STOCHASTIC HOMOGENISATION OF SINGULARLY PERTURBED INTEGRAL FUNCTIONALS

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Abstract. We study the relative impact of small-scale random inhomogeneities and singular perturbations in nonlinear elasticity. More precisely, we analyse the asymptotic behaviour of the energy functionals

$$F_\varepsilon(\omega)(u) = \int_A \Big(f\Big(\omega, \frac{x}{\varepsilon}, Du\Big) + \varepsilon^2|\Delta u|^2\Big)\, dx,$$

where ω is a random parameter and $\varepsilon > 0$ denotes a typical length-scale associated with the variations in the elastic properties of the body. For f stationary and ergodic, we show that when $\varepsilon \to 0$ the randomly inhomogeneous material described by $F_\varepsilon(\omega)$ behaves (almost surely) like a homogeneous deterministic material. The limit stored energy density is given in terms of an asymptotic cell formula in which the Laplacian perturbation explicitly appears.

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1. Introduction

In this note we deal with nonlinear elastic materials characterised by a stored energy density $f(\omega, x/\varepsilon, Du)$ depending on a random parameter ω, on the local position x/ε, and on the deformation gradient Du. In this random framework ε can be regarded as a parameter that sets the scale of the microstructure but does not represent in general a characteristic length. The elastic energy of a piece of material occupying the region A and deformed by the map $u\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$E_\varepsilon(\omega)(u) = \int_A f\Big(\omega, \frac{x}{\varepsilon}, Du\Big)\, dx, \tag{1.1}$$

where ω belongs to a probability space Ω and labels the realisations of the microstructure.

In a deterministic setting, *i.e.*, when no dependence on ω occurs, the classical variational theory of homogenisation [5, 20, 6] asserts that when f is periodic in the spatial variable and satisfies appropriate growth and coercivity conditions, the Γ-limit of E_ε is given by an integral functional whose integrand depends only on the deformation gradient. In other words, the macroscopic or "average" behaviour of a *periodic* material is that of a homogeneous one.

To draw a parallel between periodic and random homogenisation we have to restrict our analysis to those random materials which present some kind of self-repeating structure. Then, in a random setting the analogue of periodicity is *stationarity* which can be interpreted as a stochastic periodicity or a *periodicity in law*, using the terminology introduced by Dal Maso and Modica in [10]. More precisely, let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\tau_z\colon \Omega \to \Omega$ be a group of transformations parametrised by a discrete index $z \in \mathbb{Z}^n$. We assume that $(\tau_z)_{z\in\mathbb{Z}^n}$ preserves the measure P; *i.e.,*

$$P(\tau_z E) = P(E) \quad \forall\, E \in \mathscr{F}, \quad \forall\, z \in \mathbb{Z}^n. \tag{1.2}$$

Loosely speaking, the translation invariance (1.2) means that a given microstructure and the translated microstructure occur with the same probability.

We say that a random stored energy density $f\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is stationary with respect to $(\tau_z)_{z\in\mathbb{Z}^n}$ if for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$ and for every $z \in \mathbb{Z}^n$

$$f(\omega, x+z, \xi) = f(\tau_z\omega, x, \xi) \quad P\text{-almost surely.} \tag{1.3}$$

(See Section 2.2 for a precise definition.) Stationarity guarantees that, in a statistical sense, parts of the material located at different positions share the same elastic properties. In other words, the statistical properties of the medium are invariant under translations which are now understood as the transformations $(\tau_z)_{z\in\mathbb{Z}^n}$. Notice that in the deterministic setting (1.3) reduces to periodicity in the spatial variable. We say moreover that a stationary f is *ergodic* if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic; *i.e.,* any $(\tau_z)_{z\in\mathbb{Z}^n}$-invariant set $E \in \mathscr{F}$ has probability 0 or 1.

Functionals of type (1.1) model manifold materials *e.g.* materials with a random chessboard structure or materials with randomly distributed impurities (see e.g. [11, Section 3] and [19, Section 5] for some prototypical f). If f is stationary and ergodic and satisfies standard growth and coercivity conditions of order $p > 1$, it is by now well-known that *almost surely* the functionals $E_\varepsilon(\omega)$ Γ-converge with respect to the weak $W^{1,p}(A)$-topology to a deterministic homogeneous integral functional.

The above Γ-convergence result has been proved in the scalar convex case by Dal Maso and Modica in [10, 11] where for the first time ergodic theory has been used in combination with Γ-convergence. More precisely, in [11] the Γ-convergence of $E_\varepsilon(\omega)$ is obtained by combining a compactness result for convex integral functionals with p-growth and a subadditive ergodic theorem due to Akcoglu and Krengel [2]. Building upon similar ideas, the Γ-convergence of (1.1) has been afterwards also established in the vectorial nonconvex setting by Messaoudi and Michaille [19], and for $p = 1$ by Abddaimi, Michaille, and Licht [1].

The object of the present paper is the asymptotic analysis of a (singular) perturbation of $E_\varepsilon(\omega)$. Namely we study the limit as ε tends to zero of the following sequence of vectorial random functionals

$$F_\varepsilon(\omega)(u) = \int_A \Big(f\Big(\omega, \frac{x}{\varepsilon}, Du\Big) + \varepsilon^2 |\Delta u|^2 \Big)\, dx, \tag{1.4}$$

defined for $u \in W^{1,p}(A;\mathbb{R}^m) \cap W^{2,2}(A;\mathbb{R}^m)$, where, as above, $p > 1$ is related to the growth conditions satisfied by f (see Section 2 for a detailed description of the model).

The interest in functionals as in (1.4) is mainly motivated by a study carried out in the deterministic setting by Francfort and Müller. In [13], among other things, the authors analyse the limit behaviour of the periodic analogue of (1.4) in connection with certain pathologies exhibited by some nonlinear elastic materials subject to severe loadings. One of such pathologies is the formation of the so-called shear bands. In mathematical terms this corresponds to a loss of strict convexity of the stored energy density along rank-one connections (or equivalently to a loss of strict strong ellipticity of the associated Hessian tensor). Shear-band instabilities have been systematically investigated by Geymonat, Müller, and Triantafyllidis in [14] where, among other, it is shown that strict strong ellipticity is in general not preserved by periodic homogenisation (thus allowing for shear-bands instabilities in the homogenised material). The theory developed in [14] has been recently extended to the stationary stochastic setting by Gloria and Neukamm (see [16, Section 3.5] and [17]).

Then, in the very same spirit of [13], we add to the microscopic energy (1.1) a second-order perturbation. Indeed the latter, if suitably tuned in, may lead to equilibrium equations which remain always elliptic and, at the same time, could yield an effective model which qualitatively behaves like the unperturbed one, but whose characteristics would now depend on the strength of the singular perturbation and could therefore be easier to detect. Clearly this second-order perturbation has to be scaled according to the scale of the microstructure ε so as not to be dominant. Then, arguing as in the periodic setting, we end up with the singularly perturbed random functionals $F_\varepsilon(\omega)$.

We remark here that although our motivation remains valid, to our knowledge a thorough analysis of the singularly-perturbed periodic model proposed in [13] in the spirit of [14] is still outstanding.

We now come back to the object of this note, that is the study the Γ-convergence of the random functionals $F_\varepsilon(\omega)$. If f is stationary and ergodic, following the approach of Dal Maso and Modica, we show that when A varies in the class of Lipschitz open bounded subset of $\mathbb{R}^n$, the set function

$$\mu_A^\xi(\omega) = \inf\left\{\int_A \left(f(\omega,x,Du+\xi)+|\Delta u|^2\right)dx\colon\; u\in W_0^{1,p}(A;\mathbb{R}^m)\cap W_0^{2,2}(A;\mathbb{R}^m)\right\}$$

defines, for every fixed $\xi\in\mathbb{R}^{m\times n}$, a subadditive process on $(\Omega,\mathscr{F},P)$. Then the Akcoglu and Krengel ergodic Theorem applied (for every fixed $\xi\in\mathbb{R}^{m\times n}$) to μ_A^ξ allows us to prove the existence of a deterministic homogeneous energy density $f_{\rm hom}$ which turns out to be the integrand of the Γ-limit of $F_\varepsilon(\omega)$ (see Propositions 3.1 and 3.2). The main result of this paper is Theorem 3.4 which extends [13, Theorem 2.1] to the stochastic setting. More precisely, in Theorem 3.4 we prove that, P-almost surely, the functionals $F_\varepsilon(\omega)$ Γ-converge with respect to the weak $W^{1,p}(A;\mathbb{R}^m)$-topology to the deterministic functional given by

$$F_{\rm hom}(u) = \int_A f_{\rm hom}(Du)\,dx \quad \text{for } u\in W^{1,p}(A;\mathbb{R}^m),$$

where

$$f_{\rm hom}(\xi) = \lim_{k\to+\infty}\int_\Omega \frac{1}{k^n}\min\bigg\{\int_{(0,k)^n}\left(f(\omega,x,Du+\xi)+|\Delta u|^2\right)dx\colon \\ u\in W_0^{1,p}((0,k)^n;\mathbb{R}^m)\cap W_0^{2,2}((0,k)^n;\mathbb{R}^m)\bigg\}\,dP.$$

As an application, choosing $m=1$, $p=2$, and

$$f\Big(\omega,\frac{x}{\varepsilon},\xi\Big) = \sigma\Big(\omega,\frac{x}{\varepsilon}\Big)\xi\cdot\xi,$$

for some symmetric and elliptic matrix σ, we obtain that for every $g\in W^{-1,2}(A)$, P-almost surely, the unique solution to

$$\begin{cases}\varepsilon^2\Delta^2 u_\varepsilon - \operatorname{div}(\sigma_\varepsilon\nabla u_\varepsilon) = g & \text{in } A\times\Omega,\\ u_\varepsilon(\omega,\cdot)\in W_0^{2,2}(A) & \text{in } \Omega,\end{cases}$$

converges weakly in $W^{1,2}(A)$ to the unique solution of the deterministic problem

$$\begin{cases}-\operatorname{div}(\sigma_{\rm hom}\nabla u) = g & \text{in } W^{-1,2}(A),\\ u\in W_0^{1,2}(A),\end{cases}$$

where $\sigma_{\rm hom}$ is a constant symmetric elliptic matrix (see Proposition 4.1).

We finally remark that the asymptotic analysis carried out in the present paper can be also seen as a preliminary study for a multi-scale treatment of singularly-perturbed random nonlinear problems in the spirit of [8]. In fact in Subsection 3.1 we show on a simple one-dimensional (yet meaningful) example that in this nonlinear setting the homogenised energy density $f_{\rm hom}$ can be quite degenerate. As a consequence, the description given by $F_{\rm hom}$ can be too coarse to *completely* characterise the asymptotic behaviour of $F_\varepsilon(\omega)$. Then the idea is that the computation of $F_{\rm hom}$ is only the first step in the description of the asymptotic behaviour of $F_\varepsilon(\omega)$ and that additional information can be obtained by iteration of the Γ-limit procedure following the approach of Braides and Truskinovsky [7].

2. Notation and preliminaries

In this section we set a few notation and recall some preliminaries we are going to use in what follows.

2.1. **Setting of the problem.** Let $m,n>1$. Let $1<p<+\infty$ and $0<\alpha\le\beta$; we denote by $\mathcal{I}(p,\alpha,\beta)$ the class of all Carathéodory functions $f\colon\mathbb{R}^n\times\mathbb{R}^{m\times n}\to[0,+\infty)$ satisfying the two following conditions:

- (growth condition of order p) for a.e. $x\in\mathbb{R}^n$ and for every $\xi\in\mathbb{R}^{m\times n}$
$$\alpha|\xi|^p\le f(x,\xi)\le\beta(1+|\xi|^p); \tag{2.1}$$
- (local Lipschitz continuity) there exists $L>0$ such that
$$|f(x,\xi)-f(x,\eta)|\le L(1+|\xi|^{p-1}+|\eta|^{p-1})|\xi-\eta|, \tag{2.2}$$
for a.e. $x\in\mathbb{R}^n$, and for every $\xi,\eta\in\mathbb{R}^{m\times n}$.

