

ON THE SHAPE OF COMPACT HYPERSURFACES WITH ALMOST CONSTANT MEAN CURVATURE

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ABSTRACT. The distance of an almost constant mean curvature boundary from a finite family of disjoint tangent balls with equal radii is quantitatively controlled in terms of the oscillation of the scalar mean curvature. This result allows one to quantitatively describe the geometry of volume-constrained stationary sets in capillarity problems.

1. INTRODUCTION

We investigate the geometry of compact boundaries with almost constant mean curvature in \mathbb{R}^{n+1} , $n \geq 2$. Beyond its intrinsic geometric interest, this problem is motivated by the description of equilibrium shapes (volume-constrained stationary sets) of the classical Gauss free energy used in capillarity theory [Fin86], and consisting of a dominating surface tension energy plus a potential energy term. Our analysis leads to new stability estimates describing in a quantitative way the distance of these shapes from compounds of tangent balls of equal radii.

1.1. Main result. Given a connected bounded open set $\Omega \subset \mathbb{R}^{n+1}$ ($n \geq 2$) with C^2 -boundary, we denote by H the scalar mean curvature of $\partial\Omega$ with respect to the outer unit normal ν_Ω to Ω (normalized so that $H = n$ if $\Omega = B = \{x \in \mathbb{R}^{n+1} : |x| < 1\}$), and we introduce the *Alexandrov's deficit* of Ω ,

$$\delta(\Omega) = \frac{\|H - H_0\|_{C^0(\partial\Omega)}}{H_0}, \quad \text{where} \quad H_0 = \frac{n P(\Omega)}{(n+1)|\Omega|}. \quad (1.1)$$

This is a scale invariant quantity (i.e. $\delta(\Omega) = \delta(\lambda\Omega)$ for every $\lambda > 0$) with the property that $\delta(\Omega) = 0$ if and only if Ω is a ball (Alexandrov's theorem). The motivation for the particular value of H_0 considered in the definition of $\delta(\Omega)$ is that if H is constant on $\partial\Omega$, then by the divergence theorem it must be $H = H_0$ (see (2.16) below). Here and in the following, \mathcal{H}^k stands for the k -dimensional Hausdorff measure on \mathbb{R}^{n+1} , $|\Omega|$ is the Lebesgue measure (volume) of Ω , and $P(E)$ is the distributional perimeter of a set of finite perimeter $E \subset \mathbb{R}^{n+1}$ (so that $P(E) = \mathcal{H}^n(\partial E)$ whenever E is an open set with Lipschitz boundary).

Motivated by applications to geometric variational problems (see section 1.2) we want to describe the shape of sets Ω with small Alexandrov's deficit. This is a classical question in convex geometry, where the size of $\delta(\Omega)$ for Ω convex has been related to the Hausdorff distance of $\partial\Omega$ from a single sphere in various works, see [Sch90, Arn93, Koh00]. However one should keep in mind that, as soon as convexity is dropped off, the smallness of $\delta(\Omega)$ does not necessarily imply proximity to a single ball. Indeed, by slightly perturbing a given number of spheres of equal radii connected by short catenoidal necks, one can construct open sets $\{\Omega_h\}_{h \in \mathbb{N}}$ with the property that, as $h \rightarrow \infty$, $\delta(\Omega_h) \rightarrow 0$, while the necks contract to points and the sets Ω_h converge to an array of tangent balls, see [But11, BM12] (and, more generally for this kind of construction, the seminal papers [Kap90, Kap91]). At the same time, if $\delta(\Omega)$ is small enough in terms of n and the largest principal curvature of $\partial\Omega$, then Ω must be close to a single ball. More precisely, denoting by A the second fundamental form of $\partial\Omega$, in [CV15] it is proved the existence of $c(n, \|A\|_{C^0(\partial\Omega)}) > 0$ such that if $\delta(\Omega) \leq c(n, \|A\|_{C^0(\partial\Omega)})$, then the in-radius and out-radius of

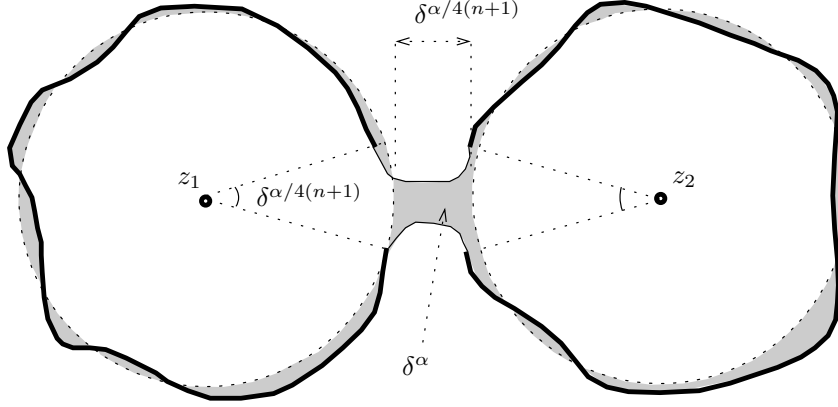


FIGURE 1. The situation in Theorem 1.1, with $\delta = \delta(\Omega)$ and $\alpha = 1/2(n+2)$. The grey region depicts $\Omega \Delta G$ (whose area is of order δ^α), while $(\text{Id} + \psi_G \nu_G)(\Sigma)$ is depicted by a bold line. The spheres $\partial B_{z_j, 1}$ are at a distance of order $\delta^{\alpha/4(n+1)}$, while Σ is obtained from ∂G by removing two spherical caps of diameter $\delta^{\alpha/4(n+1)}$.

Ω must satisfy

$$\frac{r^{\text{out}}(\Omega)}{r^{\text{in}}(\Omega)} - 1 \leq C(n, \|A\|_{C^0(\partial\Omega)}) \delta(\Omega), \quad (1.2)$$

where the linear control in terms of $\delta(\Omega)$ is sharp, as shown for example by taking a sequence of almost-round ellipsoids. In light of the examples from [But11], an assumption like $\delta(\Omega) \leq c(n, \|A\|_{C^0(\partial\Omega)})$ is necessary in order to expect Ω to be close to a single ball.

Our goal here is to address the situation when a different kind of smallness assumption on $\delta(\Omega)$ is considered. Indeed, we are just going to assume that $\delta(\Omega)$ is small with respect to the scale invariant quantity

$$Q(\Omega) = \frac{P(\Omega)^{n+1}}{(n+1)^{n+1} |\Omega|^n |B|} = \left(\frac{P(\Omega)}{P(B)} \right)^{n+1} \left(\frac{|B|}{|\Omega|} \right)^n.$$

Notice that by the Euclidean isoperimetric inequality

$$P(\Omega) \geq (n+1) |B|^{1/(n+1)} |\Omega|^{n/(n+1)} = P(B) \left(\frac{|\Omega|}{|B|} \right)^{n/(n+1)}, \quad (1.3)$$

one always has $Q(\Omega) \geq 1$, and that

$$Q(\text{a union of } L \text{ disjoint balls of equal radii}) = L, \quad \forall L \in \mathbb{N}, L \geq 1.$$

Hence, one may expect the integer part of $Q(\Omega)$ to indicate the number of balls of radius n/H_0 that should be approximating Ω : and indeed, given $L \in \mathbb{N}$, $L \geq 1$, and $a \in [0, 1)$, in Theorem 1.1 we are going to prove that if $Q(\Omega) \leq L + 1 - a$ (so that the normalized perimeter of Ω is a tad less than the normalized perimeter of $(L+1)$ -many balls) and $\delta(\Omega) \leq \delta(n, L, a)$, then Ω is close (in the various ways specified below, and quantitatively in terms of powers of $\delta(\Omega)$) to a compound of at most L -many mutually tangent balls of radius n/H_0 .

Before stating Theorem 1.1 it seems convenient to rescale Ω in such a way that the reference balls have unit radius, that is, we rescale Ω (as we can always do) in such a way that

$$H_0 = n \quad \text{and thus} \quad P(\Omega) = (n+1)|\Omega|, \quad Q(\Omega) = \frac{|\Omega|}{|B|} = \frac{P(\Omega)}{P(B)}.$$

Here and in the following we also set $B_{x,r} = \{y \in \mathbb{R}^{n+1} : |y-x| < r\}$ (so that $B = B_{0,1}$) and, given two compact sets K_1, K_2 in \mathbb{R}^{n+1} , we define their Hausdorff distance as

$$\text{hd}(K_1, K_2) = \max \left\{ \max_{x \in K_1} \text{dist}(x, K_2), \max_{x \in K_2} \text{dist}(x, K_1) \right\}.$$

Moreover, we let

$$\alpha = \frac{1}{2(n+2)}, \quad (1.4)$$

and we refer readers to the beginning of section 2 for our conventions about constants.

Theorem 1.1. *Given $n, L \in \mathbb{N}$ with $n \geq 2$ and $L \geq 1$, and $a \in (0, 1]$, there exists a positive constant $c(n, L, a) > 0$ with the following property. If Ω is a bounded connected open set with C^2 -boundary in \mathbb{R}^{n+1} such that $H > 0$ and*

$$H_0 = n, \quad P(\Omega) \leq (L+1-a)P(B), \quad \delta(\Omega) \leq c(n, L, a),$$

then there exists a finite family $\{B_{z_j, 1}\}_{j \in J}$ of mutually disjoint balls with $\#J \leq L$ such that if we set

$$G = \bigcup_{j \in J} B_{z_j, 1},$$

then

$$\frac{|\Omega \Delta G|}{|\Omega|} \leq C(n) L^2 \delta(\Omega)^\alpha, \quad (1.5)$$

$$\frac{|P(\Omega) - \#J P(B)|}{P(\Omega)} \leq C(n) L^2 \delta(\Omega)^\alpha, \quad (1.6)$$

$$\frac{\max_{x \in \partial G} \text{dist}(x, \partial\Omega)}{\text{diam}(\Omega)} \leq C(n) L \delta(\Omega)^\alpha, \quad (1.7)$$

$$\frac{\text{hd}(\partial\Omega, \partial G)}{\text{diam}(\Omega)} \leq C(n) L^{3/n} \delta(\Omega)^{\alpha/4n^2(n+1)}. \quad (1.8)$$

Moreover, there exists an open subset Σ of ∂G and a function $\psi : \Sigma \rightarrow \mathbb{R}$ with the following properties. The set $\partial G \setminus \Sigma$ consists of at most $C(n) L$ -many spherical caps whose diameters are bounded by $C(n) \delta(\Omega)^{\alpha/4(n+1)}$. The function ψ is such that $(\text{Id} + \psi \nu_G)(\Sigma) \subset \partial\Omega$ and

$$\|\psi\|_{C^{1,\gamma}(\Sigma)} \leq C(n, \gamma), \quad \forall \gamma \in (0, 1), \quad (1.9)$$

$$\frac{\|\psi\|_{C^0(\Sigma)}}{\text{diam}(\Omega)} \leq C(n) L \delta(\Omega)^\alpha, \quad \|\nabla \psi\|_{C^0(\Sigma)} \leq C(n) L^{2/n} \delta(\Omega)^{\alpha/8n(n+1)}, \quad (1.10)$$

$$\frac{\mathcal{H}^n(\partial\Omega \setminus (\text{Id} + \psi \nu_G)(\Sigma))}{P(\Omega)} \leq C(n) L^{4/n} \delta(\Omega)^{\alpha/4n(n+1)}, \quad (1.11)$$

where $(\text{Id} + \psi \nu_G)(x) = x + \psi(x) \nu_G(x)$ and ν_G is the outer unit normal to G . Finally:

- (i) if $\#J \geq 2$, then for each $j \in J$ there exists $\ell \in J$, $\ell \neq j$, such that

$$\frac{\text{dist}(\partial B_{z_j, 1}, \partial B_{z_\ell, 1})}{\text{diam}(\Omega)} \leq C(n) \delta(\Omega)^{\alpha/4(n+1)}, \quad (1.12)$$

that is to say, each ball in $\{B_{z_j, 1}\}_{j \in J}$ is close to be tangent to another ball from the family;

- (ii) if there exists $\kappa \in (0, 1)$ such that

$$|B_{x,r} \setminus \Omega| \geq \kappa |B| r^{n+1}, \quad \forall x \in \partial\Omega, r < \kappa, \quad (1.13)$$

and $\delta(\Omega) \leq c(n, L, \kappa)$, then $\#J = 1$, that is, Ω is close to a single ball.

A first consequence of Theorem 1.1 is that examples of the kind constructed in [But11] are actually the only possible examples of boundaries with almost-constant mean curvature which are not close to a single sphere. Conversely, the examples of [But11] show that Theorem 1.1 provides a qualitatively optimal information on sets with small Alexandrov's deficit. But of course, the strongest aspect of Theorem 1.1 is its quantitative nature. It is precisely this last feature which

is needed in order to obtain explicit (although arguably non-sharp) orders of magnitude in the description of capillarity droplets, see Proposition 1.2 below and the discussion after it.

A second remark is that, thanks to conclusion (i) and up to a translation of the balls $B_{z_j,1}$ of the order of $\delta(\Omega)$ appearing in (1.12), one can work with a reference configuration G such that ∂G is connected, that is to say, for every $j \in J$ one can assert that $\partial B_{z_j,1}$ is tangent to $\partial B_{z_\ell,1}$ for some $\ell \neq j$. Of course, in doing so, the various smallness estimates (1.5)–(1.11) will be of the same order of $\delta(\Omega)$ as in (1.12).

The use of the constant a should help to stress the “quantization” effect of the perimeter/energy (and of the volume) that happens under the small deficit assumption. Depending on the situation, one could be already satisfied of working with the simpler statement corresponding to the choice $a = 1$.

We also comment on assumption (ii). Our idea here is to provide more robust smallness criteria for proximity to a single ball than $\delta(\Omega) \leq \delta(n, \|A\|_{C^0(\partial\Omega)})$. The first criterion just amounts in asking that $P(\Omega) \leq (2 - a)P(B)$, for $a \in (0, 1]$. The interest of the second criterion is immediately understood if one considers that local minimizers of the capillarity energy satisfy uniform volume density estimates. Of course, a third criterion for proximity to a single ball is requiring the perimeter upper bound $P(\Omega) \leq 2 - a$ (which corresponds to taking $L = 1$).

We now illustrate the proof of Theorem 1.1. Our argument is based on Ros’ proof of Alexandrov’s theorem [Ros87], which follows closely the ideas of Reilly [Rei77], and is based on the following *Heintze-Karcher inequality* [HK78]: if Ω is a bounded connected open set with C^2 -boundary in \mathbb{R}^{n+1} with $H > 0$, then

$$\int_{\partial\Omega} \frac{n}{H} d\mathcal{H}^n \geq (n+1)|\Omega|. \quad (1.14)$$

Now, if H is constant, then it must be $H = H_0 = nP(\Omega)/(n+1)|\Omega|$, so that Ω must be an equality case in (1.14). By exploiting Reilly’s identity [Rei77], Ros proves that if equality holds in (1.14), then the solution f of

$$\begin{cases} \Delta f = 1 & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $\nabla^2 f = \text{Id}/(n+1)$ on Ω and $|\nabla f| = n/H_0(n+1)$ on $\partial\Omega$. By $\nabla^2 f = \text{Id}/(n+1)$ on Ω , there exist $x_0 \in \mathbb{R}^{n+1}$ and $c < 0$ such that $f(x) = c + |x - x_0|^2/2(n+1)$ for every $x \in \Omega$, i.e. Ω is the ball of center x_0 and radius $r = \sqrt{-2(n+1)c}$, while, by $|\nabla f| = n/H_0(n+1)$ on $\partial\Omega$, it must be $r = n/H_0$. When H is not constant, one can still infer from the proof of (1.14) that

$$C(n)|\Omega|\delta(\Omega)^{1/2} \geq \int_{\Omega} \left| \nabla^2 f - \frac{\text{Id}}{n+1} \right|, \quad (1.15)$$

$$C(n) \left(\frac{n}{H_0} \right)^2 P(\Omega) \delta(\Omega) \geq \int_{\partial\Omega} \left| \frac{n/H_0}{n+1} - |\nabla f| \right|^2, \quad (1.16)$$

where the second estimate holds if $\delta(\Omega) \leq 1/2$, and where $\nabla f = |\nabla f| \nu_\Omega \neq 0$ on $\partial\Omega$.

The problem of exploiting (1.15) and (1.16) in the description of Ω has some analogies with the quantitative analysis of Serrin’s overdetermined problem [Ser71] addressed in [BNST08]. In our terminology, the main result from [BNST08] states that, if $H_0 = n$ and for some $t > 0$ one has

$$\int_{\partial\Omega} \left| \frac{1}{n+1} - |\nabla f| \right| \leq P(\Omega)t, \quad \|\nabla f\|_{C^0(\partial\Omega)} \leq \frac{1+t}{n+1}, \quad (1.17)$$

then there exist finitely many disjoint balls $\{B_{x_i, r_i}\}_{i=1}^m$ such that

$$\left| \Omega \Delta \bigcup_{i=1}^m B_{x_i, r_i} \right|^{(n+1)/2} + \max_{1 \leq i \leq m} |r_i - 1| \leq C(n, \text{diam}(\Omega)) t^\beta, \quad \beta = \frac{1}{4n+13}. \quad (1.18)$$

Because of the uniform upper bound on $|\nabla f|$ in (1.17), it is not clear if one can take advantage of this result in the proof of Theorem 1.1. At the same time, we have a different condition at our disposal, namely (1.15), and by combining (1.15) with a global Lipschitz estimate for f (which is based on [CGS94], and exploits the geometric assumption that $H > 0$ on $\partial\Omega$), we obtain a more precise control than (1.18) on the distance of Ω from a finite family of balls. (Indeed, the power α in (1.5) is larger than the power $2\beta/(n+1)$ appearing in (1.18).) Finally, in Serrin's overdetermined problem the limiting balls need not to be tangent (sets Ω with small t may contain arbitrarily long connecting necks) and one does not expect Hausdorff estimates like (1.7) and (1.8) to hold (a set Ω with small t may contain small inclusions of large mean curvature). In other words, although (1.18) provides a qualitatively sharp information in the context of Serrin's problem, in the case of Alexandrov's theorem one expects, and thus wants to obtain, stronger information on Ω .

Coming back to the proof of Theorem 1.1, the first step consists in proving a qualitative result, see Theorem 2.5 below. Indeed, by combining a compactness argument, (1.15) and (1.16) with Reilly's identity, Pohozaev's identity, and Allard's regularity theorem (for integer rectifiable varifolds with bounded distributional mean curvature) one comes to prove the following fact: if $\{\Omega_h\}_{h \in \mathbb{N}}$ is a sequence of open, bounded and connected sets with C^2 -boundary in \mathbb{R}^{n+1} , $n \geq 2$, such that for some $L \in \mathbb{N}$, $L \geq 1$,

$$\lim_{h \rightarrow \infty} \delta(\Omega_h) = 0, \quad \sup_{h \in \mathbb{N}} Q(\Omega_h) < L + 1,$$

then, setting

$$\lambda_h = \frac{P(\Omega_h)}{(n+1)|\Omega_h|}, \quad \Omega_h^* = \lambda_h \Omega_h,$$

and up to translations, one has

$$\lim_{h \rightarrow \infty} \text{hd}(\partial\Omega_h^*, \partial G) + |P(\Omega_h^*) - P(G)| = 0,$$

where G is the union of at most L -many disjoint balls with unit radii, and with ∂G connected. Moreover, for every h large enough there exist open sets $\Sigma_h \subset \partial G$ (obtained by removing from ∂G at most $C(n)L$ -many spherical caps) and functions $\psi_h \in C^{1,\gamma}(\Sigma_h)$ for every $\gamma \in (0, 1)$ such that $(\text{Id} + \psi_h \nu_G)(\Sigma_h) \subset \partial\Omega_h^*$, and

$$\lim_{h \rightarrow \infty} \text{hd}(\Sigma_h, \partial G) = 0, \quad \|\psi_h\|_{C^{1,\gamma}(\Sigma_h)} \leq C(n, \gamma), \quad \lim_{h \rightarrow \infty} \|\psi_h\|_{C^1(\Sigma_h)} = 0.$$

This qualitative stability result, Theorem 2.5, is not needed in the proof of Theorem 1.1, and of course it is actually a corollary of it. We have nevertheless opted for including a direct discussion of it for the following reasons. First of all, it is a result of independent interest and possible usefulness, so it seems interesting to have a shorter proof of it. Secondly, by having Theorem 2.5 at hand one is able to clean up to some later quantitative arguments and obtain better estimates. Thirdly, Theorem 1.1 is actually proved by quantitatively revisiting the proof of Theorem 2.5, and therefore the separate treatment of the latter should makes more accessible the argument used in proving the former.

