SHARP GEOMETRIC AND FUNCTIONAL INEQUALITIES IN METRIC MEASURE SPACES WITH LOWER RICCI CURVATURE BOUNDS

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Abstract. For metric measure spaces satisfying the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ we prove a series of sharp functional inequalities under the additional assumption of essentially non-branching. Examples of spaces entering this framework are (weighted) Riemannian manifolds satisfying lower Ricci curvature bounds and their measured Gromov Hausdorff limits, Alexandrov spaces satisfying lower curvature bounds and more generally $\mathsf{RCD}^*(K,N)$-spaces, Finsler manifolds endowed with a strongly convex norm and satisfying lower Ricci curvature bounds.

In particular we prove the Brunn-Minkowski inequality, the p-spectral gap (or equivalently the p-Poincaré inequality) for any $p \in [1,\infty)$, the log-Sobolev inequality, the Talagrand inequality and finally the Sobolev inequality.

All the results are proved in a sharp form involving an upper bound on the diameter of the space; all our inequalities for essentially non-branching $\mathsf{CD}^*(K,N)$ spaces take the same form as the corresponding sharp ones known for a weighted Riemannian manifold satisfying the curvature-dimension condition $\mathsf{CD}(K,N)$ in the sense of Bakry-Émery. In this sense inequalities are sharp. We also discuss the rigidity and almost rigidity statements associated to the p-spectral gap.

Finally let us mention that for essentially non-branching metric measure spaces, the local curvature-dimension condition $\mathsf{CD}_{loc}(K,N)$ is equivalent to the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$. Therefore we also have shown that the sharp Brunn-Minkowski inequality in the *global* form can be deduced from the *local* curvature-dimension condition, providing a step towards (the long-standing problem of) globalization for the curvature-dimension condition $\mathsf{CD}(K,N)$.

To conclude, some of the results can be seen as answers to open problems proposed in the Optimal Transport book of Villani [75].

1. Introduction

The theory of metric measure spaces satisfying a synthetic version of lower curvature and upper dimension bounds is nowadays a rich and well-established theory; nevertheless some important functional and geometric inequalities are in some cases still not proven and in others not proven in a sharp form. The scope of this note is to generalize several functional inequalities known for Riemannian manifolds satisfying a lower bound on the Ricci curvature to the more general case of metric measure spaces satisfying the so-called curvature-dimension condition $\mathsf{CD}(K,N)$ as defined by Lott-Villani [51] and Sturm [72, 73]. More precisely our results will hold under the *reduced* curvature dimension condition $\mathsf{CD}^*(K,N)$ introduced by Bacher-Sturm [7] (which is, a priori, a weaker assumption than the classic $\mathsf{CD}(K,N)$) coupled with an essentially non-branching assumption on geodesics. We refer to Section 2.1 for the precise definitions; here let us recall that remarkable examples of essentially non-branching $\mathsf{CD}^*(K,N)$ spaces are (weighted) Riemannian manifolds satisfying lower Ricci curvature bounds and their measured Gromov Hausdorff limits, Alexandrov spaces satisfying lower curvature bounds and more generally $\mathsf{RCD}^*(K,N)$-spaces, Finsler manifolds endowed with a strongly convex norm and satisfying lower Ricci curvature bounds.

Remark 1.1. To avoid technicalities in the introduction, all the results will be stated for $N > 1$; nevertheless everything holds (and will be proved in the paper) also for $N = 1$, but in this case $\mathsf{CD}^*(K,N)$ has to be replaced by $\mathsf{CD}_{loc}(K,N)$. The two conditions are equivalent for $N > 1$ and for $N = 1, K \geq 0$, but in case $N = 1, K < 0$ the $\mathsf{CD}_{loc}(K,N)$ condition is strictly stronger (see Section 2.1 for more details).

Before committing a paragraph to each of the functional inequalities we will consider in this note, we underline that most of the proofs contained in this note are based on L^1 optimal transportation theory and in particular on one-dimensional localization. This technique, having its roots in a work

Key words and phrases. optimal transport; Ricci curvature lower bounds; metric measure spaces; Brunn-Minkowski inequality; log-Sobolev inequality; spectral gap; Sobolev inequality; Talagrand inequality.

of Payne-Weinberger [64] and developed by Gromov-Milman [38], Lovász-Simonovits [53] and Kannan-Lovász-Simonovits [42], consists in reducing an n-dimensional problem to a one dimensional one via tools of convex geometry. Recently Klartag [45] found an L^1-optimal transportation approach leading to a generalization of these ideas to Riemannian manifolds; the authors [17], via a careful analysis avoiding any smoothness assumption, generalized this approach to metric measure spaces.

It is also convenient to introduce here the family of one-dimensional measures that will be used several times for comparison:

$$\mathcal{F}_{K,N,D} := \left\{\mu \in \mathcal{P}(\mathbb{R})\colon \mathrm{supp}(\mu) \subset [0,D],\, \mu = h_\mu \cdot \mathcal{L}^1,\, h_\mu \in C^2((0,D)),\, (\mathbb{R}, |\cdot|, \mu) \in \mathsf{CD}(K,N)\right\},$$

where $(\mathbb{R}, |\cdot|, \mu) \in \mathsf{CD}(K,N)$ stands for: the metric measure space $(\mathbb{R}, |\cdot|, \mu)$ verifies $\mathsf{CD}(K,N)$ or equivalently

$$\left(h_\mu^{\frac{1}{N-1}}\right)'' + \frac{K}{N-1}\, h_\mu^{\frac{1}{N-1}} \leq 0.$$

1.1. **Brunn-Minkowski inequality.** The celebrated Brunn-Minkowski inequality estimates from below the measure of the t-intermediate points between two given subsets A_0 and A_1 of X, for $t \in [0,1]$. For metric measure spaces satisfying the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ (see Section 2.1 for a brief account of different versions of the curvature-dimension condition) almost by definition for any $A_0, A_1 \subset X$

$$\mathfrak{m}(A_t)^{1/N} \geq \sigma_{K,N}^{(1-t)}(\theta)\, \mathfrak{m}(A_0)^{1/N} + \sigma_{K,N}^{(t)}(\theta)\, \mathfrak{m}(A_1)^{1/N}, \tag{1.1}$$

where A_t is the set of t-intermediate points between A_0 and A_1, that is

$$A_t = \mathrm{e}_t\Big(\{\gamma \in \mathrm{Geo}(X)\colon \gamma_0 \in A_0, \gamma_1 \in A_1\}\Big),$$

(see Section 2 for the definition of e) θ is the minimal/maximal length of geodesics from A_0 to A_1:

$$\theta := \begin{cases} \inf_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K \geq 0,\\ \sup_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K < 0,\end{cases}$$

and $\sigma_{K,N}^{(t)}(\theta)$ is defined in (2.3). Nevertheless (1.1) is not sharp. Indeed if $(X,\mathsf{d},\mathfrak{m})$ is a weighted Riemannian manifold satisfying $\mathsf{CD}^*(K,N)$, then (1.1) holds but with better interpolation coefficients, that is with $\tau_{K,N}^{(t)}(\theta), \tau_{K,N}^{(1-t)}(\theta)$ replacing $\sigma_{K,N}^{(t)}(\theta)$ and $\sigma_{K,N}^{(1-t)}(\theta)$, respectively. Indeed for a weighted Riemannian manifold the two (a priori) different definitions of $\mathsf{CD}^*(K,N)$ and $\mathsf{CD}(K,N)$ coincide and then again almost by definition [73] one can obtain the improved (and sharp) Brunn-Minkowski inequality (let us mention that a direct proof of the Brunn-Minkowski inequality in the smooth setting was done earlier by Cordero-Erausquin, McCann and Schmuckenschläger [27]).

A first main result of this paper is to establish the sharp inequality for essentially non-branching $\mathsf{CD}^*(K,N)$ metric measure spaces.

Theorem 1.2 (Theorem 3.1)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ with $\mathfrak{m}(X) < \infty$ verify $\mathsf{CD}^*(K,N)$ for some $K,N \in \mathbb{R}$ and $N \in (1,\infty)$. Assume moreover $(X,\mathsf{d},\mathfrak{m})$ to be essentially non-branching. Then it satisfies the following sharp Brunn-Minkowski inequality:*

for any $A_0, A_1 \subset X$

$$\mathfrak{m}(A_t)^{1/N} \geq \tau_{K,N}^{(1-t)}(\theta)\, \mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\, \mathfrak{m}(A_1)^{1/N},$$

where A_t is the set of t-intermediate points between A_0 and A_1 and θ the minimal/maximal length of geodesics from A_0 to A_1.

Remark 1.3. The remarkable feature of Theorem 1.2 is that the sharp Brunn-Minkowski inequality in the *global* form can be deduced from the *local* curvature-dimension condition, providing a step towards (the long-standing problem of) globalization for the curvature-dimension condition $\mathsf{CD}(K,N)$. For an account and for partial results about this problem we refer to [6, 7, 16, 18, 68, 75].

1.2. p**-Spectral gap.** In the smooth setting, a spectral gap inequality establishes a bound from below on the first eigenvalue of the Laplacian. More generally, for any $p \in (1,\infty)$ one can define the positive real number $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})}$ as follows

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} := \inf \left\{ \frac{\int_X |\nabla f|^p \,\mathfrak{m}}{\int_X |f|^p \,\mathfrak{m}} : f \in \mathrm{Lip}(X) \cap L^p(X,\mathfrak{m}),\ f \neq 0,\ \int_X f|f|^{p-2}\,\mathfrak{m} = 0 \right\},$$

where $|\nabla f|$ is the slope (also called local Lipshitz constant) of the Lipschitz function f. The name is motivated by the fact that in case $(X,\mathsf{d},\mathfrak{m})$ is the m.m.s corresponding to a smooth compact Riemannian manifold then $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})}$ coincides with the first positive eigenvalue of the problem

$$\Delta_p f = \lambda |f|^{p-2} f,$$

on $(X,\mathsf{d},\mathfrak{m})$, where $\Delta_p f := -\mathrm{div}(|\nabla f|^{p-2}\nabla f)$ is the so called p-Laplacian.

We now state the main theorem of this paper on p-spectral gap inequality.

Theorem 1.4 (Theorem 4.4)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space satisfying $\mathsf{CD}^*(K,N)$ for some $K,N \in \mathbb{R}$ and $N \in (1,\infty)$ and assume moreover it is essentially non-branching. Let $D \in (0,\infty)$ be the diameter of X.*

Then for any $p \in (1,\infty)$ it holds

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \;\geq\; \lambda^{1,p}_{K,N,D},$$

where $\lambda^{1,p}_{K,N,D}$ is defined by

$$\lambda^{1,p}_{K,N,D} := \inf_{\mu \in \mathcal{F}_{K,N,D}} \inf \left\{ \frac{\int_{\mathbb{R}} |u'|^p \,\mu}{\int_{\mathbb{R}} |u|^p \,\mu} : u \in \mathrm{Lip}(\mathbb{R}) \cap L^p(\mu),\ \int_{\mathbb{R}} u|u|^{p-2}\mu = 0,\ u \neq 0 \right\}.$$

In other terms for any Lipschitz function $f \in L^p(X,\mathfrak{m})$ with $\int_X f|f|^{p-2}\,\mathfrak{m}(dx) = 0$ it holds

$$\lambda^{1,p}_{K,N,D} \int_X |f(x)|^p \,\mathfrak{m}(dx) \leq \int_X |\nabla f|^p(x)\,\mathfrak{m}(dx).$$

For more about the quantity $\lambda^{1,p}_{K,N,D}$ the reader is referred to Section 4.1 where the model spaces are discussed in detail. From the last formulation of the statement, it is clear that the sharp p-spectral gap above is equivalent to a sharp p-Poincaré inequality.

Let us now give a brief (and incomplete) account on the huge literature about the spectral gap. When the ambient metric measure space is a smooth Riemannian manifold equipped with the volume measure, the study of the first eigenvalue of the Laplace-Beltrami operator has a long history going back to Lichnerowicz [50], Cheeger [21], Li-Yau [49], etc. For an overview the reader can consult for instance the book by Chavel [20], the survey by Ledoux [47], or Chapter 3 in Shoen-Yau's book [71], and references therein.

We mention that the estimate of Theorem 1.4 in the case $p = 2$ started with Payne-Weinberger [64] for convex domains in $\mathbb{R}^n$ where diameter-improved spectral gap inequality for the Laplace operator was originally proved. Later it was generalized to Riemannian manifolds with non-negative Ricci curvature by Yang-Zhong [77], and by Bakry-Qian [9] for manifolds with densities. The generalization to arbitrary $p \in (1,\infty)$ has been proved by Valtorta [74] for $K = 0$ and Naber-Valtorta [61] for any $K \in \mathbb{R}$. All of these results hold for Riemannian manifolds.

Regarding metric measure spaces, the sharp Lichnerowitz spectral gap for $p = 2$ was proved by Lott-Villani [52] under the $\mathsf{CD}(K,N)$ condition. Jiang-Zhang [41] recently showed, still for $p = 2$, that the improved version under an upper diameter bound holds for $\mathsf{RCD}^*(K,N)$ metric measure spaces. For Ricci limit spaces, in the case $K > 0$ and $D = \pi\sqrt{(N-1)/K}$, the p-spectral gap above has been recently obtained by Honda [40] via proving the stability of $\lambda^{1,p}$ under mGH convergence of compact Riemannian manifolds; this approach was inspired by the celebrated work of Cheeger-Colding [25] where, in particular, it was shown the stability of $\lambda^{1,2}$ under mGH convergence. We also obtain the *almost rigidity* for the p-spectral gap: if an almost equality in the p-spectral gap holds, then the space must have almost maximal diameter.

Theorem 1.5 (Theorem 4.5)**.** *Let $N > 1$, and $p \in (1,\infty)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,N,p)$ such that the following holds.*

Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space satisfying $\mathsf{CD}^*(N-1-\delta, N+\delta)$. If $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta$, then $\mathrm{diam}\,(X) \geq \pi - \varepsilon$.

As a consequence, by a compactness argument and using the Maximal Diameter Theorem proved recently for $\mathsf{RCD}^*(K, N)$ by Ketterer [43], we have the following p-Obata and almost p-Obata Theorems.

Corollary 1.6 (p-Obata Theorem)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(N-1, N)$ space for some $N \geq 2$, and let $1 < p < \infty$. If*

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} = \lambda^{1,p}_{N-1,N,\pi} (= \lambda^{1,p}(S^N)),$$

then $(X, \mathsf{d}, \mathfrak{m})$ is a spherical suspension, i.e. there exists an $\mathsf{RCD}^*(N-2, N-1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ such that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to $[0, \pi] \times^{N-1}_{\sin} Y$.

Corollary 1.7 (Almost p-Obata Theorem)**.** *Let $N \geq 2$, and $p \in (1, \infty)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N, p) > 0$ such that the following holds.*

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(N-1-\delta, N+\delta)$ space. If

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta,$$

then there exists an $\mathsf{RCD}^*(N-2, N-1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ such that

$$\mathsf{d}_{mGH}\left((X, \mathsf{d}, \mathfrak{m}), [0, \pi] \times^{N-1}_{\sin} Y\right) \leq \varepsilon.$$

Let us mention that the classical Obata's Theorem for $\mathsf{RCD}^*(K, N)$-spaces, i.e. the version of Corollary 1.6 for $p = 2$, was recently obtained by Ketterer [44] (see also [41]) with different methods.

Finally we recall that the case $p = 1$ can be attacked using the identity $h_{(X,\mathsf{d},\mathfrak{m})} = \lambda^{1,1}_{(X,\mathsf{d},\mathfrak{m})}$, where $h_{(X,\mathsf{d},\mathfrak{m})}$ is the so-called Cheeger isoperimetric constant, see Section 5.1. Therefore Theorem 1.4, Theorem 1.5, Corollary 1.6 and Corollary 1.7 for the case $p = 1$ follow from the analogous results proved for the isoperimetric profile in [17]. Nevertheless for reader's convenience, the case $p = 1$ will be discussed in detail in Section 5.

1.3. **Log-Sobolev and Talagrand inequality.** Given a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, we say that it supports the Log-Sobolev inequality with constant $\alpha > 0$ if for any Lipschitz function $f : X \to [0, \infty)$ with $\int_X f(x)\, \mathfrak{m}(dx) = 1$ it holds

$$2\alpha \int_X f \log f \, \mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \mathfrak{m}. \tag{1.2}$$

The largest constant α, such that (1.2) holds for any Lipschitz function $f : X \to [0, \infty)$ with $\int_X f(x)\, \mathfrak{m}(dx) = 1$, will be called Log-Sobolev constant of $(X, \mathsf{d}, \mathfrak{m})$ and denoted with $\alpha^{LS}_{(X,\mathsf{d},\mathfrak{m})}$.

Log-Sobolev inequality is already known [75, Theorem 30.22] for essentially non-branching metric measure spaces satisfying $\mathsf{CD}(K, \infty)$ with $K > 0$ with sharp constant $\alpha = K$, but it is an open problem (see for instance [75, Open Problem 21.6]) to get the sharp dimensional constant $\alpha_{K,N} = \frac{KN}{N-1}$ for metric measure spaces with N-Ricci curvature bounded below by K. This is the goal of the next result.

