The line-tension approximation as the dilute limit of linear-elastic dislocations

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We prove that the classical line-tension approximation for dislocations in crystals, i.e., the approximation that neglects interactions at a distance between dislocation segments and accords dislocations energy in proportion to their length, follows as the Γ-limit of regularized linear-elasticity as the lattice parameter becomes increasingly small or, equivalently, as the dislocation measure becomes increasingly dilute. We consider two regularizations of the theory of linear-elastic dislocations: a core-cutoff and a mollification of the dislocation measure. We show that both regularizations give the same energy in the limit, namely, an energy defined on matrix-valued divergence-free measures concentrated on lines. The corresponding self-energy per unit length $\psi(b,t)$, which depends on the local Burgers vector and orientation of the dislocation, does not, however, necessarily coincide with the self-energy per unit length $\psi_0(b,t)$ obtained from the classical theory of the prelogarithmic factor of linear-elastic straight dislocations. Indeed, microstructure can occur at small scales resulting in a further relaxation the classical energy down to its $\mathcal{H}^1$-elliptic envelope.

1 Introduction

Dislocations are topological defects in crystals which mediate plastic deformation and store the attendant energy of cold work (cf., e.g., [33, 37] for authoritative reviews). Owing to this fundamental role, dislocations have been extensively studied by theoretical, experimental and computational means. The bulk of this extensive body of literature regards dislocations as line defects in otherwise linear-elastic crystals (e.g., [10, 47] for accounts of linear-elastic dislocation theory). Within this framework, the notion of *line tension*, i.e., the presumption that, to leading order, well-separated dislocations store energy proportionally to their length, plays a pervasive—and perhaps even fundamental—role.[1] Indeed, many models of yielding and hardening that, to this day, constitute the backbone of metal plasticity are based on the line-tension approximation. Examples include Taylor's formula for the dependence of the yield stress on dislocation density [63], Orowan's mechanism of precipitate strengthening [49], Read's theory of dislocation kinks [51] and Saada's theory of dislocation junctions [54], among others. The assumption that dislocations store internal energy in proportion to their length is also widely used as an indirect means of measuring dislocation densities in crystals through annealing and calorimetry [71].

There is an extensive mathematical literature concerned with variational models of linear-elastic dislocations. Ariza and Ortiz [9] have formalized the theory of discrete dislocations in general harmonic lattices using concepts of algebraic topology and lattice elasticity and have discussed the continuum limit of the discrete theory under different scaling scenarios. The simplest special case of the theory of Ariza and Ortiz [9] concerns a distribution of parallel screw dislocations in a cubic lattice, which reduces to a scalar two-dimensional model of point dislocations in the plane orthogonal to the dislocations. In this special case, Ponsiglione [50] has shown that, under logarithmic scaling, the energy converges—in the sense of Γ-convergence—to a continuum energy proportional to the number of dislocations. Subtracting this limit from the energy, Alicandro *et al.* [4] have obtained leading-order approximations of the interaction energies between parallel screw dislocations by Γ-expansion.

The case of parallel edge dislocations requires consideration of vectorial elasticity in the orthogonal plane. A linear-elastic model of a finite number of well-separated dislocations with a core-radius regularization was studied by De Luca *et al.* [22] and extended by Scardia and Zepperi [58] to nonlinear elasticity with subquadratic growth. The relation of the model to Ginzburg-Landau and spin models was discussed by Alicandro *et al.* [3]. Second-order expansions of the energy were employed by Leoni and Cermelli [17] to derive leading-order approximations of the interaction energies. A different energy scaling leading to a continuous distribution of dislocations was studied by Garroni *et al.* [27] within the geometrically linear theory and by Müller *et al.* [46] within the geometrically nonlinear theory, both with a core-radius regularization and assuming diluteness. The relation between the distribution of dislocations and the decomposition of the strain into an elastic and a plastic part in geometrically nonlinear models was addressed by Reina and Conti [52]. Dislocation pile-ups have been studied by Focardi and Garroni [25] and by Hall *et al.* [31] using

[1]It should be carefully noted that, due to directional dependence, the line tension of a dislocation line differs from its self-energy per unit length in general (cf. [68], also [33], p. 176), though the terms are sometimes used interchangeably. Thus, the term *'line-tension approximation'* should be more properly replaced by *'self-energy approximation'*. Despite this misnomer, in keeping with common practice we retain the more conventional designation of 'line-tension approximation' to refer to approximations in which the interaction part of the dislocation energy is neglected.

formal asymptotics, and by Geers *et al.* [30] using Γ-convergence.

Three-dimensional line dislocations restricted to a slip plane have been also studied within the framework of the Koslowski, Cuitiño and Ortiz (KCO) [40, 41] model of planar slip in a piecewise-quadratic Peierls potential. Macroscopic line-tension models have been rigorously derived from the KCO model for single slip [28, 29] for multiple-slip [16, 19] and for multi-planar slip [20]. A remarkable result of the analysis of Conti *et al.* [19] is that the effective self-energy may be strictly smaller than the classical self-energies of straight dislocations (cf., e.g., [11]) as a result of fine reconstruction of the dislocation line. These models also lead to energies concentrated on lines. However, the restriction of slip to one or more individual planes affords substantial simplifications in kinematics and in compactness properties. Thus, if slip is localized to the plane the dislocation line constitutes its jump set. The slip is then a BV function and BV-compactness conveniently applies. There is also a sizeable mathematical literature that focuses on non-variational approaches (cf., e.g., [23] and the references therein).

Despite this rapidly building mathematical literature and despite the pervasive and foundational role of the line-tension approximation within linear-elastic dislocation theory and physical metallurgy, a mathematical derivation of the line-tension approximation as a well-defined limit—and a clear demarcation of the conditions under which the limit is attained—have been unavailable for general three-dimensional distributions of dislocations. In this paper, we address this gap and show that the line-tension approximation indeed furnishes the Γ-limit of suitably regularized linear-elastic dislocation energy functions in the limit of dilute distributions of discrete dislocations in a fixed crystal lattice—the *dilute limit*—or, equivalently, in the zero lattice-constant limit—the *continuum limit*. Since these two approaches only differ by a rescaling we focus, for notational simplicity, to the limit as the lattice spacing tends to zero with a distribution of discrete dislocations weakly converging to a given asymptotic distribution of dislocations. The requisite sequence of energy functions whose limit we investigate is defined by regularization of linear elasticity on the scale ε of the lattice. We specifically use two conventional regularizations, namely: i) a core-cutoff regularization in which a narrow core of material is perforated around the dislocation line, and ii) a mollification of the dislocation density that accounts for the delocalized structure of dislocation cores on the scale of the lattice.

Relative to two-dimensional models of linear-elastic dislocation mechanics, such as cited in the foregoing, several additional difficulties arise in three dimensions. One first difficulty concerns the definition of the cell problem that determines the unrelaxed dislocation self-energy. This self-energy is determined by minimizing the elastic energy of a long hollow cylinder coaxial with a straight dislocation. The classical calculation of this self-energy is based on an *ad hoc* generalized plane-strain *ansatz* [11, 53]. The resulting dislocation self-energy per unit length diverges logarithmically in both the internal and external radii of the cylinder. The attendant prelogarithmic factor is quadratic in the Burgers vector and otherwise depends solely on the elastic moduli of the crystal. In the present work, we eliminate this *ad hoc* generalized plane-strain *ansatz*. To this end, we begin by furnishing a variational characterization of the elastic field in the hollow cylinder through minimizers of a problem defined on the unit circle S^1. We then consider the boundary-value problem on a cylinder of inner radius ε, outer radius R and height h coaxial with a dislocation line $\mathbb{R}t$. For this boundary-value problem, we show that, for many boundary conditions, the solutions are indeed well-approximated by the classical generalized plane-strain solution. In addition, the attendant elastic energy is proportional to the length of the cylinder and diverges as $\log(R/\varepsilon)$, with proportionality factor identical to the classical prelogarithmic factor.

Another difficulty resides in the structure and kinematics of the limiting functional. Whereas parallel straight dislocations and co-planar dislocations may be characterized as jump sets of BV functions, in three dimensions dislocations are rectifiable curves with vector-valued multiplicity and, for example, proving compactness is more challenging. Compactness and relaxation of functionals defined on curves have been studied by Conti *et al.* [18] within the framework of integral vector-valued currents (cf. [57] for a similar approach). In particular, a relaxation formula for the line-tension energy is derived by Conti *et al.* [18]. This relaxation formula generalizes the concept of BV-elliptic envelope pertaining to partition problems [5, 6], which is central to the study of dislocations in the plane.

Finally, compactness in three dimensions is more difficult than in two dimensions because the energy of two parallel dislocations with opposite Burgers vectors, or *dipole*, is very small if they are close, so the number of dipoles is not controlled by the energy. This difficulty may be overcome in two dimensions by recourse to a subtle energy lower bound that follows from methods developed for the study of Ginzburg-Landau vortices [55, 38, 2, 56]. An extension of these techniques to three-dimensional elasticity is beyond the scope of this work. Conveniently, the potential compactness deficit associated with dipoles is obviated in the limit of interest here, which is concerned with dilute distributions of dislocations. In this limit, dislocations are well-separated, away from junctions, on a scale that is intermediate between the lattice scale ε and the length scale of the domain Ω.

Our diluteness assumption means that dislocation structures have a characteristic length scale h which is intermediate between the lattice scale ε and the sample size L. This is a geometric constraint that can be understood as a bound on the curvature of dislocation lines and on the number of branching points. For simplicity, we implement the constraint by requiring that the dislocation lines be polygonals on the length scale h. Other means of enforcing the constraint are also possible and lead to identical results. We stress that we only assume $\varepsilon \ll h \ll L$, with $h \to 0$ as $\varepsilon \to 0$, so that, in the limit, dislocations can be located on any rectifiable curve γ.

Our main result is that, after logarithmic rescaling, the elastic energy converges, in the sense of Γ-convergence, to an energy of the form

$$E(\mu) := \int_\gamma \psi^*(b,t) d\mathcal{H}^1 \,, \tag{1.1}$$

where $\mu = b \otimes t \mathcal{H}^1 \llcorner \gamma$ is the divergence-free, matrix-valued, dislocation density, and ψ^* is the relaxation of the classical self-energy per unit length of straight dislocations, depending on the local Burgers vector b and orientation t of the dislocation curve γ, see Theorem 3.2 below for details. As already noted, the rescaling under consideration corresponds to the *dilute limit* of well-separated dislocations over a fixed lattice or, equivalently, to the *continuum limit* in a crystal of vanishingly small lattice parameter. Remarkably, in these limits dislocation interactions and interactions with the boundary give higher-order contributions and can be neglected to a first approximation.

This result is of both theoretical and practical significance. Thus, the result bears out the long-standing tenet that, to first order, well-separated linear-elastic dislocations store energy in proportion to their length. However, it bears emphasis that the dislocation self-energy per unit length may differ from the classical value for straight dislocations in general. Thus, depending on the elastic moduli of the crystal, the geometry of the lattice and the direction of the tangent relative to the Burgers vector, the dislocation line may relax by developing fine structure, with the result that the effective self-energy per unit length be less than the classical value. In other cases, this fine reconstruction does not occur and the effective dislocation self-energy per unit length coincides with the classical value. In this manner, the classical theory of the prelogarithmic factor and dislocation self-energies is both partially validated and invalidated by the analysis.

A far-reaching practical consequence of Theorem 3.2 concerns computational dislocation dynamics. Thus, a common approximation scheme in that field consists of discretizing the dislocation line into straight segments and then following their motion by means of physics-based mobility laws (cf., e.g., [15] and references therein). In the conventional implementation of the scheme, the linear-elastic interactions between every pair of segments are evaluated, which results in $O(N^2)$ calculations. However, in a well-annealed metal, the dislocation line density per unit volume may be as low as 10^4 mm^{-2} [37], with a mean distance between dislocations of the order of 10 μm. This mean dislocation distance is greatly in excess of the crystal lattice parameter, which is in the Angstrom scale. Even after considerable plastic deformation, the mean distance between dislocations is typically much larger than the lattice parameter, which potentially places the dislocation ensemble within the line-tension approximation regime. In this scenario, Theorem 3.2 implies that the macroscopic behavior of a crystal sample computed using the line-tension approximation, an $O(N)$ operation, is indistinguishable from the same macroscopic behavior computed accounting for all pairwise linear-elastic segment interactions at-a-distance, an

$O(N^2)$ operation. The rare intersections of dislocation lines that can occur with multiple active slip systems do not affect the energetics to leading order, but may nevertheless have an important effect on mechanical properties. This paradigm shift, when applicable, should result in considerable computational savings, especially for large dislocation ensembles and crystal samples.

This paper is organized as follows. In Section 2 we define precisely the elastic problem that we consider, including, in particular, the kinematics and the core regularization. In Section 3 we define our diluteness condition and state the main results. In Section 4, we study the solvability of the equilibrium problem for the elastic deformation in the presence of dislocations. We prove the standard $1/r$-asymptotic behavior of the elastic strain with distance r to the dislocation line and show that an elastic strain field exists in $L^{3/2}$ for all dislocation distributions. In addition, we produce an example of a dislocation distribution for which the strain field is not in L^p_{loc} for any $p > 3/2$, thus proving the optimality of the $3/2$ exponent. The cell problem of a hollow cylinder is studied in Section 5, where, in particular, we show that the solution of the corresponding one-dimensional problem on S^1 is indeed approximating. The proofs of the main results are then given in Section 6.

2 Regularized linear-elastic dislocations in crystals

We begin by summarizing the main elements of the geometrical theory of dislocations and the theory of linear-elastic dislocations with the aim of motivating the definitions given in Section 2.3 of the equilibrium problem of linear elasticity with dislocations and the appropriate space of dislocation measures. We consider throughout a free-standing crystal occupying a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. Since the equilibrium problem for a distribution of Volterrra dislocations is not variational, owing to the logarithmic divergence of the energy, we additionally define in Section 2.4 two regularizations of the problem that render energies finite and restore a variational structure. These regularizations insert, in an analytically tractable manner, additional physics into the problem beyond linear elasticity, namely, the discreteness of the lattice and the finiteness and structure of the dislocation core.

2.1 The geometrical theory of dislocations

Dislocations, such as considered here, are the result of crystallographic slip. The extent of crystallographic slip undergone by a crystal is described by a measure of the form

$$\beta^p = \delta \otimes \nu \, \mathcal{H}^2 \llcorner \Sigma, \tag{2.1}$$

where Σ is a 2-rectifiable subset of $\mathbb{R}^3$, ν is a unit normal field that orients Σ, $\delta : \Sigma \to \mathcal{B}$ is $\mathcal{H}^2 \llcorner \Sigma$-measurable, where $\mathcal{B}$ is the lattice consisting of all integer linear combinations of Burgers vectors. Equivalently, β^p can be interpreted as a rectifiable vector 2-current (cf., e.g., [44, 24] for the relevant definitions). In (2.1), Σ represents the slip surface, i.e., the surface of discontinuity in the displacement field u of the crystal, and δ is the displacement jump $[u]$ across Σ, defined in accordance with its orientation. In general, the possible slip surfaces and local Burgers vectors are constrained by crystallography, with Σ confined to certain *slip planes* characteristic of the crystal class and δ a translation vector of the lattice. A compilation of the commonly observed slip systems in a number of crystallographic classes may be found in [33].

We recall that the classical topological theory of dislocations (e.g., [64, 39, 43, 65, 34, 35, 36]) detects a dislocation of Burgers vector b supported on a closed curve γ through the *Burgers-circuit test*. Specifically, a line γ is a Volterra dislocation of Burgers vector b if the circulation obeys

$$\oint_C \beta^p dx + \mathrm{Link}(C,\gamma)\, b = 0, \tag{2.2}$$

for all oriented closed curves C, where $\mathrm{Link}(C_1, C_2)$ denotes the linking number of two oriented loops C_1 and C_2 in $\mathbb{R}^3$ (cf., e.g., [12] for definitions of the linking number and its relation to cohomology). The test circuits C used in (2.2) to detect the presence of a dislocation are known as Burgers circuits. The density of such dislocations is in turn described by the Nye dislocation measure μ [48].[2] The Nye dislocation measure is related to the plastic deformation β^p through Kröner's formula [42], namely,

$$\mu := -\,\mathrm{curl}\,\beta^p \tag{2.3}$$

where, following conventional terminology and notation for $\mathbb{R}^3$, $\mathrm{curl}\,\beta^p$ denotes the row-wise distributional curl of the measure β^p, which can also be interpreted as the boundary of the corresponding current. For sufficiently regular crystallographic plastic deformations of the type (2.1), the corresponding dislocation density is a rectifiable vector measure of the form

$$\mu = b \otimes t\, \mathcal{H}^1 \llcorner \gamma,$$

where γ is a 1-rectifiable line, $t : \gamma \to S^2$ is its tangent vector, $b : \gamma \to \mathcal{B}$ is $\mathcal{H}^1 \llcorner \gamma$-measurable, and Kröner's formula (2.3) specifically requires that

$$\int_\gamma b \cdot \varphi t \, d\mathcal{H}^1 = \int_\Sigma \delta \cdot (\mathrm{curl}\, \varphi)\nu \, d\mathcal{H}^2,$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^{3\times 3})$. It follows from the definition (2.3) that μ is exact and

$$\mathrm{div}\, \mu = 0$$

[2]In the theory of continuously distributed dislocations, μ is assumed to be absolutely continuous with respect to the Lebesgue measure and is expressed as $\mu = \alpha\, \mathcal{L}^3$, where α is Nye's dislocation density.

distributionally, i.e., the dislocation density is closed, or divergence-free. Here, following again standard terminology and notation for $\mathbb{R}^3$, $\operatorname{div}\mu$ is the distributional, row-wise divergence of μ. The measure μ can also be interpreted as a 1-current, as was done for example in [18], then the condition $\operatorname{div}\mu = 0$ implies that μ has no boundary.

2.2 The theory of linear-elastic dislocations

So far, the description of dislocations has been strictly geometrical. In order to characterize the equilibrium configurations of the crystal, a proper energy must additionally be associated with every dislocation measure. To this end, we note from Kröner's formula (2.3) that the dislocation measure μ effectively measures the failure of the plastic deformation β^p to be a gradient. Due to this incompatibility of the plastic deformation, the crystal must distort elastically in order to admit a placement in $\mathbb{R}^3$, represented by a displacement field $u : \Omega \to \mathbb{R}^3$. By virtue of this additional elastic distorsion, we have

$$Du = \beta^e + \beta^p,$$

where Du is the distributional derivative of u, presumed to be a measure, and β^e is the elastic distortion of the lattice. As a consequence of the elastic distortion, the crystal develops elastic stresses $\mathbb{C}\beta^e$, where the elastic moduli $\mathbb{C}$ define a symmetric linear map from $\mathbb{R}^{3\times 3}$ to itself, which vanishes on skew-symmetric matrices and is strictly positive definite on symmetric ones, in the sense specified in (2.10) below.

Let β^p be a plastic deformation with slip surface $\Sigma \subset \Omega$. The corresponding equilibrium problem for a free-standing, traction-free, crystal with prescribed plastic deformation β^p consists of finding displacement fields $u \in SBV(\Omega;\mathbb{R}^3)$, the space of special functions of bounded variation (cf., e.g., [7] for relevant background) with jump set $J_u \subset \Sigma$ and such that

$$\int_{\Omega\setminus\Sigma} \mathbb{C}Du \cdot D\varphi \, dx = 0, \qquad \forall \varphi \in C^1(\overline{\Omega};\mathbb{R}^3), \tag{2.4a}$$

$$[u] = \delta, \qquad \text{a. e. on } \Sigma, \tag{2.4b}$$

should any such displacement field exist. We note that, for any such equilibrium solution, Du is a measure and its action on $D\varphi$ is well-defined. Its singular part $D^s u$ is supported on Σ, while $\beta^e = Du - \beta^p = \beta\mathcal{L}^3$ is the regular part of Du. Here and below we denote by $\beta \in L^1(\Omega;\mathbb{R}^{3\times 3})$ the density of β^e with respect to $\mathcal{L}^3$, i.e., the elastic strain field. From (2.4a) and (2.4b), we have

$$\operatorname{div}\mathbb{C}\beta = 0 \quad \text{in} \quad \Omega\,, \tag{2.5}$$

and

$$(\mathbb{C}\beta)n = 0 \quad \text{on} \quad \partial\Omega\,, \tag{2.6}$$

where n is the outward unit normal on $\partial\Omega$. In particular, at equilibrium, the elastic stresses are in traction equilibrium across the slip surface Σ and on $\partial\Omega$. Under these conditions, from $\operatorname{curl} Du = 0$ distributionally, recalling the definition (2.3) we have the identity

$$\operatorname{curl} \beta^e = \mu,$$

in the sense of measures. In addition, the traction-free boundary conditions (2.5) can also be expressed in terms of the elastic strain β directly.

We thus conclude that, for a free-standing crystal in the presence of plastic slip, the equilibrium problem (2.4) can be expressed directly in terms of the elastic deformations as

$$\int_\Omega \mathbb{C}\beta \cdot D\varphi \, dx = 0, \qquad \forall \varphi \in C^1(\overline{\Omega}; \mathbb{R}^3), \tag{2.7a}$$

$$\int_\Omega \beta \cdot \operatorname{curl} \varphi \, dx = -\int_\Omega \varphi d\mu, \qquad \forall \varphi \in C^1(\overline{\Omega}; \mathbb{R}^{3\times 3}), \tag{2.7b}$$

to be solved for $\beta \in L^1(\Omega; \mathbb{R}^{3\times 3})$, should any such solutions exist.

The great advantage of problem (2.7) is that it is formulated entirely in terms of the dislocation measure μ. In particular, the slip surface Σ drops out of the problem. A similar reduction of the equilibrium problem can be effected for a crystal cell subject to periodic boundary conditions, and all subsequent results of analysis apply *mutatis mutandi* to the resulting periodic problem. However, it should be carefully noted that the calculation of the displacement field from the elastic deformations requires the specification of the slip surface. In addition, the displacement field cannot be dropped from the equilibrium problem when displacements are prescribed on part of the boundary. Furthermore, stronger-than-logarithmic singularities are to be expected in general when dislocations lie on the displacement boundary. These stronger singularities invalidate the subsequent analysis, unless a safe offset distance is enforced between dislocations and the displacement boundary.

We close this section by noting the connection between the reduced equilibrium problem (2.7) and elliptic problems with measure data. Such connection arises by treating problem (2.7) by duality. Formally, we begin by satisfying equilibrium, eq. (2.7a), by recourse to a stress potential $\psi : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\mathbb{C}\beta = \operatorname{curl} \psi,$$

subject to the additional gauge condition

$$\operatorname{div} \psi = 0. \tag{2.8}$$

The remaining curl constraint (2.7b) then becomes

$$\operatorname{curl} \mathbb{C}^{-1} \operatorname{curl} \psi + \mu = 0.$$

In addition, the traction free condition (2.5) and the gauge condition (2.8) jointly require that

$$\psi = 0 \qquad \text{on } \partial\Omega.$$

For instance, in the simple case of parallel screw dislocations, the stress potential is scalar, the gauge condition becomes trivial and the dual problem reduces to the Poisson equation with Dirichlet boundary conditions and measure data in the form of a collection of Dirac measures. There is an extensive literature on such problems (cf., e.g., [61, 21, 14] and references therein) that provides useful insights into the expected existence and regularity properties of the solutions.