We would like to remark that the growth condition (2.1) is too restrictive since for realistic nonlinear elastic materials we expect that $f(x,\xi)\to+\infty$ as $\det(\xi)\to0$. On the other hand, once (2.1) is assumed, the p-Lipschitz condition (2.2) is quite natural as it is automatically satisfied by quasiconvex (or even rank-one convex) functions with p-growth (see *e.g.* [6, Remark 4.13]).

Denote by $\mathcal{A}_0$ the class of all open and bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary and let $F\colon W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)\times\mathcal{A}_0\longrightarrow[0,+\infty]$ be the localised functional defined as
$$F(u,A):=\begin{cases}\displaystyle\int_A\big(f(x,Du)+|\Delta u|^2\big)\,dx & \text{if } u\in W^{2,2}(A;\mathbb{R}^m),\\ +\infty & \text{otherwise in } W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m),\end{cases} \tag{2.3}$$
where $f\in\mathcal{I}(p,\alpha,\beta)$.

For every $\xi\in\mathbb{R}^{m\times n}$ let ℓ_ξ be the linear vector-valued function whose gradient is ξ; *i.e.,* $\ell_\xi(x):=\xi x$; moreover, for $A\in\mathcal{A}_0$, F as in (2.3), and $\xi\in\mathbb{R}^{m\times n}$ set
$$m_A(F,\xi):=\inf\{F(u+\ell_\xi,A)\colon u\in W^{1,p}_0(A;\mathbb{R}^m)\cap W^{2,2}_0(A;\mathbb{R}^m)\}. \tag{2.4}$$

Proposition 2.1. *The infimum in* (2.4) *is attained.*

Proof. Set
$$F_\xi(u):=F(u+\ell_\xi,A)=\int_A\big(f(x,Du+\xi)+|\Delta u|^2\big)\,dx.$$
Let $(u_k)\subset W^{1,p}_0(A;\mathbb{R}^m)\cap W^{2,2}_0(A;\mathbb{R}^m)$ be a minimizing sequence for F_ξ; then
$$\lim_{k\to+\infty}F_\xi(u_k)=\inf F_\xi\le F_\xi(0)\le\beta(1+|\xi|^p)|A|<+\infty,$$
hence in particular there exists $C>0$ such that
$$\|Du_k\|_{L^p(A;\mathbb{R}^{m\times n})}\le C\quad\text{and}\quad\|\Delta u_k\|_{L^2(A;\mathbb{R}^m)}\le C, \tag{2.5}$$
for every $k\in\mathbb{N}$. Since the L^2-norm of the Laplacian is an equivalent norm in $W^{2,2}_0(A;\mathbb{R}^m)$, from (2.5) we can readily deduce that (u_k) is bounded in $W^{2,2}(A;\mathbb{R}^m)$; then, up to subsequences (not relabelled), $u_k\rightharpoonup u$ in $W^{2,2}(A;\mathbb{R}^m)$. As a consequence the Rellich Theorem ensures that $u_k\to u$ strongly in $W^{1,2}(A;\mathbb{R}^m)$ while (2.5) yields $u\in W^{1,p}_0(A;\mathbb{R}^m)\cap W^{2,2}_0(A;\mathbb{R}^m)$. Moreover,

up to a possible further extraction of a subsequence, $Du_k \to Du$ a.e. in A. Then in view of (2.2), invoking the Fatou Lemma gives

$$\liminf_{k\to+\infty} \int_A f(x, Du_k + \xi)\, dx \geq \int_A f(x, Du + \xi)\, dx, \tag{2.6}$$

while the weak sequential lower semicontinuity of the L^2-norm yields

$$\liminf_{k\to+\infty} \int_A |\Delta u_k|^2\, dx \geq \int_A |\Delta u|^2\, dx. \tag{2.7}$$

Hence, by combining (2.6) and (2.7) we find

$$\begin{aligned} \inf F_\xi = \lim_{k\to+\infty} F_\xi(u_k) &\geq \liminf_{k\to+\infty} \int_A f(x, Du_k + \xi)\, dx + \liminf_{k\to+\infty} \int_A |\Delta u_k|^2\, dx \\ &\geq \int_A f(x, Du + \xi)\, dx + \int_A |\Delta u|^2\, dx = F_\xi(u), \end{aligned}$$

and thus the thesis. □

2.2. **Ergodic theory.** Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $(\tau_z)_{z\in\mathbb{Z}^n}$ be a group of P-preserving transformations on $(\Omega, \mathscr{F})$; *i.e.*, $(\tau_z)_{z\in\mathbb{Z}^n}$ is a family of mappings $\tau_z \colon \Omega \to \Omega$ satisfying the following properties:

- (measurability) τ_z is $\mathscr{F}$-measurable for every $z \in \mathbb{Z}^n$;
- (group property) $\tau_z \circ \tau_{z'} = \tau_{z+z'}$, $\tau_{-z} = \tau_z^{-1}$, for every $z, z' \in \mathbb{Z}^n$;
- (mass invariance) $P(\tau_z E) = P(E)$, for every $E \in \mathscr{F}$ and every $z \in \mathbb{Z}^n$.

If in addition every set $E \in \mathscr{F}$ which satisfies

$$\tau_z E = E \quad \text{for every } z \in \mathbb{Z}^n,$$

has probability 0 or 1, then $(\tau_z)_{z\in\mathbb{Z}^n}$ is called *ergodic*.

We recall that the easiest way to prove the ergodicity of a group of P-preserving transformations $(\tau_z)_{z\in\mathbb{Z}^n}$ is to verify a so-called mixing condition (or independence at large distances).

Definition 2.2 (Strong mixing). *Let $(\tau_z)_{z\in\mathbb{Z}^n}$ be a group of P-preserving transformations on $(\Omega, \mathscr{F})$; then $(\tau_z)_{z\in\mathbb{Z}^n}$ is called strong mixing if*

$$\lim_{|z|\to+\infty} \big|P(\tau_z E \cap E') - P(E)P(E')\big| = 0 \quad \textit{for every } E, E' \in \mathscr{F}. \tag{2.8}$$

Clearly strong mixing implies ergodicity. Indeed if $E \in \mathscr{F}$ is an invariant set; *i.e.*, $\tau_z E = E$ for every $z \in \mathbb{Z}^n$, then $P(\tau_z E \cap E') = P(E \cap E')$ for every $z \in \mathbb{Z}^n$, which when we plugged in (2.8) implies $P(E \cap E') = P(E)P(E')$ for every $E' \in \mathscr{F}$. Then choosing $E' = E$ gives $P(E) = P(E)^2$ which in its turn yields $P(E) \in \{0, 1\}$, and therefore $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic.

Definition 2.3 (Random integrand). *Let $\mathscr{B}$, $\mathscr{B}^n$, and $\mathscr{B}^{m\times n}$ denote the Borel σ-algebra on $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$, respectively. A function $f \colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, +\infty)$ is called a random integrand if:*

- *f is $(\mathscr{F} \otimes \mathscr{B}^n \otimes \mathscr{B}^{m\times n}, \mathscr{B})$-measurable;*
- *$f(\omega, \cdot, \cdot) \in \mathcal{I}(p, \alpha, \beta)$ for P-a.e. $\omega \in \Omega$.*

Definition 2.4 (Stationary random integrand). *A random integrand f is stationary with respect to the group of P-preserving transformations $(\tau_z)_{z\in\mathbb{Z}^n}$, if for all $z \in \mathbb{Z}^n$, for almost every $x \in \mathbb{R}^n$, and for every $\xi \in \mathbb{R}^{m\times n}$*

$$f(\omega, x + z, \xi) = f(\tau_z \omega, x, \xi) \quad \textit{for } P\textit{-a.e. } \omega \in \Omega.$$

Notice that a $(0,1)^n$-periodic function can be viewed as a deterministic stationary function.

Remark 2.5 (Continuous stationarity)**.** We could consider on $(\Omega,\mathscr{F},P)$ a group of P-preserving transformations parametrised by a continuous parameter $y\in\mathbb{R}^n$ and define stationarity accordingly; *i.e.,* $f\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^{m\times n}\to[0,+\infty)$ is stationary with respect to $(\tau_y)_{y\in\mathbb{R}^n}$, if for all $y\in\mathbb{R}^n$, for almost every $x\in\mathbb{R}^n$, and for every $\xi\in\mathbb{R}^{m\times n}$

$$f(\omega,x+y,\xi)=f(\tau_y\omega,x,\xi)\quad\text{for } P\text{-a.e. }\omega\in\Omega. \tag{2.9}$$

In contrast to the case of discrete stationarity, we notice that now to show that periodicity is a particular case of (continuous) stationarity we need to exploit the random character of (2.9). Indeed choosing $\Omega=[0,1]^n$ and P equal to the Lebesgue measure on $[0,1]^n$, we have that $\tau_y\omega:=\omega+y$ (mod 1) defines a group of P-preserving transformations on Ω (which is also ergodic). Then any $g\colon\Omega\times\mathbb{R}^{m\times n}\to[0,+\infty)$ (that is any $(0,1)^n$-periodic function) corresponds to the stationary f given by

$$f(\omega,x,\xi)=g(\omega+x,\xi).$$

Definition 2.6 (Ergodic random integrand)**.** *A stationary random integrand f is ergodic if the corresponding group of P-preserving transformations $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic.*

Let $(\tau_z)_{z\in\mathbb{Z}^n}$ be a group of P-preserving transformations on $(\Omega,\mathscr{F})$. Let $\mu\colon\mathcal{A}_0\to L^1(\Omega,\mathscr{F},P)$ be a set function satisfying the following properties:

(i) (covariance) for every $A\in\mathcal{A}_0$ and $z\in\mathbb{Z}^n$

$$\mu_{A+z}=\mu_A\circ\tau_z;$$

(ii) (subadditivity) for every $A\in\mathcal{A}_0$ and for every *finite* family $(A_i)_{i\in I}\subset\mathcal{A}_0$ of pairwise disjoint sets such that $A_i\subset A$ for every $i\in I$ and $|A\setminus\cup_{i\in I}A_i|=0$, we have

$$\mu_A\le\sum_{i\in I}\mu_{A_i};$$

(iii) (boundedness) there exists $c>0$ such that for every $A\in\mathcal{A}_0$

$$0\le\mu_A\le c|A|.$$

Definition 2.7 (Subadditive process)**.** *Any set function $\mu\colon\mathcal{A}_0\to L^1(\Omega,\mathscr{F},P)$ satisfying (i)-(iii), for some group of P-preserving transformations $(\tau_z)_{z\in\mathbb{Z}^n}$, is called a subadditive process. Moreover, μ is called ergodic if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic.*

We now state a variant of the pointwise ergodic Theorem [2, Theorem 2.7 and Remark p. 59] which is suitable for our purposes.

Theorem 2.8. [11, Proposition 1] *Let $\mu\colon\mathcal{A}_0\to L^1(\Omega,\mathscr{F},P)$ be a subadditive process. Then there exist a $\mathscr{F}$-measurable function $\phi\colon\Omega\to[0,+\infty)$ and a set $\Omega'\in\mathscr{F}$ with $P(\Omega')$=1 such that*

$$\lim_{t\to+\infty}\frac{\mu_{tQ}(\omega)}{|tQ|}=\phi(\omega),$$

for every $\omega\in\Omega'$ and for every cube Q in $\mathbb{R}^n$. If in addition μ is ergodic, then ϕ is constant.

We conclude this section recalling the Birkhoff ergodic Theorem (see *e.g.* [18]).

Theorem 2.9 (Birkhoff's ergodic Theorem)**.** *Let $h\in L^\infty(\mathbb{R}^n,L^1(\Omega,\mathscr{F},P))$ be an ergodic stationary random variable; i.e., for all $z\in\mathbb{Z}^n$, for almost every $x\in\mathbb{R}^n$*

$$h(\omega,x+z)=h(\tau_z\omega,x)\quad\textit{for } P\textit{-a.e. }\omega\in\Omega,$$

with $(\tau_z)_{z\in\mathbb{Z}^n}$ *ergodic. Then, for* P*-a.e.* $\omega \in \Omega$

$$h\Big(\omega, \frac{x}{\varepsilon}\Big) \overset{*}{\rightharpoonup} \int_\Omega \bigg(\int_{(0,1)^n} h(\omega, y)\, dy\bigg)\, dP \quad in \;\; L^\infty(\mathbb{R}^n),$$

as $\varepsilon \to 0$.

