# GAGLIARDO-NIRENBERG INEQUALITIES FOR DIFFERENTIAL FORMS IN HEISENBERG GROUPS

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ABSTRACT. The $L^1$-Sobolev inequality states that for compactly supported functions $u$ on the Euclidean $n$-space, the $L^{n/(n-1)}$-norm of a compactly supported function is controlled by the $L^1$-norm of its gradient. The generalization to differential forms (due to Lanzani & Stein and Bourgain & Brezis) is recent, and states that a the $L^{n/(n-1)}$-norm of a compactly supported differential $h$-form is controlled by the $L^1$-norm of its exterior differential $du$ and its exterior codifferential $\delta u$ (in special cases the $L^1$-norm must be replaced by the $\mathcal{H}^1$-Hardy norm). We shall extend this result to Heisenberg groups in the framework of an appropriate complex of differential forms.

## 1. INTRODUCTION

The $L^1$-Sobolev inequality (also known as Gagliardo-Nirenberg inequality) states that for compactly supported functions $u$ on the Euclidean $n$-space,

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq c\|\nabla u\|_{L^1(\mathbb{R}^n)}. \tag{1}$$

The generalization to differential forms is recent (due to Bourgain & Brezis and Lanzani & Stein), and states that the $L^{n/(n-1)}$-norm of a compactly supported differential $h$-form is controlled by the $L^1$-norm of its exterior differential $du$ and its exterior codifferential $\delta u$ (in special cases the $L^1$-norm must be replaced by the $\mathcal{H}^1$-Hardy norm). We shall extend this result to Heisenberg groups in the framework of an appropriate complex of differential forms.

1.1. **The Euclidean theory.** In a series of papers ([10], [11], [12]), Bourgain and Brezis establish new estimates for the Laplacian, the div-curl system, and more general Hodge systems in $\mathbb{R}^n$ and they show in particular that if $\vec{F}$ is a compactly supported smooth vector field in $\mathbb{R}^n$, with $n \geq 3$, and if curl $\vec{F}=\vec{f}$ and div $\vec{F}=0$, then there exists a constant $C > 0$ so that

$$\| \vec{F} \|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq \| \vec{f} \|_{L^1(\mathbb{R}^n)} . \tag{2}$$

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This result does not follow straightforwardly from Calderòn-Zygmund theory and Sobolev inequality. Indeed, suppose for sake of simplicity $n = 3$ and let $\vec{F}$ be a compactly supported smooth vector field, and consider the system

$$
\begin{cases} \text{curl } \vec{F} = \vec{f} \\ \text{div } \vec{F} = 0\,. \end{cases} \tag{3}
$$

It is well known that $\vec{F} = (-\Delta)^{-1}\text{curl } \vec{f}$ is a solution of (3). Then, by Calderón-Zygmund theory we can say that

$$
\|\nabla \vec{F}\|_{L^p(\mathbb{R}^3)} \le C_p \|\vec{f}\|_{L^p(\mathbb{R}^3)}\,, \quad \text{for} \quad 1 < p < \infty.
$$

Then, by Sobolev inequality, if $1 < p < 3$ we have:

$$
\|\vec{F}\|_{L^{p*}(\mathbb{R}^3)} \le \|\vec{f}\|_{L^p(\mathbb{R}^3)}\,,
$$

where $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$. When we turn to the case $p = 1$ the first inequality fails. The second remains true. This is exactly the result proved by Bourgain and Brezis.

In [22] Lanzani & Stein proved that (1) is the first link of a chain of analogous inequalities for compactly supported smooth differential $h$-forms in $\mathbb{R}^n$, $n \ge 3$,

$$
\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\left(\|du\|_{L^1(\mathbb{R}^n)} + \|\delta u\|_{L^1(\mathbb{R}^n)}\right) \quad \text{if } h \ne 1, n-1; \tag{4}
$$

$$
\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\left(\|du\|_{L^1(\mathbb{R}^n)} + \|\delta u\|_{\mathcal{H}^1(\mathbb{R}^n)}\right) \quad \text{if } h = 1; \tag{5}
$$

$$
\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\left(\|du\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|\delta u\|_{L^1(\mathbb{R}^n)}\right) \quad \text{if } h = n-1, \tag{6}
$$

where $d$ is the exterior differential, and $\delta$ (the exterior codifferential) is its formal $L^2$-adjoint. Here $\mathcal{H}^1(\mathbb{R}^n)$ is the real Hardy space (see e.g. [31]). In other words, the main result of [22] provides a priori estimates for a div-curl systems with data in $L^1(\mathbb{R}^n)$. We stress that inequalities (5) and (6) fail if we replace the Hardy norm with the $L^1$-norm. Indeed (for instance), the inequality

$$
\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\left(\|du\|_{L^1(\mathbb{R}^n)} + \|\delta u\|_{L^1(\mathbb{R}^n)}\right) \tag{7}
$$

is false for 1-forms. The counterexample is given by E.M. Stein in [30], p. 191. Indeed, take $f_k \in \mathcal{D}(\mathbb{R}^n)$ such that $\|f\|_{L^1(\mathbb{H}^n)} = 1$ for all $k \in \mathbb{N}$ and such that $(f_k)_{k\in\mathbb{N}}$ tends to the Dirac $\delta$ in the sense of distribution. Set now $v_k := \Delta^{-1} f_k$. Then estimate (7) would yield that $\{|\nabla v_k|\ ;\ k \in N\}$ is bounded in $L^{n/(n-1)}(\mathbb{R}^n)$, and then, taking the limit as $k \to \infty$ that $|x|^{-n} \in L^1(\mathbb{R}^n)$.

1.2. **The Heisenberg setting.** Recently, in [14], Chanillo & Van Schaftingen extended Bourgain-Brezis inequality to a class of vector fields in Carnot groups. Some of the results of [14] are presented in Theorems 4.2 and 4.3 below in the setting of Heisenberg groups. These are the main tool that allows us to give a Heisenberg version of Lanzani & Stein's result. We describe now the operators that will enter our theorem.

We denote by $\mathbb{H}^n$ the $n$-dimensional Heisenberg group. It is well known that the Lie algebra $\mathfrak{h}$ of the left-invariant vector fields admits the stratification $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. We shall refer to the elements of $\mathfrak{h}_1$ as to the *horizontal derivatives* on $\mathbb{H}^n$.

Heisenberg groups admit a one-parameter group of automorphisms called dilations. Whereas, in Euclidean space, all exterior forms of degree $h$ have homogeneity $h$ under Euclidean dilations, on the contrary, because of the stratification of $\mathfrak{h}$, $h$-forms on Heisenberg groups split into two weight spaces, with weights $h$ and $h+1$. This leads to a modification $(E_0^*, d_c)$ of the de Rham complex introduced by Rumin. Bundles of covectors are replaced by subbundles $E_0^h$ and the exterior differentials by differential operators $d_c$ on spaces $\Gamma(E_0^h)$ of smooth sections of these subbundles.

It turns out that this complex, which is both invariant under left-translations and dilations, is easier to work with that ordinary differential forms.

The core of Rumin's theory relies on the following result.

**Theorem 1.1.** *If $0 \le h \le 2n+1$ there exists a linear map*

$$d_c : \Gamma(E_0^h) \to \Gamma(E_0^{h+1})$$

*such that*

i) $d_c^2 = 0$ *(i.e. $E_0 := (E_0^*, d_c)$ is a complex);*
ii) *the complex $E_0$ is exact;*
iii) $d_c : \Gamma(E_0^h) \to \Gamma(E_0^{h+1})$ *is an homogeneous differential operator in the horizontal derivatives of order 1 if $h \neq n$, whereas $d_c : \Gamma(E_0^n) \to \Gamma(E_0^{n+1})$ is an homogeneous differential operator in the horizontal derivatives of order 2;*
iv) *if $0 \le h \le n$, then $*E_0^h = E_0^{2n+1-h}$;*
v) *the operator $\delta_c := (-1)^{h(2n+1)} * d_c *$ is the formal $L^2$-adjoint of $d_c$.*

**Definition 1.2.** If $0 \le h \le 2n+1$, $1 \le p \le \infty$, we denote by $L^p(\mathbb{H}^n, E_0^h)$ the space of all sections of $E_0^h$ such that their components with respect a given left-invariant basis belong to $L^p(\mathbb{H}^n)$, endowed with its natural norm. Clearly, this definition is independent of the choice of the basis itself. If $h = 0$, we write $L^p(\mathbb{H}^n)$ for $L^p(\mathbb{H}^n, E_0^0)$.

The notations $\mathcal{D}(\mathbb{H}^n, E_0^h)$, $\mathcal{S}(\mathbb{H}^n, E_0^h)$, $\mathcal{E}(\mathbb{H}^n, E_0^h)$, and $\mathcal{H}^1(\mathbb{H}^n, E_0^h)$ have an analogous meaning (here $\mathcal{H}^1$ is the Hardy space in $\mathbb{H}^n$ defined in [17], p.75).

Now can state our main result that generalizes the results of [4] to all Heisenberg groups.

**Theorem 1.3.** *Denote by $(E_0^*, d_c)$ the Rumin's complex in $\mathbb{H}^n$, $n > 2$ (for the cases $n = 1, 2$ we refer to [4]). Then there exists $C > 0$ such that for any $h$-form $u \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, $0 \le h \le 2n+1$, such that*

$$\begin{cases} d_c u = f \\ \delta_c u = g \end{cases}$$

*we have:*

i) *if* $h = 0, 2n+1$, *then*

$$\begin{aligned} \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n)} &\le C\|f\|_{L^1(\mathbb{H}^n, E_0^1)}; \\ \|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n+1})} &\le C\|g\|_{L^1(\mathbb{H}^n, E_0^{2n})}; \end{aligned}$$

ii) *if* $h = 1, 2n$, *then*

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^1)} \le C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^2)} + \|g\|_{\mathcal{H}^1(\mathbb{H}^n)}\big);$$
$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{2n})} \le C\big(\|f\|_{\mathcal{H}^1(\mathbb{H}^n, E_0^{2n+1})} + \|g\|_{L^1(\mathbb{H}^n, E_0^{2n-1})}\big);$$

iii) *if* $1 < h < 2n$ *and* $h \neq n, n+1$, *then*

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^h)} \le C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^{h+1})} + \|g\|_{L^1(\mathbb{H}^n, E_0^{h-1})}\big);$$

iv) *if* $h = n, n+1$, *then*

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n)} \le C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})} + \|d_c g\|_{L^1(\mathbb{H}^n, E_0^n)}\big);$$
$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n, E_0^{n+1})} \le C\big(\|\delta_c f\|_{L^1(\mathbb{H}^n, E_0^{n+1})} + \|g\|_{L^1(\mathbb{H}^n, E_0^n)}\big);$$
$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n)} \le C\|g\|_{L^1(\mathbb{H}^n, E_0^{n-1})} \quad \textit{if } f = 0;$$
$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{n+1})} \le C\|f\|_{L^1(\mathbb{H}^n, E_0^{n+2})} \quad \textit{if } g = 0.$$

The proof of Theorem 1.3 follows the lines of the proofs in [4] of the corresponding results for $\mathbb{H}^1$ and $\mathbb{H}^2$. The new crucial contribution of the present paper in contained in Theorem 5.1 that, roughly speaking, states that the components with respect to a given basis of closed forms in $E_0^h$ are linear combinations of the components of a horizontal vector field with vanishing "generalized horizontal divergence". This is obtained by proving that the symbol of the intrinsic differential $d_c$ is left-invariant and invertible (see Corollary 5.5 and Proposition 5.6).

