THE INTERACTION BETWEEN BULK ENERGY AND SURFACE ENERGY IN MULTIPLE INTEGRALS

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Abstract

This paper is devoted to the study of integral functionals defined on the space $SBV(\Omega;\mathbb{R}^k)$ of vector-valued special functions with bounded variation on the open set $\Omega \subset \mathbb{R}^n$, of the form

$$F(u) = \int_\Omega f(\nabla u(x))dx + \int_{S_u\cap\Omega} g((u^+ - u^-) \otimes \nu_u)\, d\mathcal{H}^{n-1}.$$

On f we suppose only that it is finite at one point, and on g we assume that it is positively 1-homogeneous, and that it is locally bounded on the sets $\mathbb{R}^k \otimes \nu_m$, where $\{\nu_1, \dots, \nu_n\} \subset S^{n-1}$ is a basis of $\mathbb{R}^n$. We prove that the lower semicontinuous envelope of F in the $L^1(\Omega;\mathbb{R}^k)$-topology is finite and with linear growth on the whole $BV(\Omega;\mathbb{R}^k)$, and that it admits the integral representation

$$\overline{F}(u) = \int_\Omega \varphi(\nabla u(x))dx + \int_\Omega \varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u|$$

A formula for φ is given, which takes into account the interaction between the bulk energy density f and the surface energy density g.

1. Introduction

Many problems in Mathematical Physics, Computer Vision, and Mechanical Engineering involve surface energies on some “free boundary” or “free discontinuity” set. These energies account for several phenomena such as crack growth and crack initiation in the theory of brittle fracture, interface formation between different phases of Cahn-Hilliard fluids, surface tension

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between small drops of liquid crystals, and are utilized for pattern recognition in computer vision to determine surfaces corresponding to sudden changes in the image (*e.g.*, edges of objects, shadows, changes in colour).

We are interested in a variational formulation for some of the *static* free discontinuity problems, in the light of recent research on functionals which depend on discontinuous functions. From the point of view of the calculus of variations, a rather complete theory has been developed by L. Ambrosio & A. Braides [5] and [6], in the case of absence of a "volume" counterpart of the surface energy, in the framework of partitions of sets of finite perimeter. When we allow the presence of a bulk energy, it is natural to take into account spaces of functions of bounded variation. We recall that if Ω is an open set in $\mathbb{R}^n$, a function u belongs to $BV(\Omega;\mathbb{R}^k)$ if it is an integrable function, and its distributional derivative Du is a finite (matrix-valued) Radon measure on Ω. It turns out that the Lebesgue decomposition of this measure can be written as $Du = \nabla u\, dx + D_s u$, where the density of the absolutely continuous part of Du is denoted by ∇u since it can be interpreted as an approximate differential for u. For a function $u \in BV(\Omega;\mathbb{R}^k)$ it is possible moreover to define a *set of jump points* S_u where u is approximately discontinuous, and on which a "normal" ν_u together with the *traces* u^+, u^- of u on both sides are well-defined. Recently, E. De Giorgi & L. Ambrosio [18] have introduced the subspace $SBV(\Omega;\mathbb{R}^k)$ of *special* functions of bounded variation, that are characterized by the property that the singular part of Du can be written as

$$D_s u = (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1}{}_{|S_u} \tag{1.1}$$

($\mathcal{H}^{n-1}{}_{|S_u}$ denotes the restriction to S_u of the $(n-1)$-dimensional Hausdorff measure). Remark that in general $D_s u$ contains also a diffuse "Cantor" part. On the space $SBV(\Omega;\mathbb{R}^k)$ it is natural to consider functionals of the form

$$\int_\Omega f(x, u(x), \nabla u(x))\, dx + \int_{S_u\cap\Omega} g(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}. \tag{1.2}$$

These integrals model many of the problems so far considered in the literature, and provide a good functional setting for problems that had been considered before only under additional un-natural hypotheses, imposed to obtain *a priori* smoothness on S_u.

A natural question for the functionals above concerns the possibility of application of the so-called Direct Method of the Calculus of Variations, that is summarized in the equation

$$\text{lower semicontinuity} + \text{ compactness } = \text{ existence of minimizers},$$

and hence the study of necessary and sufficient conditions for their lower semicontinuity in suitable topologies. A general lower semicontinuity result is not yet available. Partial results are due to L. Ambrosio [4], which assure the lower semicontinuity when f is *convex* (in the last variable) and has a *superlinear growth* at infinity, and g is BV*-elliptic* and has a *superlinear growth* for $|u^+ - u^-| \to 0$ (for example if $g \geq c > 0$). These conditions guarantee compactness separately for the bulk and jump part of the derivative, so that the two integrals in (1.2) can be dealt with separately. This result has been recently extended in [**] to the case of *vector-valued* u , under the natural assumption of *quasiconvexity* on f in the sense of C. B. Morrey (see [26], [25], [15], [1], [22]). If $M^{k\times n}$ denotes the space of $k \times n$ matrices, we recall that a continuous function $f : M^{k\times n} \to [0, +\infty[$ is said to be quasiconvex if for every $\xi \in M^{k\times n}$, A bounded subset of $\mathbb{R}^n$, and $v \in \mathcal{C}^1_0(A; \mathbb{R}^k)$ we have the inequality

$$|A| f(\xi) \leq \int_A f(\xi + \nabla v(x))\, dx.$$

In this framework, an interesting result, due to L. Ambrosio & G. Dal Maso [7], is the L^1-lower semicontinuity of the integral defined on $BV(\Omega; \mathbb{R}^k)$ by

$$\int_\Omega f(\nabla u(x))\, dx + \int_\Omega f^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u|, \tag{1.3}$$

under the assumption of f being quasiconvex and with linear growth (f^∞ is the recession function of f and $\dfrac{D_s u}{|D_s u|}$ denotes the Radon-Nikodym derivative of the measure $D_s u$ with respect to its total variation $|D_s u|$). This result has been recently generalized by I. Fonseca & S. Müller [23], allowing the dependence of f also on x and u. Considering the restriction of the functional (1.3) to $SBV(\Omega; \mathbb{R}^k)$, we have

$$g(u^+, u^-, \nu) = f^\infty((u^+ - u^-) \otimes \nu_u). \tag{1.4}$$

The condition (1.4) is verified in some models, but in general it is not possible to obtain the *effective* surface energy density by simply considering the volume energy density.

Purpose of this work is to give a *relaxation* and *integral representation* result on the special yet meaningful class of functionals defined on $SBV(\Omega;\mathbb{R}^k)$ by integrals of the form

$$F(u)=\int_\Omega f(\nabla u(x))\,dx+\int_{S_u\cap\Omega} g((u^+-u^-)\otimes\nu_u)\,d\mathcal{H}^{n-1}. \tag{1.5}$$

The lower semicontinuous envelope in the L^1-topology of the functional F; *i.e.,* the greatest L^1-lower semicontinuous functional less than or equal to F, is defined by relaxation as

$$\overline{F}(u)=\inf\{\liminf_h F(u_h)\ :(u_h)\text{ in }SBV(\Omega;\mathbb{R}^k),u_h\to u\text{ in }L^1(\Omega,\mathbb{R}^k)\}.$$

We prove, under only the hypotheses (besides the necessary measurability conditions)

$$\begin{aligned}&f:M^{k\times n}\to[0,+\infty]\text{ finite at one point, say at }0,\\&g:M^{k\times n}\to[0,+\infty]\text{ positively 1-homogeneous}\\&\ \text{and locally bounded in }n\text{ independent directions of }\mathbb{R}^n,\end{aligned}$$

that $\overline{F}$ can be represented as an integral on the whole $BV(\Omega;\mathbb{R}^k)$. In this case, the relaxation of F takes into account, both in its volume and in its surface part, the combined effect of f and g, and it can be written

$$\overline{F}(u)=\int_\Omega\varphi(\nabla u(x))\,dx+\int_\Omega\varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u| \tag{1.6}$$

for $u\in BV(\Omega;\mathbb{R}^k)$. The function φ is a quasiconvex function with linear growth (whatever be the growth conditions satisfied by f), and it satisfies the formula

$$\begin{aligned}\varphi(\xi)=\sup\{\psi(\xi)\ :\ \psi\text{ quasiconvex, }\ &\psi\le f\text{ on }M^{k\times n},\\&\psi^\infty(w)\le g(w)\text{ if }rank(w)\le 1\}.\end{aligned} \tag{1.7}$$