3. Almost sure Γ-convergence

Let f be a random integrand in the sense of Definition 2.3 and for $\omega \in \Omega$ consider the random integral functional $F(\omega)\colon W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m) \times \mathcal{A}_0 \longrightarrow [0,+\infty]$ defined as

$$F(\omega)(u,A) := \begin{cases} \displaystyle\int_A \big(f(\omega,x,Du) + |\Delta u|^2\big)\, dx & \text{if } u \in W^{2,2}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise in } W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m). \end{cases} \tag{3.1}$$

For every $\varepsilon > 0$ consider the random integral functional $F_\varepsilon(\omega)\colon W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m) \times \mathcal{A}_0 \longrightarrow [0,+\infty]$ given by

$$F_\varepsilon(\omega)(u,A) := \begin{cases} \displaystyle\int_A \Big(f\Big(\omega,\frac{x}{\varepsilon},Du\Big) + \varepsilon^2|\Delta u|^2\Big)\, dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^m)\cap W^{2,2}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise in } W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m). \end{cases} \tag{3.2}$$

Notice that

$$F_\varepsilon(\omega)(u,A) = \varepsilon^n F(\omega)\Big(u(\varepsilon x)/\varepsilon, A/\varepsilon\Big)$$

and consequently

$$\frac{m_A(F_\varepsilon(\omega),\xi)}{|A|} = \frac{m_{A/\varepsilon}(F(\omega),\xi)}{\varepsilon^{-n}|A|}, \tag{3.3}$$

for every $A \in \mathcal{A}_0$ and every $\xi \in \mathbb{R}^{m\times n}$.

In what follows we analyse the asymptotic behaviour of the sequence of random functionals (F_ε). The main tool to perform such an analysis is the subadditive ergodic theorem, Theorem 2.8, that we apply (for every fixed $\xi \in \mathbb{R}^{m\times n}$) to the map $A \to \mu^\xi_A$ defined as $\mu^\xi_A(\omega) := m_A(F(\omega),\xi)$. To this end, we start showing that if f is stationary then the map as above is a subadditive process.

Proposition 3.1. *Let* f *be a stationary random integrand and let* F *be as in* (3.1)*. Let* $\xi \in \mathbb{R}^{m\times n}$ *and set*

$$\mu^\xi_A := m_A(F,\xi) \quad \textit{for every } A \in \mathcal{A}_0,$$

where $m_A(F,\xi)$ *is as in* (2.4)*. Then for every* $\xi \in \mathbb{R}^{m\times n}$ *the set function* μ^ξ *is a subadditive process on* $(\Omega,\mathscr{F},P)$*. Moreover, for every* $\xi \in \mathbb{R}^{m\times n}$ *and* $A \in \mathcal{A}_0$

$$0 \le \mu^\xi_A(\omega) \le \beta(1+|\xi|^p)|A|, \tag{3.4}$$

for P*-a.e.* $\omega \in \Omega$.

Proof. Let $\xi \in \mathbb{R}^{m\times n}$ and $A \in \mathcal{A}_0$; we first show that μ^ξ_A belongs to $L^1(\Omega,\mathscr{F},P)$. To this end fix $u \in W^{1,p}(A;\mathbb{R}^m)\cap W^{2,2}(A;\mathbb{R}^m)$, then the function $(\omega,x) \to f(\omega,x,Du+\xi) + |\Delta u|^2$ is $\mathscr{F}\otimes\mathcal{L}^n$-measurable on $\Omega\times\mathbb{R}^n$; hence by Fubini's Theorem

$$\omega \to F(\omega)(u+\ell_\xi, A) = \int_A \big(f(\omega,x,Du+\xi) + |\Delta u|^2\big)\, dx$$

is $\mathscr{F}$-measurable. Consider the metric space $\big(W^{1,p}_0(A;\mathbb{R}^m)\cap W^{2,2}_0(A;\mathbb{R}^m), d\big)$ where the distance d is defined as

$$d(u,v) := \|Du - Dv\|_{L^p(A;\mathbb{R}^{m\times n})} + \|\Delta u - \Delta v\|_{L^2(A;\mathbb{R}^m)};$$

notice that such a space is separable. It is immediate to check that by virtue of (2.2) the map $u \to F(u+\ell_\xi, A)$ is continuous with respect to d. Then we deduce the existence of a d-dense countable set $D \subset W^{1,p}_0(A;\mathbb{R}^m)\cap W^{2,2}_0(A;\mathbb{R}^m)$ such that

$$\mu^\xi_A(\cdot) = \inf_{u_k\in D} F(\cdot)(u_k+\ell_\xi, A);$$

hence the map $\omega \to \mu^\xi_A(\omega)$ is $\mathscr{F}$-measurable. Moreover by (2.1) we also have that

$$0 \le m_A(F,\xi) \le \beta(1+|\xi|^p)|A|,$$

hence we immediately deduce both (3.4) and that $\mu^\xi_A \in L^1(\Omega,\mathscr{F},P)$.

We now show that for every fixed $\xi\in\mathbb{R}^{m\times n}$ the set function μ^ξ is covariant; *i.e.,*

$$\mu^\xi_{A+z}(\omega) = \mu^\xi_A(\tau_z\omega) \quad \text{for } P\text{-a.e. } \omega\in\Omega, \tag{3.5}$$

for every $A\in\mathcal{A}_0$, and for every $z\in\mathbb{Z}^n$. Indeed a change of variable and the stationarity of f yield

$$\begin{aligned} m_{A+z}(F(\omega),\xi) &= \min\left\{ \int_A \big(f(\omega,x+z,Du+\xi)+|\Delta u|^2\big)\,dx \colon u\in W^{1,p}_0(A)\cap W^{2,2}_0(A)\right\} \\ &= \min\left\{ \int_A \big(f(\tau_z\omega,x,Du+\xi)+|\Delta u|^2\big)\,dx \colon u\in W^{1,p}_0(A)\cap W^{2,2}_0(A)\right\} = m_A(F(\tau_z\omega),\xi), \end{aligned}$$

thus (3.5) follows.

Then, it only remains to prove that μ^ξ is subadditive, for every $\xi\in\mathbb{R}^{m\times n}$. To this end let $A\in\mathcal{A}_0$ and let $(A_i)_{i\in I}$ be a finite family of pairwise disjoint sets in $\mathcal{A}_0$ with $A_i\subset A$ for every $i\in I$, and $|A\setminus \cup_{i\in I}A_i| = 0$. Let $u_i \in W^{1,p}_0(A_i;\mathbb{R}^m)\cap W^{2,2}_0(A_i;\mathbb{R}^m)$ be such that $\mu^\xi_{A_i}(\omega) = m_{A_i}(F(\omega),\xi) = F(\omega)(u_i+\ell_\xi, A_i)$ and define $u\in W^{1,p}_0(A;\mathbb{R}^m)\cap W^{2,2}_0(A;\mathbb{R}^m)$ by setting $u=u_i$ in A_i, for every $i\in I$. Then, by the additivity and the locality of F we have

$$\mu^\xi_A(\omega) \le F(\omega)(u+\ell_\xi,A) = \sum_{i\in I} F(\omega)(u_i+\ell_\xi,A_i) = \sum_{i\in I} m_{A_i}(F(\omega),\xi) = \sum_{i\in I}\mu^\xi_{A_i}(\omega),$$

which proves the subadditivity and eventually that μ^ξ is a subadditive process. □

On account of Proposition 3.1 we now prove the existence of the random integrand $f_{\rm hom}$ which will appear in the definition of the Γ-limit of (F_ε).

Proposition 3.2. *Let f be a stationary random integrand. Then there exist a set $\Omega'\in\mathscr{F}$ with $P(\Omega')=1$ and a homogeneous random integrand $f_{\rm hom}\colon \Omega\times\mathbb{R}^{m\times n}\to[0,+\infty)$, such that for every $\omega\in\Omega'$, for every $\xi\in\mathbb{R}^{m\times n}$, and for every open cube Q in $\mathbb{R}^n$,*

$$f_{\rm hom}(\omega,\xi) := \lim_{t\to+\infty}\frac{m_{tQ}(F(\omega),\xi)}{t^n|Q|}.$$

If in addition f is ergodic then $f_{\rm hom}$ does not depend on ω and

$$\begin{aligned} f_{\rm hom}(\xi) &= \lim_{k\to+\infty}\frac{1}{k^n}\int_\Omega m_{(0,k)^n}(F(\omega),\xi)\,dP \\ &= \inf_{k\in\mathbb{N}^*}\frac{1}{k^n}\int_\Omega m_{(0,k)^n}(F(\omega),\xi)\,dP, \end{aligned} \tag{3.6}$$

for every $\xi\in\mathbb{R}^{m\times n}$.

Proof. Appealing to Theorem 2.8 and Proposition 3.1, for every fixed $\xi \in \mathbb{R}^{m\times n}$ we deduce the existence of a set $\Omega^\xi \in \mathscr{F}$ with $P(\Omega^\xi) = 1$ and of a $\mathscr{F}$-measurable function $\phi^\xi \colon \Omega \to [0, +\infty)$ such that

$$\phi^\xi(\omega) = \lim_{t\to+\infty} \frac{\mu^\xi_{tQ}(\omega)}{|tQ|} = \lim_{t\to+\infty} \frac{m_{tQ}(F(\omega), \xi)}{|tQ|}, \tag{3.7}$$

for every $\omega \in \Omega^\xi$ and for every cube Q in $\mathbb{R}^n$.

Now let $t > 0$, $Q_t := (-t, t)^n$, and $f_{\mathrm{hom}} \colon \Omega \times \mathbb{R}^{m\times n} \to [0, +\infty)$ be the function defined as

$$f_{\mathrm{hom}}(\omega, \xi) := \limsup_{t\to+\infty} \frac{\mu^\xi_{Q_t}(\omega)}{|Q_t|} \quad \text{for every } (\omega, \xi) \in \Omega \times \mathbb{R}^{m\times n}.$$

We now show that by virtue of (2.1) and (2.2) the functions

$$\xi \mapsto \frac{\mu^\xi_A(\omega)}{|A|} \quad (\omega \in \Omega, A \in \mathcal{A}_0)$$

are locally Lipschitz continuous with Lipschitz constant $L' = L'(p, \alpha, \beta, L)$. To this end let $\xi, \eta \in \mathbb{R}^{m\times n}$ and let $u \in W^{1,p}_0(A; \mathbb{R}^m) \cap W^{2,2}_0(A; \mathbb{R}^m)$ be such that

$$m_A(F(\omega), \eta) = F(\omega)(u + \ell_\eta, A).$$

It is convenient to introduce the following shorthand notation

$$m(\xi) := \frac{\mu^\xi_A(\omega)}{|A|} = \frac{m_A(F(\omega), \xi)}{|A|}.$$

Then invoking (2.2) and the Hölder inequality we get

$$\begin{aligned}
m(\xi) - m(\eta) &\le \frac{1}{|A|}\big(F(\omega)(u + \ell_\xi, A) - F(\omega)(u + \ell_\eta, A)\big)\\
&\le \frac{1}{|A|}\int_A \big|f(\omega, x, \nabla u + \xi) - f(\omega, x, \nabla u + \eta)\big|\, dx\\
&\le \frac{1}{|A|}\int_A L(1 + |\nabla u + \xi|^{p-1} + |\nabla u + \eta|^{p-1})|\xi - \eta|\, dx\\
&\le \left(\frac{1}{|A|}\int_A L(1 + |\nabla u + \xi|^{p-1} + |\nabla u + \eta|^{p-1})^{\frac{p}{p-1}}\, dx\right)^{\frac{p-1}{p}} |\xi - \eta|\\
&\le \left(\frac{1}{|A|}\int_A CL(1 + |\xi|^p + |\eta|^p + |\nabla u + \eta|^p)\, dx\right)^{\frac{p-1}{p}} |\xi - \eta|,
\end{aligned} \tag{3.8}$$

where $C > 0$ depends only on p. Moreover, by (2.1) and (3.4) we deduce that

$$\int_A |\nabla u + \eta|^p\, dx \le \frac{1}{\alpha} F(u + \ell_\eta, A) = |A|\frac{1}{\alpha} m(\eta) \le |A|\frac{\beta}{\alpha}(1 + |\eta|^p), \tag{3.9}$$

hence plugging (3.9) into (3.8) gives

$$m(\xi) - m(\eta) \le L'(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|, \tag{3.10}$$

with $L' > 0$ depending on p, α, β, and L. Finally, the other inequality follows by interchanging the role of ξ and η. Thus it is straightforward to check that $f_{\mathrm{hom}}(\omega, \cdot)$ satisfies the same locally Lipschitz condition as in (3.10).