In Section 2 we fix the notations we shall use throughout this paper. In Section 3 we gather some more or less known results about tensor analysis in Heisenberg groups. Section 4 recalls results borrowed to Chanillo and Van Schaftingen. Section 5 contains Theorem 5.1 together with several auxiliary results. The proof of Theorem 1.3 is completed in Section 6. Section 7 contains a variant of Theorem 1.3 where no differential operator occurs on the right hand side.

## 2. Notations and definitions

As above, we denote by $\mathbb{H}^n$ the $n$-dimensional Heisenberg group identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If $p$ and $p' \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot p' = (x + x', y + y', t + t' + \frac{1}{2}\sum_{j=1}^n (x_j y_j' - y_j x_j')).$$

In particular for any $p \in \mathbb{H}^n$ there is a familiy of (left) translations $\tau_p : \mathbb{H}^n \to \mathbb{H}^n$ defined by

$$\tau_p q := p \cdot q, \quad q \in \mathbb{H}^n.$$

For a general review on Heisenberg groups and their properties, we refer to [31], [21], [8], and to [33]. We limit ourselves to fix some notations, following [19].

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. As customary, $\mathfrak{h}$ is identified with the tangent space $T_e\mathbb{H}^n$ at the origin. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} - \frac{1}{2} y_i \partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2} x_i \partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_j, Y_j] = T$, for $j = 1, \dots, n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. Coherently, from now on, we refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as to the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of $T$, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of non-isotropic dilations $\delta_\lambda$, $\lambda > 0$ in $\mathbb{H}^n$. The homogeneous dimension of $\mathbb{H}^n$ with respect to $\delta_\lambda$, $\lambda > 0$ is

$$Q = 2n + 2.$$

The vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and $T$ orthonormal.

Throughout this paper, to avoid cumbersome notations, we write also

$$W_i := X_i, \quad W_{i+n} := Y_i, \quad W_{2n+1} := T, \quad \text{for } i = 1, \cdots, n. \tag{8}$$

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ where

$$\theta := dt - \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$$

is called the *contact form* in $\mathbb{H}^n$.

We indicate as $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta)$ an orthonormal basis. The same notation will be used to denote the scalar product in $\bigwedge_1 \mathfrak{h}$ that makes $(X_1, \dots, X_n, T)$ an orthonormal basis.

Coherently with the previous notation (8), we set

$$\omega_i := dx_i, \quad \omega_{i+n} := dy_i, \quad \omega_{2n+1} := \theta, \quad \text{for } i = 1, \cdots, n.$$

We put $\bigwedge_0 \mathfrak{h} := \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le k \le 2n+1$,

$$\bigwedge\nolimits_k \mathfrak{h} := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\} =: \operatorname{span} \Psi_k,$$

$$\bigwedge\nolimits^k \mathfrak{h} := \operatorname{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\} =: \operatorname{span} \Psi^k.$$

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{h}$ and to $\bigwedge^k \mathfrak{h}$ making both bases $\Psi_k$ and $\Psi^k$ orthonormal.

If $1 \le k \le 2n+1$, we denote by $*$ the Hodge isomorphism

$$* : \bigwedge\nolimits^k \mathfrak{h} \longleftrightarrow \bigwedge\nolimits^{2n+1-k} \mathfrak{h}$$

associated with the scalar product $\langle\cdot,\cdot\rangle$ and the volume form

$$dV := \omega_1 \wedge \cdots \wedge \omega_{2n} \wedge \theta.$$

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal $k$-vectors* and *horizontal $k$-covectors*

$$\textstyle\bigwedge_k \mathfrak{h}_1 := \mathrm{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\}$$
$$\textstyle\bigwedge^k \mathfrak{h}_1 := \mathrm{span}\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\}.$$

It is well known that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space at the origin $e = 0$ of $\mathbb{H}^n$, and hence the horizontal layer $\mathfrak{h}_1$ can be identified with a subspace of $T\mathbb{H}^n_e$ that we can still denote by $\bigwedge_1 \mathfrak{h}_1$.

In addition, the symplectic 2-form

$$-d\theta = \sum_{i=1}^n dx_i \wedge dy_i$$

induces on $\mathfrak{h}_1$ a symplectic structure. We point out that this symplectic structure is compatible with our fixed scalar product $\langle\cdot,\cdot\rangle$ and with the canonical almost complex structure on $\mathfrak{h}_1 \equiv \mathbb{C}^n$.

Horizontal $k$-vectors can be identified with skew-symmetric $k$-tensor in $\otimes^k \mathfrak{h}_1$.

To fix our notations, we remind the following definition.

**Definition 2.1.** If $V, W$ are finite dimensional linear vector spaces and $S : V \to W$ is a linear isomorphism, we define a map

$$\otimes_r S : \otimes^r V \to \otimes^r W$$

as the linear map defined by

$$(\otimes_r S)(v_1 \otimes \cdots \otimes v_r) = S(v_1) \otimes \cdots \otimes S(v_r),$$

and a map

$$\otimes^s S : \otimes^s W^* \to \otimes^s V^*$$

as the linear map defined by

$$\langle (\otimes^s S)(\alpha) | v_1 \otimes \cdots \otimes v_s \rangle = \langle \alpha | (\otimes_s S)(v_1 \otimes \cdots \otimes v_s) \rangle$$

for any $\alpha \in \otimes^s W^*$ and any $s$-tensor $v_1 \otimes \cdots \otimes v_s \in \otimes^s V$. Finally, we define

$$(\otimes^s_r S) : (\otimes^r V) \otimes (\otimes^s V^*) \to (\otimes^r W) \otimes (\otimes^s W^*)$$

as follows:

$$(\otimes^s_r S)(v \otimes w) := (\otimes_r S)(v) \otimes (\otimes^s S^{-1})(w).$$

Throughout this paper, we shall deal with $(r, s)$-tensors in

$$\big(\otimes^r \mathfrak{h}\big) \otimes \big(\otimes^s \mathfrak{h}^*\big),$$

with $r, s \in \mathbb{Z}$, $r, s \ge 0$, that, in turn define a left-invariant fiber bundle over $\mathbb{H}^n$, that we still denote by $\big(\otimes^r \mathfrak{h}\big) \otimes \big(\otimes^s \mathfrak{h}^*\big)$ as follows: first we identify $\big(\otimes^r \mathfrak{h}\big) \otimes \big(\otimes^s \mathfrak{h}^*\big)$ with a subspace of $\big(\otimes^r T_e\mathbb{H}^n\big) \otimes \big(\otimes^s T^*_e\mathbb{H}^n\big)$ that we denote by

$$\big(\otimes^r \mathfrak{h}\big)_e \otimes \big(\otimes^s \mathfrak{h}^*\big)_e.$$

Then the fiber of $\big(\otimes^r \mathfrak{h}\big)_p \otimes \big(\otimes^s \mathfrak{h}^*\big)_p$ of $\big(\otimes^r \mathfrak{h}\big) \otimes \big(\otimes^s \mathfrak{h}^*\big)$ over $p \in \mathbb{H}^n$ is

$$\big(\otimes^r \mathfrak{h}\big)_p \otimes \big(\otimes^s \mathfrak{h}^*\big)_p := (\otimes^h d\tau_p(e))\big(\otimes^r \mathfrak{h}\big)_e \otimes (\otimes^h d\tau_{p^{-1}}(p))\big(\otimes^s \mathfrak{h}^*\big)_e.$$

The elements of the space of smooth sections of this bundle, i.e.

$$\Gamma\big(\mathbb{H}^n, \big(\otimes^r \mathfrak{h}\big) \otimes \big(\otimes^s \mathfrak{h}^*\big)\big),$$

are called $(r, s)$-tensors fields.

A special instance will be the *horizontal* tensors belonging to

$$\big(\otimes^r \mathfrak{h}_1\big) \otimes \big(\otimes^s \mathfrak{h}_1^*\big).$$

The horizontal $(r, 0)$-tensors fields will be called also horizontal $r$-vector fields. The skew-symmetric horizontal $(0, s)$-tensors fields are identified with the horizontal differential forms.

Moreover, to avoid cumbersome notations, from now on, when dealing with a vector bundle $\mathcal{N}$ over $\mathbb{H}^n$, if there is no way to misunderstand we shall write also

$$\Gamma(\mathcal{N}) \quad \text{for} \quad \Gamma(\mathbb{H}^n, \mathcal{N}).$$

Finally, a subbundle $\mathcal{N}$ of $\big(\otimes^r \mathfrak{h}\big) \otimes \big(\otimes^s \mathfrak{h}^*\big)$ is said *left invariant* if

$$\mathcal{N}_p = (\otimes_r^s d\tau_p(e))\mathcal{N}_e. \tag{9}$$

It is customary (see e.g. [34], Chapter I) to denote by

$$L : \bigwedge^h \mathfrak{h}_1 \to \bigwedge^{h+2} \mathfrak{h}_1$$

the Lefschetz operator defined by $L\alpha := d\theta \wedge \alpha$, and by $\Lambda$ its dual operator with respect to $\langle \cdot, \cdot \rangle$. If $2 \le h \le 2n$, we denote by $P^h \subset \bigwedge^h \mathfrak{h}_1$ the space of primitive $h$-covectors defined by

$$P^1 := \bigwedge^1 \mathfrak{h}_1 \quad \text{and} \quad P^h := \ker \Lambda \cap \bigwedge^h \mathfrak{h}_1, \quad 2 \le h \le 2n. \tag{10}$$

Following [27], [28], for $h = 0, 1, \dots, 2n + 1$ we define a linear subspace $E_0^h$, of $\bigwedge^h \mathfrak{h}$ as follows:

**Definition 2.2.** We set

- if $1 \le h \le n$ then $E_0^h = P^h$;
- if $n < h \le 2n + 1$ then $E_0^h = \{\alpha = \beta \wedge \theta,\ \beta \in \bigwedge^{h-1} \mathfrak{h}_1,\ L\beta = 0\}$.

**Remark 2.3.** Definition 2.2 is not the original definition due to M. Rumin, but is indeed a characterization of Rumin's classes that is proved in [27] (see also [26], [4] and [3]).

By (9), the spaces $E_0^*$ define a family of left-invariant subbundles (still denoted by $E_0^h$, $h = 0, \dots, 2n+1$). It turns out that we can identify $E_0^h$ and $(E_0^h)_e$.

## 3. Basic facts on tensor analysis in Heisenberg groups

The following decomposition theorem holds.