2.3 The space of dislocation densities and the elastic energy

Based on the discussion in Section 2.1 we define the set of dislocation densities $\mathcal{M}_{\mathcal{B}}(\Omega)$:

Definition 2.1. *$\mathcal{M}_{\mathcal{B}}(\Omega)$ is the set of all divergence-free bounded measures $\mu \in \mathcal{M}(\Omega;\mathbb{R}^{3\times3})$ of the form*

$$\mu = b \otimes t\,\mathcal{H}^1 \llcorner \gamma, \tag{2.9}$$

with γ a 1-rectifiable subset of Ω, $t : \gamma \to S^2$ its tangent vector, and $b \in L^1(\gamma;\mathcal{B};\mathcal{H}^1\llcorner\gamma)$ the Burgers-vector field. Here $\Omega \subset \mathbb{R}^3$ is an open set and $\mathcal{B} \subset \mathbb{R}^3$ a discrete lattice.

The set $\mathcal{B}$ is an affine image of $\mathbb{Z}^3$ and will be fixed for the entire discussion in this paper. For technical reasons it is sometimes convenient to interpret the measures defined above as currents. These are tensor-valued currents and fall outside of the scope of the conventional theory of currents (e.g., [44, 24]). They have been recently studied by Conti *et al.* [18], a summary of relevant results from their work is presented in Section 6.1.

The divergence-free constraint specifically requires that

$$\int_\gamma b \cdot (D\varphi)t\, d\mathcal{H}^1 = 0$$

for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^3)$. This condition expresses a *conservation of Burgers vector* along the dislocation line and, in particular, implies that γ must be the union of a countable number of Lipschitz curves, b must be constant on each connected component of γ away from branching points, and in each branching point the oriented sum of Burgers vectors b must be zero [18, Th. 2.5], which corresponds to Frank's classical rule for dislocation nodes (cf., e.g., [33]).

Next we define the elastic energy associated to an elastic strain field β.

Definition 2.2. *We fix a linear map $\mathbb{C} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ such that*

$$\mathbb{C} = \mathbb{C}^T\,, \quad \mathbb{C}A\cdot A \geq c_0|A+A^T|^2 \text{ and } \mathbb{C}(A-A^T) = 0 \text{ for all } A \in \mathbb{R}^{3\times3}\,, \tag{2.10}$$

and, for any open set $\Omega \subset \mathbb{R}^3$ *and* $\beta \in L^2(\Omega; \mathbb{R}^{3\times 3})$*, define*

$$\mathcal{E}[\beta, \Omega] := \int_\Omega \frac{1}{2} \mathbb{C}\beta \cdot \beta \, dx \,. \tag{2.11}$$

Here and below $A \cdot B = \sum_{ij} A_{ij} B_{ij}$ denotes the Euclidean scalar product of (vectors and) matrices.

2.4 Regularized theories of linear-elastic dislocations

Problem (2.7) belongs to a classical class of problems in linear elasticity known as 'cut-surfaces problems', which can be solved directly using Green's functions [47, 10]. By an appeal to the J-integral [53] or by direct analysis of straight dislocations [11], it is found that dislocations are, necessarily, lines of singularity of the elastic field. Specifically, the stress and strain fields diverge as $1/r$ close to the dislocation line. Owing to this singular character, the total elastic energy of the crystal diverges logarithmically, which impedes efforts to characterize the linear-elastic field of dislocations variationally. Next, we introduce two commonly employed regularizations of the energy, based on a core cutoff and on mollification, that render energies finite and endow the problem with variational structure.

2.4.1 Core cutoff regularization

A conventional regularization consists of excluding in the computation of the strain energy a small tube of material, or core, around the dislocation line of radius ε, the 'cutoff radius' (cf., e.g., [33, 37]). The resulting strain energies then diverge logarithmically in ε and, if the dislocation distribution is not Burgers-vector neutral, in the size of the body. The introduction of a core is intended to account for the discreteness of the crystal lattice and its relaxation in the vicinity of the dislocation line. The cutoff radius is an ad-hoc parameter extraneous to linear elasticity that must be determined by fitting to experiment or to atomistic calculations (cf., e.g., [69, 70] for examples).

The regularized energy is

$$\mathcal{E}[\beta, \Omega^\varepsilon(\mu)] = \int_{\Omega^\varepsilon(\mu)} \frac{1}{2} \mathbb{C}\beta \cdot \beta \, dx,$$

where $\Omega^\varepsilon(\mu) := \{x \in \Omega : \ \mathrm{dist}(x, \operatorname{supp} \mu) > \varepsilon\}$. The energy of a dislocation measure μ is obtained minimizing over all admissible elastic deformations compatible with μ, namely,

$$E^c[\mu, \Omega] := \inf\{\mathcal{E}[\beta, \Omega^\varepsilon(\mu)] : \beta \in L^1(\Omega; \mathbb{R}^{3\times 3}), \ \mathrm{curl}\, \beta = \mu \text{ in } \Omega\} \,.$$

This minimization is meaningful provided that μ is sufficiently regular, for otherwise $\Omega^\varepsilon(\mu)$ may be empty. This requisite regularity is subsequently enforced by appending an appropriate diluteness condition. Equivalently, $E^c[\mu,\Omega]$ follows by minimizing $\mathcal{E}[\beta,\Omega^\varepsilon(\mu)]$ over all $\beta \in L^1(\Omega^\varepsilon(\mu);\mathbb{R}^{3\times 3})$ such that $\operatorname{curl}\beta = 0$ in $\Omega^\varepsilon(\mu)$, subject to appropriate circulation conditions of the form (2.2) (cf. Lemma 3.4).

2.4.2 Regularization by mollification

An alternative regularization of linear-elastic dislocations consist of distributing the dislocation core over a finite area, which corresponds to replacing the dislocation measure by a suitable mollification thereof. For instance, the body-centered-cubic (bcc) screw dislocation core undergoes a symmetry-breaking reconstruction resulting in a core that is distributed on three distinct planes [32, 67]. Dislocation cores can also undergo dissociation into partial dislocations separated by a stacking fault, anti-phase boundaries and other structures. In general, the core structure is not rigid by may depend on the dislocation character, the extent of loading, close-range interactions with other dislocations, obstacles, and other effects. These core effects are an important part of a number of dislocation structures and mechanisms such as the structure of dislocation nodes, core-constriction during cross slip, and others.

Here, we adopt a simple model of delocalized core structure by fixing a mollification kernel $\eta \in C_c^\infty(B_1)$ with $\int_{B_1} \eta\, dx = 1$, setting $\eta_\varepsilon(x) = \varepsilon^{-3}\eta(x/\varepsilon)$, and defining, for $\mu \in \mathcal{M}_\mathcal{B}(\Omega)$,

$$\begin{aligned} E^m[\mu,\Omega] := \inf\big\{\mathcal{E}[\beta,\Omega] : \beta \in L^1(\Omega;\mathbb{R}^{3\times 3}),\ \hat\mu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^n), \\ \mu = \hat\mu \llcorner \Omega \ \text{ and } \ \operatorname{curl}\beta = \hat\mu * \eta_\varepsilon \text{ in } \Omega\big\}. \end{aligned} \tag{2.12}$$

Notice that in this definition we minimize over both the strain β and the extension $\hat\mu$ of μ. Alternatively, one could fix an extension operator T and require $\hat\mu = T\mu$, or enforce $\operatorname{curl}\beta = \mu$ and compute the energy of a mollification of β, $\mathcal{E}[\beta * \eta_\varepsilon, \hat\Omega^{(\varepsilon)}]$, where $\hat\Omega^{(\varepsilon)} = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$. By linearity, these variants of the mollification model are equivalent up to boundary effects. In this work, we focus on (2.12) for definiteness.

Physically, the mollification of μ reflects the fact that the dislocation core cannot be described by linear elasticity. However, since the Γ-limit does not depend on the details of the mollification, it follows that, to leading-order, the energy does not depend on the precise structure of the core, e. g., whether the dislocations split into partials, spreads into planes in the Burgers vector zone or undergo some other type of reconstruction.

For dilute dislocations, the core-radius and mollification regularizations differ only in the core region, which does not influence the leading-order behavior. We shall make this equivalence precise by showing that, to leading order, both regularizations converge to the same limiting energy. However, it should be carefully noted that a finer analysis, e.g., including the next-order term in the Γ-expansion, would depend on the details of the core energy and, therefore, would be different for the two regularizations.

3 Statement of the main results

We begin by giving a precise definition of diluteness.

Definition 3.1. *Given two positive parameters $\alpha, h > 0$, a dislocation measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$, with $\Omega \subset \mathbb{R}^3$ open, is said to be (h,α)-*dilute *if there are finitely many closed segments $\gamma_j \subset \Omega$ and vectors $b_j \in \mathcal{B}$, $t_j \in S^2$ (with t_j tangent to γ_j) such that*

$$\mu = \sum_j b_j \otimes t_j \mathcal{H}^1 \llcorner \gamma_j$$

where the closed segments γ_j satisfy the properties:

(i). each γ_j has length at least h;

(ii). if γ_j and γ_k are disjoint then their distance is at least αh;

(iii). if the segments γ_j and γ_k are not disjoint then they share an endpoint, and the angle between them is at least α.

We let $\mathcal{M}_{\mathcal{B}}^{h,\alpha}(\Omega)$ be the space of all measures in $\mathcal{M}_{\mathcal{B}}(\Omega)$ which are (h,α)-dilute.

In what follows, the diluteness parameters h and α are chosen much larger then the core radius ε, in the sense that

$$\lim_{\varepsilon\to 0} \frac{\log(1/(\alpha_\varepsilon h_\varepsilon))}{\log(1/\varepsilon)} = \lim_{\varepsilon\to 0} \alpha_\varepsilon = \lim_{\varepsilon\to 0} h_\varepsilon = 0\,. \tag{3.1}$$

We restrict the energy to dilute dislocations, and scale it by $\log(1/\varepsilon)$ to extract the leading-order contribution. In the notation of Section 2.3 and Section 2.4, we define

$$F_\varepsilon^c[\mu,\Omega] := \begin{cases} \dfrac{1}{\log(1/\varepsilon)} E_\varepsilon^c[\mu,\Omega] & \text{if } \mu \in \mathcal{M}_{\mathcal{B}}^{h_\varepsilon,\alpha_\varepsilon}(\Omega), \\ \infty & \text{otherwise}\,, \end{cases} \tag{3.2}$$

and correspondingly $F_\varepsilon^m[\mu,\Omega]$.

Our main result is that both F_ε^c and F_ε^m Γ-converge to

$$F_0[\mu,\Omega] := \begin{cases} \displaystyle\int_\gamma \psi_0^{\mathrm{rel}}(b(x),t(x))\, d\mathcal{H}^1(x) & \text{if } \mu = b\otimes t\,\mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathcal{B}}(\Omega)\,, \\ \infty & \text{otherwise}\,, \end{cases} \tag{3.3}$$

where ψ_0^{rel} is the $\mathcal{H}^1$-elliptic envelope of ψ_0, defined as

$$\begin{aligned} \psi_0^{\mathrm{rel}}(b,t) := \inf\Big\{ \int_\gamma \psi_0(\theta(x),\tau(x))\, d\mathcal{H}^1(x) :\ & \nu = \theta\otimes\tau\mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathcal{B}}(B_{1/2}(0))\,, \\ & \mathrm{supp}(\nu - b\otimes t\mathcal{H}^1 \llcorner (\mathbb{R}t \cap B_{1/2}(0))) \subset\subset B_{1/2}(0)\Big\}. \end{aligned} \tag{3.4}$$

Here ψ_0 is the scaled self-energy per unit length of a straight dislocation, as defined in (3.5) below, see also Proposition 3.3. In particular, ψ_0 coincides with the self-energy per unit length from the classical theory of the prelogarithmic factor of linear-elastic dislocations (cf., e.g., [11]).

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz set, $\mathbb{C}$ as in (2.10). For every α_ε and h_ε obeying (3.1), the following holds:*

(i). (Compactness). If $F^c_\varepsilon[\mu_\varepsilon, \Omega] \le C$ or $F^m_\varepsilon[\mu_\varepsilon, \Omega] \le C$ for infinitely many ε, then there exist a subsequence, still denoted by μ_ε, and a measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$, such that μ_ε weakly-$*$ converges to μ.

(ii). (Γ-convergence). The energies F^c_ε and F^m_ε Γ-converge to F_0, in the sense that for any sequence $\varepsilon_j \to 0$, $\mu_j \overset{*}{\rightharpoonup} \mu$ one has

$$F_0[\mu, \Omega] \le \liminf_{j\to\infty} F^c_{\varepsilon_j}[\mu_j, \Omega]$$

and for any $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$ and any sequence $\varepsilon_j \to 0$ there is a sequence $\mu_j \overset{*}{\rightharpoonup} \mu$ such that

$$\limsup_{j\to\infty} F^c_{\varepsilon_j}[\mu_j, \Omega] \le F_0[\mu, \Omega]\,,$$

and the same for F^m_ε.

Proof. The compactness and the lower bound follow from Proposition 6.6 and the upper bound from Proposition 6.8 in Section 6. □

Finally, we turn to the characterization of the self-energy per unit length of a straight dislocation. For a given direction $t \in S^2$, we let Q_t be a rotation that transforms e_3 in t. For a given Burgers vector $b \in \mathbb{R}^3$, the unrelaxed self-energy per unit length is obtained solving the one-dimensional problem

$$\psi_0(b,t) := \min\left\{ \int_0^{2\pi} \frac{1}{2}\mathbb{C}G(\theta)\cdot G(\theta) d\theta \right\}. \tag{3.5}$$

The minimum is taken over all functions $G : (0, 2\pi) \to \mathbb{R}^{3\times3}$ of the form $G(\theta) := f(\theta) \otimes Q_t e_\theta + g \otimes Q_t e_r$, for some $f : (0, 2\pi) \to \mathbb{R}^3$ with $\int_0^{2\pi} f(\theta)\, d\theta = b$ and $g \in \mathbb{R}^3$, see Lemma 5.1. Here e_r, e_θ and e_3 denote the local basis in cylindrical coordinates, as defined in (5.3) below.

The self-energy per unit length can be also computed by considering the elastic energy of a dislocation in a cylinder. For $R, h > 0$, we consider the cylinder $T^h_R := Q_t(B'_R \times (0, h))$, where B'_R is the two-dimensional open ball of radius R centered in the origin.

Proposition 3.3. *For any $b \in \mathcal{B}$, $t \in S^2$, $R > 0$, one has*

$$\lim_{h\to\infty} \lim_{\varepsilon\to0} \frac{1}{h \log(R/\varepsilon)} E^c_\varepsilon[b \otimes t\mathcal{H}^1 \llcorner (\mathbb{R}t \cap T^h_R), T^h_R] = \psi_0(b,t)$$

and the same for E^m_ε.

Proof. By the definition (5.13) and Lemma 5.4 we have

$$E_\varepsilon^m[\mu, T_R^h] \geq E_\varepsilon^c[\mu, T_R^h] = \psi(b,t,h,\varepsilon,R)h \log \frac{R}{\varepsilon},$$

where $\mu = b \otimes t\mathcal{H}^1 \llcorner (\mathbb{R}t \cap T_R^h)$. The lower bounds follow then from Lemma 5.5. The upper bound for E^c is immediate from Lemma 5.1, and follows alternatively from Lemma 5.8; the upper bound for E^m is a special case of Lemma 5.11, taking $\beta = \beta_{b,t}$ (as defined in Lemma 5.1). $\square$

We conclude this introductory section by showing that the definition of E_ε^c can be also given using strains β defined only on $\Omega^\varepsilon(\mu)$, if the condition on the curl is replaced by a corresponding circulation condition. We make the condition precise by considering the full-space solution β^μ to the problem $\operatorname{div} \mathbb{C}\beta = 0$, $\operatorname{curl} \beta = \mu$, as defined in Theorem 4.1, and requiring that $\beta - \beta^\mu$ is a gradient field in $\Omega^\varepsilon(\mu)$.

Lemma 3.4. *If Ω is simply connected and $\mu \in \mathcal{M}_\mathcal{B}^{h_\varepsilon,\alpha_\varepsilon}(\Omega)$, for ε sufficiently small, then the functional $E_\varepsilon^c[\mu,\Omega]$ equals*

$$\inf\{\mathcal{E}[\beta, \Omega^\varepsilon(\mu)] : \beta \in L^1(\Omega^\varepsilon(\mu); \mathbb{R}^{3\times 3}), \beta - \beta^\mu = Dw, w \in W^{1,1}(\Omega^\varepsilon(\mu); \mathbb{R}^3)\}.$$

Proof. Let A be the quantity given in the statement. If $\beta \in L^1(\Omega; \mathbb{R}^{3\times 3})$ satisfies $\operatorname{curl} \beta = \mu$, then $\operatorname{curl}(\beta - \beta^\mu) = 0$ on Ω, and therefore it is a gradient field. This proves $A \leq E_\varepsilon^c[\mu, \Omega]$.

Consider now a $\beta \in L^1(\Omega^\varepsilon(\mu); \mathbb{R}^{3\times 3})$ with $\beta - \beta^\mu = Dw$ for some $w \in W^{1,1}(\Omega^\varepsilon(\mu); \mathbb{R}^3)$. By the diluteness condition, for sufficiently small ε the domain $\Omega^\varepsilon(\mu)$ is Lipschitz. Therefore, w can be extended to a function $\hat{w} \in W^{1,1}(\Omega; \mathbb{R}^3)$. We define $\hat{\beta} := \beta^\mu + D\hat{w}$, which then satisfies $\operatorname{curl} \hat{\beta} = \mu$, and obtain $E_\varepsilon^c[\mu, \Omega] \leq A$. $\square$

4 Construction of the strain field

In this section we show that, for every dislocation measure μ, it is possible to construct a strain field β with $\operatorname{curl} \beta = \mu$ and $\operatorname{div} \mathbb{C}\beta = 0$. We first study the problem in all of $\mathbb{R}^3$ (Theorem 4.1), where a representation formula in Fourier space is possible, and show that a solution $\beta \in L^{3/2}$ exists for any bounded measure μ. For measures concentrated on finitely many lines, such as those representing the dilute dislocations of interest, a higher integrability is possible, namely, L^p_{loc} for all $p < 2$. However, we produce measures $\mu \in \mathcal{M}_\mathcal{B}$ whose corresponding strain is not in L^p for any $p > 3/2$. We then proceed to treat the Neumann problem in a bounded domain (Proposition 4.2) .

Theorem 4.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ be a bounded measure with $\operatorname{div} \mu = 0$, $\mathbb{C}$ as in (2.10).*

(i). There is a unique $\beta \in L^{3/2}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ such that

$$\operatorname{div} \mathbb{C}\beta = 0 \text{ and } \operatorname{curl}\beta = \mu \tag{4.1}$$

distributionally. The solution β satisfies

$$\|\beta\|_{L^{3/2}(\mathbb{R}^3)} \le c|\mu|(\mathbb{R}^3)\,.$$

(ii). The solution β to (4.1) additionally obeys

$$|\beta(x)| \le c\frac{|\mu|(\mathbb{R}^3)}{\operatorname{dist}^2(x, \operatorname{supp}\mu)}\,.$$

(iii). If additionally $\mu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^3)$ and $\mu = \sum_i b_i \otimes t_i \mathcal{H}^1 \llcorner \gamma_i$ for countably many segments γ_i, then for $x \notin \operatorname{supp}\mu$

$$|\beta(x)| \le c\sum_i \frac{|b_i|}{\operatorname{dist}(x, \gamma_i)}\,. \tag{4.2}$$

If the number of segments is finite then $\beta \in L^p(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ for all $p \in [3/2, 2)$.

(iv). There is a measure $\mu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^3)$ such that the solution β to (4.1) does not belong to $L^p(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ for any $p > 3/2$.

The constant c depends only on $\mathbb{C}$.

Proof. (i): For $f \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ and $\operatorname{div} f = 0$ we seek a solution β to

$$\operatorname{div} \mathbb{C}\beta = 0 \text{ and } \operatorname{curl}\beta = f \text{ in } \mathbb{R}^3. \tag{4.3}$$

We pass to Fourier space and write the problem as

$$\begin{cases} \sum_{j,k,l=1}^3 \xi_j \mathbb{C}_{ijkl}\hat\beta_{kl} = 0 \\ \sum_{i,l=1}^3 \xi_i \epsilon_{ilm}\hat\beta_{kl} = \hat f_{km}\,, \end{cases} \tag{4.4}$$

where $\epsilon_{123} = 1$, $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj}$. The first identity gives 3 equations, the second 9. However, in the second identity only 6 equations can be independent, since both the left and the right-hand-side have zero divergence. Therefore, (4.4) can be reduced to 9 independent equations in the 9 unknowns $\hat\beta_{kl}$. The coercivity of $\mathbb{C}$ ensures that, for any $\xi \neq 0$, the linear system (4.4) is indeed uniquely solvable for $\hat\beta$. Indeed, should this not be the case, there would be a $\xi_* \neq 0$ and $\hat\beta \in \mathbb{R}^{3\times 3}$ such that $\xi_*\mathbb{C}\hat\beta = 0$ and $\xi_*\epsilon\hat\beta = 0$ in the sense of (4.4). In particular, the second equation means that $\hat\beta$, seen as a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$, maps all vectors orthogonal to ξ_* to zero. This property in turn means that $\hat\beta = \hat u \otimes \xi_*$ for some $\hat u \in \mathbb{R}^3$.

Taking the scalar product of $\xi_*\mathbb{C}\hat{\beta} = 0$ with $\hat{u}$, we obtain $\mathbb{C}(\hat{u}\otimes\xi_*)\cdot(\hat{u}\otimes\xi_*) = 0$, hence $\hat{u}\otimes\xi_*$ must be skew-symmetric. But there is no rank-one matrix which is skew-symmetric. Therefore, $\hat{\beta} = 0$.

We define $\omega : S^2 \to \mathbb{R}^{3\times3\times3\times3}$ as the inverse of the coefficient matrix, after dividing by $|\xi|$, so that the solution to (4.4) takes the form

$$\hat{\beta}_{kl}(\xi) = \frac{1}{|\xi|}\sum_{m,n=1}^{3} \omega_{klmn}\left(\frac{\xi}{|\xi|}\right)\hat{f}_{mn}(\xi)\,. \tag{4.5}$$

The function ω is smooth, since it arises by inverting an invertible matrix which depends smoothly on ξ. Since f has compact support, its Fourier transform $\hat{f}$ is smooth. Then (4.5) shows that $\hat{\beta}$ diverges at most as $1/|\xi|$ at small ξ. In particular, it is integrable. Therefore, the inverse Fourier transform of this expression defines a solution β to (4.3) for any $f \in C_c^\infty$ with $\operatorname{div} f = 0$.

