-minimizing 3-clusters $\mathcal{F}_j$ in $\mathbb{R}^3$ such that

$$\Sigma(\mathcal{F}_j) \cap B_2 = \emptyset \qquad \forall j \in \mathbb{N}\,, \qquad \lim_{j\to\infty} \mathrm{d}_{B_R}(\mathcal{F}_j, \mathcal{Y}) = 0 \qquad \forall R > 0\,,$$

where $\mathcal{Y} = \{\mathcal{Y}(i)\}_{i=1}^3$ is a reference 3-cluster in $\mathbb{R}^3$ such that $\partial\mathcal{Y} = Y$. Notice that Theorem 1.1 can be applied to describe the structure of $\partial\mathcal{F}_j$ and that Theorem 3.1 can be used to describe the convergence of $\partial\mathcal{F}_j$ to $\partial\mathcal{Y}$. Assuming without loss of generality that

$$\partial\mathcal{Y}(1) \cap \partial\mathcal{Y}(2) = \big\{x \in \mathbb{R}^3 : x_3 = 0\,, x_1 \ge 0\big\}\,, \qquad \Sigma(\mathcal{Y}) = \big\{x \in \mathbb{R}^3 : x_1 = x_3 = 0\big\}$$

let us consider, for $0 < \rho < r$, the two-dimensional half-disk

$$D_{r,\rho} = \Big(\partial\mathcal{Y}(1) \cap \partial\mathcal{Y}(2) \cap B_r\Big) \setminus I_\rho(\Sigma(\mathcal{Y})) = \big\{x \in \mathbb{R}^3 : x_3 = 0\,, x_1 \ge \rho\,, x_1^2 + x_2^2 < r\big\}\,.$$

By Theorem 3.1-(iii) there exists $\rho_0 > 0$ such that for every $\rho < \rho_0$ there exist $j_0 \in \mathbb{N}$, $\varepsilon < \rho$, and $\{\psi_j\}_{j \ge j_0} \subset C^1(D_{2,\rho})$ such that

$$N_\varepsilon(D_{2,\rho}) \cap \partial\mathcal{F}_j = (\mathrm{Id} + \psi_j\, e_3)(D_{2,\rho})\,, \qquad \lim_{j\to\infty} \|\psi_j\|_{C^1(D_{2,\rho})} = 0\,,$$

where of course $N_\varepsilon(D_{2,\rho}) = \{x \in \mathbb{R}^3 : (x_1, x_2, 0) \in D_{2,\rho}\,, |x_3| < \varepsilon\}$. By Theorem 1.1, there exists a unique $S_j \in \mathcal{S}(\mathcal{F}_j)$ such that

$$N_\varepsilon(D_{2,\rho}) \cap S_j = N_\varepsilon(D_{2,\rho}) \cap \partial\mathcal{F}_j = (\mathrm{Id} + \psi_j\, e_3)(D_{2,\rho})\,. \tag{3.16}$$

Notice that S_j is a connected topological surface with boundary in $\mathbb{R}^3$, $S_j \setminus \Sigma_T(\mathcal{F}_j)$ is a $C^{1,\alpha}$-surface with boundary in $\mathbb{R}^3$, and

$$\mathrm{bd}_{\,\tau}(S_j) \cap B_2 \subset \Sigma(\mathcal{F}_j) \cap B_2 = \emptyset\,.$$

Hence, if T_j denotes the 2-dimensional multiplicity-one integral current T_j associated with (one of the two possible orientations of) S_j, then $\mathrm{spt}(\partial T_j) \subset \mathrm{bd}_{\,\tau}(S_j)$, so that, in particular, $\partial T_j \llcorner B_2 = 0$.

(Here and in the following, if T is a current, then ∂T denotes the boundary of T in the sense of currents.) Let us consider the Lipschitz function

$$f(x) = \max\{(x_1^2 + x_2^2)^{1/2}, |x_3|\}\,, \qquad x \in \mathbb{R}^3\,,$$

so that $f^{-1}(r)$ is the boundary of a cylinder along the x_3 axis, centered at the origin, of height $2r$ and radius r. For a.e. $r > 0$ let us denote by $\Gamma_j^r = \langle T_j, f, r\rangle$ the slicing of T_j by f at r, see [Sim83, Definition 28.4]. By definition, $\mathrm{spt}(\Gamma_j^r) \subset S_j \cap f^{-1}(r)$ and moreover for a.e. $0 < r < 1$ we have

$$\partial\Gamma_j^r \llcorner \{f < 1\} = 0\,. \tag{3.17}$$

Indeed $\{f < 1\} \subset B_2$, $\partial T_j \llcorner B_2 = 0$ and, by [Sim83, Lemma 28.5],

$$\partial\Gamma_j^r = \partial\langle T_j, f, r\rangle = -\langle \partial T_j, f, r\rangle\,, \qquad \text{for a.e. } r > 0.$$

Let us now fix $r < 1$ such that (3.17) holds, and let us consider $\rho < \rho_0$ with $10\rho < r$. By Theorem 3.1-(i), up to further increasing the value of j_0 we have

$$\mathrm{spt}(\Gamma_j^r) \subset S_j \cap f^{-1}(r) \subset \partial\mathcal{F}_j(1) \cap \partial\mathcal{F}_j(2) \cap f^{-1}(r) \subset I_\varepsilon(D_{2,0}) \cap f^{-1}(r)\,, \tag{3.18}$$

where thanks to $0 < \varepsilon < \rho$ one has

$$I_\varepsilon(D_{2,0}) \cap f^{-1}(r) \subset A_1 \cup A_2 \cup A_3\,, \tag{3.19}$$

for

$$A_1 = B_{(0,r,0),2\rho}\,, \qquad A_2 = B_{(0,-r,0),2\,\rho}\,, \qquad A_3 = \big\{x \in \mathbb{R}^3 : |x_3| < \varepsilon\,, x_1^2 + x_2^2 = r\,, x_1 > \rho\big\}\,.$$

Let ω be any compactly supported smooth 0-form such that

$$\omega = 1 \text{ on } B_{(0,r,0),3\rho} \supset A_1 \text{ and } \omega = 2 \text{ on } B_{(0,-r,0),3\rho} \supset A_2\,. \tag{3.20}$$

(Such ω exists as soon as $3\rho < r$, whence it follows that $B_{(0,r,0),3\rho}$ and $B_{(0,-r,0),3\rho}$ are at positive distance.) In this way $d\omega = 0$ on $A_1 \cup A_2$, and thus, by also taking (3.17) into account

$$0 = \int_{\Gamma_j^r} d\omega = \int_{\Gamma_j^r \llcorner (A_3 \setminus (A_1 \cup A_2))} d\omega\,. \tag{3.21}$$

Now by (3.16), the inclusion $\mathrm{spt}(\Gamma_j^r) \subset S_j \cap f^{-1}(r)$ and the definition of A_3, there exists a C^1-curve with boundary γ such that, if T_γ denotes the one-dimensional multiplicity-one integral current associated with (one of the two orientations of) γ, then

$$\Gamma_j^r \llcorner (A_3 \setminus (A_1 \cup A_2)) = T_\gamma\,.$$

Let $\mathrm{bd}\,(\gamma) = \{p_1, p_2\}$, then by construction we can assume $p_1 \in B_{(0,r,0),3\rho}$ and $p_2 \in B_{(0,-r,0),3\rho}$. By (3.21), and up to reversing the orientation of γ, we thus find the contradiction

$$0 = \int_{T_\gamma} d\omega = \omega(p_2) - \omega_(p_1) = 1\,.$$

This completes the proof of the first part of (3.7).

Step three: We are left to prove that if $R > 0$, then $\mathrm{hd}_{B_R}(\Sigma_T(\mathcal{E}_k), \Sigma_T(\mathcal{E})) \to 0$ as $k \to \infty$. We first prove that $x_k \in \Sigma_T(\mathcal{E}_k)$ with $x_k \to x$, then $x \in \Sigma_T(\mathcal{E})$. For sure $x \in \Sigma(\mathcal{E})$ thanks to step two. We may thus assume, arguing by contradiction, that $x \in \Sigma_Y(\mathcal{E})$. If this is the case, then there exists $r_x > 0$ and an injective map $\sigma : \{1,2,3\} \to \{0,...,N\}$ such that $|\mathcal{E}(h) \cap B_{x,r_x}| = 0$ if $h \neq \sigma(i)$, $i = 1,2,3$. In particular, there exists $k_0 \in \mathbb{N}$ such that, if $k \geq k_0$, then $|\mathcal{E}_k(h) \cap B_{x_k,r_x}| < \eta_0\, r_x^n$ whenever $h \neq \sigma(i)$, $i = 1,2,3$, and with η_0 as in [CLM14, Lemma 4.5]; in particular, by that lemma, $|\mathcal{E}_k(h) \cap B_{x_k,r_x/2}| = 0$ for $h \neq \sigma(i)$, $i = 1,2,3$. At the same time, since $x_k \in \Sigma_T(\mathcal{E}_k)$, there exist $r_k > 0$ with $r_k \to 0$ as $k \to \infty$ and injective maps $\sigma_k : \{1,2,3,4\} \to \{0,...,N\}$ such that $|B_{x_k,r_k} \cap \mathcal{E}_k(\sigma_k(i))| = (1/4)|B_{x_k,r_k}| + o(r_k^n)$ for every $i = 1,2,3,4$. We have thus reached a contradiction, and proved our claim.

We are thus left to show that if $x \in \Sigma_T(\mathcal{E})$, then there exists $x_k \in \Sigma_T(\mathcal{E}_k)$ such that $x_k \to x$ as $k \to \infty$. To this end, we may directly consider the existence of $\varepsilon > 0$ and $x \in \Sigma_T(\mathcal{E})$ such that $\Sigma_T(\mathcal{E}_k) \cap B_{x,\varepsilon} = \emptyset$ for every $k \in \mathbb{N}$. By step one there exist $x_k \in \Sigma_Y(\mathcal{E}_k)$ such that $x_k \to x$ as $k \to \infty$. By arguing as in the proof of [CLM14, Lemma 4.19] we find a sequence $\delta_j \to 0$ as $j \to \infty$ and $(\delta_j, \delta_j^{-1})$-minimizing 4-clusters $\mathcal{F}_j$ in $\mathbb{R}^3$ such that

$$\Sigma_T(\mathcal{F}_j) \cap B_2 = \emptyset \qquad \forall j \in \mathbb{N}\,, \qquad \lim_{j\to\infty} \mathrm{d}_{B_R}(\mathcal{F}_j, \mathcal{T}) = 0 \qquad \forall R > 0\,,$$

where $\mathcal{T} = \{\mathcal{T}(i)\}_{i=1}^4$ is a reference 4-cluster in $\mathbb{R}^3$ such that $\partial\mathcal{T} = T$. Let us then denote by ℓ one of the four closed half-lines contained in $\Sigma(\mathcal{T})$. By step one and step two, for every $y \in \ell \setminus B_{1/2} \subset \Sigma_Y(\mathcal{T})$ we can find $s_y > 0$ and $y_j \in \Sigma_Y(\mathcal{F}_j)$ such that $y_j \to y$ as $j \to \infty$ and there exist $C^{1,\alpha}$-diffeomorphisms Φ and Φ_j satisfying (3.9) and (3.10) (with $\mathcal{T}$, $\mathcal{F}_j$, y, y_j and s_y in place of $\mathcal{E}$, $\mathcal{E}_k$, x, x_k and s_x). As a consequence,

$$B_{y,s_y} \cap \Sigma(\mathcal{F}_j) = B_{y,s_y} \cap \Sigma_Y(\mathcal{F}_j) = B_{y,s_y} \cap \Phi_j(B_{0,2s_y} \cap \sigma_Y(T_y\mathcal{T}))\,,$$

so that $B_{y,s_y} \cap \Sigma(\mathcal{F}_j)$ is $C^{1,\alpha}$-diffeomorphic to $(0,1)$. By (3.9), (3.10), and by the connectedness of the curves in $\Gamma(\mathcal{F}_j)$ (see Theorem 1.1 for the notation used here) we see that there exist $\delta > 0$ and $\gamma_j \in \Gamma(\mathcal{F}_j)$ such that

$$\Sigma(\mathcal{F}_j) \cap I_\delta\big(\ell \cap (B \setminus B_{1/2})\big) = \gamma_j \cap (B \setminus B_{1/2})$$

and $\gamma_j^* = \gamma_j \cap (B \setminus B_{1/2})$ is $C^{1,\alpha}$-diffeomorphic to $(0,1)$. Let ω be a smooth 0-form with $\omega = 1$ on $B_{2/3}$ and $\mathrm{spt}\omega \subset\subset B$. By Stokes theorem, up to a change in orientation,

$$\int_{\gamma_j} d\omega = \int_{\gamma_j^*} d\omega = 1\,.$$

By $\Sigma_T(\mathcal{F}_j) \cap B_2 = \emptyset$ we have $\mathrm{bd}\,(\gamma_j) \cap B_2 = \emptyset$, which combined with $\mathrm{spt}\omega \subset\subset B$ gives us

$$\int_{\gamma_j} d\omega = 0\,.$$

We have thus reached a contradiction, and completed the proof of the theorem. □

4. Stratified boundary convergence

In this section we fix $\Lambda \ge 0$, $r_0 > 0$, and assume that (recall Remark 1.2 and compare with (3.1))

$$\begin{aligned}
&\mathcal{E} \text{ is a bounded } (\Lambda, r_0)\text{-minimizing cluster in } \mathbb{R}^3 \text{ of class } C^{2,1}\,,\\
&\{\mathcal{E}_k\}_{k\in\mathbb{N}} \text{ is a sequence of } (\Lambda, r_0)\text{-minimizing clusters in } \mathbb{R}^3\,,\\
&\mathrm{d}(\mathcal{E}_k, \mathcal{E}) \to 0 \text{ as } k \to \infty\,.
\end{aligned} \tag{4.1}$$

We also let α and C_0 be as in Theorem A. We then start proving a series of theorems and lemmas which will eventually lead us to prove Theorem 1.3.

We shall often refer to the following consequence of Theorem 3.2: if (4.1) holds and $n = 3$, then for every $\delta > 0$ we can find $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\Sigma(\mathcal{E}) &\subset I_\delta(\Sigma(\mathcal{E}_k))\,, & \Sigma_T(\mathcal{E}) &\subset I_\delta(\Sigma_T(\mathcal{E}_k))\,,\\
\Sigma(\mathcal{E}_k) &\subset I_\delta(\Sigma(\mathcal{E}))\,, & \Sigma_T(\mathcal{E}_k) &\subset I_\delta(\Sigma_T(\mathcal{E}))\,,
\end{aligned} \qquad \forall k \ge k_0\,. \tag{4.2}$$

Moreover, by exploiting the finiteness of $\Sigma_T(\mathcal{E})$, we have that, for some $\delta_0 > 0$,

$$\mathcal{H}^0(B_{x,\delta_0} \cap \Sigma_T(\mathcal{E}_k)) = 1\,, \qquad \forall x \in \Sigma_T(\mathcal{E})\,, k \ge k_0\,. \tag{4.3}$$

In the next lemma we parameterize $\partial\mathcal{E}$ and $\partial\mathcal{E}_k$ around nearby singular points at comparable scales through Theorem A.