In this direction the main difficulty arises in the application of the area excess regularity criterion of Allard, which is needed to parameterize a large portion of $\partial\Omega$ over a large portion of ∂G . A key point here is quantifying the size of $\mathcal{H}^n(\partial\Omega \cap B_{x,r})$ on a range of scales r proportional to a suitable power of $\delta(\Omega)$ and at points $x \in \partial\Omega$ sufficiently close to ∂G . We address this issue by carefully partitioning \mathbb{R}^{n+1} into suitable polyhedral regions associated to the balls $B_{z_j,1}$, and by then performing inside each of these regions a calibration type argument with respect to the corresponding ball $B_{z_j,1}$ (see, in particular, step six of the proof of Theorem 1.1).

Summarizing, Theorem 1.1 is proved by combining a mix of different ideas from elliptic PDE theory, global geometric identities, and geometric measure theory, and it contains a quantitative (and qualitatively sharp) description of boundaries with almost-constant mean curvature.

1.2. An application to capillarity surfaces. The study of the basic capillarity-type energy functional in \mathbb{R}^{n+1} leads to consider sets Ω with small Alexandrov's deficit. Indeed, given a potential energy density $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, in capillarity theory one considers the free energy

$$\mathcal{F}(\Omega) = P(\Omega) + \int_{\Omega} g(x) dx,$$

and its volume-constrained stationary points and local/global minimizers. Capillarity phenomena are characterized by the dominance of surface tension over potential energy, which is the case when the volume parameter $m = |\Omega|$ is small (as surface tension is of order $m^{n/(n+1)}$, while potential energy is typically of order m). Under mild assumptions on g (essentially, coercivity at infinity, $g(x) \rightarrow \infty$ as $|x| \rightarrow \infty$), one can show the existence of volume-constrained global minimizers of \mathcal{F} of any fixed volume. In particular, if Ω_m is such a global minimizer with $|\Omega_m| = m$, then by comparison with a ball $B^{(m)}$ of volume m one sees that

$$P(\Omega_m) \leq P(B^{(m)}) + \int_{B^{(m)} \setminus \Omega_m} g(x) dx,$$

that is, the *isoperimetric deficit* $\delta_{\text{iso}}(\Omega_m)$ of Ω_m is small in terms of m ,

$$\delta_{\text{iso}}(\Omega_m) = \frac{P(\Omega_m)}{P(B^{(m)})} - 1 \leq \frac{C(n)}{m^{n/(n+1)}} \int_{B^{(m)} \setminus \Omega_m} g(x) dx \approx C(n, g) m^{1/(n+1)}.$$

By the quantitative isoperimetric inequality [FMP08, FMP10], one finds $x_m \in \mathbb{R}^{n+1}$ such that

$$\left(\frac{|\Omega_m \Delta (x_m + B^{(m)})|}{m} \right)^2 \leq C(n) \delta_{\text{iso}}(\Omega_m),$$

so that, in conclusion, Ω_m has to be close (in a normalized L^1 -sense) to a ball of volume m . This observation is the starting point of the analysis performed in [FM11], where the proximity of Ω_m to a ball of volume m is quantified, under increasingly stronger smoothness assumptions on g , in increasingly stronger ways. For example, if $g \in C_{\text{loc}}^1(\mathbb{R}^{n+1})$ and $m \leq m_0(n, g)$, then Ω_m is shown to be convex and $\partial\Omega_m$ is proved to be a $C^{2,\gamma}$ -small normal deformation of $x_m + \partial B^{(m)}$, with explicit quantitative bounds on the $C^{2,\gamma}$ -norm of this deformation in terms of m .

When dealing with volume-constrained local minimizers or stationary points of \mathcal{F} one cannot rely anymore on the quantitative isoperimetric inequality, as one is not given the energy comparison inequality with $B^{(m)}$. However, in this more general context, Alexandrov's deficit turns out to be small in terms of the volume parameter m , thus opening the way for the application of Theorem 1.1.

Let us recall that given a vector field $X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, and denoted by f_t the flow generated by X , then the first variation of \mathcal{F} at Ω along X is defined as

$$\delta\mathcal{F}(\Omega)[X] = \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(f_t(\Omega)). \quad (1.19)$$

One says that a set of finite perimeter $\Omega \subset \mathbb{R}^{n+1}$ is a *volume-constrained stationary point* of \mathcal{F} if $\delta\mathcal{F}(\Omega)[X] = 0$ for every $X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ such that $|f_t(\Omega)| = |\Omega|$ for every t small enough. The following proposition, combined with Theorem 1.1, provides a complete description of such stationary boundaries, and its simple proof is presented in section 3.

Proposition 1.2. *Let $g \in C_{\text{loc}}^1(\mathbb{R}^{n+1})$, $R_0 > 0$, and Ω be an open set with C^2 -boundary such that $\Omega \subset B_{R_0}$. If Ω is a volume-constrained stationary point of \mathcal{F} with $|\Omega| = m$, then*

$$\delta(\Omega) \leq C_*(n) \|g\|_{C^1(B_{R_0})} m^{1/(n+1)}, \quad (1.20)$$

for some constant $C_*(n)$.

Under the assumptions of Proposition 1.2, let us now pick $L \in \mathbb{N}$, $L \geq 1$, and $a \in (0, 1]$, define $c(n, L, a)$ as in Theorem 1.1, and assume that

$$Q(\Omega) \leq L + 1 - a, \quad m \leq \left(\frac{c(n, L, a)}{C_*(n) \|g\|_{C^1(B_{R_0})}} \right)^{n+1}.$$

In this way, by Theorem 1.1 and (1.20), there exists a finite family $\{B_{z_j, 1}\}_{j \in J}$ of disjoint balls such that, looking for example at (1.5) and setting $G = \bigcup_{j \in J} B_{z_j, 1}$,

$$\frac{|\Omega \Delta G_*|}{|\Omega|} = \frac{|\Omega^* \Delta G|}{|\Omega^*|} \leq C(n) L^2 \|g\|_{C^1(B_{R_0})}^\alpha m^{\alpha/(n+1)},$$

where $\Omega^* = (H_0/n)\Omega$, and thus $G_* = \bigcup_{j \in J} B_{w_j, n/H_0}$. Notice that this proves a quantization of the volume of Ω , in the sense that $|\Omega^*|$ is close to $\#J|B|$ with an error of order $C m^{\alpha/(n+1)}$, where $C = C(n, L, \|g\|_{C^1(B_{R_0})})$. Similar results carrying different geometric information are obtained from the other estimates appearing in Theorem 1.1.

In conclusion, the quantitative side of Theorem 1.1 significantly strengthens the purely qualitative analysis that one could obtain by exploiting compactness arguments only, as it provides explicit orders of magnitude for the errors one makes in approximating Ω^* with a unit balls compound.

We finally notice that Ω^* will be close to a single ball as soon as volume-constrained stationarity is strengthened into some *local* minimality property. For example, it will suffice to require that $\mathcal{F}(\Omega) \leq \mathcal{F}(E)$ whenever $|E| = |\Omega|$ and $\partial E \subset I_\sigma(\partial\Omega) = \{x \in \mathbb{R}^{n+1} : \text{dist}(x, \partial\Omega) < \sigma\}$, with $\sigma = \sigma_0 |\Omega|/P(\Omega)$ for some $\sigma_0 > 0$. Notice that although $E = B^{(m)}$ is not an admissible competitor in this local minimality condition, thus ruling out the possibility of applying [FMP08], Ω will nevertheless be a volume-constrained stationary set for \mathcal{F} . Moreover, by a standard argument exploiting the local minimality of Ω , one obtains volume density estimates for Ω^* which make possible to apply statement (ii) in Theorem 1.1.

1.3. Organization of the paper. The proof of Theorem 1.1 and of Proposition 1.2 are discussed, respectively, in section 2 and section 3. In Appendix A we discuss the relation of the Alexandrov's stability problem with the study of almost-umbilical surfaces initiated by De Lellis and Müller in [DLM05].

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2. PROOF OF THEOREM 1.1

We begin by gathering various assumptions, preliminaries, and conventions.

Constants: The symbol C denotes a generic positive constant whose value is independent from n and Ω . We use the symbols C_0, C_1 , etc. for constants whose specific value is referred to in multiple occasions (see, for instance (2.18) below). We denote by $C(n)$ and $c(n)$ generic positive constants whose value does depend on n , but is independent from Ω , with the idea that $C(n)$ stands for a ‘‘large’’ constant, and $c(n)$ stands for a ‘‘small’’ constant. Similar conventions hold for $C(n, L)$, etc.

Assumptions on Ω : Thorough this section we always assume that

$$\begin{aligned} \Omega \subset \mathbb{R}^{n+1}, \quad n \geq 2, \text{ is a bounded connected open set} \\ \text{with } C^2\text{-boundary with } H > 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

From a certain point of our argument on we shall assume that (as one can always do up to a scaling)

$$(n+1)|\Omega| = P(\Omega). \quad (2.2)$$

Recall that, by the Euclidean isoperimetric inequality (see (1.3)), (2.2) implies

$$|B| \leq |\Omega|, \quad P(B) \leq P(\Omega), \quad (2.3)$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Moreover, (2.2) is equivalent to $H_0 = n$, so that

$$\delta(\Omega) = n^{-1} \|H - n\|_{C^0(\Omega)},$$

with $\delta(\Omega) = 0$ if and only if Ω is a ball of unit radius. Indeed, our convention for the scalar mean curvature H is that

$$\int_{\partial\Omega} \operatorname{div}^{\partial\Omega} X \, d\mathcal{H}^n = \int_{\partial\Omega} (X \cdot \nu_\Omega) H \, d\mathcal{H}^n, \quad \forall X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}),$$

and thus the scalar mean curvature of B is equal to n . In addition to (2.2) we also assume that

$$P(\Omega) \leq (L+1-a)P(B), \quad \text{where } L \in \mathbb{N}, L \geq 1, a \in (0, 1]. \quad (2.4)$$

Note that, by combining (2.4) with (2.2) one finds

$$|\Omega| \leq (L+1-a)|B|. \quad (2.5)$$

We shall work under the assumption that $\delta(\Omega) \leq c(n, L, a)$ for a suitably small positive constant $c(n, L, a) \leq 1/2$: in particular,

$$\frac{n}{2} \leq H(x) \leq 2n, \quad \forall x \in \partial\Omega. \quad (2.6)$$

By Topping's inequality [Top08], one has

$$\operatorname{diam}(\Omega) \leq C(n) \int_{\partial\Omega} |H|^{n-1},$$

so that (2.4) and (2.6) imply

$$\operatorname{diam}(\Omega) \leq C(n) L. \quad (2.7)$$

Alternatively, by the monotonicity identity (see [DL08, Theorem 2.1]) and by $H \leq 2n$ on $\partial\Omega$, one has that

$$s \in (0, \infty) \mapsto e^{2ns} \frac{\mathcal{H}^n(\partial\Omega \cap B_{x,s})}{s^n} \quad \text{is monotone increasing for every } x \in \mathbb{R}^{n+1}. \quad (2.8)$$

If $x \in \partial\Omega$, then this function converges to $\mathcal{H}^n(\{z \in \mathbb{R}^n : |z| < 1\})$ as $s \rightarrow 0^+$, and thus one obtains the uniform lower perimeter estimate

$$\mathcal{H}^n(\partial\Omega \cap B_{x,s}) \geq c(n) s^n, \quad \forall x \in \partial\Omega, s \in (0, 1). \quad (2.9)$$

We notice that this last fact can be used jointly with $P(\Omega) \leq C(n)L$ to infer (2.7). Finally, whenever Ω satisfies (2.1) we define the *Heintze-Karcher deficit* of Ω as

$$\eta(\Omega) = \frac{\int_{\partial\Omega} \frac{n}{H} - (n+1)|\Omega|}{\int_{\partial\Omega} \frac{n}{H}} = 1 - \frac{(n+1)|\Omega|}{\int_{\partial\Omega} \frac{n}{H}}. \quad (2.10)$$

Just like $\delta(\Omega)$, this is a scale invariant quantity such that $\eta(\Omega) = 0$ if and only if Ω is a ball. One has

$$\eta(\Omega) \leq \delta(\Omega). \quad (2.11)$$

Indeed,

$$\begin{aligned}\eta(\Omega) &= 1 - \frac{(n+1)|\Omega|}{\int_{\partial\Omega} \frac{n}{H}} = \frac{(n+1)|\Omega|}{\int_{\partial\Omega} \frac{n}{H_0}} - \frac{(n+1)|\Omega|}{\int_{\partial\Omega} \frac{n}{H}} \\ &= \frac{(n+1)|\Omega|}{n} \frac{\int_{\partial\Omega} \frac{1}{H} - \int_{\partial\Omega} \frac{1}{H_0}}{\int_{\partial\Omega} \frac{1}{H_0} \int_{\partial\Omega} \frac{1}{H}} \leq \frac{(n+1)|\Omega|}{n} \frac{\delta(\Omega) \int_{\partial\Omega} \frac{1}{H}}{\int_{\partial\Omega} \frac{1}{H_0} \int_{\partial\Omega} \frac{1}{H}} = \delta(\Omega).\end{aligned}$$

Torsion potential: We denote by f and u the smooth functions defined on Ω by setting

$$\begin{cases} \Delta f = 1 & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega, \end{cases} \quad u = -f. \quad (2.12)$$

Note that $f < 0$ on Ω , with $\nabla f = |\nabla f| \nu_\Omega$ on $\partial\Omega$, and $\nabla_\nu f = \nu_\Omega \cdot \nabla f = |\nabla f| > 0$ on $\partial\Omega$ by Hopf's lemma. We shall use two integral identities involving f , namely, the *Reilly's identity* (see, e.g., [Ros87, Equation (3)])

$$\int_{\partial\Omega} H |\nabla f|^2 = \int_{\Omega} (\Delta f)^2 - |\nabla^2 f|^2, \quad (2.13)$$

and the *Pohozaev's identity*, see e.g. [AM07, Theorem 8.30],

$$(n+3) \int_{\Omega} (-f) = \int_{\Omega} (x \cdot \nu_\Omega) |\nabla f|^2. \quad (2.14)$$

The first one quickly leads to prove Alexandrov's theorem and the Heintze-Karcher inequality, as shown in [Ros87].

Lemma 2.1. *If Ω and f are as in (2.1) and (2.12), then*

$$\begin{aligned}\frac{|\Omega|}{n+1} \left(\int_{\partial\Omega} \frac{n}{H} - (n+1)|\Omega| \right) &= \int_{\partial\Omega} \frac{1}{H} \int_{\Omega} |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n+1} \\ &\quad + \int_{\partial\Omega} \frac{1}{H} \int_{\partial\Omega} |\nabla f|^2 H - \left(\int_{\partial\Omega} |\nabla f| \right)^2,\end{aligned} \quad (2.15)$$

In particular, (1.14) holds, and if H is constant on $\partial\Omega$, then $H = H_0 > 0$ and Ω is a ball.

Proof. By the divergence theorem and by Hölder's inequality,

$$|\Omega|^2 = \left(\int_{\partial\Omega} \nabla_\nu f \right)^2 = \left(\int_{\partial\Omega} \frac{\sqrt{H} |\nabla f|}{\sqrt{H}} \right)^2 \leq \int_{\partial\Omega} \frac{1}{H} \int_{\partial\Omega} |\nabla f|^2 H.$$

Thanks to (2.13),

$$\int_{\partial\Omega} H |\nabla f|^2 = \frac{n}{n+1} |\Omega| + \int_{\Omega} \frac{(\Delta f)^2}{n+1} - |\nabla^2 f|^2 \leq \frac{n}{n+1} |\Omega|,$$

where we have used the Cauchy-Schwartz inequality

$$(\text{tr}M)^2 = (M : \text{Id})^2 \leq |\text{Id}|^2 |M|^2 = (n+1) |M|^2, \quad \forall M \in \mathbb{R}^n \otimes \mathbb{R}^n.$$

(Here and in the following, we denote by $:$ the scalar product on $\mathbb{R}^n \otimes \mathbb{R}^n$, and by $|\cdot|$ the corresponding Hilbert norm on $\mathbb{R}^n \otimes \mathbb{R}^n$.) This proves (2.15). Let us now assume that H is constant on $\partial\Omega$, then by applying the divergence theorem on $\partial\Omega$ and on Ω , one finds

$$\begin{aligned}\int_{\partial\Omega} \frac{n}{H} &= \frac{nP(\Omega)}{H} = \frac{\int_{\partial\Omega} \text{div}^{\partial\Omega}(x) d\mathcal{H}_x^n}{H} \\ &= \frac{\int_{\partial\Omega} (x \cdot \nu_\Omega) H d\mathcal{H}_x^n}{H} = \int_{\partial\Omega} x \cdot \nu_\Omega d\mathcal{H}_x^n = \int_{\Omega} \text{div}(x) dx = (n+1)|\Omega|,\end{aligned} \quad (2.16)$$

so that $H = H_0$ and equality holds in (1.14). In particular, (2.15) gives that $\nabla^2 f(x) = \text{Id}/(n+1)$ for every $x \in \Omega$, so that, being Ω connected,

$$f(x) = c + \frac{|x - x_0|^2}{2(n+1)},$$

for some $c < 0$ and $x_0 \in \mathbb{R}^n$. Since $f = 0$ on $\partial\Omega$, we find that Ω is the ball of center x_0 and radius $\sqrt{-2(n+1)c}$. \square

We now exploit [Tal76] and [CGS94] to obtain universal estimates on f .

Lemma 2.2. *If Ω and f are as in (2.1) and (2.12), then*

$$\|f\|_{C^0(\Omega)} \leq \frac{1}{2(n+1)} \left(\frac{|\Omega|}{|B|} \right)^{2/(n+1)} \leq C |\Omega|^{2/(n+1)}, \quad (2.17)$$

$$\|\nabla f\|_{C^0(\Omega)} \leq \sqrt{2} \|f\|_{C^0(\Omega)}^{1/2} \leq C_0 |\Omega|^{1/(n+1)}, \quad (2.18)$$

$$\|\nabla^2 f\|_{L^2(\Omega)} \leq |\Omega|^{1/2}. \quad (2.19)$$

Proof. By a classical result of Talenti [Tal76], the radially symmetric decreasing rearrangement $(-f)^*$ of $-f$ satisfies the pointwise estimate

$$(-f)^*(x) \leq \frac{R^2 - |x|^2}{2(n+1)}, \quad \text{where} \quad R = \left(\frac{|\Omega|}{|B|} \right)^{1/(n+1)}, \quad (2.20)$$

so that the first inequality in (2.17) follows immediately. The second inequality in (2.17) is then obtained by recalling that

$$|\{z \in \mathbb{R}^k : |z| < 1\}| = \frac{\pi^{k/2}}{\Gamma(1 + (k/2))}, \quad \lim_{t \rightarrow \infty} \frac{\Gamma(1+t)}{\sqrt{2\pi t} (t/e)^t} = 1. \quad (2.21)$$

(Thus (2.17) does not need the assumption that $H > 0$ on $\partial\Omega$). Moreover, we immediately deduce (2.19) from $\Delta f = 1$ and (2.13), so that we are left to prove (2.18). With $u = -f$ we set

$$p = |\nabla u|^2 + 2(u - \|u\|_{C^0(\Omega)}),$$

and aim to prove that $p \leq 0$. The key fact is the observation that

$$|\nabla u|^2 \Delta p + 2 \nabla u \cdot \nabla p \geq \frac{|\nabla p|^2}{2}, \quad \text{on } \{|\nabla u| > 0\}, \quad (2.22)$$

see [CGS94, Equation (2.7)]. Given (2.22), we argue by contradiction and assume that the maximum p_0 of p in $\bar{\Omega}$ is positive. We first claim that p_0 is achieved on $\partial\Omega$. Indeed, let $U = \{x \in \Omega : p(x) = p_0\}$, then U is obviously closed. If $x \in U$, then $|\nabla u(x)|^2 \geq p(x) = p_0 > 0$ and so p satisfies

$$\Delta p + T \cdot \nabla p \geq 0, \quad \text{in a neighborhood of } x,$$

where the vector-field

$$T = 2 \frac{\nabla u}{|\nabla u|^2} \quad (2.23)$$

is bounded on that same neighborhood. By the strong maximum principle, p must be constant in that neighborhood. This shows that U is open, so that $U = \Omega$ by connectedness. At the same time, there exists $x^* \in \Omega$ such that $\nabla u(x^*) = 0$, so that $p_0 = p(x^*) = 2(u(x^*) - \|u\|_{C^0(\Omega)}) \leq 0$ a contradiction. This shows that there exists $x_0 \in \partial\Omega$ such that

$$p(x_0) = p_0 > p(x), \quad \forall x \in \Omega.$$

By Hopf's lemma, $\nabla_\nu u(x_0) < 0$, so that

$$\Delta p + T \cdot \nabla p \geq 0, \quad \text{in a neighborhood of } x_0 \text{ in } \Omega,$$

where once again the vector-field T (defined as in (2.23) above) is bounded. By Hopf's lemma,

$$\nabla_\nu p(x_0) > 0. \quad (2.24)$$

At the same time one has

$$\nabla_\nu p(x_0) = 2\left(\nabla u(x_0) \cdot \nabla(\nabla_\nu u)(x_0) + \nabla_\nu u(x_0)\right).$$