In order to produce quantitative estimates for β in terms of f, we write the linear map in (4.5) as the composition of two maps. The first map, called Φ, is $(-\Delta)^{-1}\operatorname{curl}$. For this map there are precise estimates. In particular, it maps L^1 into $L^{3/2}$, see [13, 66, 14]. The outer map, called N, depends on the detailed elastic constants $\mathbb{C}$. However, it is of order zero and therefore maps $L^{3/2}$ into itself. Precisely, we define

$$\hat{N}_{klmn}(\xi) := -\sum_{i,j=1}^{3}\omega_{klmi}\left(\frac{\xi}{|\xi|}\right)\frac{\epsilon_{nji}\xi_j}{|\xi|}\,, \qquad \hat{\Phi}_{np}(\xi) := \sum_{h=1}^{3}\frac{\xi_h\epsilon_{nph}}{|\xi|^2}\,,$$

so that, recalling that $\operatorname{div} f = 0$,

$$\hat{\beta}_{kl}(\xi) = \sum_{m,n,p=1}^{3}\hat{N}_{klmn}(\xi)\hat{\Phi}_{np}(\xi)\hat{f}_{mp}(\xi)\,.$$

To verify this identity, it is helpful to recall that for any vector w with $\xi\cdot w = 0$ we have

$$\xi\times(\xi\times w) = \xi(\xi\cdot w) - (\xi\cdot\xi)w = -|\xi|^2 w\,.$$

We define g by $\hat{g}_{ij} := \sum_k \hat{\Phi}_{jk}\hat{f}_{ik}$. Then $g = (-\Delta)^{-1}\operatorname{curl} f$ satisfies $\operatorname{div} g = 0$ and $\operatorname{curl} g = f$, and by [13, Theorem 2] we have $g \in L^{3/2}(\mathbb{R}^3)$ with

$$\|g\|_{L^{3/2}(\mathbb{R}^3)} \le c\|f\|_{L^1(\mathbb{R}^3)}\,.$$

Since $\hat{N}$ is a zero-homogeneous smooth multiplier, by [62, Theorem 3, page 96] the associated linear map $\mathcal{F}^{-1}\hat{N}\mathcal{F}$ maps L^p to L^p for all $p \in (1,\infty)$ and, in particular,

$$\|\beta\|_{L^{3/2}(\mathbb{R}^3)} \le c\|g\|_{L^{3/2}(\mathbb{R}^3)} \le c\|f\|_{L^1(\mathbb{R}^3)}$$

for any divergence-free $f \in C_c^\infty$. Both maps can be extended by density to the case in which μ is a measure. Hence, we find that, for any divergence-free bounded measure μ, the corresponding β obeys

$$\|\beta\|_{L^{3/2}(\mathbb{R}^3)} \le c|\mu|(\mathbb{R}^3)\,.$$

It remains to prove uniqueness. If the solution were not unique, there would exist a non-constant $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$ with $Du \in L^{3/2}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ and $\operatorname{div} \mathbb{C}Du = 0$. After mollification, we can assume $u \in C^\infty$. For $R > 0$, let a_R be the average of u over $C_R = B_{2R} \setminus B_R$ and fix $\varphi \in C^1_c(B_{2R})$ with $\varphi = 1$ on B_R, $|D\varphi| \le 2/R$. Testing the equation with $(u - a_R)\varphi$ gives

$$c\|Du + Du^T\|^2_{L^2(B_R)} \le \frac{c}{R}\|Du\|_{L^{3/2}(C_R)}\|u - a_R\|_{L^3(C_R)}\,.$$

By the Sobolev embedding theorem, $\|u - a_R\|_{L^3(C_R)} \le c\|Du\|_{L^{3/2}(C_R)}$. Therefore, taking $R \to \infty$ proves that $Du + Du^T = 0$ and, hence, u is constant.

(ii): We now investigate in more detail the regularity of β. As above, we set $g = (-\Delta)^{-1} \operatorname{curl} \mu$, or equivalently $\hat{g} = \hat{\Phi}\hat{\mu}$, so that $\hat{\beta} = \hat{N}\hat{g}$. Since $\hat{N}$ is zero-homogeneous and smooth away from the origin, by [62, Theorem 6, page 75] we have that

$$\hat{\beta}_{ij} = \sum_{k,l} \hat{N}_{ijkl}\hat{g}_{kl},$$

is equivalent to

$$\beta_{ij}(x) = \gamma g_{ij}(x) + \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3\setminus B_\varepsilon(x)} \sum_{k,l} G_{ijkl}(x-y) g_{kl}(y) dy,$$

for some $\gamma \in \mathbb{R}$ and some function of the form

$$G_{ijkl}(x) = \frac{1}{|x|^3} G_{ijkl}\left(\frac{x}{|x|}\right)\,,$$

which is smooth away from the origin and has average zero on S^2. Simultaneously, it is classical that

$$g_{kl}(x) = \lim_{\delta\to 0} \int_{\mathbb{R}^3\setminus B_\delta(x)} \sum_m \Phi_{lm}(x-y) d\mu_{km}(y),$$

with $\Phi_{lm}(x) := -\sum_k \epsilon_{lmk} x_k/(4\pi|x|^3)$. Since $\Phi \in L^1_{\mathrm{loc}}$ and μ is a bounded measure, we can take $\delta = 0$. In particular, for almost every x,

$$\begin{aligned}\beta_{ij}(x) = &\sum_{k,l,m} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3\setminus B_\varepsilon(x)} \int_{\mathbb{R}^3} G_{ijkl}(x-y)\Phi_{lm}(y-z) d\mu_{km}(z) dy \\ &+ \gamma \sum_m \int_{\mathbb{R}^3} \Phi_{jm}(x-y) d\mu_{im}(y)\,.\end{aligned}$$

We swap the integrals in the first line, in order to convolve a global kernel with μ. By Fubini's theorem,

$$\beta_{ij}(x) = \sum_{k,m} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} (N^*_\varepsilon)_{ijkm}(x-z) d\mu_{km}(z) + \gamma \sum_m \int_{\mathbb{R}^3} \Phi_{jm}(x-y) d\mu_{im}(y),$$

where we have defined

$$(N^*_\varepsilon)_{ijkm}(x) := \sum_l \int_{\mathbb{R}^3\setminus B_\varepsilon(x)} G_{ijkl}(x-y)\Phi_{lm}(y)dy\,.$$

The function N^*_ε is continuous and satisfies

$$N^*_\varepsilon(x) = \lambda^2 N^*_{\lambda\varepsilon}(\lambda x) \text{ for any } \lambda, \varepsilon > 0,\, x \in \mathbb{R}^3. \tag{4.6}$$

If $0 < \delta < \varepsilon \le \frac{1}{2}|x|$, then, since each component of G has average zero on S^2,

$$\begin{aligned}|N^*_\delta - N^*_\varepsilon|(x) &= \left|\int_{B_\varepsilon(x)\setminus B_\delta(x)} G(x-y)\Phi(y)dy\right| \\ &= \left|\int_{B_\varepsilon(x)\setminus B_\delta(x)} G(x-y)(\Phi(y)-\Phi(x))dy\right| \le \frac{c\varepsilon}{|x|^3}\|G\|_{L^\infty(S^2)}\,,\end{aligned}$$

where we have used $|\Phi(y)-\Phi(x)| \le c|y-x|/|x|^3$. Therefore $\varepsilon \mapsto N^*_\varepsilon(x)$ converges for all $x \neq 0$. We define

$$N^*(x) := \lim_{\varepsilon\to 0} N^*_\varepsilon(x)$$

Assume now that $0 < \delta < \frac{1}{2}|x| < \varepsilon$. Then,

$$\begin{aligned}|N^*_\delta - N^*_\varepsilon|(x) \le &\left|\int_{B_\varepsilon(x)\setminus B_{|x|/2}(x)} G(x-y)\Phi(y)dy\right| \\ &+ \left|\int_{B_{|x|/2}(x)\setminus B_\delta(x)} G(x-y)(\Phi(y)-\Phi(x))dy\right|\,.\end{aligned}$$

In the first term, we proceed to estimate the integrand by $c\|G\|_{L^\infty(S^2)}|x-y|^{-3}|y|^{-2}$. To this end, we separate the contribution in $B_{2|x|}(x)$ from the rest. If $|y-x| \ge 2|x|$, then also $|y-x| \le 2|y|$ and, therefore,

$$\int_{B_\varepsilon(x)\setminus B_{2|x|}(x)} \frac{1}{|x-y|^3|y|^2}dy \le \int_{B_\varepsilon(x)\setminus B_{2|x|}(x)} \frac{4}{|x-y|^5}dy \le \frac{c}{|x|^2}.$$

If $\varepsilon < 2|x|$, then the integral is zero and the estimate is still true. The second part is

$$\int_{B_{2|x|}(x)\setminus B_{|x|/2}(x)} \frac{1}{|x-y|^3|y|^2}dy \le \frac{8}{|x|^3}\int_{B_{3|x|}(0)} \frac{1}{|y|^2}dy \le \frac{c}{|x|^2}\,.$$

Finally,

$$\int_{B_{|x|/2}(x)} |G|(x-y)|\Phi(y)-\Phi(x)|dy \le \int_{B_{|x|/2}(x)} \frac{c\|G\|_{L^\infty(S^2)}}{|x-y|^2|x|^3}dy \le \frac{c\|G\|_{L^\infty(S^2)}}{|x|^2}\,.$$

We conclude,

$$|N^* - N_\varepsilon^*|(x) \le c \min\left\{\frac{\varepsilon}{|x|^3}, \frac{1}{|x|^2}\right\}$$

which by dominated convergence implies $N_\varepsilon^* \to N^*$ in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$, in $L^p(\mathbb{R}^3)$ for any $p \in (1, 3/2)$, and locally uniformly in $\mathbb{R}^3 \setminus \{0\}$. Therefore,

$$\beta(x) = \int_{\mathbb{R}^3} N(x-y) d\mu(y)\,, \tag{4.7}$$

where

$$N(x) = \gamma\Phi(x) + N^*(x) = \gamma\Phi(x) + \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3 \setminus B_\varepsilon(x)} G(x-y)\Phi(y) dy$$

is continuous away from the origin. Setting $\lambda = 1/|x|$ in (4.6) and taking the limit $\varepsilon \to 0$, readily gives

$$N(x) = \frac{1}{|x|^2} N\left(\frac{x}{|x|}\right)$$

and, therefore, (4.7) yields

$$|\beta|(x) \le \frac{\|N\|_{L^\infty(S^2)}\, |\mu|(\mathbb{R}^3)}{\mathrm{dist}^2(x, \operatorname{supp}\mu)}\,.$$

(iii) Assume now that $\mu = \sum_i b_i \otimes t_i \mathcal{H}^1 \llcorner \gamma_i$, with $\gamma_i = a_i + [0, l_i] t_i$, $t_i \in S^2$, $l_i \in [0, \infty)$, $a_i \in \mathbb{R}^3$. Then, (4.7) implies

$$|\beta|(x) \le \|N\|_{L^\infty(S^2)} \sum_i |b_i| \int_0^{l_i} \frac{1}{|x - (a_i + s t_i)|^2} ds\,.$$

For every i, let x_i be the projection of x onto the straight line $a_i + \mathbb{R}t_i$. If $x_i \in \gamma_i$, then $|x - x_i| = \mathrm{dist}(x, \gamma_i) = \mathrm{dist}(x, a_i + \mathbb{R}t_i)$ and

$$\begin{aligned}\int_0^{l_i} \frac{1}{|x - (a_i + s t_i)|^2} ds &\le \int_{\mathbb{R}} \frac{1}{|x - (a_i + s t_i)|^2} ds \\ &= \int_{\mathbb{R}} \frac{1}{|x - x_i|^2 + s^2} ds = \frac{\pi}{\mathrm{dist}(x, \gamma_i)}.\end{aligned}$$

Otherwise, we can assume $a_i = x_i + s_0 t_i$, with $s_0 > 0$. Then, $x - x_i$ is orthogonal to t_i and we can estimate

$$\begin{aligned}\int_0^{l_i} \frac{1}{|x - (a_i + s t_i)|^2} ds &= \int_{s_0}^{s_0 + l_i} \frac{1}{|x - (x_i + s t_i)|^2} ds \\ &\le \int_{s_0}^{\infty} \frac{1}{|x - x_i|^2 + s^2} ds = \frac{1}{|x - x_i|}\left(\frac{\pi}{2} - \mathrm{atn}\,\frac{s_0}{|x - x_i|}\right).\end{aligned}$$

If $s_0 \le |x-x_i|$, then $\mathrm{dist}(x,\gamma_i) \le \sqrt{2}|x-x_i|$ and

$$\int_0^{l_i} \frac{1}{|x-a_i+st_i|^2} ds \le \frac{\pi}{2|x-x_i|} \le \frac{\pi}{\mathrm{dist}(x,\gamma_i)}\,.$$

Otherwise, since $\theta \le \tan\theta$ for $\theta \in [0,\pi/2)$ implies $\frac{\pi}{2} - \mathrm{atn}\, t \le 1/t$ for all $t>0$, we have

$$\int_0^{l_i} \frac{1}{|x-(a_i+st_i)|^2} ds \le \frac{1}{s_0} \le \frac{1}{\mathrm{dist}(x,\gamma_i)}\,.$$

Collecting terms,

$$|\beta|(x) \le \|N\|_{L^\infty(S^2)} \sum_i \frac{\pi |b_i|}{\mathrm{dist}(x,\gamma_i)}\,.$$

Using Fubini one readily checks that $\int_{B_2} \mathrm{dist}^{-p}(x,[0,1]e_3)dx < \infty$ for all $p \in [1,2)$, which after a change of variables implies $\beta \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$ if the sum is finite.

(iv): It suffices to construct μ such that $g = (-\Delta)^{-1}\mathrm{curl}\mu \notin L^p(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ for every $p > 3/2$. Indeed, in the notation of the proof of (i), the linear operator $\omega(\xi)$ in (4.5) is by definition invertible, therefore so is $\hat{N}(\xi)$, for every $\xi \ne 0$, an invertible linear operator and $\mathcal{F}^{-1}\hat{N}^{-1}\mathcal{F}$ is a bounded linear map from L^p to L^p for all $p \in (1,\infty)$.

We first consider a measure μ_r concentrated on a circle of radius $r > 0$ in the x_1-x_2 plane, centered in the origin, with Burgers vector $b \in \mathbb{Z}^3$, $\mu_r := b \otimes e_\theta^\perp \mathcal{H}^1 \llcorner \{re_\theta : \theta \in [0.2\pi)\}$, where $e_\theta := (\cos\theta, \sin\theta, 0)$ and $e_\theta^\perp := (-\sin\theta,\cos\theta,0)$. The corresponding solution g_r is given by

$$g_r(x) := br \otimes \int_0^{2\pi} \frac{(x-re_\theta)\wedge e_\theta^\perp}{4\pi|x-re_\theta|^3} d\theta\,.$$

By series expansion and using that the average of $x \wedge e_\theta^\perp$ over the circle is zero, one easily verifies that

$$|g_r(x)| \le \frac{cr^2|b|}{|x|^3} \qquad \text{if } |x| \ge 2r\,.$$

Simultaneously, $g_r \in C^0(\mathbb{R}^3 \setminus B_{2r};\mathbb{R}^{3\times 3})$ and, for every $t \in \mathbb{R}$,

$$g_r(te_3) = br \otimes \int_0^{2\pi} \frac{te_3 \wedge e_\theta^\perp - re_3}{4\pi(t^2+r^2)^{3/2}} d\theta = -b \otimes e_3 r^2 \frac{1}{2(t^2+r^2)^{3/2}}\,,$$

since the average of $e_\theta^\perp$ is zero. By continuity we obtain that $|g_r|(|x|) \ge c|b|/r$ in a neighbourhood of $[2r,3r]e_3$ and, therefore,

$$\int_{B_{3r}\setminus B_{2r}} |g_r|^p dx \ge c|b|^p r^{3-p}\,, \tag{4.8}$$

where, by scaling, the constant does not depend on r.

We choose a sequence of radii $r_k = 2^{-k}/k^2$ and choose points y_k such that the balls $B(y_k, 3r_k^{1/3})$ are disjoint and fix $b_k = 2^k e_1$. We let μ_k be a measure concentrated on a circle of radius r_k centered in y_k and orthogonal to e_3, with multiplicity b_k, and let g_k be the corresponding solution as discussed above, $\mu := \sum_k \mu_k$ and $g := \sum_k g_k$. It is easy to see that $|\mu|(\mathbb{R}^3) = \sum_k 2\pi|b_k|r_k < \infty$ and $\mu \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3)$.

For any given k and any $x \in B_{3r_k}(y_k)$ we have

$$\sum_{j\neq k} |g_j| \leq c \sum_{j\neq k} \frac{r_j^2|b_j|}{(|y_k - y_j| - 3r_k)^3} .$$

Since $|y_k - y_j| \geq 3r_k^{1/3} + 3r_j^{1/3} \geq 3r_k + 3r_j^{1/3}$, we have

$$\sum_{j\neq k} |g_j| \leq c \sum_{j\neq k} \frac{r_j^2|b_j|}{r_j} \leq c \sum_j r_j|b_j| = M < \infty ,$$

with $M = c'|\mu|(\mathbb{R}^3)$. Therefore, recalling (4.8),

$$\int_{B_{3r_k}(y_k)\setminus B_{2r_k}(y_k)} |g|^p dx \geq c|b_k|^p r_k^{3-p} - c'' r_k^3 M^p .$$

Summing over all k, we conclude

$$\begin{aligned}\int_{\mathbb{R}^3} |g|^p dx &\geq c \sum_k |b_k|^p r_k^{3-p} - c'' M^p \sum_k r_k^3 \\ &= c \sum_k \frac{2^{(2p-3)k}}{k^{2(3-p)}} - c'' M^p \sum_k \frac{2^{-3k}}{k^6} = \infty.\end{aligned}$$

□

In the next Proposition we show how the method of duality solutions (see [61]) can be used to solve the equilibrium equations in the presence of dislocations in a bounded domain.

Proposition 4.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^3; \mathbb{R}^{3\times3})$, $\Omega \subset \mathbb{R}^3$ be Lipschitz, simply connected, bounded, $\mathbb{C}$ as in (2.10). For every $p \in (1, 3/2]$, the problem*

$$\begin{cases} -\operatorname{div} \mathbb{C}\beta = 0 & \text{in } \Omega \\ \operatorname{curl} \beta = \mu & \text{in } \Omega \\ \mathbb{C}\beta \cdot n = 0 & \text{on } \partial\Omega \end{cases} \tag{4.9}$$

has a unique distributional solution $\beta \in L^p(\Omega; \mathbb{R}^{3\times3})$. All such solutions coincide and

$$\|\beta\|_{L^{3/2}(\Omega)} \leq c|\mu|(\mathbb{R}^3) .$$

The constant c depends only on Ω and $\mathbb{C}$.

Proof. Let $\beta^\mu \in L^{3/2}(\mathbb{R}^3;\mathbb{R}^{3\times3})$ be the $\mathbb{R}^3$ solution to $\operatorname{curl}\beta = \mu$, $\operatorname{div}\mathbb{C}\beta = 0$ from Theorem 4.1. Since $\operatorname{curl}(\beta-\beta^\mu) = 0$, β solves the system (4.9) if and only if $\beta = \beta^\mu + Du$ for some $u \in W^{1,1}(\Omega;\mathbb{R}^3)$ such that

$$\begin{cases} -\operatorname{div}\mathbb{C}Du = 0 & \text{in } \Omega\,, \\ \mathbb{C}Du\cdot n = -\mathbb{C}\beta^\mu\cdot n & \text{on } \partial\Omega\,. \end{cases} \tag{4.10}$$

If $\mathbb{C}\beta^\mu n \in H^{-1/2}(\partial\Omega)$, then the system (4.10) has a unique classical weak solution $u \in W^{1,2}(\Omega;\mathbb{R}^3)$. Therefore, we obtain a solution $\beta \in L^{3/2}(\Omega)$ of (4.9) provided that $\mu \in \mathcal{M}_\mathcal{B}$ and μ is transversal to the boundary, in the sense that no segment of its support is tangent to $\partial\Omega$. However, this argument does not lead to a uniform estimate for the norm of β.

In general, we require a weaker concept of solution to (4.10). Since $\beta^\mu \in L^{3/2}$, the operator

$$h \mapsto L(h) := \int_\Omega \mathbb{C}\beta^\mu Dh\,dx\,, \qquad h \in W^{1,3}(\Omega;\mathbb{R}^3)\,, \tag{4.11}$$

defines a bounded linear map on $W^{1,3}$ and therefore an element of its dual space. We easily see that

$$\|L\|_{(W^{1,3}(\Omega;\mathbb{R}^3))^*} \le c\|\beta^\mu\|_{L^{3/2}} \le c|\mu|(\mathbb{R}^3)\,.$$

Here, $(W^{1,p})^*$ denotes the dual of the Banach space $W^{1,p}$ and $\langle\cdot,\cdot\rangle_{(1,p),(1,p)^*}$ denotes the corresponding duality pairing. Note that $(W^{1,p})^* \subset W^{-1,p'} = (W^{1,p}_0)^*$, where p' is defined by $1/p' + 1/p = 1$.

We say that $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ is a p-duality solution of (4.10), for $p \in (1,3/2]$, if

$$\langle u,\psi\rangle_{(1,p),(1,p)^*} = \int_\Omega \mathbb{C}\beta^\mu D\varphi\,dx$$

for all $\psi \in (W^{1,p}(\Omega;\mathbb{R}^3))^*$ and $\varphi \in W^{1,p'}(\Omega;\mathbb{R}^3)$ being the unique solution of

$$\begin{cases} -\operatorname{div}\mathbb{C}D\varphi = \psi & \text{in } \Omega\,, \\ \mathbb{C}D\varphi\cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

with $\int_\Omega \varphi\,dx = 0$, in the sense that

$$\int_\Omega \mathbb{C}D\varphi D\eta\,dx = \langle \eta,\psi\rangle_{(1,p),(1,p)^*} \quad \forall \eta \in W^{1,p}(\Omega;\mathbb{R}^3) \text{ with } \int_\Omega \eta\,dx = 0\,. \tag{4.13}$$

The solution φ exists and is unique, as can be shown easily by minimizing the corresponding functional, and depends linearly on ψ. Further, by the results in [1, 45] (see Lemma 4.3 below) it obeys

$$\|\varphi\|_{W^{1,p'}(\Omega;\mathbb{R}^3)} \le c\|\psi\|_{(W^{1,p}(\Omega;\mathbb{R}^3))^*}\,.$$

Since $p \leq 3/2$, we have $p' \geq 3$ and, therefore, we can define $T : (W^{1,p})^* \to \mathbb{R}$ by

$$T(\psi) := L(\varphi) = \int_\Omega \mathbb{C}\beta^\mu D\varphi \, dx \, .$$

Since φ depends linearly on ψ, the map T is linear and, with

$$|T(\psi)| = |L(\varphi)| \leq \|L\|_{(W^{1,p'})^*} \|\varphi\|_{W^{1,p'}} \leq c|\mu|(\mathbb{R}^3)\|\psi\|_{(W^{1,p})^*},$$

it follows that T is a linear bounded functional on $(W^{1,p})^*$. Therefore, there is a unique $u \in W^{1,p}$ such that

$$T(\psi) = \langle u, \psi \rangle_{(1,p),(1,p)^*} \, , \tag{4.14}$$

which is then called a duality solution. Evidently, for fixed p the duality solution exists and is unique.