Lemma 4.1. *If (4.1) holds, then for every $\delta > 0$ one can find $k_0 \in \mathbb{N}$ and finite sets $\{x^i\}_{i\in I} \subset \Sigma(\mathcal{E})$, $\{x^i_k\}_{i\in I} \subset \Sigma(\mathcal{E}_k)$ and $\{t^i\}_{i\in I} \subset (0, \delta/2)$ such that, for every $k \ge k_0$ and $i \in I$,*

$$\Sigma_T(\mathcal{E}) \subset \{x^i\}_{i\in I}\,, \quad \Sigma_T(\mathcal{E}_k) \subset \{x^i_k\}_{i\in I}\,, \quad \lim_{k\to\infty} x^i_k = x^i\,, \quad \theta_{\partial\mathcal{E}_k}(x^i_k) = \theta_{\partial\mathcal{E}}(x^i)\,, \tag{4.4}$$

$$\Sigma(\mathcal{E}) \subset \bigcup_{i\in I} B_{x^i,t^i/3}\,, \qquad \Sigma(\mathcal{E}_k) \subset \bigcup_{i\in I} B_{x^i_k,2t^i/3}\,, \tag{4.5}$$

and such that for every $r \le t^i$ there exists a $C^{1,\alpha}$-diffeomorphism $\Phi^i_r : B_{0,2r} \to A^i_r = \Phi^i_r(B_{0,2\,r})$

$$\begin{aligned}
&\Phi^i_r(0) = x^i,\ B_{x^i,r} \subset A^i_r,\ B_{0,r/C_0} \subset (\Phi^i_r)^{-1}(B_{x^i,r})\,,\\
&\Phi^i_r(B_{0,2r} \cap T_{x^i}\partial\mathcal{E}) \cap B_{x^i,r} = B_{x^i,r} \cap \partial\mathcal{E}\,,\\
&\Phi^i_r(B_{0,2r} \cap \sigma_Y(T_{x^i}\partial\mathcal{E})) \cap B_{x^i,r} = B_{x^i,r} \cap \Sigma_Y(\mathcal{E})\,,\\
&\|\Phi^i_r\|_{C^{1,\alpha}(B_{0,2r})} + \|(\Phi^i_r)^{-1}\|_{C^{1,\alpha}(A^i_r)} \le C_0\,,
\end{aligned} \tag{4.6}$$

and there exists a $C^{1,\alpha}$-diffeomorphism $\Phi^i_{r,k} : B_{0,2r} \to A^i_{r,k} = \Phi^i_{r,k}(B_{0,2r})$ with

$$\begin{aligned}
&\Phi^i_{k,r}(0) = x^i_k,\ B_{x^i_k,r} \subset \Phi^i_{k,r}(B_{0,2r}),\ B_{0,r/C_0} \subset (\Phi^i_{k,r})^{-1}(B_{x^i_k,r})\,,\\
&\Phi^i_{k,r}(B_{0,2r} \cap T_{x^i_k}\partial\mathcal{E}_k) \cap B_{x^i_k,r} = B_{x^i_k,r} \cap \partial\mathcal{E}_k\,,\\
&\Phi^i_{k,r}(B_{0,2r} \cap \sigma_Y(T_{x^i_k}\partial\mathcal{E}_k)) \cap B_{x^i_k,r} = B_{x^i_k,r} \cap \Sigma_Y(\mathcal{E}_k)\,,\\
&\|\Phi^i_{k,r}\|_{C^{1,\alpha}(B_{0,2r})} + \|(\Phi^i_k)^{-1}\|_{C^{1,\alpha}(A^i_{k,r})} \le C_0\,.
\end{aligned} \tag{4.7}$$

Moreover, $\{x^i\}_{i\in I}$ can be chosen in such a way that for every $\gamma \in \Gamma(\mathcal{E})$ and $S \in \mathcal{S}(\mathcal{E})$, one has

$$\gamma \subset \bigcup_{i\in I(\gamma)} B_{x^i,t^i/3}\,, \qquad \Sigma_T(\mathcal{E}) \cap \gamma \subset \{x^i\}_{i\in I(\gamma)}\,, \tag{4.8}$$

$$S \cap \Sigma(\mathcal{E}) \subset \bigcup_{i\in I(S)} B_{x^i,t^i/3}\,, \qquad \Sigma_T(\mathcal{E}) \cap S \subset \{x^i\}_{i\in I(S)}\,. \tag{4.9}$$

where $I(\gamma) = \{i \in I : x^i \in \gamma\}$ and $I(S) = \{i \in I : x^i \in S\}$.

Remark 4.2. By considering (4.6) at $r = t^i$ we infer that $B_{x^i,t^i} \cap \Sigma_Y(\mathcal{E})$ is homeomorphic to $(0,1)$. This fact alone does not imply, of course, that $B_{x^i,r} \cap \Sigma_Y(\mathcal{E})$ is homeomorphic to $(0,1)$ for every $r < t^i$. The latter property is guaranteed by the fact that (4.6) holds for every $r \le t^i$.

Proof. Given $\delta > 0$ and $x \in \Sigma(\mathcal{E})$ let $t_x = \min\{s_x, \delta/2\}$ for s_x as in Theorem 3.2. Since $\partial\mathcal{E}$ is bounded, so is $\Sigma(\mathcal{E})$, while $\Sigma_T(\mathcal{E})$ is finite. By Theorem 3.2 and by compactness we can find $\{x^i\}_{i\in I} \subset \Sigma(\mathcal{E})$ finite with $\Sigma_T(\mathcal{E}) \subset \{x^i\}_{i\in I}$, such that the first inclusion in (4.5) holds, namely

$$\Sigma(\mathcal{E}) \subset \bigcup_{i\in I} B_{x^i,t^i/3}\,, \qquad t^i = t_{x^i}\,, \tag{4.10}$$

and such that (4.6) holds. By (3.7) in Theorem 3.2 for every $i \in I$ there exists $x^i_k \in \Sigma(\mathcal{E}_k)$ with $\theta_{\partial\mathcal{E}_k}(x^i_k) = \theta_{\partial\mathcal{E}}(x^i)$ and $x^i_k \to x^i$ as $k \to \infty$. If $t^* = \min\{t^i : i \in I\}$, then, up to further increase the value of k_0 we can entail $\Sigma(\mathcal{E}_k) \subset I_{t^*/6}(\Sigma(\mathcal{E}))$ and $|x^i - x^i_k| < t^*/6$ for every $i \in I$ and $k \ge k_0$, so that by (4.10)

$$\Sigma(\mathcal{E}_k) \subset \bigcup_{i\in I} B_{x^i,t^i/2} \subset \bigcup_{i\in I} B_{x^i_k,2t^i/3}\,.$$

This proves (4.4) and (4.5), while (4.7) follows by (3.10) in Theorem 3.2 up to further increase the value of k_0. □

We now introduce some further notation (in addition to the one set in Lemma 4.1) to be used in the rest of this section. Since $\partial\mathcal{E}$ is bounded (as assumed in (4.1)), thanks to Theorem 1.1 we find that the sets $\Sigma_T(\mathcal{E})$, $\Gamma(\mathcal{E})$ and $\mathcal{S}(\mathcal{E})$ are finite. We consider the partition $\{\Gamma_T(\mathcal{E}),\Gamma_Y(\mathcal{E})\}$ of $\Gamma(\mathcal{E})$ defined by

$$\begin{aligned}\Gamma_T(\mathcal{E}) &= \big\{\gamma\in\Gamma(\mathcal{E}):\gamma\cap\Sigma_T(\mathcal{E})\neq\emptyset\big\}\,,\\ \Gamma_Y(\mathcal{E}) &= \big\{\gamma\in\Gamma(\mathcal{E}):\gamma\cap\Sigma_T(\mathcal{E})=\emptyset\big\}=\big\{\gamma\in\Gamma(\mathcal{E}):\gamma\subset\Sigma_Y(\mathcal{E})\big\}\,,\end{aligned}$$

(so that each $\gamma\in\Gamma(\mathcal{E})$ is either diffeomorphic to $\mathbb{S}^1$ or to $[0,1]$ depending on whether $\gamma\in\Gamma_Y(\mathcal{E})$ or $\gamma\in\Gamma_T(\mathcal{E})$) and the partition $\{\mathcal{S}_\Sigma(\mathcal{E}),\mathcal{S}_*(\mathcal{E})\}$ of $\mathcal{S}(\mathcal{E})$ obtained by letting

$$\begin{aligned}\mathcal{S}_\Sigma(\mathcal{E}) &= \big\{S\in\mathcal{S}(\mathcal{E}):S\cap\Sigma(\mathcal{E})\neq\emptyset\big\}\,,\\ \mathcal{S}_*(\mathcal{E}) &= \big\{S\in\mathcal{S}(\mathcal{E}):S\cap\Sigma(\mathcal{E})=\emptyset\big\}=\big\{S\in\mathcal{S}(\mathcal{E}):S\subset\partial^*\mathcal{E}\big\}\,.\end{aligned}$$

In the next lemma we associate to every curve $\gamma\in\Gamma(\mathcal{E})$ a corresponding curve $\gamma_k\in\Gamma(\mathcal{E}_k)$ in such a way that $\mathrm{hd}(\gamma,\gamma_k)\to0$. This correspondence will be used in the rest of the proof of Theorem 1.3.

Lemma 4.3. *If* (4.1) *holds, then there exists $k_0\in\mathbb{N}$ with the following property: to every $\gamma\in\Gamma(\mathcal{E})$ and $k\ge k_0$ one can associate $\gamma_k\in\Gamma(\mathcal{E}_k)$ in such a way that $\gamma_k\in\Gamma_T(\mathcal{E}_k)$ if and only if $\gamma\in\Gamma_T(\mathcal{E})$ and*

$$\lim_{k\to\infty}\mathrm{hd}(\gamma,\gamma_k)+\mathrm{hd}(\mathrm{bd}\,(\gamma),\mathrm{bd}\,(\gamma_k))=0\,,\tag{4.11}$$

$$\Sigma(\mathcal{E}_k)=\bigcup_{\gamma\in\Gamma(\mathcal{E})}\gamma_k\,,\qquad\Sigma_T(\mathcal{E}_k)=\bigcup_{\gamma\in\Gamma_T(\mathcal{E})}\mathrm{bd}\,(\gamma_k)\,,\qquad\forall k\ge k_0\,.\tag{4.12}$$

Proof. We choose δ_0 to be such that

$$\begin{aligned}I_{\delta_0}(\Sigma_T(\mathcal{E}))\cap I_{\delta_0}(\gamma)&=I_{\delta_0}(\mathrm{bd}\,(\gamma))\,, &&\forall\gamma\in\Gamma_T(\mathcal{E})\,,&(4.13)\\ I_{\delta_0}(\gamma)\cap I_r(\Sigma(\mathcal{E}))&=I_r(\gamma)\,, &&\forall\gamma\in\Gamma_Y(\mathcal{E})\,,\forall r\in[0,\delta_0]\,.&(4.14)\end{aligned}$$

Given $\delta\in(0,\delta_0)$, define I, k_0, $\{x^i\}_{i\in I}\subset\Sigma(\mathcal{E})$, $\{x^i_k\}_{i\in I}\subset\Sigma(\mathcal{E}_k)$ and $\{t^i\}_{i\in I}\subset(0,\delta/2)$ as in Lemma 4.1. Note that one can always assume

$$t^*=\min\big\{t^i:i\in I\big\}\,,\qquad|x^i-x^i_k|<\frac{t^*}{6}\,,\qquad\forall k\ge k_0\,,\quad i\in I\,,\tag{4.15}$$

where of course $t^*>0$ as I is finite. With reference to (4.8), given $\gamma\in\Gamma(\mathcal{E})$ let us set $I(\gamma)=\{0,...,m\}$, so that (4.8) implies

$$\gamma\subset\bigcup_{i=0}^m B_{x^i,t^i/3}\,,\qquad x^i\in\gamma\text{ for }i=0,...,m\,.\tag{4.16}$$

We now divide the proof in three steps.

Step one: We show that to each $\gamma\in\Gamma_Y(\mathcal{E})$ and $k\ge k_0$ one can associate $\gamma_k\in\Gamma_Y(\mathcal{E}_k)$ in such a way that

$$\Sigma(\mathcal{E}_k)\cap I_{t^*/2}(\gamma)\subset\gamma_k\subset I_\delta(\gamma)\,,\qquad\gamma\subset I_\delta(\gamma_k)\,.\tag{4.17}$$

Indeed, by (4.6) and by $x^i\in\gamma\subset\Sigma_Y(\mathcal{E})$, one has that $B_{x^i,t^i}\cap\Sigma(\mathcal{E})$ is $C^{1,\alpha}$-diffeomorphic to $(0,1)$, so that $x^i\in\gamma$ and the connectedness of γ imply

$$B_{x^i,t^i}\cap\Sigma(\mathcal{E})=B_{x^i,t^i}\cap\gamma\text{ is homeomorphic to }(0,1)\text{ for every }i=0,...,m\,.\tag{4.18}$$

Since γ is homeomorphic to $\mathbb{S}^1$, by (4.16) and (4.18), up a relabeling in the index i and up to possibly discard some balls B_{x^i,t^i}, one can entail that (setting $x^{m+1}=x^0$, $x^{m+1}_k=x^0_k$ and $t^{m+1}=t^0$)

$$B_{x^i,t^i}\cap B_{x^{i+1},t^{i+1}}\cap\gamma\neq\emptyset\,,\qquad\forall i=0,...,m\,.\tag{4.19}$$

If $z^i \in B_{x^i,t^i} \cap B_{x^{i+1},t^{i+1}} \cap \gamma$, then for some $\varepsilon^i > 0$ we have $B_{z^i,\varepsilon^i} \subset B_{x^i,t^i} \cap B_{x^{i+1},t^{i+1}}$, while by Theorem 3.2 there exist $z^i_k \in \Sigma(\mathcal{E}_k)$ such that $z^i_k \to z^i$ as $k \to \infty$, so that (4.19) implies

$$B_{x^i_k,t^i} \cap B_{x^{i+1}_k,t^{i+1}} \cap \Sigma(\mathcal{E}_k) \neq \emptyset\,, \qquad \forall i = 0, ..., m\,. \tag{4.20}$$

By (4.7), for every i and $k \ge k_0$, $B_{x^i_k,t^i} \cap \Sigma(\mathcal{E}_k)$ is $C^{1,\alpha}$-diffeomorphic to $(0,1)$, so that for each i there exists a unique $\gamma^i_k \in \Gamma(\mathcal{E}_k)$ such that $B_{x^i_k,t^i} \cap \Sigma(\mathcal{E}_k) = B_{x^i_k,t^i} \cap \gamma^i_k$. By (4.20) and by connectedness of each curve in $\Gamma(\mathcal{E}_k)$, it must be $\gamma^i_k = \gamma^{i+1}_k$ for every $i = 0, ..., m$. In other words, there exists $\gamma_k \in \Gamma(\mathcal{E}_k)$ such that

$$B_{x^i_k,t^i} \cap \Sigma(\mathcal{E}_k) = B_{x^i_k,t^i} \cap \gamma_k \text{ is homeomorphic to } (0,1) \text{ for every } i = 0, ..., m\,. \tag{4.21}$$

By (4.16), there exists $s \in (0, \delta_0)$ such that

$$I_s(\gamma) \subset\subset A = \bigcup_{i=0}^m B_{x^i,t^i/3}\,.$$

In particular, by (4.21), by Theorem 3.2 (so that $\Sigma(\mathcal{E}_k) \subset I_s(\Sigma(\mathcal{E}))$ for $k \ge k_0$), by $\mathrm{cl}\,(A) \subset I_{\delta_0}(\gamma)$ and by (4.14), one finds

$$\mathrm{cl}\,(A) \cap \gamma_k \;=\; \mathrm{cl}\,(A) \cap \Sigma(\mathcal{E}_k) \;\subset\; \mathrm{cl}\,(A) \cap I_s(\Sigma(\mathcal{E}_k)) \;\subset\; I_{\delta_0}(\gamma) \cap I_s(\Sigma(\mathcal{E})) \;=\; I_s(\gamma)\,,$$

so that

$$\gamma_k = (\gamma_k \cap \mathrm{cl}\,(A)) \cup (\gamma_k \setminus \mathrm{cl}\,(A)) = (\gamma_k \cap I_s(\gamma)) \cup (\gamma_k \setminus \mathrm{cl}\,(A))\,.$$

Since $I_s(\gamma)$ and $\mathbb{R}^n \setminus \mathrm{cl}\,(A)$ are disjoint open sets, by connectedness of γ_k, we conclude that $\gamma_k \subset I_s(\gamma)$. This implies that, for k large enough, $\gamma_k \in \Gamma_Y(\mathcal{E}_k)$: for otherwise, there would be a sequence $w_k \in \Sigma_T(\mathcal{E}_k) \cap \gamma_k$ such that $w_k \to w \in \gamma \subset \Sigma_Y(\mathcal{E})$, a contradiction to Theorem 3.2. Thus,

$$\gamma_k = \bigcup_{i=0}^m B_{x^i_k,t^i} \cap \Sigma(\mathcal{E}_k) \in \Gamma_Y(\mathcal{E}_k)\,. \tag{4.22}$$

By (4.16), (4.15) and $x^i_k \subset \gamma_k$ we find

$$\gamma \subset \bigcup_{i=0}^m B_{x^i,t^i/3} \subset \bigcup_{i=0}^m B_{x^i_k,t^i/3+t^*/6} \subset I_{t^*}(\gamma_k)\,.$$

Similarly, one proves that $\gamma_k \subset \Sigma(\mathcal{E}_k) \cap I_\delta(\gamma)$, and actually by (4.16), (4.15), and (4.22) one has

$$\Sigma(\mathcal{E}_k) \cap I_{t^*/2}(\gamma) \;\subset\; \Sigma(\mathcal{E}_k) \cap \bigcup_{i=0}^m B_{x^i,t^*/2+t^i/3} \;\subset\; \Sigma(\mathcal{E}_k) \cap \bigcup_{i=0}^m B_{x^i_k,t^i} \;=\; \gamma_k\,,$$

so that the proof of (4.17) is complete.

