

Nonlinear Elliptic Systems and Mean Field Games*

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Abstract

We consider a class of quasilinear elliptic systems of PDEs consisting of N Hamilton-Jacobi-Bellman equations coupled with N divergence form equations, generalising to $N > 1$ populations the PDEs for stationary Mean-Field Games first proposed by Lasry and Lions. We provide a wide range of sufficient conditions for the existence of solutions to these systems: either the Hamiltonians are required to be at most quadratic in the gradients, or they must grow faster than linearly and not oscillate too much in the space variables, in a suitable sense. We show the connection of these systems with the classical strongly coupled systems of Hamilton-Jacobi-Bellman equations of the theory of N -person stochastic differential games studied by Bensoussan and Frehse. We also prove the existence of Nash equilibria in feedback form for some N -person games.

Keywords: Nonlinear elliptic systems, stochastic differential games, N -person games, Nash equilibria, Mean Field Games.

1 Introduction

This paper deals with systems of partial differential equations of the following type

$$\begin{cases} \mathcal{L}^i v_i + H^i(x, Dv_i) + \lambda_i = V^i[m] & \text{in } Q, \\ \mathcal{L}^{i*} m_i - \operatorname{div}(g^i(x, Dv_i)m_i) = 0 & \text{in } Q := \mathbb{T}^d, \\ \int_Q m_i(x)dx = 1, \quad m_i > 0, \quad \int_Q v_i(x)dx = 0, \quad i = 1, \dots, N, \end{cases} \quad (1)$$

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where the unknowns are the constants $\lambda = (\lambda_1, \dots, \lambda_N)$, the functions $v = (v_1, \dots, v_N)$ and the densities of probability measures $m = (m_1, \dots, m_N)$, at least continuous on the d -dimensional torus $Q := \mathbb{T}^d$. The operators

$$\mathcal{L}^i := -\text{tr}(a^i(x)D^2), \quad i = 1, \dots, N,$$

are second-order uniformly elliptic with \mathbb{Z}^d -periodic and Lipschitz coefficients $a^i(\cdot)$, i.e., for some $\nu, C > 0$,

$$a^i(x) \geq \nu I_d, \quad a^i(x) = a^i(x+k) \quad \forall k \in \mathbb{Z}^d, \quad |a^i(x) - a^i(x+y)| \leq C|y| \quad \forall x, y \in \mathbb{R}^d, \quad (2)$$

where I_d is the identity $d \times d$ matrix, and

$$\mathcal{L}^{i*} v := - \sum_{h,k} D_{hk}^2(a_{hk}^i(x)v)$$

are their formal adjoints. The Hamiltonians $H^i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ are \mathbb{Z}^d -periodic in x and satisfy *either one* of the following sets of conditions.

C1. For all $i = 1, \dots, N$, $H^i = H^i(x, p)$ is locally Lipschitz, superlinear in p uniformly in x , i.e.,

$$\inf_{x \in Q} |H^i(x, p)|/|p| \rightarrow +\infty \quad \text{as} \quad |p| \rightarrow \infty, \quad (3)$$

and $\exists \theta^i \in (0, 1)$, $C > 0$, such that

$$\text{tr}(a^i)D_x H^i \cdot p + \theta^i (H^i)^2 \geq -C|p|^2 \quad \text{for} \quad |p| \text{ large, and for a.e. } x \in Q. \quad (4)$$

C2. For all $i = 1, \dots, N$, for some $\alpha \in (0, 1)$, H^i is locally α -Hölder continuous and grows at most quadratically

$$|H^i(x, p)| \leq C_1|p|^2 + C_2 \quad \forall x \in Q, p \in \mathbb{R}^d, \quad (5)$$

for some $C_1, C_2 > 0$, the so-called *natural growth* condition.

We denote by $C(Q)$ the set of \mathbb{Z}^d -periodic continuous functions on \mathbb{R}^d , by $P(Q)$ the set of probability measures on Q endowed with the weak*-topology of the dual $C(Q)^*$ of $C(Q)$, and assume that the given operators $V^i : P(Q)^N \rightarrow C(Q)$ satisfy

$$V^i[m_n] \rightarrow V^i[m] \quad \text{uniformly in } Q \text{ for all } m_n \rightarrow m, m_n, m \in C(Q)^N \cap P(Q)^N, \quad (6)$$

and

$$\|V^i[m]\|_{C^{0,\beta}(Q)} \leq C \quad \forall m \in C(Q)^N \cap P(Q)^N, \quad (7)$$

where $\beta = 1$ if H^i verifies condition C1 above, and $\beta = \alpha$ if condition C2 holds. Finally, we assume that

$$g^i : Q \times \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \text{are measurable, locally bounded, and continuous in } p. \quad (8)$$

Under the above conditions we show the existence of a solution $\lambda_i \in \mathbb{R}$, $v_i \in C^{2,\alpha}(Q)$, $m_i \in W^{1,p}(Q)$, for all $1 \leq p < \infty$, $i = 1, \dots, N$, to the system (1), where α is the Hölder exponent appearing in condition C2 on H^i if such condition holds, and it is any number in $(0, 1)$ if

condition C1 is assumed instead. We do not address the uniqueness of the solution here. It is known to hold if $N = 1$ under some additional conditions (in particular, some monotonicity of V^i) introduced by Lasry and Lions [34, 36]. If $N > 1$ one may expect uniqueness only in some very special cases, see Cirant [21, 22] where Neumann boundary conditions are also treated.

There are two main motivations for studying systems of the form (1). The first is the theory of stationary Mean Field Games (briefly, MFG) as formulated by J-M. Lasry, P-L. Lions [34, 36]. In fact, for $N = 1$, $\mathcal{L}^i = -\Delta$, $H^i = H$ smooth and $g^i = D_p H$, (1) reduces to the stationary MFG PDEs introduced and studied in [34, 36]. This is a model of the equilibrium distribution of a large population of identical agents whose dynamics is subject to white noise and seeking to minimise a given individual cost functional. For $N > 1$ the system (1) with $g^i = D_p H^i$ can be associated to Mean Field Games with N different populations of players, see Section 3.6 and [21] for further motivations. Our assumptions allow a rather general controlled dynamics for the generic agent of each population; in particular, condition C2 on the Hamiltonians does not require any coercivity and allows the control set to be bounded, which in several applications can be more realistic than the model problem treated in [34, 36], see Sections 3.3 and 3.4 for some examples. Mean-Field Games were also introduced independently by Huang, Caines, and Malhamé and studied by different methods [31, 32], see Section 3.6.

The second motivation is the synthesis of Nash equilibria in feedback form for N -person stochastic differential games with cost functionals of ergodic type. This is usually based on the solution of a strongly coupled system of N Hamilton-Jacobi-Bellman (briefly, HJB) equations, the system (42) in Section 3.1, following a classical observation of Friedman [25]. A systematic study of such nonlinear elliptic systems was pursued by Bensoussan and Frehse starting with [10], see [12] for problems with ergodic cost and their book [13] for more references. In [34, 36] Lasry and Lions propose instead a system of N HJB and N Kolmogorov-Fokker-Planck equations of the form (1) for games where the players are coupled only via the cost functionals. An advantage of this system is the weaker coupling among the HJB equations. They give an existence result in the case $\mathcal{L}^i = -\nu^i \Delta$, $\nu^i > 0$, H^i smooth, satisfying (3) and (4) with $C = 0$, and $g^i = D_p H^i$, and show how to synthesize a feedback Nash equilibrium for the N -person game from the solution of such system. In Section 3.1 we show a more precise connection among the classical systems of [10, 12] and systems of the form (1), which is related to the adjoint methods for Hamilton-Jacobi equation explored by Evans [23]. In Section 3 we also generalise the result in [34, 36] on the synthesis of the equilibrium and give several examples of classes of differential games to which our abstract results apply.

We recall that the pioneering papers [34, 36] prove also the convergence, in a suitable sense, of N -person games to a Mean-Field Game with a single population as the number of players tends to infinity. Some estimates of the present paper were used by Feleqi [24] to prove the same result in the case of several interacting populations and under more general conditions.

The proof of the existence of solutions to (1) under condition C1 follows the suggestion in [34, 36] about getting a priori estimates for Dv_i by the Bernstein method (see also [1], [37] and [38]). On the other hand, rather than relying on estimates for Bellman equations, we mostly use more classical local a priori estimates for linear equations. We also included some of their proofs for the reader's convenience when we were not able to find an explicit reference in the literature, in the attempt to make the paper reasonably self-contained and readable by a wide audience. The proof in this case allows some variants to conditions C1

and C2 and can be adapted, for instance, to Hamiltonian behaving like $|p_1| + |p_2|^\gamma$, $\gamma > 1$, as $p = (p_1, p_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ tends to infinity, see Remark 2.4 and Example 3.9.

Under condition C2 the proof is different: we approximate the additive eigenvalue problem (1) with systems with zero-th order terms, as in Bensoussan and Frehse [11, 13], and for such systems we employ some deep estimates by Ladyzhenskaya and Uraltseva [33] and other tools of the elliptic theory [27]. In this generality our results are new even in the scalar case $N = 1$.

We conclude with some bibliographical remarks. In [14] Bensoussan and Frehse also weaken the standard quadratic growth condition on the Hamiltonians but their results do not apply to our system. The difficulties arising from constraints on the controls were treated by Mannucci [39] for parabolic equations. The existence of Nash equilibria for some stochastic N -person differential games was also proved by probabilistic methods, see, e.g., [18, 26] and the references therein. For a general presentations of Mean-Field Games and their applications we refer to the lecture notes by Guéant, Lasry, and Lions [30] and Cardaliaguet [19] and the survey paper by Gomes and Sáude [29]. Evolutive MFG were first studied in [35, 36] and [31, 32]. The justification of stationary MFG via long-time asymptotics is in [20], extended stationary MFG were studied in [28] and numerical methods in [2], see also the references therein. For other recent contributions on MFG see also the courses of P.-L. Lions at Collège de France, the monograph [15] and the two special issues of Dynamic Games and Applications [6, 7].

