New bounds for the inhomogenous Burgers and the Kuramoto-Sivashinsky equations

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June 10, 2015

Abstract

We give a substantially simplified proof of the near-optimal estimate on the Kuramoto-Sivashinsky equation from [15], at the same time slightly improving the result. The result in [15] relied on two ingredients: a regularity estimate for capillary Burgers and an a novel priori estimate for the *inhomogeneous inviscid* Burgers equation, which works out that in many ways the *conservative transport* nonlinearity acts as a *coercive* term. It is the proof of the second ingredient that we substantially simplify by proving a modified Kármán-Howarth-Monin identity for solutions of the inhomogeneous inviscid Burgers equation. We show that this provides a new interpretation of the results obtained in [8].

1 Introduction

1.1 The Kuramoto-Sivashinsky equation

We consider the one-dimensional Kuramoto-Sivashinsky equation:

$$\partial_t u + u\partial_x u + \partial_x^2 u + \partial_x^4 u = 0. \tag{K-S}$$

This equation appears in many physical contexts, in particular in the modeling of surface evolutions. Sivashinsky used it to describe flame fronts [17], wavy flow of viscous liquids on inclined planes [18] and crystal growth [7]. Although the solutions

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of (K-S) are smooth and even analytic [11], they display a chaotic behavior for sufficiently large systems size L (see [10] and Figure 1.1). The structure of the Kuramoto-Sivashinsky equation has some similarities with the Navier-Stokes equation. Therefore, it is sometimes possible to apply similar techniques to study both equations (see [16]).

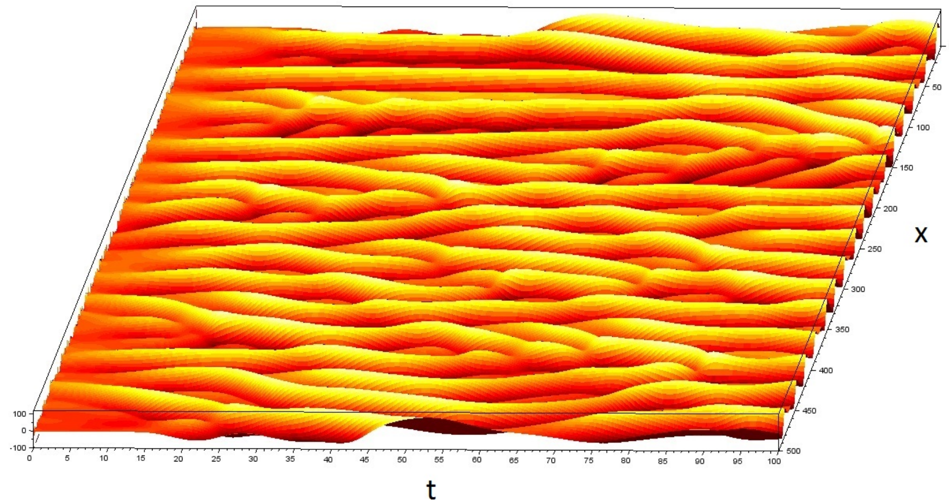


Figure 1: Chaotic behavior of u

For a given system size $L > 0$, we will consider L-periodic solutions of (K-S). Since the spatial average $\int_0^L u(t,x)dx$ is constant over time, and since the equation is invariant under the Galilean transformation:

$$t = t', \qquad x = x' + Ut, \qquad u = u' + U,$$

it is not restrictive to assume that $\int_0^L u(t,x)dx = 0$ for all $t \geq 0$.
We can artificially cut the equation in two parts and consider separately the two mechanisms involved in (K-S):

$$\partial_t u + \partial_x^2 u + \partial_x^4 u = 0, \tag{1}$$

$$\partial_t u + u\partial_x u = 0. \tag{2}$$

The first equation (1) is linear and can be seen in Fourier space as:

$$\partial_t \mathscr{F} u = (\xi^2 - \xi^4)\mathscr{F} u. \tag{3}$$

The fourth partial derivative term $\partial_x^4 u$ decreases the short wavelength part of the energy spectrum whereas the second derivative term $\partial_x^2 u$ amplifies the long wavelength part. The second equation (2) corresponds to Burgers equation. It is nonlinear and develops shocks in finite time for non-trivial initial data. Nevertheless, as we will see later, this term has some mild regularizing effect. It is worth mentioning that for (2), the energy

$$\int_0^L u^2 dx,$$

is conserved. Therefore, one can intuitively say that in (K-S) the linear terms transport the energy from long wavelengths to short ones. Numerical simulations suggest (see the article of Wittenberg and Holmes [20]) that the time-averaged power spectrum

$$\lim_{T\to\infty} (LT)^{-1} \int_0^T |\mathscr{F}(u)(t,\xi)|^2 \, dt$$

is independent of L for $L \gg 1$. Moreover, this quantity is independent of $|\xi|$ and $L \gg 1$ in the long wavelength regime $L^{-1} \ll |\xi| \ll 1$ and decays exponentially in the short wavelength regime $|\xi| \gg 1$. In line with this, numerical simulations suggest that for all $\alpha \geq 0$:

$$\limsup_{T\to\infty} (LT)^{-1} \int_0^T \int_0^L \left(|\partial_x|^\alpha u\right)^2 dx dt = O(1).$$

This conjecture is supported by a universal bound on all stationary periodic solutions of (K-S) with mean 0, due to Michelson [13].

1.2 Known bounds

A first energy bound was obtained in the 80's by Nicolaenko, Scheurer and Temam [14], who established by the "background flow method" that for every odd (in space) solution u of (K-S):

$$\limsup_{t\to\infty} \left(\frac{1}{L}\int_0^L u^2 dx\right)^{1/2} = O\left(L^p\right),$$

with $p = 2$. This has been later generalized and improved to $p = 1$ (see [9, 2, 3]). Using an entropy method, Giacomelli and the third author [6] improved this result by showing that:

$$\limsup_{t\to\infty} \left(\frac{1}{L}\int_0^L u^2 dx\right)^{1/2} = o(L).$$

The proof is based on the fact that the dispersion relation $\xi^2 - \xi^4$ in (3) vanishes for $\xi \to 0$ and it implies that for every $\alpha \in [0,2]$, we have:

$$\limsup_{T\to\infty} \left(\frac{1}{TL}\int_0^T \int_0^L \left(|\partial_x|^\alpha u\right)^2 dx dt\right)^{1/2} = o(L),$$

by using the energy identity,

$$\partial_t \int_0^L (u(t,x))^2\,dx = \int_0^L (\partial_x u(t,x))^2\,dx - \int_0^L \left(\partial_x^2 u(t,x)\right) dx.$$

In a more recent paper [15], the third author proved that, for all $\alpha \in (1/3, 2)$,

$$\limsup_{T\to\infty} \left(\frac{1}{TL} \int_0^T \int_0^L \left(|\partial_x|^\alpha\, u \right)^2 dx\,dt \right)^{1/2} = O\left(\ln^{5/3+}(L)\right),^{1} \tag{4}$$

by using two ingredients: an *a priori* estimate for the capillary Burgers equation $\partial_t u + u\partial_x u + \partial_x^4 u = |\partial_x| g$ and an *a priori* estimate for the inhomogeneous Burgers equation, that is $\partial_t u + u\partial_x u = |\partial_x| g$. More precisely, the result of [15] states that, for every solution u of (K-S),

$$\|u\|_{\mathbb{B}^{1/3}_{3,3}} = O\left(\ln^{5/3+}(L)\right), \tag{5}$$

where $\|\cdot\|_{\mathbb{B}^s_{p,r}}$ denotes a Besov norm (see the appendix).

1.3 Main result

In this paper, we improve and simplify the result of the third author by showing that:

Theorem 1.1. *Let $L > 2$. For u a smooth L-periodic solution with zero average of the equation*

$$\partial_t u + u\partial_x u + \partial_x^2 u + \partial_x^4 u = 0,$$

there holds

$$\limsup_{T\to\infty} \left(\sup_{h>0} \frac{1}{LT} \int_0^T \int_0^L \frac{|u(t,x+h) - u(t,x)|^3}{h} dx\,dt \right)^{1/3} = O\left(\ln^{1/2+}(L)\right). \tag{6}$$

This result is indeed slightly stronger than the previous one, since by (19), it implies an improvement of the exponent in (5) from 5/3+ to 5/6+. However, this is not the main contribution of this paper. It is rather a simplified proof of the a priori estimate for inhomogeneous Burgers equation, which was one of the main tool for proving (5). For this purpose, we derive a modified Kármán-Howarth-Monin formula (see (14)). We also show how the proof of Golse and Perthame [8] (based on the kinetic formulation of Burgers equation) of a similar estimate for the homogeneous Burgers equation can be reinterpreted in this light. Since we work with slightly different Besov norms compared to [15], we need also to adapt most of the other steps to get (6). Besides Proposition 2.5, which we borrow directly from [15], we give here

[1] we use the notation $O\left(\ln^{5/3+}(L)\right)$ to indicate that for every $\kappa > 5/3$, there exists $c > 0$ such that the left-hand side is bounded by $c\ln^\kappa(L)$.

self-contained proofs.

The structure of the paper is the following: In Section 2, we enunciate the main theorem and give the structure of the proof. It has several ingredients: a Besov estimate for the inhomogeneous inviscid equation (Proposition 2.3), a regularity estimate for the capillary Burgers equation (Proposition 2.5) and an inverse estimate for Besov norms on solutions of (K-S) (Proposition 2.7). The following sections (*i.e.* Section 3, 4 and 5) are devoted to the proofs. In the appendix, we recall definitions and a few classical results regarding Besov spaces.

General notations

We denote by D^h the finite-difference operator $D^h : u \mapsto u(x+h) - u(x)$, by L^p the space $L^p([0,T]\times[0,L])$ and for $k \in \mathbb{N}, L > 0$, by $\mathscr{C}_L^k = \left\{f \in \mathscr{C}^k(\mathbb{R}), f \text{ is } L-\text{periodic}\right\}$. For an L-periodic function u, the spatial Fourier transform is defined by:

$$\mathscr{F}(u)(\xi) = L^{-1}\int_0^L \exp(-i\,\xi\,x)u(x)\,dx$$

and for a Schwartz function ϕ:

$$\mathscr{F}(\phi)(\xi) = \int_{\mathbb{R}} \exp(-i\,\xi\,x)\phi(x)\,dx.$$

For $v \in \mathbb{R}$, we let $v_+ = \max(v,0)$ (and similarly, $v_- = \max(-v,0)$).