Step two: We show that to each $\gamma \in \Gamma_T(\mathcal{E})$ and $k \ge k_0$ one can associate $\gamma_k \in \Gamma_T(\mathcal{E}_k)$ in such a way that

$$\gamma_k \subset I_\delta(\gamma)\,, \qquad \gamma \subset I_\delta(\gamma_k)\,, \qquad \mathrm{bd}\,(\gamma_k) = \Sigma_T(\mathcal{E}_k) \cap I_\delta(\gamma)\,. \tag{4.23}$$

Indeed, by (4.8) we can assume without loss of generality that (4.16) holds with $\mathrm{bd}\,(\gamma) = \{x^0, x^m\}$ and $x^i \in \Sigma_Y(\mathcal{E})$ for $i = 1, ..., m-1$. In particular, by (4.6), if $i = 1, ..., m-1$, then by arguing as in the proof of (4.18) one finds

$$B_{x^i,t^i} \cap \Sigma(\mathcal{E}) = B_{x^i,t^i} \cap \gamma \text{ is homeomorphic to } (0,1) \text{ for every } i = 1, ..., m-1\,. \tag{4.24}$$

Similarly,

$$\left.\begin{aligned} &B_{x^i,t^i} \cap \Sigma(\mathcal{E}) \text{ is homeomorphic to } B \cap \sigma(T)\,,\\ &B_{x^i,t^i} \cap \gamma \text{ is homeomorphic to } [0,1)\,, \end{aligned}\right. \qquad \text{if } i = 0, m\,. \tag{4.25}$$

Since γ is homeomorphic to $[0,1]$, by (4.16), (4.24), and (4.25) we can prove that, up to a relabeling in the index i, and up to possibly discard some balls B_{x^i,t^i}, one has

$$B_{x^i,t^i}\cap B_{x^{i+1},t^{i+1}}\cap\gamma\neq\emptyset\,,\qquad\forall i=0,...,m-1\,, \tag{4.26}$$

from which we deduce, by arguing as in the previous case, that

$$B_{x_k^i,t^i}\cap B_{x_k^{i+1},t^{i+1}}\cap\Sigma(\mathcal{E}_k)\neq\emptyset\,,\qquad\forall i=0,...,m-1\,. \tag{4.27}$$

By exploiting again (4.7) we thus find $\gamma_k\in\Gamma_T(\mathcal{E}_k)$ such that $\mathrm{bd}\,(\gamma_k)=\{x_k^0,x_k^m\}$ and

$$B_{x_k^i,t^i}\cap\Sigma(\mathcal{E}_k)=B_{x_k^i,t^i}\cap\gamma_k \text{ is homeomorphic to } (0,1) \text{ for every } i=1,...,m-1\,. \tag{4.28}$$

$$B_{x_k^i,t^i}\cap\gamma_k \text{ is homeomorphic to } [0,1) \text{ if } i=0,m\,. \tag{4.29}$$

The inclusion $\gamma\subset I_\delta(\gamma_k)$ follows by (4.16), (4.15) and $x_k^i\in\gamma_k$. By (4.28) and (4.29), and by arguing as in the previous step, one finds

$$\gamma_k=\bigcup_{i=0}^m\gamma_k\cap B_{x_k^i,t^i}\,,$$

which in particular entails $\gamma_k\subset I_\delta(\gamma)$ thanks to (4.15). By (4.2) and (4.13)

$$\Sigma_T(\mathcal{E}_k)\cap I_\delta(\gamma)\;\subset\;I_\delta(\Sigma_T(\mathcal{E}))\cap I_\delta(\gamma)\;\subset\;I_{\delta_0}(\mathrm{bd}\,(\gamma))\;=\;B_{x^0,\delta_0}\cup B_{x^m,\delta_0}\,.$$

By (4.3), $\Sigma_T(\mathcal{E}_k)\cap B_{x^0,\delta_0}$ consists of a single point, which must be x_k^0 thanks to (4.15). Thus $\Sigma_T(\mathcal{E}_k)\cap I_\delta(\gamma)=\{x_k^0,x_k^m\}=\mathrm{bd}\,(\gamma_k)$, and the proof of (4.23) is complete.

Step three: We prove (4.11) and (4.12). Indeed, (4.17), (4.23) and $t^*<\delta$ immediately imply that $\mathrm{hd}(\gamma,\gamma_k)+\mathrm{hd}(\mathrm{bd}\,(\gamma),\mathrm{bd}\,(\gamma_k))<\delta$ (with the convention that $\mathrm{hd}(\emptyset,\emptyset)=0$). By (4.2) and (4.23) one has

$$\Sigma_T(\mathcal{E}_k)\;=\;\Sigma_T(\mathcal{E}_k)\cap I_\delta(\Sigma_T(\mathcal{E}))\;\subset\;\Sigma_T(\mathcal{E}_k)\cap\bigcup_{\gamma\in\Gamma_T(\mathcal{E})}I_\delta(\gamma)\;=\;\bigcup_{\gamma\in\Gamma_T(\mathcal{E})}\mathrm{bd}\,(\gamma_k)\;\subset\;\Sigma_T(\mathcal{E}_k)\,.$$

To prove the first identity in (4.12), we need to show that if $\tilde\gamma\in\Gamma(\mathcal{E}_k)$ then $\tilde\gamma=\gamma_k$ for some $\gamma\in\Gamma(\mathcal{E})$. Indeed, if $\tilde\gamma\in\Gamma_T(\mathcal{E}_k)$, then $\mathrm{bd}\,(\tilde\gamma)\subset\Sigma_T(\mathcal{E}_k)$ and thus $\tilde\gamma=\gamma_k$ for some $\gamma\in\Gamma_T(\mathcal{E})$ thanks to the second identity in (4.12); if, instead, $\tilde\gamma\in\Gamma_Y(\mathcal{E}_k)$, then, provided k_0 is large enough to entail $\Sigma_Y(\mathcal{E}_k)\subset I_{t^*/2}(\Sigma_Y(\mathcal{E}))$ for every $k\ge k_0$, one has

$$\tilde\gamma\;\subset\;\Sigma_Y(\mathcal{E}_k)\cap I_{t^*/2}(\Sigma_Y(\mathcal{E}))\;=\;\bigcup_{\gamma\in\Gamma_Y(\mathcal{E})}\Sigma(\mathcal{E}_k)\cap I_{t^*/2}(\gamma)\;\subset\;\bigcup_{\gamma\in\Gamma_Y(\mathcal{E})}\gamma_k\,,$$

where the last inclusion is based on (4.17). $\square$

From now on, k_0 will be always assumed large enough to have the correspondence $\gamma\mapsto\gamma_k$ established in Lemma 4.3 in place for every $k\ge k_0$. We recall that under this correspondence, $\gamma\in\Gamma_T(\mathcal{E})$ if and only if $\gamma_k\in\Gamma_T(\mathcal{E}_k)$. Moreover, given $v\in\mathbb{R}^n$ we set $\mathbb{R}\,v=\{t\,v:t\in\mathbb{R}\}$ and $\mathbb{R}_+v=\{t\,v:t\ge0\}$.

Lemma 4.4. *If (4.1) holds, then there exist L (depending on $\mathcal{E}$ and Λ only) and $k_0\in\mathbb{N}$ with the following property. For every $\gamma\in\Gamma(\mathcal{E})$ and $k\ge k_0$ there exist vector fields $\tau,\nu^{(j)}:\gamma\to\mathbb{S}^2$ and $\tau_k,\nu_k^{(j)}:\gamma_k\to\mathbb{S}^2$ $(j=1,2,3)$ such that*

$$T_x\gamma=\mathbb{R}\tau(x)\,,\qquad T_x\partial\mathcal{E}=\mathbb{R}\tau(x)+\sum_{j=1}^3\mathbb{R}_+\nu^{(j)}(x)\,,\qquad\forall x\in\mathrm{int}\,(\gamma)\,, \tag{4.30}$$

$$T_x\gamma_k=\mathbb{R}\tau_k(x)\,,\qquad T_x\partial\mathcal{E}_k=\mathbb{R}\tau_k(x)+\sum_{j=1}^3\mathbb{R}_+\nu_k^{(j)}(x)\,,\qquad\forall x\in\mathrm{int}\,(\gamma_k)\,, \tag{4.31}$$

while

$$\begin{cases} |\tau(x)-\tau(y)|+|\nu^{(j)}(x)-\nu^{(j)}(y)|\le L\,|x-y|^{\alpha}\,, \\ |\nu^{(j)}(x)\cdot(x-y)|\le L\,|x-y|^{1+\alpha}\,, \end{cases} \qquad \forall x,y\in\gamma\,, j=1,2,3\,, \tag{4.32}$$

$$\begin{cases} |\tau_k(x)-\tau_k(y)|+|\nu_k^{(j)}(x)-\nu_k^{(j)}(y)|\le L\,|x-y|^{\alpha}\,, \\ |\nu_k^{(j)}(x)\cdot(x-y)|\le L\,|x-y|^{1+\alpha}\,, \end{cases} \qquad \forall x,y\in\gamma_k\,, j=1,2,3\,. \tag{4.33}$$

Finally, set

$$\lambda_k^{(1)}(x)=\nu_k^{(1)}(x)\,,\qquad \lambda_k^{(2)}(x)=\frac{\nu^{(3)}(x)-\nu^{(2)}(x)}{|\nu^{(3)}(x)-\nu^{(2)}(x)|}\,,\qquad x\in\gamma_k\,,$$

so that $\{\lambda_k^{(1)}(x),\lambda_k^{(2)}(x)\}$ *is an orthonormal basis of* $\tau_k(x)^{\perp}$ *for every* $x\in\gamma_k$ *with*

$$\begin{cases} |\lambda_k^{(j)}(x)-\lambda_k^{(j)}(y)|\le L\,|x-y|^{\alpha}\,, \\ |\lambda_k^{(j)}(x)\cdot(y-x)|\le L\,|y-x|^{1+\alpha}\,, \end{cases} \qquad \forall x,y\in\gamma_k\,, j=1,2\,, \tag{4.34}$$

that is $\|\gamma_k\|_{C^{1,\alpha}}\le L$*.*

Proof. The inclusion $\mathrm{int}\,(\gamma)\subset\Sigma_Y(\mathcal{E})$ implies that $T_x\partial\mathcal{E}$ is isometric to Y for every $x\in\mathrm{int}\,(\gamma)$, so that the existence of vector fields such that (4.30) and (4.31) hold is immediate. Note that the set of the three vectors $\{\nu^{(j)}(x)\}_{j=1}^3$ is uniquely determined by (4.30) at every $x\in\mathrm{int}\,(\gamma)$, as these three vectors must be the inner conormals to the three surfaces in $\mathcal{S}(\mathcal{E})$ meeting along γ, while, for each $x\in\mathrm{int}\,(\gamma)$, (4.30) determines $\tau(x)$ only modulo multiplication by ±1.

Let us now cover γ by the family of balls $\{B_{x^i,t^i/3}\}_{i=0}^m$ considered in the proof of Lemma 4.3 in correspondence, say, to the value $\delta=\delta_0/2$. (In particular, if $\gamma\in\Gamma_Y(\mathcal{E})$, then (4.18) and (4.19) hold, while if $\gamma\in\Gamma_T(\mathcal{E})$, then (4.24), (4.25) and (4.26) hold.) Let τ_0 and $\nu_0^{(j)}$ be unit vectors such that our reference cone Y takes the form

$$Y=\mathbb{R}\,\tau_0+\sum_{j=1}^3\mathbb{R}_+\,\nu_0^{(j)}=\bigcup_{j=1}^3\Pi^{(j)}\,,$$

where $\Pi^{(j)}=\mathbb{R}\tau_0+\mathbb{R}_+\nu_0^{(j)}$ is an half-plane. By applying (4.6) with $r=t^i$, in the case $\gamma\in\Gamma_Y(\mathcal{E})$ or $\gamma\in\Gamma_T(\mathcal{E})$ with $i=1,...,m-1$, we find an open interval J^i containing 0 such that $\gamma\cap B_{x^i,t^i}=\{\Phi^i(s\,\tau_0):s\in J^i\}$ (where Φ^i stands for Φ^i_r with $r=t^i$); while in the case $\gamma\in\Gamma_T(\mathcal{E})$ and $i\in\{0,m\}$, we find an half-open/half-closed interval J^i containing 0 as an end-point, such that $\gamma\cap B_{x^i,t^i}=\{\Phi^i(s\,\tau_0):s\in J^i\}$. As a consequence

$$\tau(x)=\frac{\nabla\Phi^i(s\tau_0)[\tau_0]}{|\nabla\Phi^i(s\tau_0)[\tau_0]|}\,,\qquad x=\Phi^i(s\,\tau_0)\,,\quad s\in J^i\,, \tag{4.35}$$

defines a unit tangent vector field to $\gamma\cap B_{x^i,t^i}$. Note that $|\nabla\Phi^i(s\tau_0)[\tau_0]|>0$ for every $s\in J^i$ as Φ^i is a diffeomorphism, and that this procedure defines τ as a continuous vector field on the whole γ thanks to (4.19) and (4.26) up to possibly switching the sign in (4.35). Now let $x,y\in\gamma$, so that $x\in\gamma\cap B_{x^i,t^i/3}$ for some i. If $y\in\gamma\setminus B_{x^i,t^i}$, then $|x-y|\ge 2t^i/3\ge 2t^*/3$ for t^* defined as in (4.15), and thus $|\tau(x)-\tau(y)|\le C\,|x-y|^{\alpha}$ for a constant depending on α and t^* only. If, instead, $y\in B_{x^i,t^i}\cap\gamma$, then there exist $s,t\in J^i$ such that $x=\Phi^i(s\,\tau_0)$ and $y=\Phi^i(t\,\tau_0)$, and by exploiting $\|\Phi^i\|_{C^{0,\alpha}}\le C_0$ and $\mathrm{Lip}\,(\Phi^i)^{-1}\le C_0$ we obtain from (4.35) that $|\tau(x)-\tau(y)|\le C\,|s-t|^{\alpha}\le C\,|x-y|^{\alpha}$ for C depending on C_0 only. Since γ_k is covered by the balls $\{B_{x^i_k,2t^i/3}\}_{i=0}^m$, by (4.7) and by an entirely similar argument we come to prove the existence of vector fields τ and τ_k as in (4.32) and (4.33).

We now show that the vector fields $\nu^{(j)}$ and $\nu_k^{(j)}$ satisfy (4.32) and (4.33) respectively. Clearly, it suffices to discuss this for $\nu^{(j)}$. Moreover, we shall only detail the case $\gamma \in \Gamma_Y(\mathcal{E})$, as giving details on the case $\gamma \in \Gamma_T(\mathcal{E})$ would require the introduction of additional notation while being entirely analogous. This said, if $\gamma \in \Gamma_Y(\mathcal{E})$, then by (4.6) there exists $\{S^j\}_{j=1}^3 \subset \mathcal{S}(\mathcal{E})$ such that $\Phi^i(B_{0,2t^i} \cap \Pi^j) \cap B_{x^i,t^i} = B_{x^i,t^i} \cap S^j$ for each $j = 1,2,3$. In particular, since $\nu_0^{(j)}$ points inward Π^j and $T_x S^j = \mathbb{R}\tau(x) + \mathbb{R}_+\nu^{(j)}(x)$ for every $x \in S^j \cap \gamma$, we see that

$$\nabla\Phi^i(s\tau_0)[\nu_0^{(j)}] \in \mathbb{R}\tau(x) + \mathbb{R}_+\nu^{(j)}(x)\,, \qquad x = \Phi^i(s\tau_0)\,, \quad s \in J^i\,. \tag{4.36}$$

Since $\nabla\Phi^i(s\tau_0)[\tau_0]$ is parallel to $\tau(x)$, $\tau(x)$ and $\nu^{(j)}(x)$ are orthogonal, and $\nabla\Phi^i(s\tau_0)$ is invertible, it must actually be

$$\nabla\Phi^i(s\tau_0)[\nu_0^{(j)}] \cdot \nu^{(j)}(x) > 0\,, \qquad x = \Phi^i(s\tau_0)\,, \quad s \in J^i\,. \tag{4.37}$$

By (4.36) and (4.37) we find

$$\nu^{(j)}(x) = \frac{\nabla\Phi^i(s\tau_0)[\nu_0^{(j)}] - (\nabla\Phi^i(s\tau_0)[\nu_0^{(j)}] \cdot \tau(x))\tau(x)}{|\nabla\Phi^i(s\tau_0)[\nu_0^{(j)}] - (\nabla\Phi^i(s\tau_0)[\nu_0^{(j)}] \cdot \tau(x))\tau(x)|}\,, \qquad x = \Phi^i(s\tau_0)\,, \quad s \in J^i\,.$$

By exploiting again the fact that $\{B_{x^i,t^i/3}\}_{i=0}^m$ covers γ we conclude as in the previous case that $|\nu^{(j)}(x) - \nu^{(j)}(y)| \le C|x-y|^\alpha$ for every $x, y \in \gamma$. Again by the covering property, we are left to show that

$$|\nu^{(j)}(x) \cdot (x-y)| \le C\,|s-t|^{1+\alpha}\,, \qquad x = \Phi^i(s\tau_0)\,, y = \Phi^i(t\tau_0)\,, \quad s,t \in J^i\,. \tag{4.38}$$