The paper is organized as follows. Section 2 contains the statements and proofs of the existence results for (1) and for related systems with zero-th order terms arising from discounted infinite horizon problems. In Section 3 we describe the connections of the system (1) with N -person and Mean-Field games. We also apply the results of Section 2 to show the existence of Nash equilibria for such games in the cases of long-time-average costs and discounted infinite horizon costs. Finally, the Appendix contains the proofs of some technical lemmas.

2 Existence of solutions to elliptic systems

We will use the notations $\text{tr } b$ for the trace of a square matrix b , and $a \cdot b = \text{tr } ab^t$, $|b| := (b \cdot b)^{1/2}$. The adjoint operator \mathcal{L}^{i*} will be interpreted in the sense of distributions:

$$\langle \mathcal{L}^{i*} v, \phi \rangle = \int_Q v \mathcal{L} \phi dx \quad \forall \phi \in C^\infty(Q).$$

2.1 The additive eigenvalue problem

Theorem 2.1. *Assume (2) and (8), H^i satisfy condition C1, and V^i verify (6) and (7) with $\beta = 1$. Then there exist $\lambda_1, \dots, \lambda_N \in \mathbb{R}$, $v_1, \dots, v_N \in C^{2,\alpha}(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, for all $0 < \alpha < 1$, $1 \leq p < \infty$, which solve the system (1).*

We need the following two lemmas for linear equations. We believe they are well-known, but for lack of a precise reference we give their proofs in the Appendix.

Lemma 2.2. *Let*

$$\mathcal{L} = -a_{hk}(x)D_{hk} + b_h(x)D_h$$

be a second-order uniformly elliptic linear differential operator in $Q = \mathbb{T}^d$ with coefficients $a_{hk}, b_h, \in C^\alpha(Q)$, $h, k = 1, \dots, d$, $0 < \alpha < 1$. Then, for any $f \in C^\alpha(Q)$, the problem

$$\begin{cases} \mathcal{L}v + \lambda &= f \\ \int_Q v(x)dx &= 0 \end{cases} \quad (9)$$

has a unique solution $(v, \lambda) \in C^{2,\alpha}(Q) \times \mathbb{R}$. Moreover,

$$|\lambda| \leq \|f\|_\infty, \quad (10)$$

$$\|v\|_{C^{1,\alpha}(Q)} \leq C\|f\|_\infty, \quad (11)$$

$$\|v\|_{C^{2,\alpha}(Q)} \leq C\|f\|_{C^\alpha(Q)} \quad (12)$$

for some constant $C > 0$ which depends only on (the coefficients of) \mathcal{L} .

Lemma 2.3. *Let*

$$\mathcal{L} = -a_{hk}(x)D_{hk}, \quad g : Q \rightarrow \mathbb{R}^d$$

be a (symmetric) second-order uniformly elliptic linear differential operator in $Q = \mathbb{T}^d$ with coefficients $a_{hk} \in C^{0,1}(Q)$, $h, k = 1, \dots, d$, and a bounded measurable vector field, respectively. Then the problem

$$\begin{cases} \mathcal{L}^*m - \operatorname{div}(g(x)m) &= 0 \\ \int_Q m(x)dx &= 1 \end{cases} \quad (13)$$

has a unique solution $m \in W^{1,p}(Q)$ for all $1 \leq p < \infty$. Moreover, m is positive and

$$\|m\|_{W^{1,p}(Q)} \leq C(\|g\|_\infty), \quad (14)$$

for some constant $C(\|g\|_\infty)$ which depends (continuously) on g only through $\|g\|_\infty$ (and also on p and the coefficients a_{hk}).

Proof of Theorem 2.1. The proof is based on Schauder's fixed point theorem (see for instance [41, Theorem 4.1.1, p. 25] or [27, Corollary 11.2, p. 280]) and on a priori estimates for the gradients of v_i that are obtained by Bernstein's method, as suggested in [34, 36].

We first assume, instead of (3) and (4), that the Hamiltonians are bounded, that is,

$$\exists M > 0 \quad \text{such that} \quad |H^i(x, p)| \leq M \quad \forall i, x, p.$$

Let

$$\mathcal{B} = \left\{ u = (u_1, \dots, u_N) \in (C^{1,\alpha}(Q))^N : \int_Q u dx = 0 \right\},$$

which is a Banach space as a closed subspace of $C^{1,\alpha}(Q)^N$. We define an operator

$$T : \mathcal{B} \rightarrow \mathcal{B},$$

according to the scheme

$$u \mapsto m \mapsto (v, \lambda) \mapsto v,$$

as follows. Given $u = (u_1, \dots, u_N) \in \mathcal{B}$, we plug it in place of v in the second N scalar linear equations of the system (1) and solve those equations for the unknowns $m = (m_1, \dots, m_N)$, requiring that these unknowns satisfy conditions in the third line of (1). That these $\{m_i\}_{i=1}^N$

exist, are uniquely defined and have the required properties, is a consequence of Lemma 2.3. Then, with these $\{m_i\}$, we solve the N scalar linear equations obtained from the first equations in (1) after plugging u_i into H^i in place of v_i , i.e., by solving the N uncoupled linear equations

$$\mathcal{L}^i v_i + H^i(x, Du_i) + \lambda_i = V^i[m]$$

for the unknowns $v = (v_1, \dots, v_N)$, $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$. Since we require in addition that the v_i have zero mean, then $v = (v_1, \dots, v_N) \in C^{2,\alpha}(Q)$ and $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ are uniquely defined, as a consequence of Lemma 2.2. We set $Tu = v$. By (10), (12), (14), and a standard embedding theorem, T is continuous and compact. Moreover, the $C^{1,\alpha}$ -estimate (11) and the boundedness of H^i and V^i gives

$$\|v\|_{C^{1,\alpha}(Q)} \leq C$$

for some $C > 0$ independent of v ; thus $T\mathcal{B}$ is bounded. Therefore, by Schauder's fixed point theorem, T has a fixed point (in the convex hull of the closure of $T\mathcal{B}$).

Now we turn to consider Hamiltonians H^i that satisfy the assumptions of the theorem. We introduce the truncated Hamiltonians H_R^i defined as follows

$$H_R^i(x, p) = \begin{cases} H^i(x, p), & \text{if } |p| \leq R, \\ H^i\left(x, R \frac{p}{|p|}\right), & \text{if } |p| > R, \end{cases} \quad x \in Q, p \in \mathbb{R}^d, \quad (15)$$

where the parameter $R > 0$ is to be fixed in the sequel sufficiently large. Let $R_1 > 0$ be such that (4) is verified for all $x \in Q$, $|p| \geq R_1$. Then

$$\inf_{x \in Q} ((\text{tr } a^i) D_x H_R^i \cdot p + \theta^i (H_R^i)^2) \geq -C|p|^2 \quad \text{for all } R \geq |p| \geq R_1, \quad (16)$$

with the same θ^i ($i = 1, \dots, N$) and C as in (4). Clearly the H_R^i are bounded and Lipschitz continuous. So let $\lambda_1^R, \dots, \lambda_N^R \in \mathbb{R}$, $v_1^R, \dots, v_N^R \in C^{2,\alpha}(Q)$, $m_1^R, \dots, m_N^R \in W^{1,p}(Q)$ ($0 < \alpha < 1$, $1 \leq p < \infty$) be a solution of (1) with H_R^i in the place of H^i .

The crucial step of the proof is an a priori estimate for $\|Dv_i^R\|_\infty$ uniform in R , obtained by Bernstein's method. We drop the indices i and R in the following estimates. Let $w = Dv$ and $\psi = (1/2)|w|^2$. We have these identities

$$D\psi = wD^2v \quad (17)$$

$$D^2\psi = \sum_h w_h D^2 w_h + (D^2v)^2 \quad (18)$$

$$\mathcal{L}\psi = -a \cdot D^2\psi = -\sum_h w_h (a \cdot D^2 w_h) - a \cdot (D^2v)^2. \quad (19)$$

Let us assume temporarily that u is of class C^3 and a_{hk} , H , F are of class C^1 (clearly the truncation of H above could have been done smoothly). We apply to the first equations in (1) the operator $w \cdot D$ and use (19), (17) to obtain

$$\mathcal{L}\psi + a \cdot (D^2v)^2 - \delta a \cdot D^2v + D_x H \cdot w + D_p H \cdot D\psi = G \cdot w, \quad (20)$$

where

$$\delta a := (\delta a_{hk})_{h,k=1,\dots,d} \quad \text{with} \quad \delta a_{hk} := \sum_l D_l a_{hk} w_l \quad (21)$$

and G is a function whose L^∞ -norm does not exceed a universal constant which does not depend on v (recall the assumptions on operators V^i). We use the following inequalities which are simple consequences of Cauchy-Schwarz inequality: for any a, b symmetric matrices with $a \geq 0$ and $c, e \in \mathcal{M}^{d \times m_i}$

$$(a \cdot b)^2 \leq (a \cdot b^2) \operatorname{tr} a \quad \text{and} \quad (\operatorname{tr} c e^t b)^2 \leq |e|^2 c c^t \cdot b^2. \quad (22)$$

Assume that

$$a = \frac{1}{2} \sigma \sigma^t, \quad (23)$$

where the matrix $\sigma(x)$ is Lipschitz; such a decomposition is always possible for $a(x)$ is Lipschitz and its smallest eigenvalue (which is positive) is bounded away from zero uniformly for $x \in Q$. By (23) and (21) we have

$$\delta a = (\delta \sigma) \sigma^t.$$

Using this identity, the second of inequalities (22) and (23), we obtain

$$\begin{aligned} \delta a \cdot D^2 v &\leq |\delta \sigma| (\sigma \sigma^t \cdot (D^2 v)^2)^{1/2} = \sqrt{2} |\delta \sigma| (a \cdot (D^2 v)^2)^{1/2} \\ &\leq \varepsilon a \cdot (D^2 v)^2 + \frac{1}{2\varepsilon} |\delta \sigma|^2, \end{aligned} \quad (24)$$

for any $0 < \varepsilon < 1$. On the other hand, using the first of inequalities (22) and the first equations in (1), we have

$$\begin{aligned} (a \cdot (D^2 v)^2) \operatorname{tr} a &\geq (a \cdot D^2 v)^2 = (\mathcal{L} v)^2 \\ &\geq (\lambda + H - V[m])^2 \\ &\geq \omega H^2 - c_\omega \end{aligned} \quad (25)$$

for any $0 < \omega < 1$ and a constant c_ω independent of R . The last inequality is obtained by the boundedness of operators V and constants λ . In fact, looking at the minima and maxima of v in the first equations of (1) we obtain

$$|\lambda| \leq \sup_{x \in Q} (|H(x, 0)| + |V[m](x)|).$$

Multiplying (20) by $\operatorname{tr} a$, and using (24), (25), we get

$$(\operatorname{tr} a)(\mathcal{L} \psi + D_p H D \psi + D_x H \cdot w) + (1 - \varepsilon) \omega H^2 \leq (\operatorname{tr} a) \left(G \cdot w + \frac{1}{2\varepsilon} |\delta \sigma|^2 \right) + c_\omega$$