Set $\Omega' := \bigcap_{\xi\in\mathbb{Q}^{m\times n}} \Omega^\xi$; clearly $P(\Omega')=1$, moreover (3.7) holds true for every $\xi\in\mathbb{Q}^{m\times n}$ and $\omega\in\Omega'$. For $\xi\in\mathbb{R}^{m\times n}$ let $(\xi_k)\subset\mathbb{Q}^{m\times n}$ be such that $\xi_k\to\xi$, as $k\to+\infty$. Fix $\omega\in\Omega'$; in view of (3.10) we have

$$\begin{aligned}\left|f_{\rm hom}(\omega,\xi)-\frac{\mu^\xi_{tQ}(\omega)}{|tQ|}\right| &\leq |f_{\rm hom}(\omega,\xi)-f_{\rm hom}(\omega,\xi_k)|\\ &+\left|f_{\rm hom}(\omega,\xi_k)-\frac{\mu^{\xi_k}_{tQ}(\omega)}{|tQ|}\right|+\left|\frac{\mu^{\xi_k}_{tQ}(\omega)}{|tQ|}+\frac{\mu^{\xi}_{tQ}(\omega)}{|tQ|}\right|\\ \leq \left|f_{\rm hom}(\omega,\xi_k)-\frac{\mu^{\xi_k}_{tQ}(\omega)}{|tQ|}\right| &+2L'(1+|\xi|^{p-1}+|\xi_k|^{p-1})|\xi-\xi_k|.\end{aligned}$$

Hence appealing to (3.7), letting first $t\to+\infty$ and then $k\to+\infty$, yields

$$f_{\rm hom}(\omega,\xi)=\lim_{t\to+\infty}\frac{\mu^\xi_{tQ}(\omega)}{|tQ|},$$

for every $\omega\in\Omega'$ and every $\xi\in\mathbb{R}^{m\times n}$.

Then, it only remains to show that $f_{\rm hom}(\omega,\cdot)$ satisfies the same growth conditions as in (2.1), for every $\omega\in\Omega'$. The growth condition from below follows from

$$F(\omega)(u+\ell_\xi,A)=\int_A\big(f(\omega,x+z,Du+\xi)+|\Delta u|^2\big)\,dx\geq\alpha\int_A|Du+\xi|^p\,dx$$

and from the convexity of $z\mapsto|z|^p$; while the growth condition from above readily follows from (3.4).

If f is ergodic (or equivalently $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic) then μ^ξ is ergodic, hence Theorem 2.8 ensures that $f_{\rm hom}$ does not depend on ω. Finally, (3.4) together with the dominated convergence Theorem gives the first equality in (3.6) while the second equality is a consequence of [2, Lemma 3.4]. □

Remark 3.3. If we only assume stationarity on f, appealing to *e.g.* [3, Theorem 12.4.3] we may deduce that the random integrand $f_{\rm hom}$ can be expressed through the following formula

$$f_{\rm hom}(\omega,\xi)=\inf_{k\in\mathbb{N}^*}\frac{1}{k^n}\mathbb{E}\left[m_{(0,k)^n}(F(\omega),\xi)|\mathscr{J}\right], \tag{3.11}$$

where $\mathscr{J}$ denotes the σ-algebra of $(\tau_z)_{z\in\mathbb{Z}^n}$-invariant sets of $\mathscr{F}$ and $\mathbb{E}\left[m_{(0,k)^n}(F(\omega),\xi)|\mathscr{J}\right]$ is the conditional expectation of $\omega\mapsto m_{(0,k)^n}(F(\omega),\xi)$ given $\mathscr{J}$, that is, the unique $\mathscr{J}$-measurable function satisfying

$$\int_E\mathbb{E}\left[m_{(0,k)^n}(F(\omega),\xi)|\mathscr{J}\right]\,dP=\int_E m_{(0,k)^n}(F(\omega),\xi)\,dP\quad\text{for every } E\in\mathscr{J}.$$

Notice that if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic then $\mathscr{J}$ is trivial; *i.e.,* $P(E)\in\{0,1\}$ for all $E\in\mathscr{J}$, and (3.11) reduces to (3.6).

Let $A\in\mathcal{A}_0$, hence in particular ∂A is Lipschitz. Then any $u\in W^{1,p}(A;\mathbb{R}^m)$ can be extended to a function $\tilde u\in W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$ and the value of $F_\varepsilon(\omega)(\tilde u,A)$ does not depend on this extension. Hence each $F_\varepsilon(\omega)$ as in (3.2) defines, for every $A\in\mathcal{A}_0$, a functional $F^A_\varepsilon(\omega)\colon W^{1,p}(A;\mathbb{R}^m)\longrightarrow[0,+\infty]$. With a slight abuse of notation, in all that follows we still denote by $F_\varepsilon(\omega)$ the functionals $F^A_\varepsilon(\omega)$; *i.e.,* in what follows

$$F_\varepsilon(\omega)(u,A):=\begin{cases}\displaystyle\int_A\Big(f\Big(\omega,\frac{x}{\varepsilon},Du\Big)+\varepsilon^2|\Delta u|^2\Big)\,dx & \text{if } u\in W^{1,p}(A;\mathbb{R}^m)\cap W^{2,2}(A;\mathbb{R}^m),\\ +\infty & \text{otherwise in } W^{1,p}(A;\mathbb{R}^m).\end{cases} \tag{3.12}$$

We are now ready to prove the main result of this note, namely the almost sure Γ-convergence of the sequence of random functionals $\big(F_\varepsilon(\omega)\big)_{\varepsilon>0}$.

The following theorem extends [13, Theorem 2.1] to the stochastic stationary setting.

Theorem 3.4 (Γ-convergence)**.** *Let f be a stationary random integrand and let $F_\varepsilon(\omega)$ be as in (3.12). Then there exists $\Omega' \in \mathscr{F}$ with $P(\Omega') = 1$ such that for every $\omega \in \Omega'$ and every $A \in \mathcal{A}_0$ the functionals $F_\varepsilon(\omega)(\cdot, A)$ Γ-converge with respect to the weak $W^{1,p}(A;\mathbb{R}^m)$-topology to the random integral functional $F_{\rm hom}(\omega)(\cdot, A)\colon W^{1,p}(A) \longrightarrow [0,+\infty)$ given by*

$$F_{\rm hom}(\omega)(u,A) := \int_A f_{\rm hom}(\omega, Du)\,dx,$$

where $f_{\rm hom}$ is as in Proposition 3.2. If in addition f is ergodic then $F_{\rm hom}$ is a deterministic functional with integrand

$$f_{\rm hom}(\xi) = \lim_{k\to+\infty} \int_\Omega \frac{1}{k^n} \min\bigg\{ \int_{(0,k)^n} \big(f(\omega,x,Du+\xi) + |\Delta u|^2\big)\,dx\colon \\ u \in W^{1,p}_0((0,k)^n;\mathbb{R}^m) \cap W^{2,2}_0((0,k)^n;\mathbb{R}^m)\bigg\}\,dP. \quad (3.13)$$

Proof. Let Ω' be the set in $\mathscr{F}$ whose existence is established in Proposition 3.2; throughout the proof ω is a fixed element in Ω'.

In all that follows C denotes a strictly positive constant which may vary from line to line within the same formula. We explicit the dependence of C on the different parameters only when relevant.

Liminf inequality. We divide the proof of the liminf inequality into three steps.

Step 1: let $u = \ell_\xi$ and let $A = Q$ be an open cube in $\mathbb{R}^n$. Let $(u_\varepsilon) \subset W^{1,p}(Q;\mathbb{R}^m)$ be any sequence such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(Q;\mathbb{R}^m)$ and $\liminf_{\varepsilon\to0} F_\varepsilon(\omega)(u_\varepsilon, Q) < +\infty$. Hence up to subsequences (not relabelled)

$$\sup_{\varepsilon>0} F_\varepsilon(\omega)(u_\varepsilon, Q) < +\infty, \quad (3.14)$$

thus in particular $(u_\varepsilon) \subset W^{1,p}(Q;\mathbb{R}^m) \cap W^{2,2}(Q;\mathbb{R}^m)$. Assume moreover that $(u_\varepsilon - \ell_\xi) \in W^{1,p}_0(Q;\mathbb{R}^m) \cap W^{2,2}_0(Q;\mathbb{R}^m)$. Then, in view of (3.3), Proposition 3.2 gives

$$F_{\rm hom}(\omega)(u,Q) = |Q|\, f_{\rm hom}(\omega,\xi) = |Q| \lim_{\varepsilon\to0} \frac{m_{Q/\varepsilon}(F(\omega),\xi)}{\varepsilon^{-n}|Q|} \\ |Q| \lim_{\varepsilon\to0} \frac{m_Q(F_\varepsilon(\omega),\xi)}{|Q|} \le \liminf_{\varepsilon\to0} F_\varepsilon(\omega)(u_\varepsilon, Q), \quad (3.15)$$

where in the last inequality we used the fact that $u_\varepsilon - \ell_\xi$ is a test function for $m_Q(F_\varepsilon(\omega),\xi)$.

We now remove the restriction $(u_\varepsilon - \ell_\xi) \in W^{1,p}_0(Q;\mathbb{R}^m) \cap W^{2,2}_0(Q;\mathbb{R}^m)$ by applying a classical argument of De Giorgi (see also the proof of [13, Theorem 2.1] and that of [20, Lemma 2.1]). Denote by l the side-length of Q and let Q_0 be an open cube concentric with Q, having the same orientation, and with side-length $l_0 < l$. Set $R := l - l_0$; let $N \in \mathbb{N}$, $N > 1$ and denote by $(Q_i)_{i=1,\dots,N}$ the finite family of open cubes concentric with Q, having the same orientation of Q, with side-length l_i satisfying

$$l_i = l_0 + \frac{(i-1)}{N}\,R,$$

for every $i = 1, \dots, N$.

Consider the cut-off functions $\varphi_i \in C_0^\infty(Q)$ such that $0 \leq \varphi_i \leq 1$, $\varphi_i = 1$ in Q_{i-1}, $\varphi_i = 0$ in $Q \setminus Q_i$ and

$$\|\nabla \varphi_i\|_\infty \leq \frac{N+1}{R}, \quad \|\Delta \varphi_i\|_\infty \leq \frac{(N+1)^2}{R^2}, \tag{3.16}$$

for every $i = 1, \ldots, N$. Set

$$u_\varepsilon^i := \ell_\xi + \varphi_i(u_\varepsilon - \ell_\xi);$$

clearly $(u_\varepsilon^i - \ell_\xi) \in W_0^{1,p}(Q;\mathbb{R}^m) \cap W_0^{2,2}(Q;\mathbb{R}^m)$; moreover $u_\varepsilon^i \rightharpoonup \ell_\xi$ in $W^{1,p}(Q;\mathbb{R}^m)$ and by (3.15)

$$F_{\hom}(\omega)(\ell_\xi, Q) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(\omega)(u_\varepsilon^i, Q), \tag{3.17}$$

for every $i = 1, \ldots, N$. From

$$F_\varepsilon(\omega)(u_\varepsilon^i, Q) = \int_{Q_{i-1}} \Big(f\Big(\omega, \frac{x}{\varepsilon}, Du_\varepsilon\Big) + \varepsilon^2 |\Delta u_\varepsilon|^2 \Big)\, dx$$
$$+ \int_{Q_i \setminus Q_{i-1}} \Big(f\Big(\omega, \frac{x}{\varepsilon}, Du_\varepsilon^i\Big) + \varepsilon^2 |\Delta u_\varepsilon^i|^2 \Big)\, dx + \int_{Q \setminus Q_i} f\Big(\omega, \frac{x}{\varepsilon}, \xi\Big)\, dx,$$

in view of (2.1) and (3.16) we obtain

$$F_\varepsilon(\omega)(u_\varepsilon^i, Q) \leq \int_Q \Big(f\Big(\omega, \frac{x}{\varepsilon}, Du_\varepsilon\Big) + \varepsilon^2 |\Delta u_\varepsilon|^2 \Big)\, dx$$
$$+ C \int_{Q_i \setminus Q_{i-1}} \Big(1 + |\xi|^p + \Big(\frac{N+1}{R}\Big)^p |u_\varepsilon - \ell_\xi|^p + |Du_\varepsilon - \xi|^p \Big)\, dx$$
$$+ C \int_{Q_i \setminus Q_{i-1}} \varepsilon^2 \left(\Big(\frac{N+1}{R}\Big)^4 |u_\varepsilon - \ell_\xi|^2 + \Big(\frac{N+1}{R}\Big)^2 |Du_\varepsilon - \xi|^2 + |\Delta u_\varepsilon|^2 \right)\, dx$$
$$+ \beta(1 + |\xi|^p)|Q \setminus Q_i|, \tag{3.18}$$

where $C > 0$ depends only on α and p.