As already done above, let us introduce the model constant for the one-dimensional case. Given $K \in \mathbb{R}$, $N \geq 1$, $D \in (0, +\infty)$ we denote with $\alpha^{LS}_{K,N,D} > 0$ the maximal constant α such that

$$2\,\alpha \int_{\mathbb{R}} f \log f \, \mu \leq \int_{\{f>0\}} \frac{|f'|^2}{f} \mu, \quad \forall \mu \in \mathcal{F}_{K,N,D}, \tag{1.3}$$

for every Lipschitz $f : \mathbb{R} \to [0, \infty)$ with $\int f \, \mu = 1$.

Remark 1.8. If $K > 0$ and $D = \pi\sqrt{\frac{N-1}{K}}$, it is known that the corresponding optimal Log-Sobolev constant is $\frac{KN}{N-1}$ (for more details see the discussion in Section 6.1). It is an interesting open problem, that we don't address here, to give an explicit expression of the quantity $\alpha^{LS}_{K,N,D}$ for general $K \in \mathbb{R}, N \geq 1$, $D \in (0, \infty)$.

Theorem 1.9 (Sharp Log-Sobolev inequality, Theorem 6.2)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space with diameter $D \in (0, \infty)$ and satisfying $\mathsf{CD}^*(K, N)$ for some $K \in \mathbb{R}, N \in (1, \infty)$. Assume moreover it is essentially non-branching.*

Then for any Lipschitz function $f : X \to [0,\infty)$ *with* $\int_X f\,\mathfrak{m} = 1$ *it holds*

$$2\,\alpha^{LS}_{K,N,D} \int_X f \log f\,\mathfrak{m} \le \int_{\{f>0\}} \frac{|\nabla f|^2}{f}\mathfrak{m}.$$

In other terms it holds $\alpha^{LS}_{(X,\mathsf{d},\mathfrak{m})} \ge \alpha^{LS}_{K,N,D}$.

As a consequence, if $K > 0$ *and no diameter upper bound is assumed or* $D = \pi\sqrt{\frac{N-1}{K}}$, *then* $\alpha^{LS}_{K,N} = \frac{KN}{N-1}$ *i.e. for any Lipschitz function* $f : X \to [0,\infty)$ *with* $\int_X f\,\mathfrak{m} = 1$ *it holds*

$$\frac{2KN}{N-1} \int_X f \log f\,\mathfrak{m} \le \int_{\{f>0\}} \frac{|\nabla f|^2}{f}\mathfrak{m}.$$

In order to state the Talagrand inequality let us recall that the relative entropy functional $Ent_{\mathfrak{m}} : \mathcal{P}(X) \to [0,+\infty]$ with respect to a given $\mathfrak{m} \in \mathcal{P}(X)$ is defined to be

$$Ent_{\mathfrak{m}}(\mu) = \int_X \varrho \log \varrho\,\mathfrak{m}, \quad \text{ if } \mu = \varrho\mathfrak{m}$$

and $+\infty$ otherwise. Otto-Villani [62] proved that for smooth Riemannian manifolds the Log-Sobolev inequality with constant $\alpha > 0$ implies the Talagrand inequality with constant $\frac{2}{\alpha}$ preserving sharpness. The result was then generalized to arbitrary metric measure spaces by Gigli-Ledoux [34].

Combining this result with Theorem 1.9 we get the following corollary which improves the Talagrand constant $2/K$, which is sharp for $\mathsf{CD}(K,\infty)$ spaces, by a factor $N-1/N$ in case the dimension is bounded above by N. This constant is sharp for $\mathsf{CD}^*(K,N)$ (or $\mathsf{CD}_{loc}(K,N)$) spaces, indeed it is sharp already in the smooth setting [75, Remark 22.43]. Since both our proof of the sharp Log-Sobolev inequality and the proof of Theorem 6.4 are essentially optimal transport based, the following can be seen as an answer to [75, Open Problem 22.44].

Theorem 1.10 (Sharp Talagrand inequality)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be a metric measure space with diameter* $D \in (0,\infty)$, *satisfying* $\mathsf{CD}^*(K,N)$ *for some* $K \in \mathbb{R}, N \in (1,\infty)$, *and assume moreover it is essentially non-branching and* $\mathfrak{m}(X) = 1$.

Then it supports the Talagrand inequality with constant $\frac{2}{\alpha^{LS}_{K,N,D}}$, *where* $\alpha^{LS}_{K,N,D}$ *was defined in* (1.3), *i.e. it holds*

$$W_2^2(\mu,\mathfrak{m}) \le \frac{2}{\alpha^{LS}_{K,N,D}} Ent_{\mathfrak{m}}(\mu) \quad \textit{for all } \mu \in \mathcal{P}(X).$$

In particular, if $K > 0$ *and no upper bound on the diameter is assumed or* $D = \pi\sqrt{\frac{N-1}{K}}$, *then*

$$W_2^2(\mu,\mathfrak{m}) \le \frac{2(N-1)}{KN} Ent_{\mathfrak{m}}(\mu) \quad \textit{for all } \mu \in \mathcal{P}(X),$$

the constant in the last inequality being sharp.

1.4. **Sobolev inequality.** Sobolev inequalities have been studied in many different contexts and many papers and books are devoted to this family of inequalities. Here we only mention two references mainly dealing with them in the Riemannian manifold case and the smooth CD condition case, respectively [39] and [46].

We say that $(X,\mathsf{d},\mathfrak{m})$ supports a (p,q)-Sobolev inequality with constant $\alpha^{p,q}$ if for any $f : X \to \mathbb{R}$ Lipschitz function it holds

$$\frac{\alpha^{p,q}}{p-q}\left\{\left(\int_X |f|^p\,\mathfrak{m}\right)^{\frac{q}{p}} - \int_X |f|^q\,\mathfrak{m}\right\} \le \int_X |\nabla f|^q\,\mathfrak{m}, \tag{1.4}$$

and the largest constant $\alpha^{p,q}$ such that (1.4) holds for any Lipschitz function f will be called the (p,q)-Sobolev constant of $(X,\mathsf{d},\mathfrak{m})$and will be denoted by $\alpha^{p,q}_{(X,\mathsf{d},\mathfrak{m})}$.

A Sobolev inequality is known to hold for essentially non-branching m.m.s. satisfying $\mathsf{CD}(K,N)$, provided $K < 0$, see [75, Theorem 30.23] and other Sobolev-type inequalities have been obtained in [52] for $\mathsf{CD}(K,N)$ spaces. Let us also mention [66] where the sharp $(2^*,2)$-Sobolev inequality has been established for $\mathsf{RCD}^*(K,N)$-spaces, $K > 0$, $N \in (2,\infty)$. The goal here is to give a Sobolev inequality with sharp constant for essentially non-branching $\mathsf{CD}^*(K,N)$ spaces, $K \in \mathbb{R}, N > 1$, taking also into account an upper diameter bound.

Theorem 1.11 (Sharp Sobolev inequality, Theorem 7.1)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space with diameter $D \in (0, \infty)$ and satisfying $\mathsf{CD}^*(K, N)$ for some $K \in \mathbb{R}, N \in (1, \infty)$. Assume moreover it is essentially non-branching.*

Then for any Lipschitz function it holds

$$\frac{\alpha^{p,q}_{K,N,D}}{p-q}\left\{\left(\int_X |f(x)|^p\,\mathfrak{m}(dx)\right)^{\frac{q}{p}} - \int_X |f(x)|^q\,\mathfrak{m}(dx)\right\} \leq \int_X |\nabla f(x)|^q\,\mathfrak{m}(dx),$$

where $\alpha^{p,q}_{K,N,D}$ is defined as the supremum among $\alpha > 0$ such that

$$\frac{\alpha}{p-q}\left\{\left(\int_X |f|^p\,\mu\right)^{\frac{q}{p}} - \int_X |f|^q\,\mu\right\} \leq \int_X |\nabla f|^q\,\mu, \ \forall\ f \in \mathrm{Lip},\ \forall\ \mu \in \mathcal{F}_{K,N,D}.$$

In particular, if $K > 0, N > 2$ and no upper bound on the diameter is assumed or $D = \pi\sqrt{\frac{N-1}{K}}$, then for any Lipschitz function f it holds

$$\frac{KN}{(p-2)(N-1)}\left\{\left(\int_X |f|^p\,\mathfrak{m}\right)^{\frac{2}{p}} - \int_X |f|^2\,\mathfrak{m}\right\} \leq \int_X |\nabla f|^2\,\mathfrak{m},$$

for any $2 < p \leq 2N/(N-2)$; in other terms it holds $\alpha^{p,2}_{(X,\mathsf{d},\mathfrak{m})} \geq \frac{KN}{N-1}$.

This last result can be seen as a solution to [75, Open Problem 21.11].

Acknowledgements

The authors wish to thank the Hausdorff center of Mathematics of Bonn, where part of the work has been developed, for the excellent working conditions and the stimulating atmosphere during the trimester program "Optimal Transport". A.M. is partly supported by the Swiss National Science Foundation.

2. Prerequisites

In what follows we say that a triple $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space, m.m.s. for short, if (X, d) is a complete and separable metric space and $\mathfrak{m}$ is positive Radon measure over X. For this note we will only be concerned with m.m.s. with $\mathfrak{m}$ probability measure, that is $\mathfrak{m}(X) = 1$, or at most with $\mathfrak{m}(X) < \infty$ which will be reduced to the probability case by a constant rescaling. The space of all Borel probability measure over X will be denoted with $\mathcal{P}(X)$.

A metric space is a geodesic space if and only if for each $x, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ so that $\gamma_0 = x, \gamma_1 = y$, with

$$\mathrm{Geo}(X) := \{\gamma \in C([0, 1], X) : \mathsf{d}(\gamma_s, \gamma_t) = (s - t)\mathsf{d}(\gamma_0, \gamma_1), s, t \in [0, 1]\}.$$

Recall that for complete geodesic spaces local compactness is equivalent to properness (a metric space is proper if every closed ball is compact). We directly assume the ambient space (X, d) to be proper. Hence from now on we assume the following: the ambient metric space (X, d) is geodesic, complete, separable and proper and $\mathfrak{m}(X) = 1$.

We denote with $\mathcal{P}_2(X)$ the space of probability measures with finite second moment endowed with the L^2-Wasserstein distance W_2 defined as follows: for $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ we set

$$W_2^2(\mu_0, \mu_1) = \inf_{\pi} \int_X \mathsf{d}^2(x, y)\,\pi(dxdy), \tag{2.1}$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal. Assuming the space (X, d) to be geodesic, also the space $(\mathcal{P}_2(X), W_2)$ is geodesic.

Any geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}_2(X), W_2)$ can be lifted to a measure $\nu \in \mathcal{P}(\mathrm{Geo}(X))$, so that $(\mathrm{e}_t)\,\sharp\,\nu = \mu_t$ for all $t \in [0, 1]$. Here for any $t \in [0, 1]$, e_t denotes the evaluation map:

$$\mathrm{e}_t : \mathrm{Geo}(X) \to X, \qquad \mathrm{e}_t(\gamma) := \gamma_t.$$

Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ for which $(\mathrm{e}_0, \mathrm{e}_1)\,\sharp\,\nu$ realizes the minimum in (2.1). If (X, d) is geodesic, then the set $\mathrm{OptGeo}(\mu_0, \mu_1)$ is non-empty for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$. It is worth also introducing the subspace of $\mathcal{P}_2(X)$ formed by all those measures absolutely continuous with respect to $\mathfrak{m}$: it is denoted by $\mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$.

2.1. **Geometry of metric measure spaces.** Here we briefly recall the synthetic notions of lower Ricci curvature bounds, for more detail we refer to [7, 51, 72, 73, 75].

In order to formulate curvature properties for $(X,\mathsf{d},\mathfrak{m})$ we introduce the following distortion coefficients: given two numbers $K,N\in\mathbb{R}$ with $N\geq 1$, we set for $(t,\theta)\in[0,1]\times\mathbb{R}_+$,

$$\sigma_{K,N}^{(t)}(\theta):=\begin{cases}\infty, & \text{if } K\theta^2\geq N\pi^2,\\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0<K\theta^2<N\pi^2,\\ t & \text{if } K\theta^2<0 \text{ and } N=0, \text{ or if } K\theta^2=0,\\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2\leq 0 \text{ and } N>0.\end{cases} \tag{2.2}$$

We also set, for $N\geq 1, K\in\mathbb{R}$ and $(t,\theta)\in[0,1]\times\mathbb{R}_+$

$$\tau_{K,N}^{(t)}(\theta):=t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}. \tag{2.3}$$

As we will consider only the case of essentially non-branching spaces, we recall the following definition.

Definition 2.1. A metric measure space $(X,\mathsf{d},\mathfrak{m})$ is *essentially non-branching* if and only if for any $\mu_0,\mu_1\in\mathcal{P}_2(X)$ which are absolutely continuous with respect to $\mathfrak{m}$ any element of $\mathrm{OptGeo}(\mu_0,\mu_1)$ is concentrated on a set of non-branching geodesics.

A set $F\subset\mathrm{Geo}(X)$ is a set of non-branching geodesics if and only if for any $\gamma^1,\gamma^2\in F$, it holds:

$$\exists\,\bar t\in(0,1):\ \gamma_t^1=\gamma_t^2,\ \forall\, t\in(0,\bar t)\quad\Longrightarrow\quad \gamma_s^1=\gamma_s^2,\quad\forall s\in[0,1].$$

Definition 2.2 (CD condition)**.** An essentially non-branching m.m.s. $(X,\mathsf{d},\mathfrak{m})$ verifies $\mathsf{CD}(K,N)$ if and only if for each pair $\mu_0,\mu_1\in\mathcal{P}_2(X,\mathsf{d},\mathfrak{m})$ there exists $\nu\in\mathrm{OptGeo}(\mu_0,\mu_1)$ such that

$$\varrho_t^{-1/N}(\gamma_t)\geq\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_0^{-1/N}(\gamma_0)+\tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_1^{-1/N}(\gamma_1),\qquad \nu\text{-a.e. } \gamma\in\mathrm{Geo}(X), \tag{2.4}$$

for all $t\in[0,1]$, where $\mathrm{e}_t\,\sharp\,\nu=\varrho_t\mathfrak{m}$.

For the general definition of $\mathsf{CD}(K,N)$ see [51, 72, 73]. It is worth recalling that if (M,g) is a Riemannian manifold of dimension n and $h\in C^2(M)$ with $h>0$, then the m.m.s. $(M,g,h\,vol)$ verifies $\mathsf{CD}(K,N)$ with $N\geq n$ if and only if (see Theorem 1.7 of [73])

$$Ric_{g,h,N}\geq Kg,\qquad Ric_{g,h,N}:=Ric_g-(N-n)\frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}}.$$

In particular if $N=n$ the generalized Ricci tensor $Ric_{g,h,N}=Ric_g$ makes sense only if h is constant. In particular, if $I\subset\mathbb{R}$ is any interval, $h\in C^2(I)$ and $\mathcal{L}^1$ is the one-dimensional Lebesgue measure, the m.m.s. $(I,|\cdot|,h\mathcal{L}^1)$ verifies $\mathsf{CD}(K,N)$ if and only if

$$\left(h^{\frac{1}{N-1}}\right)''+\frac{K}{N-1}h^{\frac{1}{N-1}}\leq 0, \tag{2.5}$$

and verifies $\mathsf{CD}(K,1)$ if and only if h is constant.

We also mention the more recent Riemannian curvature dimension condition RCD^* introduced in the infinite dimensional case in [4, 2, 1] and in the finite dimensional case in [28, 5]. We refer to these papers and references therein for a general account on the synthetic formulation of Ricci curvature lower bounds for metric measure spaces. Here we only mention that $\mathsf{RCD}^*(K,N)$ condition is an enforcement of the so called reduced curvature dimension condition, denoted by $\mathsf{CD}^*(K,N)$, that has been introduced in [7]: in particular the additional condition is that the Sobolev space $W^{1,2}(X,\mathfrak{m})$ is an Hilbert space, see [3, 4].

The reduced $\mathsf{CD}^*(K,N)$ condition asks for the same inequality (2.4) of $\mathsf{CD}(K,N)$ but the coefficients $\tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))$ are replaced by $\sigma_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\sigma_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))$, respectively.

Hence while the distortion coefficients of the $\mathsf{CD}(K,N)$ condition are formally obtained imposing one direction with linear distortion and $N-1$ directions affected by curvature, the $\mathsf{CD}^*(K,N)$ condition imposes the same volume distortion in all the N directions.

It was proved in [69] that the $\mathsf{RCD}^*(K,N)$ condition implies the essentially non-branching property, so this is a fairly natural assumption in the framework of m.m.s. satisfying lower Ricci bounds.