If we choose ε and ω such that $(1 - \varepsilon) \omega > \theta$, where θ is the constant appearing in (16), using (16) we get

$$(\operatorname{tr} a)(\mathcal{L} \psi + D_p H D \psi) + ((1 - \varepsilon) \omega - \theta) H^2 \leq (\operatorname{tr} a) \left(G \cdot w + \frac{1}{2\varepsilon} |\delta \sigma|^2 \right) + C |w|^2 + c_\omega \quad (26)$$

for $R \geq |w| > R_1$. At a maximum point of ψ_i^R , (now we reintroduce i and R in order to avoid any possible confusion), say x_R^i , taking into account that $|\delta \sigma^i|$ is at most linear in w_i^R , we have

$$((1 - \varepsilon^i) \omega^i - \theta^i) (H_R^i)^2 (x_R^i, w_i^R(x_R^i)) \leq C (|w_i^R(x_R^i)|^2 + 1) \quad (27)$$

for some $C > 0$ (independent of R) and $R \geq |w_i^R(x_R^i)| \geq R_1$. But by (3), the left-hand side above is superlinear in $|w_i^R(x_R^i)|$ and thus $|w_i^R(x_R^i)|$ must be bounded by some constant independent of R .

Thus, we have shown that

$$\|Du_i^R\|_\infty \leq R_2 \quad (28)$$

for some $R_2 > 0$ independent of $R > R_1$. So if take any $R > \max\{R_1, R_2\}$ in (15), we discover that $\lambda_1^R, \dots, \lambda_N^R \in \mathbb{R}$, $v_1^R, \dots, v_N^R \in C^{2,\alpha}(Q)$, $m_1^R, \dots, m_N^R \in W^{1,p}(Q)$ ($0 < \alpha < 1$, $1 \leq p < \infty$) is also a solution of the original system of PDEs (1).

To complete the proof under our general assumptions we observe that equation (20) still holds a.e. in \mathbb{R}^d . In fact $u \in W^{3,p}$, $p > n$ (by classical elliptic regularity theory for linear equations), hence u is three times differentiable in the usual sense a.e. and also the coefficients, being Lipschitz continuous, are differentiable a.e. A sort of chain rule usable for our purposes holds by a result in [4], and the maximum principle to be used is that of [17]. Alternatively, and more simply, one can proceed by regularizing the data $\{(a_{hk}^i)_{1 \leq h,k \leq d} : i = 1, \dots, N\}$, $V^i[m]$, for $m \in P(Q)^N$, and H^i , for $i = 1, \dots, N$, via smooth approximations to the identity $\{\rho_\varepsilon\}_{\varepsilon>0}$, $\{\hat{\rho}_\varepsilon\}_{\varepsilon>0}$ given by $\rho_\varepsilon(x) = \varepsilon^{-d}\rho_1(x/\varepsilon)$ and $\hat{\rho}_\varepsilon(x, p) = \rho_\varepsilon(x)\rho_\varepsilon(p)$ for all $x \in \mathbb{R}^d$, $p \in \mathbb{R}^d$, $\varepsilon > 0$, where ρ_1 is some mollification kernel in \mathbb{R}^d (that is, a nonnegative function of class C^∞ with support in the unit ball B of \mathbb{R}^d and $\int_B \rho_1(z) dz = 1$). Noting that $\|\rho_\varepsilon \star a_{hk}^i\|_{C^1(Q)} \rightarrow \|a_{hk}^i\|_{C^{0,1}(Q)}$, $\|\rho_\varepsilon \star V^i[m]\|_{C^1(Q)} \rightarrow \|V^i[m]\|_{C^{0,1}(Q)}$ and $\inf D_x(\hat{\rho}_\varepsilon \star H^i) \rightarrow \text{ess inf } D_x H^i$ as $\varepsilon \rightarrow 0$ for all h, k, i , $m \in P(Q)^N$, one deduces estimates of type (28) with R_2 independent of ε for ε small enough. \square

Remark 2.4. (i) In dimension $d = 1$ condition (4) is not needed. Note that $\frac{d^2 v_i}{dx^2}$ is bounded from below if H^i is bounded from below; since it has zero mean in $(0, 1)$, it is bounded in L^1 norm. Therefore $\frac{dv_i}{dx}$ is bounded.

(ii) Theorem 2.1 still holds if (3) is weakened to

$$\exists \nu^i > 0 \text{ such that } \liminf_{|p| \rightarrow \infty} \frac{\inf_{x \in Q} |H^i(x, p)|}{|p|} = \nu^i > 0, \quad (29)$$

provided that we substitute (4) with the stronger condition: $\exists \theta^i, \eta^i \in (0, 1)$ such that

$$\liminf_{|p| \rightarrow \infty} \frac{1}{|p|^2} \inf_{x \in Q} \left(\theta^i (\text{tr } a) D_x H^i \cdot p + (1 - \theta^i) \theta^i \eta^i (H^i)^2 - |p|^2 \frac{(\text{tr } a^i) |D\sigma^i|^2}{2} \right) \geq 0, \quad (30)$$

where $a^i = (1/2)\sigma\sigma^t$, $D\sigma^i = (D\sigma_{hk}^i)$ is a matrix (whose entries are vectors) and $|D\sigma^i|^2 = \sum_{h,k} |D\sigma_{hk}^i|^2$. To see this take $\varepsilon^i = \theta^i$ in (24), multiply (26) by θ^i and choose ω^i in (25) so that $\omega^i > \eta^i$. Let $s > 0$, $R_s > 0$ be such that the quantity under the \liminf sign in the left-hand side of (30) is $> -s$ for all $|p| \geq R_s$. Noting that $|\delta\sigma^i| \leq |D\sigma^i| |w_i^R|$, we deduce

$$(\omega^i - \eta^i)(1 - \theta^i) \theta^i (H_R^i)^2 (x_R^i, w_i^R(x_R^i)) - s |w_i^R(x_R^i)|^2 \leq C(|w_i^R(x_R^i)| + 1)$$

for some $C > 0$ (independent of R and s) and for $R \geq |w_i^R(x_R^i)| \geq R_s$. Now if we choose $s > 0$ so small that $s < \nu^i(\omega^i - \eta^i)(1 - \theta^i) \theta^i$, by (29) we find that $|w_i^R(x_R^i)|$ is bounded uniformly in R .

(iii) The arguments of the proof of Theorem 2.1, with some obvious modifications, prove also the solvability of the problem

$$\mathcal{L}v + \lambda + H(x, Dv) = 0$$

for the unknowns $v \in C^2(Q)$, $\lambda \in \mathbb{R}$, where \mathcal{L} satisfies (2) and the Hamiltonian H is locally Lipschitz continuous and satisfies (3) and (4) or, alternatively, any condition mentioned in the remarks above. This result is known for Dirichlet and Neumann boundary value problems with H satisfying similar conditions, see [37]. If \mathcal{L} is degenerate elliptic the existence of a viscosity solution $v \in C^{0,1}(Q)$ can be found in [38].

These existence results for (1), under condition C1, may be interpreted as follows. The Hamiltonians H^i can grow arbitrarily provided that they “do not oscillate too much in x ” which rigorously means that they should satisfy the technical condition (4). On the other hand, if the Hamiltonians have at most quadratic growth as in condition C2 of the Introduction, the so called natural growth condition, we do not need any additional assumption of the aforementioned type in order to ensure existence. Indeed, we have the following result.

Theorem 2.5. *Assume (2) and (8), H^i satisfy condition C2, and V^i verify (6) and (7) with $\beta = \alpha$. Then there exist $\lambda_1, \dots, \lambda_N \in \mathbb{R}$, $v_1, \dots, v_N \in C^{2,\alpha}(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, for all $1 \leq p < \infty$, which solve the system (1).*

The proof is obtained as a limit in a system of equations with zero-th order terms, based on Theorem 2.6, and it is therefore postponed to the next section.

2.2 Equations with zero-th order terms

Theorem 2.6. *Assume (2) and (8), H^i satisfy condition C2, and V^i verify (6) and (7) with $\beta = \alpha$. Let ρ^1, \dots, ρ^N be positive constants. Then there exist $v_1, \dots, v_N \in C^{2,\alpha}(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, for all $1 \leq p < \infty$, $i = 1, \dots, N$, which solve*

$$\begin{cases} \mathcal{L}^i v_i + H^i(x, Dv_i) + \rho^i v_i = V^i[m] & \text{in } Q, \\ \mathcal{L}^{i*} m_i - \operatorname{div}(g^i(x, Dv_i)m_i) = 0 & \text{in } Q, \\ \int_{Q^d} m_i(x) dx = 1, \quad m_i > 0, \quad i = 1, \dots, N, \end{cases} \quad (31)$$

For the proof we need the following lemma which is proved in the Appendix.