2 Main theorem and structure of the proof

In this section, we state the main theorem and the results on which it is based (see the appendix for the definition and main properties of Besov spaces).

2.1 Main theorem

Theorem 2.1. *Let $L > 2$. For a smooth L-periodic solution u with zero average of* (K-S)*, there holds*

$$\|u\|_{\mathbb{B}^{1/3}_{3,\infty}} + \|u\|_{\mathbb{B}^{2}_{2,2}} = O(\ln^{1/2+}(L)). \tag{7}$$

From this theorem, we derive by interpolation (57) the following corollary:

Corollary 2.2. *Let $L > 2$. For a smooth L-periodic solution u with zero average of* (K-S) *and for indices p, s and r related by*

$$p \in [1/3, 2], \qquad s = 10/p - 3, \qquad 1/r = 3(1/p - 1/3),$$

we have

$$\|u\|_{\mathbb{B}^s_{p,r}} = O\big(\ln^{1/2+}(L)\big).$$

2.2 Structure of the proof

The proof of Theorem 2.1 uses four important ingredients: a regularity result for Burgers equation (Proposition 2.3), a higher regularity estimate for the capillary Burgers equation (Proposition 2.5), an energy estimate (Lemma 2.6), and a result which allows us to "increase" the r index of Besov spaces (Proposition 2.7). Let us now sketch the proof, discarding lower-order terms (in particular all the terms containing $g = -|\partial_x|^{-1}\partial_x^2 u$) and taking borderline exponents in the estimates[2]. The strategy is graphically represented in Figure 2. The starting point is Proposition 2.3, which for $s = 1$, $p = 5/2$, $r = 5/2$ and $\xi = -|\partial_x|^{-1}\partial_x^4 u$ (recall also (59)), roughly says that

$$\|u\|^3_{\mathbb{B}^{1/3}_{3,\infty}} \lesssim \|u\|_{\mathbb{B}^1_{5/2,5/2}} \|u\|_{\mathbb{B}^3_{5/3,5/3}}.$$

Using then the interpolation inequality (57), we get

$$\|u\|_{\mathbb{B}^3_{5/3,5/3}} \lesssim \|u\|^{1/3}_{\mathbb{B}^5_{5/4,5/4}} \|u\|^{2/3}_{\mathbb{B}^2_{2,2}}.$$

Proposition 2.5 for $\alpha = 2$, $p = 5/4$, $q = 5/2$ and therefore $\alpha' = 1$, indicates that

$$\|u\|_{\mathbb{B}^5_{5/4,5/4}} \lesssim \|u\|^2_{\mathbb{B}^1_{5/2,5/2}}. \tag{8}$$

Using the interpolation inequality (57) once again, we find

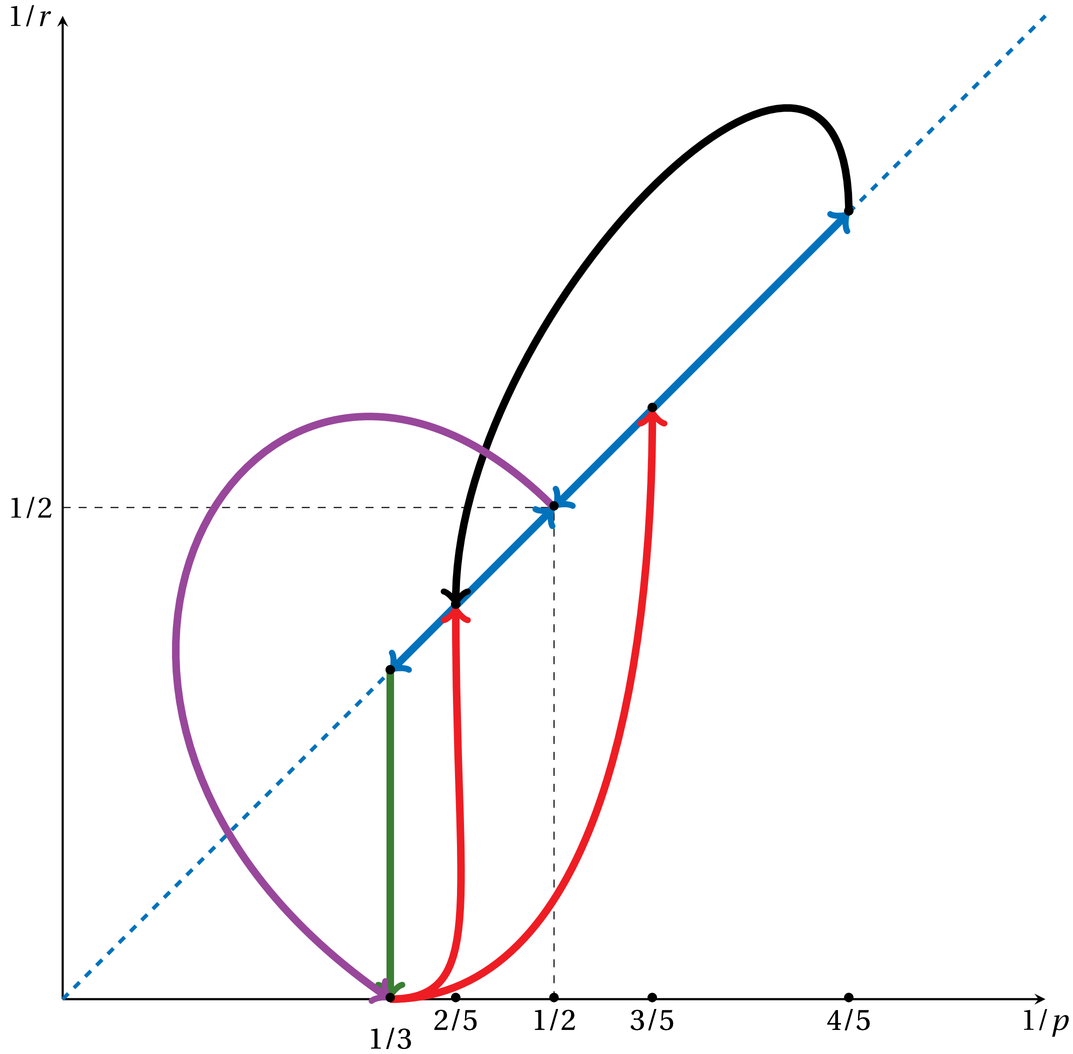
$$\|u\|_{\mathbb{B}^1_{5/2,5/2}} \lesssim \|u\|^{3/5}_{\mathbb{B}^{1/3}_{3,3}} \|u\|^{2/5}_{\mathbb{B}^2_{2,2}}. \tag{9}$$

From Lemma 2.6, we obtain

$$\|u\|_{\mathbb{B}^2_{2,2}} \lesssim \|u\|_{\mathbb{B}^{1/3}_{3,\infty}}.$$

[2]Let us stress that we cannot reach these exponents since some of the constants (in particular the one in (8)) explode.

Figure 2: Strategy of the proof



Legend:

- Interpolation
- Energy estimate
- Higher regularity
- Increase of the index r
- Estimate of the inhomogeneous Burgers equation

At this point, we see that we could have buckled the estimates if in (9), the Besov norm $\|u\|_{\mathbb{B}^{1/3}_{3,3}}$ was replaced by the stronger norm $\|u\|_{\mathbb{B}^{1/3}_{3,\infty}}$. Unfortunately, this seems not doable with our method of proof. Therefore, we need Proposition 2.7 in order to control $\|u\|_{\mathbb{B}^{1/3}_{3,3}}$ by $\|u\|_{\mathbb{B}^{1/3}_{3,\infty}}$. It is at this last stage that we lose a logarithm since (19) gives

$$\|u\|_{\mathbb{B}^{1/3}_{3,3}} \lesssim \ln^{1/3}(L)\,\|u\|_{\mathbb{B}^{1/3}_{3,\infty}}.$$

Putting all these estimates together, we find

$$\|u\|_{\mathbb{B}^{1/3}_{3,\infty}} \lesssim \ln^{1/2}(L),$$

which is (7).

As mentioned, the first ingredient is an estimate for the inhomogeneous Burgers equation. A similar estimate was obtained in [15, Prop. 1]. A related inequality for the homogeneous Burgers equation has been recently derived in [8].

Let us consider the following inhomogeneous Burgers equation:

$$\partial_t u + u\partial_x u = |\partial_x|\, g + |\partial_x|\,\xi. \tag{10}$$

Proposition 2.3 (Besov estimate for the inhomogeneous Burgers equation)**.** *Let ξ, g be smooth L-periodic functions. Then, for any smooth L-periodic solution u of* (10)*, there holds: For $s \in]0,1[$, $r, r', p, p' \in [1,+\infty]$ verifying $\frac{1}{r}+\frac{1}{r'}=1$ and $\frac{1}{p}+\frac{1}{p'}=1$, there exists a constant $c>0$ just depending on s, r, p such that:*

$$\|u\|^3_{B^{1/3}_{3,\infty}} \leq c\left(\|u\|_{B^{1/3}_{3,\infty}}\left\|g\right\|_{B^{2/3}_{3/2,1}} + \|u\|_{B^{s}_{p,r}}\|\xi\|_{B^{1-s}_{p',r'}} + \|u(0,\cdot)\|^2_{L^2[0,L]}\right). \tag{11}$$

Therefore, taking the time-space average, it holds:

$$\|u\|^3_{\mathbb{B}^{1/3}_{3,\infty}} \leq c\left(\|u\|_{\mathbb{B}^{1/3}_{3,\infty}}\left\|g\right\|_{\mathbb{B}^{2/3}_{3/2,1}} + \|u\|_{\mathbb{B}^{s}_{p,r}}\|\xi\|_{\mathbb{B}^{1-s}_{p',r'}}\right). \tag{12}$$

The proof of (11) is based on a modified Kármán-Howarth-Monin identity:

Lemma 2.4. *Let η be a smooth L-periodic function and let u be a smooth $L-$periodic solution with zero average of*

$$\partial_t u + u\partial_x u = \eta, \tag{13}$$

then for $h \in \mathbb{R}$,

$$\partial_t\left(\frac{1}{2}\int_0^L |D^h u| D^h u\,dx\right) + \partial_h \frac{1}{6}\int_0^L |D^h u|^3\,dx = \int_0^L D^h\eta |D^h u|\,dx. \tag{14}$$

The usual Kármán-Howarth-Monin identity [5] states that

$$\partial_t\left(\frac{1}{2}\int_0^L \left(D^h u\right)^2 dx\right) + \partial_h \frac{1}{6}\int_0^L \left(D^h u\right)^3 dx = \int_0^L D^h\eta D^h u\,dx. \tag{15}$$

This formula can be easily checked by using equation (13) and the periodicity. The main difference between (15) and (14) is that in the latter, the coercive term $\int_0^L |D^h u|^3 dx$ replaces the non-coercive term $\int_0^L \left(D^h u\right)^3 dx$.
We will give two proofs of (14). The first is by a direct computation and the second uses the kinetic formulation of Burgers equation following ideas of [8]. Therefore, this second proof gives a new, and hopefully interesting, interpretation of the arguments of [8].