We use density in order to show that the duality solution is also a distributional solution. Precisely, let $\varphi_k \in C_c^\infty(B_{1/k})$ be a mollifier and define $g_k = \mathbb{C}\beta^\mu * \varphi_k$. Then, $g_k \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times3})$, it converges to $\mathbb{C}\beta^\mu$ in $L^{3/2}(\mathbb{R}^3)$ and satisfies $\operatorname{div} g_k = 0$ everywhere. We define the linear maps L_k from $W^{1,p'}$ to $\mathbb{R}$ by

$$L_k(h) := \int_\Omega g_k Dh \, dx$$

and observe that, recalling the map L defined in (4.11),

$$|L_k(h) - L(h)| \leq \int_\Omega |g_k - \mathbb{C}\beta^\mu| \, |Dh| \, dx \leq \|g_k - \mathbb{C}\beta^\mu\|_{L^p(\Omega)} \|Dh\|_{L^{p'}(\Omega)} \to 0$$

provided that $p \leq 3/2$. Furthermore, the same computation shows that $L_k \to L$ as elements of $(W^{1,p'})^*$.

Let $u_k \in W^{1,2}(\Omega)$ be a classical weak solution of the problem

$$\begin{cases} \operatorname{div}(\mathbb{C}Du) = \operatorname{div} g_k & \text{in } \Omega \, , \\ \mathbb{C}D\varphi \cdot n = 0 & \text{on } \partial\Omega \, , \end{cases}$$

which obeys

$$\int_\Omega \mathbb{C}Du_k D\eta \, dx = \int_\Omega g_k D\eta \, dx \quad \forall \eta \in W^{1,2}(\Omega; \mathbb{R}^3) \, . \tag{4.15}$$

Choose $\psi \in (W^{1,p}(\Omega; \mathbb{R}^3))^*$ and define from it φ using (4.12). Then $\varphi \in W^{1,p'} \subset W^{1,2}$ and, hence, it can be used as a test function in (4.15). We obtain

$$\int_\Omega \mathbb{C}Du_k D\varphi \, dx = \int_\Omega g_k D\varphi \, dx \, . \tag{4.16}$$

By the symmetry of $\mathbb{C}$ and since $u_k \in W^{1,2} \subset W^{1,p}$, inserting $\eta = u_k$ in (4.13) and using the definition of L_k, (4.16) reduces to

$$\langle u_k, \psi \rangle_{(1,p),(1,p)^*} = L_k(\varphi) \, . \tag{4.17}$$

Passing to the limit we obtain, for any fixed ψ and φ, that $L_k(\varphi) \to L(\varphi)$ and therefore

$$\lim_{k\to\infty} \langle u_k, \psi\rangle_{(1,p),(1,p)^*} = L(\varphi) = \langle u, \psi\rangle_{(1,p),(1,p)^*} .$$

Since this identiy holds for all ψ, we have $u_k \rightharpoonup u$ weakly in $W^{1,p}$, i.e., u_k converges weakly to the duality solution u. Since the weak limit of distributional solutions is a distributional solution, it follows that u is also a distributional solution.

Assume now that another distributional solution $v \in W^{1,p}$ exists for some $p > 1$. Then

$$\int_\Omega \mathbb{C} Dv D\eta \, dx = \int_\Omega \mathbb{C}\beta^\mu D\eta \, dx \text{ for all } \eta \in W^{1,p'}(\Omega;\mathbb{R}^3) .$$

Then, the same argument as in (4.16-4.17) shows that v is also the unique p-duality solution. Since $W^{1,p} \subset W^{1,\tilde p}$ for $\tilde p < p$, all the duality solutions coincide. □

Lemma 4.3. *Let Ω be bounded, Lipschitz, $f \in L^p(\Omega;\mathbb{R}^3)$, $p \geq 2$. Then, there is a unique solution of*

$$\begin{cases} \operatorname{div} \mathbb{C} Du = \operatorname{div} f & \text{in } \Omega \\ \mathbb{C} Du \cdot n = 0 & \text{in } \partial\Omega \end{cases}$$

and the solution satisfies the bound

$$\|u\|_{W^{1,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)} ,$$

with c depending on Ω and $\mathbb{C}$.

Proof. These properties follow from theorem [45, Theorem 6.4.8], which is based on [1]. In the notation of [45], we have $h = -1$, $t_k = 2$, $s_j = 0$, $m_j = 1$, $\gamma = 0$, $p_r = 0$, $h_r = 2$. The elliptic operator is $l_{ij}(x,\partial) = \mathbb{C}_{ikjl}\partial_k\partial_l$ and the boundary operator $B_{hj}(x,\partial) = \mathbb{C}_{jlhk} n_k(x)\partial_l$. The operator l is elliptic and the boundary data are complementing, see for example [59, 60]. □

5 The cell problem

This section is devoted to the analysis of the variational problem that defines the limiting self-energy per unit length of linear-elastic dislocations. The analysis proceeds at three increasingly constrained levels of description: a minimization on finite hollow cylinders $(B'_R \setminus B'_r) \times (0, h)$, where the energy is expected to behave as $h \log R/r$; a minimization on annular regions $B'_R \setminus B'_r$ for deformations which do not depend on x_3, which leads to energies proportional to $\log R/r$; and a minimization on S^1, which gives the unrelaxed self-energy per unit length ψ_0 entering limiting problem in Theorem 3.2. We proceed to characterize the solutions of the three problems, starting from the one-dimensional one, and to show that they are equivalent. The key elements of this section are: the analysis of the one-dimensional problem that defines ψ_0, Lemma 5.1; the reduction from two dimensions to one dimensions, Lemma 5.3; the reduction from three dimensions to two dimensions, Lemma 5.5; and the modification of the boundary values in Lemmas 5.10 and 5.11.

We start by fixing some notation. As in the remainder of the paper, $\mathbb{C}$ is as in (2.10), all constants may depend on $\mathbb{C}$. For $t \in S^2$, we fix a matrix

$$Q_t \in \mathrm{SO}(3) \text{ such that } Q_t e_3 = t \tag{5.1}$$

and let

$$\Phi_t(r, \theta, z) := Q_t(r\cos\theta, r\sin\theta, z) \tag{5.2}$$

be the change of variables to cylindrical coordinates with axis t. The local basis in standard cylindrical coordinates is denoted by

$$e_r := (\cos\theta, \sin\theta, 0)\,, \quad e_\theta := (-\sin\theta, \cos\theta, 0)\,, \quad e_3 := (0, 0, 1)\,, \tag{5.3}$$

whence the rotated basis is $\{Q_t e_r, Q_t e_\theta, Q_t e_3\}$. Finally, B'_r denotes the open ball of radius r in $\mathbb{R}^2$ centered in the origin and B_r that in $\mathbb{R}^3$. For a function $u \in SBV_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$, we denote by $\nabla u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^{3\times 3})$ the part of the gradient that is absolutely continuous with respect to the Lebesgue measure and write Du for the distributional gradient.

5.1 The 1D cell problem

We start by studying the one-dimensional reduction of the cell problem. We consider a straight dislocation line $\mu_{b,t} := b \otimes t\mathcal{H}^1 \llcorner \mathbb{R}t$ and look for full-space equilibrium solutions $\beta \in L^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ with $\operatorname{curl}\beta = \mu_{b,t}$ that are one-dimensional in the sense that, in cylindrical coordinates coaxial with the axis t, they depend solely on the angular variable, in a sense made precise in (5.5) below. This property effectively reduces the problem to a minimization over S^1. In order to express the condition on the curl simply and to pave the way for the several interpolation steps that are used in the following subsections, we derive the corresponding deformation $u \in SBV_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$. The independent variables are, then, the components of the strain in S^1, which we denote by f and g. Since, here, the discreteness of the Burgers vector plays no role, we take $b \in \mathbb{R}^3$. This extension proves convenient in reducing to a compact set in Lemma 5.8.

Lemma 5.1. *Let $b \in \mathbb{R}^3$, $t \in S^2$.*

(i). For $f \in L^2((0, 2\pi); \mathbb{R}^3)$ and $g \in \mathbb{R}^3$, we define $u^{f,g} : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$u^{f,g}(\Phi_t(r, \theta, z)) := \int_0^\theta f(s)ds + g\log r\,.$$

Then, $u^{f,g} \in SBV_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$ *and the problem*

$$(f,g) \mapsto \int_0^{2\pi} \frac{1}{2}\mathbb{C}\nabla u^{f,g}(\Phi_t(1,\theta,0)) \cdot \nabla u^{f,g}(\Phi_t(1,\theta,0))d\theta \tag{5.4}$$

has a unique minimizer in the set of $(f,g) \in L^2((0,2\pi);\mathbb{R}^3) \times \mathbb{R}^3$ *such that* $\int_0^{2\pi} f(s)ds = b$.

(ii). Let $u_{b,t}$ *be the function* $u^{f,g}$ *constructed from the minimizing* f *and* g *and let* $\beta_{b,t} := \nabla u_{b,t}$*. Then,*

$$\beta_{b,t}(\Phi_t(r,\theta,z)) = \frac{1}{r}(f(\theta) \otimes Q_t e_\theta + g \otimes Q_t e_r)\,, \tag{5.5}$$

$$|\beta_{b,t}|(x) \le c\frac{|b|}{\mathrm{dist}(x,\mathbb{R}t)}\,, \tag{5.6}$$

$$\mathrm{curl}\,\beta_{b,t} = b \otimes t\mathcal{H}^1 \llcorner (\mathbb{R}t) \ \textit{in}\ \mathbb{R}^3 \tag{5.7}$$

and

$$\mathrm{div}\,\mathbb{C}\beta_{b,t} = 0 \ \textit{in}\ \mathbb{R}^3\,. \tag{5.8}$$

(iii). Let $\psi_0(b,t)$ *be the minimum in (5.4), i.e.,*

$$\begin{aligned}\psi_0(b,t) :=& \min_{f,g} \int_0^{2\pi} \frac{1}{2}\mathbb{C}\nabla u^{f,g}(\Phi_t(1,\theta,0)) \cdot \nabla u^{f,g}(\Phi_t(1,\theta,0))d\theta \\ =& \int_0^{2\pi} \frac{1}{2}\mathbb{C}\beta_{b,t}(\Phi_t(1,\theta,0)) \cdot \beta_{b,t}(\Phi_t(1,\theta,0))d\theta\,.\end{aligned} \tag{5.9}$$

Then, ψ_0 *is continuous and satisfies*

$$c_0|b|^2 \le \psi_0(b,t) \le c_1|b|^2\,. \tag{5.10}$$

For any $t \in S^2$ *the map* $\mapsto \psi_0(b,t)$ *is quadratic.*

(iv). For any $h > 0$, $R > r > 0$, *one has*

$$\mathcal{E}[\beta_{b,t}, \Phi_t((B'_R \setminus B'_r) \times (0,h))] = h \log\frac{R}{r}\psi_0(b,t)\,.$$

Proof. (i): From the definition of $u^{f,g}$, we compute

$$\nabla u^{f,g}(\Phi_t(r,\theta,z)) = \frac{1}{r}(f \otimes Q_t e_\theta + g \otimes Q_t e_r)\,. \tag{5.11}$$

Indeed, $f = \partial_\theta(u^{f,g} \circ \Phi_t) = (\nabla u^{f,g} \circ \Phi_t)Q_t r e_\theta$ and analogously for ∂_r and ∂_z. We define $G(\theta) := \nabla u^{f,g}(\Phi_t(1,\theta,0)) = f(\theta) \otimes Q_t e_\theta(\theta) + g \otimes Q_t e_r(\theta)$ and denote by $A(G)$ the integral in (5.4). By the coercivity of $\mathbb{C}$, we have

$$\int_0^{2\pi} |G + G^T|^2 d\theta \le cA(G)\,.$$

Using the orthonormal basis $(Q_te_r, Q_te_\theta, Q_te_3)$ at every θ, we deduce

$$\int_0^{2\pi} |Q_te_3 \cdot g|^2 + |Q_te_r \cdot g|^2 d\theta \le cA(G)$$

and, since g, Q_t and e_3 are independent of θ,

$$\begin{aligned}\pi|g|^2 &\le 2\pi(Q_t^Tg)_3^2 + \pi(Q_t^Tg)_1^2 + \pi(Q_t^Tg)_2^2 \\ &= \int_0^{2\pi} |e_3 \cdot Q_t^Tg|^2 + |e_r \cdot Q_t^Tg|^2 d\theta \le cA(G)\,.\end{aligned}$$

Since $f \otimes Q_te_\theta$ is a rank-one matrix, we conclude

$$\int_0^{2\pi} |f|^2(\theta)d\theta \le cA(G)\,,$$

with the constant depending only on $\mathbb{C}$. Therefore the variational problem in (5.4) is a positive-definite quadratic problem and it has a unique minimizer in the given space. This proves (i).

(iii): By the side condition on f and Jensen's inequality, we infer that $|b|^2 \le cA(G)$ for all admissible G, thus proving the lower bound in (5.10). The upper bound follows by using the test function $f = b/(2\pi)$, $g = 0$, in the definition. To prove the continuity of ψ_0 in t, we observe that the given variational problem is equivalent to the minimization of

$$\int_0^{2\pi} \sum_{i,j=1}^3 \left(\frac{1}{2}A_{ij}(\theta)f_i(\theta)f_j(\theta) + B_{ij}(\theta)f_i(\theta)g_j + \frac{1}{2}C_{ij}(\theta)g_ig_j\right) d\theta\,,$$

where

$$\begin{aligned}A_{ij} &:= \sum_{k,l=1}^3 \mathbb{C}_{ikjl}(Q_te_\theta)_k(Q_te_\theta)_l\,,\\ B_{ij} &:= \sum_{k,l=1}^3 \frac{1}{2}\left(\mathbb{C}_{ikjl}(Q_te_\theta)_k(Q_te_r)_l + \mathbb{C}_{ikjl}(Q_te_r)_k(Q_te_\theta)_l\right)\,,\\ C_{ij} &:= \sum_{k,l=1}^3 \mathbb{C}_{ikjl}(Q_te_r)_k(Q_te_r)_l\,,\end{aligned}$$

subject to the constraint $\int_0^{2\pi} f(\theta)d\theta = b$. Since A, B, C depend smoothly on θ and coercivity has already been proven, it follows that the minimizer exists, is smooth, depends continuously on the data and satisfies $\|f\|_{L^\infty} + |g| \le c|b|$, with c depending solely on $\mathbb{C}$. In particular, ψ_0 is continuous. This concludes the proof of (iii).

(ii): The expression in (5.5) follows immediately from (5.11) and, recalling the bound $\|f\|_{L^\infty} + |g| \le c|b|$, implies (5.6). To prove (5.7) we observe that the function $u^{f,g}$ is in $SBV_{\rm loc}$ and, by the admissibility condition on f, its jump set is $\Phi_t((0,\infty),0,\mathbb{R})$ and the jump equals b.

It remains to prove (5.8). Let $\varphi\in C^\infty_c(\mathbb{R}^3;\mathbb{R}^3)$. Since $\beta\in L^1_{\rm loc}(\mathbb{R}^3;\mathbb{R}^{3\times3})$, we can compute in cylindrical coordinates, using (5.5) and denoting $G(\theta):=\beta_{b,t}(\Phi_t(1,\theta,0))$ as in the proof of (i) and $\hat\varphi:=\varphi\circ\Phi_t$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathbb{C}\beta_{b,t}\cdot D\varphi\,dx=\int_0^{2\pi}\int_{\mathbb{R}}\int_0^\infty \mathbb{C}G(\theta)\cdot D\varphi(\Phi_t(r,\theta,z))drdzd\theta\\ &=\int_0^{2\pi}\int_{\mathbb{R}}\int_0^\infty \mathbb{C}G(\theta)\cdot[\partial_r\hat\varphi\otimes e_r+\frac{\partial_\theta\hat\varphi\otimes e_\theta}{r}+\partial_3\hat\varphi\otimes e_3](r,\theta,z)drdzd\theta\,.\end{aligned}$$

We discuss the three terms separately. Since $\beta_{b,t}$ does not depend on z and $\hat\varphi$ has compact support, the term in $\partial_3\hat\varphi$ vanishes.

To treat the second term, we observe that $\hat\varphi(0,\theta,z)$ is independent of θ. Therefore, we may write

$$\begin{aligned}&\int_0^{2\pi}\int_{\mathbb{R}}\int_0^\infty \mathbb{C}G(\theta)\cdot\frac{\partial_\theta\hat\varphi(r,\theta,z)\otimes e_\theta}{r}drdzd\theta\\ &=\int_0^{2\pi}\int_{\mathbb{R}}\int_0^\infty \mathbb{C}G(\theta)\cdot\frac{\partial_\theta[\hat\varphi(r,\theta,z)-\hat\varphi(0,\theta,z)]\otimes e_\theta}{r}drdzd\theta\\ &=\int_0^{2\pi}\mathbb{C}G(\theta)\cdot\left[\frac{\partial}{\partial\theta}\left(\int_{\mathbb{R}}\int_0^\infty\frac{\hat\varphi(r,\theta,z)-\hat\varphi(0,\theta,z)}{r}\,drdz\right)\otimes e_\theta\right]d\theta\end{aligned}$$

and define

$$\overline\varphi(\theta):=\int_{\mathbb{R}}\int_0^\infty\frac{\hat\varphi(r,\theta,z)-\hat\varphi(0,\theta,z)}{r}drdz\,,$$

so that $\overline\varphi\in C^\infty(\mathbb{R};\mathbb{R}^3)$, 2π-periodic. Then, the equation

$$\int_0^{2\pi}\mathbb{C}G(\theta)\cdot\overline\varphi'(\theta)\otimes e_\theta d\theta=0$$

is the Euler-Lagrange equation for the functional (5.4) corresponding to the variation $(f,g)\mapsto(f+s\overline\varphi',g)$, $s\in\mathbb{R}$. Since $\overline\varphi$ is periodic this variation is admissible.

Finally, for the first term, integrating in r we obtain

$$\begin{aligned}&\int_0^{2\pi}\int_{\mathbb{R}}\int_0^\infty \mathbb{C}G(\theta)\cdot\partial_r\hat\varphi\otimes e_r drdzd\theta\\ &=-\int_0^{2\pi}\mathbb{C}G(\theta)\cdot\left[\left(\int_{\mathbb{R}}\hat\varphi(0,\theta,z)dz\right)\otimes e_r\right]d\theta\,.\end{aligned}$$

As above, the value $\hat\varphi(0,\theta,z)$ does not depend on θ and the preceding identity is the Euler-Lagrange equation of the functional (5.4) corresponding to the variation $(f,g)\mapsto(f,g+\int_{\mathbb{R}}\hat\varphi(0,0,z)dz)$ and, therefore, vanishes.

(iv): This is a simple computation using the expression in (5.5). $\square$

Remark 5.2. *Note that, in proving that $\beta_{b,t}$ satisfies equation (5.8), we have used the minimizing property of g only to deal with the central line $\mathbb{R}t$. In particular, for any vector $g \in \mathbb{R}^3$, f_g can be defined as the corresponding minimizer of (5.4) and construct a field $\beta_{b,t}$ which satisfies* $\operatorname{div} \mathbb{C}\beta_{b,t} = 0$ *on* $\mathbb{R}^3 \setminus \mathbb{R}t$*.*

5.2 The 2D cell problem

Next, we show that the two-dimensional problem defined by minimization over the annulus $B'_R \setminus B'_r$ is equivalent to the problem on S^1 studied above. This equivalence may be regarded as a rigorous characterization of Saint-Venant's principle for the case of a point dislocation in the plane. For the case of two-dimensional planar elasticity, a similar characterization is proved in [27]. The case of three-dimensional elasticity requires consideration of deformations with three independent components. The restriction to two dimensions then consists of requiring that the deformation be constant in the axial direction. The corresponding displacement gradient may be represented by a 3×2 matrix, but this representation interacts in a notationally cumbersome way with the coercivity of $\mathbb{C}$ and Korn's inequality. Therefore, we represent deformations in three dimensions but restrict them to be constant in the axial direction.

Lemma 5.3. *Let $b \in \mathbb{R}^3$, $t \in S^2$. For $0 < r \leq \frac{1}{2}R$, we define*

$$\psi^{2D}_{r,R}(b,t) := \frac{1}{\log \frac{R}{r}} \inf \left\{ \mathcal{E}[\beta_{b,t} + Dv, T] : \, v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3); D_t v = 0 \right\}.$$

where $T = Q_t((B'_R \setminus B'_r) \times (0,1))$. Then

$$\psi_0(b,t) - \frac{c|b|^2}{\log \frac{R}{r}} \leq \psi^{2D}_{r,R}(b,t) \leq \psi_0(b,t)$$

with c depending only on $\mathbb{C}$.

The rotation Q_t was defined in (5.1), $D_t v = 0$ means that v is constant in the t direction, and $\beta_{b,t}$ was defined in Lemma 5.1.

Proof. The bound $\psi^{2D}_{r,R}(b,t) \leq \psi_0(b,t)$ follows immediately from Lemma 5.1(iv), using $v = 0$.

The proof of the remaining bound follows the same strategy used in [27]. Precisely, we start from an admissible function v with

$$\mathcal{E}[\beta_{b,t} + Dv, Q_t((B'_R \setminus B'_r) \times (0,1))] \leq \psi^{2D}_{r,R}(b,t) \log \frac{R}{r} + |b|^2 \,, \tag{5.12}$$

then modify it so that it coincides with an isometries on both the inner and the outer boundary, $\partial B'_r \times \mathbb{R}$ and $\partial B'_R \times \mathbb{R}$ respectively. Finally, we show that the isometry can be taken to be zero. Since the one-dimensional solution solves the Euler-Lagrange equation, it is the unique minimizer with respect to its own boundary conditions. We also take $t = e_3$ for simplicity of notation.

Step 1. We modify v so that it coincides with an affine map on the inner and the outer boundary of the cylinder $(B'_R \setminus B'_r) \times \mathbb{R}$.

