ON THE EULER-LAGRANGE EQUATION OF A FUNCTIONAL BY PÓLYA AND SZEGÖ

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Abstract. We study the regularity properties of generalized solutions of the Euler-Lagrange equation of a functional involving capacity and perimeter, related to a conjecture of Pólya and Szegö.

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1. Introduction

A well known open problem in convex shape optimization is the long-standing conjecture by Pólya and Szegö on minimal electrostatic capacity sets. The *electrostatic capacity* of a compact set $K \subset \mathbb{R}^3$ is defined by

$$\mathrm{Cap}K = \inf\left\{ \int_{\mathbb{R}^3} |\nabla u|^2 \, dx : \, u \in C_0^\infty(\mathbb{R}^3), u \geq 1 \text{ in } K \right\}.$$

Equivalently, it can be defined through the *equilibrium potential* U_K of K. The function U_K, defined in $\mathbb{R}^3 \setminus K$, is the unique solution of the following equation

$$\begin{cases} \Delta U_K = 0 & \text{in } \mathbb{R}^n \setminus K; \\ U_K = 1 & \text{on } \partial K; \\ U_K(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

It is well known that U_K has the asymptotic expansion

$$U_K(x) = \gamma/(4\pi)|x|^{-1} + O(|x|^{-2}) \quad \text{as } |x| \to \infty$$

where the constant γ turns out to be precisely the electrostatic capacity $\mathrm{Cap}K$. Consider the functional $\mathcal{F} : \mathcal{K}^3 \to (0, \infty)$

$$\mathcal{F}(K) = \frac{\mathrm{Cap}^2 K}{P(K)} \tag{1.1}$$

defined in the class

$$\mathcal{K}^3 = \{K \subset \mathbb{R}^3 : \, K \text{ is convex and compact, } \mathcal{H}^2(K) > 0\}.$$

In (1.1), $P(K)$ is the *reduced perimeter* of $K \in \mathcal{K}^3$, i.e., if $K \in \mathcal{K}^3$ has nonempty interior, then $P(K) := \mathcal{H}^2(\partial K)$ is the perimeter of K, while if $K \in \mathcal{K}^3$ is a planar set, then $P(K) := 2\mathcal{H}^2(K)$. In the latter case, $P(K) = \lim_{i\to\infty} P(K_i)$ for any sequence $\{K_i\} \subset \mathcal{K}^3$ such that K_i has nonempty interior and $K_i \to K$ in the Hausdorff metric. Here and in the following, by a planar set we mean a set that lies in a hyperplane in $\mathbb{R}^3$. Note that the functional $\mathcal{F}$ is scaling invariant.

Pólya and Szegö [15, I.1.18] raised the following conjecture. The conjecture says that the planar disk has the minimal capacity among all convex sets with prescribed reduced perimeter.

Conjecture 1.1. *Let D be a planar disc in $\mathbb{R}^3$. Then*

$$\mathcal{F}(K) \geq \mathcal{F}(D)$$

for all $K \in \mathcal{K}^3$. Moreover, the equality holds if and only if K is a planar disc.

The conjecture is based on the following result proved by Pólya and Szegö in [14], stating that the disc has minimal electrostatic capacity among all planar sets, not necessarily convex, of given area. The main tool used by Pólya and Szegö to prove this result is the Steiner symmetrization.

Theorem 1.2. *Let K be a planar set such that $\mathcal{H}^2(K) = \mathcal{H}^2(D)$, where D is a planar disc. Then* $\mathrm{Cap}K \geq \mathrm{Cap}D$*. Moreover, the equality holds if and only if K is a planar disc.*

Moreover, by projecting a set $K \in \mathcal{K}^3$ on a hyperplane and then using this theorem one can easily get a positive lower bound for $\mathcal{F}$ in $\mathcal{K}^3$, see [15, p.165].

Though the conjecture is still open, some progress towards its proof has been made in recent years, see e.g. [3, 7, 2]. In particular, Crasta, Fragalà and Gazzola,using the John ellipsoid theorem [12], proved in [3] that the functional $\mathcal{F}$ has a minimum in $\mathcal{K}^3$.

Theorem 1.3. *There exists $K \in \mathcal{K}^3$ such that $\mathcal{F}(K) = \min_{L\in\mathcal{K}^3} \mathcal{F}(L)$.*

Thus, Theorems 1.2 and 1.3 imply that the conjecture of Pólya and Szegö is equivalent to the following one.

Conjecture 1.4. *The functional $\mathcal{F}$ does not have a minimizer in $\mathcal{K}^3_0 \subset \mathcal{K}^3$, where*

$$\mathcal{K}^3_0 = \{K \in \mathcal{K}^3 : K \textit{ has nonempty interior}\}.$$

That is, $\mathcal{F}(K) > \min_{L\in\mathcal{K}^3} \mathcal{F}(L)$ for all $K \in \mathcal{K}^3_0$.

In order to prove this conjecture one could argue as follows. Assume by contradiction that there exists a minimizer K with nonempty interior and that K is a *smooth convex set with positive Gaussian curvature.* Then it is easily seen that K must satisfy the following Euler-Lagrange equation

$$H_K(x) = \lambda|\nabla U_K(x)|^2 \quad \text{for all } x \in \partial K, \tag{1.2}$$

where $\lambda = 2P(K)/\mathrm{Cap}K$ is the Lagrange multiplier and $H_K(x)$ is the mean curvature of K at x. Then, by writing down the second variation of the functional $\mathcal{F}$ and by exploiting this equation, it is not too hard to show that there are always directions along which the second variation is strictly negative, hence K cannot be a local minimizer. This is precisely the argument used by Bucur, Fragalà and Lamboley in [2] to prove the following result, although in their paper this theorem is only stated for the case of minimizers.

Theorem 1.5. *Let $K \in \mathcal{K}^3_0$ be a convex body. If the boundary of K contains an open set of class C^3 with positive Gaussian curvature, then K is not a local minimizer of $\mathcal{F}$.*

In this paper, we study Conjecture 1.4 and identify a class of not necessarily smooth or strictly convex sets such that a minimizer of $\mathcal{F}$ cannot belong to this class. To state our result, we need some notation. Let $K \in \mathcal{K}_0^3$. Denote by $\operatorname{reg} K$ the set of all *regular points* $x \in \partial K$, i.e., points with a unique unit exterior normal $u(x)$. In a dual way, denote by $\operatorname{regn} K$ the set of all *regular normal vectors* of K, i.e., the set of all $u \in \mathbb{S}^2$ such that there exists a unique $x \in \partial K$ such that u is an exterior normal to K at x. In other words, if $u \in \operatorname{regn} K$, then the supporting plane to K normal to u touches ∂K only at one point. On the other hand, if $u \in \mathbb{S}^2 \setminus \operatorname{regn} K$, then the corresponding supporting plane contains at least a line segment of points of ∂K. The set of all such points, also called *flat points*, coincides with $\tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K)$, where τ_K is the multivalued map that associates at each $u \in \mathbb{S}^2$ the set of all points x of ∂K such that u is an exterior normal at x. Having fixed this notation, our main result can be stated as follows.

Theorem 1.6. *Let $K \in \mathcal{K}_0^3$ be a convex body such that*

$$\mathcal{H}^1(\partial K \setminus \operatorname{reg} K) = 0 \quad \text{and} \quad \mathcal{H}^1(\mathbb{S}^2 \setminus \operatorname{regn} K) = 0.$$

Assume also that the set $\tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K)$ is closed. Then K is not a local minimizer of $\mathcal{F}$.

No smoothness or strict convexity assumptions are required in Theorem 1.6, and thus convex sets with irregular and flat points are allowed. However, we do require a bound on the size of these sets. Recall in fact that for a general convex set one can only say that $\mathcal{H}^2(\partial K \setminus \operatorname{reg} K) = 0$ and $\mathcal{H}^2(\mathbb{S}^2 \setminus \operatorname{regn} K) = 0$, see [16, Th. 2.2.5 and Th. 2.2.11]. For instance, cubes do not satisfy the above assumptions since in this case $\partial K \setminus \operatorname{reg} K$ and $\mathbb{S}^2 \setminus \operatorname{regn} K$ have both strictly positive and finite $\mathcal{H}^1$-measure.

The main difficulty in the proof of the theorem is that without a strict convexity assumption it does not seem possible to derive an even weak formulation of the Euler-Lagrange equation, let alone the fact that in the end some regularity on K is needed to give a meaning to (1.2). On the other hand, our analysis shows that if one knows that a convex set satisfies the Euler-Lagrange equation (1.2), then K cannot even be a local minimizer. Therefore, our strategy is aimed to show that under the above assumptions on a local minimizer K the Euler-Lagrange equation holds in a weak form. To state precisely the result, we recall that for every convex set $K \in \mathcal{K}_0^3$ one can define a *mean curvature measure* $C_1(K)$ that coincides with the usual mean curvature when K is smooth. We refer to Section 2.1 for the precise definition.

Theorem 1.7. *Let $K \in \mathcal{K}_0^3$ be a local minimizer of $\mathcal{F}$ satisfying the assumptions of Theorem 1.6. Then for any Borel set $B \subset \partial K$*

$$C_1(K)(B) = \frac{2P(K)}{\operatorname{Cap} K} \int_{\partial K \cap B} |\nabla U_K|^2 \, d\mathcal{H}^2.$$

The proof of Theorem 1.7 is quite involved. We build it up step by step. Firstly, we prove in Proposition 3.1 that

$$C_1(K)(B) \leq \frac{4P(K)}{\operatorname{Cap} K} \int_{\partial K \cap B} |\nabla U_K|^2 \, d\mathcal{H}^2$$

for any Borel set $B \subset \partial K$. To get this inequality, the results obtained by Jerison in [9] play an important role. In particular, we use his notion of capacitary measure and a convergence result of the support functions of suitable variations of a convex set, see Lemma 2.5. Secondly, we show in Proposition 3.2 that ∂K is locally a graph of a $W^{2,p}$ convex function for some $p > 1$. Here we apply some classical results on harmonic measures of Lipschitz domains, due to Dahlberg [4], see Theorem 2.7. Then, by another application of Dahlberg results, we show that indeed the regular set $\operatorname{reg} K$ is open and of the class $C^{1,\alpha}$ for all $0 < \alpha < 1$. Finally, with this regularity in hand, we conclude the proof of Theorem 1.7 thanks to a uniform convergence result for the support functions of certain variations of a convex set proved in Lemma 2.6.

At this point, using the Euler-Lagrange equation stated in Theorem 1.7 and the $C^{1,\alpha}$ regularity of $\operatorname{reg} K$, we are able to show that ∂K contains a smooth region with positive Gaussian curvature. Then Theorem 1.6 follows from Theorem 1.5.

Note that the arguments leading to the proof of Theorem 1.6 can be localized and we leave the precise statement to the reader. As a particular case the following simpler version of the local theorem holds.