The paper is divided as follows. Section 2 is devoted to the preliminaries about spaces of functions of bounded variation to some relaxation results in BV and Sobolev spaces, and to the statement of our main result, Theorem 2.1. In Section 3 we prove the theorem in several steps. The first one is to establish that under the very weak hypotheses on f and g the relaxation

$\overline{F}$ is of linear growth, and indeed finite, on the whole $BV(\Omega;\mathbb{R}^k)$. Then we prove by a measure theoretical approach, and localization technique, that the study of the relaxed functional at a fixed u can be reduced to the study of a regular Borel measure on Ω. This fact allows us to use some integral representation arguments by G.Buttazzo & G. Dal Maso [13] and to write the restriction of $\overline{F}$ to $\mathrm{W}^{1,1}$ as an integral. The final step is to use the lower semicontinuity results by L. Ambrosio & G. Dal Maso [7] in order to obtain upper and lower bounds for $\overline{F}$ on the whole $BV(\Omega;\mathbb{R}^k)$; the use of formula (1.7) shows that these bounds coincide, and gives (1.6). Let us remark that in the scalar case (*i.e.,* when $k=1$) this result can be obtained, together with a simpler formula for φ, using a direct construction of the "recovery sequences" for $\overline{F}(u)$ (see [11], Theorem 2.1); this approach is not possible in the vector-valued case. Finally in Section 4 we specialize formula (1.7) in the case of special f and g, and we provide some applications of our result.

2. Preliminaries and Statement of the Main Result

2.1. *Notation*

The natural numbers n, k will be fixed. We denote with $\{e_i\}$ the canonical basis of $\mathbb{R}^k$, and with $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^n$; $|\cdot|$ will be the usual euclidean norm. We shall denote by $M^{k\times n}$ the space of $k\times n$ matrices (k rows, n columns), and by $M_1^{k\times n}$ the subset of $M^{k\times n}$ of all matrices with rank less than or equal to one. We shall identify $M^{k\times n}$ with $\mathbb{R}^{kn}$. If $a\in\mathbb{R}^k$ and $b\in\mathbb{R}^n$ the *tensor product* $a\otimes b\in M_1^{k\times n}$ is the matrix whose entries are a_ib_j with $i=1,\dots,k$ and $j=1,\dots,n$. Conversely if a matrix ξ has rank one, there are two vectors $a\in\mathbb{R}^k, b\in\mathbb{R}^n$ such that $\xi=a\otimes b$. If $A\subset\mathbb{R}^k$ and $b\in\mathbb{R}^n$ we will set $A\otimes b=\{a\otimes b : a\in A\}\subset M_1^{k\times n}$; remark that $|a\otimes b|=|a|\,|b|$ (the norms are taken in the proper spaces).

Let Ω be a bounded open subset of $\mathbb{R}^n$; we shall denote with $\mathcal{A}(\Omega)$ (resp. $\mathcal{B}(\Omega)$) the family of the open (resp. Borel) subsets of Ω. We shall use standard notation for the Sobolev and Lebesgue spaces $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^k)$ and $\mathrm{L}^p(\Omega;\mathbb{R}^k)$. When $k=1$ we shall drop the target space $\mathbb{R}^k$ in the notation, and write just $\mathrm{W}^{1,p}(\Omega)$, $\mathrm{L}^p(\Omega)$, and the like.

If u is a scalar function defined on Ω, we shall sometimes use the shorter notation $\{u<t\}$ for $\{x\in\Omega : u(x)<t\}$ (and similar) when no confusion is possible.

The Lebesgue measure and the Hausdorff $(n-1)$-dimensional measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}_n$ and $\mathcal{H}^{n-1}$ respectively. We shall use also the notation $|E|$ for $\mathcal{L}_n(E)$, the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$.

Let X be a set, and $E \subset X$; we define the *characteristic function* of E as

$$\mathbf{1}_E(z) = \begin{cases} 1 & \text{if } z \in E \\ 0 & \text{if } z \in X \setminus E, \end{cases}$$

and the *indicator function* of E as

$$\chi_E(z) = \begin{cases} 0 & \text{if } z \in E \\ +\infty & \text{if } z \in X \setminus E. \end{cases}$$

If $N \geq 1$ is an integer and $f : \mathbb{R}^N \to [0, +\infty]$ is a convex function, we define $f^\infty : \mathbb{R}^N \to [0, +\infty]$, the *recession function* of f, by setting

$$f^\infty(z) = \lim_{t \to +\infty} \frac{f(tz)}{t}. \tag{2.1}$$

It is immediate to see that the limit in (2.1) exists for all z; we remark that f^∞ is a Borel function, which is convex and positively homogeneous of degree one.

The symbol $[t]$ will denote the *integral part* of the number $t \in \mathbb{R}$. The letter c will denote throughout the paper a strictly positive constant, whose value may vary from line to line, and which is independent of the parameters of the problems each time considered.

2.2. *Functions of bounded variation*

Let $n, k \geq 1$ be natural numbers, and Ω be a bounded open subset of $\mathbb{R}^n$. We say that $u \in \mathrm{L}^1(\Omega; \mathbb{R}^k)$ is a *function of bounded variation* (and we write $u \in BV(\Omega; \mathbb{R}^k)$) if for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n\}$ there is a measure μ_i^j with finite total variation in Ω such that

$$\int_\Omega u^{(i)} \frac{\partial g}{\partial x_j}\, dx = -\int g d\mu_i^j \qquad \forall g \in C_0^1(\Omega),$$

where $C_0^1(\Omega)$ denotes the space of C^1 functions with compact support in Ω.

We denote by Du the $M^{k \times n}$-valued measure whose components are the μ_i^j, and by $|Du|$ its total variation.

We denote by S_u the complement of the Lebesgue set of u, that will be sometimes referred to as the set of *jump points* of the function u; *i.e.,* $x \notin S_u$ if and only if

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |u - z| \, dx = 0$$

for some $z \in \mathbb{R}^k$. If such a z exists, it is unique, and we denote it by $\tilde{u}(x)$, the *approximate limit* of u at x. For any function $u \in \mathrm{L}^1(\Omega; \mathbb{R}^k)$ the set S_u is negligible and $\tilde{u}$ is a Borel function equal to u almost everywhere. Moreover if $u \in BV(\Omega; \mathbb{R}^k)$, there is a countable sequence of C^1 hypersurfaces Γ_i which covers $\mathcal{H}^{n-1}$-almost all of S_u, *i.e.*,

$$\mathcal{H}^{n-1}\big(S_u \setminus \bigcup_{i=1}^{\infty} \Gamma_i\big) = 0.$$

Furthermore, for $\mathcal{H}^{n-1}$-almost every $x \in S_u$ it is possible to find a, $b \in \mathbb{R}^k$ and $\nu \in \mathrm{S}^{n-1}$ such that

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho^\nu(x)} |u - a| \, dx = 0, \qquad \lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho^{-\nu}(x)} |u - b| \, dx = 0, \tag{2.2}$$

where $B_\rho^\nu(x) = \{y \in B_\rho(x) : \langle y - x, \nu \rangle > 0\}$. The triplet (a, b, ν) is uniquely determined up to a change of sign of ν and an interchange of a and b, and it will be denoted by $(u^+(x), u^-(x), \nu_u(x))$.

In general, for a function $u \in BV(\Omega; \mathbb{R}^k)$, we have the Lebesgue decomposition

$$Du = D_a u + D_s u = \nabla u \cdot \mathcal{L}_n + D_s u, \tag{2.3}$$

where we denote by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure; the notation is motivated by the fact that ∇u can be interpreted as an *approximate differential.* The *singular part* of Du with respect to the Lebesgue measure can be further decomposed into to mutually singular measures as

$$D_s u = (u^+ - u^-) \otimes \nu_u \cdot \mathcal{H}^{n-1}{}_{|S_u} + C_u, \tag{2.4}$$

where $(u^+ - u^-) \otimes \nu_u \cdot \mathcal{H}^{n-1}{}_{|S_u}$ is the *Hausdorff part* and C_u the *Cantor part* of Du. We indicate by $\dfrac{D_s u}{|D_s u|}$ the Radon-Nikodym derivative of $D_s u$

with respect to its total variation. The integral on Ω of a function ψ with respect to the measure $|D_s u|$ will be denoted simply by $\int_\Omega \psi |D_s u|$.