For both CD-CD^* definitions there is a local version that is of some relevance for our analysis. Here we state only the local formulation $\mathsf{CD}(K,N)$, the one for $\mathsf{CD}^*(K,N)$ being similar.

Definition 2.3 (CD_{loc} condition)**.** An essentially non-branching m.m.s. $(X,\mathsf{d},\mathfrak{m})$ satisfies $\mathsf{CD}_{loc}(K,N)$ if for any point $x \in X$ there exists a neighborhood $X(x)$ of x such that for each pair $\mu_0,\mu_1 \in \mathcal{P}_2(X,\mathsf{d},\mathfrak{m})$ supported in $X(x)$ there exists $\nu \in \mathrm{OptGeo}(\mu_0,\mu_1)$ such that (2.4) holds true for all $t \in [0,1]$. The support of $\mathrm{e}_t\,\sharp\,\nu$ is not necessarily contained in the neighborhood $X(x)$.

One of the main properties of the reduced curvature dimension condition is the globalization one: under the essentially non-branching property, $\mathsf{CD}^*_{loc}(K,N)$ and $\mathsf{CD}^*(K,N)$ are equivalent (see [7, Corollary 5.4]). Let us mention that the local-to-global property is satisfied also by the $\mathsf{RCD}^*(K,N)$ condition, see [6].

We also recall few relations between CD and CD^*. It is known by [32, Theorem 2.7] that, if $(X,\mathsf{d},\mathfrak{m})$ is a non-branching metric measure space satisfying $\mathsf{CD}(K,N)$ and $\mu_0,\mu_1 \in \mathcal{P}(X)$ with μ_0 absolutely continuous with respect to $\mathfrak{m}$, then there exists a unique optimal map $T : X \to X$ such that $(id,T)\,\sharp\,\mu_0$ realizes the minimum in (2.1) and the set $\mathrm{OptGeo}(\mu_0,\mu_1)$ contains only one element. The same proof holds if one replaces the non-branching assumption with the more general one of essentially non-branching, see for instance [69].

Remark 2.4 ($\mathsf{CD}^*(K,N)$ Vs $\mathsf{CD}_{loc}(K,N)$)**.** Results of [7] imply the following chain of implications: if $(X,\mathsf{d},\mathfrak{m})$ is a proper, essentially non-branching, metric measure space, then

$$\mathsf{CD}_{loc}(K,N) \iff \mathsf{CD}^*_{loc}(K,N) \iff \mathsf{CD}^*(K,N),$$

provided $K,N \in \mathbb{R}$ with $N > 1$ or $N = 1$ and $K \geq 0$. Let us remark that on the other hand $\mathsf{CD}^*(K,1)$ does not imply $\mathsf{CD}_{loc}(K,1)$ for $K < 0$: indeed it is possible to check that $(X,\mathsf{d},\mathfrak{m}) = ([0,1],|\cdot|,c\sinh(\cdot)\mathcal{L}^1)$ satisfies $\mathsf{CD}^*(-1,1)$ but not $\mathsf{CD}_{loc}(-1,1)$ which would require the density to be constant. Hence $\mathsf{CD}^*(K,N)$ and $\mathsf{CD}_{loc}(K,N)$ are equivalent if $1 < N < \infty$ or $N = 1$ and $K \geq 0$, but for $N = 1$ and $K < 0$ the $\mathsf{CD}_{loc}(K,N)$ condition is strictly stronger than $\mathsf{CD}^*(K,N)$.

Note also that many results presented in [7] are for metric measure spaces verifying $\mathsf{CD}(K-,N)$ (and its local version), that is they verify the $\mathsf{CD}(K',N)$ condition for all $K' < K$. Thanks to uniqueness of geodesics in $(\mathcal{P}_2(X),W_2)$ guaranteed by the essentially non-branching assumption, $\mathsf{CD}(K-,N)$ is equivalent to $\mathsf{CD}(K,N)$.

As a final comment we also mention that, for $K > 0$, $\mathsf{CD}^*(K,N)$ implies $\mathsf{CD}(K^*,N)$ where $K^* = K(N-1)/N$. For a deeper analysis on the interplay between CD^* and CD we refer to [16, 18].

2.2. **Measured Gromov-Hausdorff convergence and stability of $\mathsf{RCD}^*(K,N)$.** Let us first recall the notion of measured Gromov-Hausdorff convergence, mGH for short. Since in this work we will apply it to compact m.m. spaces endowed with probability measures having full support, we will restrict to this framework for simplicity (for a more general treatment see for instance [35]).

Definition 2.5. A sequence $(X_j,\mathsf{d}_j,\mathfrak{m}_j)$ of compact m.m. spaces with $\mathfrak{m}_j(X_j) = 1$ and $\mathrm{supp}(\mathfrak{m}_j) = X_j$ is said to converge in the measured Gromov-Hausdorff topology (mGH for short) to a compact m.m. space $(X_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$ with $\mathfrak{m}_\infty(X) = 1$ and $\mathrm{supp}(\mathfrak{m}_\infty) = X_\infty$ if and only if there exists a separable metric space (Z,d_Z) and isometric embeddings $\{\iota_j : (X,\mathsf{d}_j) \to (Z,\mathsf{d}_Z)\}_{i\in\bar{\mathbb{N}}}$ such that for every $\varepsilon > 0$ there exists j_0 such that for every $j > j_0$

$$\iota_\infty(X_\infty) \subset B^Z_\varepsilon[\iota_j(X_j)] \qquad \text{and} \qquad \iota_j(X_j) \subset B^Z_\varepsilon[\iota_\infty(X_\infty)],$$

where $B^Z_\varepsilon[A] := \{z \in Z : \mathsf{d}_Z(z,A) < \varepsilon\}$ for every subset $A \subset Z$, and

$$\int_Z \varphi\left((\iota_j)_\sharp(\mathfrak{m}_j)\right) \qquad \to \qquad \int_Z \varphi\left((\iota_\infty)_\sharp(\mathfrak{m}_\infty)\right) \qquad \forall \varphi \in C_b(Z),$$

where $C_b(Z)$ denotes the set of real valued bounded continuous functions in Z.

The following theorem summarizes the compactness/stability properties we will use in the proof of the almost rigidity result (notice these hold more generally for every $K \in \mathbb{R}$ by replacing mGH with *pointed*-mGH convergence).

Theorem 2.6 (Metrizability and Compactness)**.** *Let $K > 0, N > 1$ be fixed. Then the mGH convergence restricted to (isomorphism classes of) $\mathsf{RCD}^*(K,N)$ spaces is metrizable by a distance function d_{mGH}. Furthermore every sequence $(X_j,\mathsf{d}_j,\mathfrak{m}_j)$ of $\mathsf{RCD}^*(K,N)$ spaces admits a subsequence which mGH-converges to a limit $\mathsf{RCD}^*(K,N)$ space.*

The compactness follows by the standard argument of Gromov, indeed for fixed $K>0, N>1$, the spaces have uniformly bounded diameter, moreover the measures of $\mathsf{RCD}^*(K,N)$ spaces are uniformly doubling, hence the spaces are uniformly totally bounded and thus compact in the GH-topology; the weak compactness of the measures follows using the doubling condition again and the fact that they are normalized. For the stability of the $\mathsf{RCD}^*(K,N)$ condition under mGH convergence see for instance [7, 28, 35]. The metrizability of mGH convergence restricted to a class of uniformly doubling normalized m.m. spaces having uniform diameter bounds is also well known, see for instance [35].

2.3. **Warped product.** Given two geodesic m.m.s. $(B,\mathsf{d}_B,\mathfrak{m}_b)$ and $(F,\mathsf{d}_F,\mathfrak{m}_F)$ and a Lipschitz function $f : B \to \mathbb{R}_+$ one can define a length function on the product $B\times F$: for any absolutely continuous curve $\gamma:[0,1]\to B\times F$ with $\gamma=(\alpha,\beta)$, define

$$L(\gamma) := \int_0^1 \Big(|\dot\alpha|^2(t) + (f\circ\alpha)^2(t)|\dot\beta|^2(t)\Big)^{1/2}\,dt$$

and define accordingly the pseudo-distance

$$|(p,x),(q,y)| := \inf\{L(\gamma)\colon \gamma_0=(p,x),\ \gamma_1=(q,y)\}.$$

Then the warped product of B with F is defined as

$$B\times_f F := \big(B\times F/_\sim, |\cdot,\cdot|\big),$$

where $(p,x)\sim(q,y)$ if and only if $|(p,x),(q,y)|=0$. One can also associate a measure and obtain the following object

$$B\times_f^N F := (B\times_f F, \mathfrak{m}_C), \qquad \mathfrak{m}_C := f^N\mathfrak{m}_B\otimes\mathfrak{m}_F.$$

Then $B\times_f^N F$ will be a metric measure space called measured warped product. For a general picture on the curvature properties of warped products, we refer to [43].

2.4. **Localization method.** The next theorem represents the key technical tool of the present paper. The roots of such a result, known in literature as localization technique, can be traced back to a work of Payne-Weinberger [64] further developed in the Euclidean space by Gromov-Milman [38], Lovász-Simonovits [53] and Kannan-Lovász-Simonovits [42]. The basic idea consists in reducing an n-dimensional problem to a one dimensional one via tools of convex geometry. Recently Klartag [45] found an L^1-optimal transportation approach leading to a generalization of these ideas to Riemannian manifolds; the authors [17], via a careful analysis avoiding any smoothness assumption, generalized this approach to metric measure spaces.

Theorem 2.7. *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching metric measure space with $\mathfrak{m}(X)=1$ satisfying $\mathsf{CD}_{loc}(K,N)$ for some $K,N\in\mathbb{R}$ and $N\in[1,\infty)$. Let $f:X\to\mathbb{R}$ be $\mathfrak{m}$-integrable such that $\int_X f\,\mathfrak{m}=0$ and assume the existence of $x_0\in X$ such that $\int_X |f(x)|\,\mathsf{d}(x,x_0)\,\mathfrak{m}(dx)<\infty$.*

Then the space X can be written as the disjoint union of two sets Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q\in Q}$, where each X_q is the image of a geodesic; moreover there exists a family of probability measures $\{\mathfrak{m}_q\}_{q\in Q}\subset\mathcal{P}(X)$ with the following properties:

- *For any $\mathfrak{m}$-measurable set $B\subset\mathcal{T}$ it holds*
$$\mathfrak{m}(B) = \int_Q \mathfrak{m}_q(B)\,\mathfrak{q}(dq),$$
where $\mathfrak{q}$ is a probability measure over $Q\subset X$.
- *For $\mathfrak{q}$-almost every $q\in Q$, the set X_q is a geodesic with strictly positive length and $\mathfrak{m}_q$ is supported on it. Moreover $q\mapsto\mathfrak{m}_q$ is a $\mathsf{CD}(K,N)$ disintegration, that is $\mathfrak{m}_q = g(q,\cdot)_\sharp\big(h_q\cdot\mathcal{L}^1\big)$, with*
$$h_q((1-s)t_0+st_1)^{\frac{1}{N-1}} \geq \sigma^{(1-s)}_{K,N-1}(t_1-t_0)h_q(t_0)^{\frac{1}{N-1}} + \sigma^{(s)}_{K,N-1}(t_1-t_0)h_q(t_1)^{\frac{1}{N-1}}, \tag{2.6}$$
for all $s\in[0,1]$ and for all $t_0,t_1\in Dom(g(q,\cdot))$ with $t_0<t_1$, where $g(q,\cdot)$ is the isometry with range X_q. If $N=1$, for $\mathfrak{q}$-a.e. $q\in Q$ the density h_q is constant.
- *For $\mathfrak{q}$-almost every $q\in Q$, it holds $\int_{X_q} f\,\mathfrak{m}_q=0$ and $f=0$ $\mathfrak{m}$-a.e. in Z.*

Remark 2.8. Inequality (2.6) is the weak formulation of the following differential inequality on h_{q,t_0,t_1}:

$$\left(h_{q,t_0,t_1}^{\frac{1}{N-1}}\right)'' + (t_1-t_0)^2 \frac{K}{N-1} h_{q,t_0,t_1}^{\frac{1}{N-1}} \leq 0, \tag{2.7}$$

for all $t_0 < t_1 \in \mathrm{Dom}\,(g(q,\cdot))$, where $h_{q,t_0,t_1}(s) := h_q((1-s)t_0 + st_1)$. It is easy to observe that the differential inequality (2.7) on h_{q,t_0,t_1} is equivalent to the following differential inequality on h_q:

$$\left(h_q^{\frac{1}{N-1}}\right)'' + \frac{K}{N-1} h_q^{\frac{1}{N-1}} \leq 0,$$

that is precisely (2.5). Then Theorem 2.7 can be alternatively stated as follows.
If $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching m.m.s. verifying $\mathsf{CD}_{loc}(K,N)$ and $\varphi : X \to \mathbb{R}$ is a 1-Lipschitz function, then the corresponding decomposition of the space in maximal rays $\{X_q\}_{q\in Q}$ produces a disintegration $\{\mathfrak{m}_q\}_{q\in Q}$ of $\mathfrak{m}$ so that for $\mathfrak{q}$-a.e. $q \in Q$,

$$\text{the m.m.s. } (\mathrm{Dom}\,(g(q,\cdot)), |\cdot|, h_q\mathcal{L}^1) \quad \text{verifies} \quad \mathsf{CD}(K,N).$$

Accordingly, from now on we will say that the disintegration $q \mapsto \mathfrak{m}_q$ is a $\mathsf{CD}(K,N)$ disintegration.

Few comments on Theorem 2.7 are in order. From (2.6) it follows that

$$\{t \in \mathrm{Dom}\,(g(q,\cdot)) \colon h_q(t) > 0\} \text{ is convex and } \quad t \mapsto h_q(t) \text{ is locally Lipschitz continuous.} \tag{2.8}$$

The measure $\mathfrak{q}$ is the quotient measure associated to the partition $\{X_q\}_{q\in Q}$ of $\mathcal{T}$ and Q its quotient set, see [17] for details.

3. Sharp Brunn-Minkowski inequality

In this section we prove sharp Brunn-Minkowski inequality for m.m.s. satisfying $\mathsf{CD}_{loc}(K,N)$. It follows from Remark 2.4 that the same result holds under $\mathsf{CD}^*(K,N)$ for any $K,N \in \mathbb{R}$, provided $N \in (1,\infty)$ or $N = 1$ and $K \geq 0$. See also Remark 1.1. The same will hold for all the inequalities proved in the paper.

Theorem 3.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ with $\mathfrak{m}(X) < \infty$ verify $\mathsf{CD}_{loc}(K,N)$ for some $N,K \in \mathbb{R}$ and $N \in [1,\infty)$. Assume moreover $(X, \mathsf{d}, \mathfrak{m})$ to be essentially non-branching. Then it satisfies the following sharp Brunn-Minkowski inequality: for any $A_0, A_1 \subset X$*

$$\mathfrak{m}(A_t)^{1/N} \geq \tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N}, \tag{3.1}$$

where A_t is the set of t-intermediate points between A_0 and A_1, that is

$$A_t = \mathrm{e}_t\Big(\{\gamma \in \mathrm{Geo}(X) \colon \gamma_0 \in A_0, \gamma_1 \in A_1\}\Big),$$

and θ the minimal/maximal length of geodesics from A_0 to A_1:

$$\theta := \begin{cases} \inf_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K \geq 0, \\ \sup_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K < 0. \end{cases}$$

Before starting the proof of Theorem 3.1 we recall the classical result of Borell [11] and Brascamp-Lieb [12] characterizing one-dimensional measures satisfying Brunn-Minkowski inequality.

Lemma 3.2. *Let η be a Borel measure defined on $\mathbb{R}$ admitting the following representation: $\eta = h \cdot \mathcal{L}^1$. The following are equivalent:*

i) The density h is (K,N)-concave on its convex support, that is

$$\left(h^{\frac{1}{N-1}}\right)'' + \frac{K}{N-1} h^{\frac{1}{N-1}} \leq 0,$$

in the weak sense, see (2.6).

ii) *For any* A_0, A_1 *subsets of* $\mathbb{R}$

$$\eta(A_t) \geq \tau_{K,N}^{(1-t)}(\theta)\, \eta(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\, \eta(A_1)^{1/N},$$

where $A_t := \{(1-t)x + ty : x \in A_0,\ y \in A_1\}$ *and* θ *is the minimal/maximal length of geodesics from* A_0 *to* A_1*:*

$$\theta := \begin{cases} \operatorname{ess\,inf}_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K \geq 0, \\ \operatorname{ess\,sup}_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K < 0. \end{cases}$$

For reader's convenience we include here a proof that *i)* implies *ii)*, which is the implication we will use later.