Corollary 1.8. *Let $K \in \mathcal{K}_0^3$ be a convex body such that $\operatorname{reg} K \cap \tau_K(\operatorname{regn} K)$ has non empty interior. Then K is not a local minimizer of $\mathcal{F}$.*

The paper is organized as follows. Section 2.1 contains the notation and some basic results of convex analysis, including the definitions of curvature measures and surface measures. Jerison's results are discussed in Section 2.2. In Section 2.3, we discuss Dahlberg's results on the harmonic measures of Lipschitz domains. In Section 3, we give the proofs of Theorem 1.6 and Theorem 1.7.

2. Notation and preliminary results

2.1. **Convex bodies, curvature measures and surface measures.** In the following we shall denote by $\mathcal{K}_0^n$ the family of all *convex bodies* in $\mathbb{R}^n$, i.e., convex compact sets, with nonempty interior. Let $k \geq 2$ be an integer. We say that a convex body K is of class C_+^k if ∂K is of class C^k and the Gaussian curvature is everywhere positive. For all the relevant definitions and the basic properties of convex bodies we shall always refer to the book of Schneider [16]. We shall also follow essentially the same notation of this book, up to some simplifications in the symbols.

If $K \in \mathcal{K}_0^n$ we denote by $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$ its *support function*, that is,

$$h_K(u) = \sup\{x \cdot u : \, x \in K\}.$$

If $x \in \partial K$ the set of all unit exterior normals to K at x is denoted by $N_K(x)$. Then we denote by $\operatorname{reg} K$ the set of all points in ∂K for which there exists a unique unit exterior normal u. The map $g_K : \operatorname{reg} K \mapsto \mathbb{S}^{n-1}$ associating to each $x \in \operatorname{reg} K$ the unique normal in $N_K(x)$ is called the *Gauss map*. If X is any subset of ∂K we denote by $\sigma_K(X)$ the *spherical image* of K at points of X, that is,

$$\sigma_K(X) := \{u \in \mathbb{S}^{n-1} : \, u \in N_K(x) \text{ for some } x \in X\}.$$

In a dual way, the set $\operatorname{regn} K$ denotes the set of all *regular normal vectors* of K, i.e., the set of all $u \in \mathbb{S}^{n-1}$ such that there exists a unique $x \in \partial K$ for which $u \in N_K(x)$. If ω

is any subset of $\mathbb{S}^{n-1}$ the *reverse spherical image* of ω, denoted by $\tau_K(\omega)$, is defined by setting

$$\tau_K(\omega) := \{x \in \partial K : \text{ there exists } u \in N_K(x) \cap \omega\}.$$

Finally, the *normal bundle* $\operatorname{Nor} K$ is the subset of $\mathbb{R}^n \times \mathbb{S}^{n-1}$ consisting of all pairs (x, u) with $x \in \mathbb{R}^n$ and $u \in N_K(x)$. We recall, see [16, Sec. 4.2], that if K is a convex body in $\mathbb{R}^n$ for any $m = 0, \cdots, n-1$ one may define the *m-th generalized curvature* $\Theta_m(K)$ of K which is a Borel measure on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ concentrated on the normal bundle $\operatorname{Nor} K$. The corresponding marginal measures defined by setting for any Borel set $B \subset \mathbb{R}^N$ and any Borel set $\omega \subset \mathbb{S}^{n-1}$

$$C_m(K)(B) := \Theta_m(K)(B \times \mathbb{S}^{n-1}), \qquad S_m(K)(\omega) := \Theta_m(K)(\mathbb{R}^n \times \omega),$$

are called the *curvature measures* of K and the *surface measures* of K, respectively. In the following we shall mostly use the measures C_{n-2} and S_1. To this aim we recall that if $K \in C^2_+$ then for every Borel set $B \subset \mathbb{R}^n$ we have, see [16, (4.25) and (2.36)],

$$C_{n-2}(K)(B) = \int_{\partial K \cap B} H_K(x)\, d\mathcal{H}^{n-1}(x), \tag{2.1}$$

where $H_K(x)$ is the *mean curvature* at x, that is $H_K(x) := \frac{1}{n-1}(k_1(x) + \cdots + k_{n-1}(x))$, and $0 \le k_1(x) \le \cdots \le k_{n-1}(x)$ are the *principal curvatures* at x. On the other hand, by combining the representation formulae (4.26) and (2.56) in [16] one has that for any Borel subset $\omega \subset \mathbb{S}^{n-1}$

$$S_1(K)(\omega) = \int_\omega \Big(h_K(u) + \frac{1}{n-1} \Delta_{\mathbb{S}^{n-1}} h_K(u) \Big)\, d\mathcal{H}^{n-1}(u), \tag{2.2}$$

where $\Delta_{\mathbb{S}^{n-1}}$ denotes the *Laplace-Beltrami* operator on the sphere.

Remark 2.1. An interesting consequence of the previous formula is that if $K, L \in \mathcal{K}^n_0$, then

$$\int_{\mathbb{S}^{n-1}} h_K\, dS_1(L)(u) = \int_{\mathbb{S}^{n-1}} h_L\, dS_1(K)(u). \tag{2.3}$$

Indeed, if $K, L \in C^2_+$, this equality follows immediately from the representation formula (2.2) using that

$$\int_{\mathbb{S}^{n-1}} h_K \Delta_{\mathbb{S}^{n-1}} h_L\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} h_L \Delta_{\mathbb{S}^{n-1}} h_K\, d\mathcal{H}^{n-1}.$$

In the general case (2.3) follows from the approximation Theorem 2.2 below.

A natural metric in the set $\mathcal{K}^n_0$ is provided by the *Hausdorff distance*. Recall that if K, L are two compact subsets in $\mathbb{R}^n$, the Hausdorff distance $\operatorname{dist}_H(K, L)$ between them is defined as

$$\operatorname{dist}_H(K, L) := \max \Big\{ \sup_{x \in K} \inf_{y \in L} |x - y|, \sup_{x \in L} \inf_{y \in K} |x - y| \Big\}.$$

In the following, if K_h is a sequence in $\mathcal{K}^n_0$ converging to some set K in the Hausdorff distance we shall simply write $K_h \to K$ without further specifications. Next result collects some important properties of the Hausdorff convergence of convex bodies that we shall use later. For the proofs see Theorem 3.4.1 and the subsequent remarks, Lemma 1.8.12 and Theorem 4.2.1 in [16].

Theorem 2.2. *Let $K_h \in \mathcal{K}_0^n$ be a sequence of convex bodies such that $K_h \to K$. Then, the functions h_{K_h} converge uniformly to h_K in $\mathbb{S}^{n-1}$. Moreover, for any $m = 0, \dots, n-1$, the measures $C_m(K_h)$ converge weakly* to $C_m(K)$ in $\mathbb{R}^n$ and $S_m(K_h)$ converge weakly* to $S_m(K)$ in $\mathbb{S}^{n-1}$. Finally, if $K \in \mathcal{K}_0^n$, one can always find a sequence convex bodies $K_h \in C^\infty_+$ such that $K_h \to K$.*

We conclude this section with the following two other properties.

Proposition 2.3. *If $K \in \mathcal{K}_0^n$, then*

$$P(K) = \frac{1}{n-1}\int_{\mathbb{S}^{n-1}} h_K(u)\, dS_{n-2}(K)(u). \tag{2.4}$$

Proof. Let us assume that $K \in C^2_+$. In this case, see [16, p.220], the measures $S_m(K)$ are the *push-forward* of the corresponding measures $C_m(K)$ through the Gauss map g_K. Therefore

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} h_K(u)\, dS_{n-2}(K)(u) &= \int_{\partial K} h_K(g_K(x))\, dC_{n-2}(K)(x) \\ &= \int_{\partial K} x\cdot g_K(x) H_K(x)\, d\mathcal{H}^{n-1}(x),\end{aligned} \tag{2.5}$$

where the second equality follows from (2.1) and from the fact that since K is smooth and strictly convex $h_K(u) = g_K^{-1}(u)\cdot u$ for all $u \in \mathbb{S}^{n-1}$, see [16, p.115]. On the other hand, denoting by div_τ the tangential divergence on ∂K and recalling that $g_K(x)$ is the exterior normal to ∂K at x, from the divergence theorem we have immediately, observing that $\mathrm{div}_\tau\, x = n-1$,

$$\int_{\partial K} x\cdot g_K(x) H_K(x)\, d\mathcal{H}^{n-1} = \int_{\partial K} \mathrm{div}_\tau\, x\, d\mathcal{H}^{n-1} = (n-1)P(K).$$

Then (2.4) follows by combining this equality with (2.5). In the general case the result easily follows from Theorem 2.2, recalling that if $K_h \to K$ then also $P(K_h) \to P(K)$. □

Proposition 2.4. *Let $K \in \mathcal{K}_0^n$ and $m \in \{0, \dots, n-1\}$. If $X \subset \mathbb{R}^n$ and $\omega \subset \mathbb{S}^{n-1}$ are Borel sets, then*

$$S_m(K)(\sigma_K(X) \cap \mathrm{regn}\, K) \le C_m(K)(X) \le S_m(K)(\sigma_K(X)), \tag{2.6}$$

$$C_m(K)(\tau_K(\omega) \cap \mathrm{reg}\, K) \le S_m(K)(\omega) \le C_m(K)(\tau_K(\omega)). \tag{2.7}$$

Proof. The proof of (2.6) in the case X is a closed set is given in [16, Lemma 4.2.4]. To show that (2.6) also holds when X is open it is then enough to take an increasing sequence of closed sets X_i such that $X = \cup_i X_i$, to apply (2.6) to each X_i and to pass to the limit. Let us now assume that X is a Borel set. Then, for any $\varepsilon > 0$ there exists a closed set $C \subset X$ such that $C_m(X \setminus C) < \varepsilon$. Then, we have

$$C_m(K)(X) \le C_m(K)(C) + \varepsilon \le S_m(K)(\sigma_K(C)) + \varepsilon \le S_m(K)(\sigma_K(X)) + \varepsilon\,,$$

hence the second inequality in (2.6) follows letting $\varepsilon \to 0^+$. Recall now that C_m and S_m are finite measures (see [16, Th. 4.2.1]). Thus, for any $\varepsilon > 0$ there exists an open set $A \supset X$ such that $C_m(A \setminus X) < \varepsilon$. Then, the first inequality in (2.6) for X follows by applying the same inequality to A and passing to the limit as $\varepsilon \to 0^+$. The proof of (2.7) is similar. □

2.2. **Variational sets.** Let $K \in \mathcal{K}_0^n$, $n \geq 3$, be a convex body. We assume that $B_r(0) \subset K \subset B_R(0)$ where $0 < r < R$, that is, $r \leq h_K(u) \leq R$ for all $u \in \mathbb{S}^{n-1}$. Let $v \in C(\mathbb{S}^{n-1})$ be a continuous function. For $t > 0$, we set

$$K_t = \{x \in \mathbb{R}^n : x \cdot u \leq h_K(u) + tv(u) \ \ \forall u \in \mathbb{S}^{n-1}\}.$$

Then $K_t \in \mathcal{K}_0^n$ if t is small, and $K_t \to K$ in the Hausdorff distance as $t \to 0^+$. We note that

$$h_{K_t}(u) \leq h_K(u) + tv(u) \quad \forall u \in \mathbb{S}^{n-1}, \tag{2.8}$$

and they may be not equal. We have the following uniform estimate

$$\sup_{0<t\leq t_0} \frac{|h_{K_t}(u) - h_K(u)|}{t} \leq RA/r \tag{2.9}$$

for all $u \in \mathbb{S}^{n-1}$, where $A = \max_{\mathbb{S}^{n-1}} |v|$ and $t_0 = r/(2A)$. We refer to [9, Lemma 3.2] for a proof of (2.9).