Indeed we easily see that

$$|\Phi^i(s\tau_0) - \Phi^i(t\tau_0) - \nabla\Phi^i(t\tau_0)[\tau_0](s-t)| \le C\,|s-t|^{1+\alpha}\,, \qquad \forall s,t \in J^i\,,$$

while $\nabla\Phi^i(s\tau_0)[\tau_0]$ is parallel to $\tau(x)$ and $\tau(x)$ and $\nu^{(j)}(x)$ are orthogonal, so that

$$\nu^{(j)}(x) \cdot (x-y) = \nu^{(j)}(x) \cdot \big(\Phi^i(s\tau_0) - \Phi^i(t\tau_0) - \nabla\Phi^i(t\tau_0)[\tau_0](s-t)\big)\,;$$

by combining these last two fact, we prove (4.38). Finally, the assertions about the vector fields $\lambda_k^{(j)}$ follow by similar considerations. □

Lemma 4.5. *If* (4.1) *holds then for every $k \ge k_0$ and $\gamma \in \Gamma_Y(\mathcal{E})$ there exists a $C^{1,\alpha}$-diffeomorphism f_k between γ and γ_k with*

$$\lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1(\gamma)} = 0\,, \qquad \|f_k\|_{C^{1,\alpha}(\gamma)} \le C\,,$$

$$(f_k - \mathrm{Id}) \cdot \tau = 0\,, \qquad \textit{on } \gamma\,.$$

Proof. Let p_γ denote the projection of $\mathbb{R}^3$ over γ, and let $\delta_0 > 0$ be such $p_\gamma \in C^2(I_{\delta_0}(\gamma))$. For $k \ge k_0$, we have $\gamma_k \subset I_{\delta_0}(\gamma)$. We claim that

$$\lim_{k\to\infty} \|\tau \circ p_\gamma - \tau_k\|_{C^0(\gamma_k)} = 0\,. \tag{4.39}$$

Should this not be the case, then, up to extracting subsequences and thanks to Theorem 3.2 and to $\mathrm{hd}(\gamma_k, \gamma) \to 0$, we could find $\varepsilon > 0$, $y_k \in \gamma_k$, and $y_0 \in \gamma$ such that

$$\lim_{k\to\infty} y_k = y_0\,, \qquad \lim_{k\to\infty} \tau_k(y_k) = \tau(y_0) \qquad \inf_{k\ge k_0} |\tau(p_\gamma(y_k)) - \tau_k(y_k)| \ge \varepsilon\,.$$

Clearly $p_\gamma(y_k) \to y_0$, and hence $\tau(p_\gamma(y_k)) \to \tau(y_0)$ thanks to (4.32). We thus obtain a contradiction and prove (4.39). Now, by Lemma 4.4 we have

$$\begin{aligned} \mathrm{hd}_{x,r}(\gamma, x + \mathbb{R}\tau(x)) &\le C\,r^{1+\alpha}\,, \\ \mathrm{hd}_{y,r}(\gamma_k, y + \mathbb{R}\tau_k(y)) &\le C\,r^{1+\alpha}\,, \end{aligned} \qquad \forall x \in \gamma\,, y \in \gamma_k\,, r > 0\,. \tag{4.40}$$

Combining (4.39) and (4.40) we see that the restriction g_k of p_γ to γ_k is an invertible map $g_k : \gamma_k \to \gamma$. By exploiting the fact that g_k is the projection of γ_k onto γ one finds that

$$\nabla^{\gamma_k} g_k(y) = \Big(\tau(g_k(y)) \cdot \tau_k(y)\Big)\, \tau(g_k(y)) \otimes \tau_k(y)\,, \qquad \forall y \in \gamma_k\,. \tag{4.41}$$

Since, trivially, $|g_k(y) - g_k(y')| \le |y - y'|$ for every $y, y' \in \gamma_k$, by exploiting this formula together with (4.32) and (4.33), we conclude that

$$\sup_{k \ge k_0} \|g_k\|_{C^{1,\alpha}(\gamma_k)} \le C\,. \tag{4.42}$$

We also notice that, again by (4.41)

$$J^{\gamma_k} g_k = |\tau(g_k(y)) \cdot \tau_k(y)| \ge \frac{1}{2}\,, \qquad \text{on } \gamma_k \text{ for } k \ge k_0\,, \tag{4.43}$$

while $\|g_k - \mathrm{Id}\|_{C^0(\gamma_k)} \le \mathrm{hd}(\gamma, \gamma_k) \to 0$. By combining this last fact with (4.42) and (4.43) we are in the position to apply [CLM14, Theorem 2.1] and deduce that g_k is a $C^{1,\alpha}$-diffeomorphism between γ_k and γ with $\|f_k\|_{C^{1,\alpha}(\gamma)} \le C$. In order to check that $\|f_k - \mathrm{Id}\|_{C^1(\gamma)} \to 0$, it is enough to notice that, again by (4.39),

$$\|\nabla^{\gamma_k} g_k - \tau_k \otimes \tau_k\|_{C^0(\gamma_k)} \le \|(\tau \circ g_k) \cdot \tau_k - 1\|_{C^0(\gamma_k)} \to 0\,.$$

□

Lemma 4.6. *If* (4.1) *holds then there exist* $\mu_*, C_* > 0$ *with the following property. If* $\gamma \in \Gamma_T(\mathcal{E})$, $\mu < \mu_*$, *and* $k \ge k_0$ *(for some* k_0 *depending also on* μ*), then there exists a* $C^{1,\alpha}$*-diffeomorphism between* γ *and* γ_k *with* $f_k(\mathrm{bd}\,(\gamma)) = \mathrm{bd}\,(\gamma_k)$ *such that*

$$\lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1(\gamma)} = 0\,, \qquad \|f_k\|_{C^{1,\alpha}(\gamma)} \le C_*\,,$$
$$(f_k - \mathrm{Id}) \cdot \tau = 0\,, \qquad \text{on } [\gamma]_\mu\,,$$
$$\|(f_k - \mathrm{Id}) \cdot \tau\|_{C^1(\gamma)} \le \frac{C_*}{\mu}\, \|f_k - \mathrm{Id}\|_{C^0(\mathrm{bd}\,(\gamma))}\,.$$

Proof. Let $\rho_0 > 0$ be such that $[\gamma]_\rho \ne \emptyset$ for $\rho < \rho_0$. We claim the existence of $L > 0$ with the following property: for every $\rho < \rho_0$ and $k \ge k_0$ (with k_0 depending also on ρ), there exists a $C^{1,\alpha}$-diffeomorphism f_k between $[\gamma]_\rho$ and $f_k([\gamma]_\rho)$ such that

$$\begin{gathered}\lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1([\gamma]_\rho)} = 0\,, \qquad \|f_k\|_{C^{1,\alpha}([\gamma]_\rho)} \le L\,,\\ [\gamma_k]_{3\rho} \subset f_k([\gamma]_\rho) \subset \gamma_k\,, \qquad (f_k - \mathrm{Id}) \cdot \tau = 0\,, \qquad \text{on } [\gamma]_\rho\,.\end{gathered} \tag{4.44}$$

Indeed, by the same argument as in the previous proof we construct a diffeomorphism f_k from $[\gamma]_\rho$ to $\gamma_k \cap N_{\delta_0}([\gamma]_\rho)$ such that (4.44) holds with $(f_k - \mathrm{Id}) \cdot \tau = 0$ on $[\gamma]_\rho$. We are thus left to prove that if $x \in \gamma_k$ with $\mathrm{dist}(x, \mathrm{bd}\,(\gamma_k)) \ge 3\rho$, then $\mathrm{dist}(p_\gamma(x), \mathrm{bd}\,(\gamma)) \ge \rho$. Indeed, let $x^0 \in \mathrm{bd}\,(\gamma)$ be such that $\mathrm{dist}(p_\gamma(x), \mathrm{bd}\,(\gamma)) = |x^0 - p_\gamma(x)|$, and let $x_0^k \in \mathrm{bd}\,(\gamma_k)$ be such that $|x_k^0 - x^0| \le \mathrm{hd}(\mathrm{bd}\,(\gamma), \mathrm{bd}\,(\gamma_k))$. Then we have,

$$\begin{aligned} 3\rho &\le \mathrm{dist}(x, \mathrm{bd}\,(\gamma_k)) \le |x - x_k^0| \le |x - p_\gamma(x)| + |p_\gamma(x) - x^0| + |x^0 - x_k^0| \\ &\le \mathrm{hd}(\gamma, \gamma_k) + \mathrm{hd}(\mathrm{bd}\,(\gamma), \mathrm{bd}\,(\gamma_k)) + \mathrm{dist}(p_\gamma(x), \mathrm{bd}\,(\gamma))\,, \end{aligned}$$

so that $\mathrm{dist}(p_\gamma(x), \mathrm{bd}\,(\gamma)) \ge \rho$ provided $\mathrm{hd}(\gamma, \gamma_k) + \mathrm{hd}(\mathrm{bd}\,(\gamma), \mathrm{bd}\,(\gamma_k)) \le 2\rho$ for $k \ge k_0$.

Now let μ_* and C_* be the positive constants associated by [CLM14, Theorem 3.5] to $\gamma \in \Gamma(\mathcal{E})$, α as in Theorem A, and L redefined to be the maximum between the constant appearing in Lemma 4.4 and the constant appearing in (4.44). By taking into account (4.44) and

$$\|\gamma_k\|_{C^{1,\alpha}} \le L\,, \qquad \lim_{k\to\infty} \mathrm{hd}(\gamma_k, \gamma) + \mathrm{hd}(\mathrm{bd}\,(\gamma_k), \mathrm{bd}\,(\gamma)) + \|\tau \circ p_\gamma - \tau_k\|_{C^0(\mathrm{bd}\,(\gamma_k))} = 0\,,$$

(which follows from Theorem 3.2, Lemma 4.3, and Lemma 4.4) we are in the position to apply [CLM14, Theorem 3.5] to complete the proof of the lemma. □

By juxtaposing the maps f_k defined in Lemma 4.5 and Lemma 4.6 we define (for $k \ge k_0$ with k_0 corresponding to a fixed value of $\mu < \mu_*$) a homeomorphism

$$f_k : \Sigma(\mathcal{E}) \to \Sigma(\mathcal{E}_k)$$

such that $f_k(\Sigma_T(\mathcal{E})) = \Sigma_T(\mathcal{E}_k)$, $f_k(\Sigma_Y(\mathcal{E})) = \Sigma_Y(\mathcal{E}_k)$, and

$$\lim_{k\to\infty} \sup_{\gamma\in\Gamma(\mathcal{E})} \|f_k - \mathrm{Id}\|_{C^1(\gamma)} = 0\,, \qquad \sup_{k\ge k_0} \sup_{\gamma\in\Gamma(\mathcal{E})} \|f_k\|_{C^{1,\alpha}(\gamma)} \le C\,.$$

Moreover, denoting by $\tau_Y \in C^{1,1}(\Sigma_Y(\mathcal{E});\mathbb{S}^2)$ the unit tangent vector field to $\Sigma_Y(\mathcal{E})$ obtained by juxtaposing the vector fields τ defined in Lemma 4.4, we have

$$(f_k - \mathrm{Id})\cdot\tau_Y = 0 \quad \text{on } [\Sigma(\mathcal{E})]_\mu\,, \qquad \|(f_k - \mathrm{Id})\cdot\tau_Y\|_{C^1([\Sigma(\mathcal{E})]_\mu)} \le \frac{C}{\mu}\,\|f_k - \mathrm{Id}\|_{C^0(\Sigma_T(\mathcal{E}))}\,,$$

where with a slight abuse of notation with respect to (2.2) we have set

$$[\Sigma(\mathcal{E})]_\mu = \Sigma(\mathcal{E}) \setminus I_\mu(\Sigma_T(\mathcal{E}))\,.$$

We also notice for future reference that f_k has the following property with respect to the boundaries of the chambers of the clusters, namely

$$f_k(\partial\mathcal{E}(i)\cap\Sigma(\mathcal{E})) = \partial\mathcal{E}_k(i)\cap\Sigma(\mathcal{E}_k)\,, \qquad \forall i = 0,...,N\,. \tag{4.45}$$

This last remark completes the picture concerning the singular sets. We now start discussing the problem of mapping $\mathcal{S}(\mathcal{E})$ into $\mathcal{S}(\mathcal{E}_k)$. In the following ρ_0 denotes the parameter introduced in Theorem 3.1. Up to further decreasing the value of ρ_0 we may assume that

$$\mathrm{dist}(S, S') \ge 2\rho_0\,, \qquad \forall S \in \mathcal{S}_*(\mathcal{E})\,, S' \in \mathcal{S}(\mathcal{E})\setminus\{S\}\,, \tag{4.46}$$

As a consequence, we find of course that

$$\mathrm{dist}(S, \Sigma(\mathcal{E})) \ge 2\rho_0\,, \qquad \forall S \in \mathcal{S}_*(\mathcal{E})\,. \tag{4.47}$$

We also assume that

$$\begin{aligned} &S \cap I_\rho(\mathrm{bd}_\tau(S)) = S \cap I_\rho(\Sigma(\mathcal{E}))\,,\\ &[S]_\rho \text{ is connected}\,, \end{aligned} \qquad \forall S \in \mathcal{S}_\Sigma(\mathcal{E})\,, \rho < \rho_0\,. \tag{4.48}$$

Finally, we fix $\nu_\mathcal{E} \in C^{1,1}(\partial^*\mathcal{E};\mathbb{S}^2)$ (recall that under (4.1) we have that $\partial^*\mathcal{E}$ is a $C^{2,1}$-surface) and set for every $S \in \mathcal{S}(\mathcal{E})$

$$\nu_S = \nu_\mathcal{E} \qquad \text{on } S \cap \partial^*\mathcal{E}\,. \tag{4.49}$$

Lemma 4.7. *If* (4.1) *holds, then to every $S \in \mathcal{S}(\mathcal{E})$ and $k \ge k_0$ one can associate $S_k \in \mathcal{S}(\mathcal{E}_k)$ in such a way that $S \in \mathcal{S}_\Sigma(\mathcal{E})$ if and only if $S_k \in \mathcal{S}_\Sigma(\mathcal{E}_k)$ and*

$$\lim_{k\to\infty} \mathrm{hd}(S, S_k) + \mathrm{hd}(\mathrm{bd}_\tau(S), \mathrm{bd}_\tau(S_k)) = 0\,. \tag{4.50}$$

Moreover, there exists $\rho_0 > 0$ such that if $\rho < \rho_0$ and $k \ge k_0$ (for k_0 that now depends also on ρ) then there exists $\psi_k \in C^{1,\alpha}([S]_\rho)$ such that

$$[S_k]_{3\rho} \subset (\mathrm{Id} + \psi_k\nu_S)([S]_\rho) \subset S_k\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1([S]_\rho)} = 0\,, \qquad \|\psi_k\|_{C^{1,\alpha}([S]_\rho)} \le C\,. \tag{4.51}$$

In particular, if $S \in \mathcal{S}_*(\mathcal{E})$, then $S_k \in \mathcal{S}_*(\mathcal{E}_k)$ and (4.51) *boils down to*

$$S_k = (\mathrm{Id} + \psi_k\nu_S)(S)\,, \qquad \lim_{k\to\infty} \|\psi_k\|_{C^1(S)} = 0\,, \qquad \|\psi_k\|_{C^{1,\alpha}(S)} \le C\,. \tag{4.52}$$

Finally,

$$\partial\mathcal{E}_k = \bigcup_{S\in\mathcal{S}(\mathcal{E})} S_k\,. \tag{4.53}$$

Proof. Step one: In this step we associate to each $S \in \mathcal{S}_\Sigma(\mathcal{E})$ a surface $S_k \in \mathcal{S}_\Sigma(\mathcal{E}_k)$ in such a way that (4.50) and (4.51) hold. Let δ_0 be defined as in the proof of Lemma 4.3, and let I, k_0, $\{x^i\}_{i\in I} \subset \Sigma(\mathcal{E})$, $\{x^i_k\}_{i\in I} \subset \Sigma(\mathcal{E}_k)$, $\{t^i\}_{i\in I}$, and t_* be likewise defined correspondingly to $\delta = \delta_0/2$. In particular, $t_* \le t^i \le \delta_0/4$ for every $i \in I$ and

$$\mathrm{bd}_\tau(S) = S \cap \Sigma(\mathcal{E}) \subset \bigcup_{i\in I(S)} B_{x^i,t^i/3}\,, \tag{4.54}$$

where $I(S) = \{i \in I : x^i \in S\}$. By compactness and by (4.54) there exists $s_* > 0$ (depending on δ_0) such that

$$I_{s_*}(\mathrm{bd}_\tau(S)) \subset \bigcup_{i\in I(S)} B_{x^i,t^i/3}\,. \tag{4.55}$$

We shall require that ρ_0, in addition to the various constraints considered so far, is small enough in terms of s_*. By Theorem 3.1 for every $\rho < \rho_0$ and $k \ge k_0$ there exists $\psi_k \in C^{1,\alpha}([\partial\mathcal{E}]_\rho)$ (where $[\partial\mathcal{E}]_\rho = \partial\mathcal{E} \setminus I_\rho(\Sigma(\mathcal{E}))$) such that

$$\partial\mathcal{E}_k \setminus I_{2\rho}(\Sigma(\mathcal{E})) \subset (\mathrm{Id} + \psi_k\nu_{\mathcal{E}})([\partial\mathcal{E}]_\rho) \subset \partial^*\mathcal{E}_k\,, \tag{4.56}$$

$$N_{\varepsilon_0}([\partial\mathcal{E}]_\rho) \cap \partial\mathcal{E}_k = (\mathrm{Id} + \psi_k\,\nu_{\mathcal{E}})([\partial\mathcal{E}]_\rho)\,, \tag{4.57}$$

$$\lim_{k\to\infty} \|\psi_k\|_{C^1([\partial\mathcal{E}]_\rho)} = 0\,, \qquad \|\psi_k\|_{C^{1,\alpha}([\partial\mathcal{E}]_\rho)} \le C\,. \tag{4.58}$$

Here $k_0 \in \mathbb{N}$ and $\varepsilon_0 \in (0,\rho)$ depend also on ρ, while C just depends on $\mathcal{E}$, α and Λ.