Choose $k \in \mathbb{N}$ to be the smallest nonnegative integer such that $\rho_k := 2^k r$ obeys $2\rho_k \le R$ and

$$\mathcal{E}[\beta_{b,t}+Dv, (B'_{2\rho_k} \setminus B'_{\rho_k}) \times (0,1)] \le \mathcal{E}[\beta_{b,t}, (B'_{2\rho_k} \setminus B'_{\rho_k}) \times (0,1)] = (\log 2)\, \psi_0(b,t)\,.$$

If none exists then the proof is concluded. Otherwise, with $T_k = (B'_{2\rho_k} \setminus B'_{\rho_k}) \times (0, \rho_k)$, we have

$$\|Dv + Dv^T\|^2_{L^2(T_k)} \le c\mathcal{E}[\beta, T_k] + c\mathcal{E}[\beta_{b,t}, T_k] \le c\rho_k\, \psi_0(b,t)\,.$$

By Korn's inequality applied to the domain T_k, there are a matrix $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$ and a vector $d \in \mathbb{R}^3$ such that

$$\rho_k^{-2}\|v - Ax - d\|^2_{L^2(T_k)} + \|Dv - A\|^2_{L^2(T_k)} \le c\rho_k|b|^2\,.$$

By scaling, the constant in Korn's inequality does not depend on ρ_k. Since v does not depend on x_3, we can assume that $Ae_3 = 0$. We construct w such that $w(x) = Ax + d$ on $B'_{\rho_k} \times \mathbb{R}$, setting

$$w(x) := v(x)(1 - \psi(x')) + (Ax + d)\psi(x')$$

for some $\psi \in C^\infty_c(B'_{2\rho_k})$ with $\psi = 1$ on B'_{ρ_k} and $|D\psi| \le c/\rho_k$. Then, $\partial_3 w = 0$ and a simple estimate leads to

$$\mathcal{E}[\beta_{b,t} + Dw, T_k] \le c|b|^2\rho_k.$$

Therefore, recalling again that all functions are constant in the e_3-direction,

$$\mathcal{E}[\beta_{b,t} + Dw, (B'_R \setminus B'_r) \times (0,1)] \le \mathcal{E}[\beta, (B'_R \setminus B'_r) \times (0,1)] + c|b|^2\,.$$

Analogously, we let h be the smallest positive integer such that $\rho^h := 2^{-h}R \ge r$ and

$$\mathcal{E}[\beta_{b,t}+Dw, (B'_{2\rho^h} \setminus B'_{\rho^h}) \times (0,1)] \le \mathcal{E}[\beta_{b,t}, (B'_{2\rho^h} \setminus B'_{\rho^h}) \times (0,1)] = (\log 2)\, \psi_0(b,t)\,.$$

By the same procedure, we obtain a function $\overline{w}$ which coincides with an affine infinitesimal isometry on both $\partial B'_R$ and $\partial B'_r$. Subtracting one of the two isometries and writing again w for the result, we have

$$w(x) = 0 \text{ for } x \in \partial B'_R \times \mathbb{R} \quad\text{and}\quad w(x) = Ax + d \text{ for } x \in \partial B'_r \times \mathbb{R},$$

for some $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$, with $Ae_3 = 0$, $d \in \mathbb{R}^3$, and

$$\mathcal{E}[\beta_{b,t} + Dw, (B'_R \setminus B'_r) \times (0,1)] \le \mathcal{E}[\beta, (B'_R \setminus B'_r) \times (0,1)] + c|b|^2\,.$$

Step 2. We show that the affine map can be taken to be zero. We consider the variational problem

$$I[z, A, d] := \mathcal{E}[\beta_{b,t} + Dz, (B'_R \setminus B'_r) \times (0, 1)]\,,$$

for $z \in W^{1,2}((B'_R \setminus B'_r) \times (0, 1); \mathbb{R}^3)$, with $D_3 z = 0$, $A \in \mathbb{R}^{3\times3}_{\text{skew}}$, $Ae_3 = 0$, $d \in \mathbb{R}^3$ subject to the boundary data $z = 0$ on $\partial B'_R \times \mathbb{R}$ and $z(x) = Ax + d$ on $\partial B'_r \times \mathbb{R}$. By Korn's inequality and the trace theorem, I is coercive in both z and the finite-dimensional variables A and d. For fixed A and d it is readily verified that I has a minimum. A minimum in all three variables additionally exists owing to the finite dimension of A and d.

Let z, A, d be a minimizer of I. We claim that $z = A = d = 0$. Indeed, the function $\beta_{b,t} \in L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^{3\times3}) \cap L^2_{\text{loc}}(\mathbb{R}^3 \setminus \mathbb{R}e_3; \mathbb{R}^{3\times3})$ satisfies $\operatorname{div} \mathbb{C}\beta_{b,t} = 0$ distributionally. Then, for a given $\theta \in C^1_c(\mathbb{R})$ with $\int_{\mathbb{R}} \theta dt = 1$, the function

$$\varphi(x) := \begin{cases} z(x)\theta(x_3) & \text{if } (x_1, x_2) \in B'_R \setminus B'_r\,, \\ 0 & \text{if } (x_1, x_2) \notin B'_R\,, \\ (Ax + d)\theta(x_3) & \text{if } (x_1, x_2) \in B'_r\,, \end{cases}$$

belongs to $W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \cap W^{1,\infty}(B'_r \times \mathbb{R}; \mathbb{R}^3)$ and, by density, can be used as test function in the equation $\operatorname{div} \mathbb{C}\beta_{b,t} = 0$, with the result

$$\int_{\mathbb{R}^3} \mathbb{C}\beta_{b,t} \cdot D\varphi\, dx = 0\,.$$

Since $\beta_{b,t}$ and z do not depend on x_3, the terms with θ' integrate to 0. Furthermore, A is antisymmetric and, therefore,

$$\int_{(B'_R \setminus B'_r) \times (0,1)} \mathbb{C}\beta_{b,t} \cdot Dz\, dx = 0\,.$$

Simultaneously, since, for $s \in \mathbb{R}$, $(1+s)(z, A, d)$ is a possible competitor for the mimimization problem, we have

$$\int_{(B'_R \setminus B'_r) \times (0,1)} \mathbb{C}(\beta_{b,t} + Dz) \cdot Dz\, dx = 0\,.$$

Taking the difference of the two identities shows that $\mathbb{C}Dz \cdot Dz = 0$ almost everywhere, which, by Korn's inequality, implies that Dz is constant. Hence, the minimizer is $z = 0$. We thus conclude that

$$\log(R/r)\psi_0(b, t) = I[0, 0, 0] \le \mathcal{E}[\beta_{b,t} + Dv, (B'_R \setminus B'_r) \times (0, 1)] + c|b|^2$$

which, recalling (5.12), implies the desired inequality. $\square$

5.3 The 3D cell problem

Next, we show that the function $\psi_0(b,t)$ can also be obtained as the limit of the elastic energy on finite cylinders. This limiting process is the main step in order to prove the lower bound in Theorem 3.2. To this end, we require a uniform Korn-Poincaré inequality, which we prove in Lemma 5.9.

For $b \in \mathbb{R}^3$, $t \in S^2$, $h, r, R \in (0,\infty)$ with $r < R \leq h$, we define

$$\psi(b,t,h,r,R) := \frac{1}{h \log \frac{R}{r}} \min \left\{ \mathcal{E}[\beta, T_r] : \beta \in L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^{3\times 3}) \, , \operatorname{curl} \beta = \mu_0 \right\} \tag{5.13}$$

where $T_r := Q_t((B'_R \setminus B'_r) \times (0,h))$, $\mu_0 := b \otimes t \mathcal{H}^1 \llcorner \mathbb{R}t$ and Q_t is defined in (5.1). By scaling, one immediately obtains

$$\psi(b,t,\lambda h, \lambda r, \lambda R) = \psi(b,t,h,r,R) \text{ for all } \lambda > 0 \, .$$

and

$$\psi(\lambda b,t,h,r,R) = \lambda^2 \psi(b,t,h,r,R) \text{ for all } \lambda > 0 \, .$$

Lemma 5.4. *For every $b \in \mathbb{R}^3$, $t \in S^2$, $R > r > 0$, $h > 0$,*

$$\begin{aligned} \psi(b,t,h,r,R) = \frac{1}{h \log \frac{R}{r}} \min \Bigg\{ & \mathcal{E}[\beta, T_r] : \beta \in L^2(T_r; \mathbb{R}^{3\times 3}) \, , \operatorname{curl} \beta = 0 \textit{ in } T_r, \\ & \int_0^{2\pi} \beta(\Phi_t(\rho,\theta,z)) Q_t e_\theta d\theta = b \textit{ for all } \rho \in (r,R), z \in (0,h) \Bigg\} \, . \end{aligned} \tag{5.14}$$

Proof. The lemma follows by the same argument as in Lemma 3.4. Indeed, every β admissible in (5.13) is admissible also in (5.14), leading to one inequality. To prove the other remaining inequality, we note that, for every β admissible in (5.14), the matrix field $\beta - \beta_{b,t}$, with $\beta_{b,t}$ as in Lemma 5.1, is exact and, therefore, the gradient of a function $w \in W^{1,2}(T_r; \mathbb{R}^3)$. This function can be extended to a function in $\hat{w} \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3)$. It suffices to define $\hat{\beta} := \beta_{b,t} + D\hat{w}$, which is admissible in (5.13). □

Lemma 5.5. *For every $b \in \mathbb{R}^3$, $t \in S^2$, $h \geq R > 0$,*

$$\liminf_{r \to 0} \psi(b,t,h,r,R) \geq \psi_0(b,t) - c|b|^2 \frac{R}{h},$$

with c depending only on $\mathbb{C}$.

Proof. Throughout the proof, b and t are fixed. We can assume $t = e_3$ and $2r < R$. We define $T_r^h = (B'_R \setminus B'_r) \times (0,h)$, let $u_{b,t}$ be the function defined in Lemma 5.1 and β be as in the definition of ψ, with $\mathcal{E}[\beta, T_r^h] \leq h \log \frac{R}{r} \psi(b,t,h,r,R) + R|b|^2 \log \frac{R}{r}$. Since $\operatorname{curl}(\beta - \beta_{b,t}) = 0$, there is $u \in W^{1,2}(T_r^h; \mathbb{R}^3)$ such that $\beta = Du + \beta_{b,t}$.

Step 1: We modify the test function close to the top and bottom boundaries to make it affine.

Choose $z_- \in (0, h-R)$ to be the smallest value such that, on the domain $T_- = (B'_R \setminus B'_r) \times (z_-, z_- + R)$,

$$\mathcal{E}[\beta, T_-] \le \mathcal{E}[\beta_{b,t}, T_-] \,. \tag{5.15}$$

If no such z_- exists, $\mathcal{E}[\beta, T_r^h] \ge \lfloor h/R \rfloor \mathcal{E}[\beta_{b,t}, T_r^R] = \lfloor h/R \rfloor R \log \frac{R}{r} \psi_0(b,t)$ and the proof is finished. Otherwise, we have $\|Du + Du^T\|^2_{L^2(T_-)} \le c\mathcal{E}[\beta_{b,t}, T_-]$ and, by the Korn's inequality, Lemma 5.9, there exist $A_- \in \mathbb{R}^{3\times 3}$ and $d_- \in \mathbb{R}^3$ such that

$$R^{-2} \|u - A_- x - d_-\|^2_{L^2(T_-)} \le c\mathcal{E}[\beta_{b,t}, T_-] = c\,\psi_0(b,t)\, R \log \frac{R}{r} \le c|b|^2\, R\, \log \frac{R}{r} \,.$$

Analogously, let $z_+ \in (R, h)$ be the largest value such that (5.15) holds for the domain $T_+ = (B'_R \setminus B'_r) \times (z_+ - R, z_+)$, which then leads to

$$R^{-2} \|u - A_+ x - d_+\|^2_{L^2(T_+)} \le c|b|^2 R \log \frac{R}{r} \,.$$

If $z_- + R \ge z_+$ then $\mathcal{E}[\beta, T_r^h] \ge (h - 2R)\mathcal{E}[\beta_{b,t}, T_r^1] = (h-2R) \log \frac{R}{r} \psi_0(b,t)$, *q. e. d.*

Let $\psi \in C^\infty(\mathbb{R})$ with $\psi = 1$ on (R, ∞) and $\psi = 0$ on $(-\infty, 0)$, with $|\psi'| \le 2/R$. Set

$$\begin{aligned} w(x) :=& \psi(x_3 - z_-)\psi(z_+ - x_3)u(x) + (1 - \psi(x_3 - z_-))(A_- x + d_-) \\ &+ (1 - \psi(z_+ - x_3))(A_+ x + d_+) \,. \end{aligned}$$

Then, $w(x) = A_- x + d_-$ for $x_3 = 0$, $w(x) = A_+ x + d_+$ for $x_3 = h$, and

$$\begin{aligned} \mathcal{E}[\beta_{b,t} + Dw, T_-] \le& 2\mathcal{E}[\beta_{b,t}, T_-] \\ &+ c\int_{T_-} \left(|Du + Du^T|^2 + |\psi'|^2 |u(x) - A_- x - d_-|^2 \right) dx \\ \le& c|b|^2 R \log \frac{R}{r} \,, \end{aligned}$$

and analogously on the other side. In particular,

$$\mathcal{E}[\beta_{b,t} + Dw, T_r^h] \le \mathcal{E}[\beta, T_r^h] + c|b|^2 R \log \frac{R}{r} \,. \tag{5.16}$$

Step 2. We average in the vertical direction.

We define $v : B'_R \setminus B'_r \to \mathbb{R}^3$ as the average of w in x_3 and $F : B'_R \setminus B'_r \to \mathbb{R}^{3\times 3}$ as the average of Dw,

$$v(x') := \frac{1}{h} \int_0^h w(x', x_3) dx_3 \,, \qquad F(x') := \frac{1}{h} \int_0^h Dw(x', x_3) dx_3 \,,$$

where $x' = (x_1, x_2)$. Jensen's inequality then gives

$$\mathcal{E}[\beta_{b,t} + F, T_r^1] \leq \frac{1}{h}\mathcal{E}[\beta_{b,t} + Dw, T_r^h]$$

where $F(x', x_3) = F(x')$. The relation between F and v can be determined from the boundary conditions on w, with the result

$$F(x') = D'v(x') + \frac{1}{h}\int_0^h \partial_3 w(x', x_3)dx_3 \otimes e_3 = D'v(x') + \frac{A_h(x', 0) + d_h}{h} \otimes e_3,$$

where $A_h = A_+ - A_- \in \mathbb{R}^{3\times 3}_{\text{skew}}$ and $d_h = hA_+e_3 + d_+ - d_- \in \mathbb{R}^3$. Therefore, the functional G defined in Lemma 5.6 obeys

$$G[v, A_h/h, d_h/h, r, R] = \mathcal{E}[\beta_{b,t} + F, T_r^1]\,,$$

where $v(x', x_3) = v(x')$. Lemma 5.6 then gives

$$\psi_0(b, t) = \liminf_{r\to 0} \frac{\inf_{z,A,d} G[z, A, d, r, R]}{\log(R/r)} \leq \liminf_{r\to 0} \frac{\mathcal{E}[\beta_{b,t} + Dw, T_r^h]}{h\log(R/r)}$$

and, recalling (5.16), the stated conclusion follows. $\square$

Lemma 5.6. *Given $b \in \mathbb{R}^3$, $t \in S^2$ and $0 < r \leq R/2$, we consider the variational problem*

$$G[z, A, d, r, R] := \mathcal{E}[\beta_{b,t} + Dz + (A(\mathrm{Id} - t\otimes t)x + d) \otimes t, Q_t(B'_R \setminus B'_r) \times (0, 1)]$$

for $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$, $d \in \mathbb{R}^3$ and $z \in W^{1,2}_{\text{loc}}(\mathbb{R}^3 \setminus \mathbb{R}t; \mathbb{R}^3)$ with $D_t z = 0$. Then,

$$\liminf_{r\to 0} \frac{1}{\log(R/r)} \inf_{z,A,d} G[z, A, d, r, R] = \psi_0(b, t)\,.$$

Proof. We fix b, t and R, work as usual in the case $t = e_3$, and define

$$\alpha := \liminf_{r\to 0} \frac{1}{\log(R/r)} \inf_{z,A,d} G[z, A, d, r, R]\,. \tag{5.17}$$

First, we show that we can restrict to $d = 0$ in this minimization, then that we can restrict to $A = 0$, and finally use Lemma 5.3 to conclude the proof.

Fix $\eta > 0$. Let $r_0 \in (0, R)$ be such that

$$\alpha - \eta \leq \inf_{z,A,d} \frac{1}{\log(R/r)} G[z, A, d, r, R] \text{ for all } r \in (0, r_0)\,. \tag{5.18}$$

Choose $r^* \in (0, \min\{r_0^2/R, R/4\})$, z^*, A^*, d^* such that

$$\frac{1}{\log(R/r^*)} G[z^*, A^*, d^*, r^*, R] < \alpha + \eta\,. \tag{5.19}$$

Let $\rho^* := \sqrt{Rr^*}$. Separate the integral in the part inside $B'_{\rho^*} \times (0,1)$ and the part outside it. The outer part is $G[z^*, A^*, d^*, \rho^*, R]$, whereas the inner part is

$$G[z^*, A^*, d^*, r^*, \rho^*] = \mathcal{E}[\beta_{b,t} + Dz^* - (A^*(x',0) + d^*) \otimes e_3, (B'_{\rho^*} \setminus B'_{r^*}) \times (0,1)] .$$

Let $\lambda := r^*/\rho^*$. By the choice of r^*, we have $\lambda \le 1/2$. Since $\beta_{b,t}(x) = \lambda \beta_{b,t}(\lambda x)$ (recall the definition in (5.5) and $D_3 z^* = 0$), the scaling $\tilde{z}(x) := z^*(\lambda x)$ leads to

$$G[z^*, A^*, d^*, r^*, \rho^*] = G[\tilde{z}, \lambda^2 A^*, \lambda d^*, \rho^*, R] .$$

Therefore (5.19) gives

$$\frac{1}{\log(R/r^*)} G[z^*, A^*, d^*, \rho^*, R] + \frac{1}{\log(R/r^*)} G[\tilde{z}, \lambda^2 A^*, \lambda d^*, \rho^*, R] \le \alpha + \eta .$$

Since $\rho^* < r_0$, from (5.18) we obtain

$$\alpha - \eta \le 2 \frac{1}{\log(R/r^*)} G[z^*, A^*, d^*, \rho^*, R]$$

and

$$\alpha - \eta \le 2 \frac{1}{\log(R/r^*)} G[\tilde{z}, \lambda^2 A^*, \lambda d^*, \rho^*, R] .$$

This implies that

$$\frac{1}{\log(R/\rho^*)} G[z^*, A^*, d^*, \rho^*, R] \le \alpha + 3\eta \tag{5.20}$$

and

$$\frac{1}{\log(R/\rho^*)} G[\tilde{z}, \lambda^2 A^*, \lambda d^*, \rho^*, R] \le \alpha + 3\eta . \tag{5.21}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s) := \frac{1}{\log \frac{R}{\rho^*}} \min\big\{ G[z, A, sd^*, \rho^*, R] - \alpha : A \in \mathbb{R}^{3\times 3}_{\text{skew}}, z \in W^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3), \partial_3 z = 0 \big\} .$$

Let z^s and A^s be the minimizer of $G[\cdot, \cdot, sd^*, \rho^*, R]$. It is easy to see that z^s and A^s depend linearly on s, therefore $f(s)$ is a second-order polynomial in s. Since $\rho^* < r_0$, by (5.18) $f(s) \ge -\eta$ for all s. By (5.20) and (5.21), both $f(1)$ and $f(\lambda)$ lie below 3η. This implies that

$$f(0) \le 35\eta .$$

To see this, observe that $(f(s) - \min f)^{1/2} = a|s - s_0|$, for some $a \ge 0$, $s_0 \in \mathbb{R}$; we know $\min f \ge -\eta$ and $\max\{a|1-s_0|, a|\lambda - s_0|\} \le T$, where $\lambda \in (0, 1/2]$ and $T = (3\eta - \min f)^{1/2}$. Then,

$$a|s_0| \le \frac{1}{1-\lambda} (\lambda a |1 - s_0| + a|\lambda - s_0|) \le 2(\frac{1}{2} a|1 - s_0| + a|\lambda - s_0|) \le 3T ,$$

and, hence, $f(0) = (a|s_0|)^2 + \min f \leq 9T^2 + \min f = 27\eta - 8\min f \leq 35\eta$. By the arbitrariness of η, we obtain that

$$\liminf_{r\to 0} \frac{1}{\log(R/r)} \inf_{z,A} G[z, A, 0, r, R] = \alpha\,.$$

At this point, we repeat the same argument as above, with this expression in plalce of (5.17). In particular, possibly reducing r_0, (5.18) holds with $d = 0$. Therefore, for any given η we find $\tilde{r}$, $\tilde{A}$ such that (5.19) holds true with r^*, A^*, and d^* replaced by $\tilde{r}$, $\tilde{A}$, and 0, respectively. In addition, (5.20) and (5.21) follow as above, with $\tilde{\rho} := \sqrt{R\tilde{r}}$ and $\lambda := \tilde{r}/\tilde{\rho}$. Then, the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(s) := \frac{1}{\log(R/\tilde{\rho})} \min\{G[z, s\tilde{A}, 0, \tilde{\rho}, R] - \alpha : z \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3), \partial_3 z = 0\}$$

satisfies $g \geq -\eta$ everywhere and both $g(1)$ and $g(\lambda^2)$ lie below 3η. Let z_s be the minimizer of $G[\cdot, s\tilde{A}, 0, \tilde{\rho}, R]$. As previously, z_s depends linearly on s, g is a second-order polynomial and therefore $g(0) \leq 35\eta$. These properties imply that

$$\liminf_{r\to 0} \frac{1}{\log(R/r)} \inf_{z} G[z, 0, 0, r, R] = \alpha\,.$$

Finally, we observe that $\inf_z G[z, 0, 0, r, R] = \log \frac{R}{r} \psi^{2D}_{r,R}(b, t)$ and recall Lemma 5.3 to conclude the proof. $\square$

Lemma 5.7. *Let $H = \{(b, t, h, R) \in \mathbb{R}^3 \times S^2 \times (0, \infty)^2 : R \leq h\}$, $K \subset H$ compact and $r_0 = \min\{R : \exists(b, t, h, R) \in K\}$. The family of functions $r \mapsto \psi(\cdot, \cdot, \cdot, r, \cdot)$, from $(0, r_0/2)$ to $\mathbb{R}$, defined in (5.13) is equicontinuous on K. In particular,*

$$\psi_0(b, t) \leq (1 + c|t - t'|)\psi_0(b, t')\,. \tag{5.22}$$

Proof. We show that $\psi(b, t, h, r, R)$ is uniformly continuous separately in each of the variables b, t, h, and R.