Next lemma is due to Jerison, see [9, Lemma 3.3]. The proof we give here is essentially the same as in [9], up to some small corrections.

Lemma 2.5. *Suppose that the Gauss map g_K for K is continuous at $x_0 \in \partial K$ and that $g_K(x_0) = u_0$. Let $v \in C(\mathbb{S}^{n-1})$. For any family of unit vectors u_t that tend to u_0,*

$$\lim_{t\to 0^+} \frac{h_{K_t}(u_t) - h_K(u_t)}{t} = v(u_0).$$

Proof. Since g_K is continuous at x_0, for any $\varepsilon_0 > 0$ there exists $\delta_0 = \delta_0(x_0, \varepsilon_0), 0 < \delta_0 < 1$, such that

$$x_0 - \varepsilon_0\delta_0 u_0 + \delta_0 y \in K \tag{2.10}$$

for every $y \in \mathbb{R}^n$ such that $|y| \leq 1$ and $y \perp u_0$. In particular we can choose $\delta_1, 0 < \delta_1 < 1$, such that

$$x_0 - \delta_1 u_0 + \delta_1 y \in K \tag{2.11}$$

for all $y \in \mathbb{R}^n$ such that $|y| \leq 1$ and $y \perp u_0$. Now we fix a parameter $\varepsilon, 0 < \varepsilon < 1/10$. We then choose $\varepsilon_0, 0 < \varepsilon_0 < \varepsilon$, sufficiently small depending on ε. For example, $\varepsilon_0 = \varepsilon^2$ will do. Let δ_0 be the constant such that (2.10) holds. We may assume that $\delta_0 \leq \delta_1/4$. Then define

$$\bar{v} = \min_{|u-u_0|\leq 4\varepsilon} v(u), \quad A = \max_{u\in\mathbb{S}^{n-1}} |v(u)|.$$

Choose $u_t \in \mathbb{S}^{n-1}$ such that $|u_t - u_0| \leq \varepsilon_0\delta_0/10$, and $x_t \in \partial K$ such that $x_t \cdot u_t = h_K(u_t)$. Denote

$$\tilde{x}_t = \delta(x_0 - \varepsilon_0\delta_0 u_t/2) + (1 - \delta)x_t,$$

where $\delta = 8At/(\varepsilon\delta_0)$. We have $\delta < 1$, provided that

$$t < \varepsilon\delta_0/(8A). \tag{2.12}$$

We will prove that

$$\tilde{x}_t + t(\bar{v} - 10\varepsilon A)u_t \in K_t, \tag{2.13}$$

that is,

$$\big(\tilde{x}_t + t(\bar{v} - 10\varepsilon A)u_t\big) \cdot \eta \leq h_K(\eta) + tv(\eta) \tag{2.14}$$

for all $\eta \in \mathbb{S}^{n-1}$. To prove (2.13), we need to prove two claims. First, we claim that

$$x_0 - \varepsilon_0\delta_0 u_t/2 + \delta_0 y/4 \in K \tag{2.15}$$

for all $|y| \le 1$ such that $y \perp u_t$. Indeed, consider $y \in \mathbb{R}^n$ such that $|y| \le 1$ and $y \perp u_t$. Then write

$$\varepsilon_0\delta_0 u_t/2 - \delta_0 y/4 = \alpha u_0 - z, \tag{2.16}$$

where $z \perp u_0$. Then

$$\begin{aligned}\alpha &= \big(\varepsilon_0\delta_0/2\big)u_t \cdot u_0 - \big(\delta_0/4\big)y \cdot u_0\\ &= \varepsilon_0\delta_0/2 + \big(\varepsilon_0\delta_0/2\big)(u_t - u_0)\cdot u_0 + \big(\delta_0/4\big)y\cdot(u_t - u_0).\end{aligned}$$

It follows that

$$|\alpha - \varepsilon_0\delta_0/2| \le \big(\varepsilon_0\delta_0/2 + \delta_0/4\big)|u_t - u_0| \le \varepsilon_0\delta_0/10,$$

since $|u_t - u_0| \le \varepsilon_0\delta_0/10$. Hence, $3/10 \le \alpha/(\varepsilon_0\delta_0) < 1$. By (2.16), we have

$$|z|^2 = (\delta_0/4)y\cdot z - (\varepsilon_0\delta_0/2)(u_t - u_0)\cdot z \le \big(\delta_0/4 + \varepsilon_0\delta_0|u_t - u_0|/2\big)|z|,$$

from which it follows that $|z| \le 3\delta_0/10$. Now we write

$$x_0 - \varepsilon_0\delta_0 u_t/2 + \delta_0 y/4 = \big(1 - \alpha/(\varepsilon_0\delta_0)\big)x_0 + \big(\alpha/(\varepsilon_0\delta_0)\big)\big(x_0 - \varepsilon_0\delta_0 u_0 + (\varepsilon_0\delta_0/\alpha)z\big)$$

and note that $x_0 - \varepsilon_0\delta_0 u_0 + (\varepsilon_0\delta_0/\alpha)z \in K$ by (2.10), since $z \perp u_0$ and $|(\varepsilon_0\delta_0/\alpha)z| \le \delta_0$. This proves claim (2.15). It follows from (2.15) that

$$\tilde{x}_t + \delta\delta_0 y/4 = \delta\big(x_0 - \varepsilon_0\delta_0 u_t/2 + \delta_0 y/4\big) + \big(1-\delta\big)x_t \in K \tag{2.17}$$

for all $y \in \mathbb{R}^n$ such that $|y| \le 1$ and $y \perp u_t$.
Then, we claim that

$$x_0 - \delta_1 u_t/2 + \delta_1 y/4 \in K \tag{2.18}$$

for all $|y| \le 1$ such that $y \perp u_t$. The proof of (2.18) is the same as that of (2.15). Indeed, in the proof of (2.15), we let $\varepsilon_0 = 1$ and replace δ_0 by δ_1. We use (2.11), instead of (2.10). We omit the details. It follows from (2.18) by letting $y = 0$ that

$$\tilde{x}_t - \delta(\delta_1 - \delta_0)u_t/2 = \delta\big(x_0 - \delta_1 u_t/2\big) + \big(1-\delta\big)x_t \in K. \tag{2.19}$$

In order to show (2.14) for all $\eta \in \mathbb{S}^{n-1}$ we divide the proof into two cases: $|\eta - u_0| \le 4\varepsilon$ and $|\eta - u_0| > 4\varepsilon$. In the first case when $|\eta - u_0| \le 4\varepsilon$, we have

$$u_t \cdot \eta = 1 + u_t\cdot(\eta - u_0) + u_t\cdot(u_0 - u_t) \ge 1 - 4\varepsilon - |u_t - u_0| \ge 1 - 5\varepsilon,$$

since $|u_t - u_0| \le \varepsilon_0\delta_0/10 < \varepsilon$. Therefore, $|u_t\cdot\eta - 1| \le 5\varepsilon$. Letting $y = 0$ in (2.17), we have $\tilde{x}_t \in K$, which implies that

$$\tilde{x}_t \cdot \eta \le h_K(\eta).$$

Thus

$$\begin{aligned}\big(\tilde{x}_t + t(\bar{v} - 10\varepsilon A)u_t\big)\cdot\eta =& \tilde{x}_t\cdot\eta + t(\bar{v} - 10\varepsilon A) + t(\bar{v} - 10\varepsilon A)(u_t\cdot\eta - 1)\\ \le& h_K(\eta) + t\bar{v} - 10\varepsilon A t + 5\varepsilon(1 + 10\varepsilon)At\\ \le& h_K(\eta) + t\bar{v} \le h_K(\eta) + tv(\eta),\end{aligned}$$

which proves (2.14) in this case, provided that $\varepsilon < 1/10$.

In the second case, when $|\eta - u_0| > 4\varepsilon$, we have $|u_t - \eta| > 3\varepsilon$, since $|u_t - u_0| \le \varepsilon_0\delta_0/10 < \varepsilon$. Then we write

$$\eta = au_t + z, \quad a = u_t \cdot \eta,$$

where $z \perp u_t$. We have two more cases: $|z| > 3\varepsilon/2$ and $|z| \le 3\varepsilon/2$. When $|z| > 3\varepsilon/2$, we use (2.17) to prove (2.14). Indeed, by (2.17), we have

$$\tilde{x}_t + \delta\delta_0 z/(4|z|) \in K,$$

which implies that

$$\big(\tilde{x}_t + \delta\delta_0 z/(4|z|)\big) \cdot \eta \le h_K(\eta).$$

Since $z \cdot \eta = |z|^2$ and $|z| > 3\varepsilon/2$, we have that

$$h_K(\eta) \ge \tilde{x}_t \cdot \eta + 3\delta\delta_0\varepsilon/8 = \tilde{x}_t \cdot \eta + 3At.$$

Now (2.14) follows from the above inequality. Indeed,

$$\begin{aligned}\big(\tilde{x}_t + t(\bar{v} - 10\varepsilon A)u_t\big) \cdot \eta &\le \tilde{x}_t \cdot \eta + (1 + 10\varepsilon)At \le \tilde{x}_t \cdot \eta + 2At \\ &\le h_K(\eta) - At \le h_K(\eta) + tv(\eta),\end{aligned}$$

which proves (2.14). When $|z| \le 3\varepsilon/2$, we claim that

$$a = u_t \cdot \eta \le -1/2.$$

The claim follows from the fact that

$$1 = |\eta|^2 = a^2 + |z|^2 \le a^2 + 9\varepsilon^2/4, \tag{2.20}$$

since $|z| \le 3\varepsilon/2$, and that

$$9\varepsilon^2 < |u_t - \eta|^2 = (1 - a)^2 + |z|^2 \le (1 - a)^2 + 9\varepsilon^2/4, \tag{2.21}$$

since $|u_t - \eta| > 3\varepsilon$. Recall that $\varepsilon < 1/10$. Then we can first deduce from (2.20) and (2.21) that $a < 0$ and thus the claim follows from (2.20). To prove (2.14) we observe that (2.19) implies that

$$\big(\tilde{x}_t - \delta(\delta_1 - \delta_0)u_t/2\big) \cdot \eta \le h_K(\eta).$$