Since $u = 0$ on $\partial\Omega$, we have $\nabla u = (\nabla_\nu u)\nu$ on $\partial\Omega$, so that the above identity becomes

$$\nabla_\nu p(x_0) = \nabla_\nu u(x_0)(\nabla_{\nu\nu} u(x_0) + 1).$$

Since $\nabla_\nu u(x_0) < 0$, (2.24) gives us

$$\nabla_{\nu\nu} u(x_0) < -1 = \Delta u(x_0). \quad (2.25)$$

We now obtain a contradiction by showing that, thanks to $H > 0$, one has

$$\Delta u(x_0) < \nabla_{\nu\nu} u(x_0). \quad (2.26)$$

Indeed, assuming without loss of generality that $x_0 = 0$ and that Ω is (locally at 0) the subgraph of a function φ on n -variables such that $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$ (so that $\nu_\Omega(0) = e_n$, and thus $-H(0) = \Delta\varphi(0)$), by differentiating $u(z, \varphi(z)) = 0$ at $z = 0$ twice along the direction z_i , one gets

$$0 = \nabla_{z_i z_i} u(z, \varphi(z)) + 2\nabla_{z_i \nu} u(z, \varphi(z)) \nabla_{z_i} \varphi(z) + \nabla_\nu u(z, \varphi(z)) \nabla_{z_i z_i} \varphi(z) + \nabla_{\nu\nu} u(z) (\nabla_{z_i} \varphi(z))^2$$

which, evaluated at $z = 0$, by $\nabla\varphi(0) = 0$ gives us

$$0 = \nabla_{z_i z_i} u(0) + \nabla_\nu u(0) \nabla_{z_i z_i} \varphi(0).$$

By adding up over $i = 1, \dots, n$, and by $H(0) > 0$ and $\nabla_\nu u(0) < 0$, we conclude that

$$0 = \Delta u(0) - \nabla_{\nu\nu} u(0) - H(0) \nabla_\nu u(0) > \Delta u(0) - \nabla_{\nu\nu} u(0),$$

so that (2.26) holds. \square

Lemma 2.3. *If Ω and f are as in (2.1) and (2.12), then*

$$C(n) |\Omega| \sqrt{\eta(\Omega)} \geq \int_\Omega \left| \nabla^2 f - \frac{\text{Id}}{n+1} \right|. \quad (2.27)$$

If, in addition, $\delta(\Omega) \leq 1/2$, then

$$C(n) \left(\frac{n}{H_0} \right)^2 P(\Omega) \delta(\Omega) \geq \int_{\partial\Omega} \left| \frac{n/H_0}{n+1} - |\nabla f| \right|^2. \quad (2.28)$$

Remark 2.4. If we define $\bar{u} = -f$ on Ω , $\bar{u} = 0$ on $\mathbb{R}^{n+1} \setminus \Omega$, then the distributional gradient $D\bar{u}$ and the distributional Hessian $D^2\bar{u}$ of \bar{u} are given by

$$\begin{aligned} D\bar{u} &= -\nabla f \mathcal{L}^{n+1} \llcorner \Omega, \\ D^2\bar{u} &= -\nabla^2 f \mathcal{L}^{n+1} \llcorner \Omega + \frac{\nabla f \otimes \nabla f}{|\nabla f|} \mathcal{H}^n \llcorner \partial\Omega, \end{aligned}$$

where \mathcal{L}^{n+1} is the Lebesgue measure on \mathbb{R}^{n+1} . Indeed, for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ one has

$$D^2\bar{u}(\varphi) = \int_{\mathbb{R}^{n+1}} \bar{u} \partial_{ij} \varphi = - \int_\Omega f \partial_{ij} \varphi = \int_\Omega \partial_i f \partial_j \varphi = \int_{\partial\Omega} (\nu_\Omega)_j \varphi \partial_i f - \int_\Omega \varphi \partial_{ij} f,$$

where $\nu_\Omega = \nabla f / |\nabla f|$ on $\partial\Omega$. Hence, under the assumption (2.2), (2.27) and (2.28) are equivalent to

$$|D^2\bar{u} - \mu|(\mathbb{R}^{n+1}) \leq C(n) (P(\Omega) \delta(\Omega) + |\Omega| \eta(\Omega)^{1/2}) \leq C(n, L) \delta(\Omega)^{1/2}, \quad (2.29)$$

where μ is the Radon measure defined by

$$\mu = -\frac{\text{Id}}{n+1} \mathcal{L}^{n+1} \llcorner \Omega + \frac{\nu_\Omega \otimes \nu_\Omega}{n+1} \mathcal{H}^n \llcorner \partial\Omega.$$

This point of view on (2.27)–(2.28) is at the basis of the proof of Theorem 2.5 below.

Proof of Lemma 2.3. We first prove (2.27). If $M_1, M_2 \in \mathbb{R}^n \otimes \mathbb{R}^n$ with $M_1, M_2 \neq 0$, then one has

$$|M_1| |M_2| - M_1 : M_2 = \frac{1}{2} \left| \mu M_1 - \frac{M_2}{\mu} \right|^2, \quad \mu = (|M_2|/|M_1|)^{1/2},$$

so that

$$|M_1|^2 |M_2|^2 - (M_1 : M_2)^2 \geq (M_1 : M_2) \left| \mu M_1 - \frac{M_2}{\mu} \right|^2.$$

By (2.15), setting $M_1 = \nabla^2 f$ (note that $\nabla^2 f \neq 0$ as $\Delta f = 1$ on Ω) and $M_2 = \text{Id}$, and noticing that $|\text{Id}|^2 = (n+1)$ and $\Delta f = \nabla^2 f : \text{Id}$, one finds

$$n |\Omega| \eta(\Omega) \geq \int_{\Omega} |\text{Id}|^2 |\nabla^2 f|^2 - (\Delta f)^2 \geq \int_{\Omega} \mu^2 \left| \nabla^2 f - \frac{\text{Id}}{\mu^2} \right|^2, \quad (2.30)$$

where we have set $\mu(x) = (|\text{Id}|/|\nabla^2 f(x)|)^{1/2}$, $x \in \Omega$. By (2.19) and (2.30), we get

$$\begin{aligned} \left(\int_{\Omega} \left| \nabla^2 f - \frac{\text{Id}}{\mu^2} \right|^2 \right) &\leq \int_{\Omega} \mu^2 \left| \nabla^2 f - \frac{\text{Id}}{\mu^2} \right|^2 \int_{\Omega} \frac{|\nabla^2 f|}{|\text{Id}|} \\ &\leq C(n) |\Omega|^{3/2} \eta(\Omega) \left(\int_{\Omega} |\nabla^2 f|^2 \right)^{1/2} \leq C(n) |\Omega|^2 \eta(\Omega), \end{aligned}$$

that is

$$\int_{\Omega} \left| \nabla^2 f - \frac{\text{Id}}{\mu^2} \right| \leq C(n) |\Omega| \sqrt{\eta(\Omega)}. \quad (2.31)$$

In particular, $|\text{tr}(M_1) - \text{tr}(M_2)| \leq |M_1 - M_2|$ and $\Delta f = 1$ give us

$$\int_{\Omega} \left| 1 - \frac{n+1}{\mu^2} \right| \leq C(n) |\Omega| \sqrt{\eta(\Omega)},$$

which leads to

$$\int_{\Omega} \left| \frac{\text{Id}}{n+1} - \frac{\text{Id}}{\mu^2} \right| = \sqrt{n+1} \int_{\Omega} \left| \frac{1}{n+1} - \frac{1}{\mu^2} \right| \leq C(n) |\Omega| \sqrt{\eta(\Omega)}.$$

We prove (2.27) by combining this last inequality with (2.31). We now prove (2.28). By (2.15) one has

$$\frac{|\Omega|}{n+1} \left(\int_{\partial\Omega} \frac{n}{H} - (n+1)|\Omega| \right) \geq 2 \int_{\partial\Omega} |\nabla f| \left(\left(\int_{\partial\Omega} \frac{1}{H} \int_{\partial\Omega} |\nabla f|^2 H \right)^{1/2} - \int_{\partial\Omega} |\nabla f| \right). \quad (2.32)$$

Since $\int_{\partial\Omega} |\nabla f| = |\Omega|$, if $\lambda > 0$ is such that

$$\lambda^4 = \left(\int_{\partial\Omega} \frac{1}{H} \right)^{-1} \int_{\partial\Omega} |\nabla f|^2 H,$$

then (2.32) gives us

$$\begin{aligned} \frac{1}{2(n+1)} \left(\int_{\partial\Omega} \frac{n}{H} - (n+1)|\Omega| \right) &\geq \left(\int_{\partial\Omega} \frac{1}{H} \int_{\partial\Omega} |\nabla f|^2 H \right)^{1/2} - \int_{\partial\Omega} |\nabla f| \\ &= \int_{\partial\Omega} \frac{\lambda^2}{2} \frac{1}{H} + \frac{1}{2\lambda^2} |\nabla f|^2 H - |\nabla f| \\ &= \int_{\partial\Omega} \frac{1}{2} \left(\frac{\lambda}{\sqrt{H}} - \frac{|\nabla f| \sqrt{H}}{\lambda} \right)^2 \geq \int_{\partial\Omega} \frac{H}{2\lambda^2} \left(\frac{\lambda^2}{H} - |\nabla f| \right)^2. \end{aligned}$$

Again, by (2.6),

$$\frac{H_0}{\lambda^2} \int_{\partial\Omega} \left(\frac{\lambda^2}{H} - |\nabla f| \right)^2 \leq C(n) \eta(\Omega) \int_{\partial\Omega} \frac{n}{H} \leq C(n) P(\Omega) \frac{n}{H_0} \eta(\Omega).$$

Finally, by (2.13) one has $\int_{\partial\Omega} H|\nabla f|^2 \leq |\Omega|$, so that

$$\lambda^4 \leq C H_0 \frac{|\Omega|}{P(\Omega)} \leq C, \quad (2.33)$$

and thus

$$\int_{\partial\Omega} \left(\frac{\lambda^2}{H} - |\nabla f| \right)^2 \leq C(n) \left(\frac{n}{H_0} \right)^2 P(\Omega) \eta(\Omega). \quad (2.34)$$

Now let $|\nabla f|_{\partial\Omega}$ denote the average of $|\nabla f|$ on $\partial\Omega$, so that $|\nabla f|_{\partial\Omega} = |\Omega|/P(\Omega)$. By (2.33) and (2.34) we thus find

$$\begin{aligned} \int_{\partial\Omega} \left(|\nabla f|_{\partial\Omega} - |\nabla f| \right)^2 &\leq \int_{\partial\Omega} \left(\frac{\lambda^2}{H_0} - |\nabla f| \right)^2 \leq 2 \int_{\partial\Omega} \left(\frac{\lambda^2}{H_0} - \frac{\lambda^2}{H} \right)^2 + 2 \int_{\partial\Omega} \left(\frac{\lambda^2}{H} - |\nabla f| \right)^2 \\ &\leq C(n) \left(\frac{n}{H_0} \right)^2 P(\Omega) \delta(\Omega), \end{aligned}$$

where in the last inequality we have used (2.11) and $\delta(\Omega) \leq 1$. We deduce (2.28) by noticing that $|\nabla f|_{\partial\Omega} = n/(n+1)H_0$. \square

We now exploit a compactness argument to show that if the Alexandrov's deficit of Ω is small enough, then Ω can be taken arbitrarily close (in various ways) to a finite family of tangent balls of unit radii.

Theorem 2.5. *Given $n, L \in \mathbb{N}$, $n \geq 2$, $L \geq 1$, $a \in (0, 1]$, and $\tau > 0$ there exists $c(n, L, a, \tau) > 0$ with the following property. If Ω satisfies (2.1), (2.2), (2.4), f is defined as in (2.12) (and then extended to 0 on $\mathbb{R}^{n+1} \setminus \Omega$) and $\delta(\Omega) \leq c(n, L, a, \tau)$, then there exists a finite family of disjoint unit balls $\{B_{z_j, 1}\}_{j \in J}$ with $\#J \leq L$ such that, setting*

$$G = \bigcup_{j \in J} B_{z_j, 1},$$

∂G is connected (that is, each sphere $\partial B_{z_j, 1}$ intersects tangentially at least another sphere $\partial B_{z_\ell, 1}$ for some $\ell \neq j$) and

$$|\Omega \Delta G| + \text{hd}(\partial\Omega, \partial G) + |P(\Omega) - P(G)| + \|f - f_G\|_{C^0(\mathbb{R}^{n+1})} \leq \tau,$$

where

$$f_G(x) = - \sum_{j \in J} \max \left\{ \frac{1 - |x - z_j|^2}{2(n+1)}, 0 \right\}, \quad x \in \mathbb{R}^{n+1}.$$

Moreover, there exist $\Sigma \subset \partial G$ and $\phi \in C^{1,\gamma}(\Sigma)$ for every $\gamma \in (0, 1)$ such that $\partial G \setminus \Sigma$ consists of at most $C(n)$ L -many spherical caps whose diameters are bounded by τ , and such that $(\text{Id} + \phi \nu_G)(\Sigma) \subset \partial\Omega$ with

$$\|\phi\|_{C^1(\Sigma)} + \mathcal{H}^n(\partial\Omega \setminus (\text{Id} + \phi \nu_G)(\Sigma)) \leq \tau, \quad \|\phi\|_{C^{1,\gamma}(\Sigma)} \leq C(n, \gamma).$$

Remark 2.6. Notice that by Theorem 2.5 and since $\|\nabla f\|_{C^0(\Omega)} \leq \sqrt{2} \|f\|_{C^0(\Omega)}^{1/2}$ thanks to (2.18), one can deduce that

$$\|f\|_{C^1(\Omega)} \leq C_0(n), \quad (2.35)$$

whenever $\delta(\Omega) \leq c(n, L, a)$. (Indeed, it is enough to pick $\tau = \tau(n)$ and use $\|f - f_G\|_{C^0(\Omega)} \leq \tau$.) As a consequence one can choose, in the proof of Theorem 1.1, if working with (2.17)–(2.18) or with (2.35). In the former case, one obtains larger powers of L but explicitly computable constants $C(n)$ in the quantitative estimates of Theorem 1.1; in the latter case, we obtain smaller powers of L but lose the ability of computing the corresponding constants $C(n)$. We shall opt for the second possibility.

Proof of Theorem 2.5. Let us consider a sequence of sets $\{\Omega_h\}_{h \in \mathbb{N}}$ satisfying (2.1), (2.2) and (2.4) (with the same L and a for every $h \in \mathbb{N}$), and correspondingly define f_h starting from Ω_h by (2.12). Assuming that $\delta(\Omega_h) \rightarrow 0$, it will suffice to prove that, up extracting subsequences,

$$\lim_{h \rightarrow \infty} |\Omega_h \Delta G| + \text{hd}(\partial\Omega_h, \partial G) + |P(\Omega_h) - P(G)| + \|f_h - f_G\|_{C^0(\mathbb{R}^{n+1})} = 0, \quad (2.36)$$

where G and f_G are associated to a family of balls $\{B_{z_j,1}\}_{j \in J}$ as in the statement, and that there exist $\Sigma_h \subset \partial G$ and $\phi_h \in C^{1,\gamma}(\Sigma_h)$ for every $\gamma \in (0,1)$ such that $\partial G \setminus \Sigma_h$ consists of at most $C(n)$ L -many spherical caps with vanishing diameters, and $(\text{Id} + \phi_h \nu_G)(\Sigma_h) \subset \partial\Omega_h$ with

$$\lim_{h \rightarrow \infty} \|\phi_h\|_{C^1(\Sigma_h)} + \mathcal{H}^n(\partial\Omega_h \setminus (\text{Id} + \phi_h \nu_G)(\Sigma_h)) = 0, \quad \sup_{h \in \mathbb{N}} \|\phi_h\|_{C^{1,\gamma}(\Sigma_h)} \leq C(n, \gamma).$$

To this end, we first note that, by (2.7), up to translating the sets Ω_h one has

$$\Omega_h \subset B_R, \quad \forall h \in \mathbb{N}, \quad (2.37)$$

where $R = R(n, L)$. By (2.37) and since $P(\Omega_h) \leq C(n, L)$ thanks to (2.4), the compactness theorem for sets of finite perimeter [Mag12, Theorem 12.26] implies that, up to extracting subsequences,

$$\lim_{h \rightarrow \infty} |\Omega_h \Delta G| = 0, \quad (2.38)$$

where $G \subset B_R$ is a set of finite perimeter in \mathbb{R}^{n+1} . Similarly, if we define $\bar{u}_h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by setting $\bar{u}_h = -f_h$ on Ω_h , and $\bar{u}_h = 0$ on $\mathbb{R}^{n+1} \setminus \Omega_h$, then by (2.18) and by (2.37) we find that, again up to extracting subsequences,

$$\lim_{h \rightarrow \infty} \|\bar{u}_h - \bar{u}\|_{C^0(\mathbb{R}^{n+1})} + \|\bar{u}_h - \bar{u}\|_{L^1(\mathbb{R}^{n+1})} = 0, \quad (2.39)$$

where $\bar{u} : \mathbb{R}^{n+1} \rightarrow [0, \infty)$ is a Lipschitz function on \mathbb{R}^{n+1} . Now, by Remark 2.4,

$$D^2 \bar{u}_h = -\nabla^2 f_h \mathcal{L}^{n+1} \llcorner \Omega_h + |\nabla f_h| \nu_{\Omega_h} \otimes \nu_{\Omega_h} \mathcal{H}^n \llcorner \partial\Omega_h. \quad (2.40)$$

In particular,

$$\begin{aligned} |D^2 \bar{u}_h|(\mathbb{R}^{n+1}) &= \int_{\Omega_h} |\nabla^2 f_h| + \int_{\partial\Omega_h} |\nabla f_h| \\ &\leq |\Omega_h|^{1/2} \|\nabla^2 f_h\|_{L^2(\Omega_h)} + \left(\int_{\partial\Omega_h} \frac{1}{H} \right)^{1/2} \left(\int_{\partial\Omega_h} H |\nabla f_h|^2 \right)^{1/2} \\ &\leq |\Omega_h| + C(n) P(\Omega_h)^{1/2} |\Omega_h|^{1/2} \leq C(n, L), \end{aligned}$$

where in the last line we have used, in the order, (2.19), (2.6), (2.13), (2.5) and (2.4); as a consequence,

$$D\bar{u} \in BV(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), \quad D^2 \bar{u}_h \xrightarrow{*} D^2 \bar{u} \quad \text{as Radon measures on } \mathbb{R}^{n+1}. \quad (2.41)$$

If $\varphi \in C_c^0(\mathbb{R}^{n+1})$, then by (2.27) and (2.38)

$$(D^2 \bar{u}_h \llcorner \Omega_h)(\varphi) = - \int_{\Omega_h} \varphi \nabla^2 f_h \rightarrow - \frac{\text{Id}}{n+1} \int_G \varphi,$$

so that

$$D^2 \bar{u}_h \llcorner \Omega_h \xrightarrow{*} - \frac{\text{Id}}{n+1} \mathcal{L}^{n+1} \llcorner G \quad \text{as Radon measures in } \mathbb{R}^{n+1}. \quad (2.42)$$

By (2.41) and (2.42), if μ denotes the weak-* limit of the Radon measures

$$\mu_h = D^2 \bar{u}_h \llcorner (\mathbb{R}^{n+1} \setminus \Omega_h) = |\nabla f_h| \nu_{\Omega_h} \otimes \nu_{\Omega_h} \mathcal{H}^n \llcorner \partial\Omega_h,$$

then we have

$$D^2 \bar{u} = - \frac{\text{Id}}{n+1} \mathcal{L}^{n+1} \llcorner G + \mu. \quad (2.43)$$

We claim that

$$|\{\bar{u} > 0\} \setminus G| = 0, \quad \text{spt}\mu \cap \{\bar{u} > 0\} = \emptyset. \quad (2.44)$$