Proof. Consider the N-entropy: for any $\mu = \rho \cdot \eta$

$$\mathcal{S}_N(\mu|\eta) := -\int \rho^{-1/N}(x)\, \mu(dx).$$

Observe that *ii)* is implied by displacement convexity of $\mathcal{S}_N$ with respect to the L^2-Wasserstein distance over $(\mathbb{R}, |\cdot|)$. Just consider $\mu_0 := \eta(A_0)^{-1}\eta\llcorner_{A_0}$ and $\mu_1 := \eta(A_1)^{-1}\eta\llcorner_{A_1}$ and use Jensen's inequality. Consider therefore a geodesic curve

$$[0,1] \ni t \mapsto \rho_t\, \eta \in W_2(\mathbb{R}, |\cdot|), \qquad T_t \sharp \rho_0 \eta = \rho_t\, \eta,$$

where $T_t = Id(1-t) + tT$ and T is the (μ_0-essentially) unique monotone rearrangement such that $T \sharp \mu_0 = \mu_1$. Thanks to approximate differentiability of T, one can use change of variable formula

$$\rho_t(T_t(x))\, h(T_t(x))\, |(1-t) + tT'|(x) = \rho_0(x) h(x)$$

and obtain the following chain of equalities:

$$\begin{aligned} \int_{\mathrm{supp}(\mu_t)} \rho_t(x)^{\frac{N-1}{N}}\, \eta(dx) &= \int_{\mathrm{supp}(\mu_t)} \rho_t(x)^{\frac{N-1}{N}} h(x)\, dx \\ &= \int_{\mathrm{supp}(\mu_0)} \rho_t(T_t(x))^{\frac{N-1}{N}} h(T_t(x)) |(1-t) + tT'|\, dx \\ &= \int_{\mathrm{supp}(\mu_0)} \rho_0(x)^{\frac{N-1}{N}} \Big(\frac{h(T_t(x))}{h(x)}\Big)^{\frac{1}{N}} |(1-t) + tT'|(x)^{\frac{1}{N}}\, \eta(dx). \end{aligned}$$

Hence the claim has became to prove that $t \mapsto J_t(x)^{\frac{1}{N}}$ is concave, where J_t is the Jacobian of T_t with respect to η and

$$J_t(x) = J_t^G(x) \cdot J_t^W(x), \quad J_t^G(x) = |(1-t) + tT'|(x), \quad J_t^W(x) = \frac{h(T_t(x))}{h(x)},$$

where J^G is the geometric Jacobian and J^W the weighted Jacobian. Since $t \mapsto J_t^G(x)$ is linear, using Hölder's inequality the claim follows straightforwardly from the (K,N)-convexity of h. □

We can now move to the proof of Theorem 3.1.

Proof of Theorem 3.1. First of all notice that up to replacing $\mathfrak{m}$ with the normalized measure $\frac{1}{\mathfrak{m}(X)}\mathfrak{m}$ we can assume that $\mathfrak{m}(X) = 1$. Let $A_0, A_1 \subset X$ be two given Borel sets of positive $\mathfrak{m}$-measure.

Step 1. Consider the function $f := \chi_{A_0}/\mathfrak{m}(A_0) - \chi_{A_1}/\mathfrak{m}(A_1)$ and observe that $\int_X f\, \mathfrak{m} = 0$. From Theorem 2.7, the space X can be written as the disjoint union of two sets Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q\in Q}$ and a corresponding disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}}$, $\{\mathfrak{m}_q\}_{q\in Q}$ such that:

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_q\, \mathfrak{q}(dq),$$

where $\mathfrak{q}$ is the quotient measure, for $\mathfrak{q}$-almost every $q \in Q$, the set X_q is a geodesic, $\mathfrak{m}_q$ is supported on it and $q \mapsto \mathfrak{m}_q$ is a $\mathsf{CD}(K,N)$ disintegration. Finally, for $\mathfrak{q}$-almost every $q \in Q$, it holds $\int_{X_q} f\, \mathfrak{m}_q = 0$ and $f = 0$ $\mathfrak{m}$-a.e. in Z. We can also consider the trivial disintegration of $\mathfrak{m}$ restricted to Z where each equivalence class is a single point:

$$\mathfrak{m}\llcorner_Z = \int_Z \delta_z \mathfrak{m}(dz),$$

where δ_z stands for the Dirac delta in z. Then define $\hat{\mathfrak{q}} := \mathfrak{q} + \mathfrak{m}\llcorner_Z$ and $\hat{\mathfrak{m}}_q = \mathfrak{m}_q$ if $q \in Q$ and $\hat{\mathfrak{m}}_q = \delta_q$ if $q \in Z$. Since $Q \cap Z = \emptyset$, the previous definitions are well posed and we have the following decomposition of $\mathfrak{m}$ on the whole space

$$\mathfrak{m} = \int_{Q\cup Z} \hat{\mathfrak{m}}_q \, \hat{\mathfrak{q}}(dq).$$

Step 2. Use the following notation $A_{0,q} := A_0 \cap X_q$, $A_{1,q} := A_1 \cap X_q$ and the set of t-intermediate points between $A_{0,q}$ and $A_{1,q}$ in X_q is denoted with $A_{t,q} \subset X_q$. Then from Lemma 3.2, for $\hat{\mathfrak{q}}$-a.e. $q \in Q$

$$\mathfrak{m}_q(A_{t,q}) \geq \left(\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}_q(A_{0,q})^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}_q(A_{1,q})^{1/N}\right)^N.$$

Since $\int f \mathfrak{m}_q = 0$ implies $\frac{\mathfrak{m}_q(A_{0,q})}{\mathfrak{m}(A_0)} = \frac{\mathfrak{m}_q(A_{1,q})}{\mathfrak{m}(A_1)}$, it follows that

$$\mathfrak{m}_q(A_{t,q}) \geq \frac{\mathfrak{m}_q(A_{0,q})}{\mathfrak{m}(A_0)} \left(\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N}\right)^N. \tag{3.2}$$

We now show that (3.2) holds also for $\hat{\mathfrak{q}}$-a.e. (or equivalently $\mathfrak{m}$-a.e.) $q \in Z$. Note that in this case $\mathfrak{m}_q$ has to be replaced by δ_q. Since by construction $0 = f = \chi_{A_0}/\mathfrak{m}(A_0) - \chi_{A_1}/\mathfrak{m}(A_1)$ on Z, then necessarily

$$\mathfrak{m}\left(Z \setminus \big((A_0 \cap A_1) \cup (X \setminus (A_0 \cup A_1)\big)\big)\right) = 0.$$

It follows that if Z does not have $\mathfrak{m}$-measure zero, we have two possibilities:

$$\mathfrak{m}\left(Z \cap (X \setminus (A_0 \cup A_1))\right) > 0, \quad \text{or} \quad \mathfrak{m}(A_0) = \mathfrak{m}(A_1) \text{ and } \mathfrak{m}\left(Z \cap (A_0 \cap A_1)\right) > 0.$$

Therefore, if $\mathfrak{m}(Z) > 0$, for $\hat{\mathfrak{q}}$-a.e. (or equivalently $\mathfrak{m}$-a.e.) $q \in Z$ we have two possibilities:

$$q \in X \setminus (A_0 \cup A_1), \quad \text{or} \quad q \in A_0 \cap A_1.$$

Interpreting the intermediate points as the point itself, in the first case (3.2) (with $\mathfrak{m}_q$ replaced by δ_q) holds trivially (i.e. we get $0 \geq 0$). In the second case it reduces to show that

$$\left(\tau_{K,N}^{(1-t)}(\theta) + \tau_{K,N}^{(t)}(\theta)\right)^N \leq 1.$$

For $K \geq 0$, since we are in the case $\mathfrak{m}(A_0 \cap A_1) > 0$, it follows that $\theta = 0$ and therefore $\tau_{K,N}^{(t)}(\theta) = t$, proving the previous inequality. For $K < 0$, recalling that $K \to \sigma_{K,N}^{(t)}(\theta)$ is non-decreasing (see [7], Remark 2.2), by Hölder's inequality

$$\left(\tau_{K,N}^{(1-t)}(\theta) + \tau_{K,N}^{(t)}(\theta)\right)^N \leq (1-t+t) \cdot \left(\sigma_{K,N-1}^{(1-t)}(\theta) + \sigma_{K,N-1}^{(t)}(\theta)\right)^{N-1} \leq 1,$$

as desired. We have therefore proved that

$$\hat{\mathfrak{m}}_q(A_{t,q}) \geq \frac{\hat{\mathfrak{m}}_q(A_{0,q})}{\mathfrak{m}(A_0)} \left(\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N}\right)^N, \tag{3.3}$$

for $\hat{\mathfrak{q}}$-a.e. $q \in Q \cup Z$. Taking the integral of (3.3) in $q \in Q \cup Z$ one obtains that

$$\begin{aligned}
\mathfrak{m}(A_t) &= \int_{Q\cup Z} \hat{\mathfrak{m}}_q(A_t \cap X_q)\, \hat{\mathfrak{q}}(dq) \\
&\geq \int_{Q\cup Z} \hat{\mathfrak{m}}_q(A_{t,q})\, \hat{\mathfrak{q}}(dq) \\
&\geq \left(\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N}\right)^N \int_{Q\cup Z} \frac{\hat{\mathfrak{m}}_q(A_{0,q})}{\mathfrak{m}(A_0)}\, \hat{\mathfrak{q}}(dq) \\
&= \left(\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N}\right)^N,
\end{aligned}$$

and the claim follows. □

4. p-Spectral gap

Given a metric space (X,d), we denote with $\mathrm{Lip}(X)$ (respectively $\mathrm{Lip}_c(X)$) the vector space of real valued Lipschitz functions (resp. with compact support). For a Lipschitz function $f: X \to \mathbb{R}$ the local Lipschitz constant $|\nabla f|$ is defined by

$$|\nabla f|(x) = \limsup_{y\to x} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)} \quad \text{if } x \text{ is not isolated, 0 otherwise.}$$

For a m.m.s. $(X,\mathsf{d},\mathfrak{m})$, for every $p \in (1,\infty)$ we define *the first eigenvalue* $\lambda_{1,p}(X,\mathsf{d},\mathfrak{m})$ *of the p-Laplacian* by

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} := \inf\left\{ \frac{\int_X |\nabla f|^p\,\mathfrak{m}}{\int_X |f|^p\,\mathfrak{m}} \;:\; f\in \mathrm{Lip}(X)\cap L^p(X,\mathfrak{m}),\ f\neq 0,\ \int_X f|f|^{p-2}\,\mathfrak{m} = 0 \right\}. \tag{4.1}$$

4.1. **p-spectral gap for m.m.s. over $(\mathbb{R},|\cdot|)$: the model spaces.** Consider the following family of probability measures

$$\begin{aligned}\mathcal{F}^s_{K,N,D} := \{\mu\in\mathcal{P}(\mathbb{R}):\ & \mathrm{supp}(\mu)\subset[0,D],\ \mu = h_\mu\mathcal{L}^1,\ h_\mu \text{ verifies (2.6) and is continuous if } N\in(1,\infty),\\ & h_\mu \equiv \text{const if } N=1\},\end{aligned} \tag{4.2}$$

where $D\in(0,\infty)$ and the corresponding *synthetic* first non-negative eigenvalue of the p-Laplacian

$${}^s\lambda^{1,p}_{K,N,D} := \inf_{\mu\in\mathcal{F}^s_{K,N,D}} \inf\left\{ \frac{\int_{\mathbb{R}} |u'|^p\,\mu}{\int_{\mathbb{R}} |u|^p\,\mu} : u\in \mathrm{Lip}(\mathbb{R})\cap L^p(\mu),\ \int_{\mathbb{R}} u|u|^{p-2}\mu = 0,\ u\neq 0\right\}.$$

The term synthetic refers to $\mu\in\mathcal{F}^s_{K,N,D}$ meaning that the Ricci curvature bound is satisfied in its synthetic formulation: if $\mu = h\cdot\mathcal{L}^1$, then h verifies (2.6).

The first goal of this section is to prove that ${}^s\lambda^{1,p}_{K,N,D}$ coincides with its smooth counterpart $\lambda^{1,p}_{K,N,D}$ defined by

$$\lambda^{1,p}_{K,N,D} := \inf_{\mu\in\mathcal{F}_{K,N,D}} \inf\left\{ \frac{\int_{\mathbb{R}} |u'|^p\,\mu}{\int_{\mathbb{R}} |u|^p\,\mu} : u\in \mathrm{Lip}(\mathbb{R})\cap L^p(\mu),\ \int_{\mathbb{R}} u|u|^{p-2}\mu = 0,\ u\neq 0\right\}, \tag{4.3}$$

where now $\mathcal{F}_{K,N,D}$ denotes the set of $\mu\in\mathcal{P}(\mathbb{R})$ such that $\mathrm{supp}(\mu)\subset[0,D]$ and $\mu = h\cdot\mathcal{L}^1$ with $h\in C^2((0,D))$ satisfying

$$\left(h^{\frac{1}{N-1}}\right)'' + \frac{K}{N-1} h^{\frac{1}{N-1}} \le 0. \tag{4.4}$$

It is easily verified that $\mathcal{F}_{K,N,D}\subset\mathcal{F}^s_{K,N,D}$.

In order to prove that ${}^s\lambda^{1,p}_{K,N,D} = \lambda^{1,p}_{K,N,D}$ the following approximation result, proved in [17, Lemma 6.2] will play a key role. In order to state it let us recall that a standard mollifier in $\mathbb{R}$ is a non negative $C^\infty(\mathbb{R})$ function ψ with compact support in $[0,1]$ such that $\int_{\mathbb{R}}\psi = 1$.

Lemma 4.1. *Let $D\in(0,\infty)$ and let $h:[0,D]\to[0,\infty)$ be a continuous function. Fix $N\in(1,\infty)$ and for $\varepsilon>0$ define*

$$h_\varepsilon(t) := [h^{\frac{1}{N-1}} * \psi_\varepsilon(t)]^{N-1} := \left[\int_{\mathbb{R}} h(t-s)^{\frac{1}{N-1}}\psi_\varepsilon(s)\,ds\right]^{N-1} = \left[\int_{\mathbb{R}} h(s)^{\frac{1}{N-1}}\psi_\varepsilon(t-s)\,ds\right]^{N-1}, \tag{4.5}$$

where $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi(x/\varepsilon)$ and ψ is a standard mollifier function. The following properties hold:

1. *h_ε is a non-negative C^∞ function with support in $[-\varepsilon, D+\varepsilon]$;*
2. *$h_\varepsilon\to h$ uniformly as $\varepsilon\downarrow 0$, in particular $h_\varepsilon\to h$ in L^1.*
3. *If h satisfies the convexity condition (2.6) corresponding to the above fixed $N>1$ and some $K\in\mathbb{R}$ then also h_ε does. In particular h_ε satisfies the differential inequality (4.4).*

Proposition 4.2. *For every $p\in(1,+\infty)$, $N\in[1,\infty)$, $K\in\mathbb{R}$, $D\in(0,\infty)$ it holds ${}^s\lambda^{1,p}_{K,N,D} = \lambda^{1,p}_{K,N,D}$.*

Proof. First of all observe that for $N = 1$ clearly we have $\mathcal{F}_{K,N,D} = \mathcal{F}^s_{K,N,D}$ since the density h_μ has to be constant. We can then assume without loss of generality that $N \in (1, \infty)$.
Since $\mathcal{F}_{K,N,D} \subset \mathcal{F}^s_{K,N,D}$ then clearly ${}^s\lambda^{1,p}_{K,N,D} \leq \lambda^{1,p}_{K,N,D}$.
Assume by contradiction the inequality is strict. Then there exists a measure $\mu = h \cdot \mathcal{L}^1 \in \mathcal{F}^s_{K,N,D}$ and $\delta > 0$ such that

$$\lambda^{1,p}_{(\mathbb{R},|\cdot|,\mu)} \leq \lambda^{1,p}_{K,N,D} - 2\delta.$$

Therefore, by the very definition of $\lambda^{1,p}_{(\mathbb{R},|\cdot|,\mu)}$, there exists $u \in \mathrm{Lip}(\mathbb{R})$, such that $u \neq 0$, $\int_{\mathbb{R}} u|u|^{p-2}\, h\, ds = 0$ and

$$\int_{\mathbb{R}} |u'(s)|^p\, h(s)\, ds \leq \Big(\lambda^{1,p}_{K,N,D} - \frac{3}{2}\delta\Big) \int_{\mathbb{R}} |u(s)|^p\, h(s)\, ds. \tag{4.6}$$

Now, Lemma 4.1 gives a sequence $h_k \in C^\infty(\mathbb{R})$ such that

$$\mathrm{supp}(h_k) \subset \left[-\frac{1}{k}, D + \frac{1}{k}\right], \qquad \mu_k := h_k \cdot \mathcal{L}^1 \in \mathcal{F}_{K,N,D+\frac{2}{k}}, \qquad h_k \to h \text{ uniformly on } [0, D]. \tag{4.7}$$

Called now $u_k := u - c_k \in \mathrm{Lip}(\mathbb{R}) \cap L^p(\mathbb{R}, h_k\mathcal{L}^1)$ where $c_k \in \mathbb{R}$ are such that $\int_{\mathbb{R}} u_k|u_k|^{p-2}\, h_k\, ds = 0$, thanks to (4.7) it holds $c_k \to 0$ and thus

$$\int_{\mathbb{R}} |u_k(s)|^p\, h_k(s)\, ds \to \int_{\mathbb{R}} |u(s)|^p\, h(s)\, ds \quad \text{and} \quad \int_{\mathbb{R}} |u_k'(s)|^p\, h_k(s)\, ds \to \int_{\mathbb{R}} |u'(s)|^p\, h(s)\, ds.$$