The second ingredient is a higher regularity result for the capillary Burgers equation (see [15, Prop. 2, p. 14]).

Proposition 2.5 (Higher regularity)**.** *Let $p, q \in [1, +\infty[$, $\alpha \in \mathbb{R}$ satisfying:*

$$p+1 \le q \le 2p, \qquad \text{and} \qquad \alpha' = (6+\alpha)p/q - 3 \in]0,1[.$$

Then, there exists $c > 0$ such that, if u, g are smooth, L-periodic in x and satisfy

$$\partial_t u + u\partial_x u + \partial_x^4 u = |\partial_x| g,$$

the following estimate holds:

$$\|u\|_{\mathbb{B}^{3+\alpha}_{p,p}} \le c\left(\|u\|^{q/p}_{\mathbb{B}^{\alpha'}_{q,q}} + \|g\|_{\mathbb{B}^{\alpha}_{p,p}}\right). \tag{16}$$

Proposition 2.5 allows to jump from higher derivatives to smaller ones in Besov spaces. The proof, which we will not provide, is based on a narrow-band Littlewood-Paley decomposition.

The third ingredient is an elementary energy estimate, which directly bounds the $L^2\left([0,T], H^2([0,L])\right) \cong B^2_{2,2}$ norm of a solution u of the inhomogeneous capillary Burgers equation.

Lemma 2.6 (Energy estimate)**.** *Let u be a smooth solution of:*

$$\partial_t u + u\partial_x u + \partial_x^4 u = |\partial_x| g. \tag{17}$$

Then, the following estimate holds:

$$\|u\|_{\mathbb{B}^2_{2,2}} \le c\,\|u\|^{1/2}_{\mathbb{B}^{1/3}_{3,\infty}} \|g\|^{1/2}_{\mathbb{B}^{2/3}_{3/2,1}}. \tag{18}$$

Proof. Since the proof is straightforward, we give it now. Integrating over the equation (17) over $[0,T]\times[0,L]$, we get:

$$\int_0^L u(T,x)^2 dx - \int_0^L u(0,x)^2 dx = \int_0^T \int_0^L u|\partial_x| g\, dx dt - \int_0^T \int_0^L (\partial_x^2 u)^2 dx dt.$$

Therefore by (58)

$$\int_0^T \int_0^L \left(\partial_x^2 u\right)^2 dx dt \le \int_0^L u(0,x)^2 dx + \|u\|_{B^{1/3}_{3,\infty}} \|g\|_{B^{2/3}_{3/2,1}},$$

taking then the time-space average yields the result. □

As already mentioned, these three estimates will not be sufficient to conclude. We will also need an estimate relating Besov norms with different exponents r. One can easily see that if $r_1 > r_2$, then $\|\cdot\|_{\mathbb{B}^s_{p,r_1}} \leq \|\cdot\|_{\mathbb{B}^s_{p,r_2}}$ (it is a consequence of convexity inequality). In fact, it is possible to reverse the inequality for *solutions* of (K-S), but this comes with a price: a logarithm of the spatial period L appears.

Proposition 2.7 (Increasing the index r)**.** *There exists $c > 0$ such that, for all $L \geq 2$, u solution of* (K-S)*, the following estimate holds:*

$$\|u\|_{\mathbb{B}^{1/3}_{3,3}} \leq c \ln^{1/3}(L)\, \|u\|_{\mathbb{B}^{1/3}_{3,\infty}}. \tag{19}$$

3 Proof of Theorem 2.1

In this section, we derive the main theorem from the above propositions. We now consider the rescaled Besov norm $\mathbb{B}^s_{p,r}$ as a point $(s, 1/p, 1/r)$ in the space $\mathbb{R}^3$. All the norms involved in our problem lie in the rectangle $\mathscr{P}$ of $\mathbb{R}^3$ defined by[3]:

$$\begin{cases} s &= 10/p - 3, \\ \frac{1}{p} &\in [0,1], \\ \frac{1}{r} &\in [0,1]. \end{cases}$$

Proof of Theorem 2.1. Let u be a solution of (K-S). It is convenient to introduce the abbreviation:

$$D(\alpha) = \|u\|_{\mathbb{B}^{10\alpha-3}_{\alpha^{-1},\alpha^{-1}}} \qquad D_*(1/3) = \|u\|_{\mathbb{B}^{1/3}_{3,\infty}}.$$

Notice that $D(1/2) = \|u\|_{\mathbb{B}^{1/2}_{2,2}}$ and $D(1/3) = \|u\|_{\mathbb{B}^{1/3}_{3,3}}$. With this notation, interpolation inequality (57) takes the form

$$D(\alpha) \leq D^{\theta}(\alpha_1) D^{1-\theta}(\alpha_2) \tag{20}$$

for $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$ and $\theta \in [0,1]$.

Letting $s = 10\alpha - 3$, $p = \alpha^{-1}$ and $r = \alpha^{-1}$ in (12) and using (59) for $\xi = -|\partial_x|^{-1}\partial_x^4 u$, (12) can be rewritten as

$$D_*^3(1/3) \leq c\Big(D(\alpha)D(1-\alpha) + D_*(1/3)\,\|g\|_{\mathbb{B}^{2/3}_{3/2,1}}\Big), \tag{21}$$

for $\alpha \in]3/10, 2/5[$. In turn, (16) with $p^{-1} = \beta$, $q^{-1} = \gamma$ and $\alpha = 10\beta - 6$ (which implies $\alpha' = 10\gamma - 3$) gives

$$D(\beta) \leq c\Big(D^{\beta/\gamma}(\gamma) + \|g\|_{\mathbb{B}^{10\beta-6}_{\beta^{-1},\beta^{-1}}}\Big), \tag{22}$$

[3] See Figure 2 in Section 3.2 which represents the strategy in $\mathscr{P}$.

for $\gamma \in]3/10, 2/5[$ and $\frac{\gamma}{1-\gamma} \le \beta \le 2\gamma$. Finally, (18) is equivalent to

$$D(1/2) \le c D_*^{1/2}(1/3) \|g\|_{\mathbb{B}^{2/3}_{3/2,1}}^{1/2}, \tag{23}$$

and (19) to

$$D(1/3) \le c \ln^{1/3}(L) D_*(1/3). \tag{24}$$

Our first goal is to argue that for $g = -|\partial_x|^{-1}\partial_x^2 u$, we can replace in the above estimates all the Besov norms involving g by $D_*(1/3)$. By (59) and (57),

$$\|g\|_{\mathbb{B}^{10\beta-6}_{\beta^{-1},\beta^{-1}}} \le c\|u\|_{\mathbb{B}^{10\beta-5}_{\beta^{-1},\beta^{-1}}} \le c\|u\|^{1/2}_{\mathbb{B}^{10\beta-3}_{\beta^{-1},\beta^{-1}}} \|u\|^{1/2}_{\mathbb{B}^{10\beta-7}_{\beta^{-1},\beta^{-1}}} = cD^{1/2}(\beta)\|u\|^{1/2}_{\mathbb{B}^{10\beta-7}_{\beta^{-1},\beta^{-1}}}.$$

Hence, in view of (23), Young's inequality and since by (59), $\|g\|_{\mathbb{B}^{2/3}_{3/2,1}} \le c\|u\|_{\mathbb{B}^{5/3}_{3/2,1}}$, it will be enough to prove that

$$\|u\|_{\mathbb{B}^{10\beta-7}_{\beta^{-1},\beta^{-1}}} + \|u\|_{\mathbb{B}^{5/3}_{3/2,1}} \le c(D_*(1/3) + D(1/2)) \tag{25}$$

for $\beta \in]11/15, 4/5[$ (which reduces the use of (22) to $\gamma \in]1/3, 2/5[$). We can indeed prove more generally that for $1/3 < s < 2$, $p \le 2$ and any $q \ge 1$, there holds

$$\|u\|_{\mathbb{B}^s_{p,q}} \le c\left(\|u\|_{\mathbb{B}^2_{2,2}} + \|u\|_{\mathbb{B}^{1/3}_{3,\infty}}\right). \tag{26}$$

Thanks to Jensen's inequality, we have $\|u\|_{\mathbb{B}^s_{p,q}} \le \|u\|_{\mathbb{B}^s_{2,q}}$ and $\|u\|_{\mathbb{B}^{1/3}_{3,\infty}} \ge \|u\|_{\mathbb{B}^{1/3}_{2,\infty}}$. By monotonicity of the Besov norms with respect to the last index, there also holds $\|u\|_{\mathbb{B}^s_{2,q}} \le \|u\|_{\mathbb{B}^s_{2,1}}$ and $\|u\|_{\mathbb{B}^2_{2,2}} \ge \|u\|_{\mathbb{B}^2_{2,\infty}}$. Therefore, we are left with proving that

$$\|u\|_{\mathbb{B}^s_{2,1}} \le c\left(\|u\|_{\mathbb{B}^{1/3}_{2,\infty}} + \|u\|_{\mathbb{B}^2_{2,\infty}}\right).$$

By definition of the Besov norms, for $1/3 < s < 2$,

$$\begin{aligned}
\|u\|_{B^s_{2,1}} &= \sum_{k\ge 0} 2^k \|u_k\|_{L^2} + \sum_{k<0} 2^k \|u_k\|_{L^2} \\
&\le \sup_k 2^{2k}\|u_k\|_{L^2} \sum_{k\ge 0} 2^{-k} + \sup_k 2^{\frac{1}{3}k}\|u_k\|_{L^2} \sum_{k\le 0} 2^{\frac{2}{3}k} \\
&\le c\left(\|u\|_{B^{1/3}_{2,\infty}} + \|u\|_{B^2_{2,\infty}}\right),
\end{aligned}$$

which after taking the average over time and space, finishes the proof of (26).

