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DISCRETE SPIN SYSTEMS ON RANDOM LATTICES AT THE BULK SCALING

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Abstract. We study by Γ-convergence the stochastic homogenization of discrete energies on a class of random lattices as the lattice spacing vanishes. We consider general bounded spin systems at the bulk scaling and prove a homogenization result for stationary lattices. In the ergodic case we obtain a deterministic limit.

1. Introduction.

In this paper we study the discrete-to-continuum limit of bounded spin systems defined on stochastic lattices at the bulk scaling. In particular we focus on the properties of these discrete systems at zero temperature where the abstract methods of Γ-convergence have already proved to be an effective tool for the asymptotic analysis (see [2], [3], [4], [5], [10], [11]). More specifically we consider systems parameterized over the points of a lattice $\mathcal{L} \subset (\mathbb{R}^d)^{\mathbb{Z}^d}$ that we see as the outcome of a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, namely $\omega \in \Omega \mapsto \mathcal{L}(\omega) = \{\mathcal{L}(\omega)(i)\}_{i\in\mathbb{Z}^d}$. On the geometry of the lattices we make two assumptions preventing arbitrarily big empty regions and cluster of points. We then scale the lattices by a small parameter ε that will be eventually sent to 0 in the continuum limit. On these admissible lattices we define a spin field $u : \varepsilon\mathcal{L}(\omega) \to K$ where $K \subset \mathbb{R}^m$ is a bounded set and consider non negative energies of the form

$$F_\varepsilon(u) = \sum_{\varepsilon x, \varepsilon y \in D} \varepsilon^d f_\varepsilon(x, y, u(\varepsilon x), u(\varepsilon y)). \tag{1}$$

Under standard coercivity, growth and long-range decay conditions on the densities f_ε we prove a compactness and integral representation result for a Γ-limit of F_ε. In this analysis the assumptions on the geometry of the lattice play an important role. In particular they allow us to regard our spin system as defined on a periodic lattice and to take advantage of the results in [5], where an analogous system on a cubic lattice was considered, to show that the limit functional fulfills the assumption of an integral representation theorem proved in [14]. As a result the compactness and integral representation Theorem 3.2 follows. In Theorem 3.4 we extend the Γ-convergence result for the functionals F_ε to the case when the spin field is constrained to realize a certain prescribed mean value. As a corollary we obtain the convergence of minimum problems under mean value constraints which turns out to be a key element in order to prove a stochastic homogenization theorem for spin

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systems on stationary random lattices. It is at this point that we need to combine the abstract methods of Γ-convergence with the sub-additive ergodic theorem in [1]. This argument has been first used in the framework of stochastic homogenization in [16] and in the context of discrete-to-continuum limits under bulk scaling in [5] for functionals defined on Sobolev spaces. Here we prove it in the context of Lebesgue spaces as a first result for the analysis of general spin systems at surface scaling (for the analysis of Ising systems on stochastic lattices at surface scaling, where the application of the ergodic theorem is more involved, we refer to [7]).

2. **Stochastic lattices and discrete energies.**

In this section we introduce the stochastic framework we will use in the rest of the paper.

2.1. **Basic notation.** Given $x \in \mathbb{R}^d$ we denote by $|x|$ the Euclidean norm on x. If $B \subset \mathbb{R}^d$ is a Borel set we denote by $|B|$ its Lebesgue measure. Given an open set $D \subset \mathbb{R}^d$ we denote by $\mathcal{B}(D)$ the Borel σ-algebra on D, by $\mathcal{A}(D)$ the class of all bounded open subsets of D and by $\mathcal{A}^R(D)$ the family of those sets in $\mathcal{A}(D)$ with Lipschitz boundary. We denote by $\dim_{\mathcal{H}}(\cdot)$ the Hausdorff dimension and by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a fixed probability space Ω with a complete σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$. In the proofs C denotes a generic constant that may change from line to line.

2.2. **Random networks.** The random lattices in terms of which we will define our discrete systems need to satisfy some geometric properties. These are listed below and characterize what we call admissible sets of points.

Definition 2.1 (admissible sets)**.** Given a countable set of points $\Sigma = \{x_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^n$, we say that Σ is admissible if

(i) there exists $R > 0$ such that $\inf_{z\in\mathbb{R}^d} \#(\Sigma \cap B(z, R)) \geq 1$ (i.e., arbitrarily big empty regions are forbidden),
(ii) there exists $r > 0$ such that $\inf\{|x - y|,\ x, y \in \Sigma,\quad x \neq y\} \geq r$ (i.e., clusters are forbidden).

To an admissible set of points we associate the Voronoi tessellation $(C(x))_{x\in\Sigma}$ defined by

$$C(x) := \{y \in \mathbb{R}^d :\ |x - y| \leq |\overline{x} - y|,\ \forall\, \overline{x} \in \Sigma\backslash\{x\}\}.$$

Two points in Σ are said to be nearest neighbors if the Voronoi cells having them as centers share a $(d-1)$-dimensional edge. Hence we define the set of nearest neighbors of Σ as

$$\mathcal{NN}(\Sigma) := \{(x, y) \in \Sigma^2 :\ \dim_{\mathcal{H}}(C(x) \cap C(y)) = d - 1\}. \tag{2}$$

The following two properties are straightforward:

Lemma 2.2 (geometric properties of admissible sets)**.** *Let Σ be an admissible set of points with constants r, R as in Definition 2.1. Then there exists a constants $M > 0$ depending only on r and R such that, for all $x \in \Sigma$,*

(i) $B_{\frac{r}{2}}(x) \subset C(x) \subset B_R(x)$,
(ii) $\#\{y \in \Sigma :\ C(x) \cap C(y) \neq \emptyset\} \leq M$.

By stochastic lattices we mean a random variable realizing almost surely sets which are admissible with respect to the same constants r, R in Definition 2.1. To describe the stochastic properties of these lattices we need some additional definition.

Definition 2.3 (group action)**.** Let $(\tau_z)_{z\in\mathbb{Z}^d}, \tau_z : \Omega \to \Omega$, be an additive group action on Ω. We say that it is measure preserving if

$$\mathbb{P}(\tau_z B) = \mathbb{P}(B) \quad \forall B \in \mathcal{F},\, z \in \mathbb{Z}^d.$$

If in addition, for all $B \in \mathcal{F}$ we have

$$(\tau_z(B) = B \quad \forall z \in \mathbb{Z}^d) \quad \Rightarrow \quad \mathbb{P}(B) \in \{0,1\},$$

then $(\tau_z)_{z\in\mathbb{Z}^d}$ is called ergodic.

Let us specify the assumptions on the random variable generating the network.

Definition 2.4 (stochastic lattice)**.** A random variable $\mathcal{L} : \Omega \to (\mathbb{R}^d)^{\mathbb{Z}^d}$, $\omega \mapsto \mathcal{L}(\omega) = \{\mathcal{L}(\omega)(i)\}_{i\in\mathbb{Z}^d}$ is called a stochastic lattice. We say that $\mathcal{L}$ is admissible if $\mathcal{L}(\omega)$ is admissible in the sense of Definition 2.1 and the constants r, R can be chosen independent of ω $\mathbb{P}$-almost surely. The stochastic lattice $\mathcal{L}$ is said to be *stationary* if there exists a measure preserving group action $(\tau_z)_{z\in\mathbb{Z}^d}$ on Ω such that, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\mathcal{L}(\tau_z\omega) = \mathcal{L}(\omega) + z. \tag{3}$$

If in addition $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then $\mathcal{L}$ is called *ergodic*.