Therefore (4.6), combined with the continuity of $\varepsilon \mapsto \lambda^{1,p}_{K,N,D+\varepsilon}$ (see Theorem 4.3 below), implies that for k large enough one has

$$\int_{\mathbb{R}} |u_k'(s)|^p\, h_k(s)\, ds \leq (\lambda^{1,p}_{K,N,D} - \delta) \int_{\mathbb{R}} |u_k(s)|^p\, h_k(s)\, ds \leq \big(\lambda^{1,p}_{K,N,D+\frac{2}{k}} - \frac{\delta}{2}\big) \int_{\mathbb{R}} |u_k(s)|^p\, h_k(s)\, ds,$$

contradicting the definition of $\lambda^{1,p}_{K,N,D+\frac{2}{k}}$ given in (4.3). □

The next goal of the section is to understand the quantity $\lambda^{1,p}_{K,N,D}$. Since now the density of the reference probability measure is smooth, we enter into a more classical framework where a number of people contributed. The sharp p-spectral gap in case $K > 0$ and without upper bounds on the diameter was obtained by Matei [54]. The case $K = 0$ and the diameter is bounded above was obtained in the sharp form by Valtorta [74]. Finally the case $K < 0$ and diameter bounded above was obtained in the sharp form by Naber-Valtorta [61]. Actually, as explained in their paper, the arguments in [61] hold in the general case $K \in \mathbb{R}$, $N \in [1, \infty)$, provided one identifies the correct model space. As usual, to describe the model space one has to examine separately the cases $K < 0$, $K = 0$ and $K > 0$; in order to unify the presentation let us denote with $\tan_{K,N}(t)$ the following function:

$$\tan_{K,N}(t) := \begin{cases} \sqrt{-K/(N-1)}\, \tanh(\sqrt{-K/(N-1)}t) & \text{if } K < 0, \\ 0 & \text{if } K = 0 \\ \sqrt{K/(N-1)}\, \tan(\sqrt{K/(N-1)}t) & \text{if } K > 0. \end{cases} \tag{4.8}$$

Now, for each $K \in \mathbb{R}, N \in [1, \infty), D \in (0, \infty)$, let $\hat{\lambda}^{1,p}_{K,N,D}$ denote the first positive eigenvalue on $[-D/2, D/2]$ of the eigenvalue problem

$$\frac{d}{dt}\Big(\dot{w}^{(p-1)}\Big) + (N-1)\tan_{K,N}(t)\, \dot{w}^{(p-1)} + \hat{\lambda}^{1,p}_{K,N,D} w^{(p-1)} = 0. \tag{4.9}$$

It is possible to show (see [61]) that $\hat{\lambda}^{1,p}_{K,N,D}$ is the unique value of $\hat{\lambda}$ such that the solution of

$$\begin{cases} \dot{\phi} = \left(\frac{\hat{\lambda}}{p-1}\right)^{1/p} + \frac{N-1}{p-1} \tan_{K,N}(t) \cos_p^{(p-1)}(\phi) \sin_p(\phi) \\ \phi(0) = 0 \end{cases}$$

satisfies $\phi(D/2) = \pi_p/2$, where $\pi_p, \cos_p$ and $\sin_p$ are defined as follows.
For every $p \in (1, \infty)$ the positive number π_p is defined by

$$\pi_p := \int_{-1}^{1} \frac{ds}{(1 - s^p)^{1/p}} = \frac{2\pi}{p \sin(\pi/p)}.$$

The $C^1(\mathbb{R})$ function $\sin_p : \mathbb{R} \to [-1,1]$ is defined implicitly on $[-\pi_p/2, 3\pi_p/2]$ by:

$$\begin{cases} t = \int_0^{\sin_p(t)} \frac{ds}{(1-s^p)^{1/p}} & \text{if } t \in \left[-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right] \\ \sin_p(t) = \sin_p(\pi_p - t) & \text{if } t \in \left[\frac{\pi_p}{2}, \frac{3\pi_p}{2}\right] \end{cases}$$

and is periodic on $\mathbb{R}$. Set also by definition $\cos_p(t) = \frac{d}{dt}\sin_p(t)$. The usual fundamental trigonometric identity can be generalized by $|\sin_p(t)|^p + |\cos_p(t)|^p = 1$, and so it is easily seen that $\cos_p^{(p-1)} \in C^1(\mathbb{R})$. Clearly, if $p = 2$ one finds the usual quantities: $\pi_2 = \pi, \sin_2 = \sin$ and $\cos_2 = \cos$.

Theorem 4.3 ([54, 74, 61])**.** *Let $K \in \mathbb{R}$, $N \in [1,\infty)$ and $D \in (0,\infty)$. Then the following hold*

1. $\lambda^{1,p}_{K,N,D} = \hat\lambda^{1,p}_{K,N,D}$, *where $\lambda^{1,p}_{K,N,D}$ was defined in (4.3) and $\hat\lambda^{1,p}_{K,N,D}$ in (4.9).*
2. *For every fixed $p \in (1,\infty)$, the map $K, N, D \mapsto \lambda^{1,p}_{K,N,D}$ is continuous.*
3. *If $K > 0$ then for every $D \in (0, \pi\sqrt{N-1/K}]$*
$$\lambda^{1,p}_{K,N,D} \geq \lambda^{1,p}_{K,N,\pi\sqrt{N-1/K}}$$
and equality holds if and only if $D = \pi\sqrt{N-1/K}$. If moreover $N \in \mathbb{N}$, then
$$\lambda^{1,p}_{K,N,\pi\sqrt{N-1/K}} = \lambda^{1,p}(S^N(\sqrt{N-1/K})),$$
i.e. $\lambda^{1,p}_{K,N,\pi\sqrt{N-1/K}}$ coincides with the first eigenvalue of the p-laplacian on the round sphere of radius $\sqrt{N-1/K}$.
4. *If $K = 0$ then $\lambda^{1,p}_{0,N,D} = (p-1)\left(\frac{\pi_p}{D}\right)^p$.*

For $K \neq 0$ and $p \neq 2$, it is not easy to give an explicit expression of the lower bound $\lambda^{1,p}_{K,N,D}$. At least one can give some lower bounds, for instance recently Li and Wang [48] obtained that

$$\lambda^{1,p}_{K,N,D} \geq \frac{1}{(p-1)^{p-1}}\left(\frac{NK}{N-1}\right)^{p/2} \quad \text{for } K > 0,\ p \geq 2. \tag{4.10}$$

4.2. **p-spectral gap for $\mathsf{CD}_{loc}(K,N)$ spaces.**

Theorem 4.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space satisfying $\mathsf{CD}_{loc}(K,N)$, for some $K, N \in \mathbb{R}$ with $N \geq 1$, and assume moreover it is essentially non-branching. Let $D \in (0,\infty)$ be the diameter of X and fix $p \in (1,\infty)$. Then for any Lipschitz function $f \in L^p(X,\mathfrak{m})$ with $\int_X f|f|^{p-2}\,\mathfrak{m}(dx) = 0$ it holds*

$$\lambda^{1,p}_{K,N,D} \int_X |f(x)|^p\,\mathfrak{m}(dx) \leq \int_X |\nabla f|^p(x)\,\mathfrak{m}(dx). \tag{4.11}$$

In other terms it holds $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \geq \lambda^{1,p}_{K,N,D}$. Notice that for $D = \pi\sqrt{(N-1)/K}$ and $N \in \mathbb{N}$, it follows that

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \geq \lambda^{1,p}(S^N((N-1)/K)).$$

Proof. Since the space (X,d) is bounded, then the $\mathsf{CD}_{loc}(K,N)$ condition implies that $\mathfrak{m}(X) < \infty$. Noting that the inequality (4.11) is invariant under multiplication of $\mathfrak{m}$ by a positive constant, we can assume without loss of generality that $\mathfrak{m}(X) = 1$. Observing that the function

$$\tilde f := f|f|^{p-2} \in \mathrm{Lip}(X) \tag{4.12}$$

verifies the hypothesis of Theorem 2.7, we can write $X = Y \cup \mathcal{T}$ with

$$\tilde f(x) = 0, \quad \mathfrak{m}\text{-a.e. } y \in Y, \qquad \mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_q\,\mathfrak{q}(dq),$$

with $\mathfrak{m}_q = g(q,\cdot)\,\sharp\left(h_q \cdot \mathcal{L}^1\right)$, where the density h_q verifies (2.6) for $\mathfrak{q}$-a.e. $q \in Q$ and

$$0 = \int_X \tilde f(z)\,\mathfrak{m}_q(dz) = \int_{\mathrm{Dom}\,(g(q,\cdot))} \tilde f(g(q,t))\cdot h_q(t)\,\mathcal{L}^1(dt) = \int_{\mathrm{Dom}\,(g(q,\cdot))} f(g(q,t))\,|f(g(q,t))|^{p-2}\cdot h_q(t)\,\mathcal{L}^1(dt)$$

for $\mathfrak{q}$-a.e. $q \in Q$. Now consider the map $t \mapsto f_q(t) := f(g(q,t))$ and note that it is Lipschitz. Since $\mathrm{diam}\,(\mathrm{Dom}\,(g(q,\cdot))) \leq D$, from the definition of $\mathcal{F}^s_{K,N,D}$ and of $\lambda^{1,p}_{K,N,D}$ we deduce that

$$\lambda^{1,p}_{K,N,D} \int_{\mathbb{R}} |f_q(t)|^p h_q(t)\,\mathcal{L}^1(dt) \leq \int_{\mathbb{R}} |f'_q(t)|^p h_q(t)\,\mathcal{L}^1(dt).$$

Noticing that $|f'_q(t)| \leq |\nabla f|(g(q,t))$ one obtains that

$$\begin{aligned}
\lambda^{1,p}_{K,N,D} \int_X |f(x)|^p\,\mathfrak{m}(dx) &= \lambda^{1,p}_{K,N,D} \int_{\mathcal{T}} |f(x)|^p\,\mathfrak{m}(dx) \\
&= \lambda^{1,p}_{K,N,D} \int_Q \left(\int_X |f(x)|^p\,\mathfrak{m}_q(dx) \right) \mathfrak{q}(dq) \\
&= \lambda^{1,p}_{K,N,D} \int_Q \left(\int_{\mathrm{Dom}\,(g(q,\cdot))} |f_q(t)|^p\,h_q(t)\,\mathcal{L}^1(dt) \right) \mathfrak{q}(dq) \\
&\leq \int_Q \left(\int_{\mathrm{Dom}\,(g(q,\cdot))} |f'_q(t)|^p\,h_q(t)\,\mathcal{L}^1(dt) \right) \mathfrak{q}(dq) \\
&\leq \int_Q \left(\int_X |\nabla f|^p(x)\,(g(q,\cdot))\,\sharp\,\big(h_q(t)\,\mathcal{L}^1\big)\,(dx) \right) \mathfrak{q}(dq) \\
&= \int_X |\nabla f|^p(x)\,\mathfrak{m}(dx),
\end{aligned}$$

and the claim follows. □

4.3. Almost rigidity for the p-spectral gap.

Theorem 4.5 (Almost equality in the p-spectral gap implies almost maximal diameter)**.** *Let $N > 1$, and $p \in (1,\infty)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N, p)$ such that the following holds.*

Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching metric measure space satisfying $\mathsf{CD}^*(N-1-\delta, N+\delta)$. If $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta$, then $\mathrm{diam}\,(X) \geq \pi - \varepsilon$.

Proof. As above, without loss of generality we can assume $\mathfrak{m}(X) = 1$. Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ we can find an essentially non-branching metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfying $\mathsf{CD}^*(N-1-\delta, N+\delta)$, with $\mathfrak{m}(X) = 1$, such that $\mathrm{diam}\,(X) \leq \pi - \varepsilon_0$ but $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta$.
The very definition of $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})}$ implies that there exists a function $f \in \mathrm{Lip}(X)$, with $\int_X f|f|^{p-2}\mathfrak{m} = 0$ and $\int_X |f|^p\,\mathfrak{m}(dx) = 1$, such that

$$\int_X |\nabla f|^p(x)\,\mathfrak{m}(dx) \leq \lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} + \delta \leq \lambda^{1,p}_{N-1,N,\pi} + 2\delta. \tag{4.13}$$

On the other hand, Theorem 4.3 ensures that there exists $\eta > 0$ such that

$$\lambda^{1,p}_{N-1,N,D} \geq \lambda^{1,p}_{N-1,N,\pi} + 2\eta, \quad \forall D \in [0, \pi - \varepsilon_0].$$

Moreover, the continuity of $K, N, D \mapsto \lambda^{1,p}_{K,N,D}$ guarantees that, for every $D_0 \in (0,1)$ there exists $\delta_0 = \delta_0(N, D_0)$ such that

$$\lambda^{1,p}_{N-1-\delta,N+\delta,D} \geq \lambda^{1,p}_{N-1,N,D} - \eta \quad \forall \delta \in [0,\delta_0],\ \forall D \in [D_0, 2\pi].$$

Since clearly by definition we have that $\lambda^{1,p}_{K,N,D} \geq \lambda^{1,p}_{0,N,D}$ for every $K > 0, N \geq 1, p \in (1,\infty)$, Theorem 4.3 gives that

$$\lim_{D\downarrow 0} \lambda^{1,p}_{N-1-\delta,N+\delta,D} \geq \lim_{D\downarrow 0} \lambda^{1,p}_{0,N+\delta,D} = +\infty$$

uniformly for $\delta \in [0, \delta_0(N)]$. The combination of the last two estimates yields

$$\lambda^{1,p}_{N-1-\delta,N+\delta,D} \geq \lambda^{1,p}_{N-1,N,\pi} + \eta \quad \forall D \in [0, \pi - \varepsilon_0],\ \forall \delta \in [0, \delta_0(N)]. \tag{4.14}$$

By repeating the proof of Theorem 4.4, and observing that by construction it holds $\mathrm{diam}\,(\mathrm{Dom}\,(g(q,\cdot)) \leq \pi - \varepsilon_0$, we then obtain

$$\begin{aligned}\int_X |\nabla f|^p(x)\,\mathfrak{m}(dx) &= \int_Q \left(\int_X |\nabla f|^p(x)\,(g(q,\cdot))\,\sharp\left(h_q(t)\,\mathcal{L}^1\right)(dx)\right)\mathfrak{q}(dq)\\ &\geq \int_Q \left(\int_{\mathrm{Dom}\,(g(q,\cdot))} |f_q'(t)|^p\,h_q(t)\,\mathcal{L}^1(dt)\right)\mathfrak{q}(dq)\\ &\geq \int_Q \lambda^{1,p}_{N-1-\delta,N+\delta,\mathrm{diam}\,(\mathrm{Dom}\,(g(q,\cdot)))}\left(\int_{\mathrm{Dom}\,(g(q,\cdot))} |f_q(t)|^p\,h_q(t)\,\mathcal{L}^1(dt)\right)\mathfrak{q}(dq)\\ &\geq (\lambda^{1,p}_{N-1,N,\pi}+\eta)\int_Q \left(\int_{\mathrm{Dom}\,(g(q,\cdot))} |f_q(t)|^p\,h_q(t)\,\mathcal{L}^1(dt)\right)\mathfrak{q}(dq)\\ &= \lambda^{1,p}_{N-1,N,\pi}+\eta.\end{aligned}$$

Contradicting (4.13), once chosen $\delta < \eta/2$. □

Corollary 4.6 (Almost equality in the p-spectral gap implies mGH-closeness to a spherical suspension)**.** *Let $N \geq 2$, and $p \in (1,\infty)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N, p) > 0$ such that the following holds.*

Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}^*(N-1-\delta, N+\delta)$ space. If

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta,$$

then there exists an $\mathsf{RCD}^*(N-2,N-1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ such that

$$\mathsf{d}_{mGH}\left((X,\mathsf{d},\mathfrak{m}), [0,\pi]\times^{N-1}_{\sin} Y\right) \leq \varepsilon.$$

Proof. Fix $N \in [2,\infty), p \in (1,\infty)$ and assume by contradiction there exist $\varepsilon_0 > 0$ and a sequence $(X_j,\mathsf{d}_j,\mathfrak{m}_j)$ of $\mathsf{RCD}^*(N-1-\frac{1}{j}, N+\frac{1}{j})$ spaces such that $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \frac{1}{j}$, but

$$\mathsf{d}_{mGH}(X_j, [0,\pi]\times^{N-1}_{\sin} Y) \geq \varepsilon_0 \quad \text{for every } j \in \mathbb{N} \tag{4.15}$$

and every $\mathsf{RCD}^*(N-2,N-1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y)=1$. Observe that Theorem 4.5 yields

$$\mathrm{diam}\,((X_j,\mathsf{d}_j)) \to \pi. \tag{4.16}$$

By the compactness/stability property of $\mathsf{RCD}^*(K,N)$ spaces recalled in Theorem 2.6 we get that, up to subsequences, the spaces X_j mGH-converge to a limit $\mathsf{RCD}^*(N-1,N)$ space $(X_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$. Since the diameter is continuous under mGH convergence of uniformly bounded spaces, (4.16) implies that $\mathrm{diam}\,((X_\infty,\mathsf{d}_\infty)) = \pi$. But then by the Maximal Diameter Theorem [43] we get that $(X_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$ is isomorphic to a spherical suspension $[0,\pi]\times^{N-1}_{\sin} Y$ for some $\mathsf{RCD}^*(N-2,N-1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y)=1$. Clearly this contradicts (4.15) and the thesis follows. □

Corollary 4.7 (p-Obata Theorem)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}^*(N-1,N)$ space for some $N \geq 2$, and let $1 < p < \infty$. If*

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} = \lambda^{1,p}_{N-1,N,\pi} \quad (= \lambda^{1,p}(S^N),\ \textit{if } N \in \mathbb{N}),$$

then $(X,\mathsf{d},\mathfrak{m})$ is a spherical suspension, i.e. there exists an $\mathsf{RCD}^*(N-2,N-1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ such that $(X,\mathsf{d},\mathfrak{m})$ is isomorphic to $[0,\pi]\times^{N-1}_{\sin} Y$.