We will say that a set E is of *finite perimeter in* Ω if $\mathbf{1}_E \in BV(\Omega; \mathbb{R})$. We will set $\partial^* E \cap \Omega = S_{\mathbf{1}_E} \cap \Omega$ the *reduced boundary of* E in Ω. Remark that $|D\mathbf{1}_E|(\Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$ for every E of finite perimeter in Ω. It is easy to check that this notion of perimeter coincides with the elementary one in the smooth case, in particular when E is a polyhedron. A result of E. De Giorgi [19] shows that if E is a set of finite perimeter in Ω, then there exists a sequence of polyhedra (P_h) such that $|((P_h \setminus E) \cup (E \setminus P_h)) \cap \Omega| \to 0$, and

$$\mathcal{H}^{n-1}(\partial^* E \cap \Omega) = \lim_h \mathcal{H}^{n-1}(\partial P_h \cap \Omega). \tag{2.5}$$

This result demonstrates that the measure theoretical notion of perimeter is a sensible extension of the elementary definition.

We recall that if $u \in BV(\Omega; \mathbb{R})$, then for a.e. $t \in \mathbb{R}$ the set $\{u > t\}$ is of finite perimeter in Ω, and we have the so-called *coarea formula*:

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap \Omega)\, dt. \tag{2.6}$$

We recall also the *Fleming & Rishel coarea formula*. Let u be a Lipschitz function; then for every $v \in BV(\Omega)$ we have that

$$\int_\Omega v |\nabla u|\, dx = \int_{-\infty}^{+\infty} \int_{\partial^* \{u > t\} \cap \Omega} \widetilde{v}\, d\mathcal{H}^{n-1}\, dt \tag{2.7}$$

(∇u is the a.e. gradient of the function u). Analogous formulas hold with $\{u < t\}$ instead of $\{u > t\}$.

We say that u is a *special* function of bounded variation, and we write $u \in SBV(\Omega; \mathbb{R}^k)$, if $u \in BV(\Omega; \mathbb{R}^k)$ and $C_u \equiv 0$. The space $SBV(\Omega; \mathbb{R}^k)$ has been introduced by E. De Giorgi and L. Ambrosio [18].

For the general exposition of the theory of functions of bounded variation we refer to [21], [24], [30] and [31]. For an introduction to the properties of the space SBV we refer to [18], [3], [4].

2.3. *Relaxation*

We recall the notion of *relaxed functional*. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a functional on a metric space (X, τ). The relaxed functional $\overline{F}$ of F, or *relaxation* of F, (in the τ-topology) is the greatest τ-lower semicontinuous functional less than or equal to F; *i.e.,* the greatest functional such that $\overline{F} \leq F$ and $\overline{F}(u) \leq \liminf_h \overline{F}(u_h)$ for every sequence (u_h) converging to u in the τ-topology. Throughout the paper we shall consider relaxations in the L^1-topology. We point out here only that the relaxed functional $\overline{F}$ allows to describe the behaviour of minimizing sequences for F; indeed minimizing sequences for problems involving F converge, up to a subsequence, to solutions for the corresponding problems for $\overline{F}$. For a general treatment of this subject we refer to the books by G. Buttazzo [12], and by G. Dal Maso [17].

2.4. *Quasiconvexity and rank one convexity*

We recall the notion of *quasiconvex function* (cf. *e.g.* C. B. Morrey [26],[25], B. Dacorogna [15],[16]). We say that a continuous function $\varphi : M^{k\times n} \to [0, +\infty[$ is quasiconvex if for every $\xi \in M^{k\times n}$, A bounded subset of $\mathbb{R}^n$, and $u \in \mathcal{C}_0^1(A; \mathbb{R}^k)$ we have the inequality

$$|A|\varphi(\xi) \leq \int_A \varphi(\xi + \nabla u(x))\, dx.$$

This property is a well-known necessary and sufficient condition for the lower semicontinuity of multiple integrals in Sobolev spaces (cf. Acerbi & Fusco [1], Dacorogna [15]).

Every quasiconvex function $\varphi : M^{k\times n} \to [0, +\infty[$ is *rank one convex*; *i.e.*, it verifies

$$\varphi(\lambda\xi + (1-\lambda)\zeta) \leq \lambda\varphi(\xi) + (1-\lambda)\varphi(\zeta)$$

for every $\xi, \zeta \in M^{k\times n}$ such that $rank(\xi - \zeta) \leq 1$, and every $\lambda \in [0, 1]$ (cf. Dacorogna [15],[16]). A recent result by V. Šverák shows that the converse is not true (see [29]).

2.5. *The Main Result*

We are in a position to state the main result of the paper, which characterizes the relaxation in the L^1-topology of some functionals defined in $SBV(\Omega; \mathbb{R}^k)$.

Theorem 2.1. *Let $f : M^{k\times n} \to [0,+\infty]$ a positive Borel function such that $f(0) \neq +\infty$, and let $g : M_1^{k\times n} \to [0,+\infty]$ be a positively 1-homogeneous Borel function. On the function g we suppose moreover that there exist n linearly independent vectors $\nu_1,\dots,\nu_n$ in S^{n-1} such that g is locally bounded on $\mathbb{R}^k \otimes \nu_m$ for all $m = 1,\dots,n$.*

Let Ω be a bounded open subset of $\mathbb{R}^n$, and let us define the functional $\mathcal{F} : BV(\Omega;\mathbb{R}^k) \to [0,+\infty]$ by setting

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u(x))\,dx + \int_{S_u\cap\Omega} g((u^+ - u^-)\otimes\nu_u)\,d\mathcal{H}^{n-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in SBV(\Omega;\mathbb{R}^k) \\ +\infty \qquad\qquad\qquad\qquad\qquad \text{if } u \in BV(\Omega;\mathbb{R}^k)\setminus SBV(\Omega;\mathbb{R}^k). \end{cases}$$

Then the lower semicontinuous envelope of $\mathcal{F}$ in the $L^1(\Omega;\mathbb{R}^k)$-topology is given by

$$\overline{\mathcal{F}}(u) = \int_\Omega \varphi(\nabla u(x))\,dx + \int_\Omega \varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u| \tag{2.8}$$

for every $u \in BV(\Omega;\mathbb{R}^k)$, where the function $\varphi : M^{k\times n} \to [0,+\infty[$ is given by

$$\begin{aligned}\varphi(\xi) = \sup\{\psi(\xi)\ :\ &\psi \text{ quasiconvex},\ \psi \leq f \text{ on } M^{k\times n}, \\ &\psi^\infty(w) \leq g(w) \text{ if } rank(w) \leq 1\},\end{aligned} \tag{2.9}$$

and verifies $0 \leq \varphi(\xi) \leq c(1+|\xi|)$ for all $\xi \in M^{k\times n}$.

Remark 2.2. Let us suppose that, in addition, the function f is convex and the function g is rank one convex on $\mathbb{R}^k\otimes\nu$ for every $\nu \in \mathrm{S}^{n-1}$, which means that

$$g(\lambda a\otimes\nu + (1-\lambda)b\otimes\nu) \leq \lambda g(a\otimes\nu) + (1-\lambda)g(b\otimes\nu),$$

for every $a, b \in \mathbb{R}^k$, $\lambda \in [0,1]$, and $\nu \in \mathrm{S}^{n-1}$. Such a property is veryfied, for instance, if g is (the restriction to $M_1^{k\times n}$ of) a quasiconvex function.

The functional F is convex on $SBV(\Omega;\mathbb{R}^k)$; it is immediate to check that its relaxation $\overline{F}$ must be convex too, and hence also the integrand φ. Then we have the formula

$$\varphi(\xi) = \sup\{\psi(\xi)\ :\ \psi \text{ convex},\ \psi \leq f \text{ on } M^{k\times n}, \psi^\infty \leq g \text{ on } M_1^{k\times n}\}.$$

As a corollary to Theorem 2.1, in the scalar case (*i.e.,* $k = 1$) we get the following result, which generalizes the relaxation Theorem 2.1 of [11], where a simpler formula for φ is obtained using a direct construction of the "recovery sequences" for $\overline{\mathcal{F}}(u)$ (see also [10]).