To prove the first part of (2.44), we note that if $\bar{u}(x) > 0$, then by uniform convergence $\bar{u}_h \geq \bar{u}(x)/2$ on B_{x,s_x} for every $h \geq h_x$ and for some $s_x > 0$, so that $B_{x,s_x} \subset \Omega_h$ for every $h \geq h_x$. This implies that $|B_{x,s_x} \setminus G| = 0$ (thus the first part of (2.44)), and also that $B_{x,s_x} \cap \text{spt}\mu_h = \emptyset$: since $\text{spt}\mu$ is contained in the set of the accumulation points of sequences $\{x_h\}_{h \in \mathbb{N}}$ with $x_h \in \partial\Omega_h$, we have proved (2.44). By combining (2.43) and (2.44) we deduce that

$$D^2\bar{u}_- \{\bar{u} > 0\} = -\frac{\text{Id}}{n+1} \mathcal{L}^{n+1}_- \{\bar{u} > 0\}. \quad (2.45)$$

Now let $\{A_j\}_{j \in J}$ denote the connected components of the open set $\{\bar{u} > 0\}$, then by (2.45) we can find $z_j \in \mathbb{R}^{n+1}$ and $c_j \in \mathbb{R}$ such that

$$\bar{u}(x) = c_j - \frac{|x - z_j|^2}{2(n+1)}, \quad \forall x \in A_j,$$

and since $\bar{u} \geq 0$ it must be

$$c_j \geq 0, \quad A_j \subset B_{z_j, s_j} \quad \text{where } s_j = (2(n+1)c_j)^{1/2},$$

thus $c_j > 0$ because A_j is open. In conclusion,

$$\{\bar{u} > 0\} = \bigcup_{j \in J} A_j \subset \bigcup_{j \in J} B_{z_j, s_j} \subset \{\bar{u} > 0\},$$

that is, $\bar{u} = -f_G$,

$$\bar{u}(x) = \sum_{j \in J} \max \left\{ \frac{s_j^2 - |x - z_j|^2}{2(n+1)}, 0 \right\}, \quad A_j = B_{z_j, s_j}. \quad (2.46)$$

We now want to prove that $|G \Delta \{\bar{u} > 0\}| = 0$ and that J is finite with $s_j = 1$ for every $j \in J$. To this end we first notice that $s_j \leq 1$ for every $j \in J$. Indeed, by (2.46) we have that

$$\{\bar{u} > \varepsilon\} = \bigcup_{j \in J} B_{z_j, \sqrt{(s_j^2 - 2(n+1)\varepsilon)_+}}, \quad \forall \varepsilon > 0,$$

so that, by uniform convergence,

$$\bigcup_{j \in J} B_{z_j, \sqrt{(s_j^2 - 2(n+1)\varepsilon)_+}} \subset \{u_h > \frac{\varepsilon}{2}\} \subset \Omega_h, \quad \forall h \geq h_\varepsilon.$$

In particular, if we fix $j \in J$, pick $\varepsilon < s_j^2/2(n+1)$, and let $h \geq h_{\varepsilon, j}$, then by the previous inclusion there exists $y \in \partial\Omega_h$ such that

$$\frac{n}{\sqrt{s_j^2 - 2(n+1)\varepsilon}} \geq H_{\partial\Omega_h}(y) \geq n(1 - \delta(\Omega_h)),$$

that is, letting $h \rightarrow \infty$, $s_j^2 - 2(n+1)\varepsilon \leq 1$. By the arbitrariness of ε , we conclude that $s_j \leq 1$. We now apply Pohozaev's identity (2.14) to f_h to find

$$(n+3) \int_{\mathbb{R}^{n+1}} \bar{u}_h = (n+3) \int_{\Omega_h} (-f_h) = \int_{\partial\Omega_h} (x \cdot \nu_{\Omega_h}) |\nabla f_h|^2,$$

so that by (2.39), (2.28), and the divergence theorem we find

$$(n+3) \int_{\mathbb{R}^{n+1}} \bar{u} = \lim_{h \rightarrow \infty} \int_{\partial\Omega_h} \frac{(x \cdot \nu_{\Omega_h})}{(n+1)^2} = \frac{|G|}{n+1} \geq \frac{|B|}{n+1} \sum_{j \in J} s_j^{n+1}.$$

At the same time, by (2.46) and a simple computation we find

$$(n+1)(n+3) \int_{\mathbb{R}^{n+1}} \bar{u} = |B| \sum_{j \in J} s_j^{n+3},$$

so that

$$\sum_{j \in J} s_j^{n+1} (1 - s_j^2) \leq 0.$$

Since $s_j \in (0, 1]$ for every $j \in J$, we conclude that $s_j = 1$ for every $j \in J$. As a consequence, $\#J \leq L$, because of

$$(L+1-a)|B| \geq \lim_{h \rightarrow \infty} |\Omega_h| = |G| \geq |\{\bar{u} > 0\}| = \#J|B|.$$

Since J is finite we deduce from (2.46) that

$$D^2 \bar{u} = -\frac{\text{Id}}{n+1} \mathcal{L}^{n+1} \llcorner \bigcup_{j \in J} B_{z_j, 1} + \sum_{j \in J} \frac{\nu_{B_{z_j, 1}} \otimes \nu_{B_{z_j, 1}}}{n+1} \mathcal{H}^n \llcorner \partial B_{z_j, 1}.$$

By comparing this formula with (2.43) we conclude that $|G \Delta \{\bar{u} > 0\}| = 0$, provided we can show that the measure μ appearing in (2.43) is singular with respect to \mathcal{L}^{n+1} , of course. To this end, it suffices to consider the multiplicity one varifolds V_h associated to $\partial \Omega_h$. Since (in the notation and terminology of [Sim83, Chapter 8]) the varifolds $\{V_h\}_{h \in \mathbb{N}}$ have uniformly bounded masses (as $\mathbf{M}(V_h) = \mathcal{H}^n(\partial \Omega_h)$) and uniformly bounded generalized mean curvatures (thanks to (2.6)), by [Sim83, Theorem 42.7, Remark 42.8] there exists an integer multiplicity rectifiable n -varifold V such that $V_h \xrightarrow{*} V$ as varifolds. In particular, if V is supported on the n -rectifiable set M , and if θ denotes the integer multiplicity of V , then, denoting by ν_M a Borel vector-field such that $\nu_M(x)^\perp = T_x M$ for \mathcal{H}^n -a.e. $x \in M$, we get

$$\int_M \varphi \theta \nu_M \otimes \nu_M d\mathcal{H}^n = \lim_{h \rightarrow \infty} \int_{\partial \Omega_h} \varphi \nu_{\Omega_h} \otimes \nu_{\Omega_h} d\mathcal{H}^n, \quad \forall \varphi \in C_c^0(\mathbb{R}^{n+1}).$$

Hence, by (2.28) and by definition of μ_h and μ we conclude that

$$\mu = \frac{\theta}{n+1} \nu_M \otimes \nu_M \mathcal{H}^n \llcorner M.$$

As explained this shows that $|G \Delta \{\bar{u} > 0\}| = 0$, and thus, from now on we directly set

$$G = \bigcup_{j \in J} B_{z_j, 1}.$$

Let us prove that $P(\Omega_h) \rightarrow P(G)$. By the divergence theorem,

$$|(n+1)|\Omega_h| - P(\Omega_h)| = \left| \int_{\partial \Omega_h} \left(1 - \frac{H_{\partial \Omega_h}}{n}\right) (x \cdot \nu_{\Omega_h}) \right| \leq \text{diam}(\Omega_h) \delta(\Omega_h),$$

while at the same time $(n+1)|G| = P(G)$, so that

$$|P(\Omega_h) - P(G)| \leq (n+1)|\Omega_h| - |G| + \text{diam}(\Omega_h) \delta(\Omega_h) \leq (n+1)|\Omega_h \Delta G| + \text{diam}(\Omega_h) \delta(\Omega_h), \quad (2.47)$$

and $P(\Omega_h) \rightarrow P(G)$, as claimed. This last fact implies in particular that

$$\mathcal{H}^n \llcorner \partial \Omega_h \xrightarrow{*} \mathcal{H}^n \llcorner \partial G \quad \text{as Radon measures in } \mathbb{R}^{n+1}. \quad (2.48)$$

By (2.48), (2.9) and a classical argument we immediately prove that $\text{hd}(\partial \Omega_h, \partial G) \rightarrow 0$. Since $\partial \Omega_h$ is connected for every h , $\text{hd}(\partial \Omega_h, \partial G) \rightarrow 0$ implies that ∂G is connected. We are thus left to prove the existence of sets Σ_h and maps ϕ_h with the claimed properties. To this end we put the proof of the theorem on hold, and recall some basic useful facts from the regularity theory for integer rectifiable varifolds. \square

Given $x \in \mathbb{R}^{n+1}$, $\nu \in S^n$ and $r > 0$ we set

$$\begin{aligned} \mathbf{C}_{x,r}^\nu &= \{y \in \mathbb{R}^{n+1} : |\mathbf{p}_\nu(y-x)| < r, |(y-x) \cdot \nu| < r\}, & \mathbf{C}_r &= \mathbf{C}_{0,r}^{e_n}, & \mathbf{C} &= \mathbf{C}_1, \\ \mathbf{D}_{x,r}^\nu &= \{y \in \mathbb{R}^{n+1} : |\mathbf{p}_\nu(y-x)| < r, (y-x) \cdot \nu = 0\}, & \mathbf{D}_r &= \mathbf{D}_{0,r}^{e_n}, & \mathbf{D} &= \mathbf{D}_1, \end{aligned}$$

where $\mathbf{p}_\nu(v) = v - (v \cdot \nu)\nu$ for every $v \in \mathbb{R}^{n+1}$. Given $u \in C^{k,\gamma}(\mathbf{D}_r)$, it will be useful to consider, along with the standard $C^{k,\gamma}$ -norms on \mathbf{D}_r , the scaled norms

$$\|u\|_{C^{k,\gamma}(\mathbf{D}_r)}^* = \sum_{j=0}^k r^{j-1} \|D^j u\|_{C^0(\mathbf{D}_r)} + r^{k-1+\gamma} \|D^k u\|_{C^{0,\gamma}(\mathbf{D}_r)},$$

which are invariant by scaling in the sense that, if we set $\lambda_r(u)(x) = r^{-1} u(rx)$ for $x \in \mathbf{D}$, then

$$\|\lambda_r(u)\|_{C^{k,\gamma}(\mathbf{D})} = \|\lambda_r(u)\|_{C^{k,\gamma}(\mathbf{D}_r)}^* = \|u\|_{C^{k,\gamma}(\mathbf{D}_r)}^*, \quad \forall r > 0.$$

We shall need the following technical lemma, which just amounts to a simple application of the implicit function theorem, and whose proof can be found in [CLM14, Lemma 4.3]. In the statement, given $u : \mathbf{D}_{4r} \rightarrow \mathbb{R}$ with $|u| < 4r$ on \mathbf{D}_{4r} , we set

$$\Gamma_r(u) = (\text{Id} + u e_n)(\mathbf{D}_{4r}) \subset \mathbf{C}_{4r}.$$

Lemma 2.7. *Given $n \geq 1$, $M > 0$ and $\gamma \in [0, 1]$ there exist positive constants $\kappa_0 = \kappa_0(n, M, \gamma) < 1$ and $\kappa_1 = \kappa_1(n, M, \gamma)$ with the following property. If $u_1 \in C^{2,1}(\mathbf{D}_{4r})$, $u_2 \in C^{1,\gamma}(\mathbf{D}_{4r})$, and*

$$\max_{i=1,2} \|u_i\|_{C^1(\mathbf{D}_{4r})}^* \leq \kappa_0, \quad \max \{ \|u_1\|_{C^{2,1}(\mathbf{D}_{4r})}^*, \|u_2\|_{C^{1,\gamma}(\mathbf{D}_{4r})}^* \} \leq M,$$

then there exists $\psi \in C^{1,\gamma}(\mathbf{C}_{2r} \cap \Gamma_r(u_1))$ such that

$$\mathbf{C}_r \cap \Gamma_r(u_2) \subset (\text{Id} + \psi \nu)(\mathbf{C}_{2r} \cap \Gamma_r(u_1)) \subset \Gamma_r(u_2),$$

$$\begin{aligned} \frac{\|\psi\|_{C^0(\mathbf{C}_{2r} \cap \Gamma_r(u_1))}}{r} + \|\nabla \psi\|_{C^0(\mathbf{C}_{2r} \cap \Gamma_r(u_1))} + r^\gamma \|\nabla \psi\|_{C^{0,\gamma}(\mathbf{C}_{2r} \cap \Gamma_r(u_1))} &\leq \kappa_1, \\ \frac{\|\psi\|_{C^0(\mathbf{C}_{2r} \cap \Gamma_r(u_1))}}{r} + \|\nabla \psi\|_{C^0(\mathbf{C}_{2r} \cap \Gamma_r(u_1))} &\leq \kappa_1 \|u_1 - u_2\|_{C^1(\mathbf{D}_{4r})}. \end{aligned}$$

Here, $\nu \in C^{1,1}(\Gamma_r(u_1); S^n)$ is the normal unit vector field to $\Gamma_r(u_1)$ defined by

$$\nu(z, u_1(z)) = \frac{(-\nabla u_1(z), 1)}{\sqrt{1 + |\nabla u_1(z)|^2}}, \quad \forall z \in \mathbf{D}_{4r}.$$

Next, let S be a \mathcal{H}^n -rectifiable set in \mathbb{R}^{n+1} with bounded generalized mean curvature in some open set V , that is, there exists $\mathbf{H} \in L^\infty(V; \mathcal{H}^n \llcorner S)$ such that

$$\int_S \text{div}^S X d\mathcal{H}^n = \int_S X \cdot \mathbf{H} d\mathcal{H}^n, \quad \forall X \in C_c^1(V; \mathbb{R}^{n+1}),$$

and assume that $S = \text{spt}(\mathcal{H}^n \llcorner S)$, i.e., $\mathcal{H}^n(S \cap B_{x,r}) > 0$ for every $x \in S$, $r > 0$. Set

$$\sigma(S, x, r) = r \|\mathbf{H}\|_{L^\infty(B_{x,r}; \mathcal{H}^n \llcorner S)} + \max \left\{ \frac{\mathcal{H}^n(S \cap B_{x,r})}{\omega_n r^n} - 1, 0 \right\}, \quad x \in S, r > 0, \quad (2.49)$$

where $\omega_n = \mathcal{H}^n(B \cap \{x_1 = 0\})$. Then for every $\gamma \in (0, 1)$, Allard's regularity theorem [All72] (as stated in [Sim83, Theorem 24.2] – see also [DL12, Theorem 3.2]) gives us positive constants $\sigma_0(n, \gamma) < 1$ and $C(n, \gamma)$ with the following property:

Allard's theorem: *With S as above, if $x \in S$ and $r > 0$ are such that $B_{x,r} \subset\subset V$ and*

$$\sigma(S, x, r) \leq \sigma_0(n, \gamma), \quad (2.50)$$

then there exist $\nu \in S^n$ and a Lipschitz map $u : (x + \nu^\perp) \rightarrow \mathbb{R}$ with $u(x) = 0$ such that

$$S \cap \mathbf{C}_{x,\sigma_0 r}^\nu = \{z + u(z)\nu : z \in \mathbf{D}_{x,\sigma_0 r}^\nu\}, \quad \|u\|_{C^{1,\gamma}(\mathbf{D}_{x,\sigma_0 r}^\nu)}^* \leq C(n, \gamma) \sigma(S, x, r)^{1/4n}.$$

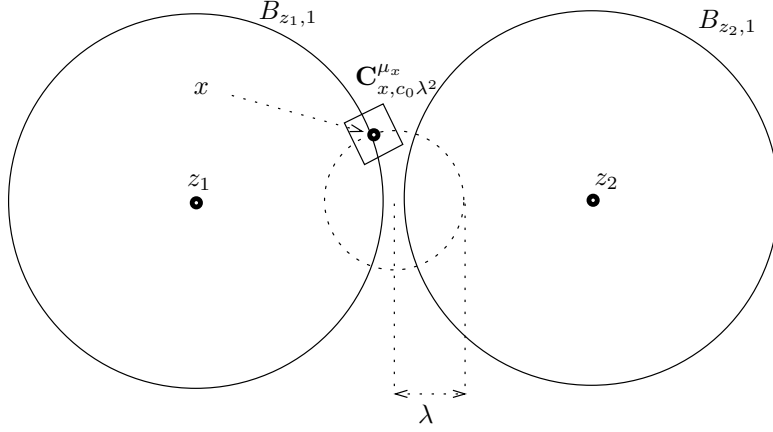


FIGURE 2. If $x \in \Sigma_\lambda \cap \partial B_{z_j,1}$, then $\partial G \cap \mathbf{C}_{x,c_0(n)\lambda^2}^{\mu_x} = \partial B_{z_j,1} \cap \mathbf{C}_{x,c_0(n)\lambda^2}^{\mu_x}$, see also (2.62). Here $\mu_x = \nu_G(x) = \nu_{B_{z_1,1}}(x)$.

(Note that this statement is a particular case of Allard's theorem in the sense that we consider only density one varifolds and we restrict to the codimension one case.) In the following we shall apply this theorem with $\gamma = 1/4n$. Correspondingly, we simply set

$$\sigma_0(n) = \sigma_0\left(n, \frac{1}{4n}\right) < 1.$$

We now prove a technical lemma which will be useful in the proof of Theorem 1.1 too.

Lemma 2.8. *There exist positive constants $\lambda(n) < 1$ and $c_0(n)$ with the following property. Let Ω satisfy (2.1), (2.2), and (2.4), let $\{B_{z_j,1}\}_{j \in J}$ be a disjoint family of unit balls, and set*

$$G = \bigcup_{j \in J} B_{z_j,1}, \quad \Sigma_\lambda = \partial G \setminus \bigcup_{j,\ell \in J, j \neq \ell} B_{(z_j+z_\ell)/2,\lambda} \quad \lambda > 0. \quad (2.51)$$

Assume that to each $\lambda \leq \lambda(n)$ and $x \in \Sigma_\lambda$ one can associate $\rho_x \in (0,1)$ and $y \in \partial\Omega$ in such a way that

$$\frac{c_0(n)\lambda^2}{2} \leq \rho_x \leq c_0(n)\lambda^2, \quad (2.52)$$

$$|x - y| = \text{dist}(x, \partial\Omega) \leq \frac{\sigma_0(n)\rho_x^2}{2}, \quad (2.53)$$

$$\sigma(\partial\Omega, y, \rho_x) \leq \sigma_0(n)\lambda(n), \quad (2.54)$$

$$|\Omega \Delta G| \leq C(n)\rho_x^{n+1}\sigma(\partial\Omega, y, \rho_x)^{1/4n}. \quad (2.55)$$

Then for every $\lambda \leq \lambda(n)$ there exists $\psi^\lambda : \Sigma_\lambda \rightarrow \mathbb{R}$ such that

$$\|\psi^\lambda\|_{C^{1,\gamma}(\Sigma_\lambda)} \leq C(n, \gamma), \quad \forall \gamma \in (0,1), \quad (2.56)$$

$$\lambda^{-2} \|\psi^\lambda\|_{C^0(\Sigma_\lambda)} + \|\nabla \psi^\lambda\|_{C^1(\Sigma_\lambda)} \leq C(n) \max_{x \in \Sigma_\lambda} \sigma(\partial\Omega, y, \rho_x)^{1/4n}, \quad (2.57)$$

$$(\text{Id} + \psi^\lambda \nu_G)(\Sigma_\lambda) \subset \partial\Omega. \quad (2.58)$$

Remark 2.9. Note that we do not assume ∂G to be connected. In other words, the balls $B_{z_j,1}$ need not to be tangent, although they may be arbitrarily close or even mutually tangent, and actually this last case will somehow be the “worst” case to keep in mind. We also note that for $\lambda(n)$ small enough, if $\lambda \leq \lambda(n)$, then $\partial G \setminus \Sigma_\lambda$ consists of finitely many (precisely, at most $C(n)\# J$ -many) spherical caps of diameters bounded by $C\lambda$.