Proof. Theorem 4.5 implies that $\mathrm{diam}\,((X,\mathsf{d})) = \pi$ and the thesis then follows by the Maximal Diameter Theorem [43]. □

Remark 4.8. The Obata's Theorem for $p=2$ in $\mathsf{RCD}^*(N-1,N)$ spaces has been recently obtained by Ketterer [43] by different methods (see also [41]); the approach proposed here has the double advantage of length and of being valid for every $p \in (1,\infty)$.

5. The case $p = 1$ and the Cheeger constant

It is well known (see for instance [40, 76]) that an alternative way of defining $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})}$ which extends also to $p = 1$ is the following. For every $p \in [1, \infty)$ and every $f \in L^p(X)$ let

$$c_p(f) := \inf_{c \in \mathbb{R}} \left(\int_X |f - c|^p \, \mathfrak{m} \right)^{1/p}.$$

For every $p \in (1, \infty)$ it holds that [40, Corollary 2.11]

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} = \inf \left\{ \int_X |\nabla f|^p \, \mathfrak{m} : f \in \mathrm{Lip} \cap L^p(X),\, c_p(f) = \|f\|_{L^p} = 1 \right\}.$$

It is then natural to set

$$\lambda^{1,1}_{(X,\mathsf{d},\mathfrak{m})} = \inf \left\{ \int_X |\nabla f| \, \mathfrak{m} : f \in \mathrm{Lip} \cap L^1(X),\, c_1(f) = \|f\|_{L^1} = 1 \right\}. \tag{5.1}$$

Assuming that $\mathfrak{m}(X) = 1$, recall that a number $M_f \in \mathbb{R}$ is a median for f if and only if

$$\mathfrak{m}(\{f \geq M_f\}) \geq \frac{1}{2} \qquad \text{and} \qquad \mathfrak{m}(\{f \leq M_f\}) \geq \frac{1}{2}.$$

It is not difficult to check that (see for instance [19, Section VI]) for every $f \in L^1(X)$ there exists a median of f, and moreover

$$\int_X |f - M_f| \, \mathfrak{m} = c_1(f)$$

holds for every median M_f of f. This link between $c_1(f)$ and M_f is useful to prove the equivalence between the Cheeger constant and $\lambda^{1,1}_{(X,\mathsf{d},\mathfrak{m})}$. Recall that the Cheeger constant $h_{(X,\mathsf{d},\mathfrak{m})}$ is defined by

$$h_{(X,\mathsf{d},\mathfrak{m})} := \inf \left\{ \frac{\mathfrak{m}^+(E)}{\mathfrak{m}(E)} \ : \ E \subset X \text{ is Borel and } \mathfrak{m}(E) \in (0, 1/2] \right\},$$

where

$$\mathfrak{m}^+(E) := \liminf_{\varepsilon \downarrow 0} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon}$$

is the (outer) Minkowski content. As usual $E^\varepsilon := \{x \in X : \exists y \in E \text{ such that } \mathsf{d}(x, y) < \varepsilon\}$ is the ε-neighborhood of E with respect to the metric d. The next result, due to Maz'ya [55] and Federer-Fleming [29] (see also [10] for a careful derivation, [56, Lemma 2.2] and [40, Proposition 2.13] for the present formulation), rewrites Cheeger's isoperimetric inequality in functional form.

Proposition 5.1. *Assume that $(X, \mathsf{d}, \mathfrak{m})$ is a m.m.s with $\mathfrak{m}(\{x\}) = 0$ for every $x \in X$, i.e. $\mathfrak{m}$ is atomless. Then*

$$h_{(X,\mathsf{d},\mathfrak{m})} = \lambda^{1,1}_{(X,\mathsf{d},\mathfrak{m})}.$$

It is then clear that the comparison and almost rigidity theorems for $\lambda^{1,1}$ will be based on the corresponding isoperimetric ones obtained by the authors in [17]. To this aim in the next subsection we briefly recall the model Cheeger constant for the comparison.

5.1. The model Cheeger constant $h_{K,N,D}$.

If $K > 0$ and $N \in \mathbb{N}$, by the Lévy-Gromov isoperimetric inequality we know that, for N-dimensional smooth manifolds having Ricci curvature bounded below by K, the Cheeger constant i is bounded below by the one of the N-dimensional round sphere of the suitable radius. In other words the *model* Cheeger constant is the one of $\mathbb{S}^N$. For $N \geq 1, K \in \mathbb{R}$ arbitrary real numbers the situation is more complicated, and just recently E. Milman [57] discovered what is the model Cheeger constant (more precisely he discovered the model isoperimetric profile, which in turn implies the model Cheeger constant). In this short section we recall its definition.

Given $\delta > 0$, set

$$s_\delta(t) := \begin{cases} \sin(\sqrt{\delta} t)/\sqrt{\delta} & \delta > 0 \\ t & \delta = 0 \\ \sinh(\sqrt{-\delta} t)/\sqrt{-\delta} & \delta < 0 \end{cases}, \quad c_\delta(t) := \begin{cases} \cos(\sqrt{\delta} t) & \delta > 0 \\ 1 & \delta = 0 \\ \cosh(\sqrt{-\delta} t) & \delta < 0 \end{cases}.$$

Given a continuous function $f:\mathbb{R}\to\mathbb{R}$ with $f(0)\geq 0$, we denote by $f_+:\mathbb{R}\to\mathbb{R}^+$ the function coinciding with f between its first non-positive and first positive roots, and vanishing everywhere else, i.e. $f_+ := f\chi_{[\xi_-,\xi_+]}$ with $\xi_- = \sup\{\xi\leq 0; f(\xi)=0\}$ and $\xi_+ = \inf\{\xi>0; f(\xi)=0\}$.

Given $H,K\in\mathbb{R}$ and $N\in[1,\infty)$, set $\delta := K/(N-1)$ and define the following (Jacobian) function of $t\in\mathbb{R}$:

$$J_{H,K,N}(t) := \begin{cases} \chi_{\{t=0\}} & N=1, K>0 \\ \chi_{\{Ht\geq 0\}} & N=1, K\leq 0 \\ \left(c_\delta(t)+\frac{H}{N-1}s_\delta(t)\right)_+^{N-1} & N\in(1,\infty) \end{cases}.$$

As last piece of notation, given a non-negative integrable function f on a closed interval $L\subset\mathbb{R}$, we denote with $\mu_{f,L}$ the probability measure supported in L with density (with respect to the Lebesgue measure) proportional to f there. In order to simplify a bit the notation we will write $h_{(L,f)}$ in place of $h_{(L,|\cdot|,\mu_{f,L})}$. The model Cheeger constant for spaces having Ricci curvature bounded below by $K\in\mathbb{R}$, dimension bounded above by $N\geq 1$ and diameter at most $D\in(0,\infty]$ is then defined by

$$h_{K,N,D} := \inf_{H\in\mathbb{R}, a\in[0,D]} h_{([-a,D-a],J_{H,K,N})}. \tag{5.2}$$

The formula above has the advantage of considering all the possible cases in just one equation, but probably it is also instructive to isolate the different cases in a more explicit way. Indeed one can check [57, Section 4] that:

- **Case 1**: $K>0$ and $D<\sqrt{\frac{N-1}{K}}\pi$,
$$h_{K,N,D} = \inf_{\xi\in\left[0,\sqrt{\frac{N-1}{K}}\pi-D\right]} h_{\left([\xi,\xi+D],\sin(\sqrt{\frac{K}{N-1}}t)^{N-1}\right)}.$$
- **Case 2**: $K>0$ and $D\geq\sqrt{\frac{N-1}{K}}\pi$,
$$h_{K,N,D} = h_{\left([0,\sqrt{\frac{N-1}{K}}\pi],\sin(\sqrt{\frac{K}{N-1}}t)^{N-1}\right)}.$$
- **Case 3**: $K=0$ and $D<\infty$,
$$\begin{aligned} h_{K,N,D} &= \min\left\{\begin{array}{c} \inf_{\xi\geq 0} h_{([\xi,\xi+D],t^{N-1})}\ , \\ h_{([0,D],1)} \end{array}\right\} \\ &= \frac{N}{D}\inf_{\xi\geq 0, v\in(0,1/2)} \frac{\left(\min(v,1-v)(\xi+1)^N+\max(v,1-v)\xi^N\right)^{\frac{N-1}{N}}}{v\left[(\xi+1)^N-\xi^N\right]}. \end{aligned}$$
- **Case 4**: $K<0$, $D<\infty$:
$$h_{K,N,D} = \min\left\{\begin{array}{c} \inf_{\xi\geq 0} h_{\left([\xi,\xi+D],\ \sinh(\sqrt{\frac{-K}{N-1}}t)^{N-1}\right)}\ , \\ h_{\left([0,D],\exp(\sqrt{-K(N-1)}t)\right)}\ , \\ \inf_{\xi\in\mathbb{R}} h_{\left([\xi,\xi+D],\ \cosh(\sqrt{\frac{-K}{N-1}}t)^{N-1}\right)} \end{array}\right\}.$$
- In all the remaining cases, the model Cheeger constant trivializes: $h_{K,N.D}=0$.

5.2. Sharp comparison and almost rigidity for $\lambda^{1,1}=h$.

Theorem 5.2. *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching $\mathsf{CD}_{loc}(K,N)$-space for some $K\in\mathbb{R}, N\in[1,\infty)$, with $\mathfrak{m}(X)=1$ and having diameter $D\in(0,+\infty]$. Then*

$$h_{(X,\mathsf{d},\mathfrak{m})}\geq h_{K,N,D}. \tag{5.3}$$

Moreover, for $K>0$ the following holds: for every $N>1$ and $\varepsilon>0$ there exists $\bar\delta=\bar\delta(K,N,\varepsilon)$ such that, for every $\delta\in[0,\bar\delta]$, if $(X,\mathsf{d},\mathfrak{m})$ is an essentially non-branching $\mathsf{CD}^*(K-\delta,N+\delta)$-space such that

$$h_{(X,\mathsf{d},\mathfrak{m})}\leq h_{K,N,\pi\sqrt{(N-1)/K}}+\delta \quad \left(= h(S^N(\sqrt{(N-1)/K}))+\delta,\ \textit{if } N\in\mathbb{N}\right), \tag{5.4}$$

then $\operatorname{diam}(X)\geq\pi\sqrt{(N-1)/K}-\varepsilon$.

Proof. Recall that the isoperimetric profile of $(X,\mathsf{d},\mathfrak{m})$ is the largest function $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})} : [0,1] \to \mathbb{R}^+$ such that for every Borel subset $E \subset X$ it holds $\mathfrak{m}^+(E) \geq \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mathfrak{m}(E))$. As discovered in [57] (see also [17, Section 2.5] for the present notation), for every $K \in \mathbb{R}, N \in [1,\infty)$, $D \in (0,\infty]$ there exists a model isoperimetric profile $\mathcal{I}_{K,N,D} : [0,1] \to \mathbb{R}^+$; it is straightforward to check that

$$h_{(X,\mathsf{d},\mathfrak{m})} = \inf_{v\in(0,1/2)} \frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v)}{v} \quad \text{and} \quad h_{K,N,D} = \inf_{v\in(0,1/2)} \frac{\mathcal{I}_{K,N,D}(v)}{v}.$$

Since in our previous paper [17, Theorem 1.2] we proved that for every $v > 0$ it holds

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) \geq \mathcal{I}_{K,N,D}(v), \tag{5.5}$$

the first claim (5.3) follows.
In order to prove the second part of the theorem, note (5.4) implies that there exists $\bar v \in (0,1/2)$ such that

$$\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\bar v)}{\bar v} \leq h_{(X,\mathsf{d},\mathfrak{m})} + \delta \leq h_{K,N,\pi\sqrt{(N-1)/K}} + 2\delta \leq \frac{\mathcal{I}_{K,N,\pi\sqrt{(N-1)/K}}(\bar v)}{\bar v} + 2\delta.$$

Multiplying by $\bar v$, we get

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\bar v) \leq \mathcal{I}_{K,N,\pi\sqrt{(N-1)/K}}(\bar v) + 2\delta\bar v \leq \mathcal{I}_{K,N,\pi\sqrt{(N-1)/K}}(\bar v) + \delta.$$

The thesis then follows by direct application of [17, Theorem 1.5].

□

Before stating the result let us observe that if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}^*(K,N)$ space for some $K > 0$ then, called $\mathsf{d}' := \sqrt{\frac{K}{N-1}}\,\mathsf{d}$, we have that $(X,\mathsf{d}',\mathfrak{m})$ is $\mathsf{RCD}^*(N-1,N)$; in other words, if the Ricci lower bound is $K > 0$ then up to scaling we can assume it is actually equal to $N-1$.

Arguing as in the proof of Corollaries 4.6-4.7 we get the following result.

Corollary 5.3. *For every $N \in [2,\infty)$, $\varepsilon > 0$ there exists $\bar\delta = \bar\delta(N,\varepsilon) > 0$ such that the following hold. For every $\delta \in [0,\bar\delta]$, if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}^*(N-1-\delta,N+\delta)$-space with $\mathfrak{m}(X)=1$, satisfying*

$$h_{(X,\mathsf{d},\mathfrak{m})} \leq h_{N-1,N,\pi} + \delta \quad (= h(S^N) + \delta, \text{ if } N \in \mathbb{N}),$$

then there exists an $\mathsf{RCD}^*(N-2,N-1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y)=1$ such that

$$\mathsf{d}_{mGH}(X,[0,\pi]\times_{\sin}^{N-1} Y) \leq \varepsilon.$$

In particular, if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}^*(N-1,N)$-space satisfying $h_{(X,\mathsf{d},\mathfrak{m})} = h_{N-1,N,\pi} = h(S^N)$, then it is isomorphic to a spherical suspension; i.e. there exists an $\mathsf{RCD}^*(N-2,N-1)$ space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y)=1$ such that $(X,\mathsf{d},\mathfrak{m})$ is isomorphic to $[0,\pi]\times_{\sin}^{N-1} Y$.

6. Sharp Log-Sobolev and Talagrand inequalities

6.1. **Sharp Log-Sobolev in diameter-curvature-dimensional form.** Recall that a m.m.s. $(X,\mathsf{d},\mathfrak{m})$ supports the Log-Sobolev inequality with constant $\alpha > 0$ if for any Lipschitz function $f : X \to [0,\infty)$ with $\int_X f\,\mathfrak{m} = 1$ it holds

$$2\alpha \int_X f\log f\,\mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f}\mathfrak{m}. \tag{6.1}$$

The largest constant α, such that (6.1) holds for any Lipschitz function $f : X \to [0,\infty)$ with $\int_X f\,\mathfrak{m} = 1$, will be called Log-Sobolev constant of $(X,\mathsf{d},\mathfrak{m})$ and denoted with $\alpha^{LS}_{(X,\mathsf{d},\mathfrak{m})}$.

As before we will reduce to the one-dimensional case. Given $K \in \mathbb{R}$, $N \geq 1$, $D \in (0,+\infty]$ we denote with $\alpha^{LS}_{K,N,D} > 0$ the maximal constant α such that

$$2\,\alpha \int_{\mathbb{R}} f\log f\,\mu \leq \int_{\{f>0\}} \frac{|f'|^2}{f}\mu, \quad \forall \mu \in \mathcal{F}^s_{K,N,D}, \tag{6.2}$$

for every Lipschitz $f : \mathbb{R} \to [0,\infty)$ with $\int f\,\mu = 1$.

Remark 6.1. If $K>0$ and $D=\pi\sqrt{\frac{N-1}{K}}$, it is known that the corresponding optimal Log-Sobolev constant is $\frac{KN}{N-1}$ (see the discussion below). It is an interesting open problem, that we don't address here, to give an explicit expression of the quantity $\alpha^{LS}_{K,N,D}$ for general $K\in\mathbb{R}, N\geq 1$, $D\in(0,\infty)$.