To sum up, we now have that (21), (22) and (23) together with (25) imply

$$D_*^3(1/3) \le c\left(D(\alpha)D(1-\alpha) + D_*^2(1/3)\right) \tag{27}$$

for $\alpha \in]3/10, 2/5[$,

$$D(\beta) \leq c\left(D^{\beta/\gamma}(\gamma) + D_*(1/3)\right) \tag{28}$$

for $\gamma \in]1/3, 2/5[$ and $\frac{\gamma}{1-\gamma} \leq \beta \leq 2\gamma$ and

$$D(1/2) \leq cD_*(1/3). \tag{29}$$

We now gather the above estimates in order to bound $D_*(1/3)$. Passing to the logarithm in the above inequalities, we see that optimizing the parameters to get the best power of $\ln(L)$ is equivalent to a linear programming problem. Its solution thus lie at the boundaries of the admissible domain. It is not hard to see that in particular, we want to take $\frac{\beta}{2} = \gamma = \alpha$ with α as close as possible to $2/5$. Let $\theta, \eta \in]0,1[$ be such that

$$(1-\alpha) = \theta\beta + (1-\theta)\frac{1}{2} \qquad \text{and} \qquad \alpha = \eta\frac{1}{2} + (1-\eta)\frac{1}{3},$$

so that θ is close to $1/3$ and η is close to $2/5$. Thanks to (20),

$$D(1-\alpha) \leq D^{\theta}(\beta)D^{1-\theta}(1/2) \qquad \text{and} \qquad D(\alpha) \leq D^{\eta}(1/2)D^{1-\eta}(1/3). \tag{30}$$

Since we can assume that $D_*(1/3) \geq 1$, we get from (27), (30) and (28),

$$\begin{aligned} D_*^3(1/3) &\leq cD(\alpha)\left(D^{2\theta}(\alpha) + D_*^{\theta}(1/3)\right)D^{1-\theta}(1/2) \\ &\leq c\left(D^{1+2\theta}(\alpha)D_*^{1-\theta}(1/3) + D(\alpha)D_*(1/3)\right), \end{aligned}$$

where in the last inequality, we used (29). From (30), (29) and (24), we deduce

$$D_*^3(1/3) \leq c\left(D_*^{2+\theta}(1/3)\ln^{\frac{1}{3}(1+2\theta)(1-\eta)}(L) + D_*^2(1/3)\ln^{\frac{1-\eta}{3}}(L)\right).$$

Dividing by $D_*^2(1/3)$ this inequality and noticing that for η close to $2/5$, $\frac{1-\eta}{3}$ is close to $1/5$, we obtain that if $D_*(1/3) \geq \ln^{\frac{1-\eta}{3}}(L)$, then

$$D_*(1/3) \leq cD_*^{\theta}(1/3)\ln^{\frac{1}{3}(1+2\theta)(1-\eta)}(L),$$

which gives finally

$$D_*(1/3) \leq c\ln^{\frac{1}{3}\frac{(1+2\theta)(1-\eta)}{1-\theta}}(L)$$

and thus the result since $\lim_{\theta\uparrow 1/3, \eta\uparrow 2/5} \frac{1}{3}\frac{(1+2\theta)(1-\eta)}{1-\theta} = 1/2$. □

4 Proof of Proposition 2.3

For the reader's convenience, let us recall the statement of Proposition 2.3. Let u be a smooth solution of the following inhomogeneous Burgers equation:

$$\partial_t u + u\partial_x u = |\partial_x|\, g + |\partial_x|\,\xi\,. \tag{31}$$

Proposition. *Let ξ, g be smooth L-periodic functions. Then, for any smooth L-periodic solution u of* (31)*, there holds: For $s \in]0,1[$, $r, r', p, p' \in [1,+\infty]$ verifying $\frac{1}{r}+\frac{1}{r'}=1$ and $\frac{1}{p}+\frac{1}{p'}=1$, there exists a constant $c \in \mathbb{R}_+^*$ just depending on s, r, p such that:*

$$\|u\|^3_{B^{1/3}_{3,\infty}} \leq c\left(\|u\|_{B^{1/3}_{3,\infty}}\left\|g\right\|_{B^{2/3}_{3/2,1}} + \|u\|_{B^s_{p,r}}\|\xi\|_{B^{1-s}_{p',r'}} + \|u(0,\cdot)\|^2_{L^2[0,L]}\right).$$

Before proceeding further, let us remark that, by approximation, this applies to any (possibly non smooth) entropy solution of Burgers equation (31). Indeed, if we consider a solution u of

$$\partial_t u - u\partial_x u - \varepsilon\partial_x^2 u = 0,$$

then it is a smooth solution of (31) with $g = 0$ and $\xi = \varepsilon\,|\partial_x|\,u$. A careful inspection of the proof of Proposition 2.3 shows that for $p = r = 2$ it extends to $s = 1$, yielding

$$\|u\|^3_{B^{1/3}_{3,\infty}} \leq c\left(\|u\|_{B^1_{2,2}}\|\xi\|_{B^0_{2,2}} + \|u(0,\cdot)\|^2_{L^2[0,L]}\right),$$

that is

$$\|u\|^3_{B^{1/3}_{3,\infty}} \leq c\left(\varepsilon\,\|\partial_x u\|^2_{L^2} + \|u(0,\cdot)\|^2_{L^2[0,L]}\right).$$

Combining this with the energy inequality: $\varepsilon\,\|\partial_x u\|^2_{L^2} \leq c\,\|u(0,\cdot)\|^2_{L^2[0,L]}$, gives

$$\|u\|^3_{B^{1/3}_{3,\infty}} \leq c\,\|u(0,\cdot)\|^2_{L^2[0,L]},$$

which passes to the limit as $\varepsilon \to 0$. The indices are optimal in the light of the result of De Lellis and Westdickenberg [4] which states that we cannot hope to have more regularity, in the sense that the Besov index s cannot be better than $1/3$.

As already pointed out, the proof of the aimed estimate is based on a modified Kármán-Howarth-Monin identity:

Lemma 4.1. *Let η be a smooth L-periodic function and let u be a smooth $L-$periodic solution with zero average of*

$$\partial_t u + u\partial_x u = \eta \tag{32}$$

then for $h \in \mathbb{R}$,

$$\partial_t\left(\frac{1}{2}\int_0^L |D^h u| D^h u\, dx\right) + \partial_h \frac{1}{6}\int_0^L \left|D^h u\right|^3 dx = \int_0^L D^h\eta\left|D^h u\right| dx. \tag{33}$$

Proof. By periodicity, (33) will be a direct consequence of the following pointwise identity:

$$\frac{1}{2}\partial_t\left(|D^h u|D^h u\right)+\frac{1}{6}\partial_h\left|D^h u\right|^3+\frac{1}{2}\partial_x\left(u|D^h u|D^h u+\frac{1}{3}|D^h u|^3\right)=D^h\eta\left|D^h u\right|. \tag{34}$$

For simplicity, let us introduce the notation $u^h(x)=u(x+h)$ (so that $D^h u=u^h-u$). Using (32) we get:

$$\begin{aligned}\frac{1}{2}\partial_t\left(|D^h u|D^h u\right)+\frac{1}{6}\partial_h\left|D^h u\right|^3&=|D^h u|\partial_t(D^h u)+\frac{1}{2}\left|D^h u\right|D^h u\partial_x u^h\\&=\left|D^h u\right|D^h\left(\eta-u\partial_x u\right)+\frac{1}{2}\left|D^h u\right|\left(u^h\partial_x u^h-u\partial_x u^h\right)\\&=D^h\eta\left|D^h u\right|+\frac{1}{2}\left|D^h u\right|\left(2u\partial_x u-u\partial_x u^h-u^h\partial_x u^h\right).\end{aligned}$$

It remains to prove that

$$|D^h u|\left(2u\partial_x u-u\partial_x u^h-u^h\partial_x u^h\right)=-\partial_x\left(u|D^h u|D^h u+\frac{1}{3}|D^h u|^3\right). \tag{35}$$

We start with

$$|D^h u|2u\partial_x u=-u^2\partial_x|D^h u|+\partial_x\left(|D^h u|u^2\right)$$

and

$$\begin{aligned}|D^h u|u\partial_x u^h&=-\partial_x\left(|D^h u|u\right)u^h+\partial_x\left(|D^h u|uu^h\right)\\&=-uu^h\partial_x|D^h u|-u^h|D^h u|\partial_x u+\partial_x\left(|D^h u|uu^h\right),\end{aligned}$$

to get

$$\begin{aligned}|D^h u|\left(2u\partial_x u-u\partial_x u^h-u^h\partial_x u^h\right)&=-u^2\partial_x|D^h u|+\partial_x\left(|D^h u|u^2\right)+uu^h\partial_x|D^h u|\\&\quad+u^h|D^h u|\partial_x u-\partial_x\left(|D^h u|uu^h\right)-|D^h u|u^h\partial_x u^h\\&=-\partial_x\left(u|D^h u|D^h u\right)+uD^h u\partial_x|D^h u|-u^h|D^h u|\partial_x D^h u.\end{aligned}$$

But since

$$D^h u\partial_x|D^h u|=|D^h u|\partial_x D^h u,$$

then

$$\begin{aligned}|D^h u|\left(2u\partial_x u-u\partial_x u^h-u^h\partial_x u^h\right)&=-\partial_x\left(u|D^h u|D^h u\right)-|D^h u|D^h u\partial_x D^h u\\&=-\partial_x\left(u|D^h u|D^h u+\frac{1}{3}|D^h u|^3\right),\end{aligned}$$

which concludes the proof of (35). □

Remark. Arguing along the same lines, one can prove that more generally, if a is non-negative and if u is a smooth solution with zero average of

$$\partial_t u + \partial_x [a(u)] = \eta \tag{36}$$

then

$$\partial_t \left(\frac{1}{2} \int_0^L |D^h u| D^h u dx \right) + \partial_h \left(\int_0^L |D^h u| (a(u) + a(u^h)) - 2|A(u^h) - A(u)| dx \right) = \int_0^L D^h \eta \left| D^h u \right| dx,$$

where $A' = a$. Notice that Burgers equation corresponds to (36) with $a(u) = \frac{1}{2} u^2$. If a is C^1 and monotone in the sense that there exist $\beta \geq 1$ and $C > 0$, such that for $v \geq w$,

$$a'(v) - a'(w) \geq C(v - w)^\beta,$$

then one can obtain a similar estimate to (11) by using that for $\bar{u} \geq u$

$$\begin{aligned}(\bar{u} - u)(a(\bar{u}) + a(u)) - 2(A(\bar{u}) - A(u)) &= \int_u^{\bar{u}} \int_w^{\bar{u}} (a'(v) - a'(w)) dv dw \\ &\geq C|\bar{u} - u|^{\beta+2}.\end{aligned}$$

In this way, one can fully recover the results from [8].