By the first condition in (4.48), if $S \in \mathcal{S}_\Sigma(\mathcal{E})$, then $[S]_\rho = [\partial\mathcal{E}]_\rho \cap S$, and thus, thanks to (4.57) and provided $\|\psi_k\|_{C^0([\partial\mathcal{E}]_\rho)} \le \varepsilon_0$, we get

$$(\mathrm{Id} + \psi_k\nu_S)([S]_\rho) = N_{\varepsilon_0}([S]_\rho) \cap (\mathrm{Id} + \psi_k\nu_{\mathcal{E}})([\partial\mathcal{E}]_\rho) = N_{\varepsilon_0}([S]_\rho) \cap \partial\mathcal{E}_k\,. \tag{4.59}$$

Since $(\mathrm{Id} + \psi_k\nu_S)([S]_\rho)$ is connected by the second condition in (4.48), we find that there exists a unique $S_k \in \mathcal{S}(\mathcal{E}_k)$ such that

$$(\mathrm{Id} + \psi_k\nu_S)([S]_\rho) \subset \mathrm{int}\,(S_k) \subset S_k\,. \tag{4.60}$$

We notice that $S_k \in \mathcal{S}_\Sigma(\mathcal{E}_k)$: indeed, for each $i \in I(S)$ and provided k_0 is large enough with respect to ρ, we have that

$$(\mathrm{Id} + \psi_k\nu_S)([S]_\rho) \cap B_{x^i_k,2t^i/3} \ne \emptyset\,; \tag{4.61}$$

at the same time, by construction, $\partial\mathcal{E}_k \cap B_{x^i_k,2t^i/3}$ consists of the intersection with $B_{x^i_k,2t^i/3}$ of exactly three or four surfaces from $\mathcal{S}_\Sigma(\mathcal{E})$. Hence $S_k \in \mathcal{S}(\mathcal{E}_k)$. Now let us set

$$M_k = M^1_k \cup M^2_k \quad \text{where} \quad \begin{cases} M^1_k = (\mathrm{Id} + \psi_k\,\nu_S)([S]_\rho)\,, \\ M^2_k = S_k \cap \mathrm{cl}\left(I_{s_*}(\mathrm{bd}_\tau(S))\right). \end{cases}$$

We claim that $M_k = S_k$. Since, trivially, M_k is a compact subset of S_k, by the connectedness of S_k it will suffice to prove that M_k is a topological surface with boundary $\mathrm{bd}_\tau(M_k) \subset \mathrm{bd}_\tau(S_k)$. Indeed, M_k is locally homeomorphic to an open disk at every $x \in M_k$ such that

$$\begin{aligned} &\text{either} \quad x \in (\mathrm{Id} + \psi_k\,\nu_S)\Big(S \setminus \mathrm{cl}\left(I_\rho(\mathrm{bd}_\tau(S))\right)\Big)\,, \\ &\text{or} \quad x \in \mathrm{int}\,(S_k) \cap I_{s_*}(\mathrm{bd}_\tau(S))\,, \\ &\text{or} \quad x \in (\mathrm{Id} + \psi_k\nu_S)\Big(\{y \in S : \mathrm{dist}(y, \mathrm{bd}_\tau(S)) = \rho\}\Big)\,, \\ &\text{or} \quad x \in S_k\,, \qquad \mathrm{dist}(x, \mathrm{bd}_\tau(S)) = s_*\,; \end{aligned} \tag{4.62}$$

and M_k is locally homeomorphic to a open half-disc union its diameter at every $x \in M_k$ such that

$$x \in \mathrm{bd}_\tau(S_k) \cap I_{s_*}(\mathrm{bd}_\tau(S))\,. \tag{4.63}$$

The first two cases in (4.62) and (4.63) are trivial (as we are localizing the topological surface with boundary S_k by intersecting it with certain open sets). In the third case, $x = y+\psi_k(y)\nu_S(y)$ with $\mathrm{dist}(y, \mathrm{bd}_\tau(S)) = \rho$, so that

$$\mathrm{dist}(x, \mathrm{bd}_\tau(S)) \le \rho + \|\psi_k\|_{C^0([S]_\rho)} < s_*\,,$$

provided k_0 is large enough; this shows that

$$(\mathrm{Id} + \psi_k\nu_S)\Big(\{y \in S : \mathrm{dist}(y, \mathrm{bd}_\tau(S)) = \rho\}\Big) \subset S_k \cap I_{s_*}(\mathrm{bd}_\tau(S)) \subset M_k\,,$$

and thus addresses the third case of (4.62). In the fourth case, we fix $0 \le i < j \le N$ such that $S \subset \partial\mathcal{E}(i) \cap \partial\mathcal{E}(j)$ and notice that by (4.60) we have $S_k \subset \partial\mathcal{E}_k(i) \cap \partial\mathcal{E}_k(j)$. Hence, (3.2) implies

$$\begin{aligned}
S_k \cap \partial I_{s_*}(\mathrm{bd}_\tau(S)) &\subset \partial\mathcal{E}_k(i) \cap \partial\mathcal{E}_k(j) \cap \partial I_{s_*}(\mathrm{bd}_\tau(S))\\
&\subset I_{\varepsilon_0/2}\Big(\partial\mathcal{E}(i) \cap \partial\mathcal{E}(j) \cap \partial I_{s_*}(\mathrm{bd}_\tau(S))\Big)\\
&= I_{\varepsilon_0/2}\Big(S \cap \partial I_{s_*}(\mathrm{bd}_\tau(S))\Big) \subset I_{\varepsilon_0/2}([S]_{3\rho}) \subset N_{\varepsilon_0}([S]_{2\rho})\,.
\end{aligned}$$

In particular, by (4.59)

$$S_k \cap \partial I_{s_*}(\mathrm{bd}_\tau(S)) \subset S_k \cap N_{\varepsilon_0}([S]_{2\rho}) \subset (\mathrm{Id} + \psi_k\nu_S)([S]_{2\rho})\,,$$

so that the fourth case of (4.62) is a particular instance of the first one. We have thus shown that $M_k = S_k$, that is

$$S_k = (\mathrm{Id} + \psi_k\,\nu_S)([S]_\rho) \cup \Big(S_k \cap \mathrm{cl}\,\big(I_{s_*}(\mathrm{bd}_\tau(S))\big)\Big)\,. \tag{4.64}$$

We notice that in the process of showing (4.64) we have also proved that (see in particular (4.61))

$$\mathrm{bd}_\tau(S_k) \cap B_{x^i_k,2t^i/3} = \Sigma(\mathcal{E}_k) \cap B_{x^i_k,2t^i/3}\,, \qquad \forall i \in I(S) \text{ s.t. } x^i_k \in \Sigma_Y(\mathcal{E}_k)\,. \tag{4.65}$$

We now claim that for every $\eta > 0$ and $k \ge k_0$ (depending on η) we have

$$S_k \subset I_\eta(S)\,, \qquad \mathrm{bd}_\tau(S_k) \subset I_\eta(\mathrm{bd}_\tau(S))\,. \tag{4.66}$$

Indeed, let us repeat the argument leading to (4.64) with a suitably small $\delta = \delta(\eta)$ in place of $\delta = \delta_0/2$ (notice that, by connectedness of S_k we select the same surface from $\mathcal{S}_\Sigma(\mathcal{E}_k)$ in the process): correspondingly, we find $s_*(\eta) < \delta(\eta)$ and ρ suitably small with respect to $s_*(\eta)$ in such a way that (4.64) holds for $k \ge k_0$ and with $s_*(\eta)$ in place of s_*. As a consequence (4.66) immediately follows. We now notice that

$$f_k(\mathrm{bd}_\tau(S)) = \mathrm{bd}_\tau(S_k)\,. \tag{4.67}$$

Indeed, if $\gamma \in \Gamma(S)$, then up to adding finitely many points to the family $\{x^i\}_{i\in I}$, we can assume that there exists $i \in I(S)$ such that $x^i \in \mathrm{int}\,(\gamma) \subset \Sigma_Y(\mathcal{E})$. In particular, $x^i_k \in \Sigma_Y(\mathcal{E}_k)$ and $f_k(x^i) \in B_{x^i_k,2t^i/3}$ for k large enough, so that (4.65) implies

$$f_k(\gamma) \cap \mathrm{bd}_\tau(S_k) \ne \emptyset\,.$$

By connectedness, $f_k(\gamma) \subset \mathrm{bd}_\tau(S_k)$, and thus $f_k(\mathrm{bd}_\tau(S)) \subset \mathrm{bd}_\tau(S_k)$. To prove the converse inclusion we notice that

$$\mathrm{bd}_\tau(S) \cap B_{x^i,t^i/3} = \Sigma(\mathcal{E}) \cap B_{x^i,t^i/3}\,, \qquad \forall i \in I(S) \text{ s.t. } x^i \in \Sigma_Y(\mathcal{E})\,. \tag{4.68}$$

If we now pick $\gamma^* \in \Gamma(S_k)$, then $\gamma^* \subset I_{s_*/2}(\mathrm{bd}_\tau(S))$ thanks to (4.66). At the same time, for $k \ge k_0$, $f_k^{-1}(\gamma^*) = \gamma \in \Gamma(\mathcal{E})$ with $\gamma \subset I_{s_*}(\mathrm{bd}_\tau(S))$. In particular there exists $x^i \in \mathrm{int}\,(\gamma) \cap \mathrm{bd}_\tau(S)$, thus $\gamma \in \Gamma(S)$, that is $\gamma^* = f_k(\gamma) \subset f_k(\mathrm{bd}_\tau(S))$. This completes the proof of (4.67) and shows that

$$\lim_{k\to\infty} \mathrm{hd}(\mathrm{bd}_\tau(S_k), \mathrm{bd}_\tau(S)) = 0\,. \tag{4.69}$$

Thanks to (4.66) and a standard compactness argument in order to prove $\mathrm{hd}(S_k,S)\to 0$ we just need to check that for every $x\in S$ there exists $x_k\in S_k$ such that $x_k\to x$ as $k\to\infty$. Indeed, if $x\in\mathrm{int}\,(S)$ then $x\in[S]_\rho$ for $\rho=\rho(x)$ small enough. In particular, for $k\ge k(x)$ we have $\psi_k(x)\in S_k$ and $x_k=\psi_k(x)\to x$; if, instead, $x\in\mathrm{bd}_\tau(S)$, then we are done thanks to (4.69). We finally notice that since $\mathrm{hd}(S_k,S)\to 0$ for $k\to\infty$, given $\rho<\rho_0$ one can find k_0 depending on ρ such that

$$S_k\subset I_{2\rho}(\mathrm{bd}_\tau(S))\cup N_{\varepsilon_0}([S]_\rho)\,,\qquad\forall k\ge k_0\,,$$

and thus, thanks also to (4.59) and (4.69),

$$\begin{aligned}(\mathrm{Id}+\psi_k\nu_S)([S]_\rho) &= N_{\varepsilon_0}([S]_\rho)\cap\partial\mathcal{E}_k\supset N_{\varepsilon_0}([S]_\rho)\cap S_k\\ &\supset S_k\setminus I_{2\rho}(\mathrm{bd}_\tau(S))\supset S_k\setminus I_{3\rho}(\mathrm{bd}_\tau(S_k))=[S_k]_{3\rho}\,.\end{aligned}$$

This remark completes the proof of (4.51), thus of step one.

Step two: We now associate to each $S\in\mathcal{S}_*(\mathcal{E})$ a surface $S_k\in\mathcal{S}_*(\mathcal{E}_k)$ in such a way that (4.50) and (4.52) hold. Indeed, by (4.47) we have $[S]_\rho=S$ for every $\rho<\rho_0$. In particular, $S\subset[\partial\mathcal{E}]_\rho$. We claim that

$$N_{\varepsilon_0}(S)\cap(\mathrm{Id}+\psi_k\nu_{\mathcal{E}})([\partial\mathcal{E}]_\rho)=(\mathrm{Id}+\psi_k\nu_S)(S)\,. \tag{4.70}$$

The $\supset$ inclusion follows by $S\subset[\partial\mathcal{E}]_\rho$, provided k_0 is large enough to entail $\|\psi_k\|_{C^0([\partial\mathcal{E}]_\rho)}\le\varepsilon_0$; the $\subset$ inclusion follows from the fact that if $x+\psi_k(x)\nu_{\mathcal{E}}(x)\in N_{\varepsilon_0}(S)$ for some $x\in[\partial\mathcal{E}]_\rho\setminus S$, then $\mathrm{dist}(x,S)\le\mathrm{dist}(x+\psi_k(x)\nu_{\mathcal{E}}(x),S)+\|\psi_k\|_{C^0([\partial\mathcal{E}]_\rho)}<2\rho_0$ for some $x\in S'\in\mathcal{S}(\mathcal{E})\setminus\{S\}$, against (4.46). The same argument shows that

$$N_{\varepsilon_0}(S)\cap N_{\varepsilon_0}([\partial\mathcal{E}]_\rho)=N_{\varepsilon_0}(S)\,, \tag{4.71}$$

so that, by intersecting both sides of (3.5) with $N_{\varepsilon_0}(S)$ we find

$$N_{\varepsilon_0}(S)\cap\partial\mathcal{E}_k=(\mathrm{Id}+\psi_k\nu_S)(S)\,,\qquad\forall k\ge k_0\,.$$

Since S is connected, one has that $(\mathrm{Id}+\psi_k\nu_S)(S)$ is connected. By connectedness of the surfaces in $\mathcal{S}(\mathcal{E}_k)$, there exists a unique $S_k\in\mathcal{S}(\mathcal{E}_k)$ which intersects $(\mathrm{Id}+\psi_k\nu_S)(S)$, and thus must actually be equal to $(\mathrm{Id}+\psi_k\nu_S)(S)$ and belong to $\mathcal{S}_*(\mathcal{E}_k)$, with (4.52) in force thanks to (4.58).