**Proposition 3.1.** *The space of 2-contravariant horizontal tensors* $\otimes^2\mathfrak{h}_1$ *can be written as a direct (orthogonal) sum*

$$\otimes^2\mathfrak{h}_1 = \mathrm{Sym}\,(\otimes^2\mathfrak{h}_1) \oplus \bigwedge_2 \mathfrak{h}_1$$

*of the space* $\mathrm{Sym}\,(\otimes^2\mathfrak{h}_1)$ *of the symmetric 2-tensors and of the space* $\bigwedge_2 \mathfrak{h}_1$ *of the skew-symmetric 2-tensors.*

*An orthonormal basis of* $\mathrm{Sym}\,(\otimes^2\mathfrak{h}_1)$ *with respect to the canonical scalar product in* $\otimes^2\mathfrak{h}_1$ *is*

$$\left\{\frac{1}{2}(W_i\otimes W_j + W_j\otimes W_i)\,;\, i\le j\right\},$$

*whereas the canonical orthonormal basis* $\Psi_2$ *of* $\bigwedge_2 \mathfrak{h}_1$ *can be identified with*

$$\left\{\frac{1}{2}(W_i\otimes W_j - W_j\otimes W_i)\,;\, i< j\right\}.$$

**Definition 3.2** (See [2], Definition 1.7.16)**.** If $\phi:\mathbb{H}^n\to\mathbb{H}^n$ is a diffeomorphism, then the push-forward of a tensor field $t:=v\otimes\alpha$, with $v\in\otimes^r\mathfrak{h}$ and $\alpha\in\otimes^s\mathfrak{h}^*$, then its push-forward $\phi_* t$ at a point $p\in\mathbb{H}^n$ is defined as

$$\phi_* t(p) := \big(\otimes_r^s\, d\phi(\tilde p)\big)((v\otimes\alpha)(\tilde p)),$$

where $\tilde p := \phi^{-1}(p)$. Moreover, the pull-back $\phi^* t$ of $t$ is defined by

$$\phi^* t = (\phi^{-1})_*.$$

A tensor field

$$v\otimes\alpha\in\Gamma\big(\mathbb{H}^n, \big(\otimes^r\,\mathfrak{h}\big)\otimes\big(\otimes^s\,\mathfrak{h}^*\big)\big)$$

is said *left invariant* if

$$(\tau_q)_*\, v\otimes\alpha = v\otimes\alpha \quad \text{for all } q\in\mathbb{H}^n.$$

**Lemma 3.3.** *Let* $v\otimes\xi\in\big(\otimes^r\,\mathfrak{h}\big)_e\otimes\big(\otimes^s\,\mathfrak{h}^*\big)_e$ *be given. If* $p\in\mathbb{H}^n$*, we set*

$$\mathcal{T}_p(v\otimes\xi) := (\tau_p)_*(v\otimes\xi)\in\big(\otimes^r\,\mathfrak{h}\big)_p\otimes\big(\otimes^s\,\mathfrak{h}^*\big)_p.$$

*Then the map* $p\to\mathcal{T}_p(v\otimes\xi)$ *is left-invariant.*

*Thus, if* $V\otimes W$ *is a linear subspace of* $\big(\otimes^r\,\mathfrak{h}\big)_e\otimes\big(\otimes^s\,\mathfrak{h}^*\big)_e$*, then* $\{\mathcal{T}_p(v\otimes\xi),\ v\otimes\xi\in V\otimes W,\ p\in\mathbb{H}^n\}$ *defines a left-invariant subbundle of* $\big(\otimes^r\,\mathfrak{h}\big)\otimes\big(\otimes^s\,\mathfrak{h}^*\big)$*. In addition, if* $\{v_i\otimes\xi_j,\ i=1,\dots N, j=1,\dots M\}$ *is a basis of* $V\otimes W$*, then* $\{\mathcal{T}_p(v\otimes\xi),\ i=1,\dots N, j=1,\dots M\}$ *is a left-invariant basis of the fiber over* $p\in\mathbb{H}^n$*.*

We remind the following well-known identity (see e.g. [1], Proposition 6.16):

**Remark 3.4.** If $W\in\mathfrak{h}_1$ is identified with a first order differential operator and $\phi:\mathbb{H}^n\to\mathbb{H}^n$ is a diffeomorphism, then

$$W(u\circ\phi)(x) = [(\phi_* W)u]\,(\phi(x)).$$

Moreover, if $W,Z\in\mathfrak{h}_1$, then

$$\phi_*(W\otimes Z) = \phi_* W\otimes\phi_* Z.$$

Set $N_h := \dim E_0^h$. Given a family of left-invariant bases $\{\xi_k^h,\ k = 1,\dots N_h\}$ of $E_0^h$, $1 \le h \le n$ as in Lemma 3.3, the differential $d_c$ can be written "in coordinates" as follows.

**Proposition 3.5.** *If* $0 \le h \le 2n$ *and*

$$\alpha = \sum_k \alpha_k\, \xi_k^h \in \Gamma(E_0^h),$$

*then*

$$d_c\alpha = \sum_{I,k} P_{I,k}\alpha_k\, \xi_I^{h+1},$$

*where*

i) *if* $h \neq n$, *then the* $P_{I,k}$*'s are linear homogeneous polynomials in* $W_1,\dots,W_{2n} \in \mathfrak{h}_1$ *(that are identified with homogeneous with first order left invariant horizontal differential operators), i.e.*

$$P_{I,k} = \sum_i F_{I,k,i} W_i,$$

*where the* $F_{I,k,i}$*'s are real constants;*

ii) *if* $h = n$, *then then the* $P_{I,k}$*'s are linear homogeneous polynomials in* $W_i \otimes W_j \in \otimes^2\mathfrak{h}_1$, $i,j = 1,\dots,2n$ *(that are identified with homogeneous second order left invariant differential horizontal operators), i.e.*

$$P_{I,k} = \sum_{i,j} F_{I,k,i,j} W_i \otimes W_j,$$

*where the* $F_{I,k,i,j}$*'s are real constants.*

**Definition 3.6.** If $0 \le h < n$ we denote $\sigma(d_c)$ the symbol of the intrinsic differential $d_c$ that is a smooth field of homomorphisms

$$\sigma(d_c) \in \Gamma(\mathrm{Hom}\,(E_0^h, \mathfrak{h}_1 \otimes E_0^{h+1}))$$

defined as follows: if $p \in \mathbb{H}^n$, $\bar\alpha = \sum_k \bar\alpha_k \xi_k^h(p) \in (E_0^h)_p$, then we can assume there exists a smooth differential form $\alpha = \sum_k \alpha_k \xi_k^h \in \Gamma(E_0^h)$ such that $\bar\alpha = \alpha(p)$. Thus, if $u \in \mathcal{E}(\mathbb{H}^n)$ satisfies $u(p) = 0$, then we have

$$d_c(u\alpha)(p) = \sum_{I,k} (P_{I,k}u)(p)\alpha_k(p)\xi_I^{h+1}(p).$$

Hence we set

$$\sigma(d_c)(p)\bar\alpha := \sum_{I,k} \bar\alpha_j P_{I,k}(p) \otimes \xi_I^{h+1}(p).$$

If $h = n$, $d_c$ is now a second order differential operator in the horizontal vector fields and then its symbol $\sigma(d_c)$ can be identified with a section

$$\sigma(d_c) \in \Gamma(\mathrm{Hom}\,(E_0^n, \otimes^2\mathfrak{h}_1 \otimes E_0^{n+1}))$$

as follows: if $p \in \mathbb{H}^n$, $\bar\alpha = \sum_k \bar\alpha_k\, \xi_k^n(p) \in (E_0^n)_p$, then we we can assume there exists a smooth differential form $\alpha = \sum_k \alpha_k\, \xi_k^n \in E_0^n$ such that $\bar\alpha = \alpha(p)$. Thus, if $u \in \mathcal{E}(\mathbb{H}^n)$ satisfies $u(p) = 0$ and $W_i u(p) = 0$, $i = 1,\dots,2n$, then we have

$$d_c(u\alpha)(p) = \sum_{I,k} (P_{I,k}u)(p)\alpha_k(p)\, \xi_I^{n+1}(p).$$

Hence we set

$$\sigma(d_c)(p)\bar\alpha := \sum_{I,k} \bar\alpha_k P_{I,k}(p) \otimes \xi_I^{n+1}(p).$$

On the other hand, the canonical projection

$$p : \otimes^2 h_1 \otimes E_0^{n+1} \to \frac{\otimes^2 h_1}{\bigwedge^2 \mathfrak{h}_1} \otimes E_0^{n+1}$$

given by

$$p(W_i \otimes W_j \otimes \xi) := [W_i \otimes W_j] \otimes \xi$$

defines a new symbol (the symmetric part of the symbol)

$$\Sigma(d_c) := p \circ \sigma(d_c) \in \mathrm{Hom}\,(E_0^n, \frac{\otimes^2 h_1}{\bigwedge^2 \mathfrak{h}_1} \otimes E_0^{n+1}).$$

Clearly, since we are dealing with 2-tensors, $[W_i \otimes W_j]$ can be represented by

$$\frac{1}{2}\left(W_i \otimes W_j + W_j \otimes W_i\right).$$

Thus

$$\begin{aligned} &\Sigma(d_c)\bar\alpha \\ &= \frac{1}{2}\sum_I \sum_{i,j,k} \bar\alpha_k (F_{I,k,i,j} + F_{I,k,j,i})\,(W_i \otimes W_j + W_j \otimes W_i) \otimes \xi_I^{n+1} \\ &=: \sum_I \sum_{i,j,k} \bar\alpha_k \tilde F_{I,k,i,j}\,(W_i \otimes W_j + W_j \otimes W_i) \otimes \xi_I^{n+1}. \end{aligned} \tag{11}$$

**Remark 3.7.** If $0 \le h < n$, mimicking the usual definition of the principal symbol $\sigma(P)$ of a differential operator $P$ (see e.g. [24], Definition 3.3.13 or [25], IV.3), one can cook up a notion of horizontal principal symbol that takes into account Heisenberg homogeneity. For $d_c$, this map would belong to

$$\Gamma(\mathrm{Hom}\,(E_0^h \otimes \mathfrak{h}_1^*, E_0^{h+1})).$$

However, our notation is not misleading, since, by [15] (16.8.2.3) and (16.18.3.4),

$$\begin{aligned} \mathrm{Hom}\,(E_0^h \otimes \mathfrak{h}_1^*, E_0^{h+1}) &\cong \mathrm{Hom}\,(E_0^n, \mathrm{Hom}\,(\mathfrak{h}_1^*, E_0^{h+1})\,) \\ &\cong \mathrm{Hom}\,(E_0^h, \mathfrak{h}_1 \otimes E_0^{h+1}). \end{aligned}$$

An analogous comment applies when $h = n$. In this case, only the projection $\Sigma(d_c)$ of the symbol onto symmetric tensors will be used.

Since $\mathfrak{h}_1$, $d_c$ and $E_0^*$ are invariant under left translations, then the symbol $\sigma(d_c)$ is uniquely determined by its value at the point $p = e$. More precisely, we have:

**Proposition 3.8.** *If $1 \le h < n$ and $p \in \mathbb{H}^n$, then the following diagram is commutative:*

$$\begin{array}{ccc} (E_0^h)_p & \xrightarrow{\sigma(d_c)(p)} & (\mathfrak{h})_p \otimes (E_0^{h+1})_p \\ {\scriptstyle \otimes^h \tau_p(e)}\big\downarrow & & \big\uparrow{\scriptstyle \mathcal{T}_p} \\ (E_0^h)_e & \xrightarrow{\sigma(d_c)(e)} & (\mathfrak{h})_e \otimes (E_0^{h+1})_e \end{array} \tag{12}$$

*An analogous comment applies when $h = n$.*

## 4. Analytic facts

The proof of Theorem 1.3 consists in applying the following two results due to Chanillo & Van Schaftingen, after an algebraic reduction that will be performed in the next section.

**Definition 4.1.** If $f : \mathbb{H}^n \to \mathbb{R}$, we denote by $\nabla_{\mathbb{H}} f$ the horizontal vector field

$$\nabla_{\mathbb{H}} f := \sum_{i=1}^{2n} (W_i f) W_i,$$

whose coordinates are $(W_1 f, ..., W_{2n} f)$. If $\Phi$ is a horizontal vector field, then $\nabla_{\mathbb{H}} \Phi$ is defined componentwise.

**Theorem 4.2** ([14], Theorem 1)**.** *Let $\Phi \in \mathcal{D}(\mathbb{H}^n, \mathfrak{h}_1)$ be a smooth compactly supported horizontal vector field. Suppose $G \in L^1_{\text{loc}}(\mathbb{H}^n, \mathfrak{h}_1)$ is $\mathbb{H}$-divergence free, i.e. if*

$$G = \sum_i G_i W_i, \quad \textit{then} \quad \sum_i W_i G_i = 0 \quad \textit{in } \mathcal{D}'(\mathbb{H}^n).$$

*Then*

$$\left| \langle G, \Phi \rangle_{L^2(\mathbb{H}^n, \mathfrak{h}_1)} \right| \le C \|G\|_{L^1(\mathbb{H}^n, \mathfrak{h}_1)} \|\nabla_{\mathbb{H}} \Phi\|_{L^Q(\mathbb{H}^n, \mathfrak{h}_1)}.$$

We notice that a stronger version of this result can be found in [35], Theorem 1.9.

