A FORMULA FOR THE TOTAL VARIATION OF *SBV* FUNCTIONS

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Abstract. Following some ideas of a recent paper by Ambrosio, Bourgain, Brezis and Figalli, we prove a formula for the total variation of certain SBV functions without making use of the distributional derivatives.

Keywords: Special functions of bounded variations.

1. Introduction

The space SBV of special BV functions whose gradient measure has no Cantor part was singled out by De Giorgi and Ambrosio [4] as the natural setting to study variational problems where both volume and surface densities have to be taken into account. In fact, for an SBV function f the derivative Df is the sum of a measure $D^a f$ absolutely continuous with respect to the Lebesgue measure and a singular measure $D^s f$ concentrated on the jump set J_f, which is a countable $(n-1)$-rectifiable set. The density of $D^a f$ is equal to the approximate gradient ∇f.

In this paper, following some ideas from [2] and [3], we give a formula for the total variation of a function $f \in SBV_{loc}(\mathbb{R}^n)$ independent of the theory of distributions. Namely, we prove in Theorem 3.3 that if we define as in [2] for a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$

$$\kappa_\varepsilon(f) := \varepsilon^{n-1} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx \,,$$

where $\mathcal{G}_\varepsilon$ is a family of disjoint open cubes Q' of side length ε and arbitrary orientation, then for any $f \in SBV_{loc}(\mathbb{R}^n)$ such that either $|\bar{J}_f| = 0$ or $\nabla f \equiv 0$ we have

$$\lim_{\varepsilon \to 0^+} \kappa_\varepsilon(f) = \frac{1}{4} \int_{\mathbb{R}^n} |\nabla f| \, dx + \frac{1}{2} |D^s f|(\mathbb{R}^n) \,. \tag{1.1}$$

The above formula extends, with a different approach, the one obtained in [2] for the special case of a BV function f with values in $\mathbb{Z}$ and in particular for the characteristic function of a set of finite perimeter. Concerning the assumption $|\bar{J}_f| = 0$ its seems to us that in order to drop it one probably needs better approximation results for SBV functions than the ones available at the moment. However, observe that this mild regularity property of f is usually satisfied by the minimizers of free discontinuity problems such as the Mumford–Shah, see [5]. In the one dimensional case (1.1) holds without any further assumption on f, see Theorem 2.1, due to the fact that a one dimensional SBV function can be always split as the sum of an absolutely continuous and a jump function.

We conclude by observing that (1.1) does not hold in general if f is a BV function with a nontrivial Cantor part. In fact at the end of Section 2 we give a one dimensional example showing that for a Cantor type function $\liminf_{\varepsilon \to 0^+} \kappa_\varepsilon(f)$ can be strictly less than the right hand side of (1.1).

2. The one dimensional case

Let f be a function in $L^1_{loc}(\mathbb{R})$ and $\varepsilon > 0$. In this section we denote by $\mathcal{G}_\varepsilon$ any family of disjoint open intervals Q' of length ε. Then we set

$$\kappa_\varepsilon(f) := \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx \,.$$

We study the behavior of $\kappa_\varepsilon(f)$ as $\varepsilon \to 0^+$ and $f \in SBV_{loc}(\mathbb{R})$. To this aim we recall that if $f \in SBV_{loc}(\mathbb{R})$ then (see [1, Cor. 3.3]) f can be uniquely split, up to a constant, as the sum of two functions, $f = f^a + f^j$, where $f^a \in W^{1,1}_{loc}(\mathbb{R})$ and f^j is a *jump function*, i.e., a function such that Df^j is a purely atomic measure. Accordingly, the measure Df can be split as $Df = (f^a)'\mathcal{L}^1 + Df^j$. Clearly $Df^j = D^s f$, the singular part of Df. Given this canonical decomposition we are going to prove the following result.

Theorem 2.1. *Let f be a function in $SBV_{loc}(\mathbb{R})$. Then,*

$$\lim_{\varepsilon \to 0^+} \kappa_\varepsilon(f) = \frac{1}{4} \int_{\mathbb{R}} |(f^a)'| \, dx + \frac{1}{2} |D^s f|(\mathbb{R}) \,. \tag{2.1}$$

Proof. **Step 1.** We start by assuming that

$$f := g + h =: g + \sum_{i=0}^{N} \alpha_j \chi_{J_j} \,,$$

where $g \in C^1(\mathbb{R})$ is such that its derivative has compact support contained in an open interval $(-a, a)$, $J_j := (x_j, x_{j+1})$, $x_0 = -\infty < -a < x_1 < \cdots < x_N < a < x_{N+1} = +\infty$, and the coefficients α_j are real numbers such that $\alpha_j \neq \alpha_{j+1}$ for all $j = 0, 1, \ldots, N$. We are going to show that

$$\limsup_{\varepsilon \to 0^+} \kappa_\varepsilon(f) \leq \frac{1}{4} \int_{\mathbb{R}} |(f^a)'| \, dx + \frac{1}{2} |D^j f|(\mathbb{R}) \,. \tag{2.2}$$

To this aim, given $\sigma > 0$ we take $0 < \varepsilon < 1/2$ such that

$$-a < x_1 - \frac{\varepsilon}{2} < x_N + \frac{\varepsilon}{2} < a, \qquad x_{j+1} - x_j > \varepsilon \quad \text{for all } j = 1, \ldots, N-1, \tag{2.3}$$

$$|g'(x) - g'(y)| < \sigma \qquad \text{whenever } x, y \in \mathbb{R} \text{ with } |x - y| \leq \varepsilon \,. \tag{2.4}$$

Consider now a family $\mathcal{G}_\varepsilon$ of open intervals of length ε. We want to estimate from above the quantity

$$\sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx \,,$$

Due to the first inequalities in (2.3) and to the fact that g' has compact support in $(-a, a)$, f is constant in the interval $(-\infty, -a)$ and in the interval $(a, +\infty)$. So we may assume without loss of generality that $\mathcal{G}_\varepsilon = \{Q_1, \ldots, Q_m\}$ for some integer $m \geq 1$, where $Q_i = (a_i - \varepsilon/2, a_i + \varepsilon/2)$, and that $Q_i \cap (-a, a) \neq \emptyset$ for all i. Note that since $\varepsilon < 1/2$ this implies in particular that $m \leq (2a+1)/\varepsilon$. Note also that by the last inequality in (2.3) it follows that for all $i = 1, \ldots, m$ there exists at most one $j_i \in 1, \ldots, N$ such that $x_{j_i} \in Q_i$. Therefore we define

$$S := \{ i \in \{1, \ldots, m\} : \text{ there exists } j_i \in 1, \ldots, N \text{ such that } x_{j_i} \in Q_i \} \,.$$

Then we have

$$\sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \Big|f(x) - \fint_{Q'} f\Big|\,dx \le \sum_{i=1}^m \fint_{Q_i} \Big|g(x) - \fint_{Q_i} g\Big|\,dx + \sum_{i\in S} \fint_{Q_i} \Big|h(x) - \fint_{Q_i} h\Big|\,dx = G + H\,. \quad (2.5)$$

Note that for $x \in Q_i$ there exists $\bar{x}_i$ such that

$$g(x) = g(a_i) + g'(\bar{x}_i)(x - a_i) = g(a_i) + g'(a_i)(x - a_i) + R_i(x)\,,$$

where, by (2.4), we have $|R_i(x)| \le (\sigma\varepsilon)/2$. Therefore we have that for all $i = 1, \dots, m$

$$\begin{aligned}\fint_{Q_i} \Big|g(x) - \fint_{Q_i} g\Big|\,dx &= \fint_{Q_i} \Big|g'(a_i)(x - a_i) + R_i(x) - \fint_{Q_i} R_i\Big|\,dx \\ &\le \fint_{Q_i} |g'(a_i)(x - a_i)|\,dx + 2\fint_{Q_i} |R_i(x)|\,dx \le \frac{|g'(a_i)|\varepsilon}{4} + \sigma\varepsilon\,. \quad (2.6)\end{aligned}$$

Summing up with respect to i, recalling that $m \le (2a+1)/\varepsilon$ and taking into account (2.4) we get

$$\begin{aligned}G &\le \frac{1}{4}\sum_{i=1}^m |g'(a_i)|\varepsilon + (2a+1)\sigma \le \frac{1}{4}\sum_{i=1}^m \varepsilon \min_{Q_i} |g'| + \frac{1}{4}\sum_{i=1}^m \varepsilon\Big||g'(a_i)| - \min_{Q_i}|g'|\Big| + (2a+1)\sigma \\ &\le \frac{1}{4}\int_{\mathbb{R}} |g'(x)|\,dx + (3a+2)\sigma\,. \quad (2.7)\end{aligned}$$

In order to estimate H, for all $i \in S$ we set $x_{j_i} = a_i - \delta_i$, where $\delta_i \in (-\varepsilon/2, \varepsilon/2)$. Then, for $i \in S$ we have

$$\fint_{Q_i} h(y)\,dy = \frac{1}{\varepsilon}\Big[\alpha_{j_i-1}\Big(\frac{\varepsilon}{2} - \delta_i\Big) + \alpha_{j_i}\Big(\frac{\varepsilon}{2} + \delta_i\Big)\Big] = \frac{\alpha_{j_i} + \alpha_{j_i-1}}{2} + \frac{\delta_i}{\varepsilon}(\alpha_{j_i} - \alpha_{j_i-1})\,.$$

Therefore, using the fact that $|\delta_i/\varepsilon| \le 1/2$, we have

$$\begin{aligned}\fint_{Q_i} \Big|h(x) - \fint_{Q_i} h\Big|\,dx &= \frac{1}{\varepsilon}\Big[\Big(\frac{\varepsilon}{2} - \delta_i\Big)\Big|\alpha_{j_i-1} - \frac{\alpha_{j_i} + \alpha_{j_i-1}}{2} - \frac{\delta_i}{\varepsilon}(\alpha_{j_i} - \alpha_{j_i-1})\Big| \\ &\qquad + \Big(\frac{\varepsilon}{2} + \delta_i\Big)\Big|\alpha_{j_i} - \frac{\alpha_{j_i} + \alpha_{j_i-1}}{2} - \frac{\delta_i}{\varepsilon}(\alpha_{j_i} - \alpha_{j_i-1})\Big|\Big] \\ &= \frac{1}{\varepsilon}\Big[\Big(\frac{\varepsilon}{2} - \delta_i\Big)\Big(1 + \frac{2\delta_i}{\varepsilon}\Big) + \Big(\frac{\varepsilon}{2} + \delta_i\Big)\Big(1 - \frac{2\delta_i}{\varepsilon}\Big)\Big]\Big|\frac{\alpha_{j_i} - \alpha_{j_i-1}}{2}\Big| \\ &= \frac{1}{\varepsilon}\Big(\varepsilon - \frac{4\delta_i^2}{\varepsilon}\Big)\Big|\frac{\alpha_{j_i} - \alpha_{j_i-1}}{2}\Big| \le \Big|\frac{\alpha_{j_i} - \alpha_{j_i-1}}{2}\Big|\,. \quad (2.8)\end{aligned}$$

Therefore, summing up over all $i \in S$ we may conclude that

$$H \le \frac{1}{2}|Dh|(\mathbb{R})\,.$$

Thus, from this inequality, recalling (2.7) and (2.5) we deduce that

$$\sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \Big|f(x) - \fint_{Q'} f\Big|\,dx \le \frac{1}{4}\int_{\mathbb{R}} |g'(x)|\,dx + \frac{1}{2}|Dh|(\mathbb{R}) + (3a+2)\sigma.$$

Therefore, we get (2.2) passing to the supremum over all possible families $\mathcal{G}_\varepsilon$ and letting first $\varepsilon \to 0^+$ and then $\sigma \to 0^+$.