Recalling that $u_t \cdot \eta = a \le -1/2$, that $\delta_0 \le \delta_1/4$ and that $\delta = 8At/(\varepsilon\delta_0)$, we obtain

$$h_K(\eta) \ge \tilde{x}_t \cdot \eta + 3\delta\delta_0/4 \ge \tilde{x}_t \cdot \eta + 3At.$$

Now (2.14) follows from the above inequality. Indeed,

$$\begin{aligned}\big(\tilde{x}_t + t(\bar{v} - 10\varepsilon A)u_t\big) \cdot \eta &\le \tilde{x}_t \cdot \eta + (1 + 10\varepsilon)At \le \tilde{x}_t \cdot \eta + 2At \\ &\le h_K(\eta) - At \le h_K(\eta) + tv(\eta).\end{aligned}$$

This proves (2.14) also in this case. The proof of (2.14) is then complete.
To conclude the proof, from (2.13) we get

$$h_{K_t}(u_t) \ge \big(\tilde{x}_t + t(\bar{v} - 10\varepsilon A)u_t\big) \cdot u_t = \tilde{x}_t \cdot u_t + t(\bar{v} - 10\varepsilon A). \tag{2.22}$$

Recalling the choice of x_t and the definition of $\tilde{x}_t$, we have

$$\begin{aligned}h_K(u_t) = x_t \cdot u_t &= \tilde{x}_t \cdot u_t + (x_t - \tilde{x}_t) \cdot u_t \\ &= \tilde{x}_t \cdot u_t + \delta(x_t - x_0) \cdot u_t + \delta\varepsilon_0\delta_0/2.\end{aligned}$$

Since $x_t \cdot u_0 \le x_0 \cdot u_0 = h_K(u_0)$, we have

$$\begin{aligned}(x_t - x_0) \cdot u_t &= (x_t - x_0) \cdot u_0 + (x_t - x_0) \cdot (u_t - u_0) \\ &\le (x_t - x_0) \cdot (u_t - u_0) \le 2R|u_t - u_0| \le \varepsilon_0 \delta_0 R/5.\end{aligned}$$

Thus we arrive at

$$h_K(u_t) \le \tilde{x}_t \cdot u_t + (1/2 + R/5)\delta \varepsilon_0 \delta_0. \tag{2.23}$$

Combining (2.22) and (2.23), we obtain that

$$\frac{h_{K_t}(u_t) - h_K(u_t)}{t} \ge \bar{v} - 10\varepsilon A - 8(1/2 + R/5)A\varepsilon_0/\varepsilon \ge \bar{v} - C\varepsilon,$$

where $C = (14 + 8R/5)A$, if we choose $\varepsilon_0 = \varepsilon^2$. This inequality, together with (2.8), yields

$$\left| \frac{h_{K_t}(u_t) - h_K(u_t)}{t} - v(u_0) \right| \le |v(u_0) - \bar{v}| + C\varepsilon, \tag{2.24}$$

provided that $\delta = 8At/(\varepsilon\delta_0) < 1$, that is, $t < \varepsilon\delta_0/(8A)$. Since v is continuous and ε is arbitrary, this last inequality concludes the proof of the lemma. □

We also need the following uniform convergence result.

Lemma 2.6. *Let $K \in \mathcal{K}_0^3$ and $v \in C(\mathbb{S}^{n-1})$. Then, $\big(h_{K_t}(u) - h_K(u)\big)/t$ converges uniformly to $v(u)$, as $t \to 0$, in every closed subset $\omega \subset \sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$.*

Proof. Observe that if $\omega \subset \sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$ is closed, then $\tau_K(\omega)$ is a closed subset of $\operatorname{reg} K$, hence the Gauss map is uniformly continuous in $\tau_K(\omega)$. Therefore, given $\varepsilon_0 > 0$ there exists δ_0 such that (2.10) holds uniformly for every $x_0 \in \tau_K(\omega)$, $u_0 = g_K(x_0)$ and $y \in \mathbb{R}^n$, with $|y| \le 1$ and $y \perp u_0$. Then, the proof goes on exactly as in Lemma 2.5, choosing $u_t = u_0, x_t = x_0, \bar{v} = v(u_0)$. At the end, instead of (2.24), we will get

$$\left| \frac{h_{K_t}(u_0) - h_K(u_0)}{t} - v(u_0) \right| \le C\varepsilon \quad \text{for all } u_0 \in \omega\,,$$

where $C = (14 + 8R/5)A$, if $t < \varepsilon\delta_0/(8A)$, thus proving the lemma. □

2.3. **Capacity and harmonic measure.** Let $K \in \mathcal{K}_0^n$ be a convex body, $n \ge 3$. Let U_K be the equilibrium potential defined in $\mathbb{R}^n \setminus K$, that is, U_K is the unique solution of the following equation

$$\begin{cases} \Delta U_K = 0 & \text{in } \mathbb{R}^n \setminus K; \\ U_K = 1 & \text{on } \partial K; \\ U_K(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

The following theorem combines the statements of Theorem 1.7 and Proposition 3.2 of [10]. The proof is due to Dahlberg [4], see also [11].

Theorem 2.7. *Let $K \in \mathcal{K}_0^n$ be a convex body and U_K be the equilibrium potential defined in $\mathbb{R}^n \setminus K$. Then the nontangential limit*

$$\lim_{y \to x} \nabla U_K(y), \quad y \in \Gamma(x) = \{z \in \mathbb{R}^n \setminus K : |z - x| \le C \operatorname{dist}(z, \partial K)\}$$

exists for $\mathcal{H}^{n-1}$-almost every $x \in \partial K$. Denote this limit by $\nabla U_K(x)$. Then $|\nabla U_K| \in L^2(\partial K; \mathcal{H}^{n-1} \llcorner \partial K)$. Moreover, there exists an exponent $p > 2$, depending on K and n, such that $|\nabla U_K| \in L^p(\partial K; \mathcal{H}^{n-1} \llcorner \partial K)$.

Note that here and in the following by $L^p(A;\mu)$ we denote the usual Lebesgue space with respect to the measure μ. We shall also denote by D a Lipschitz domain in $\mathbb{R}^n, n \geq 3$ and by P a fixed point in D. Then we shall denote by ω the harmonic measure of D evaluated at P. Next result is also due to Dahlberg, [4, Th. 3].

Theorem 2.8. *Let $D \subset \mathbb{R}^n, n \geq 3$, be a bounded Lipschitz domain and denote by G the Green function of D with pole at a point $P \in D$. Set $g = G(P, \cdot)$.*

(a) *For any constant $C > 0$, the nontangential limit*

$$\lim_{y \to x} \nabla g(y), \quad y \in \Gamma(x) = \{z \in D : |z - x| \leq C \operatorname{dist}(z, \partial D)\}$$

exists for $\mathcal{H}^{n-1}$-almost every $x \in \partial D$. We denote this limit by $\nabla g(x)$.

(b) *Let k be the density of ω with respect to $\mathcal{H}^{n-1} \llcorner \partial D$. We have $|\nabla g(x)| = (n-2)\sigma_n k(x)$ for almost every $x \in \partial D$ with respect to $\mathcal{H}^{n-1} \llcorner \partial D$, where σ_n is the surface measure of $\mathbb{S}^{n-1}$.*

(c) *There is a constant $C > 0$ such that for all $y \in \partial D$ and for all $r, 0 < r < 1$, we have*

$$\mathcal{H}^{n-1}(\partial D \cap B_r(y)) \int_{\partial D \cap B_r(y)} k(x)^2 \, d\mathcal{H}^{n-1} \leq C \left(\int_{\partial D \cap B_r(y)} k(x) \, d\mathcal{H}^{n-1} \right)^2. \tag{2.25}$$

We recall the following boundary Harnack inequality for positive harmonic functions that vanish on a part of boundary.

Theorem 2.9. *Let D be a Lipschitz domain. Suppose that V is an open set such that $V \cap \partial D \neq \emptyset$. Suppose that W is a domain such that $W \subset D$ and $\bar{W} \subset V$. Let $P_0 \in W$. Then there is a constant $C > 0$ such that if u and v are non-negative harmonic functions in D that vanish on $V \cap \partial D$ and satisfy $u(P_0) \leq v(P_0)$, then $u(x) \leq C v(x)$ for all $x \in W$.*

Putting together the results of Theorems 2.8 and 2.9, and arguing as in the proof of Theorem 2.10 below, one gets the first part of the statements of Theorem 2.7. The higher integrability of $|\nabla U_K|$ then follows from inequality (2.25), thanks to Gehring's lemma [8].

Note that the integrability result in Theorem 2.7, $|\nabla U_K| \in L^p(\partial K; \mathcal{H}^{n-1} \llcorner \partial K)$ for some $p > 2$, holds if K is Lipschitz. In the following, we prove a higher integrability result, under the smallness assumption on the Lipschitz constant of the boundary. To state the result, we need some notation.

Since $K \in \mathcal{K}_0^n$, ∂K is locally given by the graph of a Lipschitz function. Precisely, if $0 \in \partial K$, there is $r_1, r_2 > 0$ such that after a suitable rotation, we have

$$(\mathbb{R}^n \setminus K) \cap L_{r_1,r_2} = \{x = (x', x_n) \in L_{r_1,r_2} : x_n > \varphi(x')\},$$

and

$$\partial K \cap L_{r_1,r_2} = \{(x', x_n) \in L_{r_1,r_2} : x_n = \varphi(x')\},$$

where L_{r_1,r_2} is the cylinder

$$L_{r_1,r_2} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < r_1, |x_n| < r_2\}, \tag{2.26}$$

and φ is a Lipschitz function in $B_{r_1}^{n-1}(0) = \{y' \in \mathbb{R}^{n-1} : |y'| < r_1\}$, such that $\varphi(0) = 0$. We shall denote the Lipschitz constant of φ by $\Lambda(\varphi)$.

Theorem 2.10. *Let $K \in \mathcal{K}_0^n$ and U_K be the equilibrium potential defined in $\mathbb{R}^n \setminus K$. Suppose that $0 \in \partial K$ and that ∂K is given in L_{r_1,r_2} by the graph of Lipschitz function φ. Then for any $q > 2$ there exists $\varepsilon_0 = \varepsilon_0(q,n) > 0$ such that if $\delta \in (0,1)$ and $\Lambda(\varphi) < \delta\varepsilon_0$, then $|\nabla U_K(x)| \in L^q(\partial K \cap L_{(1-\delta)r_1,r_2}; \mathcal{H}^{n-1} \llcorner \partial K)$.*

Note that in the above theorem, ε_0 depends only on q and n, and that the bound for the Lipschitz constant of φ depends only on δ and ε_0. All these quantities do not depend on the size of the cylinder L_{r_1,r_2}. Of course the L^q norm of $|\nabla U_K|$ does depend on r_1, r_2. Theorem 2.10 follows by combining Lemma 3.5 in [5], and the boundary Harnack inequality Theorem 2.9.