As in the Euclidean case, estimates similar to Theorem 4.2 still hold when the condition on the divergence is replaced by a condition on higher-order derivatives [32]. Similar ideas have been applied in nilpotent homogeneous groups by S. Chanillo and J. Van Schaftingen as follows.

Let $k \ge 1$ be fixed, and let $G \in L^1(\mathbb{H}^n, \otimes^k \mathfrak{h}_1)$ belong to the space of horizontal $k$-tensors. We can write

$$G = \sum_{i_1, \dots, i_k} G_{i_1, \dots, i_k} W_{i_1} \otimes \cdots \otimes W_{i_k}.$$

We remind that $G$ can be identified with the differential operator

$$u \to Gu := \sum_{i_1, \dots, i_k} G_{i_1, \dots, i_k} W_{i_1} \cdots W_{i_k} u.$$

Denoting by $\mathcal{D}(\mathbb{H}^n, \mathrm{Sym}(\otimes^k \mathfrak{h}_1))$ the subspace of compactly supported smooth symmetric horizontal $k$-tensors, we have:

**Theorem 4.3** ([14], Theorem 5)**.** *Let $k \ge 1$ and*

$$G \in L^1(\mathbb{H}^n, \otimes^k \mathfrak{h}_1), \quad \Phi \in \mathcal{D}(\mathbb{H}^n, \mathrm{Sym}(\otimes^k \mathfrak{h}_1)).$$

*Suppose that*

$$\sum_{i_1, \dots, i_k} W_{i_k} \cdots W_{i_1} G_{i_1, \dots, i_k} = 0 \quad \textit{in } \mathcal{D}'(\mathbb{H}^n).$$

*Then*

$$\left| \int_{\mathbb{H}^n} \langle \Phi, G \rangle \, dp \right| \le C \|G\|_{L^1(\mathbb{H}^n, \otimes^k \mathfrak{h}_1)} \|\nabla_{\mathbb{H}} \Phi\|_{L^Q(\mathbb{H}^n, \otimes^k \mathfrak{h}_1)}.$$

## 5. Main algebraic step

As in [4], our proof of Theorem 1.3 relies on the fact (precisely stated in Theorem 5.1 below) that the components with respect to a given basis of closed forms in $E_0^h$ can be viewed as the components of a horizontal vector field with vanishing horizontal divergence if $h \neq n, n+1$ or vanishing "generalized horizontal divergence" if $h = n, n+1$. More precisely, we have:

**Theorem 5.1.** *Let $\alpha = \sum_J \alpha_J \xi_J^h \in \Gamma(E_0^h)$, $1 \leq h \leq 2n$, be such that*

$$d_c \alpha = 0.$$

*Then*

- *if $h \neq n$ then each component $\alpha_J$ of $\alpha$, $J = 1, \dots, \dim E_0^h$, can be written as*

$$\alpha_J = \sum_{I=1}^{\dim E_0^{h+1}} \sum_{i=1}^{2n} b_{i,I}^J G_{I,i},$$

  *where the $b_{i,I}^J$'s are real constants and for any $I = 1, \dots, \dim E_0^{h+1}$ the $G_{I,i}$'s are the components of a horizontal vector field*

$$G_I = \sum_i G_{I,i} W_i$$

  *with*

$$\sum_i W_i G_{I,i} = 0, \quad I = 1, \dots, \dim E_0^{h+1}.$$

  *Moreover there exist a geometric constant $C > 0$ such that for $I = 1, \dots, \dim E_0^{h+1}$ and $1 \leq p \leq \infty$*

$$\|G_I\|_{L^p(\mathbb{H}^n, \bigwedge_1 \mathfrak{h}_1)} \leq C \|\alpha\|_{L^p(\mathbb{H}^n, E_0^h)}. \tag{13}$$

- *If $h = n$, then each component $\alpha_J$ of $\alpha$, $J = 1, \dots, \dim E_0^n$, can be written as*

$$\alpha_J = \sum_{I=1}^{\dim E_0^{n+1}} \sum_{i,j} b_{i,j,I}^J (G_I^{\mathrm{Sym}})_{i,j}.$$

  *Here the $b_{i,j,I}^J$'s are real constants and for any $I = 1, \dots, \dim E_0^{n+1}$ the $(G_I^{\mathrm{Sym}})_{i,j}$'s are the components of the symmetric part (see Proposition 3.1) of the 2-tensor*

$$G_I = \sum_i G_{I,i,j} \, W_i \otimes W_j$$

  *that satisfies*

$$\sum_{i,j} W_i W_j \, G_{I,i,j} = 0, \quad I = 1, \dots, \dim E_0^{n+1}.$$

  *Moreover there exist a geometric constant $C > 0$ such that for $I = 1, \dots, \dim E_0^{n+1}$ and $1 \leq p \leq \infty$*

$$\|G_I\|_{L^p(\mathbb{H}^n, \otimes^2 \mathfrak{h}_1)} \leq C \|\alpha\|_{L^p(\mathbb{H}^n, E_0^n)}. \tag{14}$$

The proof of this theorem requires several preliminary steps. First of all, we want to prove that the exterior differential $d_c$ is invariant (i.e. "natural") under the action of a class of intrinsic transformation of $\mathbb{H}^n$.

**Theorem 5.2.** *If $A$ belongs to the symplectic group $Sp_{2n}(\mathbb{R})$, we associate with $A$ the real $(2n+1)\times(2n+1)$ matrix*

$$f_A : \mathfrak{h} \to \mathfrak{h}, \quad f_A = \begin{pmatrix} A_{2n\times 2n} & 0_{2n\times 1} \\ 0_{1\times 2n} & 1 \end{pmatrix} . \tag{15}$$

*Then*

i) $f_A(\mathfrak{h}_1) = \mathfrak{h}_1$;
ii) $f_A$ *induces a homogeneous group isomorphism* $\exp \circ f_A \circ \exp^{-1}$ *still denoted by $f_A$ such that* $f_A : \mathbb{H}^n \to \mathbb{H}^n$;
iii) $f_A^* : \Gamma(E_0^*) \to \Gamma(E_0^*)$;
iv) *for any $h$-form* $\alpha \in \Gamma(E_0^h)$

$$d_c(f_A^*\alpha) = f_A^*(d_c\alpha);$$

*Proof.* See [29] or [20].

□

The next step consists in proving that the symbols $\sigma(d_c)$ and $\Sigma(d_c)$ are injective.

First of all, we remind that, by [34], Chapter I, Theorem 3 and Corollary at p. 28), the following proposition holds:

**Proposition 5.3.** *Let $P^h$ be space of primitive forms defined in* (10). *Then*

i) *if $1 \le h \le 2n$, then the following orthogonal decomposition holds:*

$$\bigwedge^h \mathfrak{h}_1 = \bigoplus_{i \ge (n-h)^+} L^i(P^{h-2i});$$

ii) $P^h = \{0\}$ *if* $h > n$;
iii) *the map* $L^{n-h} : \bigwedge^h \mathfrak{h}_1 \to \bigwedge^{2n-h} \mathfrak{h}_1$ *is a linear isomorphism;*
iv) *if $h \le n$, then* $P^h = \ker L^{n-h+1}$;
v) *the map* $L^{n-h} : P^h \to \bigwedge^{2n-h} \mathfrak{h}_1 \cap \ker L$ *is a linear isomorphism;*
vi) *a symplectic map $A \in Sp_{2n}(\mathbb{R})$ commutes with $L$, i.e. if $1 \le h \le 2n-2$, then* $[\otimes^h A, L] = 0$.

The injectivity of the symbols will follow from the following result.

**Proposition 5.4.** *The symplectic group $Sp_{2n}(\mathbb{R})$ acts irreducibly on $P^h$ for $h = 1, \dots, n$ and on $\ker L$ for $h = n+1, \dots, 2n-1$.*

*Proof.* If $1 \le h \le n$, then the statement is proved in [9], p. 203. Suppose now $h > n$. If $A \in Sp_{2n}(\mathbb{R})$ and $\alpha \in \ker L$, then by Proposition 5.3, vi), $\otimes^h AL \in \ker L$, so that $A$ acts on $\ker A$. On the other hand, if $V \subset \ker L \cap \bigwedge^h \mathfrak{h}_1$ is invariant under the action of $Sp_{2n}(\mathbb{R})$, then, by Proposition 5.3, v) and vi), $(L^{h-r})^{-1}V$ is also invariant under the action of $Sp_{2n}(\mathbb{R})$, and the assertion follows by the first part of the proof. □

Using Definition 2.2, we have:

**Corollary 5.5.** *If $A \in Sp_{2n}(\mathbb{R})$ and $f_A$ is defined as in* (15)*, then $f_A$ acts irreducibly on $E_0^h$ for $h = 1, \dots, 2n$.*

Since $d_c$ is equivariant under all smooth contact transformations, it is in particular $Sp_{2n}(\mathbb{R})$-equivariant. It follows that the kernels $\ker \sigma(d_c)$ and $\ker \Sigma(d_c)$ are invariant subspaces for the action of $Sp_{2n}(\mathbb{R})$, so that the injectivity will follow. In fact, we have:

**Proposition 5.6.** *Keeping in mind Definition 3.6, if $1 \le h \le 2n$ , $h \neq n$, then $\ker \sigma(d_c)(e)$ is invariant under the action of $Sp_{2n}(\mathbb{R})$, i.e. if $A \in Sp_{2n}(\mathbb{R})$, then we have:*

$$\text{if } \bar\alpha \in E_0^h \text{ and } \sigma(d_c)(e)(\bar\alpha) = 0 \quad \text{then} \quad \sigma(d_c)(e)((\otimes^h A)\ \bar\alpha) = 0.$$

*If $h = n$, then $\ker \Sigma(d_c)(e)$ is invariant under the action of $Sp_{2n}(\mathbb{R})$, i.e., $A \in Sp_{2n}\mathbb{R})$, then we have:*

$$\text{if } \bar\alpha \in E_0^h \text{ and } \Sigma(d_c)(e)(\bar\alpha) = 0 \quad \text{then} \quad \Sigma(d_c)(e)((\otimes^h A)\ \bar\alpha) = 0.$$

*Proof.* Suppose first $h < n$ and let $\alpha \in \Gamma(E_0^h)$ be a differential form such that $\alpha(e) = \bar\alpha$. Let $f_A$ be the matrix associated with $A$ as in (15). We notice first that

$$f_A^*(\alpha)(e) = (\otimes^h A)\bar\alpha, \tag{16}$$

since $f_A(e) = e$. Let now $u$ be a smooth function such that $u(e) = 0$. We set $v := u \circ f_A^{-1}$. Keeping again in mind that (16), we have also that $v(e) = 0$. By Theorem 5.2 and Remark 3.4, we have:

$$\begin{aligned} d_c(uf_A^*\alpha)(e) &= d_c(f_A^*(v\alpha))(e) = f_A^*(d_c(v\alpha))(e) \\ &= (\otimes^{h+1} f_A)(d_c(v\alpha)(e)) = (\otimes^{h+1} A)(d_c(v\alpha)(e)) \\ &= \sum_{I,k} (A^{-1}P_{I,k}u)(e)\bar\alpha_k(\otimes^{h+1}A)(\xi_I^{h+1}(e)). \end{aligned} \tag{17}$$

Hence, by (16)

$$\begin{aligned} \sigma(d_c)(e)((\otimes^h A)\bar\alpha) &= \sigma(d_c)(e)((f_A^*\alpha)(e)) \\ &= \sum_I A^{-1}\left(\sum_k \bar\alpha_k P_{I,k}(e)\right) \otimes (\otimes^{h+1}A)(\xi_I^{h+1}(e)). \end{aligned} \tag{18}$$

