On the strongly damped wave equation with constraint

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Abstract

A weak formulation for the so-called *semilinear strongly damped wave equation with constraint* is introduced and a corresponding notion of solution is defined. The main idea in this approach consists in the use of duality techniques in Sobolev-Bochner spaces, aimed at providing a suitable "relaxation" of the constraint term. A global in time existence result is proved under the natural condition that the initial data have finite "physical" energy.

Key words: wave equation, strong damping, weak solution, maximal monotone operator, duality.

AMS (MOS) subject classification: 35L05, 74D10, 47H05, 46A20.

1 Introduction

This paper is devoted to studying the so-called semilinear wave equation with strong damping, namely

$$\varepsilon u_{tt} - \delta \Delta u_t - \Delta u + f(u) = g, \tag{1.1}$$

for $\varepsilon, \delta > 0$. The equation is settled in the parabolic cylinder $Q = (0,T) \times \Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, and $T > 0$ is a given final time, and is complemented with the initial conditions for u and u_t and with homogeneous boundary conditions either of Dirichlet or of Neumann type. The *strong damping* is provided by the term $-\delta \Delta u_t$; this comes in contrast with the *weak damping* occurring when that term is replaced by $+\delta u_t$. The function g on the right hand side is a given volume forcing term (here taken of L^2-regularity), and the semilinear term $f(u)$ is assumed to take the form $f(u) = \beta(u) - \lambda u$, where β is a monotone function (more precisely, a monotone *graph*, see Section 2 below) and $\lambda \geq 0$. In particular, the internal constraint on u is enforced by the non-smooth monotone part β of f, whereas the remaining term $-\lambda u$ is related to the (possible) nonconvexity of the energy functional associated to the equation. Actually, the main novelty of this paper stands in the fact that β is assumed to be defined only in a *bounded* interval I_0 of $\mathbb{R}$ and to diverge at the extrema of I_0. A (generalized) function β with the above properties will be referred to as a *constraint* on the variable u (cf. Section 2 below for more details). It is worth noting that, up to purely technical modifications in the proofs, our techniques could be adapted to treat also the case of *unilateral constraints*, i.e., functions β whose domain is bounded only from one side.

Physically speaking, equation (1.1) appears in a number of different contexts. Let us mention here some of them. The main application refers to the study of the motion of viscoelastic materials. In this setting, u plays the role of a (scalar) displacement and (1.1) represents the momentum balance (where accelerations are included) written in a small strain regime. In particular, respectively in space dimensions one and two, the equation describes the transversal vibrations of a homogeneous string and the longitudinal vibrations of a homogeneous bar subject to viscous effects. The strong damping term $-\delta\Delta u_t$ represents the fact that the stress is decomposed in the sum of a pure elastic part (proportional to the strain) and a viscous part (proportional to the strain rate), as in a linearized Kelvin-Voigt material. We also mention that in the literature, in space dimension three, (1.1) has been introduced to model, e.g., the deviation from the equilibrium configuration of a (homogeneous and isotropic) linearly viscoelastic solid with short "rate type" memory (cf. [15] for details), in the presence of an external displacement-dependent force $g - f(u)$. We do not enter deeper in the modeling details, and we refer to [22] for a physical derivation of models describing the motion of viscoelastic media. Let us observe that it would be meaningful to consider here a vectorial (displacement) variable $\mathbf{u}$, but we preferred, just for simplicity, to study only the scalar case at least at a first stage. Indeed, the extension of our results to the vector-valued case should be possible, at least for constant isotropic diffusion, whereas the case of non-constant stiffness (and viscosity) tensors may be somehow more involved. In this framework, we also have to quote (possibly adhesive) contact models with unilateral constraints (occurring for instance in the case of Signorini conditions) on a part of the boundary. In this setting, the (vectorial) operator β would force the direction of the trace of $\mathbf{u}$ on the boundary in such a way to ensure impenetrability (cf., e.g., [4, 5, 29]). We are planning to analyze this type of models, by using the methods developed in this paper, in future works.