Given a realization $\mathcal{L}(\omega)$ of a stochastic lattice, we denote by $\mathcal{NN}(\omega)$ the corresponding nearest neighbors.

Remark 1. We stress that our geometric assumptions on the random lattice rule out many examples of stationary point processes used in stochastic geometry, e.g. homogenous Poisson point processes.

We set $\mathcal{I} = \{[a,b) : a, b \in \mathbb{Z}^d, a \neq b\}$, where $[a,b) := \{x \in \mathbb{R}^d :\ a_i \leq x_i < b_i\ \forall i\}$ and we introduce the notion of discrete subadditive stochastic process.

Definition 2.5. A function $\mu : \mathcal{I} \to L^1(\Omega)$ is said to be a discrete subadditive stochastic process if the following properties hold $\mathbb{P}$-almost surely:

(i) for every $I \in \mathcal{I}$ and for every finite partition $(I_k)_{k\in K} \subset \mathcal{I}$ of I we have

$$\mu(I,\omega) \leq \sum_{k\in K} \mu(I_k,\omega).$$

(ii) $\inf\left\{ \frac{1}{|I|} \int_\Omega \mu(I,w)\, \mathrm{d}\mathbb{P}(\omega) :\ I \in \mathcal{I} \right\} > -\infty.$

Our homogenization results are based on the following pointwise ergodic theorem (see [1]), which has already been used in the pioneering paper [16] in the context of stochastic homogenization of elliptic integral functionals and in [6] in the case of discrete hyperelastic systems.

Theorem 2.6. *Let $\mu : \mathcal{I} \to L^1(\Omega)$ be a discrete subadditive stochastic process and let $I \in \mathcal{I}$. If μ is stationary with respect to a measure preserving group action $(\tau_z)_{z\in\mathbb{Z}^d}$; i.e.,*

$$\forall I \in \mathcal{I},\ \forall z \in \mathbb{Z}^j :\ \mu(I+z,\omega) = \mu(I,\tau_z\omega) \quad \textit{almost surely},$$

then there exists $\Phi : \Omega \to \mathbb{R}$ such that, for $\mathbb{P}$-almost every ω,

$$\lim_{k\to+\infty} \frac{\mu(kI,\omega)}{|kI|} = \Phi(\omega).$$

2.3. **Discrete spin-type energies.** Having settled the stochastic environment, we are now able to define our discrete spin energies on these stochastic lattices. We restrict ourselves to pairwise interaction energies, but it is worth noticing that the core of these results can be (modulo details) generalized to multi-body interactions as well. We let $\varepsilon > 0$ be a small parameter (describing the average distance between particles), $D \subset \mathcal{A}^R(\mathbb{R}^d)$ be a regular reference domain and let $\mathcal{L}(\omega)$ be admissible.

The energies we have in mind are defined on functions $u : \varepsilon\mathcal{L}(\omega) \to \mathbb{R}^m$. We consider the u as generalized spin fields parameterized over the ε lattice $\varepsilon\mathcal{L}(\omega)$, with ε eventually going to zero in what is usually referred to as a discrete-to-continuum limit. As our proofs rely on an integral representation result, as usual in this setting we need to define a localized version of the energies. Given $A \in \mathcal{A}^R(D)$, we consider

$$F_\varepsilon(\omega)(u, A) = F_{\varepsilon,nn}(u, A)(\omega) + F_{\varepsilon,lr}(u, A)(\omega), \tag{4}$$

where the nearest neighbor interactions and long-range interactions are respectively given by functionals of the form

$$\begin{aligned} F_{\varepsilon,nn}(u, A)(\omega) &= \sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\ \varepsilon x,\varepsilon y\in A}} \varepsilon^d f_{\varepsilon,nn}(x, y, u(\varepsilon x), u(\varepsilon y)),\\ F_{\varepsilon,lr}(u, A)(\omega) &= \sum_{\substack{(x,y)\notin\mathcal{NN}(\omega)\\ \varepsilon x,\varepsilon y\in A}} \varepsilon^d f_{\varepsilon,lr}(x, y, u(\varepsilon x), u(\varepsilon y)). \end{aligned}$$

Note that the random character of our energies depend on the underlying random geometry of the lattice but not in the interaction coefficients $f_{\varepsilon,nn}, f_{\varepsilon,lr}$, which are instead considered deterministic.

As usual in the discrete-to-continuum analysis, we embed the problems in a suitable Lebesgue space (depending on the scalings). In general, note that we can identify each function $u : \varepsilon\mathcal{L}(\omega) \to \mathbb{R}^m$ with its piecewise constant interpolation on the Voronoi cells of $\varepsilon\mathcal{L}(\omega)$. Therefore, given a bounded set $K \subset \mathbb{R}^m$ let us introduce the class

$$C_\varepsilon(\omega, K) := \{u : \mathbb{R}^d \to K : \; u_{|C(\varepsilon x)} = \text{const.} \quad \forall \varepsilon x \in \varepsilon\mathcal{L}(\omega)\}.$$

Since the intersection of two Voronoi cells has zero Lebesgue-measure, this class is well defined and $C_\varepsilon(\omega, K) \subset L^p(D)$ for every $1 \le p \le +\infty$. Without specifying the exponent p for the moment we may define $F_\varepsilon(u, A)(\omega) : L^1(D) \to (-\infty, +\infty]$ as

$$F_\varepsilon(\omega)(u, A) = \begin{cases} F_\varepsilon(\omega)(u, A) & \text{if } u \in C_\varepsilon(\omega, K),\\ +\infty & \text{otherwise.}\end{cases} \tag{5}$$

Following some ideas in [6], we now introduce an auxiliary deterministic square lattice on which we will conveniently rewrite the energies F_ε. This lattice will turn out to be useful in order to provide uniform (with respect to the stochastic variable) estimates on the discrete energies.

On setting $r' = \frac{r}{\sqrt{d}}$ it follows that for all $\alpha \in r'\mathbb{Z}^d$ it holds $\#\{\mathcal{L}(\omega) \cap \{\alpha + [0, r')^d\}\} \le 1$. We now set

$$\begin{aligned} \mathcal{Z}_{r'}(\omega) &:= \{\alpha \in r'\mathbb{Z}^d : \; \#\left(\mathcal{L}(\omega) \cap \{\alpha + [0, r')^d\}\right) = 1\},\\ x_\alpha &:= \mathcal{L}(\omega) \cap \{\alpha + [0, r')^d\}, \quad \alpha \in \mathcal{Z}_{r'}(\omega) \end{aligned}$$

and, for $\xi \in r'\mathbb{Z}^d$, $U \subset \mathbb{R}^d$ and $\varepsilon > 0$,

$$R^{\xi}_{nn,\varepsilon}(U) := \{\alpha : \ \alpha, \alpha+\xi \in \mathcal{Z}_{r'}(\omega),\, \varepsilon x_\alpha, \varepsilon x_{\alpha+\xi} \in U,\, (x_\alpha, x_{\alpha+\xi}) \in \mathcal{NN}(\omega)\},$$
$$R^{\xi}_{lr,\varepsilon}(U) := \{\alpha : \ \alpha, \alpha+\xi \in \mathcal{Z}_{r'}(\omega),\, \varepsilon x_\alpha, \varepsilon x_{\alpha+\xi} \in U,\, (x_\alpha, x_{\alpha+\xi}) \notin \mathcal{NN}(\omega)\}.$$