Corollary 2.3. *Let $k = 1$. Under the hypotheses of Theorem 2.1 φ verifies the formula*

$$\varphi(z) = \big(f \wedge (f(0) + g)\big)^{**}(z)$$

for all $z \in \mathbb{R}^n$ (h^{**} denotes the greatest convex and lower semicontinuous function less than or equal to h).

Proof. Let ψ be a convex function such that $\psi \leq f$ and $\psi^\infty \leq g$ on $\mathbb{R}^n$. In particular we have $\psi(0) \leq f(0)$ and it is easy to check then that $\psi \leq f(0) + g$. Recalling Remark 2.2 we conclude that

$$\begin{aligned}\varphi(z) &= \sup\{\psi(z) : \ \psi \text{ convex },\ \psi \leq f \wedge (f(0) + g)\} \\ &= \big(f \wedge (f(0) + g)\big)^{**}(z). \qquad \square\end{aligned}$$

For additional remarks and examples, see Section 4.

2.6. *Preliminary Results on Relaxation*

In order to prove Theorem 2.1 we shall make use of some relaxation results. The first one deals with functionals defined on Sobolev spaces.

Theorem 2.4. *(G. Buttazzo & G. Dal Maso [13] Theorem 1.1 and [12] Theorem 4.3.2) Let $\Omega \subset \mathbb{R}^n$ be an open set, and $F : \mathrm{W}^{1,1}(\Omega; \mathbb{R}^k) \times \mathcal{A}(\Omega) \to [0, +\infty[$ be a functional verifying for every $u, v \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^k)$ and for every $A \in \mathcal{A}(\Omega)$:*

(i) (linear growth condition) $|F(u, A)| \leq c(|A| + \int_A |\nabla u(x)|\, dx)$;

(ii) (locality) $F(u, A) = F(v, A)$ whenever $u = v$ on A;

(iii) (semicontinuity) $F(\cdot, A)$ is $\mathrm{W}^{1,1}$-sequentially lower semicontinuous;

(iv) (translation invariance) $F(u+b, A) = F(u, A)$ for every constant vector $b \in \mathbb{R}^k$;

(v) $F(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a regular Borel measure.

Then there exists a Carathéodory function $\psi : \Omega \times M^{k\times n} \to [0, +\infty[$, quasiconvex in the second variable for a.e. $x \in \Omega$, such that the integral representation

$$F(u, A) = \int_A \psi(x, \nabla u(x))\, dx$$

holds for every $u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^k)$ *and for every* $A \in \mathcal{A}(\Omega)$.

The second result we recall is a lower semicontinuity and relaxation theorem for quasiconvex integrals on the space of vector-valued BV-functions.

Theorem 2.5. *(L. Ambrosio & G. Dal Maso [7] Theorem 4.1) Let* $\varphi : M^{k\times n} \to [0,+\infty[$ *be a quasiconvex function satisfying*

$$0 \le \varphi(\xi) \le c(1+|\xi|) \qquad \text{for every } \xi \in M^{k\times n}, \tag{2.10}$$

and let us define on $BV(\Omega;\mathbb{R}^k)$ *the functional* $\mathcal{F}$ *by setting*

$$\mathcal{F}(u) = \int_\Omega \varphi(\nabla u(x))\,dx + \int_\Omega \varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)\,|D_s u|. \tag{2.11}$$

Then $\mathcal{F}$ *is* $\mathrm{L}^1(\Omega;\mathbb{R}^k)$*-lower semicontinuous on* $BV(\Omega;\mathbb{R}^k)$*, and we have*

$$F = \overline{F + \chi_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^k)}}; \tag{2.12}$$

i.e., F *coincides with the relaxation of its restriction to the Sobolev space* $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^k)$.

Remark that in order to have a good definition of $\mathcal{F}$ in (2.11) (and of $\overline{\mathcal{F}}$ in (2.8)) we have to extend the notion of $\varphi^\infty(\xi)$ to φ quasiconvex. This quantity is well-defined by (2.1) if the rank of ξ is less than or equal to one since quasiconvex functions are convex in rank one directions. In general the limit in (2.1) does not exist for all $\xi \in M^{k\times n}$ (cf. Müller [27]). Nevertheless, a recent result by G. Alberti [2] assures that the matrix $\dfrac{D_s u}{|D_s u|}$ is of rank 1 $|D_s u|$-a.e., and hence each quantity is well-defined in (2.11) and (2.8).

3. Proof of the Main Result

We start by giving an upper bound for the functional $\overline{\mathcal{F}}$. We shall consider the functional $\mathcal{G} \ge \mathcal{F}$ defined by

$$\mathcal{G}(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u(x))\,dx + \int_{S_u\cap\Omega} g((u^+ - u^-)\otimes\nu_u)\,d\mathcal{H}^{n-1} & \\ \qquad\qquad \text{if } u \in SBV(\Omega;\mathbb{R}^k) \text{ and } \mathcal{H}^{n-1}(S_u\cap\Omega) < +\infty & \\ +\infty \qquad\qquad \text{elsewhere in } BV(\Omega;\mathbb{R}^k). & \end{cases}$$

Obviously an upper bound for $\overline{\mathcal{G}}$ will do as well.

Proposition 3.1. *We have* $\overline{\mathcal{G}}(u) \leq c(1 + |Du|(\Omega))$ *for all functions* $u \in BV(\Omega; \mathbb{R}^k)$.

Proof. We first prove the proposition under the additional hypothesis

$$g \text{ locally bounded on rank one matrices.} \tag{3.1}$$

We define then the constant $M < +\infty$ by setting

$$M = \sup\{g(a \otimes \nu) \ : \ a \in \mathrm{S}^{k-1},\ \nu \in \mathrm{S}^{n-1}\}. \tag{3.2}$$

We shall deal first with the scalar case ($k = 1$), and then extend the proof to the case of vector-valued u.

Step 1: $k = 1$. In this case g is defined on the whole $\mathbb{R}^n = \mathbb{R} \otimes \mathbb{R}^n$. Let us consider a function $u \in BV(\Omega) \cap \mathcal{C}^1(\Omega)$. Let us fix $h \in \mathbb{N}$; by the coarea formula (2.6) we have

$$|Du|(\Omega) = \sum_{j \in \mathbb{Z}} \int_{\frac{j}{h}}^{\frac{j+1}{h}} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap \Omega)\, dt.$$

Hence, by the mean value theorem, for every $j \in \mathbb{Z}$ we can find $s_j^h \in \left]\frac{j}{h}, \frac{j+1}{h}\right[$ such that

$$\frac{1}{h}\mathcal{H}^{n-1}(\partial^*\{u > s_j^h\} \cap \Omega) \leq \int_{\frac{j}{h}}^{\frac{j+1}{h}} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap \Omega)\, dt,$$

so that

$$\sum_{j \in \mathbb{Z}} \frac{1}{h}\mathcal{H}^{n-1}(\partial^*\{u > s_j^h\} \cap \Omega) \leq |Du|(\Omega). \tag{3.3}$$

Let us construct now the sequence (u_h) in $SBV(\Omega)$ by setting

$$u_h(x) = \frac{j}{h} \quad \text{on } \{s_{j-1}^h < u < s_j^h\}. \tag{3.4}$$

It is clear that for every $h \in \mathbb{N}$ we have $\nabla u_h(x) = 0$ for a.e. $x \in \Omega$,

$$S_{u_h} \cap \Omega = \bigcup_{j \in \mathbb{Z}} \partial^*\{u > s_j^h\} \cap \Omega,$$

and

$$Du_h = D_s u_h = \sum_{j\in\mathbb{Z}} \frac{1}{h}\nu_h^j \ \mathcal{H}^{n-1}{}_{|\partial^*\{u>s_j^h\}\cap\Omega},$$

where ν_h^j is defined by

$$D\mathbf{1}_{\{u>s_j^h\}} = \nu_h^j \mathcal{H}^{n-1}{}_{|\partial^*\{u>s_j^h\}}.$$

Hence we obtain

$$\begin{aligned}
\mathcal{F}(u_h) = & \int_\Omega f(\nabla u_h(x))\,dx + \int_{S_{u_h}\cap\Omega} g((u_h^+ - u_h^-)\otimes \nu_{u_h})\,d\mathcal{H}^{n-1} \\
& = f(0)|\Omega| + \sum_{j\in\mathbb{Z}} \int_{\partial^*\{u>s_j^h\}\cap\Omega} g\big(\frac{1}{h}\nu_h^j\big)\,d\mathcal{H}^{n-1} \\
& \le f(0)|\Omega| + \sum_{j\in\mathbb{Z}} \frac{1}{h} M \ \mathcal{H}^{n-1}(\partial^*\{u>s_j^h\}\cap\Omega) \\
& \le f(0)|\Omega| + M|Du|(\Omega).
\end{aligned} \tag{3.5}$$

We have made use here of (3.2), (3.3), and of the positive homogeneity of g. Remark also that for every h

$$\mathcal{H}^{n-1}(S_{u_h}\cap\Omega) = \sum_{j\in\mathbb{Z}} \mathcal{H}^{n-1}(\partial^*\{u>s_j^h\}\cap\Omega) \le h|Du|(\Omega) < +\infty,$$

hence $\mathcal{G}(u_h) = \mathcal{F}(u_h)$.