Lemma 2.7. *Let*

$$\mathcal{L} = -a_{hk}(x)D_{hk} + b_h(x)D_h + c(x)$$

be a second-order uniformly elliptic linear differential operator in $Q = \mathbb{T}^d$ with coefficients $a_{hk}, b_h, c \in C^\alpha(Q)$, $h, k = 1, \dots, d$, $0 < \alpha < 1$. Assume also that $c > 0$. Then, for any $f \in C^\alpha(Q)$, the equation

$$\mathcal{L}v = f \quad (32)$$

has one and only one solution $v \in C^{2,\alpha}(Q)$. Moreover, for some constant $C > 0$ which depends only on (the coefficients of) \mathcal{L} ,

$$\|v\|_{C^{2,\alpha}(Q)} \leq C\|f\|_{C^\alpha(Q)}. \quad (33)$$

Proof of Theorem 2.6. We define an operator

$$T : C^{1,\alpha}(Q)^N \rightarrow C^{1,\alpha}(Q)^N,$$

$$u \mapsto m \mapsto v,$$

in the following way. Given $u = (u_1, \dots, u_N)$, we solve the second N equations in (31) with u_i plugged into g^i in place of v_i and with the corresponding normalisation conditions, and find $m = (m_1, \dots, m_N)$. With these m_i and the u_i plugged into the Hamiltonians H^i , $i = 1, \dots, N$, in place of v_i we solve the first N linear equations of (31), that is,

$$\mathcal{L}^i v_i + \rho^i v_i + H^i(x, Du_i) = V^i[m], \quad i = 1, \dots, N \quad (34)$$

and find $v = (v_1, \dots, v_N) \in C^{2,\alpha}(Q)^N$. This is possible in virtue of Lemma 2.7. We set $Tu = v$.

It is standard to verify that $T : C^{1,\alpha}(Q) \rightarrow C^{1,\alpha}(Q)$, $u \rightarrow Tu$ is continuous and compact. By Schaefer's version of Leray-Schauder theorem (see [41, Theorem 4.3.2, p 29] or [27, Theorem 11.3, p. 280]), we need only look at the set of the fixed points of the operators sT , $0 \leq s \leq 1$, that is,

$$\{u \in C^{1,\alpha}(Q)^N : sTu = u \text{ for some } 0 \leq s \leq 1\}, \quad (35)$$

and prove that it is bounded in $C^{1,\alpha}(Q)^N$.

To prove such an estimate first note that if $u = sTu$ for some $0 \leq s \leq 1$, then

$$\|u_i\|_{C(Q)} = \frac{s}{\rho^i} \max_{x \in Q} (|H^i(x, 0)| + |F^i(x)|) \leq C$$

for some $C > 0$ independent of u and s , as can be seen by looking at the extrema of u_i which satisfies equation (34) with $u_i = v_i$ and H^i , F^i multiplied by s . We combine this with a classical a priori interior estimate for the gradients of solutions of elliptic quasilinear equations [33, Theorem 3.1, p. 266] to get

$$\|u_i\|_{C^1(Q)} \leq C$$

for some $C > 0$ independent of u and s . Here we are using the growth assumption (5) and assuming temporarily that the leading coefficients of the quasilinear equation are of class C^1 . This additional condition can be removed by approximation because the estimates depend only on the L^∞ -norm of the derivatives of the coefficients and not on their modulus of continuity. The last estimate combined with [27, Theorem 8.32, p. 210] yields

$$\|u_i\|_{C^{1,\alpha}(Q)} \leq C$$

for some $C > 0$ again independent of u and s . Thus T has at least one fixed point. \square

Proof of Theorem 2.5. Let $v_1^\rho, \dots, v_N^\rho \in C^{2,\alpha}(Q)$, $m_1^\rho, \dots, m_N^\rho \in W^{1,p}(Q)$ ($1 \leq p < \infty$, $i = 1, \dots, N$) be a solution of (31) with $\rho^i = \rho > 0$. Let $\langle v_i^\rho \rangle = \int_Q v_i^\rho dx$ be the mean of v_i^ρ . The crucial observation is that

$$\|v_i^\rho - \langle v_i^\rho \rangle\|_\infty \leq C$$

for some $C > 0$ independent of ρ , which can be shown by using the techniques of [11]. Then it is not difficult to complete the proof by showing that there exists a sequence $\rho_n \rightarrow 0$ such that

$$(v_i^{\rho_n} - \langle v_i^{\rho_n} \rangle, \rho_n v_i^{\rho_n}, m_i^{\rho_n}) \rightarrow (v_i, \lambda_i, m_i) \quad \text{in } C^2(Q) \times C(Q) \times C(Q),$$

where (v_i, λ_i, m_i) , $i = 1, \dots, N$, is a solution of (1).

Remark 2.8. An alternative existence result for the system (31) can be stated under condition C1 instead of C2 if (7) holds with $\beta = 1$. This requires only a slight modification of the proof of Theorem 2.1.

3 Stochastic differential games

As an application of the previous results, in this section we show the existence of Nash equilibria for a class of N -person stochastic differential games with infinite horizon, such that the state of each player evolves independently from the states of the other players and the only coupling comes through the costs. Games of this type arise in many engineering and economic problems [31, 32, 36].

Consider a control system driven by the stochastic differential equations

$$dX_t^i = f^i(X_t^i, \alpha_t^i)dt + \sigma^i(X_t^i)dW_t^i, \quad X_0^i = x^i \in \mathbb{R}^d, \quad i = 1, \dots, N \quad (36)$$

where $\{W_t^i\}_{i=1}^N$ are N independent Brownian motions in \mathbb{R}^d , $d \geq 1$, $A^i \subseteq \mathbb{R}^m$ are closed,

$$f^i : \mathbb{R}^d \times A^i \rightarrow \mathbb{R}^d \quad \sigma^i : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$$

are continuous, \mathbb{Z}^d -periodic and Lipschitz continuous in x uniformly in α , the matrix $\sigma^i(x)$ is nonsingular for any value of x , α_t^i is an *admissible control* of the i -th player, that is, a stochastic process taking values in A^i and adapted to W_t^i . In view of the assumed periodicity in x^i of all data we will often consider functions as defined on $Q = \mathbb{T}^d$.

3.1 N -person games with long-time-average cost

Consider a game where the i -th player seeks to minimize the long-time-average or ergodic cost

$$J^i(X, \alpha^1, \dots, \alpha^N) := \liminf_{T \rightarrow +\infty} \frac{1}{T} E \left[\int_0^T L^i(X_t^i, \alpha_t^i) + F^i(X_t^1, \dots, X_t^N) dt \right], \quad (37)$$

where $X = (x^1, \dots, x^N)$ is the initial position of the system (36). On the cost of the i -th player (37) we assume

$$L^i : Q \times A^i \rightarrow \mathbb{R} \quad \text{continuous}, \quad (38)$$

$$F^i : Q^N \rightarrow \mathbb{R} \quad \beta\text{-H\"older continuous} \quad (39)$$

for some $\beta \in (0, 1)$. Define

$$\mathcal{H}^i(x^i, p, \alpha) := -p \cdot f^i(x^i, \alpha) - L^i(x^i, \alpha), \quad H^i(x^i, p) := \sup_{\alpha \in A^i} \mathcal{H}^i(x^i, p, \alpha), \quad p \in \mathbb{R}^d, \quad (40)$$

and suppose the sup is attained in the definition of H^i for all x^i, p . Moreover, set

$$a^i := \sigma^i(\sigma^i)^t / 2, \quad \mathcal{L}^i := -a^i \cdot D^2. \quad (41)$$

Following the classical theory initiated by Friedman [25] and continued by Bensoussan and Frehse [10, 12, 13], we look at the Nash equilibria for the maximization of the pre-Hamiltonians with parameters $x^j, p^j \in \mathbb{R}^d, j = 1, \dots, N$, namely,

$$P^i(\alpha^1, \dots, \alpha^N) := - \sum_{j=1}^N f^j(x^j, \alpha^j) \cdot p^j - L^i(x^i, \alpha^i).$$

Clearly $(\bar{\alpha}^1, \dots, \bar{\alpha}^N)$ is a Nash equilibrium for the static game with payoffs P^1, \dots, P^N if and only if $\bar{\alpha}^i \in \operatorname{argmax}_{\alpha^i} \mathcal{H}^i(x^i, p^i, \alpha^i)$ and the value of P^i at the equilibrium is

$$P^i(\bar{\alpha}^1, \dots, \bar{\alpha}^N) = H^i(x^i, p^i) - \sum_{j \neq i} f^j(x^j, \bar{\alpha}^j) \cdot p^j.$$

Therefore the system of Bellman equations of ergodic type [12, 13] associated to the game considered here is

$$\begin{cases} \sum_{j=1}^N \mathcal{L}^j v_i + H^i(x^i, D_{x^i} v_i) + \lambda_i = F^i(X) + \sum_{j \neq i} f^j(x^j, \bar{\alpha}^j) \cdot D_{x^j} v_i & \text{in } \mathbb{R}^{dN}, \\ \bar{\alpha}^i \in \operatorname{argmax}_{\alpha^i} \mathcal{H}^i(x^i, D_{x^i} v_i, \alpha^i), & i = 1, \dots, N, \end{cases} \quad (42)$$

where the unknowns are the constants λ_i and the functions $v_i(X)$, $i = 1, \dots, N$. Note that this system is strongly coupled via the terms $f^j(x^j, \bar{\alpha}^j)$ on the right hand side of the equation for v_i , because $\bar{\alpha}^j$ depends on $D_{x^j} v_j$.