Step 1. Continuity in b. We choose $b, b' \in \mathbb{R}^3$, let β be an admissible strain in the definition of $\psi(b, t, h, r, R)$, and define

$$\beta' := \beta + \beta_{b'-b,t}$$

where $\beta_{b'-b,t}$ was defined in Lemma 5.1. Then β' is an admissibile strain in the definition of $\psi(b', t, h, r, R)$ and for any $\delta > 0$

$$\mathcal{E}[\beta', T] \leq (1 + \delta)\mathcal{E}[\beta, T] + (1 + \frac{1}{\delta})\mathcal{E}[\beta_{b'-b,t}, T]\,,$$

where $T = Q_t((B'_R \setminus B'_r) \times (0, h))$. Choosing $\delta = |b - b'|$ we get

$$\psi(b', t, h, r, R) - \psi(b, t, h, r, R) \leq |b' - b|\psi(b, t, h, r, R) + c|b' - b| + c|b' - b|^2\,,$$

which concludes the proof.

Step 2. Continuity in t. We chose $t, t' \in S^2$, fix as above β to be an admissible strain in the definition of $\psi(b,t,h,r,R)$, and define

$$\beta'(x) := S\beta(Sx)S^T\,, \qquad S := Q_{t'}Q_t^T\,.$$

Then β' is admissible for $\psi(Sb, St, h, r, R)$ and a change of variables gives

$$\mathcal{E}[\beta', ST] = \int_{ST} \frac{1}{2}\mathbb{C}\beta' \cdot \beta' dx = \int_T \frac{1}{2}\mathbb{C}'\beta \cdot \beta \, dx$$

where $\mathbb{C}'_{ijkl} := S_{ii'}S_{jj'}S_{kk'}S_{ll'}\mathbb{C}_{i'j'k'l'}$. Since $|\mathbb{C} - \mathbb{C}'| \le c|t - t'|$, $\mathbb{C}$ is positive definite on symmetric matrices, and both vanish on skew-symmetric ones, we conclude

$$\psi(Sb, t', h, r, R) \le (1 + c|t - t'|)\psi(b, t, h, r, R)\,.$$

Recalling Step 1 (5.22) and continuity in t are proven.

Step 3. Continuity in h. Let $h < h'$. Restricting the strain field immediately gives

$$\psi(b, t, h, r, R) \le \frac{h'}{h}\psi(b, t, h', r, R)\,.$$

To prove the converse inequality, we need to extend the strain field. For simplicity, we perform the construction for $t = e_3$. Let β be an admissible deformation in the definition of $\psi(b,t,h,r,R)$ (see (5.13)) and define $u \in W^{1,2}(T_r;\mathbb{R}^3)$ by $Du = \beta - \beta_{b,t}$. We apply Korn's inequality, in the form of Lemma 5.9, on the cylinder $\hat{T}_r := (B'_R \setminus B'_r) \times (0,R)$ to obtain $A \in \mathbb{R}^{3\times3}_{\mathrm{skew}}$ such that

$$\int_{\hat{T}_r} |Du - A|^2 dx \le c\int_{\hat{T}_r} |Du + Du^T|^2 dx \le c\mathcal{E}[\beta, T_r] + cR\log\frac{R}{r}\psi_0(b,t)\,.$$

Replacing β by $\beta - A$ we reduce to the case $A = 0$. Choose $y_3 \in (0,R)$ such that

$$\int_{B'_R\setminus B'_r} |Du|^2(x', y_3)dx' \le c\frac{1}{R}\mathcal{E}[\beta, T_r] + c\log\frac{R}{r}\psi_0(b,t)$$

and define

$$\tilde{u}(x', x_3) := \begin{cases} u(x', x_3) & \text{if } x_3 \in (0, y_3) \\ u(x', y_3) & \text{if } x_3 \in (y_3, y_3 + h' - h) \\ u(x', x_3 - (h' - h)) & \text{if } x_3 \in (y_3 + h' - h, h') \end{cases}$$

and $\tilde{\beta} := \beta_{b,t} + Du$. Then,

$$\mathcal{E}[\tilde{\beta}, \tilde{T}_r] \le \mathcal{E}[\beta, T_r] + c(h' - h)\int_{B'_R\setminus B'_r} |Du|^2(x', y_3)\, dx'\,,$$

where $\tilde{T}_r := Q_t((B'_R \setminus B'_r) \times (0, h'))$ and, therefore,

$$\psi(b, t, h', r, R) \le \psi(b, t, h, r, R) + c\frac{|h' - h|}{R}|b|^2\,.$$

□

Lemma 5.8. *For every $M \geq 1$, there is a function $\omega_M : (0,\infty) \to (0,\infty)$ with*

$$\lim_{r\to 0} \omega_M(r) = 0$$

such that

$$\Big(1 - \frac{c}{M} - \omega_M(\frac{r}{R})\Big)\, \psi_0(b,t) \leq \psi(b,t,h,r,R) \leq \psi_0(b,t),$$

for all $b \in \mathbb{R}^3$, $t \in S^2$, $r, R, h > 0$ such that $MR \leq h$. Furthermore, there exists $c_* > 0$ such that, for all b, t, h, r, R with $2r \leq R \leq h$,

$$c_*|b|^2 \leq \psi(b,t,h,r,R)\,. \tag{5.23}$$

Proof. The upper bound follows immediately, simply by testing $\mathcal{E}[\beta, T_r]$ with $\beta_{b,t}$ and using the definition of ψ_0.

The lower bound follows from the two preceding Lemmas with the same proof as that of the Ascoli-Arzelá theorem. By scaling, it suffices to consider $R = 1$ and $b \in S^2$. For any $N \in \mathbb{N}$, we can subdivide $(0,h)$ into N equal intervals. Choosing the interval of minimal energy, we obtain

$$\psi(b,t,\frac{h}{N},r,1) \leq \psi(b,t,h,r,1)\,.$$

Therefore, it suffices to prove the assertion for $h \in [M, 2M]$. Let $K_M := S^2 \times S^2 \times [M, 2M]$.

For any $j \in \mathbb{N}$, since $\psi(\cdot,\cdot,\cdot,r,1)$ is equicontinuous on K_M, there exists $\delta_j > 0$ such that $|\psi(b,t,h,r,1) - \psi(b',t',h',r,1)| \leq 1/j$ if $|b-b'|+|t-t'|+|h-h'| \leq \delta_j$, $(b,t,h), (b',t',h') \in K_M$, $r \in (0,1/2)$. Cover K_M with finitely many balls of radius δ_j. By Lemma 5.5, at every center (b_k, t_k, h_k) of these balls, we have

$$\liminf_{r\to 0} \psi(b_k,t_k,h_k,r,1) \geq \psi_0(b_k,t_k) - \frac{c}{M}\,.$$

Therefore, there exists $r_j \in (0, r_{j-1})$ such that, for all $r < r_j$ and all $(b,t,h) \in K_M$,

$$\psi(b,t,h,r,1) \geq \left(1 - \frac{c}{M} - \frac{2}{j}\right) \psi_0(b,t),$$

where we use that $\min_{S^2\times S^2} \psi_0 > 0$. Then, set $\omega_M(r) := 2/j$ for $r \in (r_{j+1}, r_j]$.

The last estimate follows by the coercivity of $\mathbb{C}$ and the uniform Korn estimate of Lemma 5.9. □

In closing this section we prove Korn's inequality in a punctured cylinder, with a constant which does not depend on the radius of the central hole. We formulate it in a somewhat more general way than used above (we only needed the case $h = 1$) since the proof does not change.

Lemma 5.9 (Korn with a hole)**.** *For every $h > 0$ there is a constant $c = c(h) > 0$ with the following property: Let $R > 0$, $\varepsilon \in (0, \frac{\min\{h,1\}}{2}R]$, $T_\varepsilon := (B'_R \setminus B'_\varepsilon) \times (0, hR)$, $u : T_\varepsilon \to \mathbb{R}^3$ measurable, with $Du + Du^T \in L^2(T_\varepsilon; \mathbb{R}^{3\times 3})$. Then, there exist an antisymmetric matrix $A \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ and a vector $d \in \mathbb{R}^3$ such that*

$$\|Du - A\|_{L^2(T_\varepsilon)} + \frac{1}{R}\|u - Ax - d\|_{L^2(T_\varepsilon)} \le c\|Du + Du^T\|_{L^2(T_\varepsilon)} .$$

A similar statement, with the same proof, holds for the corresponding geometrically nonlinear estimate, building upon [26] instead of the classical Korn's inequality. The dependence of c on h arises only from the final application of Korn's inequality in (5.26) and – as is known from rod theory [8] – can be shown to be quadratic in h. The case $h \in (0, 1)$ is closer to plate theory [26]. In this situation the constant turns out to be proportional to h^{-2}. The regime $\frac{h}{2}R < \varepsilon \le \frac{1}{2}R$ can also be dealt with similarly. For notational simplicity, and since we only need the Lemma for $h = 1$, we disregard this case. The statement and proof can also be directly generalized to bounds in L^p, $p \in (1, \infty)$.

Proof. The proof is similar to that presented in [58, Lemma 3.1] for the two-dimensional case, with due care given to the third direction. The key idea is to extend the function to the inner cylinder without changing the L^2 norm of $Du + Du^T$ appreciably and, then, to use Korn's inequality on the full cylinder. Whereas in the two-dimensional case the perforation in the center has the same aspect ratio for all ε, in three dimensions the perforation becomes elongated, with the consequence that the extension cannot be effected in one step. Instead, we perform the extension first in cylinders of the type $B'_{2\varepsilon} \times (0, 2\varepsilon)$ and, then, interpolate between the extensions.

By scaling, we can take $R = 1$. We set $N := \lfloor h/\varepsilon \rfloor - 1$ and $z_j := j\varepsilon$ for $j = 0, \dots, N-1$, $z_N := h - 2\varepsilon$, so that $z_N - z_{N-1} \in [0, \varepsilon)$. For every j, we use Korn's inequality on the domain $F_j := (B'_{2\varepsilon} \setminus B'_\varepsilon) \times (z_j, z_j + 2\varepsilon)$, with the result

$$\frac{1}{\varepsilon}\|u - A_j\|_{L^2(F_j)} + \|Du - DA_j\|_{L^2(F_j)} \le c\|Du + Du^T\|_{L^2(F_j)} ,$$

for an affine function $A_j : \mathbb{R}^n \to \mathbb{R}^n$ with $DA_j + DA_j^T = 0$. By scaling, the constant does not depend on ε. By a triangular inequality,

$$\frac{1}{\varepsilon}\|A_j - A_k\|_{L^2(F_j \cap F_k)} \le c\|Du + Du^T\|_{L^2(F_j \cup F_k)} \tag{5.24}$$

for all j, k. Moreover, for any j there is an extension u_j of u to $\hat{F}_j := B'_{2\varepsilon} \times (z_j, z_j + 2\varepsilon)$ such that

$$\frac{1}{\varepsilon}\|u_j - A_j\|_{L^2(\hat{F}_j)} + \|Du_j - DA_j\|_{L^2(\hat{F}_j)} \le c\|Du + Du^T\|_{L^2(F_j)} . \tag{5.25}$$

Again by scaling, the constant does not depend on ε. Next, we interpolate between the different extensions. Choose $\theta_j \in C_c^\infty(\mathbb{R})$ such that $\sum \theta_j = 1$

on $(0,h)$, $\theta_j = 0$ on $(0,h) \setminus (z_j, z_j + 2\varepsilon)$. In particular, $\theta_0(0) = \theta_N(h) = 1$. We can also choose θ_j such that $|\theta_j'| \le c/\varepsilon$. We extend u to a function in $B_1' \times (0,h)$ defining $u := \sum_j \theta_j u_j$ in $B_\varepsilon' \times (0,h)$. In order to estimate $Du + Du^T$ in $L^2(B_\varepsilon' \times (0,h))$, we write

$$Du = \sum_j \theta_j Du_j + \sum_j (u_j - A_j) \otimes D\theta_j + \sum_j A_j \otimes D\theta_j \,.$$

The contribution of the first two terms can be estimated by (5.25). For the remaining term, we observe that $\sum_j D\theta_j = 0$ implies that, for every k,

$$\sum_j A_j \otimes D\theta_j = \sum_j (A_j - A_k) \otimes D\theta_j \,.$$

Since the θ_j's have finite overlap,

$$\| \sum_j A_j \otimes D\theta_j \|^2_{L^2(B_\varepsilon' \times (0,h))} \le \frac{c}{\varepsilon^2} \sum_j \|A_j - A_{j+1}\|_{L^2(F_j \cap F_{j+1})} \,.$$

Thus we conclude, recalling (5.24),

$$\begin{aligned} \|Du + Du^T\|_{L^2(B_1' \times (0,h))} \le & \|Du + Du^T\|_{L^2((B_1' \setminus B_\varepsilon') \times (0,h))} \\ & + c \sum_j \|Du + Du^T\|_{L^2(F_j)} \\ \le & c\|Du + Du^T\|_{L^2((B_1' \setminus B_\varepsilon') \times (0,h))}, \end{aligned} \tag{5.26}$$

for a universal constant. The assertion then follows by an application of Korn's inequality on the fixed domain $B_1' \times (0,h)$. □

5.4 The 3D cell problem with boundary data

The following lemma is required in the proof of the upper bound in the Γ-convergence theorem 3.2. For purposes of this lemma, the core-cutoff and the mollification regularizations require separate consideration.

Lemma 5.10. *Let $h, R > 0$ with $R \le h$, $t \in S^2$, $b \in \mathbb{R}^3$, $T := Q_t(B_R' \times (0,h))$. Let $\beta \in L^1(T; \mathbb{R}^{3\times 3})$ be such that*

$$|\beta|(x) \le \frac{c^*|b|}{\operatorname{dist}(x, \mathbb{R}t)} \qquad \text{for all } x \in T \tag{5.27}$$

and

$$\operatorname{curl} \beta = b \otimes t \mathcal{H}^1 \llcorner \mathbb{R}t \qquad \text{in } T \,.$$

Then, there exists $\beta_\varepsilon \in L^1(T; \mathbb{R}^{3\times 3})$ such that $\operatorname{curl} \beta_\varepsilon = \operatorname{curl} \beta$ in T, $\beta_\varepsilon = \beta$ in a neighbourhood of ∂T and, for all $\varepsilon \in (0, R/3)$,

$$\mathcal{E}[\beta_\varepsilon, T_\varepsilon] \le h\psi_0(b,t) \log \frac{R}{\varepsilon} + c|b|^2 \left(h \left(\log \frac{R}{\varepsilon} \right)^{1/2} + \frac{h^3}{R^2} \right) ,$$

where $T_\varepsilon := Q_t((B'_R \setminus B'_\varepsilon) \times (0,h))$. The constant c depends only on $\mathbb{C}$ and the constant c^* in (5.27).

Proof. We set $t = e_3$ for simplicity of notation. Let $\beta_{b,t}$ be the function defined in Lemma 5.1 for the given b and t. Then, we have $\mathrm{curl}(\beta - \beta_{b,t}) = 0$ in T. Therefore, there exists $v \in W^{1,1}(T;\mathbb{R}^3)$ with average zero such that $\beta_{b,t} = \beta + Dv$. From (5.6) and (5.27) we obtain

$$|Dv|(x) \le \frac{c|b|}{\mathrm{dist}(x, \mathbb{R}t)}$$

and, therefore,

$$|v|(x', x_3) \le c|b| \left(\frac{h}{R} + \log \frac{R}{|x'|} \right) . \tag{5.28}$$

We proceed to interpolate between v and 0 using two cutoff functions, one in the radial direction, θ_R, for the boundary at R, and one in the longitudinal direction, θ_3, for the boundaries at $x_3 = 0$ and $x_3 = h$. Precisely, for given $\delta \in (0,h)$ chosen below, we fix $\theta_R \in C_c^\infty(B'_R; [0,1])$ such that $\theta_R = 1$ on $B'_{R/2}$, $|D'\theta_R| \le c/R$ and $\theta_3 \in C_c^\infty((0,h); [0,1])$ such that $\theta_3 = 1$ on $(\delta, h-\delta)$ and $|\theta_3'| \le c/\delta$. These constants are all universal. We define $\beta_\varepsilon := \beta + Du$, where

$$u(x', x_3) := v(x', x_3)\theta_R(x')\theta_3(x_3) \tag{5.29}$$

so that $u = v$ and $\beta_\varepsilon = \beta_{b,t}$ on the set $\{\theta_R\theta_3 = 1\}$ and $u = 0$ and $\beta_\varepsilon = \beta$ on the boundary of T.

The leading-order term is computed using Lemma 5.1(iv), with the result

$$\mathcal{E}[\beta_{b,t}, (B'_R \setminus B'_\varepsilon) \times (0,h)] = h \log \frac{R}{\varepsilon} \psi_0(b,t) .$$

It remains to estimate the contribution of the interpolation regions. We write

$$|Du| \le |Dv| + |v|(|D\theta_R| + |D\theta_3|)$$

and distinguish two cases. If $\delta < R$, we start from the outer interpolation region, where $\theta_3 = 1$, $|D\theta_R| \le c/R$ and $|v| \le c|b|h/R$ (recall that $R \le h$). This region is the larger interpolation region, where only the smaller gradient of θ_R plays a role. The gradient of θ_3 plays a role in the top and bottom layers only. We have

$$\begin{aligned} &\mathcal{E}[\beta + Du, (B'_R \setminus B'_{R/2}) \times (\delta, h-\delta)] \\ &\le cR^2 h \|\beta + Du\|^2_{L^\infty((B'_R \setminus B'_{R/2}) \times (\delta, h-\delta))} \le c|b|^2 h \left(\frac{h}{R}\right)^2 . \end{aligned}$$

Next, we turn to the top and bottom interpolation regions, where $\theta_3 \neq 1$. Here $|D\theta_3| + |D\theta_R| \leq c/\delta$, since $\delta \leq R$ and, recalling (5.28),

$$\begin{aligned}
&\mathcal{E}[\beta + Du, (B'_R \setminus B'_\varepsilon) \times [(0,\delta) \cup (h-\delta, h)]] \\
&\qquad \leq c|b|^2\delta \int_{B'_R\setminus B'_\varepsilon} \left(\frac{1}{|x'|^2} + \frac{1}{\delta^2} \left(\frac{h}{R} + \log \frac{R}{|x'|} \right)^2 \right) dx' \\
&\qquad \leq c|b|^2\delta \left(\log \frac{R}{\varepsilon} + \frac{h^2}{\delta^2} + \frac{R^2}{\delta^2} \right).
\end{aligned}$$

If, instead, $R \leq \delta$, the two estimates are carried out on the domains $(B'_R \setminus B'_{R/2}) \times (0,h)$ and $(B'_{R/2} \setminus B'_\varepsilon) \times [(0,\delta) \cup (h-\delta,h)]$, respectively. In both cases, adding terms we conclude

$$\mathcal{E}[\beta_\varepsilon, T_\varepsilon] \leq h \log \frac{R}{\varepsilon} \psi_0(b,t) + c|b|^2 \left(\frac{h^3}{R^2} + \delta \log \frac{R}{\varepsilon} + \frac{h^2}{\delta} + \frac{R^2}{\delta} \right).$$

Since $h \geq R$, the last term can be ignored. We choose $\delta := h(\log(R/\varepsilon))^{-1/2}$, which is admissible since $\varepsilon < R/3$, and conclude the proof. $\square$

Next, we turn to the mollification regularization.

Lemma 5.11. *Let $h, R, \varepsilon > 0$, with $3\varepsilon \leq R \leq h$, $t \in S^2$, $b \in \mathbb{R}^3$, $T := Q_t(B'_R \times (0,h))$. Let $\beta \in L^1(T;\mathbb{R}^{3\times3})$ be such that*

$$|\beta|(x) \leq \frac{c^*|b|}{\varepsilon + \operatorname{dist}(x, \mathbb{R}t)} \qquad \text{for all } x \in T \tag{5.30}$$

and

$$\operatorname{curl} \beta = \eta_\varepsilon * (b \otimes t\mathcal{H}^1 \llcorner \mathbb{R}t) \text{ in } T.$$

Then, there exists $\beta_\varepsilon \in L^1(T;\mathbb{R}^3)$ such that $\operatorname{curl} \beta_\varepsilon = \operatorname{curl} \beta$ in T, $\beta_\varepsilon = \beta$ in a neighbourhood of ∂T, and

$$\mathcal{E}[\beta_\varepsilon, T] \leq h\psi_0(b,t) \log \frac{R}{\varepsilon} + c|b|^2 \left(h \left(\log \frac{R}{\varepsilon} \right)^{1/2} + \frac{h^3}{R^2} \right).$$

The constant c depends only on $\mathbb{C}$ and the constant c^* in (5.30).

Proof. The proof is similar to that of Lemma 5.10. Again, we set $t = e_3$ for simplicity of notation. Since $\operatorname{curl}(\beta - \eta_\varepsilon * \beta_{b,t}) = 0$, as previously there exists $v \in W^{1,1}(T;\mathbb{R}^3)$ such that $\eta_\varepsilon * \beta_{b,t} = \beta + Dv$ and, from (5.30) and (5.6), we obtain

$$|\eta_\varepsilon * \beta_{b,t}|(x) + |Dv|(x) \leq \frac{c|b|}{\varepsilon + \operatorname{dist}(x, \mathbb{R}t)}.$$

Therefore, (5.28) holds in this case as well. Also as previously, we define $\beta_\varepsilon := \beta + Du$, with u as in (5.29), so that $\beta_\varepsilon = \eta_\varepsilon * \beta_{b,t}$ on the set $\{\theta_R\theta_3 = 1\}$ and

$u = 0$ and $\beta_\varepsilon = \beta$ on the boundary of T. Since $|\eta_\varepsilon * \beta_{b,t}| \le c|b|/\varepsilon$, by convexity we have

$$\begin{aligned}
&\mathcal{E}[\eta_\varepsilon * \beta_{b,t}, B'_R \times (0,h)] \\
&\le \mathcal{E}[\beta_{b,t}, (B'_{R+\varepsilon} \setminus B'_\varepsilon) \times (-\varepsilon, h+\varepsilon)] + c\frac{|b|^2}{\varepsilon^2}\mathcal{L}^3(B'_{2\varepsilon} \times (0,h)) \\
&= (h+2\varepsilon)\log\frac{R+\varepsilon}{\varepsilon}\psi_0(b,t) + 4\pi ch|b|^2 .
\end{aligned}$$

The contribution of the interpolation regions outside the inner region in $B'_\varepsilon \times (0,h)$ is estimated as in Lemma 5.10. The region $B'_\varepsilon \times ((0,\delta) \cup (h-\delta, h))$ still requires consideration. In this regard, we use that $|\beta| + |Dv| \le c|b|/\varepsilon$ and (5.28) to estimate

$$|Du|(x) \le c|b|\left(\frac{1}{\varepsilon} + \frac{h}{R\delta} + \frac{1}{\delta}\log\frac{R}{\varepsilon}\right),$$

with the result

$$\mathcal{E}[\beta_\varepsilon, B'_\varepsilon \times ((0,\delta) \cup (h-\delta,h))] \le c|b|^2\delta\varepsilon^2\left(\frac{1}{\varepsilon^2} + \frac{h^2}{R^2\delta^2} + \frac{1}{\delta^2}\left(\log\frac{R}{\varepsilon}\right)^2\right).$$

Summing all terms, we obtain

$$\begin{aligned}
\mathcal{E}[\beta_\varepsilon, T] &\le (h+2\varepsilon)\log\frac{R+\varepsilon}{\varepsilon}\psi_0(b,t) \\
&+ c|b|^2\left(h + \frac{h^3}{R^2} + \delta\log\frac{R}{\varepsilon} + \frac{h^2}{\delta} + \frac{R^2}{\delta} + \delta + \frac{h^2\varepsilon^2}{R^2\delta} + \frac{\varepsilon^2}{\delta}\left(\log\frac{R}{\varepsilon}\right)^2\right).
\end{aligned}$$

Since $x\log\frac{1}{x} < 1$ for $x \in (0,1)$, we can estimate $\varepsilon\log(R/\varepsilon) < R$, which implies that the last term is bounded by R^2/δ. Recalling $\varepsilon \le R \le h$ and $\delta \le h$, this estimate reduces to

$$\mathcal{E}[\beta_\varepsilon, T] \le h\log\frac{R}{\varepsilon}\psi_0(b,t) + c|b|^2\left(\frac{h^3}{R^2} + \delta\log\frac{R}{\varepsilon} + \frac{h^2}{\delta}\right),$$

as in Lemma 5.10. We then choose $\delta := h(\log(R/\varepsilon))^{-1/2}$, which is admissible since $\varepsilon < R/3$, and conclude the proof. □

6 Proof of the main results

Before starting, we briefly summarize and streamline the general relaxation results from [18], which constitute a key enabling element of the proofs.