Theorem 6.2. *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space with diameter $D\in(0,\infty)$ and satisfying $\mathsf{CD}_{loc}(K,N)$ for some $K\in\mathbb{R}, N\in[1,\infty)$. Assume moreover it is essentially non-branching. Then for any Lipschitz function $f:X\to[0,\infty)$ with $\int_X f\,\mathfrak{m}=1$ it holds*

$$2\,\alpha^{LS}_{K,N,D}\int_X f\log f\,\mathfrak{m}\leq\int_{\{f>0\}}\frac{|\nabla f|^2}{f}\mathfrak{m}.$$

In other terms it holds $\alpha^{LS}_{(X,\mathsf{d},\mathfrak{m})}\geq\alpha^{LS}_{K,N,D}$.

Proof. Since $\mathsf{CD}_{loc}(K,N)$ implies that the measure is locally doubling, the finiteness of the diameter implies that $\mathfrak{m}(X)<\infty$. Observing that the Log-Sobolev inequality (6.1) is invariant under a multiplication of $\mathfrak{m}$ by a constant, we can then assume without loss of generality that $\mathfrak{m}(X)=1$. Consider any Lipschitz function with $\int_X f\,\mathfrak{m}=1$ and apply Theorem 2.7 to $\hat f:=1-f$. Hence we can write $X=Y\cup\mathcal{T}$ with

$$f(y)=1,\quad \mathfrak{m}\text{-a.e. } y\in Y,\qquad \mathfrak{m}\llcorner_{\mathcal{T}}=\int_Q\mathfrak{m}_q\,\mathfrak{q}(dq),$$

with $\mathfrak{m}_q=g(q,\cdot)\,\sharp\,\big(h_q\cdot\mathcal{L}^1\big)$, the density h_q verifies (2.6) for $\mathfrak{q}$-a.e. $q\in Q$ and

$$1=\int_X f(z)\,\mathfrak{m}_q(dz)=\int_{\mathrm{Dom}\,(g(q,\cdot))}f(g(q,t))\cdot h_q(t)\,\mathcal{L}^1(dt)$$

for $\mathfrak{q}$-a.e. $q\in Q$. Now consider the map $t\mapsto f_q(t):=f(g(q,t))$ and note that it is Lipschitz. Since $\mathrm{diam}\,(\mathrm{Dom}\,(g(q,\cdot)))\leq D$, from the definition of $\mathcal{F}^s_{K,N,D}$ and of $\alpha^{LS}_{K,N,D}$ we deduce that

$$2\alpha^{LS}_{K,N,D}\int_{\mathbb{R}}f_q(t)\log(f_q(t))\,h_q(t)\,\mathcal{L}^1(dt)\leq\int_{\{f_q(\cdot)>0\}}\frac{|f_q'(t)|^2}{f_q(t)}\,h_q(t)\,\mathcal{L}^1(dt).$$

Noticing that $|f_q'(t)|\leq|\nabla f|(g(q,t))$ and that $f\log f$ vanishes over Y, one obtains that

$$\begin{aligned}
2\alpha^{LS}_{K,N,D}\int_X f\log f\,\mathfrak{m}(dx)&=2\alpha^{LS}_{K,N,D}\int_{\mathcal{T}}f\log f\,\mathfrak{m}(dx)\\
&=2\alpha^{LS}_{K,N,D}\int_Q\left(\int_X f\log f\,\mathfrak{m}_q(dx)\right)\mathfrak{q}(dq)\\
&=2\alpha^{LS}_{K,N,D}\int_Q\left(\int_{\mathrm{Dom}\,(g(q,\cdot))}f_q(t)\log(f_q(t))\,h_q(t)\,\mathcal{L}^1(dt)\right)\mathfrak{q}(dq)\\
&\leq\int_Q\left(\int_{\mathrm{Dom}\,(g(q,\cdot))\cap\{f_q(\cdot)>0\}}\frac{|f_q'(t)|^2}{f_q(t)}\,h_q(t)\,\mathcal{L}^1(dt)\right)\mathfrak{q}(dq)\\
&\leq\int_Q\left(\int_{\{f>0\}}\frac{|\nabla f|^2}{f}\,(g(q,\cdot))\,\sharp\,\big(h_q(t)\,\mathcal{L}^1\big)\,(dx)\right)\mathfrak{q}(dq)\\
&\leq\int_{\{f>0\}}\frac{|\nabla f|^2}{f}\,\mathfrak{m}(dx),
\end{aligned}$$

and the claim follows. □

If $K>0$, by Bonnet-Myers diameter bound, we know that if $(X,\mathsf{d},\mathfrak{m})$ satisfies $\mathsf{CD}_{loc}(K,N)$ then $\mathrm{diam}\,(X)\leq\pi\sqrt{\frac{N-1}{K}}$. Recalling definition (6.2), we then set $\alpha^{LS}_{K,N}:=\alpha^{LS}_{K,N,\pi\sqrt{\frac{N-1}{K}}}$ for the Log-Sobolev constant without an upper diameter bound. By applying the regularization of the measures $h\,\mathcal{L}^1\in\mathcal{F}^s_{K,N,\pi\sqrt{\frac{N-1}{K}}}$ discussed in Lemma 4.1 and arguing analogously to the proof of Proposition 4.2, we get that in the definition of $\alpha^{LS}_{K,N}$ it is equivalent to take the inf among measures in $\mathcal{F}_{K,N,\pi\sqrt{\frac{N-1}{K}}}$, defined in (4.4). But now if $\mu\in\mathcal{F}_{K,N,\pi\sqrt{\frac{N-1}{K}}}$ is a probability measure on $\mathbb{R}$ with smooth density satisfying the

$\mathsf{CD}_{loc}(K,N)$ condition for $K>0$, it is known that the sharp Log-Sobolev constant is $\alpha^{LS}_{K,N}=\frac{KN}{N-1}$ (see for instance [8, Proposition 6.6]). More precisely, as proved by Mueller-Weissler [60], for every $K>0$ and $N\geq 1$, the sharp constant is attained by the usual model probability measure on the interval $\big[0,\sqrt{\frac{N-1}{K}}\pi\big]$ proportional to $\sin\big(\sqrt{\frac{K}{N-1}}t\big)^{N-1}$; notice that for $N\in\mathbb{N}$ it corresponds to the round sphere of radius $\sqrt{\frac{N-1}{K}}$. We then have the following corollary.

Corollary 6.3 (Sharp Log-Sobolev under $\mathsf{CD}^*(K,N), K>0, N>1$)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space satisfying $\mathsf{CD}^*(K,N)$ for some $K>0,N>1$, and assume moreover it is essentially non-branching. Then for any Lipschitz function $f:X\to[0,\infty)$ with $\int_X f\,\mathfrak{m}=1$ it holds*

$$\frac{2KN}{N-1}\int_X f\log f\,\mathfrak{m}\leq\int_{\{f>0\}}\frac{|\nabla f|^2}{f}\mathfrak{m}.$$

In other terms it holds $\alpha^{LS}_{(X,\mathsf{d},\mathfrak{m})}\geq\frac{KN}{N-1}$.

Let us mention that, since the reduction to a 1-D problem is done via an L^1-optimal transportation argument, Corollary 6.3 can be seen as a solution to [75, Open Problem 21.6].

6.2. **From Sharp Log-Sobolev to Sharp Talagrand.** First of all let us recall that the relative entropy functional $Ent_{\mathfrak{m}}:\mathcal{P}(X)\to[0,+\infty]$ with respect to a given $\mathfrak{m}\in\mathcal{P}(X)$ is defined to be

$$Ent_{\mathfrak{m}}(\mu)=\int_X\varrho\log\varrho\,\mathfrak{m},\qquad\text{if }\mu=\varrho\mathfrak{m}$$

and $+\infty$ otherwise.

Otto-Villani [62] proved that for smooth Riemannian manifolds the Log-Sobolev inequality with constant $\alpha>0$ implies the Talagrand inequality with constant $\frac{2}{\alpha}$ preserving sharpness. The result was then generalized to arbitrary metric measure spaces by Gigli-Ledoux [34], so that we can state:

Theorem 6.4 (From Log-Sobolev to Talagrand, [62, 34])**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space supporting the Log-Sobolev inequality with constant $\alpha>0$. Then it also supports the Talagrand inequality with constant $\frac{2}{\alpha}$, i.e. it holds*

$$W_2^2(\mu,\mathfrak{m})\leq\frac{2}{\alpha}Ent_{\mathfrak{m}}(\mu)$$

for all $\mu\in\mathcal{P}(X)$.

Combining Theorem 6.2 with Theorem 6.4 we get Theorem 1.10 which improves the Talagrand constant $2/K$, which is sharp for $\mathsf{CD}(K,\infty)$ spaces, by a factor $N-1/N$ in case the dimension is bounded above by N. This constant is sharp for $\mathsf{CD}_{loc}(K,N)$ spaces, indeed it is sharp already in the smooth setting [75, Remark 22.43]. Since both our proof of the sharp Log-Sobolev inequality and the proof of Theorem 6.4 are essentially optimal transport based, this be seen as an answer to [75, Open Problem 22.44].

Remark 6.5 (Sharpness and estimates of the best constants)**.** Recall that for weighted smooth manifolds, the Log-Sobolev inequality implies the Talagrand inequality which in turns implies the Poincaré inequality every step without any loss in the constants [75, Theorem 22.17]. Since when we compute the comparison Log-Sobolev constant $\alpha^{LS}_{K,N,D}$ and the comparison first eigenvalue $\lambda^{1,2}_{K,N,D}$, we work with the smooth measures $\mathcal{F}_{K,N,D}$ on $\mathbb{R}$, we always have the estimate

$$\alpha^{LS}_{K,N,D}\leq\lambda^{1,2}_{K,N,D}. \tag{6.3}$$

Notice that, for $K>0$ and $D=\sqrt{\frac{N-1}{K}}\pi$ they actually coincide:

$$\frac{KN}{N-1}=\alpha^{LS}_{K,N,\sqrt{\frac{N-1}{K}}\pi}=\lambda^{1,2}_{K,N,\sqrt{\frac{N-1}{K}}\pi}. \tag{6.4}$$

An interesting question we do not address here is if this is always the case, i.e. if in (6.3) equality holds for every $K\in\mathbb{R}$, $N\geq 1$, $D\in(0,\infty)$. Since the value of $\lambda^{1,2}_{K,N,D}$ is known in many cases, it would have as a consequence the determination of the explicit value of the best constant in both the Log-Sobolev and the Talagrand inequalities in the curvature-dimension-diameter forms. This would also imply rigidity and almost-rigidity statements attached to the Log-Sobolev and Talagrand inequalities, as proven here for the Poincaré inequality. Let us note that for the almost rigidity to hold for both the Log-Sobolev and Talagrand inequalities it would be enough to prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha^{LS}_{K,N,D} \geq \alpha^{LS}_{K,N,\sqrt{\frac{N-1}{K}}\pi} + \delta = \frac{KN}{N-1} + \delta$, if $D \in \big[0, \sqrt{\frac{N-1}{K}}\varepsilon - \delta\big]$.

7. Sharp Sobolev Inequalities

Recall that $(X,\mathsf{d},\mathfrak{m})$ supports a (p,q)-Sobolev inequality with constant $\alpha^{p,q}$ if for any Lipschitz function $f : X \to \mathbb{R}$ it holds

$$\frac{\alpha^{p,q}}{p-q}\left\{\left(\int_X |f|^p\,\mathfrak{m}\right)^{\frac{q}{p}} - \int_X |f|^q\,\mathfrak{m}\right\} \leq \int_X |\nabla f|^q\,\mathfrak{m}. \tag{7.1}$$

The largest constant $\alpha^{p,q}$ such that (7.1) holds for any Lipschitz function f will be called the (p,q)-Sobolev constant of $(X,\mathsf{d},\mathfrak{m})$ and will be denoted by $\alpha^{p,q}_{(X,\mathsf{d},\mathfrak{m})}$.

Again we consider the one-dimensional case and given $K \in \mathbb{R}, N \geq 1$ and $D \in (0,\infty]$ we define ${}^s\alpha^{p,q}_{K,N,D}$ to be the maximal constant α such that

$$\frac{\alpha}{p-q}\left\{\left(\int_X |f|^p\,\mu\right)^{\frac{q}{p}} - \int_X |f|^q\,\mu\right\} \leq \int_X |\nabla f|^q\,\mu, \qquad \forall\, \mu \in \mathcal{F}^s_{K,N,D},$$

for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$. Restricting the maximization to $\mu \in \mathcal{F}_{K,N,D}$, we obtain the constant $\alpha^{p,q}_{K,N,D}$. Using the approximation Lemma 4.1 and reasoning as in Proposition 4.2 one obtains that

$${}^s\alpha^{p,q}_{K,N,D} = \alpha^{p,q}_{K,N,D}.$$

Theorem 7.1. *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space with diameter $D \in (0,\infty)$ and satisfying $\mathsf{CD}_{loc}(K,N)$ for some $K \in \mathbb{R}, N \in [1,\infty)$. Assume moreover it is essentially non-branching. Then for any Lipschitz function it holds*

$$\frac{\alpha^{p,q}_{K,N,D}}{p-q}\left\{\left(\int_X |f(x)|^p\,\mathfrak{m}(dx)\right)^{\frac{q}{p}} - \int_X |f(x)|^q\,\mathfrak{m}(dx)\right\} \leq \int_X |\nabla f(x)|^q\,\mathfrak{m}(dx),$$

In other terms, it holds $\alpha^{p,q}_{(X,\mathsf{d},\mathfrak{m})} \geq \alpha^{p,q}_{K,N,D}$.

Proof. First of all note that $\mathsf{CD}_{loc}(K,N)$ coupled with the finiteness of the diameter implies $\mathfrak{m}(X) < \infty$.

Step 1: The case $p > q$.
With a slight abuse of notation q will denote both the exponent in the Sobolev embedding and the index in the disintegration, there should be no confusion since the clearly different roles. Fix any Lipschitz function f and consider the function $\hat f(x) := 1 - c|f(x)|^p$, with $c := 1/(\int |f|^p \mathfrak{m})$. Therefore $\int \hat f\,\mathfrak{m} = 0$ and we can invoke Theorem 2.7. Hence $X = Y \cup \mathcal{T}$ with

$$\hat f(y) = 0, \quad \mathfrak{m}\text{-a.e. } y \in Y, \qquad \mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_q\,\mathfrak{q}(dq),$$

with $\mathfrak{m}_q = g(q,\cdot)\,\sharp\,\big(h_q \cdot \mathcal{L}^1\big)$, the density h_q verifies (2.6) for $\mathfrak{q}$-a.e. $q \in Q$ and

$$0 = \int_X \hat f(z)\,\mathfrak{m}_q(dz) = \int_{\mathrm{Dom}\,(g(q,\cdot))} \hat f(g(q,t)) \cdot h_q(t)\,\mathcal{L}^1(dt)$$

for $\mathfrak{q}$-a.e. $q \in Q$.

Now consider the map $t \mapsto f_q(t) := f(g(q,t))$ and note that it is Lipschitz. Since $\mathrm{diam}\,(\mathrm{Dom}\,(g(q,\cdot))) \leq D$, from the definition of $\mathcal{F}^s_{K,N,D}$ and of $\alpha^{p,q}_{K,N,D}$ we deduce that

$$\left(\int_{\mathbb{R}} |f_q(t)|^p h_q(t)\,\mathcal{L}^1(dt)\right)^{\frac{q}{p}} \leq \int_{\mathbb{R}} |f_q(t)|^q h_q(t)\,\mathcal{L}^1(dt) + \frac{p-q}{\alpha^{p,q}_{K,N,D}} \int_{\mathbb{R}} |f'(t)|^q h_q(t)\,\mathcal{L}^1(dt).$$

Since for $\mathfrak{q}$-a.e. $q \in Q$ it holds $\int \hat f\,\mathfrak{m}_q = 0$, it follows that

$$\int_X |f(x)|^p\,\mathfrak{m}_q(dx) = \frac{1}{c} = \int_X |f(x)|^p\,\mathfrak{m}(dx).$$

Therefore the previous inequality reads as

$$1 \leq \left(\frac{1}{\int |f(x)|^p \, \mathfrak{m}(dx)} \right)^{\frac{q}{p}} \left(\int_X |f_q|^q \, \mathfrak{m}_q + \frac{p-q}{\alpha^{p,q}_{K,N,D}} \int_X |f'|^q \, \mathfrak{m}_q \right).$$

Noticing that $|f_q'(t)| \leq |\nabla f|(g(q,t))$, integrating over Q one obtains that

$$\mathfrak{m}(\mathcal{T}) \leq \left(\frac{1}{\int |f(x)|^p \, \mathfrak{m}(dx)} \right)^{\frac{q}{p}} \int_{\mathcal{T}} |f(x)|^q \, \mathfrak{m}(dx) + \frac{p-q}{\alpha^{p,q}_{K,N,D}} \int_{\mathcal{T}} |\nabla f(x)|^q \, \mathfrak{m}(dx). \tag{7.2}$$

To complete the argument one should prove that for each $y \in Y$

$$1 \leq \left(\frac{1}{\int |f|^p \, \mathfrak{m}} \right)^{\frac{q}{p}} \left(|f(y)|^q + \frac{p-q}{\alpha^{p,q}_{K,N,D}} |\nabla f(y)|^q \right).$$

As for $\mathfrak{m}$-a.e. $y \in Y$ one has $|f(y)|^p = \int_X |f|^p \, \mathfrak{m}$, this last inequality holds trivially. Integrating this last inequality over Y and adding it to (7.2), we obtain the claim.