Assume there exist functions $\bar{\alpha}^i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow A^i$ such that

$$\bar{\alpha}^i(x, p) \text{ is a maximum point for } \alpha \rightarrow \mathcal{H}^i(x, p, \alpha) \quad \forall x, p, \quad (43)$$

$$\bar{\alpha}^i \text{ is locally Lipschitz and } \mathbb{Z}^d - \text{periodic in } x, \quad (44)$$

and define

$$g^i(x, p) = -f^i(x, \bar{\alpha}^i(x, p)), \quad i = 1, \dots, N. \quad (45)$$

Making this choice in (42) we get

$$\sum_{j=1}^N \mathcal{L}^j v_i + H^i(x^i, D_{x^i} v_i) + \sum_{j \neq i} g^j(x^j, D_{x^j} v_j) \cdot D_{x^j} v_i + \lambda_i = F^i(X) \quad \text{in } \mathbb{R}^{dN}, \quad (46)$$

$i = 1, \dots, N$. By a classical verification argument [12], if $(v_i, \lambda_i) \in C^2(\mathbb{T}^{dN}) \times \mathbb{R}$, $i = 1, \dots, N$, is a solution, then $\bar{\alpha}^i(\cdot, D_{x^i} v_i(\cdot))$, $i = 1, \dots, N$, is a *Nash equilibrium feedback*. More precisely, if one solves the stochastic differential equation

$$dX_t^i = f^i(X_t^i, \bar{\alpha}^i(X_t^i, D_{x^i} v_i(X_t^i))) dt + \sigma^i(X_t^i) dW_t^i, \quad X_0^i = x^i \in \mathbb{R}^d, \quad i = 1, \dots, N. \quad (47)$$

then $\bar{\alpha}_t^i := \bar{\alpha}^i(X_t^i, D_{x^i} v_i(X_t^i))$, $i = 1, \dots, N$, is a Nash equilibrium for the cost functionals (37), and λ_i , $i = 1, \dots, N$, are the values of the game corresponding to such equilibrium.

We want to derive from (46) a different system of elliptic equations of the form (1) from which we can still synthesize a Nash equilibrium feedback. We follow an idea introduced by Lasry and Lions in the seminal paper [34]. Given a solution of (46) consider the equilibrium process defined by (47). Let us assume that

$$D_{x^i} v^i \text{ depends only on } x^i \text{ for all } i. \quad (48)$$

Then the equations in (47) are decoupled and each of them defines a non-degenerate diffusion X_t^i on the torus \mathbb{T}^d , which has a unique ergodic invariant measure with density m_i . It is known that each $m_i \in C^2(\mathbb{T}^d)$ solves the Kolmogorov-Fokker-Planck equation

$$\mathcal{L}^{i*} m_i - \operatorname{div}_{x^i} (g^i(x^i, D_{x^i} v_i) m_i) = 0, \quad \int_Q m_i(x) dx = 1, \quad i = 1, \dots, N, \quad (49)$$

where \mathcal{L}^{i*} is the formal adjoint of \mathcal{L}^i .

The next result exploits the adjoint structure of (46) and (49) to prove that multiplying the i -th equation in (46) by $\prod_{j \neq i} m_j(x^j)$ and integrating over Q^{N-1} with respect to dx^j , $j \neq i$, we arrive at

$$\mathcal{L}^i v_i + H^i(x, Dv_i) + \lambda_i = V^i[m] \quad \text{in } \mathbb{R}^d, \quad (50)$$

where

$$V^i[m](x) = \int_{Q^{N-1}} F^i(x^1, \dots, x^{i-1}, x, x^{i+1}, \dots, x^N) \prod_{j \neq i} m_j(x^j) dx^j. \quad (51)$$

Note that (39) implies (7) for such V^i and then it is easy to see that also (6) holds.

Proposition 3.1. *Assume $(v_i, \lambda_i) \in C^2(\mathbb{T}^{dN}) \times \mathbb{R}$, $i = 1, \dots, N$, is a solution of (46) satisfying (48) and m_1, \dots, m_N solve (49). Then, for all i , $x \mapsto v_i(x^1, \dots, x^{i-1}, x, x^{i+1}, \dots, x^N)$ solves (50) for all x^1, \dots, x^N .*

Proof. Multiply the i -th equation in (46) by $\prod_{j \neq i} m_j(x^j)$ and integrate over Q^{N-1} with respect to $dx^j, j \neq i$. Observe that, for $k \neq i$, integrating by parts with respect to x^k we get

$$\begin{aligned} \int_{Q^{N-1}} \left(\mathcal{L}^k v_i + g^k(x^k, D_{x^k} v_k) \cdot D_{x^k} v_i \right) \prod_{j \neq i} m_j(x^j) dx^j = \\ \int_{Q^{N-2}} \int_Q v_i \left(\mathcal{L}^{k*} m_k - \operatorname{div}_{x^k} \left(g^k(x^k, D_{x^k} v_k) m_k \right) \right) dx^k \prod_{j \neq i, k} m_j(x^j) dx^j. \end{aligned}$$

Then (49) gives

$$\int_{Q^{N-1}} \sum_{j \neq i} \left(\mathcal{L}^j v_i + g^j(x^j, D_{x^j} v_j) \cdot D_{x^j} v_i \right) \prod_{j \neq i} m_j(x^j) dx^j = 0.$$

On the other hand, using the assumption (48) and $\int_Q m_j(x) dx = 1$, we also have

$$\int_{Q^{N-1}} \left(\mathcal{L}^i v_i + H^i(x^i, Dv_i) + \lambda_i \right) \prod_{j \neq i} m_j(x^j) dx^j = \mathcal{L}^i v_i + H^i(x^i, Dv_i) + \lambda_i.$$

Now considering again the i -th equation in (46) multiplied by $\prod_{j \neq i} m_j(x^j)$ and integrated over Q^{N-1} with respect to $dx^j, j \neq i$, and plugging into it the last two identities, we get that v_i restricted to the variable x^i solves (50) with V^i given by (51). \square

Remark 3.2. If the maximum point $\bar{\alpha}^i(x, p)$ of $\mathcal{H}^i(x, p, \cdot)$ is unique, then H^i is differentiable with respect to p and $D_p H^i(x, p) = -f^i(x, \bar{\alpha}^i(x, p)) = g^i(x, p)$. Then the resulting system of HJB-KFP equations can be written in the same form as in the Lasry-Lions papers [34, 36], namely,

$$\begin{cases} \mathcal{L}^i v_i + H^i(x, Dv_i) + \lambda_i = V^i[m] & \text{in } Q, \\ \mathcal{L}^{i*} m_i - \operatorname{div} (D_p H^i(x, Dv_i) m_i) = 0 & \text{in } Q, \\ \int_{Q^d} m_i(x) dx = 1, \quad m_i > 0, \quad \int_{Q^d} v_i(x) dx = 0, \quad i = 1, \dots, N. \end{cases} \quad (52)$$

These notations also show that the KFP equations are the linearizations of the HJB equations around their solution v_1, \dots, v_N .

From the theory of the previous section we get the following existence result.

Corollary 3.3. *In addition to (2) and the assumptions of this section suppose that H^i satisfy either condition C1 or condition C2. Then there exist $\lambda_1, \dots, \lambda_N \in \mathbb{R}$, $v_1, \dots, v_N \in C^2(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, $1 \leq p < \infty$, which solve (1) with V^i and g^i given by (51) and (45), respectively.*

3.2 Synthesis of Nash equilibria

Next we prove a verification result that produces Nash feedback equilibria for the N -person differential game from any solution of the HJB-KFP system of PDEs. We recall that a feedback for the i -th player is a Lipschitz map $\alpha^i : \mathbb{R}^d \rightarrow A^i$ that generates a process X_t^i solving $dX_t^i = f^i(X_t^i, \alpha^i(X_t^i))dt + \sigma^i(X_t^i)dW_t^i$, $X_0^i = x^i$, and an admissible control $\alpha_t^i = \alpha^i(X_t^i)$. As usual, when we plug a feedback in the functional J^i we mean its associated admissible control. A *Nash equilibrium* of the N -person game of Section 3.1 for the initial position $X = (x^1, \dots, x^N)$ is a vector of admissible controls $(\bar{\alpha}^1, \dots, \bar{\alpha}^N)$ such that

$$J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^{i-1}, \alpha^i, \bar{\alpha}^{i+1}, \dots, \bar{\alpha}^N) \geq J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^N) \quad \text{for all } \alpha^i \text{ admissible control.}$$

Theorem 3.4. *Let $\lambda_i, v_i, m_i, i = 1, \dots, N$ be a solution of the system (1) as in the preceding Corollary 3.3. Then*

$$\bar{\alpha}^i(x) := \bar{\alpha}^i(x, Dv_i(x)), \quad x \in \mathbb{R}^d, \quad i = 1, \dots, N, \quad (53)$$

define a feedback which is a Nash equilibrium for all initial positions $X \in Q^N$ of the control system (36). In addition, for each $X = (x^1, \dots, x^N)$,

$$\lambda_i = J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^N) = \liminf_{T \rightarrow +\infty} \frac{1}{T} E \left[\int_0^T L^i(\bar{X}_t^i, \bar{\alpha}^i(\bar{X}_t^i)) + F^i(\bar{X}_t^1, \dots, \bar{X}_t^N) dt \right], \quad (54)$$

where \bar{X}_t^i is the process associated to the feedback $\bar{\alpha}^i$ with $\bar{X}_0^i = x^i$.

Proof. We follow the outline of proof in [34, 36]. Let us first check (54). The processes \bar{X}_t^i are independent diffusions on the torus, so each of them has a unique ergodic invariant measure, which must be m_i , see Chapter 3, Section 3 of [16] or [3]. Then the joint process $(\bar{X}_t^1, \dots, \bar{X}_t^N)$ is ergodic with invariant measure $\prod_{i=1}^N m_i(x^i)$, and therefore the right hand side of (54) is

$$J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^N) = \int_Q L^i(x, \bar{\alpha}^i(x)) dm_i(x) + \int_{Q^N} F^i(x^1, \dots, x^N) \prod_{j=1}^N dm_j(x^j).$$

On the other hand, the Ito-Dynkin formula gives

$$E \left[v_i(\bar{X}_T^i) - v_i(x^i) \right] = E \left[\int_0^T \left(-g^i(\bar{X}_t^i, Dv_i(\bar{X}_t^i)) \cdot Dv_i(\bar{X}_t^i) - \mathcal{L}^i v_i(\bar{X}_t^i) \right) dt \right]$$

By (43) and (45)

$$g^i(x, p) \cdot p = H^i(x, p) + L^i(x, \bar{\alpha}(x, p)),$$

so the HJB equation (50) implies

$$\begin{aligned} E \left[v_i(\bar{X}_T^i) - v_i(x^i) \right] = \\ \lambda_i T - E \left[\int_0^T L^i(\bar{X}_t^i, \bar{\alpha}^i(\bar{X}_t^i)) + \int_{Q^{N-1}} F^i(x^1, \dots, x^{i-1}, \bar{X}_t^i, x^{i+1}, \dots, x^N) \prod_{j \neq i} dm_j(x^j) dt \right]. \end{aligned}$$

We divide by T and let $T \rightarrow \infty$. The left-hand side vanishes, whereas the right hand side tends to $\lambda_i - J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^N)$, which proves (54).