Step three: We prove (4.53). Pick $S'\in\mathcal{S}_\Sigma(\mathcal{E}_k)$, and let $0\le i<j\le N$ be such that $S'\subset\partial\mathcal{E}_k(i)\cap\partial\mathcal{E}_k(j)$. By (4.45), $f_k^{-1}(\mathrm{bd}_\tau(S'))\subset\partial\mathcal{E}(i)\cap\partial\mathcal{E}(j)\cap\Sigma(\mathcal{E})$. Thus there exists $S\in\mathcal{S}_\Sigma(\mathcal{E})$ such that $S\subset\partial\mathcal{E}(i)\cap\partial\mathcal{E}(j)$ and $f_k^{-1}(\mathrm{bd}_\tau(S'))\cap S\ne\emptyset$. Let S_k be the surface associated to S by step one, so that, by the properties proved in step one,

$$S',S_k\in\mathcal{S}_\Sigma(\mathcal{E}_k)\,,\qquad S',S_k\subset\partial\mathcal{E}_k(i)\cap\partial\mathcal{E}_k(j)\,,\qquad\mathrm{bd}_\tau(S')\cap\mathrm{bd}_\tau(S_k)\ne\emptyset\,,$$

and hence $S'=S_k$. If instead $S'\in\mathcal{S}_*(\mathcal{E}_k)$, then by the first inclusion in (4.56) it must be $S'\cap(\mathrm{Id}+\psi_k\,\nu_S)([S]_\rho)\ne\emptyset$ for some $S\in\mathcal{S}(\mathcal{E})$. Should it be $S\in\mathcal{S}_\Sigma(\mathcal{E})$, then by arguing as in step one (see in particular (4.65)) we would find $S'=S_k\in\mathcal{S}_\Sigma(\mathcal{E}_k)$, a contradiction. Thus $S\in\mathcal{S}_*(\mathcal{E})$, and $S'=(\mathrm{Id}+\psi_k\nu_S)(S)=S_k$ under the correspondence defined in step two. $\square$

We now notice that, thanks to Theorem 1.1, given $S\in\mathcal{S}(\mathcal{E})$ the vector field $\nu_S:\mathrm{int}\,(S)\to\mathbb{S}^2$ defined in (4.49) satisfies $|\nu_S(x)-\nu_S(y)|\le L\,|x-y|^\alpha$ for every $x,y\in\mathrm{int}\,(S)$ for a constant L depending on $\mathcal{E}$ only. In particular, ν_S can be uniquely extended by continuity to S in such a way that

$$\begin{aligned}|\nu_S(x)-\nu_S(y)|&\le L\,|x-y|^\alpha\,,\\ |\nu_S(x)\cdot(y-x)|&\le L\,|x-y|^{1+\alpha}\,,\end{aligned}\qquad\forall x,y\in S\,. \tag{4.72}$$

By regularity of $\mathrm{int}\,(S_k)$, there exists $\nu_{S_k}\in C^{0,\alpha}(\mathrm{int}\,(S_k);\mathbb{S}^2)$ such that $\nu_{S_k}(x)^\perp=T_xS_k$ for every $x\in\mathrm{int}\,(S_k)$. By exploiting (4.51), (4.52), and Lemma 4.1 through an argument analogous to the one used in the proof of Lemma 4.4, we find that ν_{S_k} extends by continuity to the whole S_k in such a way that

$$\begin{aligned}|\nu_{S_k}(x)-\nu_{S_k}(y)|&\le L\,|x-y|^\alpha\,,\\ |\nu_{S_k}(x)\cdot(y-x)|&\le L\,|x-y|^{1+\alpha}\,,\end{aligned}\qquad \forall x,y\in S_k\,. \tag{4.73}$$

Moreover, thanks to Theorem 3.2,

$$x_k\in S_k\,,\quad \lim_{k\to\infty}x_k=x\in S\,,\qquad\Rightarrow\qquad \lim_{k\to\infty}\nu_{S_k}(x_k)=\nu_S(x)\,.$$

In particular, arguing by contradiction, one sees that

$$\lim_{k\to\infty}\|(\nu_{S_k}\circ f_k)-\nu_S\|_{C^0(\mathrm{bd}\,_\tau(S))}=0\,. \tag{4.74}$$

Recalling that $S^*=S\setminus\Sigma_T(\mathcal{E})$ and $S_k^*=S_k\setminus\Sigma_T(\mathcal{E})$ are $C^{1,\alpha}$-surfaces with boundary, and denoting by $\nu^{co}_{S^*}$ the outer unit conormal to S^* at $\mathrm{bd}\,(S^*)$, and similarly defining $\nu^{co}_{S_k^*}$, one comes to prove by analogous arguments that

$$\lim_{k\to\infty}\|(\nu^{co}_{S_k^*}\circ f_k)-\nu^{co}_{S^*}\|_{C^0(\mathrm{bd}\,(S^*))}=0\,. \tag{4.75}$$

As the last preparatory step towards the proof of Theorem 1.3, we now prove the following extension lemma.

Lemma 4.8. *If* (4.1) *holds, $S\in\mathcal{S}(\mathcal{E})$ and $a\in C^0(\mathrm{bd}\,_\tau(S))$ is such that $a\in C^{1,\alpha}(\gamma)$ for every $\gamma\in\Gamma(S)$, then there exists $\bar a\in C^{1,\alpha}(\mathbb{R}^3)$ such that $\bar a=a$ on $\mathrm{bd}\,_\tau(S)$ and*

$$\|\bar a\|_{C^{1,\alpha}(\mathbb{R}^3)}\le C\max_{\gamma\in\Gamma(S)}\|a\|_{C^{1,\alpha}(\gamma)}\qquad \|\bar a\|_{C^1(\mathbb{R}^3)}\le C\max_{\gamma\in\Gamma(S)}\|a\|_{C^1(\gamma)}\,. \tag{4.76}$$

Proof. The lemma is proved by an application of Whitney's extension theorem, see [CLM14, Section 2.3] for the notation and terminology adopted here. Let $X=\mathrm{bd}\,_\tau(S)$, so that X is connected by rectifiable arcs and its geodesic distance dist_X satisfies $\mathrm{dist}_X(x,y)\le\omega\,|x-y|$ whenever $x,y\in X$ and for some $\omega>0$ depending on S only. We claim the existence of a continuous vector-field $\overline{F}:X\to\mathbb{R}^3$ such that

$$\begin{aligned}|\overline{F}(x)-\overline{F}(y)|&\le C\,|x-y|^\alpha\,,\\ |a(y)-a(x)-\overline{F}(x)\cdot(y-x)|&\le C\,|x-y|^{1+\alpha}\,,\end{aligned}\qquad\forall x,y\in X\,. \tag{4.77}$$

We may then apply [CLM14, Theorem 2.3] to the jet $\mathcal{F}=\{F^{\mathbf{k}}\}_{|\mathbf{k}|\le1}$ with $F^{\mathbf{0}}=a$ and $F^{e_i}=\overline{F}\cdot e_i$ in order to conclude the proof of the lemma. Since X consists of finitely many cycles lying at mutually positive distance, in the proof of (4.77) we may as well assume that X consists of a single cycle. By Theorem 1.1, either X consists of a single $C^{2,1}$-diffeomorphic image of $\mathbb{S}^1$, or $X=\bigcup_{i=1}^m\gamma_i$ where $m\ge2$ and each γ_i is a compact connected $C^{2,1}$-curve with boundary such that $\gamma_i\cap\gamma_{i+1}=\mathrm{bd}\,(\gamma_i)\cap\mathrm{bd}\,(\gamma_{i+1})=\{p_i\}$ and $|\tau_i(p_i)\cdot\tau_{i+1}(p_i)|<1$ for every $i=1,...,m$. Here $\gamma_{m+1}=\gamma_1$ and $\tau_i\in C^{0,1}(\gamma_i,\mathbb{S}^1)$ is a tangent unit vector field to γ_i, oriented so that $\tau_i(p_i)$ points outwards γ_i at p_i, and $\tau_{i+1}(p_i)$ points inwards γ_{i+1} at p_i. Clearly, we have

$$|\tau_i(x)-\tau_i(y)|\le C\,|x-y|\,,\qquad\forall x,y\in\gamma_i\,. \tag{4.78}$$

We also record for future use that

$$\max\{|x-p_i|,|y-p_i|\}\le C\,|x-y|\,,\qquad\forall x\in\gamma_i\,,y\in\gamma_{i+1}\,, \tag{4.79}$$

as it follows easily by $|\tau_i(p_i)\cdot\tau_{i+1}(p_i)|<1$.

Now, let us set

$$\alpha_i(x)=\nabla^{\gamma_i}a(x)[\tau_i(x)]\,,\qquad x\in\mathrm{int}\,(\gamma_i)\,,$$

so that, if we denote by $\gamma_i(x,y)$ the arc of γ_i joining $x,y\in\gamma_i$, then

$$\begin{aligned}|a(y)-a(x)-\alpha_i(x)\,\mathcal{H}^1(\gamma_i(x,y))|&\le C\,|x-y|^{1+\alpha}\,,\\ |\alpha_i(x)-\alpha_i(y)|&\le C\,|x-y|^{1+\alpha}\,,\end{aligned}\qquad\forall x,y\in\gamma_i\,. \tag{4.80}$$

We claim that (4.77) holds provided we set

$$\overline{F}(x)=\alpha_i(x)\tau_i(x)+\beta_i(x)\,,\qquad x\in\gamma_i\,,\tag{4.81}$$

for any choice of $\beta_i:\gamma_i\to\mathbb{R}^3$ such that

$$\begin{array}{r}\beta_i(x)\cdot\tau_i(x)=0\,,\qquad |\beta_i(x)|\le C\\ |\beta_i(x)-\beta_i(y)|\le C|x-y|^\alpha\,,\end{array}\qquad\forall x,y\in\gamma_i\,,\tag{4.82}$$

and such that the compatibility conditions

$$\alpha_i(p_i)\tau_i(p_i)+\beta_i(p_i)=\alpha_{i+1}(p_i)\tau_{i+1}(p_i)+\beta_{i+1}(p_i)\,,\qquad 1\le i\le m\,,\tag{4.83}$$

hold. (Note that (4.83) is a necessary condition for a function $\overline{F}$ defined as in (4.81) to be continuous on X, and that the existence of choices of β satisfying (4.82) and (4.83) is easily proved.) Let us check the first condition in (4.77): if $x,y\in\gamma_i$, then this is trivial by (4.78), (4.80) and (4.82); if, instead, $x\in\gamma_i$ and $y\in\gamma_{i+1}$, then by (4.79) we have

$$\begin{aligned}|\overline{F}(x)-\overline{F}(y)| &\le |\overline{F}(x)-\overline{F}(p_i)|+|\overline{F}(y)-\overline{F}(p_i)|\le|x-p_i|^\alpha+|y-p_i|^\alpha\\ &\le 2\,\max\{|x-p_i|,|y-p_i|\}^\alpha\le C\,|x-y|^\alpha\,;\end{aligned}$$

finally, if $x\in\gamma_i$ and $y\in\gamma_j$ with $j\ne i-1,i,i+1$, then one simply has $|x-y|\ge 1/C$.

We are thus left to prove the second condition in (4.77). If $x,y\in\gamma_i$, then we have

$$|a(y)-a(x)-\overline{F}(x)\cdot(y-x)|\le|a(y)-a(x)-\alpha_i(x)\tau_i(x)\cdot(y-x)|+|\beta_i(x)\cdot(x-y)|\,.\tag{4.84}$$

If $x<y$ in the orientation of γ_i induced by τ_i, then

$$|\mathcal{H}^1(\gamma_i(x,y))-\tau_i(x)\cdot(y-x)|\le C\,|y-x|^2\,,\tag{4.85}$$

while thanks to the first condition in (4.82)

$$|\beta_i(x)\cdot(x-y)|\le C\,|x-y|^2\,,\qquad\forall x,y\in\gamma_i\,.\tag{4.86}$$

By combining (4.85) and (4.86) with (4.84) we prove the second condition in (4.77) in the case $x,y\in\gamma_i$. Once again we are left to consider the case when $x\in\gamma_i$ and $y\in\gamma_{i+1}$. In this case,

$$\begin{aligned}&|a(y)-a(x)-\overline{F}(x)\cdot(y-x)|\\ \le\ &|a(y)-a(p_i)-\alpha_{i+1}(p_i)\mathcal{H}^1(\gamma_{i+1}(p_i,y))|\\ &+|a(p_i)-a_i(x)-\alpha_i(p_i)\mathcal{H}^1(\gamma_i(p_i,x))|\\ &+|\alpha_{i+1}(p_i)\mathcal{H}^1(\gamma_{i+1}(p_i,y))-\alpha_i(p_i)\mathcal{H}^1(\gamma_i(p_i,x))-\overline{F}(x)\cdot(y-x)|\,,\end{aligned}$$

so that, by (4.80), (4.85) (where $x<p_i$ in the orientation of γ_i induced by τ_i and $p_i<y$ in the orientation of γ_{i+1} induced by τ_{i+1}) and (4.79), one finds

$$\begin{aligned}&|a(y)-a(x)-\overline{F}(x)\cdot(y-x)|\le C\,|x-y|^{1+\alpha}\\ &+|\alpha_{i+1}(p_i)\,(y-p_i)\cdot\tau_{i+1}(p_i)-\alpha_i(p_i)(p_i-x)\cdot\tau_i(p_i)-\overline{F}(x)\cdot(y-x)|\,.\end{aligned}$$

Thus it suffices to show that for every $x\in\gamma_i$ and $y\in\gamma_{i+1}$ one has

$$\begin{aligned}\big|\big(\alpha_i(p_i)\tau_i(p_i)-\overline{F}(x)\big)\cdot(p_i-x)|&\le C\,|x-y|^{1+\alpha}\,,\\ \big|\big(\alpha_{i+1}(p_i)\tau_{i+1}(p_i)-\overline{F}(x)\big)\cdot(y-p_i)|&\le C\,|x-y|^{1+\alpha}\,.\end{aligned}\tag{4.87}$$

The first inequality in (4.87) descends from the fact that $\overline{F}(x)=\alpha_i(x)\tau_i(x)+\beta_i(x)$, and thus, by (4.80), (4.86), and (4.79)

$$\begin{aligned}&\big|\big(\alpha_i(p_i)\tau_i(p_i)-\overline{F}(x)\big)\cdot(p_i-x)|\\ &\le|\alpha_i(p_i)\tau_i(p_i)-\alpha_i(x)\tau_i(x)|\,|p_i-x|+|\beta_i(x)\cdot(p_i-x)|\\ &\le C\,|p_i-x|^{1+\alpha}\le C\,|x-y|^{1+\alpha}\,.\end{aligned}$$

Concerning the second inequality, by exploiting

$$\big|(y-p_i) - \big((y-p_i)\cdot\tau_{i+1}(p_i)\big)\,\tau_{i+1}(p_i)\big| \le C\,|y-p_i|^2\,, \qquad \forall y\in\gamma_{i+1}\,,$$

we find that

$$\begin{aligned}&\big|\big(\alpha_{i+1}(p_i)\tau_{i+1}(p_i)-\overline{F}(x)\big)\cdot(y-p_i)\big|\\ \le\;& C|y-p_i|^2 + \big|\alpha_{i+1}(p_i)-\overline{F}(x)\cdot\tau_{i+1}(p_i)\big|\,|y-p_i|\,;\end{aligned}$$

by projecting (4.83) on $\tau_{i+1}(p_{i+1})$ we have $\alpha_{i+1}(p_i)=\overline{F}(p_i)\,\cdot\tau_{i+1}(p_{i+1})$, so that

$$\big|\alpha_{i+1}(p_i)-\overline{F}(x)\cdot\tau_{i+1}(p_i)\big| \le |\overline{F}(p_i)-\overline{F}(x)| \le C\,|x-p_i|^\alpha\,;$$

thus, again by (4.79),

$$\big|\big(\alpha_{i+1}(p_i)\tau_{i+1}(p_i)-\overline{F}(x)\big)\cdot(y-p_i)\big| \le C\,|x-p_i|^\alpha|y-p_i| \le C\,\max\{|x-p_i|,|y-p_i|\}^{1+\alpha} \le C\,|x-y|^{1+\alpha}\,,$$

and the proof is complete. □

5. Proof of the improved convergence theorem

We now prove Theorem 1.3. For μ_0 to be determined, we fix $\mu<\mu_0$ and $\rho<\mu^2$. (We automatically entail $\rho<\rho_0$, for ρ_0 the constant determined in the previous section, up to taking μ_0 small enough.) Let us fix $S\in\mathcal{S}(\mathcal{E})$, and correspondingly let $S_k\in\mathcal{S}(\mathcal{E})$ be the surfaces associated to S as in the previous section, and let us set $S^*=S\setminus\Sigma_T(\mathcal{E})$. In order to prove the theorem it is enough to show that for $k\ge k_0$ (depending on μ) there exists an homeomorphism f_k between S and S_k such that

$$\begin{aligned}&\|f_k\|_{C^{1,\alpha}(S^*)}\le C_0\,,\\ &\lim_{k\to\infty}\|f_k-\mathrm{Id}\|_{C^1(S^*)}=0\,,\\ &\|\pi^S(f_k-\mathrm{Id})\|_{C^1(S^*)}\le\frac{C_0}{\mu}\max_{\gamma\in\Gamma(S)}\|f_k-\mathrm{Id}\|_{C^1(\gamma)}\,,\\ &\pi^S(f_k-\mathrm{Id})=0\qquad\text{on }[S]_\mu=S\setminus I_\mu(\mathrm{bd}_\tau(S))\,,\end{aligned}\tag{5.1}$$

where for every $x\in S^*$, $v\in\mathbb{R}^3$, and $h:S^*\to\mathbb{R}^3$ we set

$$\pi^S_x(v)=v-(v\cdot\nu_S(x))\,\nu_S(x)\,,\qquad \pi^S h(x)=\pi^S_x(h(x))\,.$$

If $S\in\mathcal{S}_*(\mathcal{E})$, then (5.1) is an immediate consequence of Lemma 4.7, see in particular (4.52), so that, from now on we assume $S\in\mathcal{S}_\Sigma(\mathcal{E})$. In this way, by Lemma 4.7 there exists $\rho_0>0$ such that for every $\rho<\rho_0$ and $k\ge k_0$ (depending on ρ) there exists $\psi_k\in C^{1,\alpha}([S]_\rho)$ such that

$$\begin{aligned}&[S_k]_{3\rho}\subset(\mathrm{Id}+\psi_k\nu_S)([S]_\rho)\subset S_k\,,\\ &\|\psi_k\|_{C^{1,\alpha}([S]_\rho)}\le L\,,\qquad \|\psi_k\|_{C^1([S]_\rho)}\le\rho\,.\end{aligned}\tag{5.2}$$

and moreover

$$\mathrm{hd}(S,S_k)\le\rho\,,\qquad \|S_k\|_{C^{1,\alpha}}\le L\,,\tag{5.3}$$

where the last condition is (4.73). We denote by f_k^0 the $C^{1,\alpha}$-diffeomorphism between $\mathrm{bd}_\tau(S)$ and $\mathrm{bd}_\tau(S_k)$: precisely, f_k^0 is an homeomorphism between $\mathrm{bd}_\tau(S)$ and $\mathrm{bd}_\tau(S_k)$ such that, by Lemma 4.5, Lemma 4.6, (4.74), and (4.75), and up to increasing the value of L,

$$\begin{aligned}&\|f_k^0\|_{C^{1,\alpha}(\gamma)}\le L\,,\qquad \|f_k^0-\mathrm{Id}\|_{C^1(\gamma)}\le\rho\,,\\ &\|(\nu_{S_k}\circ f_k^0)-\nu_S\|_{C^0(\mathrm{bd}_\tau(S))}\le\rho\,,\\ &\|(\nu^{co}_{S_k^*}\circ f_k^0)-\nu^{co}_{S^*}\|_{C^0(\mathrm{bd}\,(S^*))}\le\rho\,,\end{aligned}\tag{5.4}$$

for every $\gamma\in\Gamma(S)$, where $S_k^*=S_k\setminus\Sigma_T(\mathcal{E}_k)$. Our goal is now to glue together the boundary diffeomorphism f_k^0 to the normal diffeomorphisms $(\mathrm{Id}+\psi_k\nu_S)$ defined on $[S]_\rho$ in such a way to control the size of the tangential displacement $\pi^S(f_k-\mathrm{Id})$. This is exactly the construction described in [CLM14, Theorem 3.1] in the case of k-dimensional manifolds with boundary in $\mathbb{R}^n$. Here we have $k = 2$ and $n = 3$, but, unfortunately, we cannot directly apply that result because of the boundary singularities of S (that is, because $S \cap \Sigma_T(\mathcal{E})$ may be nonempty). The proof of [CLM14, Theorem 3.1] can be anyway adapted to this context and we now describe the main modifications needed to this end.