Step 2. We now prove the inequality

$$\liminf_{\varepsilon\to 0^+} \kappa_\varepsilon(f) \ge \frac{1}{4}\int_{\mathbb{R}} |(f^a)'|\,dx + \frac{1}{2}|D^j f|(\mathbb{R})\,. \quad (2.9)$$

As in the previous step we fix $\sigma > 0$ and take $0 < \varepsilon < 1/2$ so that (2.3) and (2.4) hold and moreover

$$\int_E |g'|\,dx < \sigma \qquad \text{if } E \subset \mathbb{R} \text{ is such that } |E| < \varepsilon\,. \tag{2.10}$$

We define a family $\mathcal{G}_\varepsilon$ by setting

$$\mathcal{G}_\varepsilon := \big\{(x_1 - \varepsilon/2, x_1 + \varepsilon/2), \dots, (x_N - \varepsilon/2, x_N + \varepsilon/2)\big\} \bigcup \Big(\bigcup_{j=0}^{N} \big\{Q_{1,j}, Q_{2,j}, \dots, Q_{m_j,j}\big\} \Big)$$

where, the families $\big\{Q_{1,j}, Q_{2,j}, \dots, Q_{m_j,j}\big\}$, for $j = 0, 1, \dots, N$ are defined as follows:
- if $j = 1, \dots, N-1$, we take all the intervals of the form $Q_{l,j} = (x_j + (2l-1)\varepsilon/2, x_j + (2l+1)\varepsilon/2)$, $l = 1, \dots, m_j$, contained in the interval $(x_j + \varepsilon/2, x_{j+1} - \varepsilon/2)$;
- if $j = 0$, we take all the intervals of the form $Q_{l,0} = (-a + (l-1)\varepsilon, -a + l\varepsilon)$, $l = 1, \dots, m_0$, contained in the interval $(-a, x_1 - \varepsilon/2)$;
- if $j = N$, we take all the intervals of the form $Q_{l,N} = (x_N + (2l-1)\varepsilon/2, x_N + (2l+1)\varepsilon/2)$, $l = 1, \dots, m_N$, contained in the interval $(x_N + \varepsilon/2, a)$.

Note that the union of $\mathcal{G}_\varepsilon$ covers all the interval $(-a, a)$ except possibly $N+1$ intervals of measure strictly less than ε. Therefore, from (2.10) we have that

$$\int_{\mathbb{R}\setminus\bigcup_{\mathcal{G}_\varepsilon} Q'} |g'|\,dx < \sigma(N+1)\,. \tag{2.11}$$

Then we have

$$\begin{aligned}\sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \Big| f(x) - \fint_{Q'} f \Big|\,dx &= \sum_{j=0}^{N}\sum_{l=1}^{m_j} \fint_{Q_{l,j}} \Big| g(x) - \fint_{Q_{l,j}} g \Big|\,dx \\ &\quad + \sum_{j=1}^{N} \fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2} \Big| f(x) - \fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2} f \Big|\,dx = G_1 + H_1\,.\end{aligned} \tag{2.12}$$

Arguing as in the proof of (2.6) we easily get that for all $j = 0, \dots, N$ and $l = 1, \dots, m_j$

$$\fint_{Q_{l,j}} \Big| g(x) - \fint_{Q_{l,j}} g \Big|\,dx \ge \frac{|g'(a_{l,j})|\varepsilon}{4} - \sigma\varepsilon\,,$$

where the $a_{l,j}$ are the centers of the intervals $Q_{l,j}$. Therefore, since $\sum_{j=0}^{N} m_j \le 2a/\varepsilon$, we conclude, arguing as in the proof of (2.7) and recalling (2.11) and (2.10)

$$\begin{aligned} G_1 &\ge \frac14 \sum_{j=0}^{N}\sum_{l=1}^{m_j} \varepsilon \max_{Q_{l,j}} |g'| - \frac14 \sum_{j=0}^{N}\sum_{l=1}^{m_j} \varepsilon \Big| |g'(a_{l,j})| - \max_{Q_{l,j}} |g'| \Big| - 2a\sigma \ge \frac14 \int_{\cup_{j=0}^{N}\cup_{l=1}^{m_j} Q_{l,j}} |g'(x)|\,dx - 3a\sigma \\ &\ge \frac14 \int_{\mathbb{R}} |g'|\,dx - \frac14 \int_{\mathbb{R}\setminus\bigcup_{\mathcal{G}_\varepsilon} Q'} |g'|\,dx - \frac14 \sum_{j=1}^{N} \int_{x_j-\varepsilon/2}^{x_j+\varepsilon/2} |g'|\,dx - 3a\sigma \\ &\ge \frac14 \int_{\mathbb{R}} |g'|\,dx - (3a + N + 1)\sigma\,. \end{aligned} \tag{2.13}$$

On the other hand, the calculations made in (2.8) with $\delta_i = 0$ show that for all $j = 1, \dots, N$

$$\fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2} \Big| h(x) - \fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2} h \Big|\,dx = \Big| \frac{\alpha_j - \alpha_{j-1}}{2} \Big|\,.$$

Therefore, we have arguing as in the proof of (2.13) and recalling (2.10)

$$\fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2}\Big|f(x)-\fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2}f\Big|\,dx \geq \fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2}\Big|h(x)-\fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2}h\Big|\,dx - \fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2}\Big|g(x)-\fint_{x_j-\varepsilon/2}^{x_j+\varepsilon/2}g\Big|\,dx$$

$$\geq \Big|\frac{\alpha_j-\alpha_{j-1}}{2}\Big| - \frac{|g'(x_j)|\varepsilon}{4} - \sigma\varepsilon \geq \Big|\frac{\alpha_j-\alpha_{j-1}}{2}\Big| - \frac{1}{4}\int_{x_j-\varepsilon/2}^{x_j+\varepsilon/2}|g'|\,dx - 2\sigma\varepsilon \geq \Big|\frac{\alpha_j-\alpha_{j-1}}{2}\Big| - 2\sigma\,.$$

Thus, we have proved, see (2.12), that

$$H_1 \geq \frac{1}{2}|Dh|(\mathbb{R}) - 2N\sigma\,.$$

From this inequality and from (2.13), recalling again (2.12), we conclude that

$$\sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx \geq \frac{1}{4}\int_{\mathbb{R}}|g'|\,dx + \frac{1}{2}|Dh|(\mathbb{R}) - (3a+1+3N)\sigma\,,$$

from which (2.9) follows, letting first $\varepsilon \to 0^+$ and then $\sigma \to 0^+$.

Step 3. Assume now that $f = g + h \in SBV_{loc}(\mathbb{R})$, where $g = f^a$ and $h = f^j$. Without loss of generality we may assume that $f^a(0) = 0$. Fix $a > 0$ such that neither a nor $-a$ are atoms of the measure Df^j. Given $\sigma > 0$ we may find a continuous function d_σ with compact support contained in the the interval $(-a, a)$ such that

$$\int_{-a}^{a}|g' - d_\sigma|\,dx \leq \sigma \tag{2.14}$$

and set $g_\sigma(x) := \int_0^x d_\sigma(t)\,dt$. Denote also by $h_\sigma = \sum_{j=1}^N \alpha_j\chi_{J_j}$, where $J_j := (x_j, x_{j+1})$, $x_0 = -\infty < -a < x_1 < x_2 < \cdots < x_N < a < x_{N+1} = +\infty$, and the coefficients α_j are real numbers such that $\alpha_j \neq \alpha_{j+1}$ for all $j = 0, 1, \ldots, N$ and such that

$$|D(h - h_\sigma)|(-a, a) < \sigma\,. \tag{2.15}$$

Finally, set $f_\sigma := g_\sigma + h_\sigma$ and observe that f_σ is constant on $(-\infty, -a)$ and on $(a, +\infty)$. Take $0 < \varepsilon < a$ and consider now a family $\mathcal{G}_\varepsilon$ of open cubes of sides ε and denote by $\mathcal{G}'_\varepsilon$ the subfamily of $\mathcal{G}_\varepsilon$ made by all cubes in $\mathcal{G}_\varepsilon$ with nonempty intersection with $(-a, a)$. We have, since f_σ is constant on $(-\infty, -a)$ and on $(a, +\infty)$, using the Poincaré inequality and recalling (2.14) and (2.15)

$$\begin{aligned}
\sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx &\geq \sum_{Q'\in\mathcal{G}'_\varepsilon}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx\\
&\geq \sum_{Q'\in\mathcal{G}'_\varepsilon}\fint_{Q'}\Big|f_\sigma(x)-\fint_{Q'}f_\sigma\Big|\,dx - \sum_{Q'\in\mathcal{G}'_\varepsilon}\fint_{Q'}\Big|(f-f_\sigma)(x)-\fint_{Q'}(f-f_\sigma)\Big|\,dx\\
&\geq \sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\Big|f_\sigma(x)-\fint_{Q'}f_\sigma\Big|\,dx - C\sum_{Q'\in\mathcal{G}'_\varepsilon}|D(f-f_\sigma)|(Q')\\
&\geq \sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\Big|f_\sigma(x)-\fint_{Q'}f_\sigma\Big|\,dx - 2C\sigma - C|Df|(-a-\varepsilon,-a) - C|Df|(a,a+\varepsilon)\,,
\end{aligned}$$

where C is the constant of the Poincaré inequality on an interval and the last two terms are due to the fact there could be two cubes in $\mathcal{G}'_\varepsilon$ with nonempty intersection also with the complement of $(-a, a)$. Passing to the supremum over all possible $\mathcal{G}_\varepsilon$ we then have that

$$\kappa_\varepsilon(f) \geq \kappa_\varepsilon(f_\sigma) - 2C\sigma - C|Df|(-a-\varepsilon,-a) - C|Df|(a,a+\varepsilon)\,.$$

From this inequality, letting $\varepsilon \to 0^+$ and recalling that neither $-a$ nor a are atoms of Dh we conclude, from what we proved in the two previous steps, that

$$\liminf_{\varepsilon\to 0^+} \kappa_\varepsilon(f) \geq \lim_{\varepsilon\to 0^+} \kappa_\varepsilon(f_\sigma) - 2C\sigma = \frac{1}{4}\int_{\mathbb{R}} |g'_\sigma| + \frac{1}{2}|Dh_\sigma|(\mathbb{R})| - 2C\sigma\,.$$

From this inequality, (2.14) and (2.15), letting first $\sigma \to 0^+$ and then $a \to +\infty$, it follows immediately that also f satisfies inequality (2.9). Note that this concludes the proof in the case $|Df|(\mathbb{R}) = \infty$.