Proof of Lemma 2.8. We claim that for every $\lambda \leq \lambda(n)$ and $x \in \Sigma_\lambda$ there exists

$$\psi_x \in C^{1,1/4n}(\mathbf{C}_{x,\sigma_0 \rho_x/4}^{\mu_x} \cap \partial G)$$

such that

$$\begin{aligned} \mathbf{C}_{x,\sigma_0 \rho_x/8}^{\mu_x} \cap \partial \Omega &\subset (\text{Id} + \psi_x \nu_G)(\mathbf{C}_{x,\sigma_0 \rho_x/4}^{\mu_x} \cap \partial G) \subset \mathbf{C}_{x,\sigma_0 \rho_x/2}^{\mu_x} \cap \partial \Omega, \\ \|\psi_x\|_{C^{1,1/4n}(\mathbf{C}_{x,\sigma_0 \rho_x/4}^{\mu_x} \cap \partial G)} &\leq C(n), \end{aligned} \quad (2.59)$$

$$\rho_x^{-1} \|\psi_x\|_{C^0(\mathbf{C}_{x,\sigma_0 \rho_x/4}^{\mu_x} \cap \partial G)} + \|\nabla \psi_x\|_{C^0(\mathbf{C}_{x,\sigma_0 \rho_x/4}^{\mu_x} \cap \partial G)} \leq C(n) \sigma(\partial \Omega, y, \rho_x)^{1/4n}.$$

Postponing for the moment the proof of the claim, let us show how it allows one to complete the proof of the lemma. Indeed, (2.59) implies that, for every $x_1, x_2 \in \Sigma_\lambda$, $\psi_{x_1} = \psi_{x_2}$ on the intersection of their respective domains of definition. Then, by setting

$$\psi^\lambda(z) = \psi_x(z), \quad \forall z \in \mathbf{C}_{x,\sigma_0 \rho_x/4}^{\mu_x} \cap \partial G, x \in \Sigma_\lambda,$$

one defines a function $\psi^\lambda \in C^{1,1/4n}(\Sigma_\lambda)$ such that (2.56), (2.57) and (2.58) hold. Moreover, (2.56) follows from elliptic regularity, as $\|\psi^\lambda\|_{C^{1,1/4n}(\Sigma_\lambda)} \leq C(n)$ and the mean curvature of the graph of ψ^λ over Σ_λ is the mean curvature of Ω , and thus it is bounded and continuous.

We are thus left to prove our claim. To this end we consider $r(n) > 0$ such that

$$\mathcal{H}^n(B_{z,s} \cap \partial B) \leq (1 + C(n) s^2) \omega_n s^n, \quad \forall z \in \partial B, \forall s < r(n), \quad (2.60)$$

$$\sup \{ |(p-z) \cdot \nu_B(z)| : p \in B_{z,s} \cap \partial B \} \leq C(n) s^2, \quad \forall z \in \partial B, \forall s < r(n), \quad (2.61)$$

we fix $x \in \Sigma_\lambda$, $\lambda \leq \lambda(n)$, let y and ρ_x be as in the statement, and set

$$\mu_x = \nu_G(x) = \nu_{B_{z_j,1}}(x) \quad \text{for the unique } j \in J \text{ such that } x \in \partial B_{z_j,1}.$$

If $c_0(n) \lambda(n)^2 \leq r(n)$, then by definition of μ_x there exist $\hat{C}(n)$ and a Lipschitz map $w_x : (x + \mu_x^\perp) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \partial G \cap \mathbf{C}_{x,c_0(n)\lambda^2}^{\mu_x} &= \partial B_{z_j,1} \cap \mathbf{C}_{x,c_0(n)\lambda^2}^{\mu_x} = \{z + w_x(z) \mu_x : z \in \mathbf{D}_{x,c_0(n)\lambda^2}^{\mu_x}\}, \\ \|w_x\|_{C^{2,1}(\mathbf{D}_{x,c_0(n)\lambda^2}^{\mu_x})} &\leq \hat{C}(n), \quad \|w_x\|_{C^1(\mathbf{D}_{x,r}^{\mu_x})}^* \leq \hat{C}(n) r, \quad \forall r \leq c_0(n) \lambda^2. \end{aligned} \quad (2.62)$$

By (2.54) and by Allard's theorem, there exist $\nu_x \in S^n$ and a Lipschitz map $u_x : (y + \nu_x^\perp) \rightarrow \mathbb{R}$ such that $u_x(y) = 0$,

$$\begin{aligned} \partial \Omega \cap \mathbf{C}_{y,\sigma_0 \rho_x}^{\nu_x} &= \{z + u_x(z) \nu_x : z \in \mathbf{D}_{y,\sigma_0 \rho_x}^{\nu_x}\}, \\ \|u_x\|_{C^{1,1/4n}(\mathbf{D}_{y,\sigma_0 \rho_x}^{\nu_x})}^* &\leq C(n) \sigma(\partial \Omega, y, \rho_x)^{1/4n}. \end{aligned} \quad (2.63)$$

Now we let

$$K_y = \{z \in \mathbf{C}_{y,\sigma_0 \rho_x}^{\nu_x} : (z-y) \cdot \nu_x \leq 0\}, \quad K_x = \{z \in \mathbf{C}_{x,\sigma_0 \rho_x}^{\mu_x} : (z-x) \cdot \mu_x \leq 0\}.$$

Up to switching ν_x with $-\nu_x$, and since $|u_x| \leq C(n) \rho_x \sigma(\partial \Omega, y, \rho_x)^{1/4n}$ on $\mathbf{D}_{y,\sigma_0 \rho_x}^{\nu_x}$ by (2.63) we can assume that

$$|(K_y \Delta \Omega) \cap \mathbf{C}_{y,\sigma_0 \rho_x}^{\nu_x}| \leq C(n) \rho_x^{n+1} \sigma(\partial \Omega, y, \rho_x)^{1/4n}. \quad (2.64)$$

Similarly, by (2.62) we have $|w_x| \leq C(n) \rho_x^2$ on $\mathbf{D}_{x,\sigma_0 \rho_x}^{\mu_x}$, and thus

$$|(K_x \Delta G) \cap \mathbf{C}_{x,\sigma_0 \rho_x}^{\mu_x}| \leq C(n) \rho_x^{n+2} \leq C(n) \rho_x^{n+1} \sigma(\partial \Omega, y, \rho_x)^{1/4n}, \quad (2.65)$$

as (2.49) and (2.6) imply $\rho_x \leq C(n) \sigma(\partial \Omega, y, \rho_x)$. Since $|y-x| \leq \sigma_0 \rho_x/2$ by (2.53) and $\rho_x < 1$, we find

$$B_{x,\sigma_0 \rho_x/2} \subset B_{y,\sigma_0 \rho_x} \subset \mathbf{C}_{y,\sigma_0 \rho_x}^{\nu_x}, \quad \text{as well as } B_{x,\sigma_0 \rho_x/2} \subset \mathbf{C}_{x,\sigma_0 \rho_x}^{\mu_x} \text{ of course,}$$

and thus, by (2.64) and (2.65),

$$\begin{aligned} |\Omega\Delta G| &\geq |(\Omega\Delta G) \cap B_{x,\sigma_0\rho_x/2}| \geq |(K_x\Delta K_y) \cap B_{x,\sigma_0\rho_x/2}| - C(n)\rho_x^{n+1}\sigma(\partial\Omega, y, \rho_x)^{1/4n} \\ &\geq |((K_y + x - y)\Delta K_x) \cap B_{x,\sigma_0\rho_x/2}| \\ &\quad - C(n)\rho_x^{n+1}\sigma(\partial\Omega, y, \rho_x)^{1/4n} - |(K_y + x - y)\Delta K_y|. \end{aligned} \quad (2.66)$$

On the one hand, for every $z \in \mathbb{R}^{n+1}$, $r > 0$ and $\nu, \nu' \in S^n$ one has

$$\left| \left\{ p \in \mathbf{C}_{z,r}^\nu : (p-z) \cdot \nu \leq 0 \right\} \Delta \left\{ p \in \mathbf{C}_{z,r}^{\nu'} : (p-z) \cdot \nu' \leq 0 \right\} \right| \cap B_{z,r/2} \geq c(n) |\nu - \nu'| r^{n+1}; \quad (2.67)$$

on the other hand, again by $|y - x| \leq \sigma_0\rho_x^2/2$,

$$|K_y\Delta(x - y + K_y)| \leq C(n)P(K_y)|y - x| \leq C(n)\rho_x^n|y - x| \leq C(n)\rho_x^{n+2}. \quad (2.68)$$

By (2.66), (2.67), and (2.68) we conclude that

$$c(n)|\nu_x - \mu_x|\rho_x^{n+1} \leq C(n)\rho_x^{n+1}\sigma(\partial\Omega, y, \rho_x)^{1/4n} + |\Omega\Delta G|,$$

so that (2.55) and (2.54) give us

$$|\nu_x - \mu_x| \leq C(n)\sigma(\partial\Omega, y, \rho_x)^{1/4n}. \quad (2.69)$$

By (2.53), (2.63), and (2.69), provided $\lambda(n)$ is small enough, there exist a constant $C_*(n)$ and a Lipschitz map $v_x : (x + \mu_x^\perp) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \partial\Omega \cap \mathbf{C}_{x,\sigma_0\rho_x/2}^{\mu_x} &= \{z + v_x(z)\mu_x : z \in \mathbf{D}_{x,\sigma_0\rho_x/2}^{\mu_x}\}, \\ \|v_x\|_{C^{1,1/4n}(\mathbf{D}_{x,\sigma_0\rho_x/2}^{\mu_x})} &\leq C_*(n), \quad \|v_x\|_{C^1(\mathbf{D}_{x,\sigma_0\rho_x/2}^{\mu_x})}^* \leq C_*(n)\sigma(\partial\Omega, y, \rho_x)^{1/4n}. \end{aligned}$$

By this last property, by (2.52), and by (2.62) we can apply Lemma 2.7 into the cylinder $\mathbf{C}_{x,\sigma_0\rho_x/2}^{\mu_x}$: indeed, setting by a rigid motion $x = 0$ and $\mu_x = 0$, and choosing

$$4r = \frac{\sigma_0\rho_x}{2}, \quad u_1 = w_x, \quad u_2 = v_x, \quad \gamma = \frac{1}{4n}, \quad M = \max\{\hat{C}(n), C_*(n)\},$$

we find that

$$\begin{aligned} \max_{i=1,2} \|u_i\|_{C^1(\mathbf{D}_{4r})}^* &= \max\{\|w_x\|_{C^1(\mathbf{D}_{x,\sigma_0\rho_x/2}^{\mu_x})}^*, \|v_x\|_{C^1(\mathbf{D}_{x,\sigma_0\rho_x/2}^{\mu_x})}^*\} \\ &\leq \max\{\hat{C}(n)\frac{\sigma_0\rho_x}{2}, C_*(n)\sigma(\partial\Omega, y, \rho_x)^{1/4n}\} \leq C(n)\lambda(n)^{1/4n} \leq \kappa_0(n, \gamma, M), \end{aligned}$$

provided $\lambda(n)$ is small enough. By Lemma 2.7, there exists $\psi_x \in C^{1,1/4n}(\mathbf{C}_{x,\sigma_0\rho_x/4}^{\mu_x} \cap \partial G)$ satisfying (2.59). \square

Proof of Theorem 2.5, conclusion. We now conclude the proof of Theorem 2.5. Let us recall the situation we left: we have $\{\Omega_h\}_{h \in \mathbb{N}}$ satisfying (2.1), (2.2) and (2.4) (with the same L and a) and

$$\lim_{h \rightarrow \infty} |\Omega_h\Delta G| + \text{hd}(\partial\Omega_h, \partial G) + |P(\Omega_h) - P(G)| = 0, \quad (2.70)$$

where G is the union over a finite family of disjoint unit balls $\{B_{z_j,1}\}_{j \in J}$. To complete the proof of the theorem, we need to prove the existence of $\Sigma_h \subset \partial G$ and of $\phi_h : \Sigma_h \rightarrow \mathbb{R}$ such that $\partial G \setminus \Sigma_h$ consists of at most $C(n)L$ -many spherical caps with vanishing diameters, $(\text{Id} + \phi_h\nu_G)(\Sigma_h) \subset \partial\Omega_h$, and

$$\lim_{h \rightarrow \infty} \|\phi_h\|_{C^1(\Sigma_h)} + \mathcal{H}^n(\partial\Omega_h \setminus (\text{Id} + \phi_h\nu_G)(\Sigma_h)) = 0, \quad \sup_{h \in \mathbb{N}} \|\phi_h\|_{C^{1,\gamma}(\Sigma_h)} \leq C(n, \gamma), \quad (2.71)$$

for every $\gamma \in (0, 1)$. To this end, we want to apply Lemma 2.8 to $\Omega = \Omega_h$. Let us fix $\lambda \leq \lambda(n)$, define Σ_λ as in (2.51), and for every $x \in \Sigma_\lambda$ let us set

$$\rho_x = c_0(n)\lambda^2,$$

so that (2.52) holds trivially. Let us now fix $x \in \Sigma_\lambda$, and consider $y_h \in \partial\Omega_h$ such that $|x - y_h| = \text{dist}(x, \partial\Omega_h)$. By $\text{hd}(\partial\Omega_h, \partial G) \rightarrow 0$ we have

$$|x - y_h| \leq \text{hd}(\partial\Omega_h, \partial G) \leq \frac{\sigma_0(n)c_0(n)^2 \lambda^4}{2}, \quad \forall h \geq h_\lambda,$$

provided $h_\lambda \in \mathbb{N}$ is large enough; in particular, (2.53) holds for h large enough. Next, we notice that by (2.48) and $|y_h - x| \rightarrow 0$ we have

$$\limsup_{h \rightarrow \infty} P(\Omega_h; B_{y_h, \rho_x}) \leq P(G; B_{x, \rho_x}),$$

so that (2.6), the definition of ρ_x , and (2.60) (applied with $s = c_0(n) \lambda^2 \leq r(n)$) give us

$$\begin{aligned} \limsup_{h \rightarrow \infty} \sigma(\partial\Omega_h, y_h, \rho_x) &\leq 2n\rho_x + \frac{P(G; B_{x, \rho_x})}{\omega_n \rho_x^n} - 1 \leq C(n) \left(\lambda^2 + \frac{P(G; B_{x, c_0(n) \lambda^2})}{\omega_n (c_0(n) \lambda^2)^n} - 1 \right) \\ &\leq C(n) \lambda^2 \leq \frac{\sigma_0(n) \lambda}{2} \leq \frac{\sigma_0(n) \lambda(n)}{2}; \end{aligned} \quad (2.72)$$

in particular, (2.54) holds for h large enough. Finally, (2.6) and (2.49) imply $\sigma(\partial\Omega_h, y_h, \rho_x) \geq (n/2)\rho_x \geq c(n)\lambda^2$, thus up to take h_λ large enough to entail $|\Omega_h \Delta G| \leq C(n) \lambda^{n+3}$ we find that (2.55) holds. By Lemma 2.8 we conclude that for every $\lambda \leq \lambda(n)$ there exists $\{\psi_h^\lambda\}_{h \geq h_\lambda} \subset C^{1,\gamma}(\Sigma_\lambda)$ for every $\gamma \in (0, 1)$ such that

$$(\text{Id} + \psi_h^\lambda \nu_G)(\Sigma_\lambda) \subset \partial\Omega_h, \quad \|\psi_h^\lambda\|_{C^{1,\gamma}(\Sigma_\lambda)} \leq C(n, \gamma), \quad \|\psi_h^\lambda\|_{C^1(\Sigma_\lambda)} \leq C(n) \lambda^{1/2n}, \quad (2.73)$$

where in proving the last bound we have also taken into account the first inequality in (2.72). Since

$$\begin{aligned} \mathcal{H}^n(\partial\Omega_h \setminus (\text{Id} + \psi_h^\lambda \nu_G)(\Sigma_\lambda)) &\leq P(\Omega_h) - P(G) + \mathcal{H}^n(\partial G \setminus \Sigma_\lambda) \\ &\quad + |\mathcal{H}^n(\Sigma_\lambda) - \mathcal{H}^n((\text{Id} + \psi_h^\lambda \nu_G)(\Sigma_\lambda))|, \end{aligned}$$

by $P(\Omega_h) \rightarrow P(G)$ and by $\|\psi_h^\lambda\|_{C^1(\Sigma_\lambda)} \leq C(n) \lambda^{1/2n}$ we find that

$$\limsup_{h \rightarrow \infty} \mathcal{H}^n(\partial\Omega_h \setminus (\text{Id} + \psi_h^\lambda \nu_G)(\Sigma_\lambda)) \leq \mathcal{H}^n(\partial G \setminus \Sigma_\lambda) + C(n) \mathcal{H}^n(\Sigma_\lambda) \lambda^{1/2n}.$$

We complete the proof of the theorem by first considering any $\lambda_h \rightarrow 0$, and then by setting $\phi_h = \psi_{k(h)}^{\lambda_h}$ for a properly chosen $k(h) \rightarrow \infty$. \square

We now begin the proof of Theorem 1.1, that is, we consider the problem of turning the qualitative information provided in Theorem 2.5 into quantitative estimates in terms of $\delta(\Omega)$. Recall that, as in the introduction, we set

$$\alpha = \frac{1}{2(n+2)}.$$

Proof of Theorem 1.1. Step one: With f as in (2.12), let us set

$$\varepsilon = |\Omega|^{1/(n+1)} \eta(\Omega)^\alpha, \quad \Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}, \quad f_\varepsilon = f \star w_\varepsilon, \quad (2.74)$$

where $w_\varepsilon(x) = \varepsilon^{-(n+1)} w(x/\varepsilon)$ for $w \in C_c^\infty(B)$ with $w \geq 0$, $w(x) = w(-x)$ for every $x \in \mathbb{R}^{n+1}$, and $\int_{\mathbb{R}^{n+1}} w = 1$. We claim that, if $C_0(n)$ is the constant appearing in (2.35), then

$$\|\nabla f_\varepsilon\|_{C^0(\Omega_\varepsilon)} \leq C_0(n), \quad (2.75)$$

$$\|f_\varepsilon - f\|_{C^0(\Omega_\varepsilon)} \leq C_0(n) \varepsilon, \quad (2.76)$$

$$\left\| \nabla^2 f_\varepsilon - \frac{\text{Id}}{n+1} \right\|_{C^0(\Omega_\varepsilon)} \leq C(n) \eta(\Omega)^\alpha, \quad (2.77)$$

$$\|\nabla^2 f_\varepsilon\|_{C^0(\Omega_\varepsilon)} \leq C(n), \quad \nabla^2 f_\varepsilon(x) \geq \frac{\text{Id}}{2(n+1)}, \quad \forall x \in \Omega_\varepsilon. \quad (2.78)$$

Indeed, (2.75) and (2.76) are obvious. If $x \in \Omega_\varepsilon$, then (2.77) follows by (2.27) and (1.4),

$$\left| \nabla^2 f_\varepsilon(x) - \frac{\text{Id}}{n+1} \right| \leq \frac{\|w\|_{C^0(\mathbb{R}^{n+1})}}{\varepsilon^{n+1}} \int_\Omega \left| \nabla^2 f - \frac{\text{Id}}{n+1} \right| \leq C(n) \eta(\Omega)^{(1/2)-\alpha(n+1)}.$$

Finally, (2.78) follows from (2.77) and (2.11) provided $\delta(\Omega) \leq c(n)$ for $c(n)$ small enough.