Equation (1.1) also appears in the so-called Frémond theory for phase transitions whenever microscopic accelerations are taken into account (cf., e.g., [8, 9, 16]). In that setting, the unknown u generally denotes a (scalar) phase parameter, which is related (for a first order phase transition in a binary system) to the local proportion of one of the two phases, or components, of a binary material. Then, β represents an internal constraint forcing u to take values into the physical interval whose extrema (often given by $-1, 1$) correspond to the pure states, whereas the intermediate values represent a mixture of the phases. Physically relevant choices are $\beta(r) = \log(1 + r) - \log(1 - r)$ (i.e., the derivative of the so-called *logarithmic potential* often appearing in Allen-Cahn or Cahn-Hilliard models), or $\beta(r) = \partial I_{[-1,1]}(r)$ (i.e., the *subdifferential* of the *indicator function* of the interval $[-1, 1]$, given by $I_{[-1,1]}(r) = 0$ for $r \in [-1, 1]$ and $I_{[-1,1]}(r) = +\infty$ otherwise). It is also significant to consider equation (1.1) with other kinds of nonlinearities β not having the form of a constraint. For instance, (1.1) appears in the recent theory of isothermal viscoelasticity with very rapidly fading memory (cf. [12] and references therein), in the sine-Gordon model describing the evolution of the current u in a Josephson junction (cf. [25]; there $f(u) = \sin u$), or as a Klein-Gordon-type equation occurring in quantum mechanics (then $f(u) = |u|^\gamma u$ for suitable $\gamma > 0$).

Actually, in the case when f is smooth and defined on the whole real line, the mathematical literature on equation (1.1) is very wide (we quote, without any claim of completeness, the papers [2, 17, 19, 20, 21, 24, 28, 32]). Referring to [21] for more details, we recall here that one of the first essential results on global well-posedness of (the Dirichlet problem for) (1.1) in the 3D case was obtained by Webb, who proved in [32] that, if f satisfies standard dissipativity conditions (without any growth restriction), then the problem admits a unique strong solution (u, u_t) taking values in the space $[H^2(\Omega) \cap H^1_0(\Omega)] \times L^2(\Omega)$. On the other hand, when one looks for less regular solutions, the situation seems different. In particular, it is natural to consider weaker solutions such that the "energy of the system" remains bounded (in the analytical literature this fact corresponds to require that these solutions take values in the so-called *energy space*). Indeed, this type of regularity corresponds to the a priori estimate obtained by (formally) testing (1.1) by u_t. Then, one can easily realize that, at least if the external source g is 0, the functional

$$\mathcal{E}(u, u_t) = \int_\Omega \Big(\frac{\varepsilon}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + j(u) - \frac{\lambda}{2}u^2 \Big) \,\mathrm{d}x, \tag{1.2}$$

where j is an antiderivative of β, tends to decrease in the time evolution. Usually $\mathcal{E}$ is interpreted as a physical energy. This is particularly clear in the cases when u represents a displacement (including phase-change models where u is related to the effects of displacements at microscopic scales): then the component $|u_t|^2$ of the integrand is a density of kinetic energy, whereas the other summands correspond to some kind of configurational or potential energy. Consequently, energy solutions can be defined as those solutions taking values in the energy space, or, equivalently, keeping finiteness of the energy in the course of the evolution.

From the mathematical point of view, managing this type of solutions may be delicate, especially in high space dimension, in view of the possibly fast growth of the integrand $j(u)$. Correspondingly, the literature related to this case is much more recent: Kalantarov and Zelik in [21] consider polynomial nonlinearities of the form $f(u) \sim u|u|^q$ without any restriction on the exponent $q > 0$ and prove well-posedness of the equation in the energy space. Moreover, they analyze the asymptotic behavior of solutions for large values of the time variable. More recently Pata and Zelik in [28] have extended these results to the case when $f : \mathbb{R} \to \mathbb{R}$ is any smooth function satisfying the basic dissipativity assumptions, without any restriction on the growth rate (for instance, they may take exponential, or even faster growing, terms f). At least up to our knowledge, however, the case when f is of *constraint* type has never been considered up to now. The typical example we have in mind is $f(u) = \partial I_{[-1,1]}(u) - u$ (cf. (2.4) and (2.5) below), which may describe, for example, some phase transition phenomena accounting for microscopic accelerations (cf. [16]).

As one addresses the initial-boundary value problem for (1.1) under our assumptions, the main mathematical difficulty comes from the combination of the constraint β with the second time derivative u_{tt}. Indeed, this feature strongly restricts the available a-priori bounds. To be precise, almost all information on the solution has to be extracted from the so-called "energy" estimate, i.e., testing the equation by u_t. In addition to that, one can just get some more smoothness of u by multiplying (1.1) by $-\Delta u$ (as is done, e.g., in [28]). Anyway, this does not help for controlling the term u_{tt}, which is the main issue from the point of view of regularity. Moreover, the standard procedures that one usually adopts for obtaining higher order bounds, like differentiating in time the equation, do not seem to work here, at least for a general choice of β. This seems to be, indeed, the main difference of the present problem with respect to *first order* (in time) equations with constraint, for which additional regularity of solutions can be generally deduced by differentiating in time and testing the result by u_t, whatever is the expression of β.