For fixed $\xi \in r'\mathbb{Z}^d$ we can now rewrite the energy contribution using

$$F^{\xi}_{nn,\varepsilon}(\omega)(u,A) = \sum_{\alpha \in R^{\xi}_{nn,\varepsilon}(A)} \varepsilon^d f_{\varepsilon,nn}\left(x_\alpha, x_{\alpha+\xi}, u(x_\alpha), u(\varepsilon x_{\alpha+\xi})\right),$$
$$F^{\xi}_{lr,\varepsilon}(\omega)(u,A) = \sum_{\alpha \in R^{\xi}_{lr,\varepsilon}(A)} \varepsilon^d f_{\varepsilon,lr}\left(x_\alpha, x_{\alpha+\xi}, u(\varepsilon x_\alpha), u(\varepsilon x_{\alpha+\xi})\right).$$

3. **The bulk scaling.** This section deals with the case, where $f_{\varepsilon,nn}$ and $f_{\varepsilon,lr}$ are of order 1 with respect to ε. In particular, in contrast to [6], there is no gradient structure in the energy. As a result, the variational limit of the energies $F_\varepsilon(\cdot)(\omega)$ is given by volume inegrals of the form $\int_D f(x,u(x))\,\mathrm{d}x$. We divide this section into three paragraphs. At first we show how to derive an integral representation result for the variational limit. Then we prove the convergence of minimum problems under mean-value constraints. Finally we use the first two results to prove a stochastic homogenization theorem under the assumption of stationarity for the random lattices

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In this paragraph, for a fixed bounded set K we consider positive interaction energies fulfilling the following assumptions:

Hypthesis 1 There exist $C > 0$ and a decreasing function $J_{lr} : [0,+\infty) \to [0,+\infty)$ with

$$\int_{\mathbb{R}^d} J_{lr}(|x|)\,\mathrm{d}x = J < +\infty \tag{6}$$

such that

$$0 \le f_{\varepsilon,nn}(\cdot,\cdot,u,v) \le C \quad \text{for all } (u,v) \in K \times K,$$
$$0 \le f_{\varepsilon,lr}(x,y,u,v) \le J_{lr}(|x-y|) \quad \text{for all } (u,v) \in K \times K.$$

The next proposition, whose simple proof we omit, suggests to conveniently embed $C_\varepsilon(\omega,K) \subset L^\infty(D,\overline{co(K)})$ and equip $L^\infty(D,\overline{co(K)})$ with the weak*-topology so that $L^\infty(D,\overline{co(K)})$ is a separable metric space.

Proposition 1. *Let $u_\varepsilon \in C_\varepsilon(\omega,K)$ be such that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $L^\infty(D)$. Then $u \in L^\infty(D,\overline{co(K)})$. Conversly, for every $u \in L^\infty(D,\overline{co(K)})$ there exists a sequence $u_\varepsilon \in C_\varepsilon(\omega,K)$ such that $u_\varepsilon \overset{*}{\rightharpoonup} u$.*

3.1. **Integral representation.** As in [5] the key ingredient is the following integral representation theorem (see [14]).

Theorem 3.1. *Let $p \in [1,+\infty)$ and let $F : L^p(D,\mathbb{R}^m) \times \mathcal{B}(D) \to [0,+\infty]$ be a functional satisfying:*

(i) F is local on $\mathcal{B}(D)$; i.e., for all $u,v \in L^p(D,\mathbb{R}^m)$ and $B \in \mathcal{B}(D)$ such that $u = v$ a.e. on B, then $F(u,B) = F(v,B$,

(ii) F is additive; i.e., for all $u \in L^p(D,\mathbb{R}^m)$ and $B_1, B_2 \in \mathcal{B}(D)$ such that $B_1 \cap B_2 = \emptyset$, then $F(u, B_1 \cup B_2) = F(u,B_1) + F(u,B_2)$,

(iii) *there exists $u_0 \in L^p(D,\mathbb{R}^m)$ such that $F(u_0,\cdot)$ is a Borel measure on $\mathcal{B}(D)$ which is absolutely continuous with respect to the d-dimensional Lebesgue measure,*

(iv) *for all $B \in \mathcal{B}(D)$ the functional $F(\cdot,B)$ is lower semicontinuous with respect to the strong (respectively weak) convergence of $L^p(D,\mathbb{R}^m)$.*

Then there exists a positive measurable function $f : D \times \mathbb{R}^m \to [0,+\infty]$, with the property that $f(x,\cdot)$ is lower semicontinuous (respectively convex and lower semicontinuous) for a.e. $x \in D$, such that

$$F(u,B) = \int_B f(x,u(x))\,\mathrm{d}x$$

for all $u \in L^p(D,\mathbb{R}^m)$ and $B \in \mathcal{B}(D)$.
If in addition there exists $\alpha \in L^1(D,\mathbb{R}^m)$, c, $C > 0$ such that

$$c\|u\|^p_{L^p(B)} - \|\alpha\|_{L^1(B)} \leq F(u,B) \leq C\|u\|^p_{L^p(B)} + \|\alpha\|_{L^1(B)},$$

then f is a Carathéodory function satisfying

$$c|z|^p - \alpha(x) \leq f(x,z) \leq C|z|^p + \alpha(x) \quad \textit{for all } z \in \mathbb{R}^m \textit{ and a.e. } x \in D.$$

Remark 2. (i) If $F(\cdot,A)$ is lower semicontinuous with respect to weak or strong convergence in $L^p(D,\mathbb{R}^m)$ for every open set and the additional growth hypothesis in Theorem 3.1 holds, then it is enough to prove the locality on open sets.
(ii) Given locality and additivity, it sufficies to prove lower semicontinuity on open sets.

For $u \in L^\infty(D,\overline{co(K)})$, $A \in \mathcal{A}^R(D)$ let us introduce

$$F'(\omega)(u,A) := \inf\,\{\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,A) : u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(D,\mathbb{R}^m)\},$$

$$F''(\omega)(u,A) := \inf\,\{\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,A) : u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(D,\mathbb{R}^m)\},$$

i.e. the Γ-$\liminf$ and the Γ-$\limsup$ of the energies $F_\varepsilon(\cdot,A)(\omega)$ with respect to the weak*-topology on $L^\infty(D,\overline{co(K)})$ (we refer to [9] for an introduction to Γ-convergence). With the help of the auxiliary deterministic square lattice introduced in Section 2.3, one can repeat the arguments in [5] to prove the following proposition.