We now give an alternative proof of (the integrated form of) (33) following ideas from [8]. This proof uses the kinetic formulation of the inhomogeneous Burgers equation together with the use of an interaction identity (see (43)).

Lemma 4.2. *Let η be a smooth L-periodic function and let u be a smooth $L-$periodic solution with zero average of*

$$\partial_t u + u \partial_x u = \eta, \tag{37}$$

then for $h \in \mathbb{R}$,

$$\left[\frac{1}{2} \int_0^h \int_0^L |D^\Delta u| D^\Delta u dx d\Delta \right]_0^T + \frac{1}{6} \int_0^T \int_0^L \left| D^h u \right|^3 dx dt = \int_0^T \int_0^L \int_0^h D^\Delta \eta \left| D^\Delta u \right| d\Delta dx dt. \tag{38}$$

Proof. Before starting the proof, let us point out that since we have a direct proof of (33), we will not take care of regularity issues. Nevertheless, all passages can be rigorously justified by a suitable approximation argument.

Step 1. Without loss of generality, we can assume that $h > 0$. Letting:

$$f(t,x,v) = \begin{cases} 1 & \text{if } v \le u(t,x), \\ 0 & \text{if } v > u(t,x), \end{cases}$$

equation (37) is equivalent to the following kinetic formulation (see [12]):

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = -\partial_v f(t,x,v)\eta(t,x). \tag{39}$$

Notice that since u is bounded, $D^h f$ is integrable even though f is not. We are going to compute only integrals depending on $D^h f$ and will therefore not have to deal with integrability issues. As in [8, Lem. 4.3], we define:

$$M_u(v) = \begin{cases} 1 & \text{if } v \le u, \\ 0 & \text{if } v > u. \end{cases}$$

We first claim that for all u, $\bar{u} \in \mathbb{R}$ the following equality holds:

$$\frac{1}{6}|u-\bar{u}|^3 = \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\mathbb{1}_{\mathbb{R}^+}(v-w)\right](v-w)\left(M_u(v)-M_{\bar{u}}(v)\right)\left(M_u(w)-M_{\bar{u}}(w)\right)dv\,dw. \tag{40}$$

Without loss of generality, we can suppose that $u \ge \bar{u}$. Then:

$$M_u(v) - M_{\bar{u}}(v) = \mathbb{1}_{]\bar{u},u[}(v).$$

Thus (40) follows from:

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} &\left[\mathbb{1}_{\mathbb{R}_+}(v-w)\right](v-w)\left(M_u(v)-M_{\bar{u}}(v)\right)\left(M_u(w)-M_{\bar{u}}(w)\right)dv\,dw \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\mathbb{1}_{\mathbb{R}_+}(v-w)\mathbb{1}_{]\bar{u},u[}(v)\mathbb{1}_{]\bar{u},u[}(w)\right](v-w)\,dv\,dw \\
&= \int_{\bar{u}}^{u}\int_{w}^{u}(v-w)\,dv\,dw = \frac{1}{6}|u-\bar{u}|^3.
\end{aligned}$$

Letting

$$Q(h) = \int_0^T\int_0^L\int_{\mathbb{R}\times\mathbb{R}} \left[\mathbb{1}_{\mathbb{R}_+}(v-w)\right](v-w)D^h f(t,x,v)D^h f(t,x,w)\,dv\,dw\,dx\,dt,$$

we see that proving (38) is equivalent to

$$Q(h) = \int_0^T\int_0^L\int_0^h D^\Delta\eta\left|D^\Delta u\right|d\Delta\,dx\,dt - \left[\frac{1}{2}\int_0^h\int_0^L |D^\Delta u|D^\Delta u\,dx\,d\Delta\right]_0^T. \tag{41}$$

Step 2. To cope with the quantity Q, the main tool is the following interaction identity (see [8]), which have been introduced first by Varadhan ([19, Lem. 22.1]): Let A, B, C, D, E, $F : [0,T] \times [0,L] \to \mathbb{R}$ be functions satisfying the following system

$$\begin{cases} \partial_t A + \partial_x B = C, \\ \partial_t D + \partial_x E = F, \end{cases} \tag{42}$$

and having zero spatial average. Then the following identity holds:

$$\begin{aligned} \int_0^T \int_0^L (AE - BD) = &\int_0^T \int_0^L A(t,x) \left(\int_0^x F(t,y)\,dy \right) dx\,dt \\ &+ \int_0^T \int_0^L C(t,x) \left(\int_0^x D(t,y)\,dy \right) dx\,dt \\ &- \left[\int_0^L \int_0^x A(t,x) D(t,y)\,dy\,dx \right]_{t=0}^{t=T}. \end{aligned} \tag{43}$$

Indeed, by Taylor expansion:

$$\begin{aligned} \int_0^T \int_0^L A(t,x) E(t,x)\,dx\,dt = &\int_0^T \int_0^L \int_0^x A(t,x) \partial_x E(t,y)\,dy\,dx\,dt \\ &- \int_0^T \int_0^L A(t,x) E(t,0)\,dx\,dt. \end{aligned}$$

Since A has zero spatial average, the second term vanishes. Using equation (42) to compute the first term and integrating by parts, we get:

$$\begin{aligned} \int_0^T \int_0^L \int_0^x A(t,x) \partial_x E(t,y)\,dy\,dx\,dt = &\int_0^T \int_0^L \int_0^x A(t,x) F(t,y)\,dy\,dx\,dt \\ &- \int_0^T \int_0^L \int_0^x A(t,x) \partial_t D(t,y)\,dy\,dx\,dt \\ = &\int_0^T \int_0^L \int_0^x A(t,x) F(t,y)\,dy\,dx\,dt \\ &+ \int_0^T \int_0^L \int_0^x \partial_t A(t,x) D(t,y)\,dy\,dx\,dt \\ &- \left[\int_0^L \int_0^x A(t,x) D(t,y)\,dy\,dx \right]_{t=0}^{t=T}. \end{aligned}$$

Let us now compute more precisely the second term, using (42):

$$\begin{aligned}\int_0^T\int_0^L\int_0^x \partial_t A(t,x)D(t,y)\,dy\,dx\,dt =& \int_0^T\int_0^L\int_0^x C(t,x)D(t,y)\,dy\,dx\,dt\\ &-\int_0^T\int_0^L \partial_x B(t,x)\int_0^x D(t,y)\,dy\,dx\,dt\\ =&\int_0^T\int_0^L\int_0^x C(t,x)D(t,y)\,dy\,dx\,dt\\ &+\int_0^T\int_0^L B(t,x)D(t,x)\,dx\,dt,\end{aligned}$$

which concludes the proof of (43).

Step 3. We apply the interaction identity to:

$$\begin{cases} A(t,x,v) = D^h f(t,x,v), & D(t,x,w) = A(t,x,w),\\ B(t,x,v) = vD^h f(t,x,v), & E(t,x,w) = B(t,x,w),\\ C(t,x,v) = -\partial_v D^h(f(t,x,v)\eta(t,x)), & F(t,x,w) = C(t,x,w).\end{cases}$$

Note that A,B,C,D,E,F implicitly depend on h. Multiplying each side of the identity by $\mathbb{1}_{\mathbb{R}_+}(v-w)$ and integrating it, we get:

$$\begin{aligned} Q(h) =& -\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w)\int_0^T\int_0^L \left(A(t,x,v)E(t,x,w) - B(t,x,v)D(t,x,w)\right)dx\,dt\,dv\,dw\\ =& -\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w)\int_0^T\int_0^L A(t,x,v)\left(\int_0^x F(t,y,w)\,dy\right)dx\,dt\,dv\,dw\\ &-\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w)\int_0^T\int_0^L C(t,x,v)\left(\int_0^x D(t,y,w)\,dy\right)dx\,dt\,dv\,dw\\ &+\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w)\left[\int_0^L\int_0^x A(t,x,v)D(t,y,w)\,dy\,dx\right]_{t=0}^{t=T} dv\,dw.\\ =& -\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w)\int_0^T\int_0^L A(t,x,v)\left(\int_0^x F(t,y,w)\,dy\right)dx\,dt\,dv\,dw\\ &-\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(w-v)\int_0^T\int_0^L F(t,y,w)\left(\int_0^y A(t,x,v)\,dx\right)dy\,dt\,dv\,dw\\ &+\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w)\left[\int_0^L\int_0^x A(t,x,v)D(t,y,w)\,dy\,dx\right]_{t=0}^{t=T} dv\,dw.\\ =&Q_1+Q_2+Q_3.\end{aligned}$$

But, by periodicity, $\int_0^y A(t,x,w)\,dx = -\int_y^L A(t,x,w)\,dx$, and thus

$$\int_0^L F(t,y,v)\left(\int_0^y A(t,x,w)\,dx\right)dy = -\int_0^L A(t,x,w)\left(\int_0^x F(t,y)\,dy\right)dx.$$

Therefore,

$$Q_2 = \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(w-v) \int_0^T \int_0^L A(t,x,v) \left(\int_0^x F(t,y,w)\,dy \right) dx\,dt\,dv\,dw.$$

Step 4. In the next two steps, the time variable plays no role. We will therefore consider

$$\overline{Q}_1 = -\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w) \int_0^L A(x,v) \left(\int_0^x C(y,w)\,dy \right) dx\,dv\,dw$$

and

$$\overline{Q}_2 = \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(w-v) \int_0^L A(x,v) \left(\int_0^x C(y,w)\,dy \right) dx\,dv\,dw.$$

By definition of A and C, we have

$$\begin{aligned}\overline{Q}_1 - \overline{Q}_2 &= -\int_{\mathbb{R}\times\mathbb{R}} \int_0^L D^h f(x,v) \left(\int_0^x \partial_v D^h (f(y,w)\eta(y))\,dy \right) dx\,dv\,dw \\ &= \int_0^L D^h u(x) \left(\int_0^x D^h \eta(y)\,dy \right) dx.\end{aligned}$$

The y-integral then telescopes to:

$$\overline{Q}_1 - \overline{Q}_2 = \int_0^L D^h u(x) \left(\int_x^{x+h} - \int_0^h \right) \eta(y)\,dy\,dx.$$

Since by periodicity we have,

$$\int_0^L D^h u(x)\,dx = 0,$$

this reduces to

$$\begin{aligned}\overline{Q}_1 - \overline{Q}_2 &= \int_0^L D^h u(x) \int_x^{x+h} \eta(y)\,dy\,dx \\ &= \int_0^L \eta(y) \int_{y-h}^{y} D^h u(x)\,dx\,dy \\ &= \int_0^L \int_0^h \eta(y)(u(y+\Delta) - u(y-\Delta))\,dy\,d\Delta. \qquad (44)\end{aligned}$$