Next we check that the feedback law (53) defines a Nash equilibrium. We change the control of the i -th player into an arbitrary admissible control α_t^i and get

$$\begin{aligned} \frac{1}{T} E [v_i(X_T^i) - v_i(x^i)] &= \frac{1}{T} E \left[\int_0^T (f^i(X_t^i, \alpha_t^i) \cdot Dv_i(X_t^i) - \mathcal{L}^i v_i(X_t^i)) dt \right] \\ &\geq \lambda_i - \frac{1}{T} E \left[\int_0^T L^i(X_t^i, \alpha_t^i) + \int_{Q^{N-1}} F^i(x^1, \dots, x^{i-1}, X_t^i, x^{i+1}, \dots, x^N) \prod_{j \neq i} dm_j(x^j) dt \right]. \end{aligned}$$

By the ergodicity of the joint process $(\bar{X}_t^1, \dots, \bar{X}_t^{i-1}, X_t^i, \bar{X}_t^{i+1}, \dots, \bar{X}_t^N)$ the last term on the right-hand side tends to $J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^{i-1}, \alpha^i, \bar{\alpha}^{i+1}, \dots, \bar{\alpha}^N)$ as $T \rightarrow \infty$, and then

$$J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^{i-1}, \alpha^i, \bar{\alpha}^{i+1}, \dots, \bar{\alpha}^N) \geq \lambda_i = J^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^N)$$

for all $X \in Q^N$. □

3.3 Examples with unconstrained controls

In this section we consider $A^i = \mathbb{R}^d$ for all i and the system affine in the control, i.e.,

$$f^i(x, \alpha) = \varphi^i(x) + \sum_{k=1}^d \alpha_k f_k^i(x) = \varphi^i(x) + \Phi^i(x) \alpha, \quad (55)$$

where Φ^i is a square matrix whose columns are the vector fields f_k^i , $k = 1, \dots, d$, and all vector fields are Lipschitz in Q . Then

$$H^i(x, p) := -p \cdot \varphi^i(x) + \sup_{\alpha \in \mathbb{R}^d} \{-p \cdot \Phi^i(x) \alpha - L^i(x, \alpha)\}.$$

Assume L^i is β -Hölder in x , uniformly as α varies in any bounded subset, and

$$\lim_{\alpha \rightarrow \infty} \inf_{x \in Q} L^i(x, \alpha) / |\alpha| = +\infty. \quad (56)$$

Then the sup in the definition of H^i is attained and

$$H^i(x, p) = L^{i*}(x, -\Phi^i(x)^t p) - p \cdot \varphi^i(x), \quad L^{i*}(x, q) := \max_{\alpha \in \mathbb{R}^d} \{q \cdot \alpha - L^i(x, \alpha)\}, \quad (57)$$

i.e., $L^{i*}(x, \cdot)$ is the convex conjugate of $L^i(x, \cdot)$. Moreover H^i is locally Lipschitz in p and β -Hölder in x . Now the other conditions of the existence theorems for the elliptic system and of the verification Theorem 3.4 can be checked on the last expression of H^i .

Example 3.5. If for a $\gamma > 0$

$$L^i(x, \alpha) \geq \gamma(|\alpha|^2 - 1),$$

then $L^{i*}(x, q) \leq |q|^2 / \gamma + \gamma$ and

$$|H^i(x, p)| \leq C(1 + |p|^2), \quad \forall x \in Q, p \in \mathbb{R}^d.$$

Therefore H^i satisfy condition C2 and Theorems 2.5 and 2.6 can be used. If (43) and (44) hold, then all the assumptions of Corollary 3.3 and Theorem 3.4 are verified.

It remains to check the conditions (43) and (44) on the existence of a Lipschitz argmax for the Hamiltonians. A sufficient condition for them is that L^i be differentiable with respect to α and $D_\alpha L^i(x, \cdot)$ be invertible (a fact related to the strict convexity of L^i in α) and locally Lipschitz. Then the sup in the definition of H^i is attained at a unique value

$$\bar{\alpha}^i(x, p) = (D_\alpha L^i)^{-1}(x, -\Phi^i(x)^t p),$$

which is a locally Lipschitz function of x and p . Next we give two examples where we can easily check all the conditions for the existence of a feedback Nash equilibrium and give a formula for it.

Example 3.6. Consider

$$L^i(x, \alpha) = \alpha^t B^i(x) \alpha,$$

with positive definite matrices $B^i(x)$ Lipschitz in Q , and the affine system (55). Then

$$L^{i*}(x, q) = q^t B^i(x)^{-1} q / 4,$$

so H^i grows at most quadratically in p and satisfies condition C2. Moreover

$$\bar{\alpha}^i(x, p) = -\frac{1}{2} B^i(x)^{-1} \Phi^i(x)^t p$$

verifies (43) and (44), so Remark 3.2 and Theorem 3.4 apply and the Nash equilibrium feedback is linear in $p = Dv_i$.

If H^i do not satisfy the natural growth condition (5) we must check condition C1. By assumption (56) L^{i*} is superlinear in q , so if the matrix $\Phi^i(x)$ is nonsingular also H^i is superlinear in p , i.e., it satisfies (3). The technical condition (4) is harder to check in general but can be easily seen in the next model problem.

Example 3.7. Consider

$$L^i(x, \alpha) = c^i(x) \frac{|\alpha|^\gamma}{\gamma}$$

for some Lipschitz $c^i > 0$ and $\gamma > 1$, and the affine system (55) with $\Phi^i(x)$ nonsingular for all x . Then

$$L^{i*}(x, q) = c^i(x)^{1/(1-\gamma)} |q|^{\gamma/(\gamma-1)} \frac{\gamma-1}{\gamma},$$

so we can compute $D_x H^i$ by (57) and see that (4) is satisfied for any choice of $\theta^i > 0$. If, moreover, F^i are Lipschitz, V^i defined by (51) verify (7) with $\beta = 1$, Then Theorem 2.1 applies to this case with g^i given by (45). Moreover

$$\bar{\alpha}^i(x, p) = c^i(x)^{1/(1-\gamma)} |q|^{(2-\gamma)/(\gamma-1)} q, \quad q = -\Phi^i(x)^t p,$$

satisfies (43) and (44), so also Remark 3.2 and Theorem 3.4 apply.

3.4 Examples with constrained controls

Assume first that the vector fields f^i have the general form described at the beginning of Section 3 with all control sets A^i bounded, and (38), (39) hold. Then the Hamiltonians defined by (40) satisfy

$$|H^i(x, p)| \leq C(1 + |p|), \quad \forall x \in Q, p \in \mathbb{R}^d,$$

so the sub-quadratic growth condition (5) holds. If L^i is β -Hölder in x , condition C2 is verified. Then, assuming (43) and (44), all the assumptions of Corollary 3.3 and Theorem 3.4 are verified.

The next is a simple example where the argmax of the Hamiltonians is a singleton and verifies (44).

Example 3.8. Consider the system

$$f^i(x, \alpha) = \varphi^i(x) - \alpha, \quad A^i = \{\alpha \in \mathbb{R}^d : |\alpha| \leq R_i\}, \quad \forall i,$$

with φ^i Lipschitz, $R_i > 0$, and the costs

$$L^i(x, \alpha) = c^i(x) \frac{|\alpha|^2}{2},$$

for some Lipschitz $c^i > 0$. Then

$$\begin{aligned} H^i(x, p) &= \max_{|\alpha| \leq R_i} \left\{ p \cdot \alpha - c^i(x) \frac{|\alpha|^2}{2} \right\} - p \cdot \varphi^i(x) \\ &= \begin{cases} |p|^2 / (2c^i(x)) - p \cdot \varphi^i(x), & \text{if } |p| \leq R_i c^i(x), \\ R_i |p| - c^i(x) R_i^2 / 2 - p \cdot \varphi^i(x), & \text{if } |p| > R_i c^i(x). \end{cases} \end{aligned}$$

Moreover

$$\bar{\alpha}^i(x, p) = \begin{cases} p / c^i(x), & \text{if } |p| \leq R_i c^i(x), \\ R_i p / |p|, & \text{if } |p| > R_i c^i(x) \end{cases}$$

satisfies (43) and (44). Also Remark 3.2 applies and we have again an explicit formula for the Nash equilibrium feedback. Note that H^i is unbounded from below if $\|\varphi^i\|_\infty > R_i$, in particular it does not satisfy the coercivity condition (29).

Note also that we can restrict further A^i to controls such that some components are null, say $\alpha_k = 0$ for $k = 1, \dots, d_1$, $d_1 \leq d$. Then the Hamiltonian is linear with respect to p_k , $k = 1, \dots, d_1$, but it still satisfies the assumptions of Corollary 3.3 and Theorem 3.4.

In the last example some controls are bounded and the others are unconstrained; the Hamiltonians satisfy neither condition C1 nor C2, but a solution of the system (1) exists by Remark 2.4 (ii).

Example 3.9. Take $d_1, d_2 \geq 0$ and such that $d_1 + d_2 = d$. Consider the system

$$f^i(x, \alpha) = \varphi^i(x) - \alpha, \quad A^i = \{(\alpha_1, \alpha_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : |\alpha_1| \leq R_i\}, \quad \forall i,$$

with φ^i Lipschitz, $R_i > 0$, and the costs

$$L^i(x, \alpha) = c^i \frac{|\alpha_1|^2}{2} + \frac{|\alpha_2|^\gamma}{\gamma},$$

for $c^i > 0$ and $\gamma > 1$. We write $p = (p_1, p_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and find that for $|p_1| > R_i c^i$ the Hamiltonian is

$$H^i(x, p) = R_i |p_1| + (\gamma - 1) \frac{|p_2|^{\gamma/(\gamma-1)}}{\gamma} - p \cdot \varphi^i(x) - c^i \frac{R_i^2}{2},$$

so neither the superlinearity condition (3) nor the quadratic growth (5) are satisfied if $\gamma > 2$. Denote $\varphi^i = (\varphi_1^i, \varphi_2^i) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then H^i verifies the coercivity condition (29) if $\|\varphi_1^i\|_\infty < R_i$. Moreover, H^i satisfies also (30) if R_i is large enough with respect to $\|D_{x_1} \varphi_1^i\|_\infty$ and $\|D\sigma^i\|_\infty$, as it is easy to check (for instance, (30) holds for any $R_i > 0$ if σ^i is constant and $D_{x_1} \varphi_1^i = 0$). By the calculations in the examples 3.7 and 3.8 also (43) and (44) hold true. Therefore, by Remark 2.4 (ii), the conclusions of Corollary 3.3 and Theorem 3.4 hold in this case as well.