Proposition 2. *There exists a constant $C > 0$ such that for almost every $\omega \in \Omega$*

(i) $0 \leq F'(u,A)(\omega) \leq F''(u,A)(\omega) \leq C|A|$,

(ii) *$F'(\cdot,A)(\omega)$ and $F''(\cdot,A)(\omega)$ are weak* lower semicontinuous,*

(iii) $F''(u,A) = \sup\{F''(u,A')(\omega) :\ A' \subset\subset A\}$,

(iv) $F''(u,A\cup B)(\omega) \leq F''(u,A)(\omega) + F''(u,B)(\omega)$

(v) *If $A \cap B = \emptyset$, then $F'(u,A\cup B)(\omega) \geq F'(u,A)(\omega) + F'(u,B)(\omega)$,*

(vi) *If $u,v \in L^\infty(D,\overline{co(K)})$ and $u = v$ a.e. in A, then $F''(u,A)(\omega) = F''(v,A)(\omega)$.*

From Proposition 2 we deduce our first main theorem.

Theorem 3.2. *Let $\mathcal{L}$ be an admissible stochastic lattice and let $f_{\varepsilon,nn}$ and $f_{\varepsilon,lr}$ satisfy Hypothesis 1. For $\mathbb{P}$-almost every ω and every sequence (ε_j) converging to 0, there exists a (not relabelled) subsequence (ε_j) such that $F_{\varepsilon_j}(\omega)$ Γ-converge with respect to the weak* $L^\infty(D,\overline{co(K)})$-topology to the functional $F(\omega) : L^\infty(D,\overline{co(K)}) \to [0,+\infty)$ defined by*

$$F(\omega)(u) = \int_D f(\omega;x,u(x))\,\mathrm{d}x, \tag{7}$$

where $f(\omega;\cdot,\cdot) : D \times \overline{co(K)} \to [0,+\infty)$ *is a measurable function such that* $f(\omega;x,\cdot)$ *is convex and lower semicontinuous for a.e.* $x \in D$.
A local version of the theorem holds. For all $A \in \mathcal{A}^R(D)$ *and all* $u \in L^\infty(D,\overline{co(K)})$ *we have*

$$\Gamma - \lim_j F_{\varepsilon_j}(\omega)(u,A) = \int_A f(\omega;x,u(x))\,\mathrm{d}x.$$

Proof. In order to apply Theorem 3.1, we need to define an appropriate functional on $L^p(D)$ for some $p \in [1,+\infty)$. This can be done as follows: by compactness of Γ-convergence on separable metric spaces (see Proposition 1.42 in [9]) we can construct a diagonal sequence such that there exists a Γ-limit $\tilde{F}(\omega)(u,A)$ of $F_\varepsilon(\omega)(u,A)$ for all $(u,A) \in L^\infty(D,\overline{co(K)}) \times \mathcal{A}^R(D)$ (in order to pass from countably many sets A to $\mathcal{A}^R(D)$ one can argue as in the proof of Theorem 10.3 in [12]). We extend this limit to all open sets by

$$F(\omega)(u,V) := \sup\{\tilde{F}(\omega)(u,A) : \; A \in \mathcal{A}^R(D),\, A \subset\subset V\}.$$

Refering to Proposition 2 we can apply the De Giorgi-Letta-criterion (see e.g. Theorem 1.62 in [17]) and deduce that $F(\omega)(u,\cdot)$ is the trace of a Borel measure that is absolutely continuous with respect to the Lebesgue measure (Proposition 2 (i)), so we can extend $F(\omega)(u,\cdot)$ to all Borel sets $B \in \mathcal{B}(D)$. Let us denote by $p_K : \mathbb{R}^m \to \overline{co(K)}$ the projection map onto the compact, convex set $\overline{co(K)}$. Consider the functional $F_K(\omega)(\cdot,B) : L^1(D) \to [0,+\infty)$ defined by

$$F_K(\omega)(u,B) = F(w)(p_K \circ u, B).$$

One can show that $F_K(\omega)(u,B)$ satisfies the assumptions of Theorem 3.1 with respect to strong convergence in $L^1(D)$. The convexity and lower semicontinuity of the integrand follows from well known properties of weak* lower semicontinuous integral functionals and the local version is a direct consequence of the above construction. □

Remark 3. In general the integrand $f(\omega;x,\cdot)$ in Theorem 3.2 is not continuous up to the boundary of $\overline{co(K)}$. A counter example can be constructed easily using the discontinuous, convex function on the unit ball given by the first lemma in [18]. However $f(\omega;x,\cdot)$ is continuous on one-dimensional segments contained in $\overline{co(K)}$. Thus we can fully characterize it by its values on the relative interior of $\overline{co(K)}$. In particular only countably many of these values are needed since the function is continuous on the relative interior of $\overline{co(K)}$. In the case $\overline{co(K)}$ is a convex polytope, then the results given in [18] show that continuity holds.

3.2. **Convergence of minimum problems.** In order to prove a (stochastic) homogenization result we will use a blow-up argument to characterize the limit integrands. To relate the blow-up to our discrete energy functionals we need to prove a result regarding the convergence of minimum problems under average constraint. We first introduce a notion of discrete mean value.

Definition 3.3. For $A \in \mathcal{A}^R(D)$, $\varepsilon > 0$, $\omega \in \Omega$ and $u \in C_\varepsilon(\omega,K)$ we set

$$\langle u \rangle_A^{\omega,\varepsilon} := \sum_{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap A} \frac{|C(\varepsilon x)|}{|A|}\, u(\varepsilon x).$$

Since the co-domain of our functions will be relaxed in the limit, we do not require our constrained functionals to be defined only on those functions fulfilling a precise mean value constraint. Instead we introduce a threshold $\delta>0$ and, for fixed $z\in\overline{co(K)}, \varepsilon>0$ and $\omega\in\Omega$ we define the functional $F_{\varepsilon}^{z,\delta}(\omega):L^{\infty}(D,\overline{co(K)})\times\mathcal{A}^R(D)\to[0,+\infty]$ as

$$F_{\varepsilon}^{z,\delta}(\omega)(u,A):=\begin{cases}F_{\varepsilon}(\omega)(u,A) & \text{if } \langle u\rangle_A^{\omega,\varepsilon}\in B_{\delta}(z),\\ +\infty & \text{otherwise.}\end{cases} \tag{8}$$

The followng convergence result holds true:

Theorem 3.4. *Let $\mathcal{L}$ be an admissible stochastic lattice and let $f_{\varepsilon,nn}$ and $f_{\varepsilon,lr}$ satisfy Hypothesis 1. Then, for $\mathbb{P}$-almost every ω, the following holds: for every sequence (ε_j) converging to 0, let the subsequence (ε_j) and the function f be as in Theorem 3.2. For every $z\in\overline{co(K)}$, $\delta>0$ and every $A\in\mathcal{A}^R(D)$ the functionals $F_{\varepsilon_j}^{z,\delta}(\omega)(\cdot,A)$ Γ-converge with respect to the weak* $L^{\infty}(D,\overline{co(K)})$-topology to the functional $F^{z,\delta}(\omega)(\cdot,A):L^{\infty}(D,\overline{co(K)})\to[0,+\infty]$ defined by*

$$F^{z,\delta}(\omega)(u,A)=\begin{cases}\int_A f(\omega;x,u(x))\,\mathrm{d}x & \text{if } \langle u\rangle_A\in\overline{B_{\delta}(z)},\\ +\infty & \text{otherwise.}\end{cases}$$

Proof. We start proving the lower bound inequality: if $u_j\overset{*}{\rightharpoonup}u$ in $L^{\infty}(D,\overline{co(K)})$ with equi-bounded energy, then we have $\langle u_j\rangle_A^{\omega,\varepsilon_j}\in B_{\delta}(z)$. Using Lemma 2.2 and the regularity of A it is straightforward to check that $\langle u\rangle_A\in\overline{B_{\delta}(z)}$. The claim follows from $F_{\varepsilon_j}^{z,\delta}(\omega)(u,A)\geq F_{\varepsilon_j}(\omega)(u,A)$ and Theorem 3.2.