Step 5. Here we argue that

$$\overline{Q}_1 + \overline{Q}_2 = \int_0^L \int_0^h D^\Delta \eta \left| D^\Delta u \right| d\Delta\,dx. \qquad (45)$$

For this we prove first that

$$\overline{Q}_1 = \frac{1}{2}\int_0^h \int_0^L D^\Delta \eta \left|D^\Delta u\right| + \eta(u(x-\Delta) - u(x+\Delta))\, dx d\Delta. \tag{46}$$

Indeed, combined with (44), this would give,

$$\overline{Q}_1 + \overline{Q}_2 = 2\overline{Q_1} - (\overline{Q}_1 - \overline{Q}_2) = \int_0^L \int_0^h D^\Delta \eta \left|D^\Delta u\right| d\Delta dx.$$

which is (45). By definition of A and C:

$$\begin{aligned}
\overline{Q}_1 &= \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w) \int_0^L D^h f(x,v) \left(\int_0^x \partial_v D^h \left(f(y,w)\eta(y)\right) dy\right) dx dv dw \\
&= \int_{-\infty}^{+\infty} \int_0^L D^h f(x,v) \int_{-\infty}^v \int_0^x \partial_v D^h \left(f(y,w)\eta(y)\right) dy dw dx dv \\
&= \int_{-\infty}^{+\infty} \int_0^L D^h f(x,v) \int_0^x D^h \left(f(y,v)\eta(y)\right) - D^h \eta(y) dy dx dv.
\end{aligned}$$

Arguing as in *Step 4*, we find

$$\overline{Q}_1 = \int_0^L \eta(y) \left(\int_0^h \int_{-\infty}^{+\infty} D^h f(y-x,v) \left(f(y,v)-1\right) dx dv\right) dy.$$

After changing the names of the variables to $y = \tilde{x}$ and $x = \Delta$ we obtain

$$\overline{Q}_1 = \int_0^L \eta(\tilde{x}) \left(\int_0^h \int_{-\infty}^{+\infty} D^h f(\tilde{x}-\Delta,v) \left(f(\tilde{x},v)-1\right) d\Delta dv\right) d\tilde{x}$$

Dropping the tildas, we can rewrite the inner term using the definition of f as:

$$\begin{aligned}
&\int_0^h \int_{-\infty}^{+\infty} D^h f(x-\Delta,v)(f(x,v)-1) d\Delta dv \\
&= \int_0^h \int_{-\infty}^{+\infty} (f(x+\Delta,v) - f(x-\Delta,v))(f(x,v)-1) dv d\Delta \\
&= -\int_0^h \int_{u(x)}^{u(x)\vee u(x+\Delta)\vee u(x-\Delta)} f(x+\Delta,v) - f(x-\Delta,v) dv d\Delta \\
&= -\int_0^h \left(D^\Delta u\right)_+ - \left(D^{-\Delta} u\right)_+ d\Delta.
\end{aligned}$$

We thus find

$$\overline{Q}_1 = -\int_0^L \int_0^h \eta\left(\left(D^\Delta u\right)_+ - \left(D^{-\Delta} u\right)_+\right) d\Delta dx.$$

Using that

$$\int_0^L \eta\left(D^\Delta u\right)_+ dx = \int_0^L \eta(x-\Delta)\left(D^{-\Delta} u\right)_- dx,$$

and similarly

$$\int_0^L \eta\left(D^{-\Delta} u\right)_+ dx = \int_0^L \eta(x-\Delta)\left(D^{\Delta} u\right)_- dx,$$

we obtain

$$\begin{aligned}\overline{Q}_1 =& \frac{1}{2}\int_0^h\int_0^L D^\Delta\eta\left(D^\Delta u\right)_+ - D^{-\Delta}\eta\left(D^{-\Delta}u\right)_+ + \eta^{-\Delta}D^{-\Delta}u - \eta^\Delta D^\Delta u\,dx\,d\Delta\\ =& \frac{1}{2}\int_0^h\int_0^L D^\Delta\eta\left(D^\Delta u\right)_+ - D^{-\Delta}\eta\left(D^{-\Delta}u\right)_+ + \eta(u(x-\Delta)-u(x+\Delta))\,dx\,d\Delta\\ =& \frac{1}{2}\int_0^h\int_0^L D^\Delta\eta\left|D^\Delta u\right| + \eta\left(u(x-\Delta)-u(x+\Delta)\right)dx\,d\Delta,\end{aligned}$$

which is (46). If we now integrate (45) over time, we find

$$Q_1 + Q_2 = \int_0^T\int_0^L\int_0^h D^\Delta\eta|D^\Delta u|\,d\Delta\,dx\,dt. \tag{47}$$

Step 6. We finally argue that

$$Q_3 = -\left[\frac{1}{2}\int_0^h\int_0^L |D^\Delta u| D^\Delta u\,dx\,d\Delta\right]_0^T. \tag{48}$$

Let us recall that by definition of A:

$$Q_3 = \left[\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}(v-w)\int_0^L\int_0^x D^h f(t,x,v)D^h f(t,y,w)\,dy\,dx\,dv\,dw\right]_{t=0}^{t=T}.$$

As above:

$$Q_3 = \left[-\int_0^L\int_0^h\int_{-\infty}^{+\infty}\int_{-\infty}^{v} f(t,x,v)(f(t,x+\Delta,w)-f(t,x-\Delta,w)\,dw\,dv\,d\Delta\,dx\right]_{t=0}^{t=T}$$

and,

$$\int_{-\infty}^{v}(f(t,x+\Delta,w)-f(t,x-\Delta,w)\,dw = (u(t,x+\Delta)-v\wedge 0)-(u(t,x-\Delta)-v\wedge 0).$$

Therefore

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{v} f(t,x,v)(f(t,x+\Delta,w)-f(t,x-\Delta,w)dwdv$$
$$=\int_{-\infty}^{u(t,x)}\int_{-\infty}^{v}(f(t,x+\Delta,w)-f(t,x-\Delta,w)dwdv$$
$$=\int_{-\infty}^{u(t,x)}(u(t,x+\Delta)-v\wedge 0)-(u(t,x-\Delta)-v\wedge 0)dv$$
$$=\int_{(u(t,x+\Delta)\wedge u(t,x))}^{u(t,x)} u(t,x+\Delta)-vdv$$
$$-\int_{(u(t,x-\Delta)\wedge u(t,x))}^{u(t,x)} u(t,x-\Delta)-vdv$$

which, using that for $a,b\in\mathbb{R}$,

$$\int_{(a\wedge b)}^{b}(a-v)dv=-\frac{1}{2}\left((a-b\wedge 0)\right)^2$$

gives

$$Q_3=\frac{1}{2}\left[\int_0^L\int_0^h (u(t,x+\Delta)-u(t,x)\wedge 0)^2-(u(t,x-\Delta)-u(t,x)\wedge 0)^2 d\Delta dx\right]_{t=0}^{t=T}$$
$$=\frac{1}{2}\left[\int_0^L\int_0^h \left(D^{\Delta}u\right)_-^2-\left(D^{-\Delta}u\right)_-^2 d\Delta dx\right]_{t=0}^{t=T}$$
$$=\frac{1}{2}\left[\int_0^L\int_0^h \left(D^{\Delta}u\right)_-^2-\left(D^{\Delta}u\right)_+^2 d\Delta dx\right]_{t=0}^{t=T}$$
$$=-\frac{1}{2}\left[\int_0^L\int_0^h |D^{\Delta}u|D^{\Delta}u d\Delta dx\right]_{t=0}^{t=T}$$

which proves (48). Combined with (47), this yields (41). □

We can now prove Proposition 2.3

Proof of Proposition 2.3. By linearity, it is enough proving the estimate for $g=0$. Thanks to (38) applied to $\eta=|\partial_x|\xi$, we have

$$\int_0^T\int_0^L \left|D^h u\right|^3 dxdt$$
$$=6\int_0^T\int_0^L\int_0^h D^{\Delta}\eta\left|D^{\Delta}u\right| d\Delta dxdt-\left[3\int_0^h\int_0^L |D^{\Delta}u|D^{\Delta}u dxd\Delta\right]_0^T.$$

Thanks to Theorem A.1, for $s<1$ and every function v, $\||v|\|_{B^s_{p,r}} \le c\|v\|_{B^s_{p,r}}$ (notice that it also holds for $s=1$ if $p=r=2$). Using the triangle inequality and the invariance of the Besov norms with respect to translations, we obtain that for $s\in(0,1)$, $\left\||D^\Delta u|\right\|_{B^s_{p,r}} \le c\|u\|_{B^s_{p,r}}$. Applying (58) we get

$$\begin{aligned}\int_0^T\int_0^L\int_0^h D^\Delta\eta\left|D^\Delta u\right| d\Delta dxdt &= \int_0^T\int_0^L\int_0^h \left(|\partial_x| D^\Delta\xi\right)\left|D^\Delta u\right| d\Delta dxdt\\ &\le \frac{1}{\pi}\int_0^h \left\|D^\Delta\xi\right\|_{B^{1-s}_{p',r'}} \left\|\left|D^\Delta u\right|\right\|_{B^s_{p,r}}\\ &\le ch\|\xi\|_{B^{1-s}_{p',r'}}\|u\|_{B^s_{p,r}}.\end{aligned}$$

On the other hand, since

$$\left[\int_0^h\int_0^L |D^\Delta u|D^\Delta u dxd\Delta\right]_0^T \le ch\int_0^L u(0,x)^2+u(T,x)^2dx,$$

and since multiplying the equation (31) by u and integrating gives,

$$\begin{aligned}\int_0^L \frac{1}{2}u(T,x)^2dx-\int_0^L\frac{1}{2}u(0,x)^2dx &= \int_0^T\int_0^L \partial_t\left(\frac{1}{2}u^2\right)dxdt\\ &= \int_0^T\int_0^L u|\partial_x|\xi\, dxdt\\ &\le c\|\xi\|_{B^{1-s}_{p',r'}}\|u\|_{B^s_{p,r}},\end{aligned}$$

we have

$$\left[\int_0^h\int_0^L |D^\Delta u|D^\Delta u dxd\Delta\right]_0^T \le ch\left(\|\xi\|_{B^{1-s}_{p',r'}}\|u\|_{B^s_{p,r}}+\|u(0,\cdot)\|^2_{L^2}\right).$$

Putting this together, we find

$$\frac{1}{h}\int_0^T\int_0^L\left|D^h u\right|^3 dxdt \le c\left(\|\xi\|_{B^{1-s}_{p',r'}}\|u\|_{B^s_{p,r}}+\|u(0,\cdot)\|^2_{L^2}\right)$$

which concludes the proof. □

Remark. Starting from (33), one can obtain a larger family of estimates for the inhomogeneous Burgers equation. Unfortunately, the estimate

$$\|u\|^3_{B^{1/3}_{3,3}} \le c\left(\|u\|_{B^{1/3}_{3,3}}\left\|g\right\|_{B^{2/3}_{3/2,3/2}}+\|u\|_{B^s_{p,r}}\|\xi\|_{B^{1-s}_{p',r'}}+\|u(0,\cdot)\|^2_{L^2}\right),$$

which would allow to avoid the logarithmic correction in (7), is borderline.