Since $u_h \to u$ in $\mathrm{L}^\infty(\Omega)$, by the definition of $\overline{\mathcal{G}}$ we conclude that

$$\overline{\mathcal{G}}(u) \le \liminf_h \mathcal{G}(u_h) \le f(0)|\Omega| + M\,|Du|(\Omega).$$

For a general $u \in BV(\Omega)$ it suffices to recall that there exists a sequence (v_h) in $\mathcal{C}^\infty(\Omega)\cap BV(\Omega)$ (for example obtained by convolution from u; see [24]) such that $v_h \to u$ in $\mathrm{L}^1(\Omega)$ and

$$|Du|(\Omega) = \lim_h |Dv_h|(\Omega) = \lim_h \int_\Omega |\nabla v_h|\,dx.$$

By the lower semicontinuity of $\overline{\mathcal{G}}$ we obtain then

$$\begin{aligned}
\overline{\mathcal{G}}(u) \le & \ \liminf_h \overline{\mathcal{G}}(v_h) \le \lim_h (f(0)|\Omega| + M\,|Dv_h|(\Omega)) \\
& = f(0)|\Omega| + M\,|Du|(\Omega).
\end{aligned}$$

Step 2: $k \geq 2$. In this case we can proceed "componentwise". Let us fix a function $u = (u_{(1)}, \dots, u_{(k)}) \in \mathcal{C}^1(\Omega; \mathbb{R}^k) \cap BV(\Omega; \mathbb{R}^k)$. Proceeding as in Step 1, for every $h \in \mathbb{N}$, $j \in \mathbb{Z}$, and every $i = 1, \dots, k$ we can find $s_j^{i,h} \in \left]\frac{j}{h}, \frac{j+1}{h}\right[$ such that

$$\frac{1}{h}\mathcal{H}^{n-1}(\partial^*\{u_{(i)} > s_j^{i,h}\} \cap \Omega) \leq \int_{\frac{j}{h}}^{\frac{j+1}{h}} \mathcal{H}^{n-1}(\partial^*\{u_{(i)} > t\} \cap \Omega)\, dt.$$

We can define then the function $u_h \in SBV(\Omega; \mathbb{R}^k)$ by setting

$$u_{h\,(i)}(x) = \frac{j}{h} \quad \text{on } \{s_{j-1}^{i,h} < u_{(i)} < s_j^{i,h}\}. \tag{3.6}$$

By (3.6) we have that

$$Du_h = D_s u_h = \sum_{i=1}^{k} \sum_{j \in \mathbb{Z}} \frac{1}{h} e_i \otimes \nu_{i,h}^j \mathcal{H}^{n-1}{}_{|\partial^*\{u_{(i)} > s_j^{i,h}\} \cap \Omega},$$

where $\nu_{i,h}^j(x)$ is defined by

$$D\mathbf{1}_{\{u_{(i)} > s_j^{i,h}\}} = \nu_{i,h}^j \mathcal{H}^{n-1}{}_{|\partial^*\{u_{(i)} > s_j^{i,h}\}}.$$

We can proceed now as in Step 1 and obtain

$$\overline{\mathcal{G}}(u) \leq \liminf_h \mathcal{G}(u_h) \leq f(0)|\Omega| + \sqrt{k}\, M\, |Du|(\Omega).$$

The same inequality is valid on the whole $BV(\Omega; \mathbb{R}^k)$ by approximation.

Step 3: the case g not locally bounded. Under the general hypotheses of Theorem 2.1 g is not necessarily locally bounded on $M_1^{k \times n}$, but it is on the subspaces $\mathbb{R}^k \otimes \nu_m$ for $m = 1, \dots, n$. We have to modify the proof of the previous steps in order to have jump part densities of the form $a \otimes \nu_m$. It suffices to take into account for every i, h, and j a polyhedron $P_j^{i,h}$ with

$$\{u_{(i)} > \frac{j+1}{h}\} \subset P_j^{i,h} \subset \{u_{(i)} > \frac{j}{h}\},$$

and such that

$$\mathcal{H}^{n-1}(\partial P_j^{i,h} \cap \Omega) \leq \mathcal{H}^{n-1}(\partial^*\{u_{(i)} > s_j^{i,h}\} \cap \Omega) + \frac{1}{h} 2^{-|j|}.$$

It is clear that each of these polyhedra can be approximated by polyhedra each of whose faces is orthogonal to one of the vectors $\nu_1, \dots, \nu_n$, increasing the surface area by at most a constant factor depending on this basis. We can suppose then that each of the faces of $P_j^{i,h}$ is orthogonal to some ν_m, and that

$$\mathcal{H}^{n-1}(\partial P_j^{i,h} \cap \Omega) \leq c\big(\mathcal{H}^{n-1}(\partial^* \{u_{(i)} > s_j^{i,h}\} \cap \Omega) + \frac{1}{h} 2^{-|j|}\big).$$

We can then define $u_h \in SBV(\Omega; \mathbb{R}^k)$ by setting

$$u_{h\,(i)}(x) = \frac{j}{h} \quad \text{on } P_{j-1}^{i,h} \setminus P_j^{i,h},$$

and conclude the proof as in Step 2, taking now

$$M = \sup\{\, g(a \otimes \nu_m) : a \in \mathrm{S}^{k-1}, \quad m = 1, \dots, n\}. \tag{3.7}$$

□

Remark. If we take some extra care in Step 2 of the previous proposition, we can obtain approximating sequences which jump only in the coordinate directions of the target space $\mathbb{R}^k$ (*i.e.,* their jump part densities have the form $e_i \otimes b$). It suffices to choose the $s_j^{i,h}$ so that

$$\mathcal{H}^{n-1}((\partial^* \{u_{(i)} > s_j^{i,h}\} \cap \partial^* \{u_{(l)} > s_m^{l,h}\}) \cap \Omega) = 0$$

for every $m, j \in \mathbb{Z}$ and for every $i, l \in \{1, \dots, k\}$ with $i \neq l$.

Taking into account the construction of Step 3 of the previous proposition we can obtain jump part densities of the form

$$\frac{1}{h} e_i \otimes \nu_m.$$

Hence the conclusion of Proposition 3.1 still holds true under the only hypothesis of g to be finite on the set

$$\{e_i \otimes \nu_m \;:\; i = 1, \dots, k,\; m = 1, \dots, n\}.$$

We localize the functional $\mathcal{G}$ by defining for every open subset A of Ω

$$\mathcal{G}(u,A)=\int_A f(\nabla u(x))\,dx+\int_{S_u\cap A} g((u^+-u^-)\otimes\nu_u)\,d\mathcal{H}^{n-1} \tag{3.8}$$

if $u\in SBV(\Omega;\mathbb{R}^k)$ and $\mathcal{H}^{n-1}(S_u\cap\Omega)<+\infty$, and by setting

$$\mathcal{G}(u,A)=+\infty \qquad \text{elsewhere on } BV(\Omega;\mathbb{R}^k).$$

In the same way we define

$$\overline{\mathcal{G}}(u,A)=\inf\{\liminf_h\ \mathcal{G}(u_h,A)\ :\ u_h\to u \text{ in } \mathrm{L}^1(A), u_h\in BV(\Omega;\mathbb{R}^k)\}.$$

Localizing the proof of Proposition 3.1 we get the linear growth condition

$$|\overline{\mathcal{G}}(u,A)|\le c\big(|A|+|Du|(A)\big) \tag{3.9}$$

for every $u\in BV(\Omega;\mathbb{R}^k)$ and every $A\in\mathcal{A}(\Omega)$.