Proof of Theorem 2.10. Suppose that $K \in \mathcal{K}_0^n$ and that ∂K is given near $0 \in \partial K$ in L_{r_1,r_2} by the graph of Lipschitz function φ. It is convenient to modify the function φ as follows. Let $\eta \in C_0^\infty(B_{r_1}^{n-1}(0)), 0 \le \eta \le 1$, be a cut-off function such that $\eta = 1$ in $B_{(1-\delta/2)r_1}^{n-1}(0)$ and $|\nabla \eta| \le 4/(\delta r_1)$ in $B_{r_1}^{n-1}(0)$, where $0 < \delta < 1$. Set $\tilde\varphi = \varphi\eta$. From the assumption $\Lambda(\varphi) < \delta\varepsilon_0$ we easily get that

$$\Lambda(\tilde\varphi) < 8\varepsilon_0. \tag{2.27}$$

Extending $\tilde\varphi$ to be zero outside $B_{r_1}^{n-1}(0)$, we define

$$D(\tilde\varphi) = \{(x', x_n) \in \mathbb{R}^n : |x'| < 10r_1, \tilde\varphi(x') < x_n < M\}, \tag{2.28}$$

where M is chosen large enough such that $D(\tilde\varphi)$ is star-shaped with respect to the point $P = (0, M/2)$. Note that if we take $\varepsilon_0 < 1$ then M will only depend on r_1 and n. Note also that $\tilde\varphi(x') = \varphi(x')$ when $|x'| < (1-\delta/2)r_1$. Now set

$$D = \{(x', x_n) : |x'| < (1-\delta/2)r_1, \varphi(x') < x_n < L\},$$

where $L = \min(M/4, r_2)$ and M is the constant in the definition of $D(\tilde\varphi)$ in (2.28). From the choice of L we have that $D \subset \mathbb{R}^n \setminus K$ and $D \subset D(\tilde\varphi)$.

Let U_K be the equilibrium potential defined in $\mathbb{R}^n \setminus K$ and $G(P, \cdot)$ be the Green function of $D(\tilde\varphi)$ with the pole at $P = (0, M/2)$. Note that P is far away from the domain D defined above. Then $u = 1 - U_K$ and $g = G(P, \cdot)$ are positive harmonic functions in D, and they vanish on the boundary part $V \cap \partial D$, where $V = \{(x', x_n) : |x'| < (1-\delta/2)r_1, |x_n| < L\}$. Now we may apply the boundary Harnack inequality, Theorem 2.9, with $W = \{(x', x_n) : |x'| < (1-\delta)r_1, \varphi(x') < |x_n| < L'\}$ for a suitable $L' < L$. We conclude that u and g are comparable in W. Then it follows from Theorem 2.8 that $|\nabla u|$ and $|\nabla g|$ are also comparable on $S_\delta(\varphi) = \{(x', \varphi(x')) : |x'| < (1-\delta)r_1\}$. Then by Lemma 2.11 below, we have $|\nabla g| \in L^q(S(\tilde\varphi); \mathcal{H}^{n-1} \llcorner S(\tilde\varphi))$, where $S(\tilde\varphi) = \{(x', \tilde\varphi(x')) : |x'| \le r_1\}$. Thus $|\nabla u| \in L^q(S_\delta(\varphi); \mathcal{H}^{n-1} \llcorner S_\delta(\varphi))$, which is exactly the claim of the theorem. □

Next result is contained in Lemma 3.5 of [5]. We reproduce its proof for the sake of completeness.

Lemma 2.11. *For any $q > 2$, there exists ε_0, depending only on q and n, such that the following holds: let $\tilde\varphi$ be a Lipschitz function such that $\tilde\varphi(0) = 0$ and which satisfies (2.27). Suppose that $D(\tilde\varphi)$, defined as in (2.28), is star-shaped with respect to $P = (0, M/2)$. Denote by k the density of the harmonic measure of $D(\tilde\varphi)$ evaluated at P. Then $k \in L^q(S(\tilde\varphi); \mathcal{H}^{n-1} \llcorner S(\tilde\varphi))$, where $S(\tilde\varphi) = \{(x', \tilde\varphi(x')) : |x'| \le 10r_1\}$.*

Proof. Let us fix $q > 2$. We are going to prove the lemma with

$$\varepsilon_0 = 1/(8q(q-1)(n-1)). \tag{2.29}$$

Let $D(\tilde{\varphi})$ be defined as in (2.28) and $P = (0, M/2)$. Suppose that (2.27) holds. We find r_0 small enough such that $B_{4r_0}(P)$ is compactly contained in $D(\tilde{\varphi})$. Set $D^*(\tilde{\varphi}) = D(\tilde{\varphi}) \setminus \bar{B}_{r_0}(P)$. Thus the Green function $g = G(P, \cdot)$ of $D(\tilde{\varphi})$ with pole at point P is positive and harmonic in $D^*(\tilde{\varphi})$. The essential point of Dahlberg's proof in [5] is to consider the following function

$$F_q(\nabla g) = \big(|\nabla_{x'} g|^2 + t_0((\partial/\partial x_n)g)^2\big)^{q/2} - q|\nabla_{x'} g|^2\big(|\nabla_{x'} g|^2 + t_0((\partial/\partial x_n)g)^2\big)^{(q-2)/2}, \tag{2.30}$$

where $2 < q < \infty$, $t_0 = 1/((n-1)(q-1))$ and $|\nabla_{x'} g|^2 = \sum_{i=1}^{n-1}((\partial/\partial x_i)g)^2$. A result, due to Kuran [13, Th. 5], states that $F_q(\nabla g)$ is superharmonic in $D^*(\tilde{\varphi})$. We claim that

$$F_q(\nabla g) \geq 2B|\nabla g|^q - A \quad \text{in } D^*(\tilde{\varphi}), \tag{2.31}$$

where $B = t_0^q/4$, and A is a positive number depending on q, n, r_1, and r_2. Indeed, we first prove (2.31) under the assumption that $\tilde{\varphi}$ is smooth.

Let $S(\tilde{\varphi}) = \{(x', \tilde{\varphi}(x') : |x'| \leq 10r_1\}$. By the standard Schauder estimates, there is a positive number A_1 depending on q, n, r_1, and r_2 such that $|\nabla g| \leq A_1 \quad \text{on } \partial D^*(\tilde{\varphi}) \setminus S(\tilde{\varphi})$. Thus we can find a number $A > 0$ such that

$$F_q(\nabla g) \geq 2B|\nabla g|^q - A \quad \text{on } \partial D^*(\tilde{\varphi}) \setminus S(\tilde{\varphi}).$$

Since $\tilde{\varphi}$ is smooth, we have $|\nabla_{x'} g| \leq 8\varepsilon_0 |(\partial/\partial x_n)g|$ on $S(\tilde{\varphi})$, because of (2.27). By the definition of $F_q(\nabla g)$ and our choice of ε_0 in (2.29), we have by a simple calculation that

$$F_q(\nabla g) \geq \big(t_0^q - q(8\varepsilon_0)^2 t_0^{q-2}\big)|\nabla g|^q \geq 2B|\nabla g|^q \quad \text{on } S(\tilde{\varphi}),$$

where $B = t_0^q/4$. We have showed that the function $w = F_q(\nabla g) - 2B|\nabla g|^q + A \geq 0$ on $\partial D^*(\tilde{\varphi})$. Note that $F_q(\nabla g)$ is superharmonic and $|\nabla g|^q$ subharmonic in $D^*(\tilde{\varphi})$. Thus w is superharmonic in $D^*(\tilde{\varphi})$. Then by the minimum principle, we have $w \geq 0$ in $D^*(\tilde{\varphi})$. This proves the claim in case $\tilde{\varphi}$ is smooth. If $\tilde{\varphi}$ is not smooth, we can find a sequence of smooth functions φ_i such that $\Lambda(\varphi_i) \leq 8\varepsilon_0, \varphi_i \geq \tilde{\varphi}$, and $\varphi_i \to \tilde{\varphi}$ uniformly. Denote by g_i the Green function of $D(\varphi_i)$ with pole at P. Then we have $g_i \to g$ and $\nabla g_i \to \nabla g$ uniformly on compact subset of $D(\tilde{\varphi}) \setminus \{P\}$. Hence the claim follows from the previous case, and the claim is proved.

Now we define $v = F_q(\nabla g) - B|\nabla g|^q + A$. Then v is superharmonic in $D^*(\tilde{\varphi})$, since $F_q(\nabla g)$ is superharmonic and $|\nabla g|^q$ subharmonic. By (2.31), we have

$$v \geq B|\nabla g|^q \quad \text{in } D^*(\tilde{\varphi}). \tag{2.32}$$

Now for $0 < t < 1$, let $v_t(x) = v((1-t)tx + tP)$. If t is small enough, then v_t is non-negative and superharmonic in $D^{**}(\tilde{\varphi}) = D(\tilde{\varphi}) \setminus \bar{B}_{2r_0}(P)$. Fix a point $P_1 \in D^{**}(\tilde{\varphi})$, and denote by k_1 the density of the harmonic measure of $D^{**}(\tilde{\varphi})$ evaluated at P_1. Since v_t is superharmonic and continuous in $D^{**}(\tilde{\varphi})$, we have that

$$v_t(P_1) \geq \int_{\partial D^{**}(\tilde{\varphi})} v_t(x) k_1(x)\, d\mathcal{H}^{n-1}(x) \geq \int_{S(\tilde{\varphi})} v_t(x) k_1(x)\, d\mathcal{H}^{n-1}(x),$$

and that by (2.32)

$$v_t(P_1) \geq B \int_{S(\tilde{\varphi})} |\nabla g((1-t)x + tP)|^q k_1(x)\, d\mathcal{H}^{n-1}(x). \tag{2.33}$$

By Theorem 2.8, we know that ∇g has nontangential limits a.e. on $S(\tilde{\varphi})$. Note that there is a constant $C > 0$ such that $(1-t)x + tP \in \Gamma(x) = \{z \in D(\tilde{\varphi}) : |z - x| \leq C \operatorname{dist}(z, \partial D(\tilde{\varphi}))\}$ for all $x \in S(\tilde{\varphi})$ and for all t small, because of (2.27) and our choice of P. Thus it follows from Theorem 2.8 that $\lim_{t\to 0^+} |\nabla g((1-t)x + tP)| = (n-2)\sigma_n k(x)$ for a.e. $x \in S(\tilde{\varphi})$. It follows from Theorem 2.9 that there is a constant $C > 0$ such that $Ck_1 \geq k$ a.e. on $S(\tilde{\varphi})$. Hence by Fatou's lemma, it follows from (2.33) that

$$\int_{S(\tilde{\varphi})} k^{q+1}\, d\mathcal{H}^{n-1} \leq Cv(P_1),$$

which of course proves the lemma since q is arbitrary. □

3. Local minimizers

In this section we aim to derive a necessary condition for local minimality of the functional $\mathcal{F} : \mathcal{K}^3_0 \to (0, \infty)$ defined by setting

$$\mathcal{F}(K) = \frac{\operatorname{Cap}^2 K}{P(K)}\,.$$

We say that a set $K \in \mathcal{K}^3_0$ is a *local minimizer* of $\mathcal{F}$ if there exists $\delta > 0$ such that

$$\mathcal{F}(L) \geq \mathcal{F}(K)$$

for all $L \in \mathcal{K}^3_0$ with $\operatorname{dist}_H(K, L) < \delta$. Denote by U_K the equilibrium potential of a set $K \in \mathcal{K}^3_0$.