Step 2: The case $p < q$. It follows repeating the previous localization argument and writing the Sobolev inequality in the following form

$$\left(\int_X |f(x)|^p \, \mathfrak{m}(dx) \right)^{\frac{q}{p}} \geq \int_X |f(x)|^q \, \mathfrak{m}(dx) - \frac{q-p}{\alpha} \int_X |\nabla f(x)|^q \, \mathfrak{m}(dx).$$

□

As already observed, if $K > 0$ then $\operatorname{diam}(X) \leq \pi\sqrt{(N-1)/K}$ and therefore one can define

$$\alpha^{p,q}_{K,N} := \alpha^{p,q}_{K,N,\pi\sqrt{(N-1)/K}},$$

the (p,q)-Sobolev inequality with no diameter upper bound. If $\mu \in \mathcal{F}_{K,N,\pi\sqrt{(N-1)/K}}$ with $K > 0$, it is known that the sharp $(p,2)$-Sobolev constant, verifies (see for instance [46, Theorem 3.1])

$$\alpha^{p,2}_{K,N} \geq \frac{KN}{N-1}, \qquad \text{for } \; 1 \leq p \leq \frac{2N}{N-2}.$$

Moreover, for $N \in \mathbb{N}$, it is attained on the round sphere of radius $\sqrt{\frac{N-1}{K}}$. We then have the following corollary.

Corollary 7.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space satisfying $\mathsf{CD}^*(K,N)$ for some $K > 0, N \in (2,\infty)$, and assume moreover it is essentially non-branching. Then for any Lipschitz function f it holds*

$$\frac{KN}{(p-2)(N-1)} \left\{ \left(\int_X |f|^p \, \mathfrak{m} \right)^{\frac{2}{p}} - \int_X |f|^2 \, \mathfrak{m} \right\} \leq \int_X |\nabla f|^2 \, \mathfrak{m},$$

for any $2 < p \leq 2N/(N-2)$. In other terms it holds $\alpha^{p,2}_{(X,\mathsf{d},\mathfrak{m})} \geq \frac{KN}{N-1}$.

Corollary 7.2 can be seen as a solution to [75, Open Problem 21.11].

Appendix

All the inequalities we have presented here rely on the general scheme of applying one-dimensional localization to a big family of inequalities, called 4-functions inequalities (see for instance the work of Kannan-Lovász-Simonovits [42]).

The argument goes as follows. Suppose we are interested in proving that for f_1, f_2, f_3, f_4 integrable functions and $\alpha, \beta > 0$ it holds

$$\left(\int_X f_1 \, \mathfrak{m} \right)^{\alpha} \left(\int_X f_2 \, \mathfrak{m} \right)^{\beta} \leq \left(\int_X f_3 \, \mathfrak{m} \right)^{\alpha} \left(\int_X f_4 \, \mathfrak{m} \right)^{\beta}. \tag{7.3}$$

Then consider the one-dimensional localization induced by $g := f_3 - c f_1$, with $c = (\int f_3 \mathfrak{m})/(\int f_1 \mathfrak{m})$:

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_q \mathfrak{q}(dq),$$

where $X = \mathcal{T} \cup Y$ and on Y it holds $g(x) = 0$ for $\mathfrak{m}$-a.e. $x \in Y$. Then it is sufficient to prove that

$$\left(\int_X f_1\,\mathfrak{m}_q\right)^{\alpha}\left(\int_X f_2\,\mathfrak{m}_q\right)^{\beta} \leq \left(\int_X f_3\,\mathfrak{m}_q\right)^{\alpha}\left(\int_X f_4\,\mathfrak{m}_q\right)^{\beta}, \quad \mathfrak{q}-\text{a.e. } q \in Q$$
$$f_2(x) \leq c^{\alpha/\beta} f_4(x), \qquad \mathfrak{m}-\text{a.e. } x \in Y.$$

Indeed from the localization it follows that $\int g\,\mathfrak{m}_q = 0$ for $\mathfrak{q}$-a.e. $q \in Q$ and therefore

$$\int_X f_2(x)\,\mathfrak{m}_q(dx) \leq c^{\alpha/\beta}\int_X f_4(x)\,\mathfrak{m}_q(dq), \qquad \mathfrak{q}-\text{a.e. } q \in Q.$$

Integrating over Q and adding the integral over Y, (7.3) follows.

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure, *Trans. Amer. Math. Soc.* **367**(7), (2015), 4661–4701.

[2] L. Ambrosio, N Gigli, G. Savaré, Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. *Annals of Probab.*, **43** (1), (2015), 339–404.

[3] ———, Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below, *Invent. Math.* **195**, (2014), (2), 289–391.

[4] ———, Metric measure spaces with Riemannian Ricci curvature bounded from below, *Duke Math. J.*, **163**, 1405–1490, (2014).

[5] L. Ambrosio, A. Mondino and G. Savaré, Nonlinear diffusion equations and curvature conditions in metric measure spaces, *Preprint arXiv:1509.07273.*

[6] ———, On the Bakry-Émery condition, the gradient estimates and the Local-to-Global property of $RCD^*(K,N)$ metric measure spaces, *J. Geom. Anal.,* **26**, (2016), 24–56.

[7] K. Bacher and K.-T. Sturm, Localization and tensorization properties of the curvature-dimension condition for metric measure spaces, *Journ. Funct. Analysis,* **259** (2010), 28–56.

[8] D. Bakry, L'hypercontractivité et son utilisation en théorie des semigroupes, Lectures on Probability Theory, Ecole d'Eté de Probabilité de Saint-Flour XXII-1992, *Lecture Notes in Mathematics*, Springer, **1581**, (1994), 242–411.

[9] D. Bakry and Z. Qian, Some new results on eigenvectors via dimension, diameter, and Ricci curvature, *Adv. Math.*, **155** (1), (2000), 98–153.

[10] S.G. Bobkov and C. Houdré, Isoperimetric constants for product probability measures, *Ann. Probab.* **25**(1), (1997), 184–205.

[11] C. Borell, Convex set functions in d-space, *Period. Math. Hungar.* **6**,(1975), 111–136.

[12] H.J. Brascamp and E.H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *Journ. Funct. Analysis* **22**, (1976), 366–389.

[13] Y.D. Burago and V.A Zalgaller, Geometric inequalities, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, **285**. Springer, Berlin (1988).

[14] S. Bianchini and F. Cavalletti, The Monge problem for distance cost in geodesic spaces. *Commun. Math. Phys.*, 318, 615 – 673 (2013).

[15] F. Cavalletti, Monge problem in metric measure spaces with Riemannian curvature-dimension condition, *Nonlinear Analysis TMA* **99** (2014), 136–151.

[16] ———, Decomposition of geodesics in the Wasserstein space and the globalization property. *Geom. Funct. Anal.*, **24** (2014) 493 – 551.

[17] F. Cavalletti and A. Mondino, Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. *Preprint Arxiv:*1502.06465, (2015).

[18] F. Cavalletti and K.-T. Sturm. Local curvature-dimension condition implies measure-contraction property. *J. Funct. Anal.*, **262**, 5110 – 5127, 2012.

[19] I. Chavel, Isoperimetric inequalities, *Cambridge Tracts in Math.*, **145**, Cambridge Univ. Press, Cambridge, U.K. (2001).

[20] ———, Eigenvalues in Riemannian Geometry, *Academic Press*, (1984).

[21] G. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, *Problems in analysis, Symposium in Honor of S. Bochner*, Princeton Univ. Press, Princeton, NJ, (1970), 195–199.

[22] G. Cheeger and T.H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products, *Annals of Math.*, **144**, 1, (1996), 189–237.

[23] ———, On the structure of spaces with Ricci curvature bounded below. I. *J. Diff. Geom.*, **45** (1997), 406 – 480.

[24] ———, On the structure of spaces with Ricci curvature bounded below. II. *J. Diff. Geom.*, **54**, (2000), 13–35.

[25] ———, On the structure of spaces with Ricci curvature bounded below. III. *J. Diff. Geom.*, **54**, (2000), 37 – 74.

[26] T. Colding, A. Naber, Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications *Annals of Math.*, **176**, (2012).

[27] D. Cordero-Erausquin, R.J. McCann and M. Schmuckenschlger, A Riemannian interpolation inequality la Borell, Brascamp and Lieb *Invent. Math.*, **146** (2), (2001), 219–257.

[28] M Erbar, Kuwada and K.T. Sturm, On the Equivalence of the Entropic Curvature-Dimension Condition and Bochner's Inequality on Metric Measure Space, *Invent. Math.*, **201**, 3, (2015), 993–1071.
[29] H. Federer and W.H. Fleming, Normal and integral currents, *Annals of Math.* **2**, 72, (1960), 458–520 .
[30] D.H. Fremlin, *Measure Theory*, volume 4. Torres Fremlin, 2002.
[31] N. Garofalo and A. Mondino, Li-Yau and Harnack type inequalities in $\mathsf{RCD}^*(K,N)$ metric measure spaces, *Nonlinear Analysis TMA,* **95**, (2014), 721–734.
[32] N. Gigli, Optimal maps in non branching spaces with Ricci curvature bounded from below, *Geom. Funct. Anal.*, 22 (2012) no. 4, 990–999.
[33] ———, The splitting theorem in non-smooth context, *preprint arXiv:1302.5555*, (2013).
[34] N. Gigli, M. Ledoux, From log Sobolev to Talagrand: a quick proof, *Discrete Contin. Dyn. Syst*, **33**, (5), (2013), 1927–1935.
[35] N. Gigli, A. Mondino and G. Savaré, Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows, *to appear in Proc. London Math. Soc.,* doi: 10.1112/plms/pdv047.
[36] N. Gigli, A. Mondino and T. Rajala, Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below *Journal fur die Reine und Ang. Math. (Crelle's journal)*, **705**, (2015), 233–244.
[37] M. Gromov, Metric structures for Riemannian and non Riemannian spaces, *Modern Birkhäuser Classics*, (2007).
[38] M. Gromov and V. Milman, Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces. *Compositio Math.*, **62**, 3, (1987), 263–282.
[39] E. Hebey Sobolev Spaces on Riemannian Manifold, Lecture Notes in Mathematics, vol. 1635, Springer, 1996.
[40] S. Honda, Cheeger constant, p-Laplacian, and Gromov-Hausdorff convergence, *Preprint arxiv: 1310.0304v3*, (2014).
[41] Y. Jiang, H.-C. Zhang Sharp spectral gaps on metric measure spaces, *Calc. Var. Partial Differential Equations*, **55**, (1), (2016), 55–14.
[42] R. Kannan, L. Lovász and M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, *Discrete Comput. Geom.,* **13**, 3-4, (1995), 541–559.
[43] K. Ketterer, Cones over metric measure spaces and the maximal diameter theorem. *J. Math. Pures Appl.* **103**, 5, (2015), 1228–1275.
[44] ———, Obata's rigidity theorem for metric measure spaces, *Anal. Geom. Metr. Spaces*, **3**, (2015), 278–295.
[45] B. Klartag, Needle decomposition in Riemannian geometry, *to appear in Mem. AMS.*
[46] M. Ledoux, The geometry of Markov diffusion generators, *Ann. Fac. Sci. Toulouse Math. (6) 9*, **2**, (2000), 305–366.
[47] ———, Spectral gap, logarithmic Sobolev constant, and geometric bounds, *Surveys in Differential Geometry IX: Eigenvalues of Laplacians and other geometric operators,* Edited by A. Grigor'yan and S. T. Yau, International Press, Somerville, MA, (2004), 219–240,
[48] H.Q. Li and Y.Z. Wang, First eigenvalue lower estimates for the weighted p-Laplacian on smooth metric measure spaces, *Differential Geom. Appl.*, **45**, (2016), 23–42.
[49] P. Li and S.T. Yau, Estimates of eigenvalues of a compact Riemannian manifold, *AMS Symposium on Geometry of the Laplace Operator, XXXVI Hawaii*, (1979), 205–240.
[50] A. Lichnerowicz, Géometries des Groupes des Transformations, Paris, Dunod, (1958).
[51] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math.* (2) **169** (2009), 903–991.
[52] ———, Weak curvature conditions and functional inequalities, *J. Funct. Anal.*, **245**, (2007), 311–333.
[53] L. Lovász and M. Simonovits, Random walks in a convex body and an improved volume algorithm, *Random Structures Algorithms,* **4**, 4, (1993), 359–412.
[54] A.M. Matei, First eigenvalue for the p-Laplace operator, *Nonlinear Anal. TMA,* **39**, 1051–1068, (2000).
[55] V.G. Maz'ja, Sobolev Spaces *Springer Series in Soviet Mathematics*. Springer, Berlin, (1985).
[56] E. Milman, On the role of convexity in isoperimetry, spectral gap and concentration, *Invent. Math.*, **177**, (2009), 1–43.
[57] ———, Sharp Isoperimetric Inequalities and Model Spaces for Curvature-Dimension-Diameter Condition, *J. Eur. Math. Soc.*, **17**, (5), (2015), 1041–1078.
[58] E. Milman and L. Rotem, Complemented Brunn-Minkowski Inequalities and Isoperimetry for Homogeneous and Non-Homogeneous Measures, *Advances in Math.*, **262**, 867–908, (2014).
[59] A. Mondino and A. Naber, Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds I, *preprint*, arXiv:1405.2222.
[60] C.E. Mueller and F.B. Weissler, Hypercontractivity for the Heat Semigroup for Ultraspherical Polynomials and on the n-Sphere, *Journ. Funct. Analysis*, **48**, (1982), 252–283.
[61] A. Naber and D. Valtorta, Sharp estimates on the first eigenvalue of the p-Laplacian with negative Ricci lower bound, *Math. Z.*, **277**, (2014), 867–891.
[62] F. Otto and C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality, *J. Funct. Anal.*, **173**(2), (2000), 361–400.
[63] S.I.. Ohta, Finsler interpolation inequalities, *Calc. Var. Partial Differential Equations*, **36**, (2009), 211–249.
[64] L.E. Payne and H.F. Weinberger, An optimal Poincaré inequality for convex domains *Arch. Rational Mech. Anal.*, **5**, (1960), 286–292.
[65] A. Petrunin, Alexandrov meets Lott-Sturm-Villani, *Münster J. Math.*, **4**, (2011), 53–64.
[66] A. Profeta, The Sharp Sobolev Inequality on Metric Measure Spaces with Lower Ricci Curvature Bounds, *Potential Anal.*, **43**, (3), (2015), 513–529.

[67] T. Rajala, Local Poincaré inequalities from stable curvature conditions on metric spaces, *Calc. Var. Partial Differential Equations,* **44**, (2012), 477–494.
[68] ———, Failure of the local-to-global property for $\mathsf{CD}(K,N)$ spaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci*, to appear, preprint arXiv:1305.6436, (2013).
[69] T. Rajala and K.T. Sturm, Non-branching geodesics and optimal maps in strong $\mathsf{CD}(K,\infty)$-spaces, *Calc. Var. Partial Differential Equations,* **50**, (2014), 831–846.
[70] G. Savaré, Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in $\mathrm{RCD}(K,\infty)$ metric measure spaces, *Discrete Contin. Dyn. Syst.* **34**, (4), (2014), 1641–1661.
[71] R. Schoen and S.T. Yau, Lectures on Differential Geometry, International Press, Boston, (1994).
[72] K.T. Sturm, On the geometry of metric measure spaces. I, *Acta Math.* **196** (2006), 65–131.
[73] K.T. Sturm, On the geometry of metric measure spaces. II, *Acta Math.* **196** (2006), 133–177.
[74] D. Valtorta, Sharp estimate on the first eigenvalue of the p-Laplacian, *Nonlinear Anal.* **75**, 4974–4994 (2012).
[75] C. Villani, Optimal transport. Old and new, *Grundlehren der Mathematischen Wissenschaften*,**338**, Springer-Verlag, Berlin, (2009).
[76] J.Y. Wu, E.M. Wang, and Y. Zheng, First eigenvalue of the p-Laplace operator along the Ricci flow, *Ann. global anal. and geom.* **38**, (2010), 27–55.
[77] H.C. Yang and J.Q. Zhong, On the estimate of the first eigenvalue of a compact Riemannian manifold, *Sci. Sinica Ser.*, **27** (12), (1984), 1265–1273.

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