If instead $|Df|(\mathbb{R}) < \infty$, given $\sigma > 0$ there exist a C^1 function g_σ with compact support in $\mathbb{R}$ and a function h_σ as above such that the estimates (2.14) and (2.15) hold on the whole $\mathbb{R}$, i.e.

$$\int_{\mathbb{R}} |g' - g'_\sigma|\,dx < \sigma \qquad \text{and} \qquad |D(h-h_\sigma)|(\mathbb{R}) < \sigma\,.$$

Then, using the fact that (2.1) holds for $f_\sigma = g_\sigma + h_\sigma$ and arguing as above we easily get the inequality (2.2). □

We conclude this section by showing that formula (2.1) does not hold if f has a Cantor part.

Example 2.2. Let us denote by C the well known *Cantor–Vitali* function, see for instance [1, Example 3.34]. Let us define a $BV_{loc}(\mathbb{R})$ function f by setting

$$f(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ C(x) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1\,. \end{cases}$$

We want to show that

$$\liminf_{\varepsilon\to 0^+} \kappa_\varepsilon(f) < \frac{1}{2}|D^s f|(\mathbb{R}) = \frac{1}{2}\,. \tag{2.16}$$

Thus, from this inequality it follows that formula (2.1) does not hold for f. To get (2.16) it is enough to prove that there exists $\gamma < 1/2$ such that for any interval $I \subset \mathbb{R}$ of length 1 one has

$$\int_I \left| f(x) - \int_I f \right| dx \leq \gamma |Df|(I)\,. \tag{2.17}$$

In fact, assume that (2.17) is true. Consider an interval J of length $1/3$ and observe that

$$f(x) = \frac{1}{2} f(3x) \quad \text{for all } x \in (-1/3, 2/3)\,.$$

Thus, using (2.17) and rescaling, we have immediately that if $J \subset (-1/3, 2/3)$ then

$$-\!\!\!\!\!\!\int_J \left| f(x) - -\!\!\!\!\!\!\int_J f \right| dx \leq \gamma |Df|(J)\,.$$

Similarly, observe that

$$f(x) = \frac{1}{2} + \frac{1}{2} f\big(3\big(x - \tfrac{2}{3}\big)\big) \quad \text{for all } x \in (1/3, 4/3)\,.$$

Thus, using again (2.17) and rescaling, we have that if $J \subset (1/3, 4/3)$ then

$$-\!\!\!\!\!\!\int_J \left| f(x) - -\!\!\!\!\!\!\int_J f \right| dx \leq \gamma |Df|(J)\,.$$

Finally, if $J \subset (-\infty, 0) \cup (1, \infty)$ both sides of the previous inequality are zero. Therefore, given any family $\mathcal{G}_{1/3}$ of disjoint intervals of length 1/3, we have

$$\sum_{Q' \in \mathcal{G}_{1/3}} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx \leq \gamma |Df|(\mathbb{R})$$

and thus

$$\kappa_{1/3}(f) \leq \gamma |Df|(\mathbb{R}) .$$

Then we may iterate this argument for all intervals of length $1/3^j$ using as before the self similarity properties of the Cantor–Vitali function and (2.17). Thus we get that for every integer $j \geq 1$

$$\kappa_{1/3^j}(f) \leq \gamma |Df|(\mathbb{R}) ,$$

from which (2.16) follows, since $\gamma < 1/2$.
To prove (2.17) we recall that the Poincaré inequality states that, if $f \in BV(0,1)$, then

$$\int_0^1 \left| f(x) - \int_0^1 f \right| dx \leq \frac{1}{2} |Df|(0,1) ,$$

with the equality holding if and only if f is equal to $a + b\chi_{(1/2,1)}$ for $a, b \in \mathbb{R}$. Observe also that in order to prove (2.17) it is enough to consider an interval $I \subset [-1, 2]$. Equivalently, we have to prove that there exists $\gamma < \frac{1}{2}$ such that

$$\int_x^{x+1} \left| f(y) - \int_x^{x+1} f \right| dy \leq \gamma |Df|(x, x+1) = \gamma [f(x+1) - f(x)] \quad \text{for all } x \in [-1, 1].$$

Observe that the function defined for all $x \in (-1, 1)$ by

$$R(x) := \frac{\displaystyle\int_x^{x+1} \left| f(y) - \int_x^{x+1} f \right| dy}{f(x+1) - f(x)}$$

is continuous. Moreover,

$$\lim_{x \to -1^+} R(x) = 0 = \lim_{x \to 1^-} R(x) . \tag{2.18}$$

Let us prove only the first equality. The second one can be proved in the same way. Denoting as before by C the Cantor–Vitali function, we calculate

$$\begin{aligned} \lim_{x \to -1^+} R(x) &= \lim_{\varepsilon \to 0^+} \frac{\displaystyle\int_{-1+\varepsilon}^{\varepsilon} \left| f(y) - \int_{-1+\varepsilon}^{\varepsilon} f \right| dy}{f(\varepsilon) - f(-1+\varepsilon)} = \lim_{\varepsilon \to 0^+} \frac{\displaystyle\int_{-1+\varepsilon}^{\varepsilon} \left| f(y) - \int_0^{\varepsilon} C \right| dy}{C(\varepsilon)} \\ &= \lim_{\varepsilon \to 0^+} \frac{(1-\varepsilon) \displaystyle\int_0^{\varepsilon} C(y)\, dy + \int_0^{\varepsilon} \left| C(y) - \int_0^{\varepsilon} C \right| dy}{C(\varepsilon)} . \end{aligned}$$

Now the first equality in (2.18) follows immediately by observing that

$$\int_0^{\varepsilon} C(x)\, dx \leq \varepsilon C(\varepsilon) .$$

Having extended by continuity R to the whole closed interval $[-1, 1]$ we conclude that R has a maximum point $\bar{x} \in (-1, 1)$. Then (2.17) follows by observing that $R(\bar{x}) < 1/2$. In fact if $R(\bar{x}) = 1/2$ then the restriction of f to the interval $(\bar{x}, \bar{x}+1)$ would be optimal for the Poincaré inequality and this is not true.

3. The n-dimensional case

We are now going to discuss the case of functions of n variables. To this aim in the following we denote by $Q_l(x)$ the cube in $\mathbb{R}^n$ with center in x, faces parallel to the coordinate axes and side length equal to l. If x is the origin we shall simply write Q_l, while if the side length is 1 we shall drop the subscript writing simply $Q(x)$ or Q if we refer to the unitary cube centered at the origin. We shall denote by $SO(n)$ the group of all rotations of $\mathbb{R}^n$ around the origin, while the elements of the standard base in $\mathbb{R}^n$ will be denoted by e_i, $i = 1, \dots, n$. Finally, if $x \in \mathbb{R}^n$ we denote by $x' := (x_1, \dots, x_{n-1})$ its projection over the first $n-1$ coordinate axes.

Lemma 3.1. *Let $\varphi : \mathbb{S}^{n-1} \to (0, \infty)$ be the function defined for all $\nu \in \mathbb{S}^{n-1}$ by setting*

$$\varphi(\nu) := \int_Q |x \cdot \nu| \, dx$$

Then φ attains its maximum on $\mathbb{S}^{n-1}$ at $\pm e_i$, for $i = 1, \dots, n$.

Proof. Let us fix $i \neq j \in \{1, \dots, n\}$ and denote by $\mathcal{R} \in SO(n)$ a rotation around the origin that does not move the vectors e_h for $h \neq i, j$ and rotates only the 2-dimensional plane containing e_i and e_j of an angle equal to $\pm\pi/2$ or π. Clearly, $\mathcal{R}(Q) = Q$. Therefore, given any vector $\nu \in \mathbb{S}^{n-1}$, performing the change of variable $y = \mathcal{R}x$, we have

$$\varphi(\nu) = \int_Q |\mathcal{R}^{-1} y \cdot \nu| \, dy = \int_Q |\mathcal{R}^{-1} y \cdot \mathcal{R}^{-1}(\mathcal{R}\nu)| \, dy = \int_Q |y \cdot \mathcal{R}\nu| \, dy = \varphi(\mathcal{R}\nu)\,.$$

Thus, if $\overline{\nu}$ is a maximum of φ on $\mathbb{S}^{n-1}$ also the vectors obtained by applying to $\overline{\nu}$ a rotation of the type above are maximum points. This implies in particular that

$$\max_{\mathbb{S}^{n-1}} \varphi = \max_C \varphi, \quad \text{where} \quad C := \Big\{ \nu \in \mathbb{S}^{n-1} : \nu_n \in \Big[\frac{\sqrt{2}}{2}, 1\Big]\Big\}\,.$$