The first remark is that, by arguing as in the proof of [CLM14, Theorem 3.5], in order to prove (5.1) it is enough to show that for every $\rho < \mu^2$ and $k \ge k_0$ depending on ρ there exists an homeomorphism f_k^ρ between S and S_k such that

$$\begin{aligned} &f_k^\rho = f_k^0 \quad \text{on } \mathrm{bd}_\tau(S)\,, \qquad f_k^\rho = \mathrm{Id} + \psi_k\,\nu_S \quad \text{on } [S]_\mu\,,\\ &\|f_k^\rho\|_{C^{1,\alpha}(S^*)} \le C\,, \qquad \|f_k^\rho - \mathrm{Id}\|_{C^1(S^*)} \le \frac{C}{\mu}\,\rho^\alpha\,,\\ &\|\pi^S(f_k^\rho - \mathrm{Id})\|_{C^1(S^*)} \le \frac{C}{\mu}\max_{\gamma\in\Gamma(S)}\|f_k^0 - \mathrm{Id}\|_{C^1(\gamma)}\,. \end{aligned} \tag{5.5}$$

To this end we start we start noticing that, by Remark 1.2 (see in particular (1.7)) and by applying Whitney's extension theorem as explained in [CLM14, Remark 3.4], there exists a surface $\widetilde S$ of class $C^{2,1}$ in $\mathbb{R}^3$ such that, up to increasing the value of L,

$$S \subset \widetilde S\,, \qquad \mathrm{diam}(\widetilde S) \le L\,, \qquad \mathrm{dist}_{\widetilde S}(x,y) \le L\,|x-y|\,, \qquad \forall x,y \in \widetilde S\,, \tag{5.6}$$

and there exists $\nu \in C^{1,1}(\widetilde S;\mathbb{S}^2)$ with $T_x\widetilde S = \nu(x)^\perp$ for every $x \in \widetilde S$ and

$$\|\nu\|_{C^{1,1}(\widetilde S)} \le L\,. \tag{5.7}$$

As a consequence of (5.7), one has

$$\begin{aligned} |\nu(x)\cdot(y-x)| &\le C\,|\pi_x^{\widetilde S}(y-x)|^2\,, && \forall x \in \widetilde S\,, y \in B_{x,1/C}\cap\widetilde S\,,\\ |y-x| &\le 2\,|\pi_x^{\widetilde S}(y-x)|\,, && \forall x \in \widetilde S\,, y \in B_{x,1/C}\cap\widetilde S\,,\\ \|\pi_x^{\widetilde S} - \pi_y^{\widetilde S}\| &\le C\,|x-y|\,, && \forall x,y \in \widetilde S\,. \end{aligned} \tag{5.8}$$

Finally, we exploit $\|S_k\|_{C^{1,\alpha}} \le L$ and [CLM14, Proposition 2.4] to construct $d_{S_k} \in C^{1,\alpha}(\mathbb{R}^3)$ and $\varepsilon_k > 0$ such that

$$\begin{aligned} &d_{S_k}(x) = 0 \text{ and } \nabla d_{S_k}(x) = \nu_{S_k}(x) \text{ for every } x \in S_k\,,\\ &I_{\varepsilon_k}(S_k)\cap\{d_{S_k} = 0\} \text{ is a } C^{1,\alpha}\text{-surface in } \mathbb{R}^3\,,\\ &\max\big\{\varepsilon_k^{-1}, \|d_{S_k}\|_{C^{1,\alpha}(\mathbb{R}^3)}\big\} \le C\,. \end{aligned} \tag{5.9}$$

We set $\widetilde S_k = I_{\varepsilon_k}(S_k)\cap\{d_{S_k} = 0\}$ and, for any $x \in \widetilde S$ and $\delta > 0$,

$$K_\delta = I_\delta(\mathrm{bd}_\tau(S))\cap\widetilde S\,, \qquad K_\delta^+ = I_\delta(\mathrm{bd}_\tau(S))\cap S\,.$$

We now claim that there exists η_0 depending on α and L only such that, if μ_0 is small enough with respect to η_0, then one can construct $f_k^\rho : K_{\eta_0} \to \widetilde{S}_k$ with

$$
\begin{aligned}
f_k^\rho &= f_k^0\,, && \text{on } \mathrm{bd}_\tau(S)\,, && (5.10)\\
f_k^\rho &= \mathrm{Id} + \psi_k \nu_S\,, && \text{on } K_{\eta_0}^+ \setminus K_\mu\,, && (5.11)\\
\|f_k^\rho\|_{C^{1,\alpha}(K_{\eta_0})} &\le C\,, && && (5.12)\\
\|f_k^\rho - \mathrm{Id}\|_{C^0(K_{\eta_0}^+)} &\le C\,\rho && && (5.13)\\
\|f_k^\rho - \mathrm{Id}\|_{C^1(K_{\eta_0}^+)} &\le \frac{C}{\mu}\,\rho^\alpha\,, && && (5.14)\\
\|\pi^{\widetilde{S}}(f_k^\rho - \mathrm{Id})\|_{C^1(K_{\eta_0})} &\le \frac{C}{\mu} \max_{\gamma \in \Gamma(S)} \|f_k^0 - \mathrm{Id}\|_{C^1(\gamma)}\,, && && (5.15)\\
J^{\widetilde{S}} f_k^\rho &\ge \frac{1}{2}\,, && \text{on } K_{\eta_0}\,, && (5.16)\\
\pi^{\widetilde{S}}(f_k^\rho - \mathrm{Id}) &= 0\,, && \text{on } K_{\eta_0} \setminus K_\mu\,, && (5.17)\\
f_k^\rho(K_{\eta_0}^+) &\subset S_k\,. && && (5.18)
\end{aligned}
$$

Once the claim has been proved, one defines f_k by setting $f_k = (\mathrm{Id} + \psi_k \nu_S)$ on $S \setminus K_{\eta_0}$, and then by setting $f_k = f_k^\rho$ on the rest of S. The fact that this gluing operation defines a diffeomorphism with the properties listed in (5.5) follows from (5.2), (5.8), (5.12), (5.13), (5.14), (5.16), and (5.18) thanks also to a uniform version of the inverse function theorem, see [CLM14, Theorem 2.1]. To prove the claim, we fix $\phi \in C^\infty(\mathbb{R}^3 \times (0,\infty); [0,1])$ such that, setting $\phi_\mu = \phi(\cdot, \mu)$,

$$
\begin{aligned}
&\phi_\mu \in C_c^\infty(I_\mu(\mathrm{bd}_\tau(S)))\,, \qquad \phi_\mu = 1 \text{ on } I_{\mu/2}(\mathrm{bd}_\tau(S))\,,\\
&|\nabla \phi_\mu(x)| \le \frac{C}{\mu}\,, \qquad |\nabla^2 \phi_\mu(x)| \le \frac{C}{\mu^2}\,, \qquad \forall (x,\mu) \in \mathbb{R}^3 \times (0,\infty)\,.
\end{aligned}
\tag{5.19}
$$

Next, we define $\bar{a}_k : \mathrm{bd}_\tau(S) \to \mathbb{R}$ and $\bar{b}_k : \mathrm{bd}_\tau(S) \to \mathbb{R}^3$ by setting

$$
\bar{a}_k(x) = (f_k^0(x) - x) \cdot \nu(x)\,, \qquad \bar{b}_k(x) = f_k^0(x) - x - \bar{a}_k(x)\,\nu(x)\,, \qquad x \in \mathrm{bd}_\tau(S)\,, \tag{5.20}
$$

so that by (5.4) one has

$$
\|\bar{a}_k\|_{C^{1,\alpha}(\gamma)} + \|\bar{b}_k\|_{C^{1,\alpha}(\gamma)} \le C\,, \qquad \|\bar{a}_k\|_{C^1(\gamma)} + \|\bar{b}_k\|_{C^1(\gamma)} \le C\,\|f_k^0 - \mathrm{Id}\|_{C^1(\gamma)}\,, \tag{5.21}
$$

for every $\gamma \in \Gamma(S)$. By using Lemma 4.8 (which we must use in place of [CLM14, Proposition 2.4] in order to deal with the singular points of $\mathrm{bd}_\tau(S)$), we find $a_k \in C^{1,\alpha}(\mathbb{R}^3)$ and $b_k \in C^{1,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ such that

$$
\begin{aligned}
&a_k = \bar{a}_k \text{ and } b_k = \bar{b}_k\,, \qquad \text{on } \mathrm{bd}_\tau(S)\,,\\
&\|a_k\|_{C^{1,\alpha}(\mathbb{R}^3)} + \|b_k\|_{C^{1,\alpha}(\mathbb{R}^3)} \le C\,,\\
&\|a_k\|_{C^1(\mathbb{R}^3)} + \|b_k\|_{C^1(\mathbb{R}^3)} \le C \max_{\gamma \in \Gamma(S)} \|f_k^0 - \mathrm{Id}\|_{C^1(\gamma)}\,.
\end{aligned}
\tag{5.22}
$$

Correspondingly, we define $F_k \in C^{1,\alpha}(\widetilde{S} \times (-1,1); \mathbb{R}^3)$ by setting, for $(x,t) \in \widetilde{S} \times (-1,1)$,

$$
F_k(x,t) = x + \phi_\mu(x)\, b_k(x) + (a_k(x) + t)\,\nu(x)\,, \tag{5.23}
$$

and then exploit $d_{S_k} \in C^{1,\alpha}(\mathbb{R}^3)$ to define $u_k \in C^{1,\alpha}(\widetilde{S} \times (-1,1))$ as

$$
u_k(x,t) = d_{S_k}(F_k(x,t))\,, \qquad (x,t) \in \widetilde{S} \times (-1,1)\,.
$$

By noticing that, for every $x \in \mathrm{bd}_\tau(S)$, $u_k(x,0) = 0$ (thanks to (5.9), (5.20), and (5.22)) and $\partial u_k/\partial t(x,0) \ge 1/2$ (thanks to (5.9) and (5.4)), and $\|u_k\|_{C^{1,\alpha}(\widetilde{S} \times (-1,1))} \le C$ (thanks to (5.7), (5.9), (5.21), (5.22), (5.19)) one applies a uniform version of the implicit function theorem (see [CLM14, Theorem 2.2]) in order to find $\eta_0 > 0$ and a function $\zeta_k \in C^{1,\alpha}(K_{\eta_0})$ such that

$$u_k(x, \zeta_k(x)) = 0 \quad \forall x \in K_{\eta_0}\,, \qquad \zeta_k(x) = 0 \quad \forall x \in \mathrm{bd}\,_\tau(S)\,, \tag{5.24}$$

$$\|\zeta_k\|_{C^0(K_{\eta_0})} \le C\,\eta_0\,, \qquad \|\zeta_k\|_{C^{1,\alpha}(K_{\eta_0})} \le C\,. \tag{5.25}$$

We prove the claim by setting

$$f_k(x) = F_k(x, \zeta_k(x))\,, \qquad x \in K_{\eta_0}\,. \tag{5.26}$$

Following the same argument as in [CLM14, Proof of Theorem 3.1], one shows (5.10), (5.12), (5.17), (5.15), (5.16) and

$$f_k(K_{\eta_0}) \subset \widetilde{S}_k\,, \tag{5.27}$$

as well as

$$\nabla^{\widetilde{S}} f_k(x)[\nu^{co}_{S^*}(x)] \cdot \nu^{co}_{S^*_k}(f_k(x)) \ge \frac{1}{2}\,, \qquad \forall x \in \mathrm{bd}\,(S^*)\,. \tag{5.28}$$

By (5.10), (5.27), and (5.16), one has

$$\nabla^{\widetilde{S}} f_k(x)[T_x \widetilde{S}] = T_{f_k(x)} \widetilde{S}_k\,, \qquad \forall x \in \mathrm{bd}\,_\tau(S)\,,$$

$$\nabla^{\widetilde{S}} f_k(x)[T_x(\mathrm{bd}\,(S^*))] = T_{f_k(x)}(\mathrm{bd}\,(S^*_k))\,, \qquad \forall x \in \mathrm{bd}\,(S^*)\,,$$

so that (5.28) gives at each $x \in \mathrm{bd}\,(S^*)$

$$\nabla^{\widetilde{S}} f_k(x)\Big[\big\{v \in T_x\widetilde{S} : v \cdot \nu^{co}_{S^*}(x) < 0\big\}\Big] = \big\{w \in T_{f_k(x)}\widetilde{S}_k : w \cdot \nu^{co}_{S^*_k}(f_k(x)) < 0\big\}\,.$$

By combining this fact with (5.27) we deduce (5.18) (up to possibly further decreasing η_0 in dependence of the bound in (5.12)). We are thus left to prove (5.11), (5.13) and (5.14), and this can be achieved once again by arguing exactly as in the proof of [CLM14, Proof of Theorem 3.1]. This completes the proof of Theorem 1.3.

Appendix A. Proof of Theorem 2.1

The aim of this section is proving Theorem 2.1, i.e., we want to prove that if $\mathcal{E}$ is satisfies

$$P(\mathcal{E}; B_{x,r}) \le P(\mathcal{F}; B_{x,r}) + \Lambda\,\mathrm{d}(\mathcal{E}, \mathcal{F})\,, \tag{A.1}$$

whenever $x \in \mathbb{R}^n$, $r < r_0$ and $\mathcal{E}(h)\Delta\mathcal{F}(h) \subset\subset B_{x,r}$ for every $h = 1, ..., N$, then there exists positive constants L and ρ_0 (depending on Λ, r_0, n, N and $\max_{1\le h\le N} |\mathcal{E}(h)|$ only) such that

$$\mathcal{H}^{n-1}(W \cap \partial\mathcal{E}) \le \mathcal{H}^{n-1}(f(W \cap \partial\mathcal{E})) + L\,r^n\,, \tag{A.2}$$

whenever $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitzian, $W = \{f \ne \mathrm{Id}\}$, and $\mathrm{diam}(W \cup f(W)) = r < \rho_0$. We notice that this is trivial when f is a bi-Lipschitz map. Indeed, in this case, $f(\mathcal{E}) = \{f(\mathcal{E}(h))\}_{h=1}^N$ is a N-cluster in $\mathbb{R}^n$ with $\partial^* f(\mathcal{E}) =_{\mathcal{H}^{n-1}} f(\partial^*\mathcal{E})$ (as it follows, e.g., by [Mag12, Proposition 17.1]), and thus (A.1) boils down to (A.2) if one takes $\mathcal{F} = f(\mathcal{E})$. This said, Taylor's regularity theorem is based on the possibility of testing (A.2) on non-injective Lipschitz maps f. In order to deduce (A.2) from (A.1) on such maps, one needs to construct a comparison cluster $\mathcal{F}$, admissible in (1.1), and with $P(\mathcal{F}; W) \le \mathcal{H}^{n-1}(f(W \cap \partial\mathcal{E}))$. Proposition A.1 below is crucial in achieving this goal, and in order to state it we introduce an ad hoc definition.