Step two: Next we define

$$\rho = C_0(n) \varepsilon = C_0(n) |\Omega|^{1/(n+1)} \eta(\Omega)^\alpha, \quad A = \{f_\varepsilon < -3\rho\}. \quad (2.79)$$

We claim that

$$\{f < -4\rho\} \subset A \subset \{f < -2\rho\}, \quad \{f < -\rho\} \subset \Omega_\varepsilon, \quad (2.80)$$

and that if $\{A_i\}_{i \in I}$ are the connected components of A , then each A_i is convex and there exist $x_i \in A_i$ and $0 < r_1^i \leq r_2^i < \infty$ such that

$$B_{x_i, r_1^i} \subset A_i \subset B_{x_i, r_2^i}, \quad 1 \geq \frac{r_1^i}{r_2^i} \geq 1 - C(n) \eta(\Omega)^\alpha \geq \frac{1}{2}, \quad (2.81)$$

$$r_1^i \leq 1 + C_1 \delta(\Omega), \quad r_2^i \leq 1 + C(n) \delta(\Omega)^\alpha, \quad (2.82)$$

$$\int_{\Omega \setminus \bigcup_{i \in I} B_{x_i, r_1^i}} (-f) \leq C(n) |\Omega|^{(n+2)/(n+1)} \eta(\Omega)^\alpha. \quad (2.83)$$

Let us first prove that $\{f < -\rho\} \subset \Omega_\varepsilon$: indeed, if $f(x) < -\rho$ but there exists $y \in \partial\Omega$ with $|y - x| = \text{dist}(x, \partial\Omega) \leq \varepsilon$, then the segment joining x to y is contained in Ω , and thus by (2.35)

$$-\rho > f(x) \geq f(y) - C_0(n) |x - y| \geq f(y) - \rho = -\rho,$$

a contradiction. Similarly, our choice of ρ and (2.76) imply the other inclusions in (2.80). By (2.78), $A = \{f_\varepsilon < -3\rho\}$ is an open set with convex connected components $\{A_i\}_{i \in I}$. Let $x_i \in A_i$ be such that $f_\varepsilon(x_i) \leq f_\varepsilon(x)$ for every $x \in A_i$, and define

$$g_i(x) = \frac{|x - x_i|^2}{2(n+1)} + f_\varepsilon(x_i), \quad x \in \mathbb{R}^{n+1}.$$

By (2.77),

$$|\nabla^2(f_\varepsilon - g_i)(x)| \leq C(n) \eta(\Omega)^\alpha, \quad \forall x \in \Omega_\varepsilon, \quad (2.84)$$

so that, by the convexity of A_i , $g_i(x_i) = f_\varepsilon(x_i)$, and $\nabla g_i(x_i) = \nabla f_\varepsilon(x_i) = 0$,

$$|\nabla f_\varepsilon(x) - \nabla g_i(x)| \leq C(n) \eta(\Omega)^\alpha |x - x_i|, \quad \forall x \in A_i, \quad (2.85)$$

$$|f_\varepsilon(x) - g_i(x)| \leq C(n) \eta(\Omega)^\alpha |x - x_i|^2, \quad \forall x \in A_i. \quad (2.86)$$

Let now $r_1^i \leq r_2^i$ be such that

$$r_1^i = \sup \{r > 0 : B_{x_i, r} \subset A_i\}, \quad r_2^i = \inf \{r > 0 : A_i \subset B_{x_i, r}\}.$$

By definition there are $\nu_1, \nu_2 \in S^n$ such that $x_i + r_1^i \nu_1, x_i + r_2^i \nu_2 \in \partial A_i \subset \{f_\varepsilon = -3\rho\}$: hence,

$$\begin{aligned} 0 &= f_\varepsilon(x_i + r_2^i \nu_2) - f_\varepsilon(x_i + r_1^i \nu_1) \\ &\geq g_i(x_i + r_2^i \nu_2) - g_i(x_i + r_1^i \nu_1) - C(n) \eta(\Omega)^\alpha ((r_1^i)^2 + (r_2^i)^2) \\ &= \frac{(r_2^i)^2 - (r_1^i)^2}{2(n+1)} - C(n) \eta(\Omega)^\alpha ((r_1^i)^2 + (r_2^i)^2), \end{aligned} \quad (2.87)$$

that is, setting $t = r_2^i/r_1^i \geq 1$,

$$C(n) \eta(\Omega)^\alpha \geq \frac{(r_2^i)^2 - (r_1^i)^2}{(r_1^i)^2 + (r_2^i)^2} = \frac{t^2 - 1}{t^2 + 1} \geq \frac{t - 1}{t} = 1 - \frac{r_1^i}{r_2^i},$$

thus proving (2.81). The second inequality in (2.82) follows from (2.81) and (2.11), while the first one is proved by noticing that $B_{x_i, r_1^i} \subset \Omega$, and thus for some $x_0 \in \partial\Omega$ one has $n/r_1^i \geq H(x_0) \geq H_0(1 - \delta(\Omega)) = n(1 - \delta(\Omega))$. Finally, thanks to (2.80) and (2.5),

$$\int_{\Omega \setminus A} (-f) \leq \int_{\{-f \leq 4\rho\}} (-f) \leq 4\rho|\Omega| \leq C(n) |\Omega|^{(n+2)/(n+1)} \eta(\Omega)^\alpha, \quad (2.88)$$

while (2.81), $t^{n+1} - 1 \leq 2^{n+1}(t - 1)$ for $t = r_2^i/r_1^i \in [1, 2]$, and (2.35) give us

$$\begin{aligned} \int_{A \setminus \bigcup_{i \in I} B_{x_i, r_1^i}} (-f) &\leq \|f\|_{C^0(\Omega)} |B| \sum_{i \in I} (r_2^i)^{n+1} - (r_1^i)^{n+1} \\ &\leq C(n) \eta(\Omega)^\alpha |B| \sum_{i \in I} (r_1^i)^{n+1} \\ &\leq C(n) \eta(\Omega)^\alpha |A| \leq C(n) |\Omega| \eta(\Omega)^\alpha, \end{aligned}$$

where in the last inequality we have used $|A| \leq |\Omega|$. This proves (2.83).

Step three: We show the existence of $\{B_{x_j, s_j}\}_{j \in J} \subset \{B_{x_i, r_1^i}\}_{i \in I}$ such that $\{s_j\}_{j \in J}$ satisfies

$$\frac{\max_{j \in J} |s_j - 1|}{\text{diam}(\Omega)} \leq C(n) |\Omega| \delta(\Omega)^\alpha, \quad (2.89)$$

and, if $G^* = \bigcup_{j \in J} B_{x_j, s_j}$ (so that $G^* \subset \Omega$ by construction), then

$$\frac{|\Omega \setminus G^*|}{|\Omega|} \leq C_1(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha, \quad (2.90)$$

$$\frac{|P(\Omega) - \#JP(B)|}{P(\Omega)} \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha, \quad (2.91)$$

$$\#J \leq L, \quad \#J \leq C(n) |\Omega|. \quad (2.92)$$

(Note that (2.91) implies (1.6) by (2.5) and (2.7).) Having in mind to exploit the Pohozaev's identity (2.14) (recall the proof of Theorem 2.5), we first notice that, by the divergence theorem and by (2.35),

$$\begin{aligned} \left| \frac{|\Omega|}{n+1} - \int_{\partial\Omega} (x \cdot \nu_\Omega) |\nabla f|^2 \right| &= \left| \int_{\partial\Omega} (x \cdot \nu_\Omega) \left(\frac{1}{(n+1)^2} - |\nabla f|^2 \right) \right| \\ &\leq \text{diam}(\Omega) \left(\frac{1}{n+1} + \|\nabla f\|_{C^0(\Omega)} \right) \int_{\partial\Omega} \left| \frac{1}{n+1} - |\nabla f| \right| \\ &\leq C(n) \text{diam}(\Omega) \int_{\partial\Omega} \left| \frac{1}{n+1} - |\nabla f| \right|. \end{aligned}$$

By (2.28) and $P(\Omega) \leq C(n) |\Omega|$, we thus find

$$\begin{aligned} \left| \frac{|\Omega|}{n+1} - \int_{\partial\Omega} (x \cdot \nu_\Omega) |\nabla f|^2 \right| &\leq C(n) \text{diam}(\Omega) \left(P(\Omega) \int_{\partial\Omega} \left| \frac{1}{n+1} - |\nabla f| \right|^2 \right)^{1/2} \\ &\leq C(n) \text{diam}(\Omega) P(\Omega) \delta(\Omega)^{1/2}. \end{aligned} \quad (2.93)$$

By (2.76), (2.86), and $\text{diam}(A_i) \leq 2$, one has

$$\|f - g_i\|_{C^0(A_i)} \leq C(n) |\Omega|^{1/(n+1)} \delta(\Omega)^\alpha,$$

thus, by (2.83)

$$\left| \int_{\Omega} f - \sum_{i \in I} \int_{B_{x_i, r_1^i}} g_i \right| \leq C(n) |\Omega|^{(n+2)/(n+1)} \delta(\Omega)^\alpha. \quad (2.94)$$

With $x_i + r_1^i \nu_1$ as in (2.87), by definition of ρ and by (2.86) we have

$$\begin{aligned} C |\Omega|^{1/(n+1)} \delta(\Omega)^\alpha &\geq 3\rho = -f_\varepsilon(x_i + r_1^i \nu_1) \\ &\geq -g_i(x_i + r_1^i \nu_1) - C(n) (r_1^i)^2 \eta(\Omega)^\alpha \\ &= -\frac{(r_1^i)^2}{2(n+1)} - f_\varepsilon(x_i) - C(n) (r_1^i)^2 \eta(\Omega)^\alpha, \end{aligned}$$

that is, since we definitely have $r_1^i \leq 2$,

$$-\frac{(r_1^i)^2}{2(n+1)} - f_\varepsilon(x_i) \leq C(n) |\Omega|^{1/(n+1)} \delta(\Omega)^\alpha.$$

By this last estimate and the definition of g_i ,

$$\begin{aligned} \int_{B_{x_i, r_1^i}} (-g_i) &= \int_{B_{x_i, r_1^i}} -f_\varepsilon(x_i) - \frac{|x - x_i|^2}{2(n+1)} \\ &\leq C(n) |B_{x_i, r_1^i}| |\Omega|^{1/(n+1)} \delta(\Omega)^\alpha + \int_{B_{x_i, r_1^i}} \frac{(r_1^i)^2 - |x - x_i|^2}{2(n+1)} \\ &= C(n) |B_{x_i, r_1^i}| |\Omega|^{1/(n+1)} \delta(\Omega)^\alpha + \frac{|B_{x_i, r_1^i}| (r_1^i)^2}{(n+3)(n+1)}. \end{aligned}$$

By combining (2.94) with this last inequality we find

$$\int_{\Omega} (-f) \leq \sum_{i \in I} \frac{|B_{x_i, r_1^i}| (r_1^i)^2}{(n+3)(n+1)} + C(n) |\Omega|^{(n+2)/(n+1)} \delta(\Omega)^\alpha. \quad (2.95)$$

By Pohozaev's identity (2.14), (2.93) and (2.95), and by taking into account that $|\Omega|^{1/(n+1)} \leq C(n) \text{diam}(\Omega)$ and (2.2), we find

$$\text{diam}(\Omega) P(\Omega) \delta(\Omega)^{1/2} + |\Omega|^{(n+2)/(n+1)} \delta(\Omega)^\alpha \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha,$$

and thus

$$|\Omega| \leq \sum_{i \in I} |B_{x_i, r_1^i}| (r_1^i)^2 + C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha. \quad (2.96)$$

By $|\Omega| \geq \sum_{i \in I} |B_{x_i, r_1^i}|$ we finally get

$$|B| \sum_{i \in I} (r_1^i)^{n+1} (1 - (r_1^i)^2) \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha. \quad (2.97)$$

Let us set $\varphi(r) = r^{n+1} (1 - r^2)$, $r \geq 0$, and note that

$$\varphi(r) \geq \begin{cases} \frac{3}{4} r^{n+1}, & \text{if } 0 \leq r \leq \frac{1}{2}, \\ \frac{1-r}{2^{n+1}}, & \text{if } \frac{1}{2} \leq r \leq 1. \end{cases} \quad (2.98)$$

With C_1 as in (2.82), let us now set

$$I^* = \{i \in I : 1 \leq r_1^i \leq 1 + C_1 \delta(\Omega)^\alpha\}, \quad I^{**} = \{i \in I : \frac{1}{2} \leq r_1^i \leq 1\}.$$

Since $B_{x_i, r_1^i} \subset \Omega$ for each $i \in I$ and by $\delta(\Omega) \leq c(n)$ we find

$$\# I^* \leq \frac{|\Omega|}{|B|}, \quad 0 \geq \varphi(r_1^i) \geq -C(n) \delta(\Omega)^\alpha, \quad \forall i \in I^*,$$

so that

$$-|B| \sum_{i \in I^*} \varphi(r_1^i) \leq C(n) |\Omega| \delta(\Omega)^\alpha. \quad (2.99)$$

By combining (2.99) with (2.97) one finds

$$|B| \sum_{i \in I \setminus I^*} \varphi(r_1^i) \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha. \quad (2.100)$$

Since $\varphi(r_1^i) \geq 0$ for every $i \in I \setminus I^*$, (2.98) implies that

$$\frac{3}{4} \sum_{i \in I \setminus (I^* \cup I^{**})} |B_{x_i, r_1^i}| \leq |B| \sum_{i \in I \setminus (I^* \cup I^{**})} \varphi(r_1^i),$$

and thus, by (2.100),

$$\sum_{i \in I \setminus (I^* \cup I^{**})} |B_{x_i, r_1^i}| \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha. \quad (2.101)$$

We now prove that r_1^i is close to 1 for every $i \in I^{**}$. Indeed, by exploiting again the fact that $\varphi(r_1^i) \geq 0$ for every $i \in I \setminus I^*$, together with (2.100) and (2.98), we find that

$$\frac{1}{2^{n+1}} \sum_{i \in I^{**}} (1 - r_1^i) \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha,$$

which in particular gives

$$1 \geq r_1^i \geq 1 - C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha, \quad \forall i \in I^{**}. \quad (2.102)$$

Finally, if we set $J = I^* \cup I^{**}$ and $s_j = r_1^j$ for $j \in J$, then (2.89) follows from (2.102) and the definition of I^* , while (2.96), (2.101) and (2.89) give us

$$\begin{aligned} |\Omega| &\leq \sum_{j \in J} |B_{x_j, s_j}| s_j^2 + C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha \\ &\leq (1 + C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha) |G^*| + C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha, \end{aligned}$$

i.e.

$$|\Omega \setminus G^*| \leq C(n) \text{diam}(\Omega) |\Omega| |G^*| \delta(\Omega)^\alpha \leq C(n) \text{diam}(\Omega) |\Omega|^2 \delta(\Omega)^\alpha.$$

This proves (2.90). Now by (2.89) and since $s_j \geq 1/2$

$$\begin{aligned} |P(G^*) - (n+1)|G^*|| &= (n+1)|B| \sum_{j \in J} s_j^n |s_j - 1| \leq C(n) \max_{j \in J} |s_j - 1| |B| \sum_{j \in J} s_j^{n+1} \\ &\leq C(n) \text{diam}(\Omega) |\Omega|^2 \delta(\Omega)^\alpha, \end{aligned}$$

so that (2.2) gives us

$$\begin{aligned} |P(\Omega) - P(G^*)| &= |(n+1)|\Omega| - P(G^*)| \leq C(n) \left| |\Omega| - |G^*| \right| + C(n) \text{diam}(\Omega) |\Omega|^2 \delta(\Omega)^\alpha \\ &\leq C(n) \text{diam}(\Omega) |\Omega|^2 \delta(\Omega)^\alpha, \end{aligned}$$

which proves (2.91) as $|\Omega| \leq C(n) P(\Omega)$, and since, by an entirely similar argument,

$$|P(G^*) - \#J P(B)| \leq C(n) P(\Omega) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha.$$

We conclude this step by proving (2.92): indeed, by (2.5), (2.7) and (2.89)

$$(L+1-a)|B| \geq |\Omega| \geq |G^*| \geq (1 - C(n, L) \delta(\Omega)^\alpha) |B| \#J,$$

and thus we conclude by $\delta(\Omega) \leq c(n, L, a)$.

Step four: We prove that

$$\frac{\max_{x \in \partial G^*} \text{dist}(x, \partial\Omega)}{\text{diam}(\Omega)} \leq C(n) \delta(\Omega)^\alpha. \quad (2.103)$$

We first notice that if $x_0 \in \partial A_i$, then by (2.85), $A_i \subset B_{x_i, r_2^i}$ and $r_2^i \leq 2$, one has

$$\left| \nabla f_\varepsilon(x_0) - \frac{(x_0 - x_i)}{n+1} \right| \leq C \delta(\Omega)^\alpha,$$

so that, by $|x_0 - x_i| \geq r_1^i \geq 1/2$, one finds

$$|\nabla f_\varepsilon(x_0)| \geq c_1(n), \quad \forall x_0 \in \partial A_i.$$

Let A_i^* be the set of points $x \in \Omega_\varepsilon \setminus \overline{A_i}$ such that if $x_0 \in \partial A_i$ denotes the projection of x onto the convex set A_i , then the open segment joining x to x_0 is entirely contained in Ω_ε (and thus in A_i^* : in particular, A_i^* is connected). By (2.78), $f_\varepsilon(x) \leq 0$ and $\partial A_i \subset \{f_\varepsilon = -3\rho\}$, we get

$$\begin{aligned} 3\rho &\geq f_\varepsilon(x) - f_\varepsilon(x_0) = \nabla f_\varepsilon(x_0) \cdot (x - x_0) + \frac{1}{2} \int_0^1 \nabla^2 f_\varepsilon(tx + (1-t)x_0)[x - x_0, x - x_0] dt \\ &\geq |\nabla f_\varepsilon(x_0)| |x - x_0| - C_2(n) |x - x_0|^2 \geq \frac{c_1(n)}{2} |x - x_0|, \end{aligned}$$

where we have used the fact that both $\nabla f_\varepsilon(x_0)$ and $x - x_0$ are orthogonal to ∂A_i at x_0 , and we have assumed that $|x - x_0| \leq c_1(n)/2C_2(n)$. If we denote by

$$I_d(X) = \{z \in \mathbb{R}^{n+1} : \text{dist}(z, X) < d\}, \quad X \subset \mathbb{R}^{n+1}, d > 0,$$

the d -neighborhood of a set X , then, setting $k_0(n) = c_1(n)/2C_2(n)$, we obtain

$$I_{k_0(n)}(A_i) \cap A_i^* \subset I_{6\rho/c_1(n)}(A_i).$$

By connectedness of A_i^* and by $\delta(\Omega) \leq c(n, L)$, this proves that

$$A_i^* \subset I_{6\rho/c_1(n)}(A_i).$$

Since $A_i \subset\subset \Omega_\varepsilon$ (thanks to (2.80)), for every $x \in \partial A_i$, there exists $y \in \partial\Omega_\varepsilon$ such the open segment joining x and y is entirely contained in A_i^* , and the length of this segment is bounded by $6\rho/c_1(n)$, so that

$$\partial A_i \subset I_{6\rho/c_1(n)}(\partial\Omega_\varepsilon) \subset I_{\varepsilon+(6\rho/c_1(n))}(\partial\Omega).$$

Step five: We construct a family of disjoint balls $\{B_{z_j, 1}\}_{j \in J}$ such that if we set

$$G = \bigcup_{j \in J} B_{z_j, 1},$$

then

$$\frac{|\Omega \Delta G|}{|\Omega|} \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha, \quad \frac{\max_{x \in \partial G} \text{dist}(x, \partial\Omega)}{\text{diam}(\Omega)} \leq C(n) |\Omega| \delta(\Omega)^\alpha. \quad (2.104)$$

(Note that (2.104) imply (1.5) and (1.7) thanks to (2.5) and (2.7).) Indeed if we set $s'_j = \min\{s_j, 1\}$, then $\{B_{x_j, s'_j}\}_{j \in J}$ is a family of disjoint balls such that $G' = \bigcup_{j \in J} B_{x_j, s'_j}$ satisfies $G' \subset \Omega$ and

$$\frac{|\Omega \setminus G'|}{|\Omega|} \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha, \quad \frac{\max_{x \in \partial G'} \text{dist}(x, \partial\Omega)}{\text{diam}(\Omega)} \leq C(n) |\Omega| \delta(\Omega)^\alpha, \quad (2.105)$$

thanks to (2.90), (2.103) and (2.89). Next, let us fix $j_0 \in J$ such that $1 > s_{j_0}$. By translating each x_j with $j \neq j_0$ into

$$\hat{x}_j = x_j + (1 - s_{j_0}) \frac{x_j - x_{j_0}}{|x_j - x_{j_0}|},$$

and setting $\hat{s}_j = s_j$ if $j \neq j_0$, $\hat{s}_{j_0} = 1$, $\hat{x}_{j_0} = x_{j_0}$, we find that $\{B_{\hat{x}_j, \hat{s}_j}\}_{j \in J}$ is a family of disjoint balls such that $\hat{G} = \bigcup_{j \in J} B_{\hat{x}_j, \hat{s}_j}$ satisfies

$$\frac{|\Omega \Delta \hat{G}|}{|\Omega|} \leq C(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha, \quad \frac{\max_{x \in \partial \hat{G}} \text{dist}(x, \partial\Omega)}{\text{diam}(\Omega)} \leq C(n) |\Omega| \delta(\Omega)^\alpha, \quad (2.106)$$

thanks to (2.105) and (2.89). By iteratively repeating this procedure on each $j \in J$ such that $\hat{s}_j < 1$ we finally construct a family G with the required properties.