Therefore, in order to prove the assertion it is enough to show that

$$\max_C \varphi = \varphi(e_n) = \frac{1}{4}\,. \tag{3.1}$$

To this aim, we denote the points $x \in \mathbb{R}^n$ by $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Given $\nu \in C$ we define $P^\pm := \{x \in Q : x \cdot \nu \gtrless 0\}$ and for any $x' \in \mathbb{R}^{n-1}$ we denote by $P^\pm_{x'}$ the vertical section of $P^\pm$ over x'. Denote by $\pi(P^\pm)$ the projections of $P^\pm$ over the first $n-1$ coordinate axes and observe that for all $x' \in \pi(P^+)$ we have $P^+_{x'} \subset (-x' \cdot \nu'/\nu_n, 1/2)$ and for all $x' \in \pi(P^-)$ we have $P^-_{x'} \subset (-1/2, -x' \cdot \nu'/\nu_n)$. Thus, using Fubini's theorem, we get for $\nu \in C$

$$\begin{aligned}
\int_Q |x \cdot \nu| \, dx &\leq \int_{\pi(P^+)} dx' \int_{-\frac{x' \cdot \nu'}{\nu_n}}^{\frac12} |x \cdot \nu| \, dx_n + \int_{\pi(P^-)} dx' \int_{-\frac12}^{-\frac{x' \cdot \nu'}{\nu_n}} |x \cdot \nu| \, dx_n \\
&= \int_{\pi(P^+)} dx' \int_{-\frac{x' \cdot \nu'}{\nu_n}}^{\frac12} (x' \cdot \nu' + x_n \nu_n) \, dx_n - \int_{\pi(P^-)} dx' \int_{-\frac12}^{-\frac{x' \cdot \nu'}{\nu_n}} (x' \cdot \nu' + x_n \nu_n) \, dx_n \\
&= \int_{\pi(P^+)} \Big(\frac{\nu_n}{8} + \frac{x' \cdot \nu'}{2} + \frac{(x' \cdot \nu')^2}{2\nu_n} \Big) dx' + \int_{\pi(P^-)} \Big(\frac{\nu_n}{8} - \frac{x' \cdot \nu'}{2} + \frac{(x' \cdot \nu')^2}{2\nu_n} \Big) dx'\,.
\end{aligned}$$

Since both integrands are nonnegative, we may further estimate as follows

$$\begin{aligned}\int_Q |x\cdot\nu|\,dx &\le \int_{(-\frac12,\frac12)^{n-1}}\left(\frac{\nu_n}{8}+\frac{x'\cdot\nu'}{2}+\frac{(x'\cdot\nu')^2}{2\nu_n}\right)dx' + \int_{(-\frac12,\frac12)^{n-1}}\left(\frac{\nu_n}{8}-\frac{x'\cdot\nu'}{2}+\frac{(x'\cdot\nu')^2}{2\nu_n}\right)dx' \\ &= \int_{(-\frac12,\frac12)^{n-1}}\left(\frac{\nu_n}{4}+\frac{(x'\cdot\nu')^2}{\nu_n}\right)dx' = \frac{\nu_n}{4}+\sum_{i,j=1}^{n-1}\frac{\nu_i\nu_j}{\nu_n}\int_{(-\frac12,\frac12)^{n-1}} x_i x_j\,dx' \\ &= \frac{\nu_n}{4}+\sum_{i=1}^{n-1}\frac{\nu_i^2}{\nu_n}\int_{(-\frac12,\frac12)^{n-1}} x_i^2\,dx' = \frac{\nu_n}{4}+\sum_{i=1}^{n-1}\frac{\nu_i^2}{12\nu_n} = \frac{\nu_n}{4}+\frac{1-\nu_n^2}{12\nu_n} = \frac{2\nu_n^2+1}{12\nu_n}\,.\end{aligned}$$

Since the maximum for $\nu_n \in [\sqrt{2}/2, 2]$ of the function on the right hand side is attained for $\nu_n = 1$ and is equal to $1/4$, we have proved (3.1). Hence the assertion follows. $\square$

Remark 3.2. Note that Lemma 3.1 can be equivalently restated by saying that if $Q'_l(x_0)$ is any cube centered at the x_0, of side length l and with two faces parallel to a vector $\overline{\nu} \in \mathbb{S}^{n-1}$, then

$$\int_{Q'_l(x_0)} |\nu\cdot(x-x_0)|\,dx \le \int_{Q'_l(x_0)} |\overline{\nu}\cdot(x-x_0)|\,dx = \frac{l^{n+1}}{4} \qquad \text{for all } \nu\in\mathbb{S}^{n-1}\,. \tag{3.2}$$

Given a function $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varepsilon > 0$, we denote by $\mathcal{G}_\varepsilon$ a family of disjoint open cubes Q' of side length ε and arbitrary orientation. Then we set

$$\kappa_\varepsilon(f) := \varepsilon^{n-1}\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}_\varepsilon} ⨍_{Q'}\left|f(x) - ⨍_{Q'} f\right|dx\,. \tag{3.3}$$

We now proceed to consider the case of a function $f \in SBV_{loc}(\mathbb{R}^n)$. In this case, the gradient measure Df can be split as the sum of two terms, $Df = D^a f + D^s f$, where $D^a f$ is absolutely continuous with respect to the Lebesgue measure and $D^s f$ is singular. Then, see [1, Th. 3.83], the density of $D^a f$ with respect to the Lebesgue measure is given by the approximate gradient ∇f, while $D^s f$ is concentrated on the jump set J_f of f. Note that $f \in W^{1,1}_{loc}(\mathbb{R}^n \setminus \bar{J}_f)$.

In the following, for any set $E \subset \mathbb{R}^n$ and for any $\delta > 0$ we shall denote by $I_\delta(E)$ the δ-neighborhood of E defined as $I_\delta(E) := \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < \delta\}$.

We now state the n-dimensional counterpart of Theorem 2.1

Theorem 3.3. *Let $f \in SBV_{loc}(\mathbb{R}^n)$ be such that either $\nabla f \equiv 0$ or $|\bar{J}_f| = 0$. Then*

$$\lim_{\varepsilon\to0^+}\kappa_\varepsilon(f) = \frac14\int_{\mathbb{R}^n}|\nabla f|\,dx + \frac12|D^s f|(\mathbb{R}^n)\,.$$

This result will follow by estimating the above limit separately from above and from below, see Propositions 3.4 and 3.5. We start with the estimate from above.

Proposition 3.4. *Let f be as in Theorem 3.3. Then,*

$$\limsup_{\varepsilon\to0^+}\kappa_\varepsilon(f) \le \frac14\int_{\mathbb{R}^n}|\nabla f|\,dx + \frac12|D^s f|(\mathbb{R}^n)\,. \tag{3.4}$$

Proof. We may assume without loss of generality that the right hand side of (3.4) is finite, since otherwise there is nothing to prove. Assume first that $|\bar{J}_f| = 0$ and fix $\delta > 0$. For any $\varepsilon \in (0, \delta/2\sqrt{n})$ and any family $\mathcal{G}_\varepsilon$ of disjoint open cubes of side ε and arbitrary orientation we set

$$\mathcal{G}'_\varepsilon := \{Q' \in \mathcal{G}_\varepsilon :\ Q' \cap I_{\delta/2}(\bar{J}_f) = \emptyset\}, \qquad \mathcal{G}''_\varepsilon = \mathcal{G}_\varepsilon \setminus \mathcal{G}'_\varepsilon\,.$$

Accordingly, we define

$$\kappa'_\varepsilon(f) := \varepsilon^{n-1}\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}'_\varepsilon}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx \qquad \kappa''_\varepsilon(f) := \varepsilon^{n-1}\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}''_\varepsilon}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx\,.$$

Clearly, we have $\kappa_\varepsilon(f)\le\kappa'_\varepsilon(f)+\kappa''_\varepsilon(f)$ and thus

$$\limsup_{\varepsilon\to0^+}\kappa_\varepsilon(f)\le\limsup_{\varepsilon\to0^+}\kappa'_\varepsilon(f)+\limsup_{\varepsilon\to0^+}\kappa''_\varepsilon(f) \tag{3.5}$$

and we estimate separately the last two terms in the previous inequality.

Step 1. Set now $\Omega_\delta := \{x\in\mathbb{R}^n : \operatorname{dist}(x,\bar J_f)>\delta/2\}$. Since $f\in W^{1,1}(\mathbb{R}^n\setminus\bar J_f)$, given $N>0$ we may find a function $f_N\in C^\infty(\mathbb{R}^n\setminus\bar J_f)$ such that $\|f-f_N\|_{W^{1,1}(\mathbb{R}^n\setminus\bar J_f)}\le\frac{1}{N}$. Let us fix $\sigma>0$ and $a>1$ and choose $\varepsilon\in(0,a/2\sqrt{n})$ so that

$$|\nabla f_N(x)-\nabla f_N(y)|\le\sigma \qquad \text{whenever } x,y\in\Omega_\delta\cap Q_{2a} \text{ with } |x-y|\le\sqrt{n}\varepsilon/2\,. \tag{3.6}$$

We work now with the cubes contained in the subfamily $\mathcal{G}'_\varepsilon$ and observe that, using the Poincaré inequality, we have

$$\begin{aligned}
\sum_{Q'\in\mathcal{G}'_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx &\le \sum_{Q'\in\mathcal{G}'_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\Big|f_N(x)-\fint_{Q'}f_N\Big|\,dx\\
&\quad+\sum_{Q'\in\mathcal{G}'_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\Big|(f(x)-f_N(x))-\fint_{Q'}(f-f_N)\Big|\,dx\\
&\le \sum_{Q'\in\mathcal{G}'_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\Big|f_N(x)-\fint_{Q'}f_N\Big|\,dx+C\sum_{Q'\in\mathcal{G}'_\varepsilon}\int_{Q'}|\nabla f-\nabla f_N|\,dx\\
&\le \sum_{Q'\in\mathcal{G}'_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\Big|f_N(x)-\fint_{Q'}f_N\Big|\,dx+C\int_{\Omega_\delta}|\nabla f-\nabla f_N|\,dx\,,
\end{aligned} \tag{3.7}$$

where C is a constant depending only on n. In order to estimate

$$\sum_{Q'\in\mathcal{G}'_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\Big|f_N(x)-\fint_{Q'}f_N\Big|\,dx\,,$$

we denote by Q'_i, for $i=1,\dots,m$, the cubes belonging to $\mathcal{G}'_\varepsilon$ having nonempty intersection with Q_a and by z_i the corresponding centers. The remaining cubes of the subfamily $\mathcal{G}'_\varepsilon$ will be denoted by Q'_i, for $i=m+1,\dots$. For $i\in\{1,\dots,m\}$ we have

$$f_N(x)=f_N(z_i)+\nabla f_N(\bar x_i)\cdot(x-z_i)=f_N(z_i)+\nabla f_N(z_i)\cdot(x-z_i)+R_i(x)\,,$$

where, by (3.6), we have $|R_i(x)|\le(\sqrt{n}\sigma\varepsilon)/2$. Therefore from (3.2) we have for $i=1,\dots,m$

$$\begin{aligned}
\varepsilon^{n-1}\fint_{Q'_i}\Big|f_N(x)-\fint_{Q'_i}f_N\Big|\,dx &= \varepsilon^{n-1}\fint_{Q'_i}\Big|\nabla f_N(z_i)\cdot(x-z_i)+R_i(x)-\fint_{Q'_i}R_i\Big|\,dx\\
&\le \varepsilon^{n-1}\fint_{Q'_i}|\nabla f_N(z_i)\cdot(x-z_i)|\,dx+2\varepsilon^{n-1}\fint_{Q'_i}|R_i(x)|\,dx\le\frac{\varepsilon^n|\nabla f_N(z_i)|}{4}+\sqrt{n}\sigma\varepsilon^n\,.
\end{aligned} \tag{3.8}$$