5 Proof of Proposition 2.7

Let us remind the reader the statement we want to prove:

Proposition (Comparison between Besov norms of different index r)**.** *There exists $c > 0$ such that, for all $L > 2$, for every u solution of* (K-S) *with average zero, the following estimate holds:*

$$\|u\|_{\mathbb{B}^{1/3}_{3,3}} \leq c \ln^{1/3}(L) \|u\|_{\mathbb{B}^{1/3}_{3,\infty}}. \tag{49}$$

Proof. The proof is elementary and resembles [15, Prop. 4 II), Step 2 &3]. Let us first cut the term that we want to bound in three parts:

$$\begin{aligned}\int_0^{+\infty} \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} =& \int_0^{\ell} \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} + \int_{\ell}^{L} \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} + \int_L^{+\infty} \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} \\ =& A(\ell) + B(\ell) + C.\end{aligned}$$

The large scale term C is in fact controlled by $A(\ell)$ and $B(\ell)$ by periodicity. Indeed:

$$\begin{aligned} C =& \int_L^{+\infty} \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} \\ =& \sum_{n=1}^{+\infty} \int_{nL}^{(n+1)L} \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} \\ \leq& \sum_{n=1}^{+\infty} \frac{1}{n^2} \int_0^L \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} \\ \leq& c\,(A(\ell) + B(\ell)). \end{aligned}$$

The intermediate scale term $B(\ell)$ is directly handled with thanks to $\|u\|_{B^{1/3}_{3,\infty}}$:

$$\begin{aligned} B(\ell) =& \int_{\ell}^{L} \frac{\|D^h u\|^3_{L^3}}{h} \frac{dh}{h} \\ \leq& \sup_{h \in \mathbb{R}^*_+} \frac{\|D^h u\|^3_{L^3}}{h} \int_{\ell}^{L} \frac{dh}{h} \\ \leq& \ln(L/\ell) \|u\|^3_{B^{1/3}_{3,\infty}}. \end{aligned}$$

The small scale term $A(\ell)$ is the most difficult to bound. We shall prove that:

$$A(\ell) \leq c\ell \left(\left(\int_0^L u(0,x)^2 dx \right)^{3/2} + L^{3/2} \|u\|^3_{B^{1/3}_{3,\infty}} \right). \tag{50}$$

Before proceeding with the proof of (50), let us show that it is sufficient to conclude. Indeed, fix now $\ell = L^{-3/2}$. Then, we obtain:

$$\int_0^{+\infty} \frac{\|D^h u\|_{L^3}^3}{h} \frac{dh}{h} \leq c \ln(L) \|u\|_{B^{1/3}_{3,\infty}}^3 + c\left(\frac{1}{L}\int_0^L u(0,x)^2 dx\right)^{3/2}.$$

Taking now the time-space average yields (49). It remains to prove (50).

First, we prove that:

$$\int_0^T \int_0^L \left|D^h u\right|^3 dx dt \leq ch^2 \left(\left(\int_0^L u(0,x)^2 dx\right)^{3/2} + \int_0^T \left(\int_0^L u^2 dx\right)^{3/2} dt\right). \tag{51}$$

We start by noting that:

$$\begin{aligned}\int_0^L \left|D^h u(t,x)\right|^3 dx \leq& 2\left(\sup_{x\in[0,L]} |u(t,x)|\right) \int_0^L \left|\int_x^{x+h} \partial_x u(t,y) dy\right|^2 dx \\ \leq& 2\left(\sup_{x\in[0,L]} |u(t,x)|\right) h^2 \int_0^L (\partial_x u)^2 dx.\end{aligned}$$

For the convenience of the reader, we recall the argument for

$$\sup_{x\in[0,L]} |u| \leq c\left(\int_0^L u^2 dx\right)^{1/4} \left(\int_0^L (\partial_x u)^2 dx\right)^{1/4}. \tag{52}$$

In fact, starting from:

$$u^2(t,x) = u^2(t,y) + \int_y^x \frac{1}{2} u(t,z) \partial_x u(t,z) dz \qquad \forall y \in [0,L],$$

and using that u has zero average and thus vanishes somewhere, Cauchy-Schwarz inequality immediately gives (52). Therefore,

$$\int_0^L \left|D^h u\right|^3 dx \leq ch^2 \left(\int_0^L u^2 dx\right)^{1/4} \left(\int_0^L (\partial_x u)^2 dx\right)^{5/4},$$

and using the following Sobolev inequality (which can easily be proved by using Fourier methods):

$$\int_0^L (\partial_x u)^2 dx \leq \left(\int_0^L u^2 dx\right)^{1/2} \left(\int_0^L \left(\partial_x^2 u\right)^2 dx\right)^{1/2},$$

we get:

$$\int_0^L \left|D^h u\right|^3 dx \leq ch^2 \left(\int_0^L u^2 dx\right)^{7/8} \left(\int_0^L \left(\partial_x^2 u\right)^2 dx\right)^{5/8}. \tag{53}$$

Now, we have to work a bit to extract information from the energy identity. Multiplying (K-S) by u and integrating over x, we obtain:

$$\begin{aligned}\frac{d}{dt}\int_0^L u^2\,dx =& 2\int_0^L (\partial_x u)^2\,dx - 2\int_0^L \left(\partial_x^2 u\right)^2 dx\\ \leq& 2\left(\int_0^L u^2\,dx\right)^{1/2}\left(\int_0^L \left(\partial_x^2 u\right)^2 dx\right)^{1/2} - 2\int_0^L \left(\partial_x^2 u\right)^2 dx\\ \leq& \int_0^L u^2\,dx - \int_0^L \left(\partial_x^2 u\right)^2 dx.\end{aligned}$$

In a first step, we get from this differential inequality that there exists $c > 0$ such that:

$$\int_0^L u^2(t+s,x)\,dx \leq c\int_0^L u^2(t,x)\,dx \qquad \forall s \in [0,1]. \tag{54}$$

In a second step, we get:

$$\int_0^L u^2(t+1,x)\,dx - \int_0^L u^2(t,x)\,dx \leq \int_t^{t+1}\int_0^L u^2\,dx\,ds - \int_t^{t+1}\int_0^L \left(\partial_x^2 u\right)^2 dx\,ds.$$

The combination of both implies:

$$\int_t^{t+1}\int_0^L \left(\partial_x^2 u\right)^2 dx\,ds + \sup_{s\in[t,t+1]}\int_0^L u^2(s,x)\,dx \leq c\int_0^L u^2(t,x)\,dx. \tag{55}$$

Hence, together with (53) in the form of

$$\int_t^{t+1}\int_0^L \left|D^h u\right|^3 dx\,ds \leq Ch^2\left(\sup_{s\in[t,t+1]}\int_0^L u^2(s,x)\,dx\right)^{7/8}\left(\int_t^{t+1}\int_0^L (\partial_x^2 u)^2\,dx\,ds\right)^{5/8},$$

we deduce

$$\int_t^{t+1}\int_0^L \left|D^h u\right|^3 dx\,ds \leq Ch^2\left(\int_0^L u^2(t,x)\,dx\right)^{3/2}.$$

Using this inequality for $t = 0$ and in its integrated form between zero and T, we obtain (51).

We now claim that

$$\int_0^T\left(\int_0^L u^2\,dx\right)^{3/2} dt \leq L^{3/2}\,\|u\|^3_{B^{1/3}_{3,\infty}}. \tag{56}$$

Indeed, using Hölder's inequality, the fact that u has zero average and Jensen's inequality we obtain:

$$
\begin{aligned}
\int_0^T \left(\int_0^L u^2(t,x)dx \right)^{3/2} dt \leq & L^{1/2} \int_0^T \int_0^L \left| u(t,x) - \frac{1}{L} \int_0^L u(t,h)dh \right|^3 dxdt \\
\leq & L^{-1/2} \int_0^T \int_0^L \int_0^L |u(t,x) - u(t,x+h)|^3 dhdxdt \\
\leq & L^{1/2} \int_0^L \int_0^T \int_0^L \frac{|u(t,x+h) - u(t,x)|^3}{h} dxdtdh \\
\leq & L^{3/2} \sup_{h \in \mathbb{R}_+} \int_0^T \int_0^L \frac{\left|D^h u(t,x)\right|^3}{h} dxdt.
\end{aligned}
$$

Conclusion Putting together (51) and (56), we get as desired

$$
\begin{aligned}
A(\ell) = \int_0^\ell \int_0^T \int_0^L \left|D^h u\right|^3 dxdt \frac{dh}{h^2} \leq & c \int_0^\ell dh \left(\left(\int_0^L u(0,x)^2 dx \right)^{3/2} + L^{3/2} \|u\|^3_{B^{1/3}_{3,\infty}} \right) \\
\leq & c\ell \left(\left(\int_0^L u(0,x)^2 dx \right)^{3/2} + L^{3/2} \|u\|^3_{B^{1/3}_{3,\infty}} \right).
\end{aligned}
$$

□

Remark. As in [15, Prop. 4 II), Step 2 &3], we could have used an L^∞ (in time) bound on $\int_0^L u^2 dx$ (proven for instance in [6, Prop. 4]) to get (50) directly. However, since we have a relatively simple and self-contained argument for it, we preferred to include it.

A Besov spaces

A.1 Definition of time-space Besov spaces

We recall here some basics of the theory of Besov spaces. We refer to [1, Chapter 2, p. 51-121], for the construction of a dyadic Littlewood-Paley decomposition, and most of the proofs.