On the other hand, by assumption,

$$\sum_{I,k} \bar\alpha_k P_{I,k}(e) \otimes \xi_I^{h+1}(e) = 0,$$

so that

$$\sum_k \bar\alpha_k P_{I,k}(e) = 0 \quad I = 1, \dots, \dim E_0^{h+1},$$

since the $\xi_I^{h+1}$'s are linearly independent. Thus eventually from (18)

$$\sigma(d_c)(e)((\otimes^h A)\ \bar\alpha) = 0.$$

Consider now the case $h = n$, when $d_c$ is a second order operator in the horizontal derivatives. We stress that $E_0^{n+1}$ contains only vertical forms, i.e. forms that are multiple of the contact form $\theta$. Suppose $\Sigma(d_c)(e)\bar\alpha = 0$. Then, by (11),

$$\sum_{i,j,k} \bar\alpha_k \tilde F_{I,k,i,j}\,(W_i\otimes W_j + W_j\otimes W_i) = 0, \quad I=1,\dots,\dim E_0^{h+1}. \tag{19}$$

We take now $u\in\mathcal{E}(\mathbb{H}^n)$ satisfies $u(e)=0$ and $W_iu(e)=0$, $i=1,\dots,2n$. As above, we set $v:=u\circ f_A^{-1}$. Keeping in mind that $f_A(e)=e$, we have also that $v(e)=0$ and $(W_iv)(e)=0$, $i=1,\dots,2n$. Then equations (17) become

$$\begin{aligned} d_c(uf_A^*\alpha)(e) &= d_c(f_A^*(v\alpha))(e) = f_A^*(d_c(v\alpha))(e)\\ &= (\otimes^{h+1} f_A)(d_c(v\alpha)(e))\\ &= \sum_{I,k}\sum_{i,j} F_{I,k,i,j}(A^{-1}W_i)(A^{-1}W_j)u(e)\bar\alpha_k(\otimes^{n+1}f_A)(\xi_I^{n+1}(e)). \end{aligned}$$

Therefore, keeping in mind Remark 3.4 and (16), we have:

$$\begin{aligned} &\Sigma(d_c)(e)((\otimes^h A)\bar\alpha) = \Sigma(d_c)(e)((f_A^*\alpha)(e))\\ &= \sum_I\sum_{i,j,k}\bar\alpha_k\tilde F_{I,k,i,j}\big((A^{-1}W_i)(e)\otimes(A^{-1}W_j)(e)\\ &\qquad\qquad +(A^{-1}W_j)(e)\otimes(A^{-1}W_i)(e)\big)\otimes(\otimes^{n+1}f_A)(\xi_I^{n+1}(e))\\ &= \sum_I A^{-1}\Big(\sum_{i,j,k}\bar\alpha_k\tilde F_{I,k,i,j}\big(W_i(e)\otimes W_j(e)\\ &\qquad\qquad + W_j(e)\otimes W_i(e)\big)\Big)\otimes(\otimes^{n+1}f_A)(\xi_I^{n+1}(e)) = 0, \end{aligned}$$

by (19).

Finally, the proof for $h>n$ can be carried out precisely as in the case $h<n$, with only minor changes. In particular, (16) must be replaced taking into account that a form $\alpha\in E_0^h$ has also a vertical component of the form $\beta\wedge\theta$ and that

$$f_A^*(\beta\wedge\theta)(e) = (\otimes^h A)\beta(e)\wedge\theta.$$

This completes the proof of the proposition.

□

*Proof of Theorem 5.1.* First of all, we notice that, by Corollary 5.5 and Proposition 5.6, both $\ker\sigma(d_c)(e)$ (if $h\neq n$) and $\ker\Sigma(d_c)(e)$ (if $h=n$) are the null space $\{0\}$, and hence both $\sigma(d_c)(e)$ and $\Sigma(d_c)(e)$ have a left inverse

$$B_h\in \mathrm{Hom}\,(\mathfrak{h}_1\otimes(E_0^{h+1})_e,(E_0^h)_e)\quad \text{if } h\neq n.$$

and

$$B_n\in \mathrm{Hom}\,(\mathrm{Sym}\,(\otimes^2\mathfrak{h}_1)\otimes(E_0^{n+1})_e,(E_0^n)_e)\quad \text{if } h=n.$$

By the commutativity of the diagram (12), $B_h$ and $B_n$ can be identified with constant coefficient maps

$$B_h\in \mathrm{Hom}\,(\mathfrak{h}_1\otimes E_0^{h+1},E_0^h),\quad B_n\in\mathrm{Hom}\,(\mathrm{Sym}\,(\otimes^2\mathfrak{h}_1)\otimes E_0^{n+1},E_0^n)$$

such that

$$\alpha = B_h(\sigma(d_c)\alpha)\quad \text{for all } \alpha\in\Gamma(E_0^h),\ h\neq n, \tag{20}$$

and

$$\alpha = B_n(\Sigma(d_c)\alpha) \quad \text{for all } \alpha \in \Gamma(E_0^n). \tag{21}$$

We deal first with the case $h \neq n$ and we set

$$B_h(W_i \otimes \xi_I^{h+1}) := \sum_J b_{i,I}^J \xi_J^h.$$

Then, if we write $\alpha = \sum_J \alpha_J \xi_J^h \in \Gamma(E_0^h)$ and

$$P_{I,k} = \sum_i F_{I,k,i} W_i,$$

(where the $F_{I,k,i}$'s are real constants) identity (20) becomes

$$\begin{aligned} \alpha &= B_h(\sum_{I,k} \alpha_k P_{I,k} \otimes \xi_I^{h+1}) \\ &= \sum_J \left( \sum_{I,k,i} b_{i,I}^J F_{I,k,i} \alpha_k \right) \xi_J^h, \end{aligned} \tag{22}$$

so that

$$\alpha_J = \sum_{I,k,i} b_{i,I}^J F_{I,k,i} \alpha_k, \quad \text{for } J = 1, \dots, \dim E_0^h. \tag{23}$$

Suppose now $d_c\alpha = 0$. Then, writing the identity in coordinates, if $I = 1, \dots, \dim E_0^{h+1}$, we have

$$\sum_i W_i \left( \sum_k F_{I,k,i} \alpha_k \right) = 0,$$

so that, if we denote by $G_I$ the horizontal vector field

$$G_I = \sum_i G_{I,i} W_i := \sum_i \left( \sum_k F_{I,k,i} \alpha_k \right) W_i$$

then

$$\sum_i W_i G_{I,i} = 0, \qquad I = 1, \dots, \dim E_0^{h+1}.$$

Thus (23) reads as

$$\alpha_J = \sum_{I,i} b_{i,I}^J G_{I,i},$$

achieving the proof in the case $h < n$.

We deal now with the case $h = n$ and we set

$$B_n((W_i \otimes W_j + W_j \otimes W_i) \otimes \xi_I^{n+1}) := \sum_J b_{i,j,I}^J \xi_J^n.$$

Thus, if we write $\alpha = \sum_J \alpha_J \xi_J^h \in \Gamma(E_0^n)$ by (11), identity (21) becomes

$$\begin{aligned} \alpha &= B_n(\sum_{I,k,i,j} \alpha_k \tilde{F}_{I,k,i,j} (W_i \otimes W_j + W_j \otimes W_i) \otimes \xi_I^{n+1}) \\ &= \sum_J \left( \sum_{I,k,i,j} b_{i,j,I}^J \tilde{F}_{I,k,i,j} \alpha_k \right) \xi_J^n, \end{aligned} \tag{24}$$

so that

$$\alpha_J = \sum_{I,k,i,j} b^J_{i,j,I} \tilde F_{I,k,i,j}\alpha_k, \quad \text{for } J = 1,\dots,\dim E_0^n. \tag{25}$$

Denote by $G_I$ the horizontal tensor field

$$G_I = \sum_{i,j} G_{I,i,j} W_i \otimes W_j := \sum_{i,j}\Big(\sum_k F_{I,k,i,j}\alpha_k\Big) W_i\otimes W_j. \tag{26}$$

By Proposition 3.1 we can write

$$G_I = G_I^{\rm Sym} + G_I^{\rm Skew},$$

where

$$\begin{aligned} G_I^{\rm Sym} &= \sum_{i,j} (G_I^{\rm Sym})_{i,j}(W_i\otimes W_j + W_j\otimes W_i) \\ &:= \sum_{i,j}\Big(\sum_k \tilde F_{I,k,i,j}\alpha_k\Big)(W_i\otimes W_j + W_j\otimes W_i). \end{aligned}$$

We suppose now that $d_c\alpha = 0$, that, by (26), in coordinates is

$$\sum_{i,j} W_iW_j\, G_{I,i,j} = 0, \quad I = 1,\dots,\dim E_0^{n+1}.$$

Thus (25) reads as

$$\alpha_J = \sum_{I,i,j} b^J_{i,j,I}(G_I^{\rm Sym})_{i,j} \quad \text{for } J = 1,\dots,\dim E_0^n,$$

achieving the proof in the case $h = n$.

□

## 6. Proof of Theorem 1.3

The proof follows the lines of [4]. Let us reming few facts of harmonic analysis in homogeneous groups.

A differential operator $P : \Gamma(E_0^h) \to \Gamma(E_0^k)$ is said left-invariant if for all $q \in \mathbb{H}^n$

$$P(\tau_q)_*\alpha = (\tau_q)_*(P\alpha) \quad \text{for all } \alpha \in \Gamma(E_0^h).$$

If $f$ is a real function defined in $\mathbb{H}^n$, we denote by ${}^{\rm v}f$ the function defined by ${}^{\rm v}f(p) := f(p^{-1})$, and, if $T \in \mathcal{D}'(\mathbb{H}^n)$, then ${}^{\rm v}T$ is the distribution defined by $\langle {}^{\rm v}T|\phi\rangle := \langle T|{}^{\rm v}\phi\rangle$ for any test function $\phi$.

Following e.g. [17], we can define a group convolution in $\mathbb{H}^n$: if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\rm loc}(\mathbb{H}^n)$, we set

$$f * g(p) := \int f(q) g(q^{-1}\cdot p)\, dq \quad \text{for } q \in \mathbb{H}^n. \tag{27}$$

We remind that, if (say) $g$ is a smooth function and $P$ is a left invariant differential operator, then

$$P(f * g) = f * Pg.$$

We remind also that the convolution is again well defined when $f, g \in \mathcal{D}'(\mathbb{H}^n)$, provided at least one of them has compact support. In this case the following identities hold

$$\langle f * g | \phi \rangle = \langle g | {}^{\mathrm{v}} f * \phi \rangle \quad \text{and} \quad \langle f * g | \phi \rangle = \langle f | \phi * {}^{\mathrm{v}} g \rangle \tag{28}$$

for any test function $\phi$.

As in [17], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_{2n+1})$ is a multi–index, we set $W^I = W_1^{i_1} \cdots W_{2n}^{i_{2n}}\, T^{i_{2n+1}}$. By the Poincaré–Birkhoff–Witt theorem, the differential operators $W^I$ form a basis for the algebra of left invariant differential operators in $\mathbb{H}^n$. Furthermore, we set $|I| := i_1 + \cdots + i_{2n} + i_{2n+1}$ the order of the differential operator $W^I$, and $d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}$ its degree of homogeneity with respect to group dilations.