We start by proving the following necessary condition for local minimality for which only one sided variations of the set K are considered. Therefore the function v defining the variation K_t as in Section 2.2 will be taken nonnegative.

Proposition 3.1. *Let $K \in \mathcal{K}^3_0$ be a local minimizer of $\mathcal{F}$. Assume also that*

$$\mathcal{H}^1(\partial K \setminus \operatorname{reg} K) = 0 \quad \text{and} \quad \mathcal{H}^1(\mathbb{S}^2 \setminus \operatorname{regn} K) = 0. \tag{3.1}$$

Then, $C_1(K)$ is absolutely continuous with respect to $\mathcal{H}^2 \llcorner \partial K$ and for any Borel set $B \subset \partial K$

$$C_1(K)(B) \leq \frac{4P(K)}{\operatorname{Cap} K} \int_{\partial K \cap B} |\nabla U_K|^2\, d\mathcal{H}^2.$$

Proof. Step 1. We start by fixing a continuous function $v : \mathbb{S}^2 \to [0, \infty)$ and a number $t > 0$. Then, we set

$$K_t = \{x \in \mathbb{R}^3 : \, x \cdot u \leq h_K(u) + tv(u) \;\; \forall u \in \mathbb{S}^2\}.$$

Clearly $K_t \in \mathcal{K}^n_0$ if t is small, and $K_t \to K$ in the Hausdorff distance as $t \to 0^+$. Therefore, by the local minimality of K we have that

$$\limsup_{t\to 0^+} \frac{1}{t}\left(\frac{\operatorname{Cap}^2 K_t}{P(K_t)} - \frac{\operatorname{Cap}^2 K}{P(K)}\right) \geq 0\,,$$

from which we easily infer that

$$\limsup_{t\to 0^+}\frac{1}{t}\left(\frac{P(K)\mathrm{Cap}^2K_t-P(K)\mathrm{Cap}^2K}{P(K)P(K_t)}\right)\geq \liminf_{t\to 0^+}\frac{1}{t}\left(\frac{P(K_t)\mathrm{Cap}^2K-P(K)\mathrm{Cap}^2K}{P(K)P(K_t)}\right).$$

Thus, since $P(K_t)\to P(K)$, we have that

$$\limsup_{t\to 0^+}\frac{\mathrm{Cap}^2K_t-\mathrm{Cap}^2K}{t}\geq\frac{\mathrm{Cap}^2K}{P(K)}\liminf_{t\to 0^+}\frac{P(K_t)-P(K)}{t}.$$

In turn, using the representation formula for the first variation of the capacity in $\mathcal{K}_0^n$ provided by [9, Th. 2.5], we get

$$2\int_{\mathbb{S}^2}v(u)\,d\mu_K^{\mathrm{Cap}}(u)\geq\frac{\mathrm{Cap}\,K}{P(K)}\liminf_{t\to 0^+}\frac{P(K_t)-P(K)}{t}, \tag{3.2}$$

where the measure μ_K^{Cap} is given by

$$\mu_K^{\mathrm{Cap}}=(g_K)_\#(|\nabla U_K|^2\mathcal{H}^2\llcorner\partial K) \tag{3.3}$$

and $(g_K)_\#(\mu)$ denotes the push-forward of the measure μ through the Gauss map g_K. In order to estimate the right hand side of (3.2) we observe that from (2.3) and (2.4) it follows that

$$\begin{aligned}P(K_t)-P(K)&=\frac{1}{2}\int_{\mathbb{S}^2}h_{K_t}\,dS_1(K_t)-\frac{1}{2}\int_{\mathbb{S}^2}h_K\,dS_1(K)\\&=\frac{1}{2}\int_{\mathbb{S}^2}(h_{K_t}-h_K)\,d(S_1(K_t)+S_1(K)).\end{aligned}$$

Therefore, we have in particular that

$$\liminf_{t\to 0^+}\frac{P(K_t)-P(K)}{t}\geq\frac{1}{2}\int_{\mathbb{S}^2}\liminf_{t\to 0^+}\frac{h_{K_t}(u)-h_K(u)}{t}\,dS_1(K)(u). \tag{3.4}$$

Note that in general the lim inf on the right hand side is smaller then $v(u)$. However, by Lemma 2.5, if $u\in g_K(\mathrm{reg}\,K)$ then

$$\lim_{t\to 0^+}\frac{h_{K_t}(u)-h_K(u)}{t}=v(u).$$

Thus, recalling (3.2) and (3.4) we get that

$$\frac{4P(K)}{\mathrm{Cap}K}\int_{\mathbb{S}^2}v(u)\,d\mu_K^{\mathrm{Cap}}(u)\geq\int_{\sigma_K(\mathrm{reg}\,K)}v(u)\,dS_1(K)(u). \tag{3.5}$$

Step 2. Recall now (see [16, Th. 4.5.5]) that there exist positive constants a, b such that for any convex set $K\subset\mathbb{R}^3$, any Borel set $B\subset\partial K$ and any Borel set $\omega\subset\mathbb{S}^2$ one has

$$C_1(K)(B)\leq a\mathcal{H}^1(B),\qquad S_1(K)(\omega)\leq b\mathcal{H}^1(\omega).$$

Therefore, from the first inequality above, the assumption $\mathcal{H}^1(\partial K\setminus\mathrm{reg}\,K)=0$ and (2.6) we have that $S_1(K)\big((\mathbb{S}^2\setminus\sigma_K(\mathrm{reg}\,K))\cap\mathrm{regn}\,K\big)=0$. On the other hand, using the second inequality above we have also that $S_1(K)(\mathbb{S}^2\setminus\mathrm{regn}\,K)=0$. Thus, recalling (3.5), we may conclude that

$$\frac{4P(K)}{\mathrm{Cap}K}\int_{\mathbb{S}^2}v(u)\,d\mu_K^{\mathrm{Cap}}(u)\geq\int_{\mathbb{S}^2}v(u)\,dS_1(K)(u)$$

for all nonnegative continuous functions v on $\mathbb{S}^2$. By approximation the inequality above also holds for any Borel function $v:\mathbb{S}^2\to[0,\infty)$. Now observe that from our assumption and (2.7) it follows that for any Borel set $\omega\subset\mathbb{S}^2$ we have $S_1(K)(\omega)=C_1(K)(\tau_K(\omega)\cap\operatorname{reg}K)=C_1(K)(g_K^{-1}(\omega))$. Therefore $S_1(K)=(g_K)_\#(C_1(K))$ and thus, recalling (3.3), we may rewrite the above inequality as

$$\frac{4P(K)}{\mathrm{Cap}K}\int_{\partial K}v(g_K(x))|\nabla U_K(x)|^2\,d\mathcal{H}^2(x)\geq\int_{\partial K}v(g_K(x))\,dC_1(K)(x).$$

In particular, choosing $v(u)=f(g_K^{-1}(u))\chi_{\operatorname{regn}K\cap g_K(\operatorname{reg}K)}(u)$, where $f:\partial K\to[0,\infty)$ is a Borel function, we have that

$$\begin{aligned}\frac{4P(K)}{\mathrm{Cap}K}\int_{\partial K}f(x)|\nabla U_K(x)|^2\,d\mathcal{H}^2(x)&\geq\int_{\operatorname{reg}K\cap g_K^{-1}(\operatorname{regn}K)}f(x)\,dC_1(K)(x)\\&=\int_{\partial K}f(x)\,dC_1(K)(x),\end{aligned}$$

where the last equality follows from (3.1) arguing as before. This final inequality concludes the proof. □

Next result tells us that the boundary of K can be written as a graph of a function with second derivatives and thus $C_1(K)$ can be written in the usual divergence way.

Proposition 3.2. *Let $K\in\mathcal{K}_0^3$ be a local minimizer of $\mathcal{F}$ satisfying the assumptions (3.1). There exists $p>1$ such that ∂K can be locally written as a graph of a $W^{2,p}$ convex function.*

Proof. Let us fix a point $x_0\in\partial K$. Without loss of generality we may assume that $x_0=0$ and that there exist an open set $U\subset\mathbb{R}^2$, a positive number δ and a convex function $\varphi:U\to[0,\infty)$ such that

$$\begin{aligned}K\cap\big(U\times(-\delta,\delta)\big)&=\{(x',x_3)\in U\times(-\delta,\delta):\ x_3>\varphi(x')\}.\\\partial K\cap\big(U\times(-\delta,\delta)\big)&=\{(x',x_3)\in U\times(-\delta,\delta):\ x_3=\varphi(x')\}.\end{aligned}$$