Definition A.1 (Dyadic Littlewood-Paley decomposition)**.** *Let $\left(\phi_k\right)_{k\in\mathbb{Z}}$ be a family of Schwartz functions such that their Fourier transforms $\left(\mathscr{F}\phi_k\right)_{k\in\mathbb{Z}}$ satisfy:*

$$
\begin{aligned}
&\mathscr{F}(\phi_0)(\xi) = 0 && \forall\, |\xi| \notin \left]2^{-1}, 2\right[, \\
&\mathscr{F}(\phi_k)(\xi) = \mathscr{F}(\phi_0)\left(2^{-k}\xi\right) && \forall k \in \mathbb{Z}, \forall \xi \in \mathbb{R}, \\
&\sum_{k\in\mathbb{Z}} \mathscr{F}(\phi_k)(\xi) = 1 && \forall \xi \in \mathbb{R}.
\end{aligned}
$$

Then, for a L-periodic function u, we define its Littlewood-Paley decomposition as:

$$u_k(t,\cdot) = \phi_k * u(t,\cdot)$$

where $*$ denotes the periodic convolution. This allows us to define time-space Besov space $B^s_{p,r}$ for $s \in [0,+\infty]$, $p \in [1,\infty]$, $r \in [1,\infty]$ by the set of functions such that:

$$\|u\|_{B^s_{p,r}} = \left(\sum_{k\in\mathbb{Z}} 2^{rsk} \|u_k\|^r_{L^p}\right)^{1/r} < \infty \qquad \text{if } r < \infty,$$
$$\|u\|_{B^s_{p,r}} = \sup_{k\in\mathbb{Z}} 2^{sk} \|u\|_{L^p} < \infty \qquad \text{if } r = \infty.$$

We are actually interested in a rescaled homogeneous Besov norm, defined by:

$$\|u\|_{\mathbb{B}^s_{p,r}} = \limsup_{T\to+\infty} \frac{1}{(LT)^{1/p}} \|u\|_{B^s_{p,r}}.$$

The time-space Besov norm can be replaced by an equivalent one, as stated in the following theorem (see [1, Th. 2.36]):

Theorem A.1. *Let $s \in]0,1[$ and $(p,r) \in [1,+\infty]^2$. Then there exists $c > 0$ such that:*

$$c^{-1} \left\| \frac{\|D^h u\|_{L^P}}{h^s} \right\|_{L^r\left(\mathbb{R}_+,\frac{dh}{|h|}\right)} \leq \|u\|_{B^s_{p,r}} \leq c \left\| \frac{\|D^h u\|_{L^P}}{h^s} \right\|_{L^r\left(\mathbb{R}_+,\frac{dh}{|h|}\right)}.$$

where for $r < \infty$:

$$\left\| \frac{\|D^h u\|_{L^P}}{h^s} \right\|_{L^r\left(\mathbb{R}_+,\frac{dh}{|h|}\right)} = \left(\int_0^\infty \left(\int_0^T \int_0^L \left(\frac{|D^h u|}{h^s} \right)^p dx\,dt \right)^{r/p} \frac{dh}{h} \right)^{1/r},$$

and for $r = \infty$:

$$\left\| \frac{\|D^h u\|_{L^P}}{h^s} \right\|_{L^\infty\left(\mathbb{R}_+,\frac{dh}{|h|}\right)} = \sup_{h>0} \left(\int_0^T \int_0^L \left(\frac{|D^h u|}{h^s} \right)^p dx\,dt \right)^{1/p}.$$

Besov spaces are particularly well adapted for interpolation as seen from the following theorem:

Theorem A.2 (Interpolation between Besov spaces)**.** *Let $(s,p,r), (s_1,p_1,r_1), (s_2,p_2,r_2) \in \mathbb{R}_+ \times [1,+\infty]^2$, and $u \in B^{s_1}_{p_1,r_1} \cap B^{s_2}_{p_2,r_2}$. If*

$$\begin{cases} s &= \theta s_1 + (1-\theta) s_2, \\ \frac{1}{p} &= \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \\ \frac{1}{r} &= \frac{\theta}{r_1} + \frac{1-\theta}{r_2}, \end{cases}$$

with $\theta \in]0,1[$, *then* $u \in B^s_{p,r}$ *and:*

$$\|u\|_{B^s_{p,r}} \leq \|u\|^{\theta}_{B^{s_1}_{p_1,r_1}} \|u\|^{1-\theta}_{B^{s_2}_{p_2,r_2}}. \tag{57}$$

Theorem A.2 simply follows from the definition of the Besov norms and an application of Hölder's inequality. In some lemmas that we will enunciate later, we will use the partial derivative $|\partial_x|$ which is slightly different from the classical ∂_x. It is defined via Fourier series:

$$|\partial_x| : \sum_{n\in\mathbb{Z}} a_n e^{\frac{2i\pi nx}{L}} \mapsto \sum_{n\in\mathbb{Z}} a_n \frac{2\pi}{L} |n| e^{\frac{2i\pi nx}{L}}.$$

The following theorems underlines a link between Besov spaces and the operator $|\partial_x|$:

Lemma A.3. *For all* $\phi, g \in \mathscr{C}^1_L(\mathbb{R})$*:*

$$\int_0^L \phi |\partial_x| g dx = \frac{1}{\pi} \int_0^{+\infty} \int_0^L D^h\phi D^h g dx \frac{dh}{h^2}.$$

Proof. Let us expand ϕ and g in Fourier series as

$$\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n e^{\frac{2i\pi}{L} nx} \qquad \text{and} \qquad g(x) = \sum_{n\in\mathbb{Z}} g_n e^{\frac{2i\pi}{L} nx}.$$

Therefore, one can explicitly compute on the one hand:

$$\int_0^L \phi |\partial_x| g dx = \sum_{n\in\mathbb{Z}} 2\pi |n| \phi_n g_{-n},$$

and on the other hand:

$$\begin{aligned}
\int_0^{+\infty} \int_0^L D^h\phi D^h g dx \frac{dh}{h^2} &= \int_0^{+\infty} \sum_{n\in\mathbb{Z}} L\left(\left(e^{\frac{2i\pi}{L} hn} - 1\right)\phi_n \left(e^{-\frac{2i\pi}{L} hn} - 1\right) g_{-n}\right) \frac{dh}{h^2} \\
&= \int_0^{+\infty} \sum_{n\in\mathbb{Z}} 4L \sin^2\left(\frac{\pi hn}{L}\right) \phi_n g_{-n} \frac{dh}{h^2} \\
&= 4 \sum_{n\in\mathbb{Z}} \phi_n g_{-n} \pi |n| \int_0^{+\infty} \sin^2(y) \frac{dy}{y^2} \\
&= 2\pi^2 \sum_{n\in\mathbb{Z}} \phi_n g_{-n} |n|,
\end{aligned}$$

which implies the result. □

We derive from this identity the following Besov estimate (see also [15, Step. 3 p 39]):

Proposition A.4. *Let $\phi, g \in \mathscr{C}^1_L(\mathbb{R})$. Then, for all $p, p', r,' \in [1,+\infty]$, with $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$, for all $s \in]0,1[$ the following estimate holds:*

$$\int_0^T \int_0^L \phi |\partial_x| g dx dt \leq \frac{1}{\pi} \|\phi\|_{B^s_{p,r}} \|g\|_{B^{1-s}_{p',r'}}. \tag{58}$$

Proof. Using Lemma A.3 we get:

$$\int_0^T \int_0^L \phi |\partial_x| g dx dt = \frac{1}{\pi} \int_0^{+\infty} \int_0^T \int_0^L D^h \phi D^h g dx dt \frac{dh}{h^2}.$$

Then, Hölder's inequality leads us to the result:

$$\begin{aligned}\int_0^{+\infty} \int_0^T \int_0^L D^h \phi D^h g dx dt \frac{dh}{h^2} &\leq \int_0^{+\infty} \left\| D^h \phi \right\|_{L^p} \left\| D^h g \right\|_{L^{p'}} \frac{dh}{h^2} \\ &\leq \left\| \frac{\|D^h \phi\|_{L^p}}{h^s} \right\|_{L^r\left(\mathbb{R}_+, \frac{dh}{h}\right)} \left\| \frac{\|D^h g\|_{L^{p'}}}{h^{1-s}} \right\|_{L^{r'}\left(\mathbb{R}_+, \frac{dh}{h}\right)}.\end{aligned}$$

□

We finally state a useful lemma relating Besov norms of derivatives.

Lemma A.5. *Let $s>0$, $p, r \in [1,\infty]$, $m \in \mathbb{N}$ and $u \in B^{s+m-1}_{p,r}$. Suppose $h = |\partial_x|^{-1} \partial_x^m u$. Then there exists a positive constant c depending only on (s,p,r) such that the following estimate holds:*

$$\|h\|_{B^s_{p,r}} \leq c \|u\|_{B^{s+m-1}_{p,r}}. \tag{59}$$

Proof. The proof is analogous to [15, Step 1 p. 17]. By definition, we have:

$$h_k = \phi_k * h.$$

Therefore, using the properties of convolution and of the quasi-orthogonality of the dyadic partition of unity we get:

$$\begin{aligned} h_k &= \phi_k * \sum_{k' \in [k-1;k+1]} \phi_{k'} * h \\ &= |\partial_x|^{-1} \partial_x^m \phi_k * \sum_{k' \in [k-1,k+1]} u_{k'}. \end{aligned}$$

Then, using Young's inequality, we obtain:

$$\begin{aligned} \|h_k\|_{L^p} &\leq \left(\int_{\mathbb{R}} \left| |\partial_x|^{-1} \partial_x^m \phi_k \right| dx \right) \sum_{k' \in [k-1,k+1]} \|u_{k'}\|_{L^p} \\ &\leq 2^{k(m-1)} \left(\int_{\mathbb{R}} \left| |\partial_x|^{-1} \partial_x^m \phi_0 \right| dx \right) \sum_{k' \in [k-1,k+1]} \|u_{k'}\|_{L^p} \\ &\leq c 2^{k(m-1)} \sum_{k' \in [k-1,k+1]} \|u_{k'}\|_{L^p}. \end{aligned}$$

Hence:

$$\sum_{k\in\mathbb{Z}} 2^{krs}\,\|h_k\|_{L^p}^r \le c \sum_{k\in\mathbb{Z}} 2^{kr(m-1+s)}\,\|u_k\|_{L^p}^r\,,$$

which implies the aimed inequality. □

Acknowledgment

FO acknowledges many discussions with Dorian Goldman on simplifications of the proof of [15]. MG was partially funded by a Von Humboldt fellowship. Most of this work was done while MG and MJ were hosted by the MPI-MIS Leipzig whose kind hospitality is acknowledged.

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