By (3.14), in view of the nonnegativity of f we deduce that

$$\sup_{\varepsilon > 0} \|\varepsilon \Delta u_\varepsilon\|_{L^2(Q;\mathbb{R}^m)} < +\infty, \tag{3.19}$$

moreover we clearly have $\|u_\varepsilon\|_{W^{1,p}(Q;\mathbb{R}^m)} \leq C$.

We now want to show that the term

$$\int_{Q_i \setminus Q_{i-1}} \varepsilon^2 \big(|u_\varepsilon - \ell_\xi|^2 + |Du_\varepsilon - \xi|^2 \big)\, dx$$

can be bounded independently of ε. The case $p \geq 2$ is immediate, thus we let $p < 2$. For every open set $U \subset\subset Q$ interior elliptic regularity (see *e.g.* [15, Theorem 9.11]) yields

$$\|\varepsilon u_\varepsilon\|_{W^{2,p}(U;\mathbb{R}^m)} \leq C(U)\big(\|\varepsilon u_\varepsilon\|_{L^p(Q;\mathbb{R}^m)} + \|\varepsilon \Delta u_\varepsilon\|_{L^p(Q;\mathbb{R}^m)} \big).$$

Then appealing to (3.19), the Sobolev embedding

$$W^{2,p} \hookrightarrow W^{1,p^*} \hookrightarrow L^{(p^*)^*},$$

easily gives that $\varepsilon u_\varepsilon$ is equi-bounded in $W^{2,2}_{\mathrm{loc}}(Q;\mathbb{R}^m)$. Therefore since $u_\varepsilon \rightharpoonup \ell_\xi$ in $W^{1,p}(Q;\mathbb{R}^m)$ we deduce

$$\varepsilon u_\varepsilon \rightharpoonup 0 \quad \text{weakly in } W^{2,2}_{\mathrm{loc}}(Q;\mathbb{R}^m),$$

and by Rellich's compactness Theorem

$$\varepsilon u_\varepsilon \to 0 \quad \text{strongly in } W^{1,2}_{\mathrm{loc}}(Q;\mathbb{R}^m);$$

therefore

$$\int_{Q_i\setminus Q_{i-1}} \varepsilon^2\big(|u_\varepsilon-\ell_\xi|^2+|Du_\varepsilon-\xi|^2\big)\,dx\to 0 \quad \text{as } \varepsilon\to 0, \tag{3.20}$$

for every $i=1,\dots,N$. Finally gathering (3.17), (3.18), and (3.20) gives

$$F_{\rm hom}(\omega)(u,Q)\le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,Q)+C|Q\setminus Q_0|$$
$$+C\liminf_{\varepsilon\to 0}\int_{Q_i\setminus Q_{i-1}}\big(|Du_\varepsilon-\xi|^p+\varepsilon^2|\Delta u_\varepsilon|^2\big)\,dx,$$

for every $i=1,\dots,N$. Hence, summing over i, dividing by N, letting $N\to+\infty$, and $l_0\to l$ entails

$$F_{\rm hom}(\omega)(u,Q)\le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,Q),$$

and thus the desired liminf inequality.

Step 2: let $u=\ell_\xi$ and $A\in\mathcal{A}_0$. Let $\eta>0$; there exists a finite family of pairwise disjoint open cubes $(Q_i)_{i\in I(\eta)}$, included in A and such that $|A\setminus\bigcup_{i\in I(\eta)}Q_i|\le\eta$. Appealing to the growth condition from above satisfied by $f_{\rm hom}$ and by the previous step in the proof we find

$$F_{\rm hom}(u,A)\le \sum_{i\in I(\eta)} F_{\rm hom}(u,Q_i)+\eta\beta(1+|\xi|^p)$$
$$\le \sum_{i\in I(\eta)}\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,Q_i)+\eta\beta(1+|\xi|^p)$$
$$\le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,A)+\eta\beta(1+|\xi|^p),$$

hence we conclude by the arbitrariness of η.

Step 3: let $u\in W^{1,p}(A;\mathbb{R}^m)$ and $A\in\mathcal{A}_0$. Let $(u_\varepsilon)\subset W^{1,p}(A;\mathbb{R}^m)$ be such that $u_\varepsilon\rightharpoonup u$ in $W^{1,p}(A;\mathbb{R}^m)$ and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,A)<+\infty,$$

as before we deduce that, up to subsequences not relabelled, $(u_\varepsilon)\subset W^{1,p}(A;\mathbb{R}^m)\cap W^{2,2}(A;\mathbb{R}^m)$ and $\varepsilon|\Delta u_\varepsilon|$ is bounded in $L^2(A;\mathbb{R}^m)$ independently of ε. Thanks to the Lipschitz regularity of ∂A there exists a sequence $(w_\varepsilon)\subset C^\infty(\overline{A};\mathbb{R}^m)$ such that

$$\begin{cases} w_\varepsilon\to u \ \text{ strongly in } W^{1,p}(A;\mathbb{R}^m),\\ \varepsilon\Delta w_\varepsilon\to 0 \ \text{ strongly in } L^2(A;\mathbb{R}^m).\end{cases} \tag{3.21}$$

(This is clearly always possible at the price of a suitable relabelling of the sequence ε.) Moreover, for every $\eta>0$ we can find a finite partition of A into open sets $(A_i)_{i\in I(\eta)}$ such that

$$\sum_{i\in I(\eta)}\int_{A_i}|Du-\xi_i|^p\,dx<\eta \quad\text{where}\quad \xi_i:=\frac{1}{|A_i|}\int_{A_i}Du\,dx.$$

For $x\in A_i$ define $v^i_\varepsilon:=\ell_{\xi_i}+u_\varepsilon-w_\varepsilon$. Clearly $v^i_\varepsilon\rightharpoonup\ell_{\xi_i}$ in $W^{1,p}(A_i;\mathbb{R}^m)$, hence invoking Step 3 we obtain after summation

$$\liminf_{\varepsilon\to 0}\sum_{i\in I(\eta)} F_\varepsilon(\omega)(v^i_\varepsilon,A_i)\ge \sum_{i\in I(\eta)} F_{\rm hom}(\omega)(\ell_{\xi_i},A_i). \tag{3.22}$$

Then the liminf inequality follows from two estimates. Firstly, by (2.2) and noticing that $\Delta u_\varepsilon - \Delta v_\varepsilon^i = \Delta w_\varepsilon$ in A_i, we have

$$
\begin{aligned}
&\Big|F_\varepsilon(\omega)(u_\varepsilon, A) - \sum_{i\in I(\eta)} F_\varepsilon(\omega)(v_\varepsilon^i, A_i)\Big| \\
&\leq \sum_{i\in I(\eta)} \int_{A_i} \bigg(\Big|f\Big(\omega, \frac{x}{\varepsilon}, Du_\varepsilon\Big) - f\Big(\omega, \frac{x}{\varepsilon}, Dv_\varepsilon^i\Big)\Big| + \varepsilon^2\big||\Delta u_\varepsilon|^2 - |\Delta v_\varepsilon^i|^2\big|\bigg)\,dx \\
&\leq \sum_{i\in I(\eta)} \int_{A_i} \Big(L(1+|Du_\varepsilon|^{p-1} + |Dv_\varepsilon^i|^{p-1})|Du_\varepsilon - Dv_\varepsilon^i| + \varepsilon(|\Delta u_\varepsilon| + |\Delta v_\varepsilon^i|)\,\varepsilon|\Delta w_\varepsilon|\Big)\,dx \\
&\leq C\bigg(\sum_{i\in I(\eta)} \int_{A_i} (1+|Du_\varepsilon|^p + |Dv_\varepsilon^i|^p)\,dx\bigg)^{\frac{p-1}{p}} \bigg(\sum_{i\in I(\eta)} \int_{A_i} |Dw_\varepsilon - \xi_i|^p\,dx\bigg)^{\frac{1}{p}} \\
&\quad + C\big(\|\varepsilon\Delta u_\varepsilon\|_{L^2(A;\mathbb{R}^m)} + \|\varepsilon\Delta w_\varepsilon\|_{L^2(A;\mathbb{R}^m)}\big)\|\varepsilon\Delta w_\varepsilon\|_{L^2(A;\mathbb{R}^m)} \\
&\leq C\Big(\eta^{\frac{1}{p}} + \|Dw_\varepsilon - Du\|_{L^p(A;\mathbb{R}^{m\times n})} + \|\varepsilon\Delta w_\varepsilon\|_{L^2(A;\mathbb{R}^m)}\Big).
\end{aligned}
\tag{3.23}
$$

Secondly, in view of the local Lipschitz continuity of $f_{\rm hom}$ a similar calculation gives

$$
\Big|F_{\rm hom}(\omega)(u, A) - \sum_{i\in I(\eta)} F_{\rm hom}(\omega)(\xi_i, A_i)\Big| \leq C\eta^{\frac{1}{p}}. \tag{3.24}
$$

Thus, the conclusion follows from (3.21), (3.22), (3.23), and (3.24) passing to the limit first as $\varepsilon \to 0$ and then as $\eta \to 0$.

Limsup inequality. In order to not overburden notation, in all that follows we drop the dependence on ω in the sequences of functions we are going to construct.

In view of the continuity of $F_{\rm hom}$ with respect to the strong $W^{1,p}$-topology, a standard density argument allows us to prove the limsup inequality only for those target functions $u \in C^\infty(\overline{A};\mathbb{R}^m)$.

For every $A_0 \subset\subset A$ and for every $\eta > 0$ there exists v_η piecewise affine on A_0 such that

$$
\|u - v_\eta\|_{C^1(\overline{A}_0;\mathbb{R}^m)} \leq \eta \quad \text{and} \quad \|Dv_\eta\|_\infty \leq \|Du\|_\infty \tag{3.25}
$$

(see *e.g.* [12, X. Proposition 2.1]). Then, in view of the nonnegativity and the local Lipschitz continuity of $f_{\rm hom}$, (3.25) yields

$$
F_{\rm hom}(\omega)(u, A) \geq F_{\rm hom}(\omega)(v_\eta, A_0) - C\eta. \tag{3.26}
$$

Since v_η is piecewise affine on A_0 there exist finitely many pairwise disjoint open sets $(A_i)_{i\in I(\eta)}$ such that $|A_0 \setminus \bigcup_{i\in I(\eta)} A_i| = 0$ and $Dv_{\eta|_{A_i}} = \xi_i$.