3.5 A class of N -person games with discounted costs

Consider again the controlled stochastic dynamics (36) with the same assumptions on the regularity and periodicity of the drifts f^i and diffusions σ^i , $i = 1, \dots, N$. For any set of controls $(\alpha^i)_{i=1}^N$, the corresponding solution processes X_t^i , $i = 1, \dots, N$, are ergodic with invariant measures $m_i(x) = m_i(x; \alpha_i)$, $x \in Q$, see, e.g., [16, Theorem 3.2., p. 373]. The same is true for the joint process $X_t = (X_t^1, \dots, X_t^N)$ with invariant measure $\prod_{i=1}^N m_i(x^i)$.

Each player i seeks to minimize the discounted cost functional

$$J_{\rho^i}^i(X, \alpha^1, \dots, \alpha^N) = E \left[\int_0^\infty e^{-\rho^i t} (L^i(X_t^i, \alpha_t^i) + V^i[m](X_t^i)) dt \right]. \quad (58)$$

Here L^i , V^i , $i = 1, \dots, N$, are the same as in (38), (51), respectively, and $\rho^i > 0$ is the discount rate of the i -th player. Note that only the state of the i -th player appears explicitly in the cost, whereas the states of the other players influence it only through their invariant measure.

In order to find Nash points we are led to the following Hamilton-Jacobi-Bellman system of PDEs.

$$\begin{cases} \mathcal{L}^i v_i + \rho^i v_i + H^i(x, Dv_i) = V^i[m] \\ \mathcal{L}^{i*} m_i - \operatorname{div} (g^i(x^i, Dv_i) m_i) = 0 \quad \text{in } Q, \\ \int_Q m_i(x) dx = 1, \quad m_i > 0, \quad i = 1, \dots, N, \end{cases} \quad (59)$$

where \mathcal{L}^i , H^i are given by (41), (40), respectively, we assume (43), (44), and define as before g^i by (45).

Under the assumptions of Theorem 2.6 there exists a solution $v_1, \dots, v_N \in C^2(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, $1 \leq p < \infty$ to (59). Then a verification theorem similar to Theorem 3.4 gives the existence of feedback Nash equilibria for this game.

Proposition 3.10. *Let $\rho^1, \dots, \rho^N > 0$ be given discount rates. For any solution $v_1^{\rho^1}, \dots, v_N^{\rho^N} \in C^2(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, $1 \leq p < \infty$, of the system (59), the feedback law $\bar{\alpha}^i(x) := \bar{\alpha}^i(x, Dv_i^{\rho^i}(x))$, $i = 1, \dots, N$, provides a Nash equilibrium for all initial positions $X \in Q^N$. Moreover, for each $X = (x^i)_{i=1}^N \in Q^N$,*

$$v_i^{\rho^i}(x^i) = J_{\rho^i}^i(X, \bar{\alpha}^1, \dots, \bar{\alpha}^N) = E \left[\int_0^\infty e^{-\rho^i t} (L^i(\bar{X}_t^i, \bar{\alpha}^i(\bar{X}_t^i)) + V^i[m](\bar{X}_t^i)) dt \right], \quad (60)$$

where \bar{X}_t^i is the solution of (36) associated to the feedback $\bar{\alpha}^i$.

The theorem applies to all the examples of systems and running costs L^i presented in the preceding sections 3.3 and 3.4.

3.6 Stationary Mean-Field Games

For $N = 1$ the system of PDEs (1) is associated to a stationary Mean-Field Game if the Hamiltonian has the form (40) related to the vector field g by (45). For $N > 1$ it is related to Mean-Field Games involving N distinct homogeneous populations of agents. Each population has a large number of identical players, and each of the players has a controlled nonlinear dynamics affected by white noise and seeks to minimise his individual cost functional (the noises of different agents are independent). The function m_i in system (1) describes the density of the distribution of the i -th population.

For MFG models V^i typically has the form $G^i(x, K_1^i * m_1, \dots, K_N^i * m_N)$ with K_j^i Lipschitz kernels on Q and $G^i : Q \times \mathbb{R}^N \rightarrow \mathbb{R}$ locally Lipschitz, so that the general conditions of the Introduction are satisfied. If the Hamiltonians H^i and the vector fields g^i are, respectively, of the form (40) and (45) of Section 3.1, we have existence of a solution $\lambda_i, v_i, m_i, i = 1, \dots, N$, to (1) under the same assumptions of Corollary 3.3, but with V^i verifying (7) with $\beta = 1$ under condition C1 and $\beta = \alpha$ under condition C2.

Such solution can be used for the following verification theorem. Given a vector of probability densities m consider a generic agent of the i -th population with dynamics (36) and cost functional

$$\tilde{J}^i(X_0^i, \alpha^i, m) := \liminf_{T \rightarrow +\infty} \frac{1}{T} E \left[\int_0^T L^i(X_t^i, \alpha_t^i) + V^i[m](X_t^i) dt \right], \quad (61)$$

with V^i as above.

Proposition 3.11. *Let $\lambda_i, v_i, m_i, i = 1, \dots, N$ be a solution of the system (1) with H^i, g^i, V^i as above. Then*

$$\bar{\alpha}^i(x) := \bar{\alpha}^i(x, Dv_i(x)), \quad x \in \mathbb{R}^d, \quad i = 1, \dots, N, \quad (62)$$

is an optimal feedback for any agent of the i -th population for all initial positions $X_0^i \in Q$, i.e.,

$$\tilde{J}^i(X_0^i, \bar{\alpha}^i, m) \leq \tilde{J}^i(X_0^i, \alpha^i, m) \quad \text{for all admissible controls } \alpha^i.$$

Moreover, the probability density of the position of the i -th agent using such feedback is m_i .

Proof. It is enough to use a standard verification theorem in ergodic control (whose proof is very similar to that of Theorem 3.4). In fact, the i -th HJB equation of the system (1) is the Bellman equation of such a control problem with cost functional \tilde{J}^i . The second statement follows from the i -th KFP equation of the system (1) and the choice of g^i (45). \square

The meaning of the result is that a solution of (1) provides an equilibrium for the MFG in the sense that for the generic agent of the i -th population the optimal control $\bar{\alpha}^i$ produces a probability distribution of the agent that coincides with the distribution m_i of the whole i -th population.

Let us recall the connections between Mean-Field Games and games with large populations of identical players.

• Suppose we have only one population made of \tilde{N} identical players with dynamics $f^i(x, \alpha) = \alpha$, $\sigma^i(x) = \nu > 0$, $L^i = L$ for all $i = 1, \dots, \tilde{N}$, coupled only via the running costs $F^i(x^1, \dots, x^{\tilde{N}}) = V[\frac{1}{\tilde{N}-1} \sum_{j \neq i} \delta_{x_j}](x^i)$ where V is an operator on measures with the properties described in the Introduction and δ_{x_j} are Dirac measures, so the argument of V is the empirical distribution of the players different from i . Let $\lambda_i^{\tilde{N}}, v_i^{\tilde{N}}, m_i^{\tilde{N}}, i = 1, \dots, \tilde{N}$ be the solution of (1) given by Corollary 3.3. Then Lasry and Lions [34] showed that this sequence of solutions has a convergent subsequence as $\tilde{N} \rightarrow \infty$ and that the limit of each convergent subsequence solves the MFG PDEs with $N = 1$

$$\begin{cases} -\Delta v + H(x, Dv) + \lambda = V[m] & \text{in } Q, \\ -\Delta m - \operatorname{div}(D_p H(x, Dv)m) = 0 & \text{in } Q, \\ \int_{Q^d} m(x) dx = 1, \quad m > 0, \quad \int_{Q^d} v(x) dx = 0. \end{cases}$$

For linear-quadratic models a stronger convergence was proved by completely different methods in [5, 8].

- The same kind of limit result for general dynamics and N populations of identical players, as the number of players of each population goes to infinity, was proved by Feleqi [24] using the methods of [34] and the estimates of Section 2 of the present paper.
- A basic result of the approach to MFG by Huang, Caines and Malhame [31, 32] states that the optimal feedback synthesised from the MFG PDEs is an ϵ -Nash equilibrium for the \tilde{N} -person game for \tilde{N} large enough, see also [19].

4 Appendix: proofs of some technical results

Proof of Lemma 2.2. For any solution $(v, \lambda) \in C^{2,\alpha}(Q) \times \mathbb{R}$ of (9), we immediately deduce (10) by looking at the extrema of v .

Uniqueness. Thus $f \equiv 0$ implies $\lambda = 0$. On the other hand $v \equiv 0$ otherwise the strong maximum principle (e.g., [27, Theorem 3.5, p. 35]) would be contradicted.