Let us now consider a sequence of convex bodies $K_h\in C^2_+$ converging to K in the Hausdorff metric. Let us fix a point $\bar x=(0,\bar x_3)$, with $\bar x_3>0$ and $r>0$ so that the closed ball $\bar B_r(\bar x)\subset\mathring{K}\cap\big(U\times(0,\delta)\big)$. By the convergence of K_h to K it follows easily that for h large $\bar B_r(\bar x)\subset\mathring{K}_h$. Therefore, if h is sufficiently large, for any $x'\in V:=\{|x'|<r\}$ there exists a unique number t_h such that $(x',t_h)\in\partial K_h$ and the point (x',t_h) lies below the ball $B_r(\bar x)$. Then we set $\varphi_h(x')=t_h$ for any $x'\in V$. Clearly, the functions φ_h are smooth and convex and converge uniformly to φ in V. By convexity we may conclude that their gradients $\nabla\varphi_h$ are locally equibounded in V and converge locally weakly* in L^∞ to $\nabla\varphi$. Let us now fix a number $0<\delta'<\delta$ such that for h sufficiently large $|\varphi(x')|<\delta'$ for all $x'\in V$, a nonnegative function $\eta\in C^1_c(V)$ and a cut-off function $\psi\in C^1_c(-\delta,\delta)$ with $\psi\equiv1$ in $[-\delta',\delta']$. Then, using Theorem 2.2 and denoting by H_{K_h} the mean curvature of K_h, we get by Proposition 3.1 that

$$\lim_h \int_V \eta\sqrt{1+|\nabla\varphi_h|^2}\,\mathrm{div}\left(\frac{\nabla\varphi_h}{\sqrt{1+|\nabla\varphi_h|^2}}\right)dx' = \lim_h \int_{\partial K_h} \eta(x')\psi(x_3)H_{K_h}(x)\,d\mathcal{H}^2$$
$$= \lim_h \int_{\partial K_h} \eta(x')\psi(x_3)\,dC_1(K_h) = \int_{\partial K} \eta(x')\psi(x_3)\,dC_1(K)$$
$$\leq c\int_{\partial K} \eta(x')\psi(x_3)|\nabla U_K(x)|^2\,d\mathcal{H}^2 \leq c\int_V \eta(x')|\nabla U_K(x',\varphi(x'))|^2 dx',$$

for a suitable constant c depending only on K. Therefore, we may conclude that, up to a not relabeled subsequence, there exists a Radon measure ν in V such that

$$\mathrm{div}\left(\frac{\nabla\varphi_h}{\sqrt{1+|\nabla\varphi_h|^2}}\right)\mathcal{L}^2 \llcorner V \rightharpoonup \nu$$

weakly* locally in V. Moreover denoting by g the density of ν with respect to the Lebesgue measure in V, we get that $0 \leq g \leq |c\nabla U_K|^2$. for some positive constant c. By Theorem 2.7, there exists $p > 1$ such that $|\nabla U_K|^2 \in L^p(\partial K)$. Therefore we have also that $g \in L^p(V)$.

Let us now consider again η as above and denote by $0 \leq \lambda_{h,1}(x') \leq \lambda_{h,2}(x')$ the eigenvalues of the matrix $D^2\varphi_h(x')$ for $x' \in V$. We have for any h

$$\int_V \eta\,\mathrm{div}\left(\frac{\nabla\varphi_h}{\sqrt{1+|\nabla\varphi_h|^2}}\right)dx' = \int_V \eta\left(\frac{\Delta\varphi_h(1+|\nabla\varphi_h|^2) - D_{i,j}\varphi_h D_i\varphi_h D_j\varphi_h}{(1+|\nabla\varphi_h|^2)^{3/2}}\right)dx'$$
$$= \int_V \eta\left(\frac{(\lambda_{h,1}+\lambda_{h,2})(1+|\nabla\varphi_h|^2) - \lambda_{h,2}|\nabla\varphi_h|^2}{(1+|\nabla\varphi_h|^2)^{3/2}}\right)dx'$$
$$\geq c\int_V \eta(\lambda_{h,1}+\lambda_{h,2})\,dx' = c\int_V \eta\Delta\varphi_h\,dx'$$

for some positive constant independent of h. From this estimate and from the fact that $g \in L^p(V)$ we easily infer that the distributional Laplacian of φ in V is equal to some nonnegative function $f \in L^p(V)$. Then, standard elliptic regularity immediately implies that $\varphi \in W^{2,p}(V)$. □

The following lemma shows that the regular set $\mathrm{reg}\,K$ is open and in $C^{1,\alpha}$, for all α, $0 < \alpha < 1$, and that the irregular set $\partial K \setminus \mathrm{reg}\,K$ is small. To this aim, we shall denote by $\dim_{\mathcal{H}}(B)$ the *Hausdorff dimension* of a Borel set $B \subset \mathbb{R}^3$.

Lemma 3.3. *Let $K \in \mathcal{K}_0^3$ be a local minimizer of $\mathcal{F}$ satisfying the assumptions* (3.1). *Then the set* $\mathrm{reg}\,K$ *is open and* $\mathrm{reg}\,K \in C^{1,\alpha}$ *for all α, $0 < \alpha < 1$. Furthermore, we have* $\dim_{\mathcal{H}}(\partial K \setminus \mathrm{reg}\,K) < 1$.

Proof. Fix a point $x_0 \in \mathrm{reg}\,K$. There is a unique unit exterior normal $u(x_0)$ to K at x_0. We may assume that $x_0 = 0$ by a translation and that $u(x_0) = (0,0,-1)$ by a rotation. Since $K \in \mathcal{K}_0^3$, there exist $0 < r_1' < 1$, $r_2 > 0$ such that

$$K \cap L_{r_1',r_2} = \{(x',x_3) \in L_{r_1',r_2} : x_3 \geq \varphi(x')\},$$

and that

$$\partial K \cap L_{r_1',r_2} = \{(x',x_3) \in L_{r_1',r_2} : x_3 = \varphi(x')\},$$

where $L_{r'_1,r_2}$ is the cylinder defined in (2.26) and φ is a non-negative convex function in $B^2_{r'_1} = \{y' \in \mathbb{R}^2 : |y'| < r'_1\}$ with $\varphi(0) = 0$.

We will apply Theorem 2.10 to prove the lemma. Fix q such that $4 < q < \infty$. Let $\varepsilon_0 = \varepsilon_0(q) > 0$ be the number as in Theorem 2.10. In order to apply Theorem 2.10, we will show that the boundary ∂K is flat in a small neighbourhood of the origin. We claim that there is $r_1 = r_1(p)$, $0 < r_1 < r'_1$, such that

$$\|\nabla\varphi\|_{L^\infty(B^2_{r_1})} \le \frac{\varepsilon_0}{2}\,,$$

that is, $\Lambda(\varphi) \le \varepsilon_0/2$ in $B^2_{r_1}$. We will prove the claim arguing as in the proof of Lemma 3.3 in [10]. Indeed, since $0 \in \operatorname{reg} K$, then φ is differentiable at 0. Thus there exists $r_1 = r_1(\varepsilon_0)$, $0 < r_1 < r'_1/2$, such that

$$|\varphi(x') - \varphi(0) - \nabla\varphi(0)\cdot x'| \le \frac{\varepsilon_0}{4}|x'| \tag{3.6}$$

for all $x' \in B^2_{2r_1}$. Recall that the unique unit exterior normal $u(0)$ to K at 0 is the vector $(0, 0, -1)$. Therefore, $\nabla\varphi(0) = 0$. Recall also that $\varphi(0) = 0$. Thus (3.6) becomes

$$|\varphi(x')| \le \frac{\varepsilon_0}{4}|x'| \tag{3.7}$$

for all $x' \in B^2_{2r_1}$. Let y' be a point where φ is differentiable and $|y'| < r_1$. Without loss of generality we may assume that $\nabla\varphi(y') \neq 0$, so that we may consider the unit vector $\xi = \nabla\varphi(y')/|\nabla\varphi(y')|$. Since $|y' + r_1\xi| < 2r_1$, (3.7) gives us that

$$\varphi(y' + r_1\xi) \le \varepsilon_0 r_1/2.$$

On the other hand, since φ is non-negative and convex in $B^2_{r'_1}$, we have

$$\varphi(y' + r_1\xi) \ge \varphi(y') + \nabla\varphi(y')\cdot(r_1\xi) \ge r_1|\nabla\varphi(y')|.$$

It follows from the above two inequalities that

$$|\nabla\varphi(y')| \le \varepsilon_0/2.$$

Since the above inequality holds for almost every y' in $B^2_{r_1}(0)$, the claim follows easily.

Now we can apply Theorem 2.10 to obtain the following higher integrability for the gradient of the potential,

$$|\nabla U_K| \in L^q(\partial K \cap L_{r_1/2,r_2}; \mathcal{H}^2 \llcorner \partial K).$$

Then by the same argument used in the proof of Proposition 3.2, we can prove that $\varphi \in W^{2,q/2}(B^2_{r_1/4})$. This implies that $\varphi \in C^{1,\alpha}(B^2_{r_1/4})$ with $\alpha = 1 - 4/q$, by the Sobolev embedding theorem. Thus we have proved that the regular part of ∂K, $\operatorname{reg} K$, is open and is in $C^{1,\alpha}$ for all $0 < \alpha < 1$.

It remains to prove that $\dim_{\mathcal{H}}(\partial K \setminus \operatorname{reg} K) < 1$. By Proposition 3.2, ∂K can be locally written as a graph of a convex function $\varphi \in W^{2,p}(U)$ for some $p > 1$, where $U \subset \mathbb{R}^2$ is an open set. Set $G_\varphi = \{x' \in U : \varphi \text{ is differentiable at } x'\}$ and $L_\varphi := \{x' \in U : x' \text{ is a Lebesgue point for } \nabla\varphi\}$. Since $\varphi \in W^{2,p}(U)$, we get that $\dim_{\mathcal{H}}(U \setminus L_\varphi) \le 2 - p$. On the other hand, since φ is locally Lipschitz $L_\varphi \subset G_\varphi$, see the proof of Theorem 2.14 in [1]. This shows that $\dim_{\mathcal{H}}(\partial K \setminus \operatorname{reg} K) \le 2 - p < 1$. □

In the following, we shall need an extra assumption, that is

$$\tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K) \text{ is closed.} \tag{3.8}$$

We will work on the set $\operatorname{reg} K \cap \tau_K(\operatorname{regn} K) \subset \partial K$. We have the following lemma.

Lemma 3.4. *Let $K \in \mathcal{K}_0^3$ be a local minimizer of $\mathcal{F}$ satisfying the assumptions (3.1) and (3.8). Then $\operatorname{reg} K \cap \tau_K(\operatorname{regn} K) \subset \partial K$ and $\sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K \subset \mathbb{S}^2$ are open, and the Gauss map*

$$g_K : \operatorname{reg} K \cap \tau_K(\operatorname{regn} K) \to \sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$$

is Hölder continuous and bijective. Furthermore,

$$S_1(K)\big(\mathbb{S}^2 \setminus (\sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K)\big) = 0 \quad \text{and} \quad C_1(K)\big(\partial K \setminus (\operatorname{reg} K \cap \tau_K(\operatorname{regn} K))\big) = 0.$$

Proof. First, by Lemma 3.3, the set $\operatorname{reg} K$ is open. Therefore, $\operatorname{reg} K \cap \tau_K(\operatorname{regn} K) = \operatorname{reg} K \setminus \tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K)$ is open, by assumption (3.8). We will then show that the set $\sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$ is open. We argue by contradiction. Let $u \in \sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$. Suppose that there is a sequence $\{u_i\} \subset \mathbb{S}^2 \setminus \big(\sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K\big)$ that tends to u. Then there is a sequence $\{x_i\} \subset (\partial K \setminus \operatorname{reg} K) \cup \tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K)$ such that $u_i \in N_K(x_i)$ for each i. Then the sequence x_i, up to a subsequence, tends to a point $x \in \partial K$. Since $\partial K \setminus \operatorname{reg} K$ is closed and $\tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K)$ is also closed by assumption (3.8), we have $x \in (\partial K \setminus \operatorname{reg} K) \cup \tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K)$. On the other hand, we have $u \in N_K(x)$. Recall that $u \in \sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$. This means that $x \in \operatorname{reg} K \setminus \tau_K(\mathbb{S}^2 \setminus \operatorname{regn} K)$. We reach a contradiction. This proves that $\sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$ is open.