Let $\delta > 0$ and denote by $\mathcal{Q}_\delta$ the cubic lattice in $\mathbb{R}^n$ containing $(0,\delta)^n$. For every fixed $i \in I(\eta)$ denote by $(Q_\delta^j)_{j\in J^i(\delta)}$ the family of all the open cubes in $\mathcal{Q}_\delta$ which are contained in A_i. Moreover for $i \in I(\eta)$ and $j \in J^i(\delta)$ let $v_{\varepsilon,\delta,\eta}^{i,j} \in W_0^{1,p}(Q_\delta^j;\mathbb{R}^m) \cap W_0^{2,2}(Q_\delta^j;\mathbb{R}^m)$ be such that

$$
m_{Q_\delta^j}(F_\varepsilon(\omega), \xi_i) = F_\varepsilon(\omega)(v_{\varepsilon,\delta,\eta}^{i,j} + \ell_{\xi_i}, Q_\delta^j).
$$

For every fixed $i \in I(\eta)$ set

$$
v_{\varepsilon,\delta,\eta}^i := \begin{cases} v_{\varepsilon,\delta,\eta}^{i,j} & \text{in } Q_\delta^j \quad \forall j \in J^i(\delta), \\[2ex] 0 & \text{in } A_i \setminus \bigcup_{j\in J^i(\delta)} Q_\delta^j, \end{cases}
$$

by definition $v^i_{\varepsilon,\delta,\eta} \in W^{1,p}_0(A_i;\mathbb{R}^m) \cap W^{2,2}_0(A_i;\mathbb{R}^m)$. Therefore if we set

$$v_{\varepsilon,\delta,\eta} := \begin{cases} v^i_{\varepsilon,\delta,\eta} & \text{in } A_i \quad \forall i \in I(\eta), \\ \\ 0 & \text{in } A \setminus A_0, \end{cases}$$

we obtain $v_{\varepsilon,\delta,\eta} \in W^{1,p}_0(A;\mathbb{R}^m) \cap W^{2,2}_0(A;\mathbb{R}^m)$. Thus we finally set

$$u_{\varepsilon,\delta,\eta} := v_{\varepsilon,\delta,\eta} + u.$$

Since $u \in C^\infty(\overline{A};\mathbb{R}^m)$ we deduce that $u_{\varepsilon,\delta,\eta} \in W^{1,p}(A;\mathbb{R}^m) \cap W^{2,2}(A;\mathbb{R}^m)$; moreover

$$\begin{aligned} \|u_{\varepsilon,\delta,\eta} - u\|^p_{L^p(A;\mathbb{R}^m)} &= \|v_{\varepsilon,\delta,\eta}\|^p_{L^p(A;\mathbb{R}^m)} \\ &= \sum_{i\in I(\eta)} \sum_{j\in J^i(\delta)} \|v^{i,j}_{\varepsilon,\delta,\eta}\|^p_{L^p(Q^j_\delta;\mathbb{R}^m)} \\ &\le C_p \delta^p \sum_{i\in I(\eta)} \sum_{j\in J^i(\delta)} \|Dv^{i,j}_{\varepsilon,\delta,\eta}\|^p_{L^p(Q^j_\delta;\mathbb{R}^m)} \end{aligned} \tag{3.27}$$

where $C_p > 0$ is the Poincaré constant in $W^{1,p}_0((0,1)^n;\mathbb{R}^m)$. For every fixed $i \in I(\eta)$, by definition of $v^{i,j}_{\varepsilon,\delta,\eta}$ and by (3.25) we also have

$$\begin{aligned} \|Dv^{i,j}_{\varepsilon,\delta,\eta} + \xi_i\|^p_{L^p(Q^j_\delta;\mathbb{R}^m)} &\le \frac{\beta}{\alpha}(1+|\xi_i|^p)|Q^j_\delta| \\ &\le \frac{\beta}{\alpha}(1+\|Du\|^p_\infty)|Q^j_\delta|, \end{aligned} \tag{3.28}$$

for every $j \in J^i(\delta)$. Then gathering (3.27) and (3.28) gives

$$\|u_{\varepsilon,\delta,\eta} - u\|^p_{L^p(A;\mathbb{R}^m)} \le C|A|\delta^p \to 0 \quad \text{as } \delta \to 0. \tag{3.29}$$

We have

$$F_\varepsilon(\omega)(u_{\varepsilon,\delta,\eta}, A) = F_\varepsilon(\omega)(u, A\setminus A_0) + F_\varepsilon(\omega)(u_{\varepsilon,\delta,\eta}, A_0), \tag{3.30}$$

where

$$F_\varepsilon(\omega)(u, A\setminus A_0) \le \int_{A\setminus A_0} \beta(1+|Du|^p)\,dx + \varepsilon^2 \int_{A\setminus A_0} |\Delta u|^2\,dx, \tag{3.31}$$

and in view of the local Lipschitz continuity of f

$$\begin{aligned} F_\varepsilon(\omega)(u_{\varepsilon,\delta,\eta}, A_0) &= F_\varepsilon(\omega)(v_{\varepsilon,\delta,\eta} + u, A_0) \\ = \int_{A_0} f\Big(\omega, \frac{x}{\varepsilon}, Dv_{\varepsilon,\delta,\eta} + Du\Big)\,dx &+ \varepsilon^2 \int_{A_0} |\Delta v_{\varepsilon,\delta,\eta} + \Delta u|^2\,dx \\ \le \int_{A_0} f\Big(\omega, \frac{x}{\varepsilon}, Dv_{\varepsilon,\delta,\eta} + Dv_\eta\Big)\,dx &+ \varepsilon^2 \int_{A_0} |\Delta v_{\varepsilon,\delta,\eta}|^2\,dx \\ +C\eta + \varepsilon^2 &\int_{A_0} (2|\Delta v_{\varepsilon,\delta,\eta}| + |\Delta u|)|\Delta u|\,dx \\ \le \int_{A_0} \Big(f\Big(\omega, \frac{x}{\varepsilon}, Dv_{\varepsilon,\delta,\eta} + Dv_\eta\Big) &+ \varepsilon^2 |\Delta v_{\varepsilon,\delta,\eta}|^2 \Big)\,dx \\ +C\eta + C\Big(\|\varepsilon\Delta v_{\varepsilon,\delta,\eta}\|_{L^2(A_0;\mathbb{R}^m)} + \|\varepsilon\Delta u\|_{L^2(A_0;\mathbb{R}^m)} \Big) &\|\varepsilon \Delta u\|_{L^2(A_0;\mathbb{R}^m)}\,dx. \end{aligned} \tag{3.32}$$

We now estimate the two terms

$$I_1^{\varepsilon,\delta,\eta} := \int_{A_0} \Big(f\Big(\omega, \frac{x}{\varepsilon}, Dv_{\varepsilon,\delta,\eta} + Dv_\eta\Big) + \varepsilon^2 |\Delta v_{\varepsilon,\delta,\eta}|^2 \Big)\, dx$$

and

$$I_2^{\varepsilon,\delta,\eta} := \Big(\|\varepsilon \Delta v_{\varepsilon,\delta,\eta}\|_{L^2(A_0;\mathbb{R}^m)} + \|\varepsilon \Delta u\|_{L^2(A_0;\mathbb{R}^m)} \Big) \|\varepsilon \Delta u\|_{L^2(A_0;\mathbb{R}^m)}\, dx.$$

For every fixed $i \in I(\eta)$ denote by $(Q_\delta^j)_{j \in \widetilde{J}^i(\delta)}$ the family of all the open cubes in $\mathcal{Q}_\delta$ which intersect the boundary of A_i. Notice that $v^i_{\varepsilon,\delta,\eta} = 0$ in $\bigcup_{j\in \widetilde{J}^i(\delta)} Q^j_\delta$, for every $i \in I(\eta)$. Then by definition of $v^{i,j}_{\varepsilon,\delta,\eta}$ and v_η, and by (3.25) we obtain

$$\begin{aligned} I_1^{\varepsilon,\delta,\eta} &= \sum_{i\in I(\eta)} F_\varepsilon(\omega)(v^i_{\varepsilon,\delta,\eta} + \ell_{\xi_i}, A_i) \\ &= \sum_{i\in I(\eta)} \sum_{j \in J^i(\delta)} F_\varepsilon(\omega)(v^{i,j}_{\varepsilon,\delta,\eta} + \ell_{\xi_i}, Q^j_\delta) + \sum_{i\in I(\eta)} \sum_{j \in \widetilde{J}^i(\delta)} F_\varepsilon(\omega)(\ell_{\xi_i}, Q^j_\delta) \\ &\leq \sum_{i\in I(\eta)} \sum_{j \in J^i(\delta)} |Q^j_\delta| \, \frac{m_{Q^\delta_j}(F_\varepsilon(\omega),\xi_i)}{|Q^j_\delta|} + \sum_{i\in I(\eta)} \sum_{j \in \widetilde{J}^i(\delta)} \beta(1 + \|Du\|^p_\infty)|Q^j_\delta|. \end{aligned}$$

Hence recalling (3.3) and appealing to Proposition 3.2 we have that for every fixed $i \in I(\eta)$

$$\lim_{\varepsilon\to 0} \frac{m_{Q^j_\delta}(F_\varepsilon(\omega),\xi_i)}{|Q^j_\delta|} = f_{\rm hom}(\omega,\xi_i), \quad \text{for every } j \in J^i(\delta).$$

Hence also taking into account that

$$\sum_{i\in I(\eta)} \sum_{j \in \widetilde{J}^i(\delta)} |Q^j_\delta| \leq C(\eta)\delta,$$

we get

$$\begin{aligned} \limsup_{\varepsilon\to 0} I_1^{\varepsilon,\delta,\eta} &\leq \sum_{i\in I(\eta)} \sum_{j \in J^i(\delta)} |Q^j_\delta| \lim_{\varepsilon\to 0} \frac{m_{Q^j_\delta}(F_\varepsilon(\omega),\xi_i)}{|Q^j_\delta|} + C(\eta)\delta \\ &= \sum_{i\in I(\eta)} \sum_{j \in J^i(\delta)} |Q^j_\delta| f_{\rm hom}(\omega,\xi_i) + C(\eta)\delta \\ &= \sum_{i\in I(\eta)} \sum_{j \in J^i(\delta)} \int_{Q^\delta_j} f_{\rm hom}(\omega, Dv_\eta)\, dx + C(\eta)\delta \\ &\leq \int_{A_0} f_{\rm hom}(\omega, Dv_\eta)\, dx + C(\eta)\delta, \end{aligned}$$

therefore

$$\limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} I_1^{\varepsilon,\delta,\eta} \leq \int_{A_0} f_{\rm hom}(\omega, Dv_\eta)\, dx = F_{\rm hom}(\omega)(v_\eta, A),$$

thus finally invoking (3.26) gives

$$\limsup_{\eta\to 0} \limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} I_1^{\varepsilon,\delta,\eta} \leq F_{\rm hom}(\omega)(u,A). \tag{3.33}$$

We now show that

$$\lim_{\varepsilon\to 0} I_2^{\varepsilon,\delta,\eta} = 0. \tag{3.34}$$

Since clearly $\|\varepsilon\Delta u\|_{L^2(A_0;\mathbb{R}^m)} \to 0$, to prove (3.34) it suffices to show that $\|\varepsilon\Delta v_{\varepsilon,\delta,\eta}\|_{L^2(A_0;\mathbb{R}^m)}$ is bounded uniformly in ε. By definition of $v_{\varepsilon,\delta,\eta}$, of v_η, appealing to (3.4), and to (3.25) we find

$$\begin{aligned}\varepsilon^2 \int_{A_0} |\Delta v_{\varepsilon,\delta,\eta}|^2\,dx &= \sum_{i\in I(\eta)}\sum_{j\in J^i(\delta)} \varepsilon^2 \int_{Q^j_\delta} |\Delta v^{i,j}_{\varepsilon,\delta,\eta}|^2\,dx \\ &\le \sum_{i\in I(\eta)}\sum_{j\in J^i(\delta)} m_{Q^j_\delta}(F_\varepsilon(\omega),\xi_i) \\ &\le \sum_{i\in I(\eta)}\sum_{j\in J^i(\delta)} \beta(1+|\xi_i|^p)|Q^j_\delta| \\ &\le \beta(1+\|Du\|^p_\infty)|A|,\end{aligned}$$

thus (3.34) is achieved. Then gathering (3.30)-(3.34) we obtain

$$\limsup_{\eta\to 0}\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u_{\varepsilon,\delta,\eta},A) \le F_{\rm hom}(\omega)(u,A) + \int_{A\setminus A_0} \beta(1+|Du|^p)\,dx. \tag{3.35}$$

Therefore, in view of (3.29) and (3.35) a diagonlaization argument yields the existence of two positive sequences $\delta(\varepsilon),\eta(\varepsilon)\to 0$ as $\varepsilon\to 0$, such that

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,A) \le F_{\rm hom}(\omega)(u,A) + \int_{A\setminus A_0} \beta(1+|Du|^p)\,dx, \tag{3.36}$$

and

$$\|u_\varepsilon - u\|^p_{L^p(A;\mathbb{R}^m)} \to 0,$$

where $u_\varepsilon := u_{\varepsilon,\delta(\varepsilon),\eta(\varepsilon)}$. Notice that by construction we also have $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(A;\mathbb{R}^m)$. Finally, since $u\in W^{1,p}(A;\mathbb{R}^m)$ letting $A_0 \nearrow A$ and choosing yet another diagonal sequence gives the conclusion. □

Remark 3.5. By the properties of Γ-convergence we readily deduce that $F_{\rm hom}(\omega)$ is lower semicontinuous in the weak $W^{1,p}(A;\mathbb{R}^m)$-topology and, as a consequence, that the random integrand $f_{\rm hom}$ is quasiconvex in ξ.