Proposition 3.2. *For every $u\in BV(\Omega;\mathbb{R}^k)$ the set function $\overline{\mathcal{G}}(u,\cdot)$ is (the restriction to the family of the open subsets of Ω of) a regular Borel measure on Ω.*

Proof.
Step 1: $\overline{\mathcal{G}}(u,\cdot)$ is regular; *i.e.,* for every open set $A\subset\Omega$, we have

$$\overline{\mathcal{G}}(u,A)=\sup\{\overline{\mathcal{G}}(u,A')\ :\ A' \text{ open}, A'\subset\subset A\}. \tag{3.10}$$

We shall first consider the case of g locally bounded; *i.e.,* that there exists a constant M, defined as in (3.2), such that

$$g(a\otimes\nu)\le M\,|a|.$$

Let us remark that $\overline{\mathcal{G}}(u,\cdot)$ is an increasing set function; *i.e.,* $\overline{\mathcal{G}}(u,A')\le\overline{\mathcal{G}}(u,A)$ if $A'\subset A$, hence the inequality "$\ge$" in (3.10) is trivial. Let us prove now the opposite inequality. Fixed K a compact subset of A, let us define $\delta=\frac{1}{2}\mathrm{dist}\,(\partial A,K)$, $\mathrm{d}_K(x)=\mathrm{dist}\,(x,K)$,

$$B(t)=\{x\in A\ :\ \mathrm{d}_K(x)<t\} \qquad t\in]0,\delta[,$$

and $B=B(\delta)=\{x\in A\ :\ \mathrm{d}_K(x)<\delta\}$.

Let us choose two sequences of functions $(u_h), (v_h)$ in $SBV(\Omega;\mathbb{R}^k)$ such that $u_h \to u$ in $\mathrm{L}^1(B)$, $v_h \to u$ in $\mathrm{L}^1(A\setminus K)$, and

$$\overline{\mathcal{G}}(u,B) = \lim_h \mathcal{G}(u_h,B)$$

$$\overline{\mathcal{G}}(u,A\setminus K) = \lim_h \mathcal{G}(v_h,A\setminus K).$$

Since we have $\mathcal{H}^{n-1}(S_{u_h}\cap\Omega)+\mathcal{H}^{n-1}(S_{v_h}\cap\Omega) < +\infty$, then for every h the set of $t\in]0,\delta[$ that do not verify

$$\mathcal{H}^{n-1}(S_{u_h}\cap\partial^* B(t)) + \mathcal{H}^{n-1}(S_{v_h}\cap\partial^* B(t)) = 0 \tag{3.11}$$

is at most countable (in fact, the set of t for which this quantity is larger than $\frac{1}{h}$ is finite). In the same way we have that

$$|Du_h|(\partial^* B(t)) + |Dv_h|(\partial^* B(t)) = 0 \tag{3.12}$$

except for at most a countable set of $t\in]0,\delta[$.

We can apply Fleming & Rishel coarea formula (2.7) to the integral

$$\int_{B\setminus K} |u_h - v_h|\,dx = \int_{B\setminus K} |u_h-v_h||\nabla \mathrm{d}_K|\,dx$$

$$= \int_0^\delta \int_{\partial^* B(t)} |\tilde u_h(x) - \tilde v_h(x)|\,d\mathcal{H}^{n-1}(x)\,dt$$

(recall that the a.e. gradient $\nabla\mathrm{d}_K$ of the Lipschitz function d_K has unit length a.e.). By the mean value theorem for every h we can choose $t_h\in]0,\delta[$ such that (3.11) and (3.12) hold, $B(t_h)$ is a set of finite perimeter, and

$$\int_{\partial^* B(t_h)} |\tilde u_h - \tilde v_h|\,d\mathcal{H}^{n-1} \le \frac{1}{\delta}\int_{B\setminus K} |u_h - v_h|\,dx. \tag{3.13}$$

We can define the sequence (w_h) in $\mathrm{L}^1(A)$ by setting

$$w_h = \begin{cases} u_h & \text{in } B(t_h) \\ \\ v_h & \text{in } A\setminus B(t_h). \end{cases}$$

Note that for every h we have $w_h \in SBV(\Omega;\mathbb{R}^k)$ and

$$\nabla w_h = \nabla u_h \mathbf{1}_{B(t_h)} + \nabla v_h \mathbf{1}_{A\setminus B(t_h)};$$

moreover the Hausdorff part of the measure Dw_h is given by

$$(u_h^+ - u_h^-) \otimes \nu_{u_h} \cdot \mathcal{H}^{n-1}{}_{|S_{u_h} \cap B(t_h)} + (v_h^+ - v_h^-) \otimes \nu_{v_h} \cdot \mathcal{H}^{n-1}{}_{|S_{v_h} \cap (A \setminus B(t_h))} + (\tilde{u}_h - \tilde{v}_h) \otimes \nu_{\partial^* B(t_h)} \cdot \mathcal{H}^{n-1}{}_{|\partial^* B(t_h)},$$

where $\nu_{\partial^* B(t_h)}$ denotes the normal to $\partial^* B(t_h)$ pointing inwards $B(t_h)$.

We obtain then

$$\begin{aligned}\mathcal{G}(w_h, A) &\leq \mathcal{G}(u_h, B) + \mathcal{G}(v_h, A \setminus K) + M \int_{\partial^* B(t_h)} |{w_h}^+ - {w_h}^-| \, d\mathcal{H}^{n-1} \\ &= \mathcal{G}(u_h, B) + \mathcal{G}(v_h, A \setminus K) + M \int_{\partial^* B(t_h)} |\tilde{u}_h - \tilde{v}_h| \, d\mathcal{H}^{n-1} \\ &\leq \mathcal{G}(u_h, B) + \mathcal{G}(v_h, A \setminus K) + M \frac{1}{\delta} \int_{B \setminus K} |u_h - v_h| \, dx. \qquad (3.14)\end{aligned}$$

Since $w_h \to u$ in $\mathrm{L}^1(A)$, and $(u_h - v_h) \to 0$ in $\mathrm{L}^1(B \setminus K)$, we have, by taking the limit as $h \to +\infty$,

$$\overline{\mathcal{G}}(u, A) \leq \liminf_h \mathcal{G}(w_h, A) \leq \overline{\mathcal{G}}(u, B) + \overline{\mathcal{G}}(u, A \setminus K).$$

By (3.9) we get then

$$\overline{\mathcal{G}}(u, A) \leq \overline{\mathcal{G}}(u, B) + c(|A \setminus K| + |Du|(A \setminus K)).$$

Since the last term in this inequality can be taken arbitrarily small and $B \subset\subset A$, we obtain the desired inequality in (3.10).

We can remove now the hypothesis of local boundedness of g, assuming the only hypotheses of Theorem 2.1. Remark that the only place where we make use of the local boundedness of g is the inequality (3.14). We choose now a closed polyhedron P with faces orthogonal to the directions $\nu_1, \ldots, \nu_n$ such that $K \subset P \subset\subset A$. Let us consider, in place of the usual distance, the new distance $\mathrm{dist}^\nu(x, y) = \sup_m |\langle x - y, \nu_m \rangle|$, and $\mathrm{d}_P(x) = \min\{\mathrm{dist}^\nu(x, y) : y \in P\}$. With this new definition of the distance we can proceed as above with P instead of K, remarking that the sets $B(t)$ are all polyhedra with faces orthogonal to the directions $\nu_1, \ldots, \nu_n$. It is clear that we take into account only the values of g on the sets $\mathbb{R}^k \otimes \nu_m$, $m = 1, \ldots, n$, and therefore we obtain (3.14) with M defined as in (3.7).

Step 2: $\overline{\mathcal{G}}(u, \cdot)$ is a subadditive set function; *i.e.,* we have

$$\overline{\mathcal{G}}(u, A_1 \cup A_2) \leq \overline{\mathcal{G}}(u, A_1) + \overline{\mathcal{G}}(u, A_2)$$

for every pair of open subsets A_1, A_2 of Ω.