Let us recall that an *integer rectifiable n-current* T on $\mathbb{R}^n$ is a linear functional on the vector space $\mathcal{D}^k(\mathbb{R}^n)$ of compactly supported smooth k-forms on $\mathbb{R}^n$ which can be represented by integration as

$$\langle T, \omega\rangle = \int_M \theta\,\langle \omega, \tau\rangle\, d\mathcal{H}^k\,, \qquad \forall \omega \in \mathcal{D}^k(\mathbb{R}^n)\,, \tag{A.3}$$

where M is an $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^n$, θ is a Borel measurable, integer-valued and non-negative function defined on M, and τ is a Borel orientation of M (that is, $\tau(x)$ is a simple unit k-vector defining an orientation on the approximate tangent space T_xM for $\mathcal{H}^k$-a.e. $x \in M$ such that T_xM exists). We set $\|T\| = \theta\,\mathcal{H}^k \llcorner M$ for the *total variation measure* of T, $\theta^*(x)$ for the mod 2 representative of $\theta(x)$ in $\{0,1\}$, and define the *carrier* of T as

$$\operatorname{car} T = \left\{x \in \mathbb{R}^n : \theta^*(x) = 1\right\},$$

(Here we are borrowing some concepts and terminology from [Zie62], while avoiding to use the full machinery of currents modulo 2 for the sake of simplicity.) We denote by T^* the integer rectifiable k-current (with unit multiplicity) defined by

$$\langle T^*, \omega\rangle = \int_M \theta^* \langle \omega, \tau\rangle\, d\mathcal{H}^k = \int_{\operatorname{car} T} \langle \omega, \tau\rangle\, d\mathcal{H}^k\,, \qquad \forall \omega \in \mathcal{D}^k(\mathbb{R}^n)\,,$$

so that $\|T^*\| = \mathcal{H}^k \llcorner (\operatorname{car} T)$. Notice that, with this definition, if T_1 and T_2 are two rectifiable currents, then it holds

$$\|(T_1 + T_2)^*\| \le \|T_1^*\| + \|T_2^*\|\,, \tag{A.4}$$

where the simple verification of (A.4) is left to the reader. Next, we let $e = e_1 \wedge \cdots \wedge e_n$ and $\mathbf{E}^n$ denote, respectively, the canonical orientation of $\mathbb{R}^n$ and the corresponding canonical identification of $\mathbb{R}^n$ as an n-dimensional multiplicity-one current; then we set $T_E = \mathbf{E}^n \llcorner E$ for every Borel set $E \subset \mathbb{R}^n$. If T is an integral n-current on $\mathbb{R}^n$ (that is to say, both T and ∂T are integer rectifiable currents in $\mathbb{R}^n$), then by [Fed69, 4.5.17] there exists a partition $\{G^k\}_{k\in\mathbb{Z}}$ into sets of finite perimeter such that

$$\begin{gathered} T = T^+ - T^-\,, \\ T^+ = \sum_{k\in\mathbb{N}} k\,\mathbf{E}^n \llcorner G^k\,, \qquad T^- = \sum_{k\in\mathbb{N}} k\,\mathbf{E}^n \llcorner G^{-k}\,. \end{gathered} \tag{A.5}$$

In this case, $\theta^* = 1$ a.e. on G^k if and only if k is odd (i.e., $k = 2i+1$ for some $i \in \mathbb{Z}$), and thus we obtain

$$\begin{gathered} \operatorname{car}(T^\pm) = \bigcup_{k\ge 1\,\text{odd}} G^{\pm k}\,, \qquad \operatorname{car}(T) = \operatorname{car}(T^+) \cup \operatorname{car}(T^-)\,, \\ T^* = \mathbf{E}^n \llcorner \operatorname{car}(T^+) - \mathbf{E}^n \llcorner \operatorname{car}(T^-)\,. \end{gathered} \tag{A.6}$$

In this way, if E and F are sets of finite perimeter, then $T = T_E - T_F$ is an n-dimensional integral current on $\mathbb{R}^n$ with $\operatorname{car}(T^+) = E \setminus F$, $\operatorname{car}(T^-) = F \setminus E$, and $\operatorname{car}(T) = E\Delta F$; therefore we find

$$(T_E - T_F)^* = \mathbf{E}^n \llcorner (E \setminus F) - \mathbf{E}^n \llcorner (F \setminus E)\,, \qquad \|T_E - T_F\| = \|(T_E - T_F)^*\| = \mathcal{H}^n \llcorner (E\Delta F)\,. \tag{A.7}$$

We are now ready to state and prove Proposition A.1, where the notion of push-forward of a current is used, see, e.g. [Sim83, Chapter 26].

Proposition A.1. *If E is a set of finite perimeter in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a proper Lipschitz map, and we set $F = \operatorname{car}(f_\# T_E)$, then F is a set of finite perimeter with*

$$\mathcal{H}^{n-1} \llcorner \partial^* F \le \mathcal{H}^{n-1} \llcorner f(\partial^* E) \quad \text{on Borel sets.} \tag{A.8}$$

Moreover, $\mathbf{M}((T_E - f_\# T_E)^*) = |E\Delta F|$.

Proof. Since f is a proper Lipschitz map and E is a set of finite perimeter, $f_\# T_E$ is a integral n-current in $\mathbb{R}^n$. By (A.5) and (A.6) there exists a partition $\{G^k\}_{k\in\mathbb{Z}}$ of $\mathbb{R}^n$ into sets of finite perimeter such that

$$f_\# T_E = \sum_{k\in\mathbb{Z}} k\,\mathbf{E}^n \llcorner G^k\,, \qquad F = \operatorname{car}(f_\# T_E) = \bigcup_{k\,\text{odd}} G^k\,. \tag{A.9}$$

Since $\{G^k\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}^n$ into sets of finite perimeter, we have

$$\partial^* G^k = \bigcup_{h\neq k}(\partial^* G^k \cap \partial^* G^h) \qquad \text{up to } \mathcal{H}^{n-1}\text{–null sets,} \tag{A.10}$$

$$\mathcal{H}^{n-1}(\partial^* G^k \cap \partial^* G^h \cap \partial^* G^j) = 0\,, \qquad (k,h,j \text{ distinct})\,, \tag{A.11}$$

$$\nu_{G^k}(x) = -\nu_{G^h}(x)\,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* G^k \cap \partial^* G^h \ (k\neq h)\,. \tag{A.12}$$

By exploiting [Mag12, Theorem 16.3] we thus find that, up to a $\mathcal{H}^{n-1}$-negligible set,

$$\partial^* F = \bigcup_{k\,\mathrm{odd}}\ \bigcup_{h\,\mathrm{even}} \partial^* G^k \cap \partial^* G^h\,. \tag{A.13}$$

At the same time, by (A.9), (A.10) and (A.12) we obtain

$$\begin{aligned}
\partial(f_\# T_E) &= \sum_{k\in\mathbb{Z}} k \star \nu_{G^k}\, \mathcal{H}^{n-1} \llcorner \partial^* G^k \\
&= \sum_{k\in\mathbb{Z}}\sum_{h\neq k} k \star \nu_{G^k}\, \mathcal{H}^{n-1} \llcorner \Big(\partial^* G^k \cap \partial^* G^h\Big) \\
&= \sum_{k\in\mathbb{Z}}\sum_{h<k} (k-h) \star \nu_{G^k}\, \mathcal{H}^{n-1} \llcorner \Big(\partial^* G^k \cap \partial^* G^h\Big)\,,
\end{aligned}$$

so that owing to (A.13) we get

$$\mathcal{H}^{n-1}\llcorner \partial^* F \le \|\partial(f_\# T_E)\|\,. \tag{A.14}$$

Finally, by noticing that $\partial(f_\# T_E) = f_\#(\partial T_E)$ with $\partial T_E = \star\nu_E\, \mathcal{H}^{n-1}\llcorner\partial^* E$, we find that

$$\|\partial(f_\# T_E)\| \le \mathcal{H}^{n-1}\llcorner f(\partial^* E)\,, \tag{A.15}$$

so that (A.8) immediately follows by (A.14) and (A.15). We finally notice that, since $\{G^k\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}^n$ up to $\mathcal{H}^n$–negligible sets, we have

$$\begin{aligned}
T_E - f_\# T_E &= \mathbf{E}^n \llcorner E - \sum_{k\in\mathbb{Z}} k\, \mathbf{E}^n \llcorner G^k \\
&= \sum_{k\in\mathbb{Z}} (1-k)\mathbf{E}^n \llcorner (E\cap G^k) - \sum_{k\in\mathbb{Z}} k\, \mathbf{E}^n \llcorner (G^k\setminus E)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
(T_E - f_\# T_E)^* &= \sum_{k\le 0\,\mathrm{even}} \mathbf{E}^n \llcorner (E\cap G^k) - \sum_{k\ge 2\,\mathrm{even}} \mathbf{E}^n \llcorner (E\cap G^k) \\
&\qquad + \sum_{k\ge 1\,\mathrm{odd}} \mathbf{E}^n \llcorner (G^k\setminus E) - \sum_{k\le -1\,\mathrm{odd}} \mathbf{E}^n \llcorner (G^k\setminus E)\,, \\
\mathbf{M}((T_E - f_\# T_E)^*) &= \sum_{k\,\mathrm{even}} |E\cap G^k| + \sum_{k\,\mathrm{odd}} |G^k\setminus E| = |E\setminus F| + |F\setminus E|\,,
\end{aligned}$$

and $\mathbf{M}((T_E - f_\# T_E)^*) = |E\Delta F|$, as claimed. □

Proof of Theorem 2.1. Given $\rho_0>0$ (the required constraints on ρ_0 shall be stated in the course of proof), let us consider a Lipschitz map $f:\mathbb{R}^n\to\mathbb{R}^n$ such that $\mathrm{diam}(W\cup f(W)) = r$ for some $r<\rho_0$, where $W=\{f\neq \mathrm{Id}\}$. In this way

$$\mathrm{diam}(W\cup f(W)) = r\,, \qquad W\cup f(W) \subset\subset B(x_0,3r)\,, \tag{A.16}$$

for some $x_0 \in \mathbb{R}^n$. Let us consider the integer n-currents $T_h = \mathbf{E}^n \llcorner \mathcal{E}(h)$, $0 \le h \le N$. Since $\{\mathcal{E}(h)\}_{h=0}^N$ is a partition of $\mathbb{R}^n$ up to a negligible set, we have that

$$\mathbf{E}^n = \sum_{h=0}^N T_h \,. \tag{A.17}$$

At the same time, since f is a proper Lipschitz map with $f(x) = x$ for every x outside some bounded set, for a.e. $y \in \mathbb{R}^n$ and for every $R > 0$ large enough we have

$$1 = \deg(f, B_R, y) = \int_{f^{-1}(y)} \frac{\det \nabla f(x)}{|\det \nabla f(x)|}\, d\mathcal{H}^0(x) = \int_{f^{-1}(y)} \frac{\det \nabla f(x)}{Jf(x)}\, d\mathcal{H}^0(x) \,.$$

Therefore, for every $\omega = \varphi\, dx^1 \wedge \cdots \wedge dx^n$ with compact support (contained in B_R for some large value of R), by the area formula (see, e.g. [Mag12, Corollary 8.11]) we find that

$$\langle f_\# \mathbf{E}^n, \omega\rangle = \langle \mathbf{E}^n, f^\# \omega\rangle = \int_{\mathbb{R}^n} \varphi(f(x))\, \det \nabla f(x)\, dx = \int_{\mathbb{R}^n} \varphi(y)\, \deg(f, B_R, y)\, dy = \langle \mathbf{E}^n, \omega\rangle \,,$$

that is, $\mathbf{E}^n = f_\# \mathbf{E}^n$. In particular, (A.17) gives

$$\mathbf{E}^n = \sum_{h=0}^N f_\# T_h \,. \tag{A.18}$$

By(A.4) and by (A.18) we find $\mathcal{H}^n \le \sum_{h=0}^N \|(f_\# T_h)^*\|$, which of course implies, setting for brevity

$$F_h = \text{car } f_\# T_h \,, \qquad 0 \le h \le N \,,$$

that the family of sets of finite perimeter $\{F_h\}_{h=0}^N$ covers $\mathbb{R}^n$ up to a set of Lebesgue measure zero. We now notice that, by Proposition A.1, for every $h = 0, ..., N$,

$$\begin{aligned} \mathcal{H}^{n-1} \llcorner \partial^* F_h &\le \mathcal{H}^{n-1} \llcorner f(\partial^* \mathcal{E}(h)) \,, & \text{(A.19)} \\ \mathbf{M}((T_h - f_\# T_h)^*) &= |\mathcal{E}(h) \Delta F_h| \,, & \text{(A.20)} \end{aligned}$$

and then define a partition of $\mathbb{R}^n$ into sets of finite perimeter $\{\mathcal{F}(h)\}_{h=0}^N$ (up to $\mathcal{H}^n$-negligible sets) by setting

$$\mathcal{F}(0) = F_0 \,, \qquad \mathcal{F}(h) = F_h \setminus \bigcup_{j=0}^{h-1} F_j \,, \qquad 1 \le h \le N \,. \tag{A.21}$$

Since $\mathcal{E}$ is a cluster, for each $h = 0, \dots, N$ one has

$$|\mathcal{E}(h) \Delta \mathcal{F}(h)| \le \sum_{j=0}^h |\mathcal{E}(h) \Delta F_h| = \sum_{j=0}^h \mathbf{M}((T_h - f_\# T_h)^*) \le (h+1)\, |W| \,, \tag{A.22}$$

where we have also used (A.20). In particular, for $h = 1, \dots, N$,

$$\Big| |\mathcal{E}(h)| - |\mathcal{F}(h)| \Big| \le (N+1)\, |W| \le (N+1)\, 2^n\, \omega_n\, r^n \le C(n, N)\, (\rho_0)^n \,,$$

so that, for ρ_0 suitably small with respect to n, N, and $\mathrm{vol}\,(\mathcal{E})$, we find that $|\mathcal{F}(h)| > 0$ for $h = 1, ..., N$, and thus that $\mathcal{F}$ is a N-cluster. For each $h = 0, ..., N$, thanks to (A.16), we have $\mathcal{E}(h) \Delta F_h \subset\subset W \subset B_{x_0, 3\,r}$, and thus $\mathcal{E}(h) \Delta \mathcal{F}(h) \subset\subset W \subset B_{x_0, 3r}$: hence, provided $3\rho_0 \le r_0$, we can exploit the fact that $\mathcal{E}$ is a (Λ, r_0)-minimizing cluster to find

$$P(\mathcal{E}; W) \le P(\mathcal{F}; W) + \Lambda\, \mathrm{d}(\mathcal{E}, \mathcal{F}) \,. \tag{A.23}$$

By (A.22), and since $\mathrm{spt}\,(T_h - f_\# T_h) \subset W \cup f(W)$ with $\mathrm{diam}(W \cup f(W)) < r$, we find that

$$\Lambda\, \mathrm{d}(\mathcal{E}, \mathcal{F}) \le L\, r^n \,, \tag{A.24}$$

for a suitable constant L depending on Λ, n, and N. We also claim that, if we set $S = \partial\mathcal{E}$, then

$$P(\mathcal{E};W) = \mathcal{H}^{n-1}(S\cap W)\,,\qquad P(\mathcal{F};W) \le \mathcal{H}^{n-1}(f(S\cap W))\,. \tag{A.25}$$

The first identity follows since $\mathcal{H}^{n-1}(\partial\mathcal{E}\setminus\partial^*\mathcal{E}) = 0$. Concerning the second identity, let us first notice that, by [Mag12, Theorem 16.3] and by (A.19)

$$\partial^*\mathcal{F}(h)\ \subset\ \bigcup_{j=0}^N \partial^* F_j\ \subset\ \bigcup_{j=0}^N f(\partial^*\mathcal{E}(j))\ =\ f(\partial^*\mathcal{E})\ =\ f(\partial\mathcal{E})\,,$$

where the first and second inclusions, as well as the last equality, are true up to $\mathcal{H}^{n-1}$–negligible sets; moreover, in the last identity we have used again $\mathcal{H}^{n-1}(\partial\mathcal{E}\setminus\partial^*\mathcal{E}) = 0$ and the area formula. Since $\{\mathcal{F}(h)\}_{h=0}^N$ is a partition of $\mathbb{R}^n$ into sets of finite perimeter, it turns out that $\{\partial^*\mathcal{F}(h)\cap\partial^*\mathcal{F}(k)\}_{0\le h<k\le N}$ is a family of Borel sets that are mutually disjoint up to $\mathcal{H}^{n-1}$–negligible sets, and thus, by taking also into account that $W\cap f(\partial\mathcal{E})\subset f(W\cap\partial\mathcal{E})$, we have

$$P(\mathcal{F};W) = \sum_{0\le h<k\le N}^N \mathcal{H}^{n-1}\Big(W\cap\partial^*\mathcal{F}(h)\cap\partial^*\mathcal{F}(k)\Big) \le \mathcal{H}^{n-1}(W\cap f(\partial\mathcal{E})) \le \mathcal{H}^{n-1}(f(W\cap\partial\mathcal{E}))\,,$$

and prove (A.25). By combining (A.23), (A.24), and (A.25) we finally deduce that

$$\mathcal{H}^{n-1}(W\cap\partial\mathcal{E}) \le \mathcal{H}^{n-1}(f(W\cap\partial\mathcal{E})) + L\,r^n\,,$$

and thus complete the proof of the theorem. □

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