Second, the claim that the Gauss map $g_K : \operatorname{reg} K \cap \tau_K(\operatorname{regn} K) \to \sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K$ is bijective is clear. By Lemma 3.3, we have $\operatorname{reg} K \in C^{1,\alpha}$ for all $\alpha, 0 < \alpha < 1$. Thus g_K is Hölder continuous on $\operatorname{reg} K \cap \tau_K(\operatorname{regn} K)$.

Finally, since $S_1(K)(\mathbb{S}^2 \setminus \operatorname{regn} K) = 0$ and $C_1(K)(\partial K \setminus \operatorname{reg} K) = 0$, we have

$$\begin{aligned} &S_1(K)\big(\mathbb{S}^2 \setminus (\sigma_K(\operatorname{reg} K) \cap \operatorname{regn} K)\big) \\ &\le S_1(K)\big(\mathbb{S}^2 \setminus \sigma_K(\operatorname{reg} K)\big) + S_1(K)(\mathbb{S}^2 \setminus \operatorname{regn} K) \\ &\le C_1(K)\big(\tau_K(\mathbb{S}^2 \setminus \sigma_K(\operatorname{reg} K))\big) \le C_1(K)(\partial K \setminus \operatorname{reg} K) = 0, \end{aligned}$$

where we used (2.7) of Proposition 2.4, and the fact that $\tau_K(\mathbb{S}^2 \setminus \sigma_K(\operatorname{reg} K)) \subset \partial K \setminus \operatorname{reg} K$. We also have that

$$\begin{aligned} &C_1(K)\big(\partial K \setminus (\operatorname{reg} K \cap \tau_K(\operatorname{regn} K))\big) \\ &\le C_1(K)\big(\partial K \setminus \tau_K(\operatorname{regn} K)\big) + C_1(K)\big(\partial K \setminus \operatorname{reg} K\big) \\ &\le S_1(K)\big(\sigma_K(\partial K \setminus \tau_K(\operatorname{regn} K))\big) \le S_1(K)\big(\mathbb{S}^2 \setminus \operatorname{regn} K\big) = 0, \end{aligned}$$

where we used (2.6) of Proposition 2.4, and the fact that $\sigma_K(\partial K \setminus \tau_K(\operatorname{regn} K)) \subset \mathbb{S}^2 \setminus \operatorname{regn} K$. The proof is now complete. □

Now we prove Theorem 1.7 for which we consider general variations. Hence the function v defining the variation K_t as in Section 2.2 may have any sign.

Proof of Theorem 1.7. Let $v : \mathbb{S}^2 \to \mathbb{R}$ be a continuous function. Set

$$K_t = \{x \in \mathbb{R}^3 : x \cdot u \le h_K(u) + tv(u) \ \ \forall u \in \mathbb{S}^2\}.$$

We may argue as in Proposition 3.1 by the minimality of K and obtain (3.2), that is,

$$2\int_{\mathbb{S}^2} v(u)\, d\mu_K^{\mathrm{Cap}}(u) \geq \frac{\mathrm{Cap}\, K}{P(K)} \liminf_{t\to 0^+} \frac{P(K_t)-P(K)}{t}. \tag{3.9}$$

We have

$$P(K_t)-P(K) = \frac{1}{2}\int_{\mathbb{S}^2} (h_{K_t}(u)-h_K(u))\, d(S_1(K_t)(u)+S_1(K)(u)).$$

Now we claim that

$$\liminf_{t\to 0^+}\int_{\mathbb{S}^2} \frac{h_{K_t}(u)-h_K(u)}{t}\, d(S_1(K_t)(u)+S_1(K)(u)) \geq 2\int_{\mathbb{S}^2} v(u)\, dS_1(K)(u). \tag{3.10}$$

Indeed, we have that $\sigma_K(\mathrm{reg}\, K)\cap \mathrm{regn}\, K$ is open and that $S_1(K)\big(\mathbb{S}^2\setminus(\sigma_K(\mathrm{reg}\, K)\cap \mathrm{regn}\, K)\big)=0$ by Lemma 3.4. Then for any $\varepsilon>0$, we may find an open set $\omega\subset\mathbb{S}^2$ whose closure $\bar\omega\subset\sigma_K(\mathrm{reg}\, K)\cap\mathrm{regn}\, K$ and $S_1(K)(\mathbb{S}^2\setminus\omega)<\varepsilon$. Since the sequence of measures $S_1(K_t)$ converges weakly to $S_1(K)$ and $\mathbb{S}^2\setminus\omega$ is closed, we have

$$\limsup_{t\to 0^+} S_1(K_t)(\mathbb{S}^2\setminus\omega) \leq S_1(K)(\mathbb{S}^2\setminus\omega) < \varepsilon. \tag{3.11}$$

Now recall the uniform estimate (2.9), that is,

$$\sup_{0<t\leq t_0} \frac{|h_{K_t}(u)-h_K(u)|}{t} \leq RA/r, \tag{3.12}$$

where $t_0 = r/(2A)$, $A=\max_{\mathbb{S}^2}|v|$ and r, R are positive numbers such that $B_r(0)\subset K\subset B_R(0)$. Then we conclude that

$$\limsup_{t\to 0^+}\int_{\mathbb{S}^2\setminus\omega} \frac{|h_{K_t}(u)-h_K(u)|}{t}\, d(S_1(K_t)(u)+S_1(K)(u)) \leq 2RA\varepsilon/r.$$

By Lemma 3.4 and Lemma 2.6 we have that $\big(h_{K_t}(u)-h_K(u)\big)/t$ converges uniformly to $v(u)$ in ω as t vanishes. Thus

$$\liminf_{t\to 0^+}\int_{\omega} \frac{h_{K_t}(u)-h_K(u)}{t}\, d(S_1(K_t)(u)+S_1(K)(u)) = \liminf_{t\to 0^+}\int_{\omega} v(u)\, d(S_1(K_t)(u)+S_1(K)(u)).$$

We also have

$$\begin{aligned}\liminf_{t\to 0^+}\int_\omega v(u)\, dS_1(K_t)(u) &\geq \liminf_{t\to 0^+}\int_{\mathbb{S}^2} v(u)\, dS_1(K_t)(u) - \limsup_{t\to 0^+}\int_{\mathbb{S}^2\setminus\omega}|v(u)|\, dS_1(K_t)(u)\\ &\geq \int_{\mathbb{S}^2} v(u)\, dS_1(K)(u) - A\varepsilon.\end{aligned}$$

The claim (3.10) follows from the above estimates and from the inequality

$$\int_{\mathbb{S}^2\setminus\omega}|v(u)|\, dS_1(K)(u) \leq A\varepsilon.$$

Therefore from (3.9) we get

$$\frac{2P(K)}{\mathrm{Cap}\, K}\int_{\mathbb{S}^2} v(u)\, d\mu_K^{\mathrm{Cap}}(u) \geq \int_{\mathbb{S}^2} v(u)\, dS_1(K)(u).$$

Since this inequality holds for every continuous function v, replacing v by $-v$ we conclude that for every $v \in C(\mathbb{S}^2)$

$$\frac{2P(K)}{\mathrm{Cap}\,K}\int_{\mathbb{S}^2} v(u)\,d\mu_K^{\mathrm{Cap}}(u) = \int_{\mathbb{S}^2} v(u)\,dS_1(K)(u).$$

Then we may argue in the same way as that in Step 2 of the proof of Proposition 3.1 to prove that

$$\frac{2P(K)}{\mathrm{Cap}\,K}\int_{\partial K} f(x)|\nabla U_K(x)|^2\,d\mathcal{H}^2(x) = \int_{\partial K} f(x)\,dC_1(K)(x),$$

where $f : \partial K \to \mathbb{R}$ is a Borel function. This concludes the proof. □

Remark 3.5. Note that if $K \in \mathcal{K}_0^3$ is a convex body such that there exists an open set $U \subset \mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$, then it is easily checked that also $\sigma_K(U)$ is open. Therefore the same argument used in the proof above shows that if K is a local minimizer of $\mathcal{F}$ then

$$C_1(K) \llcorner U = \frac{2P(K)}{\mathrm{Cap}K}|\nabla U_K|^2\,\mathcal{H}^2 \llcorner U\,.$$

Now we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. First, by Lemma 3.3, we know that the set $\mathrm{reg}\,K$ is not empty and open. We will show that it is smooth. To this end, we will apply the following result due to Widman: let $X \subset \partial K$ be an open set such that $X \in C^{k,\alpha}, k \geq 1$ and $0 < \alpha < 1$, then $|\nabla U_K| \in C^{k-1,\alpha}(X)$, see [17, Th. 2.4]. Since $\mathrm{reg}\,K \in C^{1,\alpha}$ for all $0 < \alpha < 1$ by Lemma 3.3, $|\nabla U_K|$ is α-Hölder continuous in the set $\mathrm{reg}\,K$. Let H_K be the density of $C_1(K)$ with respect to $\mathcal{H}^2 \llcorner \partial K$. Then by Theorem 1.7, we have

$$H_K(x) = \frac{2P(K)}{\mathrm{Cap}K}|\nabla U_K(x)|^2$$

for almost every $x \in \partial K$ with respect to $\mathcal{H}^2 \llcorner \partial K$. Thus $H_K \in C^{0,\alpha}(\mathrm{reg}\,K)$. Note that $H_K(x)$ is the mean curvature of ∂K at x. This implies that $\mathrm{reg}\,K \in C^{2,\alpha}$ and a standard bootstrap argument yields that $\mathrm{reg}\,K \in C^{k,\alpha}$ for all k.

From Lemma 3.4 we know that the set $\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$ is open. Since $C_1(K)\big(\partial K \setminus (\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K))\big) = 0$, this impies that $\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$ is not empty. We will show that $\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$ has a subregion with positive Gaussian curvature. We argue by contradiction. Suppose that the Gaussian curvature is zero at every point in $\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$. Then, for every $x \in \mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$, there exists a line segment through x, lying in $\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$, see [6, Proposition 1, p.409]. This contradicts the fact that $x \in \mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$.

Finally, we conclude the proof thanks to Theorem 1.5, since $\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$ is smooth and has a subregion with positive Gaussian curvature. □

Observe that in view of Remark 3.5 the proof above also shows that if K is a convex body such that $\mathrm{reg}\,K \cap \tau_K(\mathrm{regn}\,K)$ has non empty interior, then K cannot be a local minimizer of $\mathcal{F}$, thus proving Corollary 1.8.

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