To prove the upper bound, note that if $u\in L^{\infty}(D,\overline{co(K)})$ is such that $\langle u\rangle\in B_{\delta}(z)$ we can take the same recovery sequence as for the unconstrained functional and conclude. It remains the case when $\langle u\rangle_A\in\partial B_{\delta}(z)$. In this case we argue by approximation as follows. We have $\langle u\rangle_A=z+\delta\nu$ with $\nu\in S^{m-1}$. Then the set

$$A^1:=\{x\in A:\ (u(x)-z)^T\nu\geq 1\}$$

has positive measure. For $\eta>0$ small enough take $A_{\eta}\subset A^1$ such that $|A_{\eta}|\leq\eta$. We define $u_{\eta}\in L^{\infty}(D,\overline{co(K)})$ via

$$u_{\eta}(x)=\begin{cases}z & \text{if } x\in A_{\eta},\\ u(x) & \text{otherwise.}\end{cases}$$

We have, for η small enough,

$$|\langle u_{\eta}\rangle_A-z|^2=\left|\delta\nu-\frac{1}{|A|}\int_{A_{\eta}}u(x)-z\,\mathrm{d}x\right|^2\leq\left(\delta-\frac{|A_{\eta}|}{|A|}\right)^2+\left(\frac{C_K|A_{\eta}|}{|A|}\right)^2<\delta^2.$$

Since $F(\omega)(u_{\eta},A)\to F(\omega)(u,A)$ as $\eta\to 0$ we deduce the claim from lower semi-continuity. □

By standard arguments in the theory of Γ-convergence we obtain the following corollary about the convergence of minimum problems.

Corollary 1. *Under the assumptions of Theorem 3.4, for almost every ω, for $z\in\overline{co(K)}$, $\delta>0$ and $A\in\mathcal{A}^R(D)$, it holds that*

(i)

$$\lim_j\left(\inf_{u\in L^{\infty}(D,K)}F_{\varepsilon_j}^{z,\delta}(\omega)(u,A)\right)=\min_{u\in L^{\infty}(D,\overline{co(K)})}F^{z,\delta}(\omega)(u,A).$$

(ii) Moreover, if $(u_j)_j$ is a weakly converging sequence in $L^\infty(D,\overline{co(K)})$ such that*

$$F^{z,\delta}_{\varepsilon_j}(\omega)(u_j,A)=\inf_{u\in L^\infty(D,K)}F^{z,\delta}_{\varepsilon_j}(\omega)(u,A)+\mathcal{O}(1),$$

then its limit is a minimizer of $F^{z,\delta}(\omega)(\cdot,A)$.

3.3. **Stochastic homogenization.** This section contains the main result of the paper Theorem 3.7 about the stochastic homogenization of the bulk scaling of discrete (spin type) energies on random lattices.

As a first result needed in the proof of the stochastic homogenization Theorem 3.7 we need a lemma that ensures that we can recover the continuum limit energy density by a suitable blow-up procedure. We remark that our proof follows the same arguments of the one of Theorem 1 in [15]. At first let us recall the definition of *nicely shrinking sets*.

Definition 3.5. A familiy $(Q_\eta)_{\eta>0}$ of Borel sets shrinks nicely to $x\in\mathbb{R}^d$ if there exists a constand $c>0$ such that

$$Q_\eta\subset B_\eta(x),\qquad |Q_\eta|\geq c|B_\eta(x)|.$$

Lemma 3.6. *Let $f:D\times\overline{co(K)}\to\mathbb{R}$ be bounded function that is measurable in the first variable and convex and lower semicontinuous in the second one for almost every $x\in D$. Then there exists a null set $N\subset D$ such that*

$$f(x,z)=\lim_{\delta\to0}\lim_{\eta\to0}\frac{1}{|Q_\eta|}\inf\left\{\int_{Q_\eta}f(y,u(y))\,\mathrm{d}y:\ u\in L^\infty(D,\overline{co(K)}),\ \langle u\rangle_{Q_\eta}\in\overline{B_\delta(z)}\right\}$$

for all $x\in D\backslash N$, $z\in\overline{co(K)}$ and every familiy Q_η shrinking nicely to x.

Proof. To reduce notation we set

$$U_\delta(z):=\overline{B_\delta(z)}\cap\overline{co(K)}$$

and

$$m_\delta(z,Q_\eta):=\frac{1}{|Q_\eta|}\inf\left\{\int_{Q_\eta}f(y,u(y))\,\mathrm{d}y:\ u\in L^\infty(D,\overline{co(K)}),\ \langle u\rangle_{Q_\eta}\in\overline{B_\delta(z)}\right\}.$$

At first we prove the statement if f is Lipschitz continuous on $\overline{co(K)}$ uniformly in $x\in D$. Applying Jensen's inequality we have that

$$\frac{1}{|Q_\eta|}\inf\left\{\int_{Q_\eta}f(x,u(y))\,\mathrm{d}y:\ u\in L^\infty(D,\overline{co(K)}),\ \langle u\rangle_{Q_\eta}\in\overline{B_\delta(z)}\right\}=\inf_{v\in U_\delta(z)}f(x,v)$$

for almost every x. It follows that

$$\begin{aligned}\left|\inf_{v\in U_\delta(v)}f(x,v)-m_\delta(z,Q_\eta)\right|&\leq\frac{1}{|Q_\eta|}\sup_{u\in L^\infty(D,\overline{co(K)})}\left|\int_{Q_\eta}f(x,u(y))-f(y,u(y))\,\mathrm{d}y\right|\\&\leq\frac{1}{|Q_\eta|}\int_{Q_\eta}\sup_{q\in\overline{co(K)}}|f(x,q)-f(y,q)|\,\mathrm{d}y.\end{aligned}$$

From this estimate, using Lebesgue's differentiation theorem combined with a covering argument (see the proof of Proposition 1.1 in [15] for details) one concludes that there exists a null set $N\subset D$ such that

$$\inf_{v\in U_\delta(z)}f(x,v)=\lim_{\eta\to0}m_\delta(z,Q_\eta)\tag{9}$$

for all $x \in D\backslash N$, $z \in \overline{co(K)}$ and every family Q_η shrinking nicely to x. Letting $\delta \to 0$ in the above equality yields the claim.