By the regularity of $\overline{\mathcal{G}}(u,\cdot)$ (Step 1), it is sufficient to prove that

$$\overline{\mathcal{G}}(u,A) \le \overline{\mathcal{G}}(u,A_1) + \overline{\mathcal{G}}(u,A_2)$$

for every open set $A \subset\subset A_1 \cup A_2$. This inequality can be proved arguing as in Step 1, choosing $K = A \setminus A_2$, and

$$B = \{x \in A \ : \ \mathrm{dist}(x,K) < \frac{1}{2}\mathrm{dist}(K, A \setminus A_1)\}.$$

Moreover it is clear that $\overline{\mathcal{G}}(u,\cdot)$ is additive on disjoint sets; *i.e.,*

$$\overline{\mathcal{G}}(u,A_1 \cup A_2) = \overline{\mathcal{G}}(u,A_1) + \overline{\mathcal{G}}(u,A_2)$$

if $A_1 \cap A_2 = \emptyset$.

Step 3: $\overline{\mathcal{G}}(u,\cdot)$ is the restriction to the open subsets of Ω of a regular Borel measure. It suffices to remark that the set function $\overline{\mathcal{G}}(u,\cdot)$ verifies:

(a) $\overline{\mathcal{G}}(u,\cdot)$ is a positive and increasing set function defined on $\mathcal{A}(\Omega)$;

(b) $\overline{\mathcal{G}}(u,\cdot)$ is regular (Step 1);

(c) $\overline{\mathcal{G}}(u,\cdot)$ is subadditive, and it is additive on disjoint sets (Step 2),

and apply Theorem 5.6 by E. De Giorgi & G. Letta [20]. □

Proposition 3.3. *There exists a quasiconvex function $\psi : M^{k\times n} \to [0,+\infty[$ such that we have*

$$\overline{\mathcal{G}}(u,A) = \int_A \psi(\nabla u(x))\,dx \tag{3.15}$$

for every $A \in \mathcal{A}(\Omega)$ and $u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^k)$. The function ψ verifies

$$0 \le \psi(\xi) \le c(1+|\xi|) \tag{3.16}$$

for every $\xi \in M^{k\times n}$.

Proof. We want to apply Theorem 2.4, which assures the representation

$$\overline{\mathcal{G}}(u,A) = \int_A \psi(x,\nabla u(x))\,dx \tag{3.17}$$

for a suitable quasiconvex Carathéodory function $\psi : \Omega \times M^{k\times n} \to [0,+\infty[$. Let us check the hypotheses of Theorem 2.4. We have already proved conditions (i) and (v) (see (3.9) and Proposition 3.2). The property (ii) follows from the definition of $\overline{\mathcal{G}}$, while (iii) is verified since $\overline{\mathcal{G}}(\cdot, A)$ is L^1-lower semicontinuous. Finally, it is easy to check that $\mathcal{G}$ verifies $\mathcal{G}(u+z, A) = \mathcal{G}(u, A)$ for every $u \in BV(\Omega; \mathbb{R}^k)$, $A \in \mathcal{A}(\Omega)$, and for every constant vector $z \in \mathbb{R}^k$, and so does $\overline{\mathcal{G}}$. Hence we obtain the representation formula (3.17) for every $u \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^k)$.

In order to prove that ψ does not depend on x, we observe that, by the definition of $\mathcal{G}$ and $\overline{\mathcal{G}}$, if we compute $\overline{\mathcal{G}}$ on the linear function $u_\xi(x) = \xi\, x$, we obtain

$$\int_{B_1} \psi(x,\xi)\, dx = \int_{B_2} \psi(x,\xi)\, dx$$

on any pair of congruent balls $B_1, B_2 \subset \Omega$. This equality implies that $\psi(x,\xi) = \psi(y,\xi)$ at every pair of Lebesgue points of the function $\psi(\cdot,\xi)$. If we choose a dense sequence (ξ_h) in $M^{k\times n}$, using the continuity of $\psi(x,\cdot)$ we get the existence of a set $N \subset \Omega$ with $|N| = 0$, and such that

$$\psi(x,\xi) = \psi(y,\xi)$$

for every $\xi \in M^{k\times n}$ and for every $x, y \in \Omega \setminus N$. Hence it is not restrictive to suppose

$$\psi(x,\xi) = \psi(\xi),$$

obtaining (3.15). The inequalities in (3.16) follow from the integral representation (3.15), using estimate (3.9) and the positivity of $\overline{\mathcal{G}}$. □

Proposition 3.4. *The function ψ in Proposition 3.3 verifies*

$$\psi(\xi) \le f(\xi) \qquad \text{for every } \xi \in M^{k\times n}, \tag{3.18}$$

$$\psi^\infty(a \otimes \nu) \le g(a \otimes \nu) \qquad \text{for every } a \in \mathbb{R}^k, \nu \in \mathrm{S}^{n-1}. \tag{3.19}$$

Proof. The inequality (3.18) follows for example from

$$|\Omega|\, \psi(\xi) = \overline{\mathcal{G}}(\xi x, \Omega) \le \mathcal{G}(\xi x, \Omega) = |\Omega|\, f(\xi).$$

As for (3.19), let us fix $a \in \mathbb{R}^k$ and $\nu \in S^{n-1}$. For every $t > 0$ we can consider the linear function

$$u_t(x) = ta\langle x, \nu\rangle,$$

so that we have $Du_t = ta \otimes \nu$. We can approximate u_t in $L^1(\Omega;\mathbb{R}^k)$ with a sequence (u_t^h) in $SBV(\Omega;\mathbb{R}^k)$ defined by

$$u_t^h(x) = \frac{1}{h} ta\,[h\langle x, \nu\rangle],$$

which has jumps of size $\frac{1}{h}$ in the direction a, along hyperplanes orthogonal to ν at a regular distance of $\frac{1}{h}$. Let now Q_ν be any open cube contained in Ω with an edge parallel to ν. It is easy to see then that we have

$$\begin{aligned}\mathcal{G}(u_t^h, Q_\nu) &= f(0)|Q_\nu| + g(\frac{1}{h}\, t\, a \otimes \nu)\ \mathcal{H}^{n-1}(S_{u_t^h} \cap Q_\nu)\\ &\le f(0)|Q_\nu| + t\, g(a \otimes \nu)|Q_\nu|\end{aligned}$$

(note that we use here only the positive homogeneity of g). Hence we obtain

$$\psi(t\, a \otimes \nu) = |Q_\nu|^{-1}\overline{\mathcal{G}}(u_t, Q_\nu) \le |Q_\nu|^{-1} \liminf_h \mathcal{G}(u_t^h, Q_\nu) \le f(0) + t\, g(a \otimes \nu).$$

Dividing by t, and letting $t \to +\infty$ we obtain

$$\psi^\infty(a \otimes \nu) = \lim_{t\to+\infty} \frac{\psi(t\, a \otimes \nu)}{t} \le \lim_{t\to+\infty} \frac{f(0) + t\, g(a \otimes \nu)}{t} = g(a \otimes \nu),$$

that is, the inequality in (3.19). $\square$

Proof of Theorem 2.1 We can proceed now in the proof of Theorem 2.1. By the definition of $\overline{\mathcal{F}}$, and by (2.12), for every $u \in BV(\Omega;\mathbb{R}^k)$ we have

$$\begin{aligned}\overline{\mathcal{F}}(u) \le \overline{\mathcal{G}}(u) &\le \overline{(\overline{\mathcal{G}} + \chi_{W^{1,1}(\Omega;\mathbb{R}^k)})}(u)\\ &= \int_\Omega \psi(\nabla u(x))\, dx + \int_\Omega \psi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u|.\end{aligned}$$

Let now φ be the function defined by formula (2.9). By Proposition 3.4 we have that $\psi \le \varphi$, hence

$$\overline{\mathcal{F}}(u) \le \int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u|. \tag{3.20}$$

For the opposite inequality we have to prove that φ satisfies the hypotheses of Theorem 2.5. Inequalities $0 \le \varphi \le f$ follow immediately from the definition of φ. Let us prove that φ verifies the linear growth condition

$$0 \le \varphi(\xi) \le c(1 + |\xi|), \tag{3.21}$$

for every $\xi \in M^{k\times n}$. It suffices to show that $\phi(\xi) \le c(1+|\xi|)$ on $M^{k\times n}$ for every ϕ quasiconvex such that $\phi \le f$ on $M^{k\times n}$ and $\phi^\infty \le g$ on $M_1^{k\times n}$. Let us fix such a ϕ. By the rank one convexity of ϕ it follows that for every $\xi \in M^{k\times n}$, $a \in \mathbb{R}^k$, and $m = 1, \dots, n$ the function $t \mapsto \phi(\xi + t\, a \otimes \nu_m)$ is convex on $\mathbb{R}$ and Lipschitz with constant $\max\{g(a \otimes \nu_m), g(-a \otimes \nu_m)\}$. Since every $\xi \in M^{k\times n}$ can be uniquely decomposed as $\xi = \sum_{m=1}^n a_m \otimes \nu_m$, for suitable vectors $a_m \in \mathbb{R}^k$, by the Lipschitz condition on ϕ we get

$$\phi(\xi) \le \phi(0) + \sum_{m=1}^{n} g(a_m \otimes \nu_m) \le f(0) + cM|\xi|,$$

where M is defined as in (3.7). Hence φ verifies (3.21), and it is quasiconvex. Finally it is easy to see that $\varphi^\infty \le g$ on $M_1^{k\times n}$. Therefore we have

$$\int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u| \le \mathcal{F}(u). \tag{3.22}$$

Moreover, by Theorem 2.5 the left-hand side of (3.22) gives a L^1-lower semicontinuous functional on $BV(\Omega;\mathbb{R}^k)$, hence we obtain, by definition of relaxation,

$$\int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big)|D_s u| \le \overline{\mathcal{F}}(u).$$

This inequality, together with (3.20), concludes the proof of Theorem 2.1.