Thus, since $m \leq 2^n a^n/\varepsilon^n$, summing up over i, we conclude, using (3.6) and arguing as in the proof of (2.7), that

$$\sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \Big| f_N(x) - \fint_{Q'} f_N \Big| \, dx \leq \frac{1}{4} \sum_{i=1}^{m} |\nabla f_N(z_i)| \varepsilon^n + Ca^n\sigma + C \sum_{i>m} \int_{Q'_i} |\nabla f_N| \, dx$$
$$\leq \frac{1}{4} \int_{\Omega_\delta \cap Q_a} |\nabla f_N(x)| \, dx + Ca^n\sigma + C \int_{\Omega_\delta \setminus Q_a} |\nabla f_N(x)| \, dx \,,$$

for some positive constant C depending only on n. Therefore, recalling (3.7) and the fact that $\|f - f_N\|_{W^{1,1}(\mathbb{R}^n \setminus \bar{J}_f)} \leq \frac{1}{N}$ we have

$$\sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \Big| f(x) - \fint_{Q'} f \Big| \leq \frac{1}{4} \int_{\Omega_\delta} |\nabla f(x)| \, dx + Ca^n\sigma + C \int_{\Omega_\delta \setminus Q_a} |\nabla f(x)| \, dx + \frac{C}{N} \,.$$

Therefore, taking first the supremum over all families $\mathcal{G}_\varepsilon$, letting $\varepsilon \to 0^+$ and then letting $\sigma \to 0^+$, $a \to \infty$ and finally $N \to \infty$ we conclude that

$$\limsup_{\varepsilon\to 0^+} \kappa'_\varepsilon(f) \leq \frac{1}{4} \int_{\Omega_\delta} |\nabla f(x)| \, dx \,. \tag{3.9}$$

Step 2. We start by observing that since $\varepsilon < \delta/2\sqrt{n}$ all cubes $Q' \in \mathcal{G}''_\varepsilon$ are contained in $I_\delta(\bar{J}_f)$. Moreover, given $a > 1$, we decompose further the family $\mathcal{G}''_\varepsilon$ into two subfamilies $\mathcal{G}''_{\varepsilon,a}$ and $\mathcal{G}''_\varepsilon \setminus \mathcal{G}''_{\varepsilon,a}$, by setting $\mathcal{G}''_{\varepsilon,a} := \{Q' \in \mathcal{G}''_\varepsilon : \, Q' \subset Q_a\}$. Note that if a is sufficiently large then $Q' \cap Q_{a/2} = \emptyset$ for all $Q' \in \mathcal{G}''_\varepsilon \setminus \mathcal{G}''_{\varepsilon,a}$. We approximate f with a piecewise constant BV function in $I_\delta(\bar{J}_f) \cap Q_a$ in the following way. By the coarea formula for BV functions, [1, Th. 3.40], we have that

$$|Df|(I_\delta(\bar{J}_f) \cap Q_a) = \int_{-\infty}^{+\infty} P(\{x \in \mathbb{R}^n : \, f(x) > t\}; I_\delta(\bar{J}_f) \cap Q_a) \, dt \,.$$

Thus, for every integer j we may choose $t_{j,N} \in (\frac{j}{N}, \frac{j+1}{N})$ such that

$$P(\{x \in \mathbb{R}^n : \, f(x) > t_{j,N}\}; I_\delta(\bar{J}_f) \cap Q_a) \leq \frac{1}{N} \int_{\frac{j}{N}}^{\frac{j+1}{N}} P(\{x \in \mathbb{R}^n : \, f(x) > t\}; I_\delta(\bar{J}_f) \cap Q_a) \, dt \,.$$

Then, we set for all $j \in \mathbb{Z}$

$$\begin{aligned} E_{j,N,a} &:= \{x \in I_\delta(\bar{J}_f) \cap Q_a : \, f(x) > t_{j,N}\} \quad \text{for } j \geq 0, \\ E_{j,N,a} &:= \{x \in I_\delta(\bar{J}_f) \cap Q_a : \, f(x) \leq t_{j,N}\} \quad \text{for } j < 0, \\ h_N &:= \frac{1}{N} \sum_{j=0}^{+\infty} \chi_{E_{j,N,a}} - \frac{1}{N} \sum_{j=1}^{+\infty} \chi_{E_{-j,N,a}} \,. \end{aligned} \tag{3.10}$$

From the definition of h_N, using the coarea formula, we have $|Dh_N|(I_\delta(\bar{J}_f) \cap Q_a) \leq |Df|(I_\delta(\bar{J}_f))|$. Moreover, though the functions f and h_N are not necessarily bounded, by construction we have that $\|f - h_N\|_{L^\infty(I_\delta(\bar{J}_f) \cap Q_a)} \leq \frac{1}{N}$. Thus, using the Poincaré inequality and the Sobolev imbedding theorem, we have, for some positive constant C depending only on n,

$$\begin{aligned}
&\sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1} \fint_{Q'} \Big|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\Big|\,dx \\
&\quad\le \sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\Big(\fint_{Q'}\Big|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\Big|^{\frac{n}{n-1}}dx\Big)^{\frac{n-1}{n}} \\
&\quad\le \frac{1}{N^{1/n}} \sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\Big(\fint_{Q'}\Big|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\Big|\,dx\Big)^{\frac{n-1}{n}} \\
&\quad\le \frac{1}{N^{1/n}} \sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\fint_{Q'}\Big|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\Big|\,dx + \frac{1}{N^{1/n}}\sum_{Q'\in\mathcal{G}''_{\varepsilon,a}}\varepsilon^{n-1} \\
&\quad\le \frac{C}{N^{1/n}}|D(f-h_N)|(I_\delta(\bar J_f)\cap Q_a) + \frac{1}{\varepsilon N^{1/n}}|I_\delta(\bar J_f)\cap Q_a|\,.
\end{aligned}$$

Then, from this inequality, recalling that $|Dh_N|(I_\delta(\bar J_f)\cap Q_a)\le |Df|(I_\delta(\bar J_f))|$ and still denoting by C a constant depending only on n, we get easily that

$$\begin{aligned}
\sum_{Q'\in\mathcal{G}''_{\varepsilon}} \varepsilon^{n-1}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx &\le \sum_{Q'\in\mathcal{G}''_{\varepsilon}\setminus\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\fint_{Q'}\Big|f(x)-\fint_{Q'}f\Big|\,dx \qquad (3.11)\\
&\quad + \sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\fint_{Q'}\Big|h_N(x)-\fint_{Q'}h_N\Big|\,dx + \frac{C}{N^{1/n}}|Df|(I_\delta(\bar J_f)) + \frac{|I_\delta(\bar J_f)\cap Q_a|}{\varepsilon N^{1/n}} \\
&\le \sum_{Q'\in\mathcal{G}''_{\varepsilon}\setminus\mathcal{G}''_{\varepsilon,a}} C|Df|(Q') + \sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\fint_{Q'}\Big|h_N(x)-\fint_{Q'}h_N\Big|\,dx + \frac{C}{N^{1/n}}|Df|(I_\delta(\bar J_f)) + \frac{Ca^n}{\varepsilon N^{1/n}} \\
&\le C|Df|(\mathbb{R}^n\setminus Q_{a/2}) + \sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\fint_{Q'}\Big|h_N(x)-\fint_{Q'}h_N\Big|\,dx + \frac{C}{N^{1/n}}|Df|(I_\delta(\bar J_f)) + \frac{Ca^n}{\varepsilon N^{1/n}}\,.
\end{aligned}$$

We now recall that if $E\subset Q'$, then, see for instance [2, Sec. 5] or [6],

$$|E||Q'\setminus E| \le \frac{\varepsilon^{n+1}}{4}P(E;Q')\,.$$

Thus, we get

$$\begin{aligned}
\sum_{Q'\in\mathcal{G}''_{\varepsilon,a}} \varepsilon^{n-1}\fint_{Q'}\Big|h_N(x)-\fint_{Q'}h_N\Big|\,dx &\le \frac{1}{N}\sum_{Q'\in\mathcal{G}''_{\varepsilon,a}}\varepsilon^{n-1}\sum_{j=-\infty}^{\infty}\fint_{Q'}\Big|\chi_{E_{j,N,a}}(x)-\fint_{Q'}\chi_{E_{j,N,a}}\Big|\,dx \\
&= \frac{1}{N}\sum_{Q'\in\mathcal{G}''_{\varepsilon,a}}\frac{1}{\varepsilon^{n+1}}\sum_{j=-\infty}^{\infty}2|E_{j,N,a}\cap Q'||Q'\setminus E_{j,N,a}| \le \frac{1}{2N}\sum_{Q'\in\mathcal{G}''_{\varepsilon,a}}\sum_{j=-\infty}^{\infty}P(E_{j,N,a}\cap Q';Q') \\
&\le \frac{1}{2N}\sum_{j=-\infty}^{\infty}P(E_{j,N,a};I_\delta(\bar J_f)\cap Q_a) = \frac{1}{2}|Dh_N|(I_\delta(\bar J_f)\cap Q_a)\,.
\end{aligned}$$

Therefore, from this inequality and from (3.11), recalling again the inequality $|Dh_N|(I_\delta(\bar{J}_f)\cap Q_a) \leq |Df|(I_\delta(\bar{J}_f))|$, we have that

$$\sum_{Q'\in\mathcal{G}''_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx \leq \frac{1}{2}|Df|(I_\delta(\bar{J}_f))| + C|Df|(\mathbb{R}^n \setminus Q_{a/2}) + \frac{C}{N^{1/n}}|Df|(I_\delta(\bar{J}_f)) + \frac{Ca^n}{\varepsilon N^{1/n}} .$$

From this inequality, letting first $N \to \infty$ and then $a \to \infty$, taking the supremum over all families $\mathcal{G}_\varepsilon$ we obtain

$$\limsup_{\varepsilon\to 0^+} \kappa''_\varepsilon(f) \leq \frac{1}{2}|Df|(I_\delta(\bar{J}_f))| . \tag{3.12}$$

Step 3. Putting together inequalities (3.5), (3.9) and (3.12) we get at once

$$\limsup_{\varepsilon\to 0^+} \kappa_\varepsilon(f) \leq \frac{1}{4}\int_{\Omega_\delta} |\nabla f|\, dx + \frac{1}{2}|Df|(I_\delta(\bar{J}_f))| .$$

From this inequality, (3.4) follows observing that, since $|\bar{J}_f| = 0$, then

$$\lim_{\delta\to 0^+} |Df|(I_\delta(\bar{J}_f)) = |Df|(\bar{J}_f) = |D^s f|(\mathbb{R}^n) .$$

Finally if $\nabla f \equiv 0$, we may replace $I_\delta(\bar{J}_f)$ by the whole $\mathbb{R}^n$ and thus construct the function h_N in the whole cube Q_a. The rest of the proof goes without changes. ☐

We now prove the estimate from below, that turns out to be more delicate than the estimate from above proved in Proposition 3.4.