For a general function f let N' be the null set such that $f(x,\cdot)$ it is not convex and lower semicontinuous. We define an decreasing sequence $f_n : D \times \overline{co(K)} \to \mathbb{R}$ by

$$f_n(x,z) = \sup_{q \in \overline{co(K)}} \{f(x,q) - n|z-q|\}.$$

It is well known that $f_n(x,\cdot)$ is Lipschitz continuous with constant n. By lower semicontinuity for $x \in D\backslash N'$ the supremum defining f_n can be taken over a countable dense set $G \subset \overline{co(K)}$ ensuring the measurability of $f_n(\cdot,z)$ for every $z \in \overline{co(K)}$. Using Lebesgue's differentiation theorem one can construct a set N_n of measure zero such that

$$\lim_{\eta\to 0} \frac{1}{|Q_\eta|} \int_{Q_\eta} |f_n(y,q) - f_n(x,q)| \,\mathrm{d}y = 0 \tag{10}$$

for all $x \in D\backslash N_n$, $q \in G$ and every familiy Q_η shrinking nicely to x. Using the uniform Lipschitz continuity of f_n it is not hard to verify that (10) holds for all $z \in \overline{co(K)}$. Set $N_- = N' \cup \bigcup_{n\geq 1} N_n$ and fix $x \in D\backslash N_-$, $z \in \overline{co(K)}$ and a familiy Q_η shrinking nicely to x. By monotonicity and (10) we have

$$\limsup_{\eta\to 0} m_\delta(z,Q_\eta) \leq \limsup_{\eta\to 0} m_\delta^n(z,Q_\eta) \tag{11}$$

$$\leq \inf_{v\in U_\delta(z)} \limsup_{\eta\to 0} \frac{1}{|Q_\eta|} \int_{Q_\eta} f_n(y,v)\,\mathrm{d}y \leq \inf_{v\in U_\delta(z)} f_n(x,v). \tag{12}$$

By Remark 3 it holds that

$$\inf_{v\in U_\delta(z)} f(x,v) = \inf_{v\in \overline{B_\delta(z)} \cap \mathrm{ri}_{\mathrm{aff}}(\overline{co(K)})} f(x,v),$$

where $\mathrm{ri}_{\mathrm{aff}}$ denotes the relative interior. Let $v \in \mathrm{ri}_{\mathrm{aff}}(\overline{co(K)})$. Since $f(x,\cdot)$ is continuous in v we have

$$\lim_{n\to+\infty} f_n(x,v) = f(x,v)$$

which yields by (11)

$$\limsup_{\eta\to 0} m_\delta(v,Q_\eta) \leq \inf_{v\in U_\delta(z)} f(x,v). \tag{13}$$

In order to get the converse inequality we define $g_n : D \times \overline{co(K)} \to \mathbb{R}$ as

$$g_n(x,z) = \inf_{q\in\overline{co(K)}} \{f(x,q) + n|z-q|\}.$$

This family is increasing, convex and Lipschitz continuous with constant n in z and, again by Remark 3 we can restrict the infimum over a countable dense set ensuring measurability in x. In this case, by lower semicontinuity, monotonicity and Proposition 1.27 in [9], we have

$$\lim_n g_n(x,z) = \Gamma\text{-}\lim_n g_n(x,z) = f(x,z).$$

for all $z \in \overline{co(K)}$. By the first part of the proof, for every g_n, we can find null sets $N_n' \subset D$ such that (9) holds for all $x \in D\backslash N_n'$, $z \in \overline{co(K)}$ and every family Q_η shrinking nicely to x. Since $g_n \leq f$ we deduce that

$$\inf_{v\in U_\delta(z)} g_n(x,v) = \lim_{\eta\to 0} m_\delta^n(z,Q_\eta) \leq \liminf_{\eta\to 0} m_\delta(z,Q_\eta).$$

Using the fundamental theorem of Γ-convergence we can pass to the limit in n which implies together with (13) that

$$\lim_{\eta\to 0} m_\delta(z,Q_\eta) = \inf_{v\in U_\delta(z)} f(x,v)$$

for all $x\in N := N_-\cup\bigcup_{n\geq 1} N'_n$, $z\in\overline{co(K)}$ and every family Q_η shrinking nicely to x. Letting $\delta\to 0$ and using again the fact that $f(x,\cdot)$ is lower semicontinuous yields the claim of the proposition. □

For a bounded set $A\subset\mathbb{R}^n$, $\rho>0$, $z\in\mathbb{R}^m$ and $\varepsilon>0$ we set

$$C_\varepsilon^{z,\delta}(\omega,A) := \{u\in C_\varepsilon(\omega,K):\ \langle u\rangle_A^{\omega,\varepsilon}\in B_\delta(z)\}.$$

In this section we consider spin-type systems of a particular form. Let us suppose that $f_{nn}, f_{lr}:\mathbb{R}^n\times K\times K\to[0,+\infty)$ are measurable in the first variable, upper semicontinuous in the remaining couple of variables and such that

$$f_{\varepsilon,nn}(x,y,u,v) = f_{nn}(y-x,u,v), \tag{14}$$

$$f_{\varepsilon,lr}(x,y,u,v) = f_{lr}(y-x,u,v). \tag{15}$$

Theorem 3.7. *Let $\mathcal{L}$ be a stationary admissible stochastic lattice and assume Hypothesis 1 holds with the additional upper semicontinuity and structure assumption in (14, 15). For $\mathbb{P}$-almost every $\omega\in\Omega$ and for all $z\in\overline{co(K)}$ there exists*

$$f_{hom}(\omega;z) := \lim_{\delta\to 0}\lim_{N\to+\infty}\frac{1}{N^d}\inf\left\{F_1(\omega)(u,(0,N)^d):\ u\in C_1^{z,\delta}(\omega,(0,N)^d)\right\}.$$

The functionals $F_\varepsilon(\omega)$ Γ-converge with respect to the weak $L^\infty(D,\overline{co(K)})$-topology to the functional $F_{hom}(\omega): L^\infty(D,\overline{co(K)})\to[0,+\infty)$ defined by*

$$F_{hom}(\omega)(u) = \int_D f_{hom}(\omega;u(x))\,\mathrm{d}x.$$

If $\mathcal{L}$ is ergodic, then $f_{hom}(\cdot,z)$ is constant almost surely and it is given by

$$f_{hom}(z) = \lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{N^d}\int_\Omega \inf\left\{F_1(\omega)(u,(0,N)^d):\ u\in C_1^{z,\delta}(\omega,(0,N)^d)\right\}\mathrm{d}\mathbb{P}(\omega).$$

Remark 4. The upper semicontinuity assumption can be dropped if K is at most countable. In fact it is only needed to prove measurability of the stochastic process defined in (21).

Proof of Theorem 3.7. Let $\varepsilon_j\to 0$. By Theorem 3.2, for almost every $\omega\in\Omega$ there exists a subsequence (not relabeled) such that

$$\Gamma\text{-}\lim_j F_{\varepsilon_j}(\omega)(u,A) = \int_A f(\omega;x,u(x))\,\mathrm{d}x$$

for all $A\in\mathcal{A}^R(D)$ and $u\in L^\infty(D,\overline{co(K)})$. We divide the proof into several steps.