6.1 Relaxation of functionals defined on curves

Let $\Omega \subset \mathbb{R}^n$ be open and let $\mathcal{B} \subset \mathbb{R}^m$ be a discrete lattice. We denote by $\mathcal{M}_{\mathcal{B}}(\Omega)$ the set of divergence-free measures of the form $\mu = b \otimes t\mathcal{H}^1 \llcorner \gamma$, with $b \in L^1(\gamma; \mathcal{B}; \mathcal{H}^1 \llcorner \gamma)$ and $t \in L^\infty(\gamma; S^{n-1}; \mathcal{H}^1 \llcorner \gamma)$, the tangent to γ. We say that μ is polyhedral if γ is the union of finitely many segments and b is constant on each of them. In this section we do not restrict attention to the three-dimensional case, since dimension is irrelevant to the statements and proofs.

Theorem 6.1 (Compactness). *Let $\mu_k \in \mathcal{M}_{\mathcal{B}}(\Omega)$ be a bounded sequence. Then there exists a measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$ and a subsequence such that*

$$\mu_{k_j} \overset{*}{\rightharpoonup} \mu \,.$$

Proof. This follows from the similar result for scalar currents [24, Theorem 4.2.16] working componentwise. The limiting current has the given form by the structure theorem in [18, Theorem 2.5]. □

Measures in $\mathcal{M}_{\mathcal{B}}(\Omega)$ can be extended to measures in $\mathcal{M}_{\mathcal{B}}(\mathbb{R}^n)$ so that the total variation in a neighbourhood of Ω is not too large. Here and subsequently, we denote by

$$\Omega_\delta := \{x : \operatorname{dist}(x, \Omega) < \delta\} \,.$$

the δ-neighborhood of Ω.

Lemma 6.2 (Extension). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz open set. Then, there is a bounded operator $T : \mathcal{M}_{\mathcal{B}}(\Omega) \to \mathcal{M}_{\mathcal{B}}(\mathbb{R}^n)$ such that $\mu = (T\mu) \llcorner \Omega$ for all μ and $\lim_{\delta \to 0} |T\mu|(\Omega_\delta \setminus \Omega) = 0$.*

Proof. The existence of an extension operator follows immediately from [18, Lemma 2.3] interpreting the measures in $\mathcal{M}_{\mathcal{B}}(\mathbb{R}^n)$ as rectifiable currents. □

The following density result ensures strong approximation of a measure $\mu \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^n)$ by a deformed polyhedral measure. For $f \in \operatorname{Lip}(\mathbb{R}^n; \mathbb{R}^n)$, we define the push-forward of the measure $\mu = b \otimes t\mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^n)$ as

$$f_\sharp \mu := b \circ f^{-1} \otimes \tau \mathcal{H}^1 \llcorner f(\gamma) \,, \qquad \tau := \frac{D_t f}{|D_t f|} \circ f^{-1},$$

where $D_t f$ denotes the tangential derivative of f along γ, which exists $\mathcal{H}^1$-almost everywhere on γ since f is Lipschitz on γ. If f is differentiable in x, then $D_t f(x) = Df(x)t(x)$. We remark that this definition corresponds to the push-forward of a current and not to the usual push-forward of a measure, since the tangent vector is also transported. This notion of push-forward is the appropriate one here since it preserves the divergence-free condition.

Theorem 6.3 (Density)**.** *For $\mu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^n)$ and $\varepsilon > 0$, there exists a bijective map $f \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ and a polyhedral measure $\nu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^n)$ such that*

$$|f_\sharp\mu - \nu|(\mathbb{R}^n) \le \varepsilon$$

and

$$|Df(x) - \mathrm{Id}| + |f(x) - x| \le \varepsilon \quad \text{for all } x \in \mathbb{R}^n\,.$$

Moreover, $f(x) = x$ whenever $\operatorname{dist}(x, \operatorname{supp}\mu) \ge \varepsilon$.

Proof. This follows immediately from [18, Theorem 2.1] interpreting the measures in $\mathcal{M}_\mathcal{B}(\mathbb{R}^n)$ as rectifiable currents. □

In the proof of the upper bound we shall need a refined version of the density result, which ensures continuity of the unrelaxed energy. Precisely, for $\psi \in C^0(\mathcal{B} \times S^{n-1}; [0,\infty))$ we define

$$F[\mu,\Omega] := \int_\gamma \psi(b,t)\,d\mathcal{H}^1, \qquad \text{if } \mu \llcorner \Omega = b \otimes t\mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_\mathcal{B}(\Omega)$$

and ∞ otherwise.

Lemma 6.4. *Assume that ψ satisfies*

$$\psi(b,t) \le (1 + c|t - t'|)\psi(b,t') \tag{6.1}$$

and fix a measure $\mu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^n)$ such that

$$\lim_{\rho\to 0} |\mu|(\Omega_\rho \setminus \Omega) = 0\,.$$

Then, for every $\eta \in (0,1)$, there exists $r > 0$, a polyhedral measure $\nu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^n)$ and a bijective map $f \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$|f_\sharp\mu - \nu|(\mathbb{R}^n) \le \eta,$$

$$|Df(x) - \mathrm{Id}| + |f(x) - x| \le \eta \qquad \text{for all } x \in \mathbb{R}^n,$$

and

$$F[\nu,\Omega_r] \le (1 + c\eta)F[\mu,\Omega] + c\eta\,.$$

Furthermore, $\nu \llcorner \Omega$ is polyhedral.

Proof. Let $\mu = b \otimes t\mathcal{H}^1 \llcorner \gamma$. We choose $M_\eta > 1$ such that

$$\int_{\gamma\cap\{|b|>M_\eta\}} |b|\,d\mathcal{H}^1 \le \eta$$

and $r \in (0,\eta)$ such that

$$|\mu|(\Omega_{4r} \setminus \Omega) \le \frac{\eta^2}{M_\eta}\,.$$

We fix $\varepsilon \in (0, r)$, chosen below. We apply the deformation theorem (Theorem 6.3) to μ and find a diffeomorphism f such that

$$|Df(x) - \mathrm{Id}| + |f(x) - x| \leq \varepsilon \qquad \text{for all } x \in \mathbb{R}^n,$$

and a polyhedral measure $\hat{\mu} \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^n)$ of the form

$$\hat{\mu} = \sum_h b_h \otimes t_h \mathcal{H}^1 \llcorner \gamma_h$$

such that $|f_\sharp \mu - \hat{\mu}|(\mathbb{R}^n) \leq \varepsilon$. However, the bound on the difference does not give a bound on the energy difference, since the energy density ψ does not have linear growth. Therefore, we construct ν by modifying $\hat{\mu}$ so that the parts in which it differs from $f_\sharp \mu$ have a uniformly bounded multiplicity b. In order to deal with the boundary terms, it is helpful to assume γ_h pairwise disjoint, up to the endpoints, and $\mathcal{H}^1(\gamma_h) \leq r$ for all h, which can always be ensured by increasing the number of segments in the sum.

We start by showing that

$$F[f_\sharp \mu, f(\Omega)] \leq (1 + c\varepsilon) F[\mu, \Omega] \,. \tag{6.2}$$

To verify this bound, we use the change of variables formula, to obtain

$$F[f_\sharp \mu, f(\Omega)] = \int_{f(\gamma) \cap f(\Omega)} \psi(b \circ f^{-1}, \tau) d\mathcal{H}^1 \leq (1 + \varepsilon) \int_{\gamma \cap \Omega} \psi(b, \tau \circ f) d\mathcal{H}^1$$

where the error $(1 + \varepsilon)$ is an estimate of the Jacobian. By the properties of f, $|t - \tau \circ f| \leq 2\varepsilon$. Recalling (6.1), we conclude the proof of (6.2). Analogously, we obtain

$$\int_{f(\gamma) \cap \{|b \circ f^{-1}| > M_\eta\}} |b \circ f^{-1}| d\mathcal{H}^1 \leq 2\eta \,. \tag{6.3}$$

It remains to estimate the boundary contributions, which arise because Ω_r is not contained in $f(\Omega)$, and the error terms arising from the difference $f_\sharp \mu - \hat{\mu}$. Both are small in measure, but the energy does not have linear growth. Therefore, we first select the segments where these two effects have a significant contribution and proceed to modify the measure on those segments. For every h and $\mathcal{H}^1$-almost every $x \in \gamma_h$, we define

$$g_h(x) := \frac{d[(f_\sharp \mu) \llcorner (\gamma_h \cap f(\Omega))]}{d[\mathcal{H}^1 \llcorner \gamma_h]}(x) \,,$$

so that

$$\sum_h \int_{\gamma_h \cap f(\Omega)} |g_h - b_h \otimes t_h| d\mathcal{H}^1 \leq |f_\sharp \mu - \hat{\mu}|(f(\Omega)) \leq \varepsilon \,. \tag{6.4}$$

Notice that $g_h(x) = 0$, if $x \notin f(\Omega)$. Let $\hat{\gamma}_h := \{x \in \gamma_h : g_h(x) = b_h \otimes t_h\}$, i.e., the part of γ_h whose energy is already accounted for in $F[f_\sharp \mu, f(\Omega)]$.

We fix $\delta \in (0,1)$, chosen below, and define H as the set of indices h such that $\hat\gamma_h$ covers most of the segment γ_h, namely,

$$H := \left\{h : \mathcal{H}^1(\gamma_h \setminus \hat\gamma_h) < \delta \mathcal{H}^1(\gamma_h)\right\} .$$

If $h \in H$, then $\mathcal{H}^1(\hat\gamma_h) > (1-\delta)\mathcal{H}^1(\gamma_h)$. We compute

$$\sum_{h\in H} \psi(b_h, t_h)\mathcal{H}^1(\gamma_h) \le \sum_{h\in H} \psi(b_h, t_h)\frac{\mathcal{H}^1(\hat\gamma_h)}{1-\delta} \le \frac{1}{1-\delta} F[f_\sharp\mu, f(\Omega)] . \tag{6.5}$$

This contribution can be estimated by (6.2).

To estimate the remaining contributions, we define

$$K := \{h : h \notin H, \gamma_h \cap \Omega_{2r} \neq \emptyset\} .$$

The indices which are not in H and not in K can be ignored, since they do not contribute to the energy in Ω_{2r}. For every $h \in K$, we estimate

$$\mathcal{H}^1(\gamma_h) \le \frac{1}{\delta}\mathcal{H}^1(\gamma_h \setminus \hat\gamma_h) \le \frac{1}{c_*\delta}\int_{\gamma_h} |g_h - b_h \otimes t_h| d\mathcal{H}^1 ,$$

where we use that $|g_h(x) - b_h \otimes t_h|$ is, for $\mathcal{H}^1$-a. e. x, either zero or, at least, $c_* > 0$, since $\mathcal{B}$ is a discrete set and the tangent to a curve is a. e. uniquely defined, up to a sign. Therefore, for $h \in K$ with $|b_h| \le 2M_\eta$ we obtain

$$|b_h|\mathcal{H}^1(\gamma_h) \le c\frac{M_\eta}{\delta}\int_{\gamma_h} |g_h - b_h \otimes t_h| d\mathcal{H}^1 .$$

Simultaneously, if $|b_h| > 2M_\eta$ then a triangular inequality shows that

$$|b_h| \le 2|g_h - b_h \otimes t_h| + |g_h|\chi_{\{|g_h|>M_\eta\}}$$

pointwise. Therefore, for $h \in K$ with $|b_h| > 2M_\eta$,

$$|b_h|\mathcal{H}^1(\gamma_h) \le 2\int_{\gamma_h} |g_h - b_h \otimes t_h| d\mathcal{H}^1 + \int_{\gamma_h\cap\{|g_h|>M_\eta\}} |g_h| d\mathcal{H}^1 .$$

Adding terms, and recalling that $M_\eta/\delta > 1$,

$$\sum_{h\in K} |b_h|\mathcal{H}^1(\gamma_h) \le c\frac{M_\eta}{\delta}\sum_{h\in K}\int_{\gamma_h} |g_h - b_h \otimes t_h| d\mathcal{H}^1 + \sum_{h\in K}\int_{\gamma_h\cap\{|g_h|>M_\eta\}} |g_h| d\mathcal{H}^1 .$$

We estimate

$$\begin{aligned}
\sum_{h\in K}\int_{\gamma_h} |g_h - b_h \otimes t_h| d\mathcal{H}^1 \le & \sum_{h\in K}\int_{\gamma_h\cap f(\Omega)} |g_h - b_h \otimes t_h| d\mathcal{H}^1 \\
& + \int_{\gamma_h\setminus f(\Omega)} |b_h \otimes t_h| d\mathcal{H}^1 \\
\le & \varepsilon + |\hat\mu|(\Omega_{3r} \setminus f(\Omega)),
\end{aligned}$$

where we use (6.4) and the fact that no segment is longer than r. Furthermore,

$$\begin{aligned}|\hat\mu|(\Omega_{3r}\setminus f(\Omega)) &\le |f_\sharp\mu|(\Omega_{3r}\setminus f(\Omega)) + |\hat\mu - f_\sharp\mu|(\mathbb{R}^n)\\ &\le 2|\mu|(f^{-1}(\Omega_{3r})\setminus\Omega) + |\hat\mu - f_\sharp\mu|(\mathbb{R}^n)\\ &\le 2|\mu|(\Omega_{4r}\setminus\Omega) + |\hat\mu - f_\sharp\mu|(\mathbb{R}^n) \le 2\frac{\eta^2}{M_\eta} + \varepsilon\,.\end{aligned}$$

Analogously, and recalling (6.3),

$$\begin{aligned}\sum_{h\in K}\int_{\gamma_h\cap\{|g_h|>M_\eta\}} |g_h| d\mathcal{H}^1 &\le |f_\sharp\mu - \hat\mu| + \int_{f(\gamma)\cap\{|b\circ f^{-1}|>M_\eta\}} |b\circ f^{-1}| d\mathcal{H}^1\\ &\le \varepsilon + 2\eta\,.\end{aligned}$$

Therefore

$$\sum_{h\in K} |b_h|\mathcal{H}^1(\gamma_h) \le c\frac{M_\eta}{\delta}(2\varepsilon + 2\frac{\eta^2}{M_\eta}) + \varepsilon + 2\eta \le c\frac{M_\eta\varepsilon}{\delta} + c\frac{\eta^2}{\delta} + 3\eta\,.$$

We choose $\delta := \eta$ and $\varepsilon \le \eta\delta/M_\eta$, so that

$$\sum_{h\in K} |b_h|\mathcal{H}^1(\gamma_h) \le c\eta\,. \tag{6.6}$$

In order to transform this bound into an estimate of the energy, it only remains to modify the measure so that the multiplicities are bounded. We first choose a finite set $\mathcal{A}\subset\mathcal{B}$ such that each element of $\mathcal{B}$ is a finite sum of elements of $\mathcal{A}$. For example if $\mathcal{B} = \mathbb{Z}^m$ then it suffices that $\mathcal{A} = \{\pm e_i\}$.

We iterate the following procedure for all $h\in K$, dropping the index h from most of the quantities for notational simplicity. Assume $\gamma_h = [x, y]$, with $x, y \in \mathbb{R}^n$. Fix N vectors $d_1, \dots, d_N \in \mathcal{A}$ such that $\sum_l d_l = b_h$, and $\sum_l |d_l| \le c_{\mathcal{A}}|b_h|$.

Fix a small ball B around the midpoint of γ_h, with radius smaller than $\mathcal{H}^1(\gamma_h)$. We choose N distinct points $p_1, \dots, p_N$ in B such that the segments $[x, p_l]$, $[y, p_l]$, $\{\gamma_{h'}\}_{h'}$ are all pairwise disjoint up to the endpoints.

We set

$$\hat\mu' := \hat\mu - b_h\otimes t_h\mathcal{H}^1\llcorner[x,y] + \sum_{l=1}^N d_l\otimes\tau_l\mathcal{H}^1\llcorner[x,p_l] + \sum_{l=1}^N d_l\otimes\tau_l'\mathcal{H}^1\llcorner[p_l,y]\,,$$

where τ_l and τ_l' are tangent vectors to $[x, p_l]$ and $[p_l, y]$, with the same direction as t_h. Then, $|\hat\mu' - \hat\mu|(\mathbb{R}^n) \le c|x-y|\sum_l |d_l| \le cc_{\mathcal{A}}|b_h|\mathcal{H}^1(\gamma_h)$.

We repeat the same procedure for all indices $h\in K$. At each step, the points are chosen so that the new segments are disjoint from all the previous segments. The result is a polyhedral measure $\nu\in\mathcal{M}_{\mathcal{B}}(\mathbb{R}^n)$ of the form

$$\nu = \sum_{h\notin K} b_h\otimes t_h\mathcal{H}^1\llcorner\gamma_h + \sum_{h\in K}\sum_{l=1}^{2N_h} b_{h,l}\otimes t_{h,l}\mathcal{H}^1\llcorner\gamma_{h,l}\,,$$

where $b_{h,l} \in \mathcal{A}$. It satisfies

$$|\nu - \hat{\mu}|(\mathbb{R}^n) \le c \sum_{h\in K} |b_h| \mathcal{H}^1(\gamma_h) \,.$$

We finally estimate, recalling (6.6),

$$\sum_{h\in K} \sum_{l=1}^{2N_h} \psi(b_{h,l}, t_{h,l}) \mathcal{H}^1(\gamma_{h,l}) \le c \sum_{h\in K} |b_h| \mathcal{H}^1(\gamma_h) \le c\eta \,,$$

where the constant depends on $\max \psi(\mathcal{A} \times S^{n-1})$.

Recalling (6.5), (6.2) and (6.6) and the choices $\delta = \eta$, $\varepsilon \le \eta$, we obtain

$$\begin{aligned} F[\nu, \Omega_{2r}] &\le \sum_{h\in H} \psi(b_h, t_h) \mathcal{H}^1(\gamma_h) + \sum_{h\in K} \sum_{l=1}^{2N_h} \psi(b_{h,l}, t_{h,l}) \mathcal{H}^1(\gamma_{h,l}) \\ &\le \frac{1}{1-\eta} F[f_\sharp \mu, f(\Omega)] + c\eta \,. \end{aligned}$$

It only remains to ensure that the restriction of ν to Ω is polyhedral. This property is not automatic, since the intersection of a segment with a Lipschitz domain may consist of countably many segments. However, we shall show that almost all translations of ν give rise to a polyhedral intersection. Since the support of ν consists of finitely many segments, it suffices to prove this property for any segment. Precisely, we claim that for any given pair of directions $t, \tau \in S^{n-1}$, with $t \cdot \tau \neq 0$, for almost every $z \in \tau^\perp$ the set $z + \mathbb{R}t \cap \partial\Omega$ is finite. To very this property, we apply the coarea formula with $f : \mathbb{R}^n \to \tau^\perp$ the projection along t to obtain [7, Theorem 2.93]

$$\int_{\partial\Omega} (C_{n-1} D^{\partial\Omega} f) d\mathcal{H}^{n-1} = \int_{\tau^\perp} \mathcal{H}^0(\partial\Omega \cap f^{-1}(z)) d\mathcal{H}^{n-1}(z) \,.$$

Since f and $\partial\Omega$ are Lipschitz, the tangential differential of f at points of $\partial\Omega$, $D^{\partial\Omega} f$ and the coarea factor $C_{n-1} D^{\partial\Omega} f$ are bounded. Hence, the left-hand side is finite. Therefore, the integrand in the right-hand side is finite almost everywhere and, for almost every $z \in \tau^\perp$, the set $\partial\Omega \cap z + \mathbb{R}t$ is finite. We choose a τ not orthogonal to any of the segments composing ν and conclude that, for almost every $y \in \tau^\perp$, the measure $[(\mathrm{Id} + y)_\sharp]\nu \llcorner \Omega$ is polyhedral. Since translations leave the energy invariant, choosing one such y with length less than r and modifying f accordingly concludes the proof. $\square$

Theorem 6.5 (Relaxation)**.** *Let $\psi : \mathcal{B} \times S^{n-1} \to [0, \infty)$ be Borel-measurable with $\psi(b, t) \ge c_0 |b|$ and $\psi(0, \cdot) = 0$. Define ψ^{rel} as*

$$\begin{aligned} \psi^{\mathrm{rel}}(b, t) := \inf \Big\{ \int_\gamma \psi(\theta, \tau)\, d\mathcal{H}^1 \ :\ & \mu = \theta \otimes \tau \mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathcal{B}}(B_{1/2}) \,, \\ & \mathrm{supp}(\mu - b \otimes t \mathcal{H}^1 \llcorner (\mathbb{R}t \cap B_{1/2})) \subset\subset B_{1/2} \Big\} \,. \end{aligned} \tag{6.7}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz set. Then $F^{\mathrm{rel}} : \mathcal{M}_{\mathcal{B}}(\Omega) \to [0, \infty)$, defined by $F^{\mathrm{rel}}[b \otimes t\mathcal{H}^1 \llcorner \gamma, \Omega] := \int_\gamma \psi^{\mathrm{rel}}(b, t) d\mathcal{H}^1$, is the lower-semicontinuous envelope of F with respect to weak convergence in $\mathcal{M}_{\mathcal{B}}(\Omega)$, in the sense that

$$F^{\mathrm{rel}}[\mu, \Omega] = \inf \left\{ \liminf_{j\to\infty} F[\mu_j, \Omega] : \mu_j \in \mathcal{M}_{\mathcal{B}}(\Omega),\ \mu_j \overset{*}{\rightharpoonup} \mu \right\} .$$

In particular, F^{rel} is lower-semicontinuous.