3.1. **Homogenisation with regular cells occupied by two randomly chosen nonlinear materials.** In this subsection we discuss a simple one-dimensional example which allows us to deduce a rather explicit formula for the corresponding homogenised energy density $f_{\rm hom}$.

We start by observing that for $m=n=1$ the homogenisation formula (3.13) can be equivalently rewritten as

$$f_{\rm hom}(\xi) = \lim_{k\to+\infty}\int_\Omega \frac{1}{k}\min\Big\{\int_0^k \big(f(\omega,x,u)+(u')^2\big)\,dx\colon u\in L^p(0,k)\cap W^{1,2}(0,k),\ \fint_0^k u\,dx = \xi\Big\}\,dP, \tag{3.37}$$

where $\xi\in\mathbb{R}$.

We now consider a *nonlinear* elastic string made of two different constituents with elastic constants $\lambda,\Lambda>0$ independently chosen according to a *Bernoulli process*. Specifically, let $q\in(0,1)$ and on $\{\lambda,\Lambda\}$ define the probability measure μ as

$$\mu(\lambda)=q,\quad \mu(\Lambda)=1-q. \tag{3.38}$$

Then, for $(x,u)\in\mathbb{R}\times\mathbb{R}$ we consider the random elastic energy density given by

$$f(\omega,x,u) = a(\omega,x)\,W(u)+1, \tag{3.39}$$

where the generic random parameter ω is a sequence $(\omega_n)_{n\in\mathbb{Z}}$ with values in $\{\lambda,\Lambda\}$, the function $a(\omega,\cdot)$ is defined for every $x\in\mathbb{R}$ as

$$a(\omega,x):=\omega_j \quad \text{if} \quad x\in[j,j+1),\ j\in\mathbb{Z},$$

and W is the double-well function with zeroes at ±1 given by

$$W(u):=(|u|-1)^2.$$

(We notice that the constant 1 in (3.39) is added only to obtain an f which satisfies a growth condition from below as in (2.1).)

Then the probability space we choose is $(\Omega,\mathscr{F},P)$ where $\Omega=\{\lambda,\Lambda\}^{\mathbb{Z}}$, $\mathscr{F}$ is the product σ-algebra of $\mathcal{P}(\{\lambda,\Lambda\})$ (where $\mathcal{P}(\{\lambda,\Lambda\})$ denotes the power set of $\{\lambda,\Lambda\}$), and the probability measure P is the product measure of the measures μ defined on $\{\lambda,\Lambda\}$ as in (3.38).

For $z\in\mathbb{Z}$ we denote by τ_z the shift operator on Ω; *i.e.,* $\tau_z(\omega_n):=\omega_{n+z}$. Then $(\tau_z)_{z\in\mathbb{Z}}$ is a group of P-preserving transformations on Ω which is also ergodic. Moreover for every $z\in\mathbb{Z}$ we have

$$a(\omega,x+z)=a(\tau_z\omega,x),$$

hence f as in (3.39) is an ergodic random integrand. Then, appealing to Theorem 3.4 we know that the random functionals

$$F_\varepsilon(\omega)(u)=\int_a^b\Big(f\Big(\omega,\frac{x}{\varepsilon},u\Big)+\varepsilon^2(u')^2\Big)\,dx$$

Γ-converge (almost surely) with respect to the weak $L^2(a,b)$-convergence to the deterministic functional given by

$$F_{\rm hom}(u)=\int_a^b f_{\rm hom}(u)\,dx,$$

moreover, according to Proposition 3.2, the homogeneous integrand $f_{\rm hom}\colon\mathbb{R}\to[0,+\infty)$ is defined for every $\xi\in\mathbb{R}$ in terms of the following asymptotic formula

$$f_{\rm hom}(\xi)=\lim_{k\to+\infty}\int_\Omega\frac{1}{k}\min\Big\{\int_0^k\big(a(\omega,x)\,W(u)+(u')^2\big)\,dx\colon\ u\in W^{1,2}(0,k),\ \fint_0^k u\,dx=\xi\Big\}\,dP+1.$$

We notice that in view of the nonnegativity of a and W we have that $f_{\rm hom}(\xi)\geq1$ for every $\xi\in\mathbb{R}$, moreover

$$f_{\rm hom}(1)=f_{\rm hom}(-1)=1$$

(achieved at $u\equiv1$ and $u\equiv-1$, respectively). Then since $f_{\rm hom}$ is convex (see Remark 3.5) we must have

$$f_{\rm hom}(\xi)=1 \quad \text{for every} \quad |\xi|\leq1.$$

We now observe that for fixed $k\in\mathbb{N}$ the cost of a transition between the two zeroes of W in terms of the energy-functional

$$\int_0^k\big(a(\omega,x)\,W(u)+(u')^2\big)\,dx$$

is strictly positive. As a consequence, also taking into account the definition of a, it can be easily deduced that for $\xi>1$ we have

$$f_{\rm hom}(\xi)-1=\lim_{k\to+\infty}\int_\Omega\frac{1}{k}\min\Big\{\int_0^k\big(a(\omega,x)(u-1)^2+(u')^2\big)\,dx\colon\ u\in W^{1,2}(0,k),\ \fint_0^k u\,dx=\xi\Big\}\,dP,$$

while for $\xi<-1$ we have

$$f_{\rm hom}(\xi)-1=\lim_{k\to+\infty}\int_\Omega\frac{1}{k}\min\Big\{\int_0^k\big(a(\omega,x)(u+1)^2+(u')^2\big)\,dx\colon\ u\in W^{1,2}(0,k),\ \fint_0^k u\,dx=\xi\Big\}\,dP.$$

Therefore we get

$$f_{\rm hom}(\xi) - 1 = \begin{cases} \varphi(\xi - 1) & \text{for } \xi > 1, \\ \varphi(\xi + 1) & \text{for } \xi < 1, \end{cases}$$

where $\varphi \colon \mathbb{R} \to [0, +\infty)$ is defined by the following linear problem

$$\varphi(\xi) = \lim_{k\to+\infty} \int_\Omega \frac{1}{k} \min\left\{ \int_0^k \big(a(\omega,x)u^2 + (u')^2\big)\,dx \colon\ u \in W^{1,2}(0,k),\ \fint_0^k u\,dx = \xi \right\} dP. \quad (3.40)$$

It is immediate to check that φ is 2-homogeneous, hence

$$\varphi(\xi) = \varphi(1)\,\xi^2 \quad \text{for every } \xi \in \mathbb{R}.$$

Thus finally,

$$f_{\rm hom}(\xi) = \varphi(1)\left((|\xi| - 1)^2\right)^{**} + 1 = \begin{cases} \varphi(1)\,(\xi - 1)^2 + 1 & \text{for } \xi > 1 \\ 1 & \text{for } |\xi| \le 1 \\ \varphi(1)\,(\xi + 1)^2 + 1 & \text{for } \xi < -1 \end{cases}.$$

We now estimate the constant $\varphi(1)$. Choosing $u \equiv 1$ as a test function in the minimisation problem defining $\varphi(1)$ we immediately deduce

$$\varphi(1) \le \overline{m} := \lim_{k\to+\infty} \int_\Omega \Big(\fint_0^k a(\omega,x)\,dx \Big)\,dP = \lim_{k\to+\infty} \int_\Omega \frac{1}{k} \sum_{j=0}^{k-1} \omega_j\,dP,$$

therefore the strong law of large numbers yields

$$\overline{m} = \lambda q + \Lambda(1-q),$$

that is $\overline{m}$ is the q-weighted mean of λ and Λ.

On the other hand, from (3.40) we have

$$\underline{m} := \lim_{k\to+\infty} \int_\Omega \frac{1}{k} \min\left\{ \int_0^k a(\omega,x)u^2\,dx \colon\ u \in L^2(0,k),\ \fint_0^k u\,dx = 1 \right\} dP \le \varphi(1).$$

Further, a direct computation gives

$$\begin{aligned} &\min\left\{ \int_0^k a(\omega,x)u^2\,dx \colon\ u \in L^2(0,k),\ \fint_0^k u\,dx = 1 \right\} \\ &= \min_{\substack{s_0,\dots,s_{k-1}\colon \\ \sum_{j=0}^{k-1} s_j = k}} \left(\sum_{j=0}^{k-1} \min\left\{ \int_j^{j+1} \omega_j\,u^2\,dx \colon\ u \in L^2(j,j+1),\ \int_j^{j+1} u\,dx = s_j \right\} \right) \\ &= \min_{\substack{s_0,\dots,s_{k-1}\colon \\ \sum_{j=0}^{k-1} s_j = k}} \left(\sum_{j=0}^{k-1} \omega_j s_j^2 \right) = k^2 \left(\sum_{j=0}^{k-1} \frac{1}{\omega_j} \right)^{-1}. \end{aligned}$$

Therefore again appealing to the strong law of large numbers we find

$$\underline{m} = \left(\frac{1}{\lambda} q + \frac{1}{\Lambda}(1-q) \right)^{-1},$$

hence $\underline{m}$ is the harmonic q-weighted mean of λ and Λ.

Thus we get the following bounds

$$\left(\frac{1}{\lambda} q + \frac{1}{\Lambda}(1-q) \right)^{-1} \le \varphi(1) \le \lambda q + \Lambda(1-q).$$

We finally notice that for $q = 1/2$ the constants $\underline{m}$ and $\overline{m}$ reduce to the mean and to the harmonic mean of λ and Λ respectively, and $\varphi(1)$ satisfies the same bounds as in the deterministic periodic case.

4. Stochastic homogenisation of singularly perturbed linear operators

In this section we use the Γ-convergence result Theorem 3.4 to study the asymptotic behaviour of the following sequence of problems

$$\begin{cases} \varepsilon^2 \Delta^2 u_\varepsilon - \operatorname{div}(\sigma_\varepsilon(\omega, x)\nabla u_\varepsilon) = g & \text{in } \Omega \times A, \\ u_\varepsilon = \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \Omega \times \partial A. \end{cases} \tag{4.1}$$

In (4.1) g belongs to $W^{-1,2}(A)$ while

$$\sigma_\varepsilon(\omega, x) := \sigma(\omega, x/\varepsilon), \tag{4.2}$$

for some σ satisfying:

- σ is $(\mathscr{F} \otimes \mathcal{L}^n)$-measurable;
- $\sigma(\omega, \cdot) \in \mathcal{M}(\alpha, \beta, A)$ for P-a.e. $\omega \in \Omega$;

where for $0 < \alpha \leq \beta$ we set

$$\mathcal{M}(\alpha, \beta, A) := \{\sigma \in L^\infty(A; \mathbb{R}^{n\times n}) \colon \sigma = \sigma^t,\ \alpha|\xi|^2 \leq \sigma(x)\xi\cdot\xi \leq \beta|\xi|^2,\ \forall\, \xi \in \mathbb{R}^n,\ \text{a.e. in } A\}. \tag{4.3}$$

The deterministic analogue of (4.1) has been studied by Bensoussan, Lions, and Papanicolaou in [4, Ch. 1, Section 14] in the periodic case and by Francfort and Müller in [13, Section 1] in the general (nonsymmetric, nonperiodic) case.