Step 1 (characterization of the energy density)
Theorem 3.2 allows us to apply Lemma 3.6 to obtain a null set $N(\omega)\subset D$ depending only on ω such that

$$f(\omega;x_0,z) = \lim_{\delta\to 0}\lim_{\eta\to 0}\frac{1}{|Q_\eta|}\inf\left\{\int_{Q_\eta} f(\omega;x,u(x))\,\mathrm{d}x:\ \langle u\rangle_{Q_\eta}\in\overline{B_\delta(z)}\right\}$$

for all $x_0 \in D\backslash N$, $z \in \overline{co(K)}$ and a family of open cubes with rational vertices shrinking nicely to x_0. Using Corollary 1 and setting $t_j = \varepsilon_j^{-1}$ we deduce that

$$\begin{aligned} f(\omega; x_0, z) &= \lim_{\delta\to 0}\lim_{\eta\to 0}\lim_j \frac{1}{|Q_\eta|}\inf\left\{F_{\varepsilon_j}(\omega)(u, Q_\eta):\ u \in C^{z,\delta}_{\varepsilon_j}(\omega, Q_\eta)\right\} \\ &= \lim_{\delta\to 0}\lim_{\eta\to 0}\lim_j \frac{1}{|t_jQ_\eta|}\inf\left\{F_1(\omega)(u, t_jQ_\eta):\ u \in C^{z,\delta}_1(\omega, t_jQ_\eta)\right\}. \end{aligned}$$

Suppose we have shown that for every $\delta > 0$ there exists a function $\phi_{z,\delta}: \Omega \to \mathbb{R}$ such that

$$\phi_{z,\delta}(\omega) = \lim_{t\to+\infty}\frac{1}{|tQ|}\inf\left\{F_1(\omega)(u, tQ):\ u \in C^{z,\delta}_1(\omega, tQ)\right\} \tag{16}$$

almost surely for every open cube $Q \subset D$ with rational vertices. Then we can take the limit with respect to δ because $\phi_{z,\delta}$ is increasing when $\delta \to 0$. Taking a countable dense set $G \subset \overline{co(K)}$ we can construct a set Ω' of full probability such that for fixed $\omega \in \Omega'$ we have

$$f(\omega; x_0, z) = f_{\text{hom}}(\omega; z) \qquad \forall x_0 \in D\backslash N(\omega),\ z \in G. \tag{17}$$

Observing that $z \mapsto f_{\text{hom}}(\omega; z)$ is convex and lower semicontinuous, by Remark 3 we have that (17) holds for all $z \in \overline{co(K)}$. Thus the Γ-limit exists and the integrand is given by $f_{\text{hom}}(\omega; z)$. It remains to show (16).

Step 2 (truncation of the interaction range)
Let us fix $z \in \overline{co(K)}$ and $\delta > 0$. As a next step we limit the range of interactions as well as the possible positions of interacting points:

To this end, for $\xi \in r'\mathbb{Z}^d$ and $U \subset \mathbb{R}^d$, we set

$$I^\xi_{nn}(U) := \{\alpha \in R^\xi_{nn,1}(U):\ \exists \eta > 0:\ [B_\eta(x_\alpha)\cap U, B_\eta(x_{\alpha+\xi})\cap U] \subset U\}, \tag{18}$$

$$I^\xi_{lr}(U) := \{\alpha \in R^\xi_{lr,1}(U):\ \exists \eta > 0:\ [B_\eta(x_\alpha)\cap U, B_\eta(x_{\alpha+\xi})\cap U] \subset U\}, \tag{19}$$

where $[A, B] := \{tx + (1-t)y:\ x \in A, y \in B, 0 \le t \le 1\}$. Note that $I^\xi_{nn/lr}(U) = R^\xi_{nn/lr,1}(U)$ if U is convex. Moreover for $L \in \mathbb{N}\cup\{+\infty\}$ we define the (truncated) energy by

$$\begin{aligned} F_1^L(\omega)(v, U) := &\sum_{|\xi|<L}\sum_{\alpha\in I^\xi_{nn}(U)} f_{nn}\left(x_{\alpha+\xi} - x_\alpha, v(x_\alpha), v(x_{\alpha+\xi})\right) \\ &+ \sum_{|\xi|<L}\sum_{\alpha\in I^\xi_{lr}(U)} f_{lr}\left(x_{\alpha+\xi} - x_\alpha, v(x_\alpha), v(x_{\alpha+\xi})\right). \end{aligned}$$

Given a set $Q \subset \mathbb{R}^d$ we let

$$\mu^L(\omega; Q) := \inf\left\{F_1^L(\omega)(u, Q):\ u \in C^{z,\delta}_1(\omega, Q)\right\}.$$

By the decay assumptions on the long-range interactions of Hypothesis 1, it follows that (16) is proved if we show that for every $L \in \mathbb{N}$ there exists a function $\phi^L_{z,\delta}: \Omega \to \mathbb{R}$ with

$$\phi^L_{z,\delta}(\omega) = \lim_{t\to+\infty}\frac{1}{|tQ|}\inf\left\{F_1^L(\omega)(u, tQ):\ u \in C^{z,\delta}_1(\omega, tQ)\right\} \tag{20}$$

for every open cube $Q \subset D$ with rational vertices and all $\omega \in \Omega_L$, where Ω_L is a set of full measure. At this point note that if t_k is large enough, then there exists at least one function $u \in C^{z,\delta}_1(\omega, t_kQ)$ to test the two infimum problems. It remains to prove (20).

Step 3 (definition of the stochastic process)
As observed in the previous step, since z and δ are fixed there exists $n \in \mathbb{N}$ such that $C_1^{z,\delta}(\omega, nI) \neq \emptyset$ for all $I \in \mathcal{I}$. Let us define a discrete stochastic process $\tilde{\mu}^L : \mathcal{I} \to L^1(\Omega)$ by

$$\tilde{\mu}^L(I)(\omega) := \frac{\mu^L(\omega; nI)}{n^d} + \gamma\,\mathrm{Per}(nI, \mathbb{R}^d), \tag{21}$$

where $\gamma > 0$ is a suitable constant to be chosen. Testing the infimum problem defining μ^L with any function $u \in C_1^{z,\delta}(\omega, nI)$ we infer from Hypothesis 1 that there exists a constant $C > 0$ such that

$$\tilde{\mu}^L(I) \leq C|I| + \gamma\,\mathrm{Per}(nI, \mathbb{R}^d), \tag{22}$$

in particular $\tilde{\mu}^L(I) \in L^\infty(\Omega)$. The proof of the measurability is more involved and can be found in the appendix. For the proof of stationarity we can assume without loss of generality that $r' = \frac{1}{l}$ for some $l \in \mathbb{N}$. Then, using (3) it is straightforward to check that

$$\tilde{\mu}^L(I - z)(\omega) = \tilde{\mu}^L(I)(\tau_z \omega),$$

which yields stationarity. In order to establish subadditivity, let $\mathcal{J}$ be the set of finite unions of elements in $\mathcal{I}$ and let $I_1, I_2 \in \mathcal{J}$ be disjoint. Note that by the special structure of the sets $I_{mn}^{\xi}(U)$ and $I_{lr}^{\xi}(U)$ two lattice points $x_\alpha \in nI_1$ and $x_{\alpha+\xi} \in nI_2$ can interact only if the ray $[x_\alpha, x_{\alpha+\xi}]$ has length at most $L + r$ and intersects a $(d-1)$-dimensional side of $n(\overline{I_1} \cap \overline{I_2})$. Therefore one can argue as in Step 3 of the proof of Theorem 2 in [6] to show that the stochastic process is subadditive if we choose γ large enough. The fact that

$$\inf\left\{\frac{1}{|I|}\int_\Omega \tilde{\mu}^L(I)(\omega)\,\mathrm{d}\mathbb{P}(\omega) :\ I \in \mathcal{I}\right\} > -\infty$$

is trivial since the integrand is always positive.