Proposition 3.5. *Let f be as in Theorem 3.3. Then,*

$$\liminf_{\varepsilon\to 0^+} \kappa_\varepsilon(f) \geq \frac{1}{4}\int_{\mathbb{R}^n} |\nabla f|\, dx + \frac{1}{2}|D^s f|(\mathbb{R}^n) .$$

In order to prove the estimate from below, we introduce a local version of the quantity κ_ε defined in (3.3). To this aim, given an open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{G}_\varepsilon(\Omega)$ a family of disjoint open cubes of sides ε and arbitrary orientation contained in Ω. Then, given a function $f \in L^1_{loc}(\Omega)$ for all $\varepsilon > 0$ we set

$$\kappa_\varepsilon(f;\Omega) := \sup_{\mathcal{G}_\varepsilon(\Omega)} \sum_{Q'\in\mathcal{G}_\varepsilon(\Omega)} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx . \tag{3.13}$$

The proof of Proposition 3.5 will follow from the next local estimate from below.

Proposition 3.6. *Let Ω be a bounded open set and $f \in SBV(\Omega)\cap L^\infty(\Omega)$ such that either $\nabla f \equiv 0$ or $|\bar{J}_f \cap \Omega| = 0$. Then,*

$$\liminf_{\varepsilon\to 0^+} k_\varepsilon(f;\Omega) \geq \frac{1}{4}\int_\Omega |\nabla f|\, dx + \frac{1}{2}|D^s f|(\Omega) .$$

Given this result, the proof of Proposition 3.5 is immediate.

Proof of Proposition 3.5. Let f satisfy the assumptions of Proposition 3.5. Fix $M > 0$ and a bounded open set Ω. Set $f^M := (f \vee (-M)) \wedge M$. Then f^M satisfies in Ω the assumptions of

Proposition 3.6. Therefore, given a family $\mathcal{G}_\varepsilon(\Omega)$ of disjoint cubes of side ε contained in Ω we have, by the Poincaré inequality,

$$\sum_{Q'\in\mathcal{G}_\varepsilon(\Omega)} \varepsilon^{n-1} \fint_{Q'} \Big| f(x) - \fint_{Q'} f \Big| \, dx \geq \sum_{Q'\in\mathcal{G}_\varepsilon(\Omega)} \varepsilon^{n-1} \fint_{Q'} \Big| f^M(x) - \fint_{Q'} f^M \Big| \, dx - C|D(f-f^M)|(\Omega) \,.$$

Passing to the supremum over all possible families of cubes $\mathcal{G}_\varepsilon(\Omega)$ we get

$$\kappa_\varepsilon(f) \geq \kappa_\varepsilon(f;\Omega) \geq \kappa_\varepsilon(f^M;\Omega) - C|D(f-f^M)|(\Omega) \,.$$

Therefore from Proposition 3.6 we have

$$\liminf_{\varepsilon\to 0^+} \kappa_\varepsilon(f) \geq \frac{1}{4}\int_\Omega |\nabla f^M| \, dx + \frac{1}{2}|D^s f^M|(\Omega) - C|D(f-f^M)|(\Omega) \,.$$

Since $|D(f-f^M)|(\Omega) \to 0$ as $M \to +\infty$, the result follows from the previous inequality letting first $M \to +\infty$ and then $\Omega \uparrow \mathbb{R}^n$. $\square$

Before proving Proposition 3.6 we fix some notation. Given a point $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, we denote by $Q_r(x,\nu)$ the cube with center at x, side r, with two faces orthogonal to ν and we define two half cubes by setting

$$Q_r^+(x,\nu) := \{y \in Q_r(x,\nu) : \, (y-x)\cdot\nu \geq 0\} \qquad Q_r^-(x,\nu) := \{y \in Q_r(x,\nu) : \, (y-x)\cdot\nu \leq 0\} \,.$$

We recall, see [1, Prop. 6.39] that if $f \in BV(\Omega)$, then one can define a Borel function $\nu : J_f \to \mathbb{S}^{n-1}$ and two Borel functions $f^\pm : J_f \to \mathbb{R}$ such that for all $x \in J_f$

$$\lim_{r\to 0^+} \fint_{Q_r^\pm(x,\nu(x))} |f(y) - f^\pm(x)| \, dy = 0 \,. \tag{3.14}$$

Moreover, if $f \in SBV(\Omega)$ one has, see [1, Th. 3.78],

$$D^s f = (f^+ - f^-)\nu\mathcal{H}^{n-1} \llcorner J_f \,.$$

We now proceed to the

Proof of Proposition 3.6. Assume first that $|\bar{J}_f \cap \Omega| = 0$. Then fix $\delta > 0$ and set $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x,\bar{J}_f) > \delta$ and $\operatorname{dist}(x,\partial\Omega) > \delta\}$. Given $\varepsilon < \delta/\sqrt{n}$ we are going to construct a family $\mathcal{G}_\varepsilon$ of disjoint open cubes of side ε and arbitrary orientation which will be the union of two disjoint subfamilies, $\mathcal{G}'_\varepsilon$, $\mathcal{G}''_\varepsilon$ with the cubes of $\mathcal{G}'_\varepsilon$ all contained in Ω_δ and the cubes of $\mathcal{G}''_\varepsilon$ all contained in $I_\delta(\bar{J}_f) \cap \Omega$.

Step 1. *Estimate of the absolutely continuous part.*

In order to construct the subfamily $\mathcal{G}'_\varepsilon$ we fix $N > 0$ and consider a function $f_N \in C^\infty(\Omega \setminus \bar{J}_f)$ such that $\|f - f_N\|_{W^{1,1}(\Omega\setminus\bar{J}_f)} \leq \frac{1}{N}$. Next we fix $\sigma > 0$ and choose ε so small that

$$|\nabla f_N(x) - \nabla f_N(y)| \leq \sigma \qquad \text{whenever } \, x,y \in \Omega_\delta \text{ with } |x-y| \leq \sqrt{n}\varepsilon/2 \,. \tag{3.15}$$

For $t > 0$ we set

$$U_t := \{x \in \Omega_\delta : \, |\nabla f_N(x)| > t\} \,.$$

We claim that we may find k pairwise disjoint sets $S_j \subset \mathbb{S}^{n-1}$, relatively open in $\mathbb{S}^{n-1}$ and such that

$$\bigcup_{j=1}^{k} \overline{S}_j = \mathbb{S}^{n-1}, \qquad \operatorname{diam} S_j < \sigma \quad \text{for all } i = 1, \dots, k \tag{3.16}$$

$$\Big| \bigcup_{j=1}^{k} \Big\{ x \in U_\sigma : \frac{\nabla f_N(x)}{|\nabla f_N(x)|} \in \partial S_j \Big\} \Big| = 0 \,, \tag{3.17}$$

where, when $X \subset \mathbb{S}^{n-1}$, the symbol ∂X denotes the relative boundary of X on $\mathbb{S}^{n-1}$. To prove this claim observe that for any $h = 1, \dots, n$ there exists a countable set $E_h \subset [-1, 1]$ such that

$$\Big| \Big\{ x \in U_\sigma : \frac{\nabla f_N(x)}{|\nabla f_N(x)|} \in \mathbb{S}^{n-1} \cap \{x_h = t\} \Big\} \Big| = 0 \qquad \text{for all } t \in [-1,1] \setminus E_h \,.$$

Therefore, for any h we may choose finitely many levels $t_{i,h} \notin E_h$, $t_{0,h} < -1 < t_{1,h} < \cdots < t_{n_h-1,h} < 1 < t_{n_h,h}$, such that $t_{i,h} - t_{i-1,h} < \sigma/\sqrt{n}$. We set $S_{i,h} := \{\nu \in \mathbb{S}^{n-1} : t_{i-1,h} < \nu_h < t_{i,h}\}$, $i = 1, \dots, n_h$, and then define the sets S_j as all possible nonempty intersections of the type $S_{i_1,1} \cap S_{i_2,2} \cap \dots S_{i_n,n}$. Note that (3.16) and (3.17) hold by construction. For all $j = 1, \dots, k$ we choose $\mu_j \in S_j$ and set

$$A_j := \Big\{ x \in U_\sigma : \frac{\nabla f_N(x)}{|\nabla f_N(x)|} \in S_j \Big\} \,.$$

By construction the sets A_j are all open and by (3.17)

$$\Big| U_\sigma \setminus \bigcup_{j=1}^{k} A_j \Big| = 0 \,. \tag{3.18}$$

For $\varepsilon > 0$ we consider the family $\mathcal{F}_\varepsilon$ of all open cubes with faces parallel to the coordinate planes, side length ε, centered at all points of the form $\varepsilon(k_1 e_1 + \cdots + k_n e_n)$, with $(k_1, \dots, k_n) \in \mathbb{Z}^n$. Then for all $j = 1, \dots, k$ we denote by $\mathcal{R}_j \in SO(n)$ a rotation that takes e_1 into the unit vector $\mu_j \in S_j$. Note that in this way each cube $Q' \in \mathcal{F}_\varepsilon$ is transformed into a cube $\mathcal{R}_j(Q')$ with two faces orthogonal to the vector μ_j. Finally, we define

$$\mathcal{G}'_\varepsilon = \bigcup_{j=1}^{k} \bigcup \{ \mathcal{R}_j(Q') : Q' \in \mathcal{F}_\varepsilon \text{ and } \mathcal{R}_j(Q') \subset A_j \} \,.$$