For fixed $\varepsilon > 0$ and $\omega \in \Omega$ let $B_\varepsilon(\omega)\colon W^{2,2}_0(A) \times W^{2,2}_0(A) \to \mathbb{R}$ be defined as

$$B_\varepsilon(\omega)(u, v) := \int_A (\sigma_\varepsilon(\omega, x)\nabla u \cdot \nabla v)\, dx + \varepsilon^2 \int_A \Delta u \Delta v\, dx.$$

The fact that $\sigma_\varepsilon(\omega, \cdot)$ belongs to $\mathcal{M}(\alpha, \beta, A)$ ensures that $B_\varepsilon(\omega)$ defines a continuos and coercive bilinear form on $W^{2,2}_0(A)$; hence the existence and uniqueness of a solution $\bar u_\varepsilon(\omega, \cdot)$ to (4.1) is guaranteed by the Lax-Milgram Lemma.

In this section we show that under suitable hypothesis on the random symmetric matrices σ_ε the sequence of solutions $(\bar u_\varepsilon(\omega, \cdot))$ converges for P-a.e. $\omega \in \Omega$ to some $\bar u$ satisfying some related elliptic PDE.

We say that the random matrix-valued function σ is *ergodic* if for almost every $y \in \mathbb{R}^n$ and for every $z \in \mathbb{Z}^n$

$$\sigma(\omega, y + z) = \sigma(\tau_z\omega, y) \quad \text{for } P\text{-a.e. } \omega \in \Omega,$$

where $(\tau_z)_{z\in\mathbb{Z}}$ is an *ergodic* group of P-preserving transformations on $(\Omega, \mathscr{F})$.

For a given $g \in W^{-1,2}(A)$ we consider the random functionals $G_\varepsilon(\omega)\colon W^{1,2}(A) \to [0, +\infty]$ defined as

$$G_\varepsilon(\omega)(u) := \begin{cases} \displaystyle\int_A \Big(\sigma_\varepsilon(\omega, x)\nabla u \cdot \nabla u + \varepsilon^2|\Delta u|^2\Big)\, dx - 2\langle g, u\rangle & \text{if } u \in W^{2,2}_0(A), \\ +\infty & \text{otherwise in } W^{1,2}(A), \end{cases} \tag{4.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,2}(A)$ and $W^{1,2}_0(A)$.

The following result is a consequence of Theorem 3.4 and of the fundamental property of Γ-convergence and extends [4, Theorem 14.1] to the stochastic ergodic setting.

Proposition 4.1. *Let $g \in W^{-1,2}(A)$ and let $\sigma\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be an ergodic random matrix-valued function. For every $\varepsilon > 0$ let σ_ε be as in* (4.2) *and let u_ε be the unique solution to*

$$\begin{cases} \varepsilon^2\Delta^2 u_\varepsilon - \operatorname{div}(\sigma_\varepsilon \nabla u_\varepsilon) = g & \text{in } W^{-1,2}(A), \text{ for } P\text{-a.e. } \omega \in \Omega, \\ u_\varepsilon(\omega,\cdot) \in W^{2,2}_0(A) & \text{for } P\text{-a.e. } \omega \in \Omega. \end{cases} \tag{4.5}$$

Then, there exists $\Omega' \in \mathscr{F}$ with $P(\Omega') = 1$ such that for every $\omega \in \Omega'$

$$\begin{cases} u_\varepsilon(\omega,\cdot) \rightharpoonup u & \text{in } W^{1,2}_0(A), \\ \varepsilon\Delta u_\varepsilon(\omega,\cdot) \rightharpoonup 0 & \text{in } L^2(A), \end{cases} \tag{4.6}$$

with u satisfying

$$\begin{cases} -\operatorname{div}(\sigma_{\rm hom}\nabla u) = g & \text{in } W^{-1,2}(A), \\ u \in W^{1,2}_0(A), \end{cases}$$

where the deterministic constant matrix $\sigma_{\rm hom}$ belongs to $\mathcal{M}(\alpha,\beta,A)$ and is given by

$$\sigma_{\rm hom}\xi\cdot\xi = \lim_{k\to+\infty}\int_\Omega \frac{1}{k^n}\min\left\{\int_{(0,k)^n}\big(\sigma(\omega,y)(\nabla u+\xi)\cdot(\nabla u+\xi) + |\Delta u|^2\big)\,dy\colon u \in W^{2,2}_0((0,k)^n)\right\} dP.$$

Proof. A direct computation shows that (4.5) is the Euler-Lagrange equation associated to the quadratic functional G_ε as in (4.4). For fixed $\varepsilon > 0$ and $\omega \in \Omega$ let $u_\varepsilon(\omega,\cdot) \in W^{2,2}_0(A)$ be the unique minimiser of $G_\varepsilon(\omega)$ (or equivalently the unique solution to (4.5)). Then multiplication of the first equation in (4.5) by $u_\varepsilon(\omega,\cdot)$ and integration by parts give

$$\int_A \varepsilon^2|\Delta u_\varepsilon|^2\,dx + \int_A \sigma_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon\,dx = \langle g, u_\varepsilon\rangle,$$

hence by the corecivity condition in (4.3) we get

$$\|\varepsilon\Delta u_\varepsilon\|^2_{L^2(A)} + \alpha\|\nabla u_\varepsilon\|^2_{L^2(A;\mathbb{R}^n)} \le \|g\|_{W^{-1,2}(A)}\|u_\varepsilon\|_{W^{1,2}_0(A)},$$

which in its turn implies

$$\begin{cases} \|u_\varepsilon\|_{W^{1,2}_0(A)} \le \dfrac{C}{\alpha}\|g\|_{W^{-1,2}(A)}, \\ \\ \|\varepsilon\Delta u_\varepsilon\|^2_{L^2(A)} \le \dfrac{C}{\alpha}\|g\|^2_{W^{-1,2}(A)}, \end{cases}$$

where $C > 0$ depends only on A. Therefore, up to the extraction of a subsequence (not relabelled), we have

$$u_\varepsilon(\omega,\cdot) \rightharpoonup \bar u(\omega,\cdot) \text{ in } W^{1,2}(A), \quad \varepsilon\Delta u_\varepsilon(\omega,\cdot) \rightharpoonup 0 \text{ in } L^2(A),$$

for some $\bar u(\omega,\cdot) \in W^{1,2}_0(A)$.

On the other hand, Theorem 3.4 applied with $m = 1$, $p = 2$, and

$$f\Big(\omega, \frac{x}{\varepsilon}, \xi\Big) = \sigma_\varepsilon(\omega,x)\xi\cdot\xi,$$

provides us with a $\mathscr{F}$-measurable set Ω' with $P(\Omega') = 1$ such that for every $\omega \in \Omega'$ the random functionals $F_\varepsilon(\omega)\colon W^{1,2}(A) \to [0,+\infty]$ given by

$$F_\varepsilon(\omega)(u) = \begin{cases} \displaystyle\int_A \Big(\sigma_\varepsilon(\omega,x)\nabla u\cdot\nabla u + \varepsilon^2|\Delta u|^2\Big)\,dx & \text{if } u \in W^{2,2}_0(A), \\ +\infty & \text{otherwise in } W^{1,2}(A) \end{cases}$$

Γ-converge with respect to the weak $W^{1,2}(A)$-topology to the deterministic functional

$$F_{\hom}(u) = \begin{cases} \displaystyle\int_A f_{\hom}(\nabla u)\,dx & \text{if } u \in W^{1,2}_0(A), \\ +\infty & \text{otherwise in } W^{1,2}(A), \end{cases}$$

where

$$f_{\hom}(\xi) = \lim_{k\to+\infty} \int_\Omega \frac{1}{k^n} \min\Big\{ \int_{(0,k)^n} \big(\sigma(\omega,y)(\nabla u+\xi)\cdot(\nabla u+\xi) + |\Delta u|^2\big)\,dy \colon u \in W^{2,2}_0((0,k)^n)\Big\}\,dP.$$

Here the only thing that shall be observed is that by a standard density argument it is enough to approximate target functions in $W^{2,2}_0(A)$ and that in this case a recovery sequence constructed as in Theorem 3.4 automatically belongs to $W^{2,2}_0(A)$.

Since for every $\varepsilon > 0$ the functional F_ε is a nonnegative quadratic form, appealing to [9, Theorem 11.10] we deduce that

$$f_{\hom}(\xi) = \sigma_{\hom}\xi\cdot\xi \quad \text{for every } \xi \in \mathbb{R}^n,$$

where $\sigma_{\hom} \in \mathcal{M}(\alpha,\beta,A)$. Moreover, the stability of Γ-convergence under continuous perturbations implies that for every $\omega \in \Omega'$ the random functionals $G_\varepsilon(\omega)$ Γ-converge with respect to the weak $W^{1,2}(A)$-topology to the deterministic functional given by

$$G(u) = \begin{cases} \displaystyle\int_A \big(\sigma_{\hom}\nabla u\cdot\nabla u\big)\,dx - 2\langle g,u\rangle & \text{if } u \in W^{1,2}_0(A), \\ +\infty & \text{otherwise in } W^{1,2}(A), \end{cases}$$

whose associated Euler-Lagrange equation is

$$\begin{cases} -\operatorname{div}\big(\sigma_{\hom}\nabla u\big) = g & \text{in } W^{-1,2}(A), \\ u \in W^{1,2}_0(A). \end{cases} \tag{4.7}$$

Then invoking [9, Corollary 7.20] we deduce that for every $\omega \in \Omega'$ the whole sequence of minimisers $(u_\varepsilon(\omega,\cdot))_{\varepsilon>0}$ converges to the unique (deterministic) solution u of (4.7), thus $\bar u(\omega,\cdot) = u$ for P-a.e. $\omega\in\Omega$ and the thesis is achieved. □

We conclude this section deducing simple lower and upper bounds for $\sigma_{\hom}$.

Remark 4.2. Let σ_ε be as in the statement of Proposition 4.1 and let $\overline{F}_\varepsilon(\omega)$ be the random functional defined by

$$\overline{F}_\varepsilon(\omega)(u) = \int_A (\sigma_\varepsilon(\omega,x)\nabla u\cdot\nabla u)\,dx \quad \text{for every} \quad u \in W^{1,2}(A);$$

clearly $\overline{F}_\varepsilon(\omega) \le F_\varepsilon(\omega)$. Then appealing to [11, Theorem I] and to Theorem 3.4 we can find $\Omega' \in \mathscr{F}$ with $P(\Omega') = 1$ such that for every $\omega \in \Omega'$ the Γ-limits of $\overline{F}_\varepsilon(\omega)$ and of $F_\varepsilon(\omega)$ both exist and we have

$$\overline{F} \le F_{\hom} \quad \text{on} \quad W^{1,2}(A) \tag{4.8}$$

where

$$\overline{F}(u) := \int_A \big(\bar\sigma\nabla u\cdot\nabla u\big)\,dx \quad \text{for every} \quad u \in W^{1,2}(A),$$

and

$$\bar\sigma\xi\cdot\xi := \lim_{k\to+\infty}\int_\Omega \frac{1}{k^n}\min\Big\{\int_{(0,k)^n}\big(\sigma(\omega,y)(\nabla u+\xi)\cdot(\nabla u+\xi)\big)\,dy \colon u\in W^{1,2}_0((0,k)^n)\Big\}\,dP.$$

On the other hand, invoking Theorem 2.9 we deduce the existence of an $\mathscr{F}$-measurable set Ω'', with $P(\Omega'') = 1$, such that for every $\omega \in \Omega''$ the pointwise limit of $F_\varepsilon(\omega)$ exists for every $u \in W^{2,2}(A)$ and we have

$$\lim_{\varepsilon\to 0} F_\varepsilon(\omega)(u) = \int_A \sigma_0 \nabla u \cdot \nabla u \, dx =: F_0(u),$$

where

$$\sigma_0 := \int_\Omega \left(\int_{(0,1)^n} \sigma(\omega, y)\, dy \right) dP.$$

Then, since

$$F_{\rm hom} \leq F_0 \quad \text{on } \; W^{2,2}(A) \tag{4.9}$$

(notice the Γ-convergence in Theorem 3.4 takes place also with respect to the strong $L^p(A)$-topology), choosing $u = \ell_\xi$ in (4.8) and (4.9) gives the bound

$$\bar\sigma\xi \cdot \xi \leq \sigma_{\rm hom}\xi \cdot \xi \leq \sigma_0 \xi \cdot \xi,$$

for every $\xi \in \mathbb{R}^n$.

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