Suppose now $f \in \mathcal{E}'(\mathbb{H}^n)$ and $g \in \mathcal{D}'(\mathbb{H}^n)$. Then, if $\psi \in \mathcal{D}(\mathbb{H}^n)$, we have

$$\begin{aligned} \langle (W^I f) * g | \psi \rangle &= \langle W^I f | \psi * {}^{\mathrm{v}} g \rangle = (-1)^{|I|} \langle f | \psi * (W^I\, {}^{\mathrm{v}} g) \rangle \\ &= (-1)^{|I|} \langle f * {}^{\mathrm{v}} W^I\, {}^{\mathrm{v}} g | \psi \rangle. \end{aligned} \tag{29}$$

Following [16], we remind now the notion of *kernel of order $a$*, as well as some basic properties.

**Definition 6.1.** A kernel of order $a$ is a homogeneous distribution of degree $a - Q$ (with respect to group dilations), that is smooth outside of the origin.

**Proposition 6.2.** *Let $K \in \mathcal{D}'(\mathbb{H}^n)$ be a kernel of order $a$.*

i) *${}^{\mathrm{v}} K$ is again a kernel of order $a$;*
ii) *$W_\ell K$ is a a kernel of order $a - 1$ for any horizontal derivative $W_\ell K$, $\ell = 1, \dots, 2n$;*
iii) *If $a > 0$, then $K \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$.*

**Definition 6.3.** In $\mathbb{H}^n$, following [27], we define the operator $\Delta_{\mathbb{H},h}$ on $E_0^h$ by setting

$$\Delta_{\mathbb{H},h} = \begin{cases} d_c \delta_c + \delta_c d_c & \text{if} \quad h \neq n, n+1; \\ (d_c \delta_c)^2 + \delta_c d_c & \text{if} \quad h = n; \\ d_c \delta_c + (\delta_c d_c)^2 & \text{if} \quad h = n+1. \end{cases}$$

Notice that $-\Delta_{\mathbb{H},0} = \sum_{j=1}^{2n} (W_j^2)$ is the usual sub-Laplacian of $\mathbb{H}^n$.

For sake of simplicity, once a basis of $E_0^h$ is fixed, the operator $\Delta_{\mathbb{H},h}$ can be identified with a matrix-valued map, still denoted by $\Delta_{\mathbb{H},h}$

$$\Delta_{\mathbb{H},h} = (\Delta_{\mathbb{H},h}^{ij})_{i,j=1,\dots,N_h} : \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}) \to \mathcal{D}'(\mathbb{H}^n, \mathbb{R}^{N_h}). \tag{30}$$

This identification makes possible to avoid the notion of currents: we refer to [5] for this more elegant presentation.

Combining [27], Section 3, and [6], Theorems 3.1 and 4.1, we obtain the following result.

**Theorem 6.4.** *If $0 \le h \le 2n+1$, then the differential operator $\Delta_{\mathbb{H},h}$ is hypoelliptic of order $a$, where $a = 2$ if $h \neq n, n+1$ and $a = 4$ if $h = n, n+1$ with respect to group dilations. Then*

i) *for $j = 1, \dots, N_h$ there exists*

$$K_j = \big(K_{1j}, \dots, K_{N_h j}\big), \quad j = 1, \dots N_h \tag{31}$$

*with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n) \cap \mathcal{E}(\mathbb{H}^n \setminus \{0\})$, $i, j = 1, \dots, N$;*

ii) *if $a < Q$, then the $K_{ij}$'s are kernels of type $a$ for $i, j = 1, \dots, N_h$*
*If $a = Q$, then the $K_{ij}$'s satisfy the logarithmic estimate $|K_{ij}(p)| \le C(1 + |\ln \rho(p)|)$ and hence belong to $L^1_{\mathrm{loc}}(\mathbb{H}^n)$. Moreover, their horizontal derivatives $W_\ell K_{ij}$, $\ell = 1, \dots, 2n$, are kernels of type $Q - 1$;*

iii) *when $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$, if we set*

$$\mathcal{K}\alpha := \big(\sum_j \alpha_j * K_{1j}, \dots, \sum_j \alpha_j * K_{N_h j}\big), \tag{32}$$

*then $\Delta_{\mathbb{H},h}\mathcal{K}\alpha = \alpha$. Moreover, if $a < Q$, also $\mathcal{K}\Delta_{\mathbb{H},h}\alpha = \alpha$.*

iv) *if $a = Q$, then for any $\alpha \in \mathcal{D}(\mathbb{H}^n, \mathbb{R}^{N_h})$ there exists $\beta_\alpha := (\beta_1, \dots, \beta_{N_h}) \in \mathbb{R}^{N_h}$, such that*

$$\mathcal{K}\Delta_{\mathbb{H},h}\alpha - \alpha = \beta_\alpha.$$

**Remark 6.5.** Coherently with formula (30), the operator $\mathcal{K}$ can be identified with an operator (still denoted by $\mathcal{K}$) acting on smooth compactly supported differential forms in $\mathcal{D}(\mathbb{H}^n, E_0^h)$.

*Proof of Theorem 1.3.* The case $h = 0$ is well known ([18], [13], [23]).

**Case $\mathbf{1 < h < 2n}$ and $\mathbf{h \neq n, n+1}$.** If $u, \phi \in \mathcal{D}(\mathbb{H}^n, E_0^h)$, we can write

$$\begin{aligned} \langle u, \phi\rangle_{L^2(\mathbb{H}^n, E_0^h)} &= \langle u, \Delta_{\mathbb{H},h}\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^h)} \\ &= \langle u, (\delta_c d_c + d_c\delta_c)\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^h)}. \end{aligned} \tag{33}$$

Consider now the first term in the previous sum,

$$\langle u, \delta_c d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^h)} = \langle d_c u, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{h+1})}.$$

If we write $f := d_c u$, then $d_c f = 0$. From now on, without loss of generality, for $1 \le h \le 2n+1$ we take an orthonormal basis of $E_0^h$, still denoted by $\{\xi_\ell^h\ ;\ \ell = 1, \dots, \dim E_0^h\}$. Thus, since $f, d_c\mathcal{K}\phi \in E_0^{h+1}$, we can write $f = \sum_{\ell=1}^{\dim E_0^{h+1}} f_\ell \xi_\ell^{h+1}$, $d_c\mathcal{K}\phi = \sum_{\ell=1}^{\dim E_0^{h+1}} (d_c\mathcal{K}\phi)_\ell \xi_\ell^{h+1}$, and hence we can reduce ourselves to estimate

$$\langle f_\ell, (d_c\mathcal{K}\phi)_\ell\rangle_{L^2(\mathbb{H}^n)} \quad \text{for } \ell = 1, \dots, \dim E_0^{h+1}. \tag{34}$$

By Theorem 5.1, if $h \neq n-1$ , each component $f_\ell$ of $f$ can be written as

$$f_\ell = \sum_{I=1}^{\dim E_0^{h+2}} \sum_{i=1}^{2n} b_{i,I}^\ell G_{I,i},$$

where the $b_{i,I}^\ell$'s are real constants and for any $I = 1, \dots, \dim E_0^{h+2}$ the $G_{I,i}$'s are the components of an horizontal vector field

$$G_I = \sum_i G_{I,i} W_i$$

with

$$\sum_i W_i G_{I,i} = 0, \qquad I = 1, \dots, \dim E_0^{h+2}. \tag{35}$$

On the other hand, for $h = n-1$, each component $f_\ell$ of $f$, $\ell = 1, \dots, \dim E_0^n$, can be written as

$$f_\ell = \sum_{I=1}^{\dim E_0^{n+1}} \sum_{i,j} b_{i,j,I}^\ell (G_I^{\mathrm{Sym}})_{i,j}.$$

Here the $b_{i,j,I}^\ell$'s are real constants and for any $I = 1, \dots, \dim E_0^{n+1}$ the $(G_I^{\mathrm{Sym}})_{i,j}$'s are the components of the symmetric part (see Proposition 3.1) of the 2-tensor

$$G_I = \sum_i G_{I,i,j}\, W_i \otimes W_j$$

that satisfies

$$\sum_{i,j} W_i W_j\, G_{I,i,j} = 0, \quad I = 1, \dots, \dim E_0^{n+1}. \tag{36}$$

Suppose now $h \neq n-1$. In order to estimate the terms of (34), we have to estimate terms of the form

$$\langle G_{I,i}, (d_c \mathcal{K}\phi)_\ell \rangle_{L^2(\mathbb{H}^n)} = \langle G_I, \Phi \rangle_{L^2(\mathbb{H}^n, \mathfrak{h}_1)}, \tag{37}$$

where

$$\Phi = (d_c \mathcal{K}\phi)_\ell W_i.$$

We can apply Theorem 4.2. Keeping in mind (13), we obtain

$$\big| \langle f_\ell, (d_c \mathcal{K}\phi)_\ell \rangle_{L^2(\mathbb{H}^n)} \big| \leq C \|f\|_{L^1(\mathbb{H}^n, E_0^{h+1})} \|\nabla_{\mathbb{H}} d_c \mathcal{K}\phi\|_{L^Q(\mathbb{H}^n, E_0^{h+1})}. \tag{38}$$

On the other hand, $\nabla_{\mathbb{H}} d_c \mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\phi_j * W^I \tilde{K}_{ij} \quad \text{with } d(I) = 2.$$

By Theorem 6.4, iv) and Proposition 6.2, ii) $W^I \tilde{K}_{ij}$ are kernels of type 0, so that, by [16], Proposition 1.9 we have

$$|\langle f_\ell, (d_c \mathcal{K}\phi)_\ell \rangle_{L^2(\mathbb{H}^n)}| \leq C \|f\|_{L^1(\mathbb{H}^n, E_0^{h+1})} \|\phi\|_{L^Q(\mathbb{H}^n, E_0^h)}. \tag{39}$$

The same argument can be carried out for all the components of $f$, yielding

$$|\langle f, d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^n, E_0^{h+1})}| \leq C \|f\|_{L^1(\mathbb{H}^n, E_0^{h+1})} \|\phi\|_{L^Q(\mathbb{H}^n, E_0^h)}. \tag{40}$$

Suppose now $h = n-1$ then we have to estimate terms of the form

$$\langle (G_I^{\mathrm{Sym}})_{i,j}, (d_c \mathcal{K}\phi)_\ell \rangle_{L^2(\mathbb{H}^n)} = \langle G_I, \Phi \rangle_{L^2(\mathbb{H}^n, \otimes^2 \mathfrak{h}_1)}, \tag{41}$$

$$\Phi = (d_c \mathcal{K}\phi)_\ell \big(W_i \otimes W_j + W_j \otimes W_i\big) \in \Gamma(\mathbb{H}^n, \mathrm{Sym}\,(\otimes^2 \mathfrak{h}_1)).$$

We can apply Theorem 4.3 and we obtain

$$|\langle f, d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^n, E_0^n)}| \leq C \|f\|_{L^1(\mathbb{H}^n, E_0^n)} \|\phi\|_{L^Q(\mathbb{H}^n, E_0^{n-1})}. \tag{42}$$

This achieve the estimate of the first term of (33) for all $1 < h < 2n$, $h \neq n, n+1$.

$$|\langle u, \delta_c d_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^n, E_0^h)}| \leq C \|f\|_{L^1(\mathbb{H}^n, E_0^{h+1})} \|\phi\|_{L^Q(\mathbb{H}^n, E_0^h)}. \tag{43}$$

Consider now the second term in (33)

$$\begin{aligned} \langle u, d_c \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^n, E_0^h)} &= \langle \delta_c u, \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^n, E_0^{h-1})} \\ &= \langle g, \delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^n, E_0^{h-1})} = \langle *g, *\delta_c \mathcal{K}\phi \rangle_{L^2(\mathbb{H}^n, E_0^{2n+2-h})}, \end{aligned}$$

where $*$ denotes the Hodge duality. We notice now that from $\delta_c u = g$, by Hodge duality we have $d_c * u = *g$. Hence $d_c(*g) = 0$, and thus, arguing precisely as above, we get

$$\Big|\langle u, d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^h)}\Big| \le C\|g\|_{L^1(\mathbb{H}^n,E_0^{h+1})}\|\phi\|_{L^Q(\mathbb{H}^n,E_0^h)}. \tag{44}$$