Step six: In this step we complete the proof of Theorem 1.1 up to statements (i) and (ii). To this end, we want to apply Lemma 2.8 to Ω and G . We first notice that for every $x \in \partial G$, thanks to (1.7), there exists $g(x) \in \partial\Omega$ such that

$$|x - g(x)| \leq C(n) \operatorname{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha. \quad (2.107)$$

(The point $g(x)$ will play the role of y in Lemma 2.8.) Setting $S_j = \partial B_{z_j, 1}$ for $j \in J$, we define $\{r_x\}_{x \in \partial G}$ by the rule

$$r_x = \sup \left\{ r \in (0, 2\delta(\Omega)^\beta) : B_{x,r} \cap (\partial G \setminus S_j) = \emptyset \right\}, \quad \text{if } j \in J, x \in S_j, \quad (2.108)$$

and then set

$$\Sigma^* = \{x \in \partial G : r_x \geq \delta(\Omega)^\beta\}, \quad (2.109)$$

for some $\beta = \beta(n) \in (0, \alpha)$ to be suitable chosen later on, see (2.134). With Σ_λ defined as in (2.51), see the statement of Lemma 2.8, it is clear that we can choose $c_3(n) > 0$ in such a way that

$$\Sigma_\lambda \subset \Sigma^*, \quad \text{for } \lambda = c_3(n) \delta(\Omega)^{\beta/2}. \quad (2.110)$$

In particular, by Remark 2.9 and by (2.92),

$$\begin{aligned} \partial G \setminus \Sigma \text{ consists of at most } C(n)\#J\text{-many spherical caps} \\ \text{whose diameters are bounded by } C(n)\lambda \leq C(n)\delta(\Omega)^{\beta/2}. \end{aligned} \quad (2.111)$$

We now *claim* that for every $x \in \Sigma_\lambda$, λ as in (2.110), one can find ρ_x such that,

$$\frac{c_0(n)\lambda^2}{2} \leq \rho_x \leq c_0(n)\lambda^2, \quad (2.112)$$

$$\sigma(\partial\Omega, g(x), \rho_x) \leq \sigma_0(n)\lambda(n), \quad (2.113)$$

$$|x - g(x)| \leq \frac{\sigma_0(n)\rho_x^2}{2}, \quad (2.114)$$

$$|\Omega \Delta G| \leq C(n)\rho_x^{n+1}\sigma(\partial\Omega, g(x), \rho_x)^{1/4n}, \quad (2.115)$$

where $\sigma_0(n)$ and $\lambda(n)$ are as in Lemma 2.8. In proving the claim, the harder task is accommodating (2.113), because it requires to control the perimeter convergence of Ω to G localized in balls in terms of the Alexandrov's deficit.

We now prove the claim. First of all we notice that in order to entail (2.112), and thanks to $\Sigma_\lambda \subset \Sigma^*$, (2.108) and (2.109), it is enough to pick ρ_x satisfying

$$\frac{c_4(n)}{2} r_x \leq \rho_x \leq c_4(n) r_x, \quad (2.116)$$

for a suitable constant $c_4 \in (0, 1)$. Next, we notice that by $\delta(\Omega) \leq c(n, L)$ we can entail

$$\sup_{x \in \Sigma_\lambda} r_x \leq 2\delta(\Omega)^\beta \leq r_*(n), \quad (2.117)$$

for an arbitrarily small constant $r_*(n)$. Provided $r_*(n)$ is small enough, then (2.117), (2.60) and (2.61) give us

$$P(G; B_{x,r}) \leq (1 + C(n)r^2)\omega_n r^n, \quad \forall r < r_x, \quad (2.118)$$

$$\sup \{ |(p-x) \cdot \nu_G(x)| : p \in B_{x,r} \cap \partial G \} \leq C(n)r^2, \quad \forall r < r_x. \quad (2.119)$$

(We are going to use this bounds to quantify the size of $P(G; B_{g(x), \rho_x})$, see (2.132) below.) By Chebyshev inequality and by (2.90), we can pick ρ_x satisfying (2.116) and

$$\mathcal{H}^n\left((\Omega \setminus G^*) \cap \partial B_{g(x), \rho_x}\right) \leq C(n) |\Omega|^2 \text{diam}(\Omega) \delta(\Omega)^{\alpha-\beta}, \quad (2.120)$$

$$\mathcal{H}^n((\partial\Omega \cup \partial G^* \cup \partial G) \cap \partial B_{g(x), \rho_x}) = 0. \quad (2.121)$$

(Notice that we are using G^* in place of G here, because G^* is contained in Ω , and this will simplify a key computation based on the divergence theorem.) We include a brief justification of (2.120) for the sake of clarity: let us set

$$W = \left\{ \rho \in \left(\frac{c_4 r_x}{2}, c_4 r_x \right) : \mathcal{H}^n((\Omega \setminus G^*) \cap \partial B_{g(x), \rho}) \geq K(n) |\Omega|^2 \text{diam}(\Omega) \delta(\Omega)^{\alpha-\beta} \right\},$$

then, with $C(n)$ as in (2.90) and for a suitably large value of $K(n)$, we have $\mathcal{H}^1(W) \leq (C(n)/K(n)) \delta(\Omega)^\beta < c_4 \delta(\Omega)^\beta / 2 \leq c_4 r_x / 2$. Now let us consider the open sets

$$U_j = \left\{ y \in \mathbb{R}^{n+1} : |y - z_j| < |y - z_{j'}| \quad \forall j' \neq j \right\}, \quad j \in J,$$

so that $B_{z_j, 1} \subset U_j$ for every $j \in J$, and $\{U_j\}_{j \in J}$ is a partition of \mathbb{R}^{n+1} modulo a \mathcal{H}^n -dimensional set. The boundary of each U_j is contained into finitely many hyperplanes $\{L_{j,i}\}_{i=1}^{m_j}$, where $m_j \leq \#J \leq C(n) |\Omega|$. Thus

$$\#\{(j, i) : j \in J, 1 \leq i \leq m_j\} \leq C(n) |\Omega|^2. \quad (2.122)$$

We claim the existence of $v \in S^n$ and $t^* \in \mathbb{R}$ such that, setting $L_{j,i}^* = t^* v + L_{j,i}$,

$$\mathcal{H}^n(L_{j,i}^* \cap (\Omega \setminus G^*)) \leq C(n) |\Omega|^6 \text{diam}(\Omega) \delta(\Omega)^{\alpha/2}, \quad (2.123)$$

$$|t^*| \leq \delta(\Omega)^{\alpha/2}, \quad (2.124)$$

$$\mathcal{H}^n(L_{j,i}^* \cap (\partial\Omega \cup \partial G^*)) = 0. \quad (2.125)$$

To choose v , we let $\nu_{j,i}$ be a normal vector to $L_{j,i}$, and require $v \in S^n$ to be such that

$$|v \cdot \nu_{j,i}| \geq \frac{c_2(n)}{|\Omega|^2}, \quad \forall j \in J, 1 \leq i \leq m_j. \quad (2.126)$$

(The existence of such v is deduced by observing that if $\theta > 0$, then each spherical stripe $Y_{j,i}^\theta = \{u \in S^n : |u \cdot \nu_{j,i}| < \theta\}$ satisfies $\mathcal{H}^n(Y_{j,i}^\theta) \leq C(n)\theta$ so that by (2.122)

$$\mathcal{H}^n\left(S^n \setminus \bigcup_{j \in J} \bigcup_{i=1}^{m_j} Y_{j,i}^\theta\right) \geq \mathcal{H}^n(S^n) - C(n) |\Omega|^2 \theta > 0,$$

provided $\theta = c(n)/|\Omega|^2$ for a suitably small value of $c(n)$.) We now find t^* . For a constant $M(n)$ to be properly chosen, let us set

$$I_{j,i} = \left\{ t \in \mathbb{R} : |t| < \delta(\Omega)^{\alpha/2}, \mathcal{H}^n((\Omega \setminus G^*) \cap (t v + L_{j,i})) \geq M(n) |\Omega|^6 \text{diam}(\Omega) \delta(\Omega)^{\alpha/2} \right\}.$$

If $\ell_{j,i}(y) = y \cdot \nu_{j,i}$ then $L_{j,i} = \{\ell_{j,i} = \beta_{j,i}\}$ for some $\beta_{j,i}$, while $t v + L_{j,i} = \{\ell_{j,i} = \beta_{j,i} + t v \cdot \nu_{j,i}\}$. By (2.90) and Fubini's theorem we find

$$\begin{aligned} C_1(n) |\Omega|^2 \text{diam}(\Omega) \delta(\Omega)^\alpha &\geq |\Omega \setminus G^*| = \int_{\mathbb{R}} \mathcal{H}^n((\Omega \setminus G^*) \cap \{\ell_{j,i} = s\}) ds \\ &\geq |v \cdot \nu_{j,i}| \int_{\mathbb{R}} \mathcal{H}^n((\Omega \setminus G^*) \cap \{\ell_{j,i} = \beta_{j,i} + t v \cdot \nu_{j,i}\}) dt \\ &\geq |v \cdot \nu_{j,i}| \mathcal{H}^1(I_{j,i}) M(n) |\Omega|^6 \text{diam}(\Omega) \delta(\Omega)^{\alpha/2}, \end{aligned}$$

that is, by (2.126),

$$|\Omega|^2 \mathcal{H}^1(I_{j,i}) \leq \frac{C_1(n) \delta(\Omega)^{\alpha/2}}{c_2(n) M(n)}, \quad \forall j \in J, 1 \leq i \leq m_j.$$

By combining this estimate with (2.122), we see that if $M(n)$ is large enough, then

$$\mathcal{H}^1\left((-\delta(\Omega)^{\alpha/2}, \delta(\Omega)^{\alpha/2}) \setminus \bigcup_{j \in J} \bigcup_{i=1}^{m_j} \cap I_{j,i}\right) > 0,$$

that is, there exists t^* such that (2.123), (2.124) and (2.125) hold. If we set $U_j^* = t^* v + U_j$, then $\{U_j^*\}_{j \in J}$ is a partition of \mathbb{R}^{n+1} modulo a \mathcal{H}^n -dimensional set such that ∂U_j^* is contained into the hyperplanes $\{L_{j,i}^*\}_{i=1}^{m_j}$ and such that

$$\mathcal{H}^n(\partial U_j^* \cap (\Omega \setminus G^*)) \leq C(n) |\Omega|^6 \text{diam}(\Omega) \delta(\Omega)^{\alpha/2}, \quad (2.127)$$

$$\mathcal{H}^n(\partial U_j^* \cap (\partial\Omega \cup \partial G^*)) = 0, \quad (2.128)$$

$$\mathcal{H}^n((U_j^* \cap \partial G^*) \Delta S_j) \leq C(n) \delta(\Omega)^{\alpha/2}. \quad (2.129)$$

Here, (2.127) and (2.128) are immediate from (2.123) and (2.125). To prove (2.129), let us recall that $S_j = \partial B_{x_j, s_j} \subset \overline{U_j}$, so that by translating the boundary hyperplanes of U_j by $t^* v$ with $|t^*| \leq \delta(\Omega)^{\alpha/2}$ we have possibly cut out from S_j at most m_j -many spherical caps of \mathcal{H}^n -measure bounded above by

$$C(n) |t^*|^{n/2} \leq C(n) \delta(\Omega)^{n\alpha/4},$$

that is, thanks also to $m_j \leq L$,

$$\mathcal{H}^n((U_j^* \cap S_j) \Delta S_j) \leq C(n) L \delta(\Omega)^{n\alpha/4} \leq C(n) \delta(\Omega)^{\alpha/2},$$

thanks to $\delta(\Omega) \leq c(n, L)$. By a similar argument, since $\mathcal{H}^n(S_{j'} \cap U_j) = 0$ for $j \neq j'$, we have that $\mathcal{H}^n(U_j^* \cap S_{j'}) \leq C(n) \delta(\Omega)^{\alpha/2}$, and thus (2.129) is proved.

We now apply the divergence theorem to the vector field $y \mapsto (y - z_j)/|y - z_j|$ on the set of finite perimeter $(\Omega \setminus G^*) \cap (U_j^* \setminus B_{g(x), \rho_x})$. Since $\text{div}((y - z_j)/|y - z_j|) = n/|y - z_j|$ for $y \neq z_j$ and $G^* \subset \Omega$, by exploiting [Mag12, Theorem 16.3], one finds

$$\begin{aligned} 0 &< \int_{(U_j^* \setminus B_{g(x), \rho_x}) \cap \partial\Omega} \frac{y - z_j}{|y - z_j|} \cdot \nu_\Omega(y) d\mathcal{H}_y^n - \int_{(U_j^* \setminus B_{g(x), \rho_x}) \cap \partial G^*} \frac{y - z_j}{|y - z_j|} \cdot \nu_{G^*}(y) d\mathcal{H}_y^n \\ &+ \int_{\partial(U_j^* \setminus B_{g(x), \rho_x}) \cap (\Omega \setminus G^*)} \frac{y - z_j}{|y - z_j|} \cdot \nu_{U_j^* \setminus B_{g(x), \rho_x}}(y) d\mathcal{H}_y^n \\ &\leq P(\Omega; U_j^* \setminus B_{g(x), \rho_x}) + \mathcal{H}^n((\partial U_j^* \cup \partial B_{g(x), \rho_x}) \cap (\Omega \setminus G^*)) \\ &\quad - P(G^*; U_j^* \setminus B_{g(x), \rho_x}) + C(n) \delta(\Omega)^{\alpha/2} \end{aligned}$$

where in the last inequality we have used $\nu_{G^*}(y) \cdot [(y - z_j)/(y - z_j)] = 1$ if $y \in S_j$ and (2.129). By combining this last inequality with (2.120) and (2.127) we thus find

$$\begin{aligned} P(G^*; U_j^* \setminus B_{g(x), \rho_x}) &\leq P(\Omega; U_j^* \setminus B_{g(x), \rho_x}) \\ &\quad + C(n) |\Omega|^2 \text{diam}(\Omega) \left(\delta(\Omega)^{\alpha-\beta} + |\Omega|^4 \delta(\Omega)^{\alpha/2} \right). \end{aligned}$$

By adding up over $j \in J$, and since $\# J \leq C(n) |\Omega|$, we thus find

$$\begin{aligned} P(G^*; \mathbb{R}^{n+1} \setminus B_{g(x), \rho_x}) &\leq P(\Omega; \mathbb{R}^{n+1} \setminus B_{g(x), \rho_x}) \\ &\quad + C(n) |\Omega|^3 \text{diam}(\Omega) \left(\delta(\Omega)^{\alpha-\beta} + |\Omega|^4 \delta(\Omega)^{\alpha/2} \right), \end{aligned}$$

which gives us, keeping in mind the construction used in step five to define G starting from G^* , and also thanks to (2.89) and $\delta(\Omega) \leq c(n)$,

$$\begin{aligned} P(G; \mathbb{R}^{n+1} \setminus B_{g(x), \rho_x}) &\leq P(\Omega; \mathbb{R}^{n+1} \setminus B_{g(x), \rho_x}) \\ &\quad + C(n) |\Omega|^3 \text{diam}(\Omega) \left(\delta(\Omega)^{\alpha-\beta} + |\Omega|^4 \delta(\Omega)^{\alpha/2} \right), \end{aligned} \quad (2.130)$$

which, combined with (1.6) and (2.121), gives us

$$\begin{aligned} P(\Omega; B_{g(x), \rho_x}) &= P(\Omega) - P(\Omega; \mathbb{R}^{n+1} \setminus B_{g(x), \rho_x}) \\ &\leq P(G) - P(G; \mathbb{R}^{n+1} \setminus B_{g(x), \rho_x}) + C(n) |\Omega|^3 \text{diam}(\Omega) \left(\delta(\Omega)^{\alpha-\beta} + |\Omega|^4 \delta(\Omega)^{\alpha/2} \right) \\ &= P(G; B_{g(x), \rho_x}) + C(n) |\Omega|^3 \text{diam}(\Omega) \left(\delta(\Omega)^{\alpha-\beta} + |\Omega|^4 \delta(\Omega)^{\alpha/2} \right). \end{aligned} \quad (2.131)$$

By (2.107), (2.5), (2.7), and thanks to $\delta(\Omega) \leq c(n, L)$, we entail $B_{g(x), \rho_x} \subset B_{x, r_x}$, so that, by definition of r_x ,

$$\begin{aligned} P(G; B_{g(x), \rho_x}) &= \mathcal{H}^n(B_{g(x), \rho_x} \cap \partial G) = \mathcal{H}^n(B_{g(x), \rho_x} \cap S_j) \\ &\leq (1 + C(n) \rho_x) \mathcal{H}^n(B_{x, \rho_x} \cap S_j) = (1 + C(n) \rho_x) P(G; B_{x, \rho_x}). \end{aligned}$$

By combining this inequality with (2.131), (2.118), (2.109) and (2.116) we find

$$\begin{aligned} \frac{P(\Omega; B_{g(x), \rho_x})}{\omega_n \rho_x^n} - 1 &\leq C(n) \rho_x + \frac{C(n)}{\rho_x^n} |\Omega|^3 \text{diam}(\Omega) \left(\delta(\Omega)^{\alpha-\beta} + |\Omega|^4 \delta(\Omega)^{\alpha/2} \right) \\ &\leq C(n) \left(\delta(\Omega)^\beta + |\Omega|^3 \text{diam}(\Omega) (\delta(\Omega)^{\alpha-(n+1)\beta} + |\Omega|^4 \delta(\Omega)^{(\alpha/2)-n\beta}) \right). \end{aligned} \quad (2.132)$$

By combining (2.132) with (2.49) (the definition of $\sigma(\partial\Omega, g(x), \rho_x)$), (2.6) and $\rho_x \leq 2\delta(\Omega)^\beta$, we find

$$\sigma(\partial\Omega, g(x), \rho_x) \leq C(n) \left(\delta(\Omega)^\beta + |\Omega|^3 \text{diam}(\Omega) (\delta(\Omega)^{\alpha-(n+1)\beta} + |\Omega|^4 \delta(\Omega)^{(\alpha/2)-n\beta}) \right).$$

For this estimate to be nontrivial we definitely need

$$\alpha > \max\{(n+1)\beta, 2n\beta\} = 2n\beta.$$

Under this assumption we have $\alpha - (n+1)\beta > (\alpha/2) - n\beta$, and thus

$$\begin{aligned} \sigma(\partial\Omega, g(x), \rho_x) &\leq C(n) \left(\delta(\Omega)^\beta + |\Omega|^7 \text{diam}(\Omega) \delta(\Omega)^{(\alpha/2)-n\beta} \right) \\ &\leq C(n) |\Omega|^7 \text{diam}(\Omega) \delta(\Omega)^{\alpha/2(n+1)}, \end{aligned} \quad (2.133)$$

where we have set

$$\beta = \frac{\alpha}{2(n+1)}, \quad (2.134)$$

in order to have $\beta = (\alpha/2) - n\beta$. By $\delta(\Omega) \leq c(n, L)$ and by (2.133), we have thus proved so far that for every $x \in \Sigma_\lambda$ one can find $g(x) \in \partial\Omega$ and $\rho_x \in (0, 1)$ such that (2.112) and (2.113) hold.

In order to prove our claim, we are left to prove (2.114) and (2.115). By (2.107), (2.5) and (2.7) we have $|x - g(x)| \leq C(n, L) \delta(\Omega)^\alpha$ while $\sigma_0(n) \rho_x^2 / 2 \geq c(n) \delta(\Omega)^{2\beta}$ by (2.112), so that (2.114) follows by $\delta(\Omega) \leq c(n, L)$ thanks to the fact that $\alpha > 2\beta$. Similarly, concerning (2.115) we see by (2.6) that $\sigma(\partial\Omega, g(x), \rho_x) \geq n \rho_x$ so that

$$\rho_x^{n+1} \sigma(\partial\Omega, g(x), \rho_x)^{1/4n} \geq c(n) \rho_x^{n+2} = c(n) \delta(\Omega)^{\beta(n+2)},$$

while (1.5) gives us $|\Omega \Delta G| \leq C(n, L) \delta(\Omega)^\alpha$, so that $\delta(\Omega) \leq c(n, L)$ implies (2.115) thanks to the fact that $\alpha > \beta(n+2)$.