For all $j = 1, \dots, k$ we shall denote by $\mathcal{R}_j(Q'_{h,j})$, $Q'_{h,j} \in \mathcal{F}_\varepsilon$, $h = 1, \dots, m_j$, the elements of $\mathcal{G}'_\varepsilon$ contained in A_j. By (3.18) we have that there exists $\varepsilon(\sigma)$ such that if $\varepsilon < \varepsilon(\sigma)$ then

$$\Big| U_\sigma \setminus \bigcup_{j=1}^{k} \bigcup_{h=1}^{m_j} \mathcal{R}_j(Q'_{h,j}) \Big| \le \sigma \quad \text{and (3.15) holds.} \tag{3.19}$$

Denote now by $z_{h,j}$ the centers of the cubes $\mathcal{R}_j(Q'_{h,j})$ and argue as in the proof of (3.8), indicating by $R_{h,j}$ the remainder term. Then, recalling (3.2) and the inequality in (3.16), we have

$$\begin{aligned}
&\fint_{\mathcal{R}_j(Q'_{h,j})} \Big| f_N(x) - \fint_{\mathcal{R}_j(Q'_{h,j})} f_N \Big| \, dx = \fint_{\mathcal{R}_j(Q'_{h,j})} \Big| \nabla f_N(z_{h,j}) \cdot (x - z_{h,j}) + R_{h,j}(x) - \fint_{\mathcal{R}_j(Q'_{h,j})} R_{h,j} \Big| \, dx \\
&\qquad \geq \fint_{\mathcal{R}_j(Q'_{h,j})} |\nabla f_N(z_{h,j}) \cdot (x - z_{h,j})| \, dx - 2 \fint_{\mathcal{R}_j(Q'_{h,j})} |R_{h,j}(x)| \, dx \\
&\qquad \geq \fint_{\mathcal{R}_j(Q'_{h,j})} |\nabla f_N(z_{h,j})| |\mu_j \cdot (x - z_{h,j})| \, dx \\
&\qquad\qquad - \fint_{\mathcal{R}_j(Q'_{h,j})} |\nabla f_N(z_{h,j})| \Big| \Big(\frac{\nabla f_N(z_{h,j})}{|\nabla f_N(z_{h,j})|} - \mu_j \Big) \cdot (x - z_{h,j}) \Big| \, dx - \sqrt{n}\sigma\varepsilon \\
&\qquad \geq \frac{\varepsilon |\nabla f_N(z_{h,j})|}{4} - \sqrt{n}\sigma\varepsilon (\|\nabla f_N\|_{L^\infty(\Omega_\delta)} + 1) \, .
\end{aligned}$$

Observing that $\sum_{j=1}^k m_j \leq |\Omega|/\varepsilon^n$, from the previous inequality, adding up over j and h we get, arguing as in the proof of (2.13) and recalling (3.19),

$$\begin{aligned}
\varepsilon^{n-1} \sum_{Q' \in \mathcal{G}'_\varepsilon} \fint_{Q'} \Big| f_N(x) - \fint_{Q'} f_N \Big| \, dx &\geq \sum_{j=1}^k \sum_{h=1}^{m_j} \frac{\varepsilon^n |\nabla f_N(z_{h,j})|}{4} - |\Omega| \sqrt{n}\sigma (\|\nabla f_N\|_{L^\infty(\Omega_\delta)} + 1) \\
&\geq \frac{1}{4} \sum_{j=1}^k \sum_{h=1}^{m_j} \int_{\mathcal{R}_j(Q'_{h,j})} |\nabla f_N| \, dx - C\sigma (1 + \|\nabla f_N\|_{L^\infty(\Omega_\delta)}) \\
&\geq \frac{1}{4} \int_{\Omega_\delta} |\nabla f_N| \, dx - C\sigma (1 + \|\nabla f_N\|_{L^\infty(\Omega_\delta)}) \\
&\geq \frac{1}{4} \int_{\Omega_\delta} |\nabla f| \, dx - C\sigma (1 + \|\nabla f_N\|_{L^\infty(\Omega_\delta)}) - \frac{C}{N} \, ,
\end{aligned}$$

for some positive constant C depending only on n and $|\Omega|$. Therefore, choosing σ sufficiently small, and thus also ε small enough, we conclude, arguing as in the proof of (3.7)

$$\begin{aligned}
\sum_{Q' \in \mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \Big| f(x) - \fint_{Q'} f \Big| \, dx &\geq \sum_{Q' \in \mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \Big| f_N(x) - \fint_{Q'} f_N \Big| \, dx - C \int_{\Omega_\delta} |\nabla f - \nabla f_N| \, dx \\
&\geq \frac{1}{4} \int_{\Omega_\delta} |\nabla f| \, dx - \frac{C}{N} \, . \qquad (3.20)
\end{aligned}$$

Step 2. *Singular part: construction of the cubes.*
To construct the cubes of the family $\mathcal{G}''_\varepsilon$ we start by fixing $\eta > 0$. Using the fact that $\mathcal{H}^{n-1} \llcorner J_f$ is a σ-finite measure we may find a compact $H \subset J_f$ such that $\mathcal{H}^{n-1}(H) < \infty$,

$$|Df|(J_f \setminus H)| < \eta \quad \text{and } \nu_{|H}, \, f^+_{|H}, \, f^-_{|H} \text{ are continuous} \, . \qquad (3.21)$$

Moreover, recalling (3.14), H can be chosen with the additional property that

$$\fint_{Q_r^\pm(x,\nu(x))} |f(y) - f^\pm(x)| \, dy \to 0, \qquad \text{uniformly in } H \text{ as } r \to 0^+ . \qquad (3.22)$$

Having chosen H we set

$$L(\eta) := \mathcal{H}^{n-1}(H) \, .$$

Now, we fix $\sigma > 0$. Since J_f is $(n-1)$-countably rectifiable, using [1, Prop. 2.76], we may find a compact set $K \subset H$ such that

$$|Df|(H \setminus K)| < \eta \tag{3.23}$$

with the property that K can be split as the union of finitely many disjoint compact sets $K_1, \dots, K_m$, where each K_i is contained in the graph of a C^1 function φ_i with compact support defined on a $(n-1)$-hyperplane π_i with $\|\nabla \varphi_i\|_{L^\infty(\pi_i)} < \sigma$.
The idea now is to construct m families of cubes. The i-th family of cubes will have two sides parallel to π_i and will cover K_i up to a small error. Since the construction of these families is the same for all $i = 1, \dots, m$ in order to not overburden the exposition with many indices we shall fix $i = 1$ and show how to construct the cubes for K_1.
We know that K_1 is contained in the graph of a C^1 function φ_1 with compact support defined on a $(n-1)$-hyperplane π_1 with $\|\nabla \varphi_1\|_\infty < \sigma$. Up to a rotation and a translation we may assume with no loss of generality that $\pi_1 = \{x_n = 0\}$ and thus that $\varphi_1 \in C^1_c(\mathbb{R}^{n-1})$.
Denote now by Γ the graph of φ_1 and let us extend the functions $f^+_{|K_1}$, $f^-_{|K_1}$ to the whole Γ so to get two continuous functions with compact support from Γ to $\mathbb{R}$ and L^∞-norm less than $\|f\|_{L^\infty}$. We denote these extensions by $\tilde{f}^+$, $\tilde{f}^-$, respectively. Then we denote by $\pi(K_1)$ the projection of K_1 on $\{x_n = 0\}$ and consider an open set $U \subset \mathbb{R}^{n-1}$ such that $\pi(K_1) \subset U$ and $\mathcal{H}^{n-1}(U \setminus \pi(K_1)) < \sigma \mathcal{H}^{n-1}(K_1)$. Finally, given $\varepsilon > 0$ we consider the standard covering of $\mathbb{R}^{n-1}$ by cubes of side length ε and sides parallel to the coordinate axes. Since $\mathcal{H}^{n-1}(U \setminus \pi(K_1)) < \sigma \mathcal{H}^{n-1}(K_1)$, we take ε so small that we can find k of such $(n-1)$ dimensional disjoint open cubes $Q_1^{n-1}, \dots Q_k^{n-1}$ with the property that each of them has not empty intersection with $\pi(K_1)$ and

$$\mathcal{H}^{n-1}\big(U \setminus \cup_{i=1}^k Q_i^{n-1}\big) < 2\sigma \mathcal{H}^{n-1}(K_1)\,. \tag{3.24}$$

Moreover we may choose ε so small that

$$|\tilde{f}^\pm(x) - \tilde{f}^\pm(y)| \le \sigma \qquad \text{whenever } x, y \in \Gamma \text{ with } |x-y| \le \sqrt{n}\varepsilon/2\,. \tag{3.25}$$

Finally, we may also require that

$$\fint_{Q_r^\pm(x,e_n)} |f(y) - f^\pm(x)|\,dy < C\sigma \qquad \text{for all } r < 3\varepsilon \text{ and for all } x \in K_1\,, \tag{3.26}$$

for some constant depending only on n and $\|f\|_{L^\infty}$. Indeed this inequality follows easily from (3.22) on observing that since $\|\nabla \varphi\|_\infty < \sigma$ we have

$$\begin{aligned}\fint_{Q_r^\pm(x,e_n)} |f(y) - f^\pm(x)|\,dy &\le \fint_{Q_r^\pm(x,\nu(x))} |f(y) - f^\pm(x)|\,dy + 2\|f\|_\infty \frac{|Q_r^\pm(x,e_n) \Delta Q_r^\pm(x,\nu(x))|}{r^n} \\ &\le \fint_{Q_r^\pm(x,\nu(x))} |f(y) - f^\pm(x)|\,dy + C'\|f\|_\infty \sigma\,,\end{aligned}$$

for some constant C' depending only on n.
Having decided how small ε has to be, we now consider the cubes obtained by translating vertically each cube Q_i^{n-1} in such a way that the resulting cube has center in $A := \{x = (x', \varphi(x')) : x' \in U\}$. These cubes will be denoted by Q_i and their centers by y_i. Note that ε can be chosen so small that all the cubes Q_i are contained in $I_\delta(\bar{J}_f) \cap \Omega$.