Existence. Consider (9) with $\mathcal{L}_0 = -\Delta$ instead of \mathcal{L} . It is clearly solvable for any $f \in C^\infty$ with $\lambda = \int_Q f dx$ and v that can be determined by Fourier series. Moreover, by Bessel's identity one deduces $\|D^\beta v\|_p \leq \|f\|_p$ for all $2 \leq p < \infty$ and all multiindex β with length $|\beta| \leq 2$. By letting $p \rightarrow \infty$, this estimate holds also for $p = \infty$. Thus (9) with \mathcal{L}_0 instead of \mathcal{L} is solvable for any $f \in C(Q)$ and the solution $v \in C^2(Q)$. Actually, by Schauder interior estimates (see [27, Theorem 4.8, p. 62]) $f \in C^\alpha(Q)$ implies $v \in C^{2,\alpha}(Q)$. In order to apply a continuity method, see [27, Theorem 5.2, p. 75], and deduce the solvability of (9), we introduce the operators

$$\begin{aligned} T_i : C^{2,\alpha}(Q)/\mathbb{R} \times \mathbb{R} &\rightarrow C^\alpha(Q), & i = 0, 1, \\ (u, \lambda) &\rightarrow T_i(u, \lambda) = \mathcal{L}_i u + \lambda & \mathcal{L}_1 \equiv \mathcal{L}. \end{aligned}$$

Note that

$$[v]_{2,\alpha,Q} = \sup_{x,y \in Q, |\beta|=2} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x - y|^\alpha}$$

is a norm in $C^{2,\alpha}(Q)/\mathbb{R}$ equivalent to the natural one. Define also $T_s = (1 - s)T_0 + sT_1$, $0 \leq s \leq 1$. These operators are clearly linear and bounded. T_0 is also an isomorphism of

Banach spaces. We need only prove that they are bounded from below in order to finish. This requires a careful look at Schauder interior estimates, see [27, Theorem 6.2, p. 90]. Let $\Omega \supset Q$ be a ball of diameter D containing Q and having same center as Q . We need some notation from [27]. For any function g differentiable as many times as needed, if $d_x := \text{dist}(x, \partial\Omega)$, $d_{x,y} := \min\{d_x, d_y\}$, we set

$$\begin{aligned} [g]_{2,\alpha,\Omega}^* &= \sup_{x,y \in \Omega, |\beta|=2} d_{x,y}^{2+\alpha} \frac{|D^\beta g(x) - D^\beta g(y)|}{|x-y|^\alpha}, \\ [g]_{l,\alpha,\Omega}^{(j)} &= \sup_{x,y \in \Omega, |\beta|=l} d_{x,y}^{j+l+\alpha} \frac{|D^\beta g(x) - D^\beta g(y)|}{|x-y|^\alpha} \\ [g]_{l,\Omega}^{(j)} &= \sup_{x \in \Omega, |\beta|=l} d_x^{l+j} |D^\beta g(x)|, \quad |g|_{i,\alpha,\Omega}^{(j)} = \sum_{l=1}^i [g]_{l,\Omega}^{(j)} + [g]_{i,\alpha,\Omega}^{(j)} \end{aligned}$$

for any nonnegative integers i, j, l . Let $T_s(v, \lambda) = f$. Then, by [27, Theorem 6.2, p. 90],

$$[v]_{2,\alpha,\Omega}^* \leq C(\|v\|_{C(\Omega)} + |f - \lambda|_{0,\alpha,\Omega}^{(2)}), \quad (63)$$

where C depends only on a constant of ellipticity of $\mathcal{L}_s = (1-s)\mathcal{L}_0 + s\mathcal{L}_1$ which clearly can be taken to be independent of s , and at most linearly on

$$\max_{h,k} \{ |1-s + sa_{hk}|_{0,\alpha,\Omega}^{(0)}, |sb_h|_{0,\alpha,\Omega}^{(1)} \} \leq D^{1+\alpha} \max_{h,k} \{ \|a_{hk}\|_{C(Q)}, \|b_h\|_{C^\alpha(Q)} \},$$

where D is the diameter of Ω . Therefore, taking also into account (10), by (63) we obtain

$$[v]_{2,\alpha,Q} \leq C \frac{D^{1+\alpha}}{(D - \sqrt{d})^{2+\alpha}} (\|v\|_{C(Q)} + D^{2+\alpha} \|f\|_{C^\alpha(Q)})$$

for some C independent of D . Now it is sufficient to take an Ω with a suitably large diameter D in order to obtain (12). Finally, the $C^{1,\alpha}$ -estimate (11) follows from [27, Theorem 8.32, p. 210]. This concludes the proof of this lemma. \square

Remark 4.1. An alternative proof of the existence relies on the approximation by equations with zero-th order terms $\mathcal{L}v + \rho v = f$, for $\rho > 0$. If v^ρ is the solution (see Lemma 2.7), we prove that $(v^\rho - \int_Q v^\rho dx, \rho v^\rho) \rightarrow (v, \lambda)$ in $C^2(Q) \times C(Q)$ as $\rho \rightarrow 0$, where (v, λ) is a solution of (9), see, e.g., [11], where more general nonlinear equations are treated.

Proof of Lemma 2.3. That $m \in W^{1,2}(Q)$ exists, is unique and positive is proved, e.g., in [16, Theorem 3.4, 378] or [9, Theorem 4.2, p. 133, Theorem 4.3, p. 136]. (In [9] only the case $\mathcal{L} = -\Delta$ is treated but the techniques adapt easily to our operator.) The fact that $m \in W^{1,p}(Q)$ is also known, but for lack of a reference we sketch a proof here based on ideas of [40]. Split the operator $\mathcal{L} = \mathcal{L}_0 + R$ into a (formally) selfadjoint part $\mathcal{L}_0 = -D_k(a_{hk}(x)D_h)$ and a reminder $R = D_k a_{ak} D_h$. We use the deep fact that

$$\mathcal{L}_0 : W^{2,p}(Q)/\mathbb{R} \rightarrow L^p(Q)/\mathbb{R}, \quad 1 < p < \infty$$

is an isomorphism of Banach spaces. Then, by duality and interpolation,

$$\mathcal{L}_0 : W^{1,p}(Q)/\mathbb{R} \rightarrow W^{-1,p}(Q)/\mathbb{R}, \quad 1 < p < \infty$$

is also an isomorphism of Banach spaces. So we need only show that $\mathcal{L}_0 m \in W^{-1,p}(Q)$ in order to conclude.

Note that, by Sobolev's embedding lemma, $m \in L^p(Q)$, where $p > 2$ is given by $1/p = 1/2 - 1/d$ if $d \geq 3$, or, otherwise, for any $1 \leq p < \infty$. By the equation (13), we have $\mathcal{L}_0 m = -R^* m + \operatorname{div}(gm)$. Hence, for any $\varphi \in C^\infty(Q)$,

$$\begin{aligned} | \langle \mathcal{L}_0 m, \varphi \rangle | &= \left| \int_Q m D_k a_{hk} D_h \varphi dx - \int_Q m g \cdot D \varphi dx \right| \\ &\leq (\|g\|_\infty + C) \|m\|_p \|D \varphi\|_{p'} \end{aligned}$$

for some $C > 0$ (independent of g), $p' = p/(p-1)$. Therefore, $\mathcal{L}_0 m \in W^{-1,p}(Q)$ and $m \in W^{1,p}(Q)$. We are done if $d \leq 2$. Otherwise, again Sobolev's lemma implies that $m \in L^p(Q)$, where p now is given by $1/p = 1/2 - 2/d$ if $d \geq 5$, or, otherwise, for any $1 \leq p < \infty$. In the same manner we conclude that $m \in W^{1,p}(Q)$. Thus, by a bootstrap argument, we deduce that $m \in W^{1,p}(Q)$ for all $1 \leq p < \infty$.

Moreover, by the estimates above and the fact that $\mathcal{L}_0 : W^{1,p}(Q)/\mathbb{R} \rightarrow W^{-1,p}(Q)/\mathbb{R}$ is an isomorphism, we deduce that

$$\|m\|_{W^{1,p}(Q)} \leq C_1(\|g\|_\infty + C_2) \|m\|_p.$$

for some $C_1, C_2 > 0$ independent of g . Taking also into account [9, Theorem 4.3, p. 136] which states that $\delta_1 < m < \delta_2$ for some constants $\delta_1, \delta_2 > 0$ that depend only on $\|g\|_\infty$ (and in our case also on the coefficients a_{hk} , in a way which we do not specify because we will not need it in the sequel) we obtain (14). \square

Proof of Lemma 2.7. Uniqueness is standard. For the existence, we use a continuity method, e.g., [27, Theorem 5.2, p. 75], and Schauder a priori estimates to reduce to the equation corresponding to a simpler operator, say

$$\mathcal{L}_0 = -\Delta + 1.$$

That $\mathcal{L}_0 v = f$ has a solution $v \in C^\infty(Q)$ for each $f \in C^\infty(Q)$ can be shown, e.g., by Fourier series. Moreover, $\mathcal{L}_0 v = f$ implies $\|v\|_{C(Q)} \leq \|f\|_{C(Q)}$. Then, by a Schauder estimate, see [27, Theorem 4.8, p. 62], $\|v\|_{C^{2,\alpha}(Q)} \leq C\|f\|_{C^\alpha(Q)}$. Now for an arbitrary $f \in C^\alpha(Q)$ consider a sequence $\{f_n\} \subset C^\infty(Q)$ such that $f_n \rightarrow f$ in $C^\alpha(Q)$. The sequence of the corresponding solutions $\{v_n\}$ is Cauchy in $C^{2,\alpha}(Q)$ and its limit v verifies $\mathcal{L}_0 v = f$.

Next, introduce the family of operators

$$\mathcal{L}_s : C^{2,\alpha}(Q) \rightarrow C^\alpha(Q), \quad 0 \leq s \leq 1,$$

$$u \rightarrow \mathcal{L}_s u = (1-s)\mathcal{L}_0 u + s\mathcal{L}u.$$

For all $u \in C^{2,\alpha}(Q)$, by looking at its extrema, we have

$$\|u\|_{C(Q)} \leq \max\{1, \|1/c\|_{C(Q)}\} \|\mathcal{L}_s u\|_{C(Q)}.$$

Combining this with the interior Schauder estimates, see [27, Theorem 6.2, p. 90], we obtain

$$\|u\|_{C^{2,\alpha}(Q)} \leq C \|\mathcal{L}_s u\|_{C^\alpha(Q)}$$

for all $u \in C^{2,\alpha}(Q)$, $0 \leq s \leq 1$ and some $C > 0$ independent of u and s . Since \mathcal{L}_0 is onto, by the method of continuity [27, Theorem 5.2, p. 75], \mathcal{L}_1 is also onto, which is what we wanted to prove. With these considerations we also proved (33). \square

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