Proof. This theorem has been proved in [18, Theorem 3.1]. □

6.2 Compactness and lower bound

This section is devoted to showing the compactness and the lower bound in Theorem 3.2, as stated in the following Proposition.

Proposition 6.6. *Under the assumptions of Theorem 3.2, for every sequence $\mu_\varepsilon \in \mathcal{M}_{\mathcal{B}}^{h_\varepsilon,\alpha_\varepsilon}(\Omega)$ such that $F_\varepsilon^c[\mu_\varepsilon, \Omega] \le C$ or $F_\varepsilon^m[\mu_\varepsilon, \Omega] \le C$ for infinitely many ε, there exists a subsequence that converges weakly to a measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$. Moreover, for any sequence $\mu_\varepsilon \in \mathcal{M}_{\mathcal{B}}^{h_\varepsilon,\alpha_\varepsilon}(\Omega)$ that converges weakly to some measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$, we have*

$$F_0[\mu, \Omega] \le \liminf_{\varepsilon\to 0} F_\varepsilon^c[\mu_\varepsilon, \Omega] ,$$

and the likewise for F_ε^m.

Proof. Let $\mu_\varepsilon \in \mathcal{M}_{\mathcal{B}}^{h_\varepsilon,\alpha_\varepsilon}(\Omega)$ be as in the statement. We choose $\beta_\varepsilon^c \in L^1(\Omega; \mathbb{R}^{3\times 3})$ such that $\operatorname{curl} \beta_\varepsilon^c = \mu_\varepsilon$ in Ω and

$$\mathcal{E}[\beta_\varepsilon^c, \Omega^\varepsilon(\mu_\varepsilon)] \le \log(1/\varepsilon) \left(F_\varepsilon^c[\mu_\varepsilon, \Omega] + \varepsilon\right) .$$

In the case of F_ε^m, we choose first an extension $\hat\mu_\varepsilon \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3)$ of μ_ε and then $\beta_\varepsilon^m \in L^1(\Omega; \mathbb{R}^{3\times 3})$ such that $\operatorname{curl} \beta_\varepsilon^m = \eta_\varepsilon * \hat\mu_\varepsilon$ in Ω and

$$\mathcal{E}[\beta_\varepsilon^m, \Omega] \le \log(1/\varepsilon) \left(F_\varepsilon^m[\mu_\varepsilon, \Omega] + \varepsilon\right) .$$

Since μ_ε is dilute, we have that $\mu_\varepsilon = \sum_i b_\varepsilon^i \otimes t_\varepsilon^i \mathcal{H}^1 \llcorner \gamma_\varepsilon^i$, where $\gamma_\varepsilon^i \subset \Omega$ are segments satisfying the diluteness conditions (i)–(iii) of Definition 3.1. We choose $\rho_\varepsilon := (\alpha_\varepsilon h_\varepsilon)^2$, $\delta_\varepsilon := \alpha_\varepsilon h_\varepsilon$. We let $S_\varepsilon^i \subset \mathbb{R}$ be a segment of length $\mathcal{H}^1(\gamma_\varepsilon^i) - 2\delta_\varepsilon$, A_ε^i an affine isometry that maps $S_\varepsilon^i e_3$ into γ_ε^i and the midpoint of $S_\varepsilon^i e_3$ to the midpoint of γ_ε^i. We define the cylinders

$$T_\varepsilon^i := A_\varepsilon^i(B'_{\rho_\varepsilon} \times S_\varepsilon^i) \text{ and } \hat T_\varepsilon^i := A_\varepsilon^i(B'_\varepsilon \times S_\varepsilon^i) .$$

Since disjoint segments in the family $\{\gamma_\varepsilon^i\}_i$ are separated by $\alpha_\varepsilon h_\varepsilon \gg \rho_\varepsilon$ and the angle between two non-disjoint segments is larger than α_ε, for small ε it follows that $\delta_\varepsilon \tan \frac{1}{2}\alpha_\varepsilon > \rho_\varepsilon$ and, hence, the sets $\{T^i_\varepsilon\}_i$ are pairwise disjoint and $T^i_\varepsilon \cap \gamma^j_\varepsilon = \emptyset$ for all $j \neq i$. Therefore,

$$\mathcal{E}[\beta^c_\varepsilon, \Omega^\varepsilon(\mu_\varepsilon)] \geq \sum_i \mathcal{E}[\beta^c_\varepsilon, T^i_\varepsilon \setminus \hat{T}^i_\varepsilon]$$

and

$$\mathcal{E}[\beta^m_\varepsilon, \Omega] \geq \sum_i \mathcal{E}[\beta^m_\varepsilon, T^i_\varepsilon \setminus \hat{T}^i_\varepsilon] .$$

Recalling the definition of ψ in (5.13) in the first case and Lemma 5.4 in the second, we get, denoting by $|S^i_\varepsilon| := \mathcal{H}^1(S^i_\varepsilon)$ the height of T^i_ε,

$$\begin{aligned} \mathcal{E}[\beta^z_\varepsilon, T^i_\varepsilon \setminus \hat{T}^i_\varepsilon] &\geq |S^i_\varepsilon| \log \frac{\rho_\varepsilon}{\varepsilon} \psi(b^i_\varepsilon, t^i_\varepsilon, |S^i_\varepsilon|, \varepsilon, \rho_\varepsilon) \\ &= |S^i_\varepsilon| \log \frac{\rho_\varepsilon}{\varepsilon} \psi(b^i_\varepsilon, t^i_\varepsilon, \frac{|S^i_\varepsilon|}{\rho_\varepsilon}, \frac{\varepsilon}{\rho_\varepsilon}, 1) \end{aligned} \tag{6.8}$$

for $z \in \{c, m\}$. By (5.23) we have $\psi(b^i_\varepsilon, t^i_\varepsilon, \frac{|S^i_\varepsilon|}{\rho_\varepsilon}, \frac{\varepsilon}{\rho_\varepsilon}, 1) \geq c_*|b^i_\varepsilon|^2$. Therefore, with $c' := c_*/\min\{|b| : b \in \mathcal{B} \setminus \{0\}\}$,

$$\begin{aligned} F^z_\varepsilon[\mu_\varepsilon, \Omega] + \varepsilon &\geq c_* \sum_i |S^i_\varepsilon||b^i_\varepsilon|^2 \frac{\log(\rho_\varepsilon/\varepsilon)}{\log(1/\varepsilon)} \\ &\geq c_* \sum_i (1 - 2\alpha_\varepsilon)\mathcal{H}^1(\gamma^i_\varepsilon)|b^i_\varepsilon|^2 \frac{\log(\rho_\varepsilon/\varepsilon)}{\log(1/\varepsilon)} \\ &\geq c'(1 - 2\alpha_\varepsilon)\frac{\log(\rho_\varepsilon/\varepsilon)}{\log(1/\varepsilon)}|\mu_\varepsilon|(\Omega) . \end{aligned}$$

By (3.1), we obtain

$$\lim_{\varepsilon \to 0} \frac{\log(\rho_\varepsilon/\varepsilon)}{\log(1/\varepsilon)} = \lim_{\varepsilon \to 0} \left[1 - \frac{\log(1/\rho_\varepsilon)}{\log(1/\varepsilon)}\right] = 1 .$$

Therefore, the sequence μ_ε is bounded in $\mathcal{M}_\mathcal{B}(\Omega)$ and compactness follows from Theorem 6.1.

We now turn to the lower bound. Fix $M > 1$, chosen below. For sufficiently small ε, we have $|S^i_\varepsilon| \geq M\rho_\varepsilon$ for all i. Let ω_M be the function from Lemma 5.8, so that

$$\psi(b^i_\varepsilon, t^i_\varepsilon, \frac{|S^i_\varepsilon|}{\rho_\varepsilon}, \frac{\varepsilon}{\rho_\varepsilon}, 1) \geq \left(1 - \frac{c}{M} - \omega_M(\frac{\varepsilon}{\rho_\varepsilon})\right) \psi_0(b^i_\varepsilon, t^i_\varepsilon) .$$

Inserting this estimate into (6.8) and recalling that $|S^i_\varepsilon| \geq (1 - 2\alpha_\varepsilon)\mathcal{H}^1(\gamma^i_\varepsilon)$ and $\psi_0(b^i_\varepsilon, t^i_\varepsilon) \geq \psi^{\mathrm{rel}}_0(b^i_\varepsilon, t^i_\varepsilon)$, we have

$$\begin{aligned} F^z_\varepsilon[\mu_\varepsilon, \Omega] + \varepsilon &\geq \sum_i \mathcal{H}^1(\gamma^i_\varepsilon)(1 - 2\alpha_\varepsilon) \left(1 - \frac{c}{M} - \omega_M\left(\frac{\varepsilon}{\rho_\varepsilon}\right)\right) \psi^{\mathrm{rel}}_0(b^i_\varepsilon, t^i_\varepsilon) \\ &\geq \left(1 - \frac{c}{M} - \omega_M\left(\frac{\varepsilon}{\rho_\varepsilon}\right)\right) (1 - 2\alpha_\varepsilon)F_0[\mu_\varepsilon, \Omega] . \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ gives

$$\liminf_{\varepsilon\to 0} F_\varepsilon^z[\mu_\varepsilon, \Omega] \geq (1 - \frac{c}{M}) \liminf_{\varepsilon\to 0} F_0[\mu_\varepsilon, \Omega] \,.$$

In view of Theorem 6.5, the relaxed energy F_0 is lower-semicontinuous and the lower bound follows by taking the limit as $M \to \infty$. $\square$

6.3 Construction of the recovery sequence

The upper bound is proved in two steps. In Proposition 6.7, we first show that the unrelaxed self-energy per unit length of any polygonal dislocation network μ can be recovered by a strain field β_ε. Secondly, in Proposition 6.8 we prove that the relaxed energy can be achieved by the energy of unrelaxed polygonal networks and take a diagonal subsequence.

In order to separate the two parts of the proof, we consider the unrelaxed functional, denoted by F_u and defined as

$$F_\mathrm{u}[\mu, \Omega] := \int_\gamma \psi_0(b,t) d\mathcal{H}^1 \qquad \text{if } \mu \llcorner \Omega = b \otimes t \mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_\mathcal{B}(\Omega) \tag{6.9}$$

and ∞ otherwise, where ψ_0 is the unrelaxed self-energy per unit length of Lemma 5.1.

Proposition 6.7. *Let $\mu \in \mathcal{M}_\mathcal{B}(\mathbb{R}^3)$ be polyhedral, $r > 0$. Then for any $\varepsilon > 0$, there exists $\beta_\varepsilon \in L^1(\Omega; \mathbb{R}^{3\times 3})$ such that* $\operatorname{curl} \beta_\varepsilon = \mu$ *in Ω and*

$$\limsup_{\varepsilon\to 0} \frac{1}{\log(1/\varepsilon)} \mathcal{E}[\beta_\varepsilon, \Omega^\varepsilon(\mu)] \leq F_\mathrm{u}[\mu, \Omega_r] \,. \tag{6.10}$$

Furthermore, there exists $\hat\beta_\varepsilon \in L^1(\Omega; \mathbb{R}^{3\times 3})$ such that $\operatorname{curl} \hat\beta_\varepsilon = \eta_\varepsilon * \mu$ *in Ω and*

$$\limsup_{\varepsilon\to 0} \frac{1}{\log(1/\varepsilon)} \mathcal{E}[\hat\beta_\varepsilon, \Omega] \leq F_\mathrm{u}[\mu, \Omega_r] \,. \tag{6.11}$$

Proof. The general strategy that we follow is to use the construction of Theorem 4.1 for the global structure and to improve on it using Lemma 5.10 or 5.11 around the largest part of the dislocations.

Let β^μ be the solution to $\operatorname{div} \mathbb{C}\beta = 0$, $\operatorname{curl} \beta = \mu$ in $\mathbb{R}^3$, as given by Theorem 4.1. Since μ is polyhedral, it can be written as

$$\mu = \sum_{i=1}^M b_i \otimes t_i \mathcal{H}^1 \llcorner \gamma_i$$

where each γ_i is a segment, $t_i \in S^2$ its tangent vector, and $b_i \in \mathcal{B}$. Possibly increasing the number of segments, we can assume $\mathcal{H}^1(\gamma_i) \leq r/2$ for all i. For a $\delta_\varepsilon > 0$ and $\rho_\varepsilon > 0$ chosen below, such that $\delta_\varepsilon \to 0$ and $\rho_\varepsilon/\delta_\varepsilon \to 0$, we construct cylinders T_ε^i of radius ρ_ε and $\hat T_\varepsilon^i$ of radius ε, as in the proof of the lower bound.

We start from the construction of β_ε. For every cylinder, we apply Lemma 5.10 using β^μ as boundary data and let β_ε be equal to β^μ outside the union of the cylinders and the result of Lemma 5.10 inside each cylinder.

We estimate, using Theorem 4.1(iii),

$$\mathcal{E}[\beta, \Omega^\varepsilon(\mu) \setminus \cup_i T_\varepsilon^i] \le c_b \int_{\Omega^\varepsilon(\mu)\setminus \cup_i T_\varepsilon^i} \frac{1}{\operatorname{dist}^2(x, \operatorname{supp}\mu)} dx \le c_b \sum_i \int_{\Omega \setminus T_\varepsilon^i \setminus (\gamma_i)_\varepsilon} f_i^2 dx\,,$$

where c_b depends on the $\{b_i\}$, $(\gamma_i)_\varepsilon := \{x : \operatorname{dist}(x, \gamma_i) < \varepsilon\}$ and

$$f_i(x) := \frac{1}{\operatorname{dist}(x, \gamma_i)}\,.$$

Let $R = 2\operatorname{diam}(\Omega) + 2r$. If $\operatorname{dist}(\gamma_i, \Omega) < R/2$, then, after a change of variables, we can assume $\gamma_i = (-a, a)e_3$, $T_\varepsilon^i = B'_{\rho_\varepsilon} \times (-a + \delta_\varepsilon, a - \delta_\varepsilon)$, $\Omega \subset B_R$. We estimate

$$\begin{aligned}
&\int_{\Omega \setminus T_\varepsilon^i \setminus (\gamma_i)_\varepsilon} f_i^2 dx \\
&\le 2(a - \delta_\varepsilon) \int_{B'_R \setminus B'_{\rho_\varepsilon}} \frac{1}{|x'|^2} dx' + 2\delta_\varepsilon \int_{B'_R \setminus B'_\varepsilon} \frac{1}{|x'|^2} dx' + \int_{B_R \setminus B_\varepsilon} \frac{1}{|x|^2} dx \\
&= 4\pi(a - \delta_\varepsilon) \log\frac{R}{\rho_\varepsilon} + 4\pi\delta_\varepsilon \log\frac{R}{\varepsilon} + 4\pi(R - \varepsilon)\,.
\end{aligned}$$

If instead $\operatorname{dist}(\gamma_i, \Omega) \ge R/2$, then $\int_\Omega f_i^2 dx \le |\Omega|(4/R^2) \le R$. Adding terms, we conclude

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{E}[\beta, \Omega^\varepsilon(\mu) \setminus \cup_i T_\varepsilon^i]}{\log(1/\varepsilon)} = 0\,. \tag{6.12}$$

Let $I := \{i : \gamma_i \cap \Omega_{r/2} \ne \emptyset\}$. If $\rho_\varepsilon < r/2$, no cylinder T_ε^i with $i \notin I$ intersects Ω. By Lemma 5.10,

$$\begin{aligned}
&\mathcal{E}[\beta_\varepsilon, \Omega^\varepsilon(\mu) \cap \cup_i T_\varepsilon^i] \le \sum_{i\in I} \mathcal{E}[\beta_\varepsilon, T_\varepsilon^i \setminus \hat{T}_\varepsilon^i] \\
&\le \sum_{i\in I} \psi_0(b_i, t_i)\mathcal{H}^1(\gamma^i) \log\frac{\rho_\varepsilon}{\varepsilon} + c|b_i|^2 \left(\mathcal{H}^1(\gamma^i)\left(\log\frac{\rho_\varepsilon}{\varepsilon}\right)^{1/2} + \frac{\mathcal{H}^1(\gamma^i)^3}{\rho_\varepsilon^2}\right)\,.
\end{aligned}$$

Finally, we choose $\delta_\varepsilon := (\log(1/\varepsilon))^{-1/6}$ and $\rho_\varepsilon := \delta_\varepsilon^2$, so that

$$\limsup_{\varepsilon\to 0} \frac{\log(\rho_\varepsilon/\varepsilon)}{\log(1/\varepsilon)} = \limsup_{\varepsilon\to 0} \frac{\log(1/\varepsilon) - \log(1/\rho_\varepsilon)}{\log(1/\varepsilon)} = 1.$$

Inserting into the preceding estimate gives

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{E}[\beta_\varepsilon, \Omega^\varepsilon(\mu) \cap \cup_i T_\varepsilon^i]}{\log(1/\varepsilon)} \le \sum_{i\in I} \psi_0(b_i, t_i)\mathcal{H}^1(\gamma_i)\,.$$

Recalling (6.12), we conclude

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{E}[\beta_\varepsilon, \Omega^\varepsilon(\mu)]}{\log(1/\varepsilon)} \le \sum_{i\in I} \psi_0(b_i, t_i)\mathcal{H}^1(\gamma_i)\,.$$

Since each segment has length bounded by $r/2$ and intersects $\Omega_{r/2}$, all segments with $i \in I$ are contained in Ω_r. This concludes the proof of (6.10).

The proof of (6.11) is similar. We define $\hat\beta^\mu_\varepsilon := \eta_\varepsilon * \beta^\mu$ and use Lemma 5.11 instead of Lemma 5.10. The estimate in the cylinders is identical. The estimate outside the cylinders becomes

$$\mathcal{E}[\hat\beta, \Omega\setminus \cup_i T^i_\varepsilon] \le c_b \int_{\Omega\setminus\cup_i T^i_\varepsilon} \eta_\varepsilon * \frac{1}{\operatorname{dist}^2(x, \operatorname{supp}\mu)} dx \le c_b \sum_i \int_{\Omega\setminus T^i_\varepsilon} \hat f_i^2 dx\,,$$

where

$$\hat f_i(x) = (\eta_\varepsilon * f_i)(x) \le \frac{c}{\varepsilon + \operatorname{dist}(x,\gamma_i)}\,.$$

The only additional term is

$$\int_{(\gamma_i)_\varepsilon} \hat f_i^2(x) dx \le \left(\pi\mathcal{H}^1(\gamma_i) + \frac{4\pi}{3}\varepsilon\right)\varepsilon^2 \|\hat f_i\|^2_{L^\infty}$$

leading to the same estimates. □

Proposition 6.8. *Let $\mu \in \mathcal{M}_\mathcal{B}(\Omega)$. Then, for every sequence $\varepsilon_k \to 0$ there exists a sequence $\mu_k \in \mathcal{M}_\mathcal{B}(\mathbb{R}^n)$ such that $\mu_k \llcorner \Omega$ converges to μ weak-$*$ in measures, $\mu_k\llcorner\Omega$ is $(h_{\varepsilon_k}, \alpha_{\varepsilon_k})$-dilute and for every k there exists a $\beta_k \in L^1(\Omega;\mathbb{R}^{3\times 3})$ such that $\operatorname{curl}\beta_k = \mu_k$ and*

$$\limsup_{k\to\infty} \frac{1}{\log(1/\varepsilon_k)} \mathcal{E}[\beta_k, \Omega^{\varepsilon_k}(\mu_k)] \le F_0[\mu,\Omega]\,.$$

Furthermore, there exists a $\hat\beta_k \in L^1(\Omega;\mathbb{R}^{3\times 3})$ with $\operatorname{curl}\hat\beta_k = \eta_{\varepsilon_k} * \mu_k$ in Ω and

$$\limsup_{k\to\infty} \frac{1}{\log(1/\varepsilon_k)} \mathcal{E}[\hat\beta_k, \Omega] \le F_0[\mu,\Omega]\,.$$

Proof. By Theorem 6.5, there is a sequence $\mu_j \in \mathcal{M}_\mathcal{B}(\Omega)$ that converges weak-$*$ to μ and such that $\limsup_{j\to\infty} F_\mathrm{u}[\mu_j,\Omega] \le F_0[\mu,\Omega]$, where F_u is the unrelaxed energy defined in (6.9). Let $T\mu_j \in \mathcal{M}_\mathcal{B}(\mathbb{R}^n)$ be an extension, as given by Lemma 6.2. By Lemma 5.7, ψ_0 fulfills the continuity condition (6.1). By Lemma 6.4 applied to $T\mu_j$ and $\eta := \eta_j := 1/j$, there are polyhedral measures $\nu_j \in \mathcal{M}_\mathcal{B}(\mathbb{R}^3)$ and $r_j > 0$ such that

$$\limsup_{j\to\infty} F_\mathrm{u}[\nu_j, \Omega_{r_j}] \le F_0[\mu,\Omega]$$

and Lipschitz functions f_j converging to the identity such that $|(f_j)_\sharp T\mu_j - \nu_j|(\mathbb{R}^3) \to 0$, which in turn implies $\nu_j \overset{*}{\rightharpoonup} \mu$. Every restriction $\nu_j \llcorner \Omega$ is polyhedral and, therefore, (h,α)-dilute, provided that h and α are small enough.

Next, we fix j. By Proposition 6.7, there is a sequence β_k^j such that $\operatorname{curl}\beta_k^j = \nu_j$ and

$$\limsup_{k\to\infty} \frac{1}{\log(1/\varepsilon_k)} \mathcal{E}[\beta_k^j, \Omega^{\varepsilon_k}(\nu_j)] \le F_{\mathrm{u}}[\nu_j, \Omega_{r_j}]\,.$$

To conclude the proof, it suffices to take a diagonal subsequence. Precisely, we define $k : \mathbb{N} \to \mathbb{N}$ by $k(0) := 0$ and

$$\begin{aligned} k(j) := \min\Big\{ & k > k(j-1) : \nu_j \text{ is } (h_{\varepsilon_{k'}}, \alpha_{\varepsilon_{k'}}) \text{ dilute for all } k' \ge k\,, \\ & \frac{1}{\log(1/\varepsilon_{k'})} \mathcal{E}[\beta_{k'}^j, \Omega^{\varepsilon_{k'}}(\nu_j)] \le F_{\mathrm{u}}[\nu_j, \Omega_{r_j}] + \frac{1}{j} \text{ for all } k' \ge k \Big\}\,. \end{aligned}$$

We set $\mu_k := \nu_j$ and $\beta_k := \beta_{\varepsilon_k}^j$ for all $k \in [k(j), k(j+1)) \cap \mathbb{N}$. The second statement is analogous, using the second part of Proposition 6.7 to construct $\hat\beta_k^j$. $\square$

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