Step 3. *Estimate of the singular part.* Since by construction $Q_i^{n-1} \cap \pi(K_1) \neq \emptyset$ and $\|\nabla \varphi\|_{L^\infty} < \sigma$, taking σ small, for any i we find a point $x_i \in Q_i \cap K_1$. Then, let us denote by $Q_{i,2}(x_i)$ the cube with center in x_i and side equal to 2ε. Clearly, for all $i = 1, \dots, k$ we have $Q_i \subset Q_{i,2}(x_i)$. Finally, let us set for all $i = 1, \dots, k$

$$Q_i^{\pm} := \{x \in Q_i : (x - y_i) \cdot e_n \gtrless 0\}, \quad Q_{i,2}^{\pm}(x_i) := \{x \in Q_{i,2}(x_i) : (x - x_i) \cdot e_n \gtrless 0\}.$$

We now fix i and we start by estimating

$$\begin{aligned}
&\fint_{Q_i} \left| f(x) - \fint_{Q_i} f \right| dx = \fint_{Q_i} \left| f(x) - \frac{1}{2} \fint_{Q_i^+} f - \frac{1}{2} \fint_{Q_i^-} f \right| dx \qquad (3.27)\\
&\geq \fint_{Q_i} \left| f(x) - \frac{1}{2} \fint_{Q_{i,2}^+(x_i)} f - \frac{1}{2} \fint_{Q_{i,2}^-(x_i)} f \right| dx - \frac{1}{2} \left| \fint_{Q_i^+} f - \fint_{Q_{i,2}^+(x_i)} f \right| - \frac{1}{2} \left| \fint_{Q_i^-} f - \fint_{Q_{i,2}^-(x_i)} f \right|.
\end{aligned}$$

Observe now that from (3.26) we have that

$$\left| \fint_{Q_i^{\pm}} f - \fint_{Q_{i,2}^{\pm}(x_i)} f \right| \leq \fint_{Q_i^{\pm}} \left| f(x) - \fint_{Q_{i,2}^{\pm}(x_i)} f \right| dx \leq 2^n \fint_{Q_{i,2}^{\pm}(x_i)} \left| f(x) - \fint_{Q_{i,2}^{\pm}(x_i)} f \right| dx \leq C\sigma.$$

Using this inequality and (3.26) again, from (3.27) we get

$$\begin{aligned}
&\fint_{Q_i} \left| f(x) - \fint_{Q_i} f \right| dx \geq \fint_{Q_i} \left| f(x) - \frac{1}{2} \fint_{Q_{i,2}^+(x_i)} f - \frac{1}{2} \fint_{Q_{i,2}^-(x_i)} f \right| dx - C\sigma \\
&= \frac{1}{2} \fint_{Q_i^+} \left| f(x) - \fint_{Q_{i,2}^+(x_i)} f + \frac{1}{2} \Big(\fint_{Q_{i,2}^+(x_i)} f - \fint_{Q_{i,2}^-(x_i)} f \Big) \right| dx \\
&\qquad + \frac{1}{2} \fint_{Q_i^-} \left| f(x) - \fint_{Q_{i,2}^-(x_i)} f - \frac{1}{2} \Big(\fint_{Q_{i,2}^+(x_i)} f - \fint_{Q_{i,2}^-(x_i)} f \Big) \right| dx - C\sigma \\
&\geq \frac{1}{2} \left| \fint_{Q_{i,2}^+(x_i)} f - \fint_{Q_{i,2}^-(x_i)} f \right| - \frac{1}{2} \fint_{Q_i^+} \left| f(x) - \fint_{Q_{i,2}^+(x_i)} f \right| dx - \frac{1}{2} \fint_{Q_i^-} \left| f(x) - \fint_{Q_{i,2}^-(x_i)} f \right| dx - C\sigma \\
&\geq \frac{1}{2} |f^+(x_i) - f^-(x_i)| - C\sigma.
\end{aligned}$$

Then, using the uniform continuity of the functions $\tilde{f}^{\pm}$ as stated in (3.25) we finally estimate for all $i = 1, \dots, k$

$$\fint_{Q_i} \left| f(x) - \fint_{Q_i} f \right| dx \geq \frac{1}{2} |\tilde{f}^+(y_i) - \tilde{f}^-(y_i)| - C\sigma,$$

where the y_i are the centers of the cubes Q_i. Now we sum up the previous inequality, observing that by construction $k\varepsilon^{n-1} \leq \mathcal{H}^{n-1}(U)$ and denoting by $z_i \in \mathbb{R}^{n-1}$ the centers of the cubes Q_i^{n-1}. Thus we obtain

$$\begin{aligned}
\sum_{i=1}^{k} \varepsilon^{n-1} \fint_{Q_i} \left| f(x) - \fint_{Q_i} f \right| dx &\geq \frac{1}{2} \sum_{i=1}^{k} \varepsilon^{n-1} |\tilde{f}^+(y_i) - \tilde{f}^-(y_i)| - C\sigma \mathcal{H}^{n-1}(U) \\
&= \frac{1}{2} \sum_{i=1}^{k} \mathcal{H}^{n-1}(Q_i^{n-1}) |\tilde{f}^+(z_i, \varphi(z_i)) - \tilde{f}^-(z_i, \varphi(z_i))| - C\sigma \mathcal{H}^{n-1}(U) \\
&\geq \frac{1}{2} \int_{\cup_{i=1}^k Q_i^{n-1}} |\tilde{f}^+(z, \varphi(z)) - \tilde{f}^-(z, \varphi(z))| \, d\mathcal{H}^{n-1}(z) - C\sigma \mathcal{H}^{n-1}(U),
\end{aligned}$$

where the last inequality follows from (3.25). Finally, recalling that $\|\nabla \varphi\|_{L^\infty} < \sigma$ and that $\|\tilde{f}^{\pm}\|_{L^\infty} \leq \|f\|_{L^\infty}$ we easily get that there exists a constant C depending only on n and on $\|f\|_{L^\infty}$

such that

$$\begin{aligned}\sum_{i=1}^{k}\varepsilon^{n-1}\fint_{Q_i}\left|f(x)-\fint_{Q_i}f\right|dx &\geq \frac{1}{2}\int_{\Gamma\cap\cup_{i=1}^{k}Q_i}|\tilde{f}^+(x)-\tilde{f}^-(x)|\,d\mathcal{H}^{n-1}(x)-C\sigma\mathcal{H}^{n-1}(U)\\ &\geq \frac{1}{2}\int_{K_1\cap\cup_{i=1}^{k}Q_i}|f^+(x)-f^-(x)|\,d\mathcal{H}^{n-1}(x)-C\sigma\mathcal{H}^{n-1}(U)\\ &\geq \frac{1}{2}\int_{K_1}|f^+(x)-f^-(x)|\,d\mathcal{H}^{n-1}(x)-C\sigma\mathcal{H}^{n-1}(K_1)\end{aligned}$$

where the last inequality follows immediately from (3.24) and from the inequality $\mathcal{H}^{n-1}(U\backslash\pi(K_1))\leq\sigma\mathcal{H}^{n-1}(K_1)$. Therefore, we finally get

$$\sum_{i=1}^{k}\varepsilon^{n-1}\fint_{Q_i}\left|f(x)-\fint_{Q_i}f\right|dx\geq\frac{1}{2}|D^sf|(K_1)-C\sigma\mathcal{H}^{n-1}(K_1)\,.$$

Then, we repeat for all the K_j, $j=2,\dots,m$ the construction made for K_1, thus getting for each j finitely many cubes $Q_{j,1},\dots Q_{j,k_j}$ of side ε such that

$$\sum_{i=1}^{k_j}\varepsilon^{n-1}\fint_{Q_{j,i}}\left|f(x)-\fint_{Q_{j,i}}f\right|dx\geq\frac{1}{2}|D^sf|(K_j)-C\sigma\mathcal{H}^{n-1}(K_j)\,.$$

It is clear that since we have only finitely many K_j we can choose the same side ε for all j and moreover this ε can be taken so small that any two cubes constructed for two different sets K_j do not intersect each other. Finally, we set $\mathcal{G}''_\varepsilon:=\{Q_1,\dots,Q_k\}\cup\cup_{j=2}^{m}\cup_{i=1}^{k_j}Q_{j,i}$ and from the previous two inequalities we obtain

$$\sum_{Q'\in\mathcal{G}''_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\left|f(x)-\fint_{Q'}f\right|dx\geq\frac{1}{2}|D^sf|(K)-C\sigma\mathcal{H}^{n-1}(K)\geq\frac{1}{2}|D^sf|(K)-C\sigma L(\eta)\,,$$

where C depends only on n and on $\|f\|_{L^\infty}$. Therefore, choosing σ sufficiently small, hence taking ε accordingly small, and recalling (3.21) and (3.23) we conclude that

$$\sum_{Q'\in\mathcal{G}''_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\left|f(x)-\fint_{Q'}f\right|dx\geq\frac{1}{2}|D^sf|(J_f)-3\eta\,.$$

Step 4. *Conclusion.*
From the last inequality and (3.20) we deduce that given $\eta>0$ and $N\geq 1$, for ε sufficiently small, we have, setting $\mathcal{G}_\varepsilon=\mathcal{G}'_\varepsilon\cup\mathcal{G}''_\varepsilon$

$$\kappa_\varepsilon(f;\Omega)\geq\sum_{Q'\in\mathcal{G}_\varepsilon}\varepsilon^{n-1}\fint_{Q'}\left|f(x)-\fint_{Q'}f\right|dx\geq\frac{1}{4}\int_{\Omega_\delta}|\nabla f|\,dx+\frac{1}{2}|D^sf|(\Omega)-\frac{C}{N}-3\eta\,.$$

The result then follows letting first $\varepsilon\to 0$, then $N\to\infty$ and $\eta\to 0$ and finally letting $\delta\to 0$. Note that if $\nabla f\equiv 0$, as in the proof of Proposition 3.4 we may replace $I_\delta(\bar{J}_f)$ by the whole $\mathbb{R}^n$ and the argue as in the Steps 2 and 3 without changes. □

Remark 3.7. From the proofs of Propositions 3.4 and 3.5 it is clear that Theorem 3.3 holds also locally. Namely, if $f\in SBV_{loc}(\Omega)$ and either $\nabla f\equiv 0$ in Ω or $|\bar{J}_f\cap\Omega|=0$ then

$$\lim_{\varepsilon\to 0^+}\kappa_\varepsilon(f;\Omega)=\frac{1}{4}\int_\Omega|\nabla f|\,dx+\frac{1}{2}|D^sf|(\Omega)\,,$$

where $\kappa_\varepsilon(f;\Omega)$ is defined as in (3.13).

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