Step 4 (proof of (20) and the case of ergodic lattices)
Applying Theorem 2.6 to the process $\tilde{\mu}^L(I)(\omega)$ we deduce that there exists a function $\phi_{z,\delta}^L : \Omega \to \mathbb{R}$ such that

$$\phi_{z,\delta}^L(\omega) = \lim_{k\to+\infty} \inf\left\{\frac{1}{|k\,Q|} F_1^L(\omega)(u, k\,Q) :\ u \in C_1^{z,\delta}(\omega, k\,Q)\right\} \tag{23}$$

for almost every $\omega \in \Omega$ and every $Q \in n\mathcal{I}$. The extension to arbitrary sequences and to all open cubes with rational vertices is straightforward since the differences are lower order terms. Finally note that the function $\phi_{z,\delta}^L(\omega)$ is invariant under the group action τ_z, thus ergodicity implies that it is constant as well as its limit as $L \to +\infty$. □

Appendix A. **.**

Lemma A.1. *For fixed $I \in \mathcal{I}$ the function $\tilde{\mu}^L(I)$ defined in (21) is measurable.*

Proof. Let us write $\mathbb{Z}^d = \{z_1, z_2, \dots\}$. By $\Sigma_{r,R}$ we denote the set of admissible point sets, where r, R are as in Definition 2.1. Since $\mathcal{F}$ is a complete sigma algebra we may assume that $\mathcal{L}(\omega)\in\Sigma_{r,R}$ for all $\omega\in\Omega$. Identifying every $u\in C_1(\omega,K)$ with a vector $u\in(K)^{\mathbb{Z}^d}$ we can write

$$\mu^L(\omega;I):=\inf_{\substack{u\in(K)^{\mathbb{Z}^n}\\ \sum_{i\geq 1}\frac{|C(\mathcal{L}(\omega)_{z_i})\cap I|}{|I|}u_{z_i}=z}}\left\{F_1^L(\omega)(v,I)\right\}.$$

Substep 1 At first we prove that, for fixed $i\in\mathbb{N}$, the mapping $\omega\mapsto|C(\mathcal{L}(\omega)_{z_i})|$ is measurable.
Given $\mathcal{L}(\omega)\in\Sigma_{r,R}$, there exists $M\in\mathbb{N}$ such that, for all $x=(x_{z_j})\in\Sigma_{r,R}$ with $\sup_j|\mathcal{L}(\omega)_{z_j}-x_{z_j}|\leq R$, we have

$$C(x_{z_i})=\{y\in\mathbb{R}^d:\ |y-x_{z_i}|\leq|y-x_{z_j}|\quad\forall j=1,\dots,M\}. \tag{24}$$

Indeed, let $M:=\max\{j\in\mathbb{N}:\ |\mathcal{L}(\omega)_{z_i}-\mathcal{L}(\omega)_{z_j}|\leq 6R\}$. If $y\in\mathbb{R}^d$ is such that $|y-x_{z_i}|\leq|y-x_{z_j}|$ for all $j\leq M$, we can estimate as follows ($k>M$):

$$\begin{aligned}|y-x_{z_k}|&\geq|\mathcal{L}(\omega)_{z_k}-\mathcal{L}(\omega)_{z_i}|-|\mathcal{L}(\omega)_{z_k}-x_{z_k}|-|x_{z_i}-\mathcal{L}(\omega)_{z_i}|-|y-x_{z_i}|\\&\geq 4R-|y-x_{z_i}|.\end{aligned}$$

Then (24) follows if we show that $|y-x_{z_i}|\leq 2R$. Assume by contradiction that this is not the case. Then, for $\overline{y}:=x_{z_i}+2R\frac{y-x_{z_i}}{|y-x_{z_i}|}$, there exists $j\leq M$, $j\neq i$ such that $|\overline{y}-x_{z_j}|\leq R$, where we have used that $x\in\Sigma_{r,R}$. We conclude that

$$\begin{aligned}|y-x_{z_j}|\leq R+|\overline{y}-y|&=R+\frac{|y-x_{z_i}|-2R}{|y-x_{z_i}|}|y-x_{z_i}|\\&=|y-x_{z_i}|-R<|y-x_{z_i}|,\end{aligned}$$

which leads to a contradiction. Using (24) a short arument based on dominated convergence shows that for every $\varepsilon>0$ there exists $\delta>0$ such that for all $x\in\Sigma_{r,R}$ with $\sup_j|\mathcal{L}(\omega)_{z_j}-x_{z_j}|\leq R$ the following implication holds:

$$\max_{j\leq M}|x_{z_j}-\mathcal{L}(\omega)_{z_j}|\leq\delta\quad\Rightarrow\quad||C(x_{z_i})|-|C(\mathcal{L}(\omega)_{z_i})||\leq\varepsilon.$$

Hence for an open set $U\subset\mathbb{R}$ the preimage of the mapping $\Sigma_{r,R}\ni x\mapsto|C(x_{z_i})|$ can be written as an a priori uncountable union of measurable sets of the form $\Sigma_{r,R}\cap\left(\prod_{j=1}^M B_\delta(x_{z_j})\times\prod_{j>M}B_R(x_{z_j})\right)$, where M and δ can vary. However, for fixed $M\in\mathbb{N}$, the corresponding union can be taken countable since $(\mathbb{R}^d)^M$ is second countable. Therefore the whole union can be written as a countable union of measurable sets. By the measurability of the mapping $\omega\mapsto\mathcal{L}(\omega)$ we have proven the claim of this substep.
Substep 2
From the previous substep we know that for fixed $u\in(K)^{\mathbb{Z}^d}$ the mapping

$$\omega\mapsto M(u)(\omega):=\begin{cases}0 & \text{if }\sum_{i\geq 1}\frac{|C(\mathcal{L}(\omega)_{z_i})|}{|nI|}u_{z_i}\mathbb{1}_{nI}(\mathcal{L}(\omega)_{z_i})\in B_\delta(z),\\ +\infty & \text{otherwise.}\end{cases}$$

is measurable. However the set over which we take the infimum in order to define $\mu^L(\omega;I)$ is not countable and still depends on ω. To overcome this issue we first fix $l\in\mathbb{N}$. Now take a countable dense subset G of K and pick one element $g_0\in G$. We set

$$v_{z_i}(g_1,\dots,g_l):=\begin{cases}g_i & \text{if } i\leq l,\\ g_0 & \text{otherwise,}\end{cases}$$

where $g_i \in G$. These are countably many vectors. Using upper semicontinuity of f_{nn} and f_{lr} with respect to the second two variables and the fact that we took the open ball $B_\delta(z)$ for the constraint we deduce by density that

$$\mu^L(nI)(\omega) = \lim_{l\to+\infty} \inf_{(g_1,\dots,g_l)\in G^l} \{F_1^L(\omega)(v(g_1,\dots,g_l),nI) + M(v(g_1,\dots,g_l))(\omega)\}$$

and we conclude the proof if we show that for fixed $u \in (K)^{\mathbb{Z}^d}$ the function $F_1^L(\omega)(u,nI)$ is measurable. This is shown in the appendix of [7]. □

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