We thus proved our claim: for every $x \in \Sigma_\lambda$ there exists $g(x) \in \partial\Omega$ and ρ_x satisfying (2.112)–(2.115). By (2.110) and $\delta(\Omega) \leq c(n, L)$ we have that $\lambda \leq \lambda(n)$, and thus we can apply Lemma 2.8 to find a function $\psi : \Sigma_\lambda \rightarrow \mathbb{R}$ such that,

$$\|\psi\|_{C^{1,\gamma}(\Sigma_\lambda)} \leq C(n, \gamma), \quad \forall \gamma \in (0, 1), \quad (2.135)$$

$$\lambda^{-2} \|\psi\|_{C^0(\Sigma_\lambda)} + \|\nabla\psi\|_{C^0(\Sigma_\lambda)} \leq C(n) \max_{x \in \Sigma_\lambda} \sigma(\partial\Omega, g(x), \rho_x)^{1/4n}, \quad (2.136)$$

$$(\text{Id} + \psi\nu_G)(\Sigma_\lambda) \subset \partial\Omega. \quad (2.137)$$

(Notice that (2.135) is (2.56).) Moreover, let us recall from the proof of Lemma 2.8 (see (2.59)) that if $x \in \Sigma_\lambda \cap S_j$, then the function ψ is actually defined on $\mathbf{C}_{x, \sigma_0 \rho_x/4}^{\mu_x} \cap \partial G = \mathbf{C}_{x, \sigma_0 \rho_x/4}^{\mu_x} \cap S_j$, with

$$\begin{aligned} \mathbf{C}_{x, \sigma_0 \rho_x/8}^{\mu_x} \cap \partial\Omega \subset (\text{Id} + \psi\nu_G)(\mathbf{C}_{x, \sigma_0 \rho_x/4}^{\mu_x} \cap \partial G) \subset \mathbf{C}_{x, \sigma_0 \rho_x/2}^{\mu_x} \cap \partial\Omega, \\ \|\psi\|_{C^1(\mathbf{C}_{x, \sigma_0 \rho_x/4}^{\mu_x} \cap \partial G)} \leq C(n) \sigma(\partial\Omega, g(x), \rho_x)^{1/4n}. \end{aligned} \quad (2.138)$$

Now, by (2.107), $\delta(\Omega) \leq c(n, L)$, and $\sigma_0 \rho_x/8 \geq c(n) \delta(\Omega)^\beta$ we find

$$g(x) \in \mathbf{C}_{x, \sigma_0 \rho_x/8}^{\mu_x} \cap \partial\Omega,$$

and thus, by the first inclusion in (2.138), there exists $y \in \mathbf{C}_{x, \sigma_0 \rho_x/4}^{\mu_x} \cap \partial G$ such that

$$g(x) = y + \psi(y)\nu_G(y).$$

By (2.138), (2.133), and $\delta(\Omega) \leq c(n, L)$ we can definitely ensure

$$\|\psi\|_{C^1(\mathbf{C}_{x, \sigma_0 \rho_x/4}^{\mu_x} \cap \partial G)} \leq c(n), \quad (2.139)$$

so that, taking into account that (2.119) gives $|(y-x) \cdot \nu_G(y)| \leq C(n) |y-x|^2$, we find that

$$|g(x) - x|^2 \geq |x-y|^2 + |\psi(y)|^2 - 2\|\psi\|_{C^0(\mathbf{C}_{x, \sigma_0 \rho_x/4}^{\mu_x} \cap \partial G)} |\nu_G(y) \cdot (y-x)| \geq \frac{|x-y|^2}{2} + |\psi(y)|^2.$$

Again by (2.107) we conclude $|x-y| + |\psi(y)| \leq C(n) |\Omega| \text{diam}(\Omega) \delta(\Omega)^\alpha$, and thus, provided $c(n) \leq 1$ in (2.139), $|\psi(x)| \leq C(n) |\Omega| \text{diam}(\Omega) \delta(\Omega)^\alpha$. We have thus improved the C^0 -bound on ψ in (2.136), by showing that

$$\|\psi\|_{C^0(\Sigma_\lambda)} \leq C_3(n) |\Omega| \text{diam}(\Omega) \delta(\Omega)^\alpha. \quad (2.140)$$

By combining (2.133) with (2.136) and $\text{diam}(\Omega) \leq C(n) P(\Omega) \leq C(n) |\Omega|$, we obtain

$$\|\nabla\psi\|_{C^0(\Sigma_\lambda)} \leq C(n) |\Omega|^{2/n} \delta(\Omega)^{\beta/4n}. \quad (2.141)$$

(By (2.140) and (2.141) we deduce (1.10).) Next, by (2.111) and by $\#J \leq C(n) |\Omega| \leq C(n) P(\Omega)$, we have

$$\mathcal{H}^n(\partial G \setminus \Sigma_\lambda) \leq C(n) P(\Omega) \delta(\Omega)^{n\beta/2}, \quad (2.142)$$

while by the area formula

$$\mathcal{H}^n((\text{Id} + \psi\nu_G)(\Sigma_\lambda)) = \int_{\Sigma_\lambda} \sqrt{(1+\psi)^{2n} + (1+\psi)^{2(n-1)} |\nabla\psi|^2},$$

so that

$$\begin{aligned} |\mathcal{H}^n(\Sigma_\lambda) - \mathcal{H}^n((\text{Id} + \psi\nu_G)(\Sigma_\lambda))| &\leq P(G) (\|\psi\|_{C^0(\Sigma_\lambda)} + \|\nabla\psi\|_{C^0(\Sigma_\lambda)}^2) \\ &\leq C(n) P(\Omega) |\Omega|^{4/n} \delta(\Omega)^{\beta/2n}, \end{aligned}$$

where in the last inequality we have used $P(G) \leq 2P(\Omega)$ (which follows by $\delta(\Omega) \leq c(n, L)$ and (2.91)) together with (2.140) and (2.141). By (2.91), (2.142), and

$$\begin{aligned} &\mathcal{H}^n(\partial\Omega \setminus (\text{Id} + \psi\nu_G)(\Sigma_\lambda)) \\ &\leq P(\Omega) - P(G) + \mathcal{H}^n(\partial G \setminus \Sigma_\lambda) + |\mathcal{H}^n(\Sigma_\lambda) - \mathcal{H}^n((\text{Id} + \psi\nu_G)(\Sigma_\lambda))| \end{aligned}$$

we thus obtain

$$\frac{\mathcal{H}^n(\partial\Omega \setminus (\text{Id} + \psi\nu_G)(\Sigma_\lambda))}{P(\Omega)} \leq C(n) |\Omega|^{4/n} \delta(\Omega)^{\beta/2n}. \quad (2.143)$$

Now let $x_0 \in \partial\Omega$ and $\tau > 0$ be such that

$$\tau = \max_{x \in \partial\Omega} \text{dist}(x, \partial G) = \text{dist}(x_0, \partial G),$$

and assume that $\tau \geq 2\theta$ for $\theta = C_3(n) \text{diam}(\Omega) |\Omega| \delta(\Omega)^\alpha$. By (2.140) and since $B_{x_0, \tau} \subset \mathbb{R}^{n+1} \setminus \partial G$, one has

$$B_{x_0, \tau - \theta} \subset \mathbb{R}^{n+1} \setminus I_\theta(\partial G) \subset \mathbb{R}^{n+1} \setminus (\text{Id} + \psi\nu_G)(\Sigma_\lambda),$$

so that, thanks to (2.9),

$$c(n) \min\{1, \tau - \theta\}^n \leq P(\Omega, B_{x_0, \tau - \theta}) \leq \mathcal{H}^n(\partial\Omega \setminus (\text{Id} + \psi\nu_G)(\Sigma_\lambda)) \leq C(n) P(\Omega) |\Omega|^{4/n} \delta(\Omega)^{\beta/2n}.$$

By $\delta(\Omega) \leq c(n, L)$ we thus find $\min\{1, \tau - \theta\} = \tau - \theta$, and thus

$$\tau \leq \theta + C(n) P(\Omega)^{1/n} |\Omega|^{4/n^2} \delta(\Omega)^{\beta/2n^2} \leq C(n) P(\Omega)^{1/n} |\Omega|^{4/n^2} \delta(\Omega)^{\beta/2n^2}.$$

Since $P(\Omega) \leq C(n) |\Omega| \leq C(n) \text{diam}(\Omega)^{n+1}$ and $\text{diam}(\Omega) \leq C(n) |\Omega|$, we find $P(\Omega)^{1/n} \leq C(n) \text{diam}(\Omega) |\Omega|^{1/n}$ and thus conclude that

$$\frac{\max_{x \in \partial\Omega} \text{dist}(x, \partial G)}{\text{diam}(\Omega)} \leq C(n) |\Omega|^{3/n} \delta(\Omega)^{\beta/2n^2}. \quad (2.144)$$

where we have simplified some powers on $|\Omega|$ by noticing that $(4/n^2) + (1/n) \leq 3/n$. By combining (2.104) and (2.144) we obtain (1.8).

Step seven: We complete the proof of the theorem. By contradiction, and by definition of Σ_λ , if $\#J \geq 2$ and $\text{dist}(S_j, S_\ell) \geq 4\lambda$ for every $\ell \neq j$, then $S_j \subset \Sigma_\lambda$, and thus $\Gamma_j = (\text{Id} + \psi\nu_G)(S_j) \subset \partial\Omega$, with Γ_j connected. Since $\partial\Omega$ is connected, we conclude that $\partial\Omega = \Gamma_j$, thus that $\#J = 1$. This proves that if $\#J \geq 2$, then for every $j \in J$ there is $\ell \neq j$ such that $\text{dist}(S_j, S_\ell) \leq 4\lambda$, and then (i) follows by $\text{diam}(\Omega) \geq c(n) |\Omega|^{1/(n+1)} \geq c(n)$. In order to prove (ii), let us assume that $\#J \geq 2$, and let $j \neq \ell \in J$ be such that (1.12) holds. Let us set

$$x = \frac{z_j + z_\ell}{2} = \frac{\bar{z}_j + \bar{z}_\ell}{2},$$

where \bar{z}_j and \bar{z}_ℓ belong to the closed segment joining z_j and z_ℓ and are such that

$$\partial B_{\bar{z}_j, 1} \cap \partial B_{\bar{z}_\ell, 1} = \{x\}.$$

In this way, for every $r \in (0, \kappa)$ and thanks to (1.12) we have

$$C(n) r^{n+2} \geq |B_{x, r} \setminus (B_{\bar{z}_j, 1} \cup B_{\bar{z}_\ell, 1})| \geq |B_{x, r} \setminus (B_{z_j, 1} \cup B_{z_\ell, 1})| - C(n) \delta(\Omega)^{\alpha/4(n+1)}.$$

By (1.5) we have

$$|B_{x, r} \setminus (B_{z_j, 1} \cup B_{z_\ell, 1})| \geq |B_{x, r} \setminus G| \geq |B_{x, r} \setminus \Omega| - C(n) L^3 \delta(\Omega)^\alpha,$$

so that by (1.13) we finally get

$$\kappa |B| r^{n+1} \leq C(n) (L^3 \delta(\Omega)^\alpha + \delta(\Omega)^{\alpha/4(n+1)} + r^{n+2}), \quad \forall r < \kappa.$$

We now use this inequality with $r = \vartheta \kappa$, with $\vartheta = \vartheta(n) \in (0, 1)$ to be properly chosen. By $\delta(\Omega) \leq c(n, L, \kappa)$ we can entail,

$$L^3 \delta(\Omega)^\alpha + \delta(\Omega)^{\alpha/4(n+1)} \leq \vartheta^{n+2} \kappa^{n+2},$$

thus finding $\kappa^{n+2} |B| \vartheta^{n+1} \leq C(n) \vartheta^{n+2} \kappa^{n+2}$. This is of course a contradiction if we pick ϑ small enough. The proof of Theorem 1.1 is complete. \square

3. AN APPLICATION TO CAPILLARITY-TYPE ENERGIES

Proof of Proposition 1.2. It is well-known that (1.19) implies the existence of a constant $\lambda \in \mathbb{R}$ such that

$$\int_{\partial\Omega} \operatorname{div}^{\partial\Omega} X + \int_{\Omega} g(X \cdot \nu_{\Omega}) = \lambda \int_{\partial\Omega} (X \cdot \nu_{\Omega}), \quad \forall X \in C_c^{\infty}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \quad (3.1)$$

Since $R_0 > 0$ is such that $\Omega \subset\subset B_{2R_0}$, by testing (3.1) with $X(x) = \varphi(x)x$ for some $\varphi \in C_c^{\infty}(B_{2R_0})$ with $\varphi = 1$ on $\bar{\Omega}$, one easily obtains that

$$(n+1)|\Omega|\lambda = nP(\Omega) + \int_{\Omega} \operatorname{div}(x, g(x)) dx. \quad (3.2)$$

At the same time, since $\partial\Omega$ is of class C^2 , (3.1) implies that $H + g = \lambda$ on $\partial\Omega$, which combined with (3.2) gives

$$\frac{H(x) - H_0}{H_0} = \frac{1}{H_0} \left(-g(x) + \frac{\int_{\Omega} \operatorname{div}(x, g(x)) dx}{(n+1)|\Omega|} \right) \Rightarrow \delta(\Omega) \leq \frac{C \|g\|_{C^1(B_{R_0})}}{H_0}.$$

By the isoperimetric inequality,

$$H_0 = \frac{nP(\Omega)}{(n+1)|\Omega|} \geq \frac{n(n+1)|B|^{1/(n+1)}|\Omega|^{n/(n+1)}}{(n+1)|\Omega|} = n \left(\frac{|B|}{|\Omega|} \right)^{1/(n+1)}.$$

so that $\delta(\Omega) \leq C(n) \|g\|_{C^1(B_{R_0})} m^{1/(n+1)}$, that is (1.20). \square

APPENDIX A. ALMOST-CONSTANT MEAN CURVATURE IMPLIES ALMOST-UMBILICALITY

The purpose of this appendix is to discuss the relation between the Alexandrov's deficit $\delta(\Omega)$ and the size of the traceless part \mathring{A} of the second fundamental form A of $\partial\Omega$. Having in mind the quantitative results for almost-umbilical surfaces by De Lellis-Müller [DLM05, DLM06] and Perez [Per11], we seek for a control of \mathring{A} in $L^p(\partial\Omega)$ for some $p \geq n$. In this direction, we have the following proposition, where $\eta(\Omega)$ denotes the Heintze-Karcher deficit of Ω , see (2.10).

Proposition A.1. *If $\Omega \subset \mathbb{R}^{n+1}$ ($n \geq 2$) is a bounded connected open set with C^2 -boundary, $H > 0$ on $\partial\Omega$, and $\delta(\Omega) \leq 1/2$, then*

$$C(n) (P(\Omega) \eta(\Omega))^{1/(n+1)} \|A\|_{L^{p^*}(\partial\Omega)} \geq \|\mathring{A}\|_{L^p(\partial\Omega)}, \quad \forall p \in [1, n+1], \quad (A.1)$$

where $p^* = (n+1)p/[(n+1)-p]$ if $p < n+1$, and $p^* = +\infty$ otherwise.

Before proving Proposition A.1, let us discuss (A.1) in connection with the above mentioned results for almost-umbilical surfaces. Let us consider

$$\theta_p(\Omega) = \inf_{\lambda \in \mathbb{R}} \|A - \lambda \operatorname{Id}\|_{L^p(\partial\Omega)}, \quad p \geq 1,$$

as a measure of the non-umbilicality of $\partial\Omega$. In [Per11, Theorem 1.1] it is shown that if Ω is a bounded connected open set with smooth boundary such that

$$P(\Omega) = P(B), \quad \|A\|_{L^p(\partial\Omega)} \leq K,$$

for some $p \in (n, \infty)$, then

$$\inf_{\lambda \in \mathbb{R}} \|A - \lambda \operatorname{Id}\|_{L^p(\partial\Omega)} \leq C(n, p, K) \|\mathring{A}\|_{L^p(\partial\Omega)}. \quad (A.2)$$

Moreover, in [Per11, Corollary 1.2] it is shown that for every $\varepsilon > 0$ there exists $c = c(n, p, K, \varepsilon)$ such that if

$$\|\mathring{A}\|_{L^p(\partial\Omega)} \leq c \quad \Rightarrow \quad \inf_{x \in \mathbb{R}^{n+1}} \operatorname{hd}(\partial\Omega, \partial B_{x,1}) < \varepsilon. \quad (A.3)$$

By combining (A.1) and (A.2) with $\eta(\Omega) \leq \delta(\Omega)$, we deduce that, if $P(\Omega) = P(B)$, $H > 0$ and $\delta(\Omega) \leq 1/2$, then

$$\theta_p(\Omega) \leq C(n, p, \|A\|_{L^{p^*}(\partial\Omega)}) \delta(\Omega)^{1/(n+1)}.$$

If, in addition, $\delta(\Omega) \leq c(n, p, \|A\|_{L^{p^*}(\partial\Omega)}, \varepsilon)$, then, by (A.3),

$$\inf_{x \in \mathbb{R}^{n+1}} \text{hd}(\partial\Omega, \partial B_{x,1}) < \varepsilon.$$

A similar comparison is possible with the results of De Lellis and Müller, which pertain the case $n = p = 2$. In conclusion, the use of almost-umbilicality in attacking Theorem 1.1 does not seem to provide one with a starting point as effective as the one based on the study of the torsion potential discussed in section 2. We finally prove the above proposition.

Proof of Proposition A.1. This is consequence of the proof of the Heintze-Karcher inequality by Montiel-Ros [MR91], which we now recall for the reader's convenience. For each $x \in \partial\Omega$ let $\{\kappa_i(x)\}_{i=1}^n$ be the principal curvatures of $\partial\Omega$ at x , so that, if we set $\kappa = \max_{1 \leq i \leq n} \kappa_i$, then $\kappa \geq H/n > 0$ on $\partial\Omega$. Let us consider the set

$$\Gamma = \left\{ (x, t) \in \partial\Omega \times (0, \infty) : t \leq \frac{1}{\kappa(x)} \right\},$$

and the function $g : \partial\Omega \times (0, \infty) \rightarrow \mathbb{R}^{n+1}$

$$g(x, t) = x - t \nu_\Omega(x), \quad (x, t) \in \partial\Omega \times (0, \infty).$$

We claim that $\Omega \subset g(\Gamma)$. Indeed, given $y \in \Omega$ let $x \in \partial\Omega$ be such that $|x - y| = \text{dist}(y, \partial\Omega)$. If γ is a curve in $\partial\Omega$ with $\gamma(0) = x$ and $\gamma'(0) = \tau \in S^n$, then we obtain

$$(\gamma - y) \cdot \gamma' = 0, \quad (\gamma(0) - y) \cdot \gamma''(0) + \gamma'(0)^2 \geq 0.$$

In particular, there exists $t > 0$ such that $x = y - t \nu_\Omega(y)$, and it must be $1 - \kappa(y)t \geq 0$. We now combine $\Omega \subset g(\Gamma)$ with the area formula, the arithmetic-geometric mean inequality, and $\kappa \geq H/n$, to prove the Heintze-Karcher inequality

$$\begin{aligned} |\Omega| &\leq |g(\Gamma)| \leq \int_{g(\Gamma)} \mathcal{H}^0(g^{-1}(y)) dy = \int_{\Gamma} J^\Gamma g(x, t) d\mathcal{H}^n(x) dt \\ &= \int_{\partial\Omega} d\mathcal{H}^n \int_0^{1/\kappa} \prod_{i=1}^n (1 - t \kappa_i) dt \\ &\leq \int_{\partial\Omega} d\mathcal{H}^n \int_0^{1/\kappa} \left(\frac{1}{n} \sum_{i=1}^n (1 - t \kappa_i) \right)^n dt = \int_{\partial\Omega} d\mathcal{H}^n \int_0^{1/\kappa} \left(1 - t \frac{H}{n} \right)^n dt \\ &\leq \int_{\partial\Omega} d\mathcal{H}^n \int_0^{n/H} \left(1 - t \frac{H}{n} \right)^n dt = \frac{1}{n+1} \int_{\partial\Omega} \frac{n}{H}. \end{aligned}$$

This chain of inequalities implies of course the following identity

$$\begin{aligned} \int_{\partial\Omega} \frac{n}{H} d\mathcal{H}^n \frac{\eta(\Omega)}{n+1} &= \frac{1}{n+1} \int_{\partial\Omega} \frac{n}{H} \left(1 - \frac{1}{\kappa} \frac{H}{n} \right)^{n+1} + \int_{\Gamma} \mu_A^n - \mu_G^n \\ &\quad + \int_{g(\Gamma)} (\mathcal{H}^0(g^{-1}(y)) - 1) dy + (|g(\Gamma)| - |\Omega|), \end{aligned} \tag{A.4}$$

where for every $(x, t) \in \Gamma$ we have set $\mu_i(x, t) = 1 - t \kappa_i(x)$ (note that $\mu_i \geq 0$ on Γ) and

$$\mu_A = \frac{1}{n} \sum_{i=1}^n \mu_i, \quad \mu_G = \prod_{i=1}^n \mu_i^{1/n}.$$

The first two terms on the right-hand side of (A.4) provide some control on \mathring{A} . Since $|\mathring{A}(x)| \leq C(n) |\kappa(x) - (H(x)/n)|$ for every $x \in \partial\Omega$, by looking at the first term, we get

$$C(n) \eta(\Omega) \int_{\partial\Omega} \frac{n}{H} \geq \int_{\partial\Omega} \frac{n}{H} \left(\frac{|\mathring{A}|}{\kappa} \right)^{n+1},$$

that is, by exploiting $\delta(\Omega) \leq 1/2$ to infer $H_0/2 \leq H(x) \leq 2H_0$ for every $x \in \partial\Omega$,

$$C(n) P(\Omega) \eta(\Omega) \geq \int_{\partial\Omega} \left(\frac{|\mathring{A}|}{\kappa} \right)^{n+1},$$

We thus conclude the proof by Hölder inequality. \square

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