Combining (44) with (43), we get eventually

$$|\langle u,\phi\rangle_{L^2(\mathbb{H}^n,E_0^h)}| \le C\big(\|f\|_{L^1(\mathbb{H}^n,E_0^{h+1})} + \|g\|_{L^1(\mathbb{H}^n,E_0^{h-1})}\big)\|\phi\|_{L^Q(\mathbb{H}^n,E_0^h)},$$

and hence

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n,E_0^h)} \le C\big(\|f\|_{L^1(\mathbb{H}^n,E_0^{h+1})} + \|g\|_{L^1(\mathbb{H}^n,E_0^{h-1})}\big).$$

This completes the proof of statement iii) of the theorem.
**Case $\mathbf{h = 1, 2n}$**. By Hodge duality we may restrict ourselves to the case $h = 1$. Again we write

$$\begin{aligned}\langle u,\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)} &= \langle u, \Delta_{\mathbb{H},h}\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)}\\ &= \langle u, \delta_c d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)} + \langle u, d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)}\\ &= \langle f, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)} + \langle g, \delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)}.\end{aligned} \tag{45}$$

In order to estimate the first term $\langle f, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)}$, we repeat verbatim the arguments above for the corresponding term in the case $h \neq n, n+1$. As for the second term, by Theorem 6.4, formula (32), and keeping in mind that $\delta_c$ is an operator of order 1 in the horizontal derivatives when acting on $E_0^1$ the quantity $\delta_c\mathcal{K}\phi$ can be written as a sum of terms such as

$$\phi_j * W_\ell\tilde{K}_{ij}, \quad \text{with } \ell = 1,\dots,2n.$$

On the other hand,

$$\langle g, \phi_j * W_\ell\tilde{K}_{ij}\rangle_{L^2(\mathbb{H}^n)} = \langle g * {}^{\mathrm{v}}(W_\ell\tilde{K}_{ij}), \phi_j\rangle_{L^2(\mathbb{H}^n)}$$

Notice the $W_\ell\tilde{K}_{ij}$'s and hence the ${}^{\mathrm{v}}(W_\ell\tilde{K}_{ij})$'s are kernels of type 1. Thus, by Theorem 6.10 in [17],

$$|\langle g, \phi_j * W_\ell\tilde{K}_{ij}\rangle_{L^2(\mathbb{H}^n)}| \le C\|g\|_{\mathcal{H}^1(\mathbb{H}^n)}\|\phi\|_{L^Q(\mathbb{H}^n,E_0^1)}.$$

Combining this estimate with the one in (43), we get eventually

$$|\langle u,\phi\rangle_{L^2(\mathbb{H}^n,E_0^1)}| \le C\big(\|f\|_{L^1(\mathbb{H}^n,E_0^2)} + \|g\|_{\mathcal{H}^1(\mathbb{H}^n)}\big)\|\phi\|_{L^Q(\mathbb{H}^n,E_0^1)},$$

and hence

$$\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n,E_0^1)} \le C\big(\|f\|_{L^1(\mathbb{H}^n,E_0^2)} + \|g\|_{\mathcal{H}^1(\mathbb{H}^n)}\big).$$

This completes the proof of statement ii) of the theorem.
**Case $\mathbf{h = n, n+1}$**. By Hodge duality we may restrict ourselves to the case $h = n$.

If $u, \phi \in E_0^n$ are smooth compactly supported forms, then we can write

$$\begin{aligned}\langle u,\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)} &= \langle u, \Delta_{\mathbb{H},n}\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)}\\ &= \langle u, (\delta_c d_c + (d_c\delta_c)^2)\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)}.\end{aligned} \tag{46}$$

Consider now the term

$$\langle u, \delta_c d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)} = \langle d_c u, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^{n+1})} = \langle f, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^{n+1})}.$$

Let us write $f := d_c u$. Again, $d_c f = 0$, and hence, as above, if $f = \sum_{\ell=1}^{\dim E_0^{n+1}} f_\ell \xi_\ell^{n+1}$, $d_c\mathcal{K}\phi = \sum_{\ell=1}^{\dim E_0^{n+1}} (d_c\mathcal{K}\phi)_\ell \xi_\ell^{n+1}$, and thus we can reduce ourselves to estimate

$$\langle f_\ell, (d_c\mathcal{K}\phi)_\ell\rangle_{L^2(\mathbb{H}^n)} \quad \text{for } \ell = 1,\dots,\dim E_0^{n+1}. \tag{47}$$

By Theorem 5.1, each component $f_\ell$ of $f$ can be written as

$$f_\ell = \sum_{I=1}^{\dim E_0^{n+2}} \sum_{i=1}^{2n} b_{i,I}^\ell G_{I,i},$$

where the $b_{i,I}^\ell$'s are real constants and for any $I = 1,\dots,\dim E_0^{n+2}$ the $G_{I,i}$'s are the components of an horizontal vector field

$$G_I = \sum_i G_{I,i} W_i$$

with

$$\sum_i W_i G_{I,i} = 0, \qquad I = 1,\dots,\dim E_0^{n+2}. \tag{48}$$

As in the previous cases, in order to estimate the terms of (47), we have to deal terms of the form

$$\langle G_{I,i}, (d_c\mathcal{K}\phi)_\ell\rangle_{L^2(\mathbb{H}^n)} = \langle G_I, \Phi\rangle_{L^2(\mathbb{H}^n,\mathfrak{h}_1)}, \tag{49}$$

where

$$\Phi = (d_c\mathcal{K}\phi)_\ell W_i.$$

We can apply Theorem 4.2. Again keeping in mind (13), we obtain

$$\big|\langle f_\ell, (d_c\mathcal{K}\phi)_\ell\rangle_{L^2(\mathbb{H}^n)}\big| \le C\|f\|_{L^1(\mathbb{H}^n,E_0^{n+1})}\|\nabla_{\mathbb{H}} d_c\mathcal{K}\phi\|_{L^Q(\mathbb{H}^n,E_0^{n+1})}. \tag{50}$$

On the other hand, $\nabla_{\mathbb{H}} d_c\mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\phi_j * W^I \tilde K_{ij} \quad \text{with } d(I) = 3,$$

since the differential $d_c$ on $n$-forms has order 2 in the horizontal derivatives. By Theorem 6.4, iv) and Proposition 6.2, ii) $W^I\tilde K_{ij}$ are kernels of type 1, so that, by [16], Proposition 1.11 we have

$$|\langle f_\ell, (d_c\mathcal{K}\phi)_\ell\rangle_{L^2(\mathbb{H}^n)}| \le C\|f\|_{L^1(\mathbb{H}^n,E_0^{n+1})}\|\phi\|_{L^{Q/2}(\mathbb{H}^n,E_0^n)}. \tag{51}$$

The same argument can be carried out for all the components of $f$, yielding

$$|\langle f, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^{n+1})}| \le C\|f\|_{L^1(\mathbb{H}^n,E_0^{n+1})}\|\phi\|_{L^{Q/2}(\mathbb{H}^n,E_0^n)}. \tag{52}$$

Consider now the second term in (46). We have

$$\begin{aligned}\langle u, (d_c\delta_c)^2\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)} &= \langle d_c\delta_c u, d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)}\\ &= \langle d_c g, d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)}.\end{aligned}$$

We notice now that $d_c g$ is a $d_c$-closed form in $E_0^n$, and then we can repeat the arguments leading to (42) for $f$ in the case $h = n-1$, obtaining

$$\big|\langle d_c g, d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n,E_0^n)}\big| \le \|d_c g\|_{L^1(\mathbb{H}^n,E_0^n)}\|\nabla_{\mathbb{H}} d_c\delta_c\mathcal{K}\phi\|_{L^Q(\mathbb{H}^n,E_0^n)} \tag{53}$$

As above, $\nabla_{\mathbb{H}} d_c \delta_c \mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\phi_j * W^I \tilde{K}_{ij}, \quad \text{with } d(I) = 3,$$

since $\delta_c : E_0^n \to E_0^{n-1}$ is an operator of order 1 in the horizontal derivatives, as well as $d_c : E_0^{n-1} \to E_0^n$. By Theorem 6.4, iv) and Proposition 6.2, ii) $W^I \tilde{K}_{ij}$ are kernels of type 1, so that, by [16], Proposition 1.11 we have

$$\left|\langle d_c g, d_c \delta_c \mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^n)}\right| \leq C\|d_c g\|_{L^1(\mathbb{H}^n, E_0^n)}\|\phi\|_{L^{Q/2}(\mathbb{H}^n, E_0^n)}.$$

Combining this estimate with the one in (52), we get eventually

$$|\langle u, \phi\rangle_{L^2(\mathbb{H}^n, E_0^n)}| \leq C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})} + \|d_c g\|_{L^1(\mathbb{H}^n, E_0^n)}\big)\|\phi\|_{L^{Q/2}(\mathbb{H}^n, E_0^n)},$$

and hence

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n)} \leq C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})} + \|d_c g\|_{L^1(\mathbb{H}^n, E_0^n)}\big).$$

To achieve the proof of statement iv) of the theorem we have to consider separately the cases $f = 0$ and $g = 0$. Suppose $h = n + 1$ (i.e. $g = 0$). The proof for $h = n$ (i.e. $f = 0$) follows by Hodge duality. In the case $h = n + 1$ identity (46) read as

$$\begin{aligned}\langle u, \phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})} &= \langle u, \Delta_{\mathbb{H},n+1}\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})} = \langle u, (\delta_c d_c)^2\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})} \\ &= \langle f, d_c\delta_c d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+2})}.\end{aligned}$$

Since $d_c f = 0$, by Theorem 5.1 we can apply Theorem 4.2, and we get

$$\begin{aligned}\left|\langle u, \phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})}\right| &\leq C\|f\|_{L^1(\mathbb{H}^n, E_0^{n+2})}\|\nabla_{\mathbb{H}} d_c\delta_c d_c\mathcal{K}\phi\|_{L^Q(\mathbb{H}^n, E_0^{n+2})} \\ &\leq C\|f\|_{L^1(\mathbb{H}^n, E_0^{n+2})}\|\phi\|_{L^Q(\mathbb{H}^n, E_0^{n+1})},\end{aligned}$$

by [16], Proposition 1.9, since $\nabla_{\mathbb{H}} d_c\delta_c d_c\mathcal{K}$ is a kernel of type 0. Then we can conclude by duality as of the proof of case iii), achieving the proof of statement iv) of the theorem that now is completely proved.

□

## 7. Final remarks

The estimates in Theorem 1.3 for $n$-forms and $(n+1)$-forms can be reformulated in the spirit of the estimates proved in [7]. To state our result, we must recall preliminarily few definitions of the function spaces we need for our results.

If $p, q \in [1, \infty]$, we define the space

$$L^{p,q}(\mathbb{H}^n) := L^p(\mathbb{H}^n) \cap L^q(\mathbb{H}^n)$$

endowed with the norm

$$\|u\|_{L^{p,q}(\mathbb{H}^n)} := (\|u\|^2_{L^p(\mathbb{H}^n)} + \|u\|^2_{L^q(\mathbb{H}^n)})^{1/2}.$$

We have:

- $L^{p,q}(\mathbb{H}^n)$ is a Banach space;
- $\mathcal{D}(\mathbb{H}^n)$ is dense in $L^{p,q}(\mathbb{H}^n)$.