□

4. Additional Remarks

In this section we provide some examples and remarks for some classes of special f and g.

Remark 4.1. *(Positively 1-homogeneous functionals)* If the bulk energy density f is positively 1-homogeneous, then it is easy to see that φ is positively 1-homogeneous. In fact it is immediate to check that ψ is a quasiconvex function such that $\psi \le f$ on $M^{k\times n}$ and $\psi^\infty \le g$ on $M_1^{k\times n}$ if and only if for every fixed $\lambda > 0$ such is the function $\phi(\xi) = \frac{1}{\lambda}\psi(\lambda\xi)$.

Therefore we obtain the following formula for φ:

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex and positively 1-homogeneous}, \\ \psi \le f \text{ on } M^{k\times n},\ \psi^\infty \le g \text{ on } M_1^{k\times n}\},$$

hence (recall that $\psi^\infty = \psi$ for ψ positively 1-homogeneous)

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex and positively 1-homogeneous}, \\ \psi \le f \wedge g \text{ on } M^{k\times n}\}$$

(the function g is extended to $+\infty$ on $M^{k\times n} \setminus M_1^{k\times n}$).

Note that the fact that φ is positively 1-homogeneous and quasiconvex does not imply that φ is convex (cf. Müller [27]).

Remark 4.2. *(Partitions)* Let us consider the case

$$f(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ +\infty & \text{elsewhere.} \end{cases}$$

Then the functional $\mathcal{F}$ is finite only on functions in the space $SBV(\Omega; \mathbb{R}^k)$ with $\nabla u = 0$ a.e. These functions can be identified with "partitions of Ω in sets of finite perimeter" (see Ambrosio & Braides [5], [6], Congedo & Tamanini [14]). Every such function can be expressed as

$$u = \sum_{j\in\mathbb{N}} c_j \mathbf{1}_{E_j}, \tag{4.1}$$

where $c_j \in \mathbb{R}^k$, and (E_j) is a partition of Ω in sets of finite perimeter. The functional can be rewritten then in the form

$$\mathcal{F}(u) = \sum_{i,j\in\mathbb{N}} \frac{1}{2} \int_{(\partial^* E_i \cap \partial^* E_j)\cap\Omega} g\Big((c_j - c_i) \otimes \nu_j\Big)\, d\mathcal{H}^{n-1},$$

where ν_j is the interior normal to E_j, and $\partial^* E_j$ denotes the reduced boundary of E_j.

Since f is positively 1-homogeneous, by Remark 4.1 the relaxed functional $\overline{\mathcal{F}}$ is given by

$$\overline{\mathcal{F}}(u) = \int_\Omega \varphi(Du) = \int_\Omega \varphi(\nabla u)\,dx + \int_\Omega \varphi(\frac{D_s u}{|D_s u|})|D_s u|,$$

with

$$\varphi(\xi) = \sup\{\psi(\xi) :\ \psi \text{ quasiconvex and positively 1-homogeneous,}\\ \psi \le g \text{ on } M_1^{k\times n}\}$$

(note that φ is positively 1-homogeneous, hence $\varphi = \varphi^\infty$).

Remark that we obtain as a by-product of Theorem 2.1 (but also directly from [7]; see also I. Fonseca [22]) that if g is quasiconvex then the corresponding functional defined on partitions is lower semicontinuous with respect to the L^1-convergence. Hence the integrand $\tilde g(u,v,\nu) = g((v-u)\otimes\nu)$ is *BV-elliptic* (see Ambrosio & Braides [6]).

Remark 4.3. *(Partitions in Polyhedral Sets)* As a particular case of functionals defined on partitions, we can consider a function g finite and locally bounded only for n linearly independent directions $\nu_1,\dots,\nu_n$ in $\mathbb{R}^n$; *i.e.,*

$$g(a\otimes\nu) < +\infty \Longrightarrow \nu = \nu_m \text{ for some } m \in \{1,\dots,n\},$$
$$\sup\{\ g(a\otimes\nu_m)\ :\ a \in \mathrm{S}^{k-1}, m = 1,\dots,n\} < +\infty.$$

The domain of the functional $\mathcal{F}$ is then the set $\mathcal{P}_\nu$ of all partitions of Ω of the form (4.1) into polyhedra whose faces are orthogonal to the directions $\nu_1,\dots,\nu_n$.

If for example $g(\xi) \ge c|\xi|$ on $M^{k\times n}$ (this hypothesis assures the existence of a minimum in (4.2)), we can apply Remark 4.2 and prove the equivalence between segmentation problems of the type

$$\inf\Big\{\sum_{i,j\in\mathbb{N}}\int_{\partial(E_i\cap\partial E_j)\cap\Omega} g\big((c_j-c_i)\otimes\nu_j\big)\,d\mathcal{H}^{n-1} + \sum_{j\in\mathbb{N}}\int_{E_j}|c_j-\alpha(x)|\,dx :\\ u = \sum_{j\in\mathbb{N}} c_j\mathbf{1}_{E_j} \in \mathcal{P}_\nu\Big\},$$

where α is a given L^1 function, and the corresponding minimum problems in $BV(\Omega;\mathbb{R}^k)$

$$\min\Big\{2\int_\Omega\varphi(Du) + \int_\Omega|u(x)-\alpha(x)|\,dx :\ u \in BV(\Omega;\mathbb{R}^k)\Big\}. \tag{4.2}$$

As an example, if $\{e_i\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$ and

$$g(\xi) = \begin{cases} |a| & \text{if } \xi = a \otimes e_m,\ m = 1, \dots, n \\ +\infty & \text{otherwise,} \end{cases}$$

the function φ can be computed explicitly. Indeed, by Remark 4.1 and 4.2 we have

$$\varphi(\xi) = g^{**}(\xi) = \sum_{j=1}^{n} \sqrt{\sum_{i=1}^{k} \xi_{ij}^2}.$$

A similar representation for φ can be obtained for an arbitrary choice of the basis $\nu_1, \dots, \nu_n$ in $\mathbb{R}^n$ instead of the canonical one.

Remark 4.4. *(The two-well problem)* Let A and $B \in M^{k\times n}$ be two matrices such that $rank(A - B) \geq 2$. If we take

$$f(\xi) = \begin{cases} 0 & \text{if } \xi = A \text{ or } \xi = B, \\ +\infty & \text{otherwise on } M^{k\times n}, \end{cases}$$

and

$$g(a \otimes \nu) = |a|,$$

then the function φ gived by formula (2.9) is quasiconvex but not convex.

To see this, let us denote by $Q\psi$ the *quasiconvexification* of (*i.e.*, the greatest quasiconvex function less than or equal to) the function

$$\psi(\xi) = \min\{|\xi - A|, |\xi - B|\}.$$

It is already proved (see [28]) that $Q\psi$ is zero only on A and B. Moreover it is clear that $Q\psi \leq f$ on $M^{k\times n}$ and that $(Q\psi)^\infty \leq g$ on $M_1^{k\times n}$. It follows that $\varphi \geq Q\psi$; since $\varphi(A) = \varphi(B) = 0$, we conclude that $\varphi(\xi) = 0$ if and only if $\xi = A$ or $\xi = B$, hence φ is not convex.

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