Again if $p, q \in [1, \infty]$, we can endow the vector space $L^p(\mathbb{H}^n) + L^q(\mathbb{H}^n)$ with the norm

$$\|u\|_{L^p(\mathbb{H}^n)+L^q(\mathbb{H}^n)} := \inf\{(\|u_1\|^2_{L^p(\mathbb{H}^n)} + \|u_2\|^2_{L^q(\mathbb{H}^n)})^{1/2}; \\ u_1 \in L^p(\mathbb{H}^n), u_2 \in L^q(\mathbb{H}^n), u = u_1 + u_2\}.$$

We stress that $L^p(\mathbb{H}^n) + L^q(\mathbb{H}^n) \subset L^1_{\mathrm{loc}}(\mathbb{H}^n)$. Analogous spaces of forms can be defined in the usual way.

The following characterization of $(L^{p,q}(\mathbb{H}^n))^*$ can be proved by standard arguments of functional analysis.

**Proposition 7.1.** *If $p, q \in (1, \infty)$ and $p', q'$ are their conjugate exponents, then*

i) *if $u = u_1 + u_2 \in L^{p'}(\mathbb{H}^n) + L^{q'}(\mathbb{H}^n)$, with $u_1 \in L^{p'}(\mathbb{H}^n)$ and $u_2 \in L^{q'}(\mathbb{H}^n)$, then the map*

$$\phi \to \int_{\mathbb{H}^n} (u_1\phi + u_2\phi)\, dp \quad \text{for } \phi \in L^{p,q}(\mathbb{H}^n)$$

*belongs to $(L^{p,q}(\mathbb{H}^n))^*$ and $\|u\|_{L^{p'}(\mathbb{H}^n)+L^{q'}(\mathbb{H}^n)} \geq \|F\|$;*

ii) *if $u \in L^{p'}(\mathbb{H}^n) + L^{q'}(\mathbb{H}^n)$, then there exist $u_1 \in L^{p'}(\mathbb{H}^n)$ and $u_2 \in L^{q'}(\mathbb{H}^n)$ such that $u = u_1+u_2$ and $\|u\|_{L^{p'}(\mathbb{H}^n)+L^{q'}\mathbb{H}^n)} = (\|u_1\|^2_{L^p(\mathbb{H}^n)} + \|u_2\|^2_{L^q(\mathbb{H}^n)})^{1/2}$. Moreover the functional*

$$\phi \to F(\phi) := \int_{\mathbb{H}^n} (u_1\phi + u_2\phi)\, dp \quad \text{for } \phi \in L^{p,q}(\mathbb{H}^n)$$

*belongs to $(L^{p,q}(\mathbb{H}^n))^*$ and $\|F\| \approx \|u\|_{L^{p'}(\mathbb{H}^n)+L^{q'}(\mathbb{H}^n)}$.*

iii) *reciprocally, if $F \in (L^{p,q}(\mathbb{H}^n))^*$, then there exist $u_1 \in L^{p'}(\mathbb{H}^n)$ and $u_2 \in L^{q'}(\mathbb{H}^n)$ such that*

$$F(\phi) = \int_{\mathbb{H}^n} (u_1\phi + u_2\phi)\, dp \quad \text{for all } \phi \in L^{p,q}(\mathbb{H}^n).$$

*If we set $u := u_1+u_2 \in L^{p'}(\mathbb{H}^n)+L^{q'}(\mathbb{H}^n)$, then $\|u\|_{L^{p'}(\mathbb{H}^n)+L^{q'}\mathbb{H}^n)} = \|F\|$.*

iv) *if $u_0 \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$ satisfies*

$$\left| \int_{\mathbb{H}^n} u_0\phi\; dp \right| \leq C\|\phi\|_{L^{p',q'}(\mathbb{H}^n)}$$

*for some $C > 0$ and for all $\phi \in \mathcal{D}(\mathbb{H}^n)$, then $u_0 \in L^{p'}(\mathbb{H}^n) + L^{q'}(\mathbb{H}^n)$ and*

$$\|u_0\|_{L^p(\mathbb{H}^n)+L^q(\mathbb{H}^n)} \leq C.$$

Using the function spaces defined above, we can reformulate Theorem 1.3 in the critical cases $h = n$ as follows. Since a similar formulations lacks in [4], we state and prove the theorem also of $n = 1$.

**Theorem 7.2.** *Denote by $(E_0^*, d_c)$ the Rumin's complex in $\mathbb{H}^n$, $n \geq 1$. Consider the system*

$$\begin{cases} d_c u = f \\ \delta_c u = g \end{cases}$$

*If $n \geq 2$, then there exists $C > 0$ such that*

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n)+L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n)} \leq C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})} + \|g\|_{L^1(\mathbb{H}^n, E_0^{n-1})}\big);$$

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n, E_0^{n+1})+L^{Q/(Q-1)}(\mathbb{H}^n, E_0^{n+1})} \leq C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^{n+2})} + \|g\|_{L^1(\mathbb{H}^n, E_0^{n})}\big),$$

*for any $u \in \mathcal{D}(\mathbb{H}^n, E_0^n)$ and for any $u \in \mathcal{D}(\mathbb{H}^n, E_0^{n+1})$, respectively.*

*If $n = 1$,*

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^1, E_0^1)+L^{Q/(Q-1)}(\mathbb{H}^1, E_0^1)} \leq C\big(\|f\|_{L^1(\mathbb{H}^1, E_0^2)} + \|g\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^0)}\big);$$

$$\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^1, E_0^2)+L^{Q/(Q-1)}(\mathbb{H}^1, E_0^2)} \leq C\big(\|f\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^3)} + \|g\|_{L^1(\mathbb{H}^1, E_0^1)}\big),$$

*for any $u \in \mathcal{D}(\mathbb{H}^1, E_0^1)$ and for any $u \in \mathcal{D}(\mathbb{H}^1, E_0^2)$, respectively.*

*Proof.* By Hodge duality we may restrict ourselves to the case $h = n$.

If $u, \phi \in E_0^n$ are smooth compactly supported forms, then we can write

$$\begin{aligned} \langle u, \phi\rangle_{L^2(\mathbb{H}^n, E_0^n)} &= \langle u, \Delta_{\mathbb{H},n}\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^n)} \\ &= \langle u, (\delta_c d_c + (d_c\delta_c)^2)\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^n)} \\ &= \langle u, \delta_c d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^n)} + \langle u, (d_c\delta_c)^2)\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^n)} \\ &= \langle d_c u, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})} + \langle \delta_c u, \delta_c d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n-1})} \\ &= \langle f, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})} + \langle g, \delta_c d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n-1})}. \end{aligned} \tag{54}$$

The estimate of the first term of the last line of (54) is already given in (52) and reads

$$|\langle f, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})}| \leq C\|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})}\|\phi\|_{L^{Q/2}(\mathbb{H}^n, E_0^n)}, \tag{55}$$

and, eventually,

$$\begin{aligned} &|\langle f, d_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+1})}| \\ &\quad \leq C\|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})}\left(\|\phi\|_{L^{Q/2}(\mathbb{H}^n, E_0^n)} + \|\phi\|_{L^{Q}(\mathbb{H}^n, E_0^n)}\right). \end{aligned} \tag{56}$$

Consider now the second term in the last line of (54). If $n \geq 2$, we have

$$\langle g, \delta_c d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n-1})} = \langle *g, *\delta_c d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+2})} \tag{57}$$

We notice now that $*g$ is a $d_c$-closed form in $E_0^{n+2}$. Then we can repeat the arguments of the proof of Theorem 1.3 and we get

$$\begin{aligned} &\left|\langle *g, *\delta_c d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n+2})}\right| \\ &\quad \leq \|g\|_{L^1(\mathbb{H}^n, E_0^{n-1})}\|\nabla_{\mathbb{H}}\delta_c d_c\delta_c\mathcal{K}\phi\|_{L^Q(\mathbb{H}^n, E_0^n)} \end{aligned} \tag{58}$$

As in the proof of Theorem 1.3, $\nabla_{\mathbb{H}}\delta_c d_c\delta_c\mathcal{K}\phi$ can be expressed as a sum of terms with components of the form

$$\phi_j * W^I\tilde{K}_{ij}, \quad \text{with } d(I) = 4,$$

since $\delta_c : E_0^n \to E_0^{n-1}$ is an operator of order 1 in the horizontal derivatives, as well as $d_c : E_0^{n-1} \to E_0^n$. By Theorem 6.4, iv) and Proposition 6.2, ii) $W^I\tilde{K}_{ij}$ are kernels of type 0, so that, keeping in mind (57), by Proposition 1.9 we have

$$\begin{aligned} &\left|\langle g, \delta_c d_c\delta_c\mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n-1})}\right| \\ &\quad \leq C\|g\|_{L^1(\mathbb{H}^n, E_0^{n-1})}\left(\|\phi\|_{L^{Q/2}(\mathbb{H}^n, E_0^n)} + \|\phi\|_{L^{Q}(\mathbb{H}^n, E_0^n)}\right). \end{aligned} \tag{59}$$

If $n = 1$, we write instead

$$\langle g, \delta_c d_c \delta_c \mathcal{K}\phi\rangle_{L^2(\mathbb{H}^1, E_0^0)} = \langle g * \tilde{K}, \phi\rangle_{L^2(\mathbb{H}^1, E_0^0)},$$

where $\tilde{K}$ is a kernel of type 1. By Hölder inequality and [17], Theorem 6.10,

$$\begin{aligned} &|\langle g * \tilde{K}, \phi\rangle_{L^2(\mathbb{H}^1, E_0^0)}| \\ &\le \|g * \tilde{K}\|_{L^{Q/(Q-1)}(\mathbb{H}^1, E_0^0)} \|\phi\|_{L^Q(\mathbb{H}^1, E_0^0)} \\ &\le \|g\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^0)} \|\phi\|_{L^Q(\mathbb{H}^1, E_0^0)}. \end{aligned}$$

This yields

(60)

$$\langle g, \delta_c d_c \delta_c \mathcal{K}\phi\rangle_{L^2(\mathbb{H}^1, E_0^0)} \le \|g\|_{\mathcal{H}^1(\mathbb{H}^1, E_0^0)} (\|\phi\|_{L^Q(\mathbb{H}^1, E_0^0)}\| + \|\phi\|_{L^{Q/2}(\mathbb{H}^1, E_0^0)}).$$

To conclude the proof, if $n > 1$, combining (59) with (56) or (60), we get eventually

$$\begin{aligned} \left|\langle g, \delta_c d_c \delta_c \mathcal{K}\phi\rangle_{L^2(\mathbb{H}^n, E_0^{n-1})}\right| &\le C \left( \|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})} + \|g\|_{L^1(\mathbb{H}^n, E_0^{n-1})} \right) \\ &\cdot \left( \|\phi\|_{L^{Q/2}(\mathbb{H}^n, E_0^n)} + \|\phi\|_{L^Q(\mathbb{H}^n, E_0^n)} \right). \end{aligned} \tag{61}$$

If $n = 1$ the same estimate holds with $\|g\|_{L^1}$ replaced with $\|g\|_{\mathcal{H}^1}$.

Indeed, if $n > 1$ and we replace (56) and (61) in (54), we obtain by duality (Proposition 7.1 - iv) )

$$\begin{aligned} &\|u\|_{L^{Q/(Q-2)}(\mathbb{H}^n, E_0^n) + L^{Q/(Q-1)}(\mathbb{H}^n, E_0^n)} \\ &\qquad \le C\big(\|f\|_{L^1(\mathbb{H}^n, E_0^{n+1})} + \|g\|_{L^1(\mathbb{H}^n, E_0^{n-1})}\big). \end{aligned}$$

Again, $\|g\|_{L^1}$ must be replaced by $\|g\|_{\mathcal{H}^1}$ if $n = 1$. This completes the proof of the theorem.

□

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