In view of the lack of estimates, we need to build a notion of weak solution which is sufficiently general to exist under the sole "energy" regularity. This is, indeed, a somehow delicate issue. In particular, one cannot expect to reproduce the same type of results that hold in the case of less general nonlinearities β. To say it shortly, the main novelties of our approach can be summarized in two points:

(i) a relaxed form β_w of the operator β obtained by means of duality techniques;

(ii) an integrated (both in space and in time) variational formulation where test functions are chosen in suitable Sobolev-Bochner spaces.

These choices permit us, indeed, to prove existence. However, both of them come at some price. Namely,

(i) it will not be possible to intend the equation, and the constraint in particular, in the pointwise sense;

(ii) we cannot exclude the occurrence of *jumps* of u_t. Actually, u_t may be discontinuous with respect to time (and, more precisely, is *expected* to be discontinuous, as we can show by means of examples).

However, from a physical point of view, if (1.1) comes from a variational principle (as the principle of virtual power is), the variational setting in which we introduce the solution is the natural one. In particular, the operator β (in its relaxed version β_w) stands for an internal force which is defined in duality with velocities/displacements. In addition to that, an internal constraint on the function u is still ensured by the definition of the domain of β_w. Finally, the fact that we can have jumps on the velocity u_t w.r.t. time, corresponds to the possible occurrence of internal (or external) shocks, which are expected to happen in this framework (cf, e.g., [16]).

A further drawback is concerned with the problem of uniqueness. Actually, we expect the occurrence of genuine nonuniqueness, even though some criteria for "physicality" of weak solutions may be proposed (cf. Remark 3.3 at the end).

Let us conclude by giving some more words of explanation for our method. The basic strategy of proof is, in a sense, very standard: we replace the singular function β by a smooth approximation β^ε of controlled growth at infinity (e.g., the Yosida approximation), prove existence of a solution u^ε to the regularized problem (which basically follows from results already known in the literature, cf., e.g., the quoted [17, 21, 28, 32]), and then let the approximation parameter ε go to 0. Indeed, as a consequence of the so-called "energy estimate", u^ε, at least for a subsequence, tends to some limit u which we would like to identify as a "weak" solution (where, of course, we need to state precisely what we mean with this). However, the only uniform bound available for the nonlinear term $\beta^\varepsilon(u^\varepsilon)$ is in the norm of $L^1(0,T;L^1(\Omega))$, and there is evidence coming from concrete examples that we cannot go further, at least for general β. This fact has, indeed, a number of consequences. First of all, arguing by comparison, we can obtain an $L^1(0,T;X)$-bound for u^ε_{tt}, where X is a Banach space such that $L^1(\Omega)$ and $H^1(\Omega)$ are compactly embedded into X (for example, we can take $X = H^{-2}(\Omega)$ in the 3D case). This estimate suffices, via a generalized version of the Aubin-Lions compactness lemma, to prove strong convergence of u^ε_t in $L^2(0,T;L^2(\Omega))$. However, the limit function u_t may exhibit jumps with respect to time. Secondly, the limit of $\beta^\varepsilon(u^\varepsilon)$ can be taken at least in the (weak) sense of measures. A crucial point is, as usual, concerned with the identification of its limit. In view of our assumptions it looks natural to rely on a suitable version of the so-called Minty's trick for monotone operators, i.e., to combine the weak convergence of u^ε in some (reflexive) Banach space $\mathcal{V}$, the weak convergence of $\beta^\varepsilon(u^\varepsilon)$ in the dual space $\mathcal{V}'$, and a $\limsup$-inequality. A look at the estimates suggests that an admissible choice for this procedure is (in the Dirichlet case) the Sobolev-Bochner space $\mathcal{V} = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ (in the Neumann case, $H^1_0(\Omega)$ is simply replaced by $H^1(\Omega)$ and no further difficulties arise). In such a setting, the constraint β has to be reinterpreted in a relaxed form β_w acting as a maximal monotone operator from $\mathcal{V}$ to $2^{\mathcal{V}'}$ (cf. Definition 2.13 below; see, e.g., [11, 18] for some additional background). Correspondingly, equation (1.1) has to be intended as a relation in $\mathcal{V}'$. Let us point out that, from a physical point of view, in the case when (1.1) corresponds to a mechanical balance equation (i.e., to the momentum balance equation), our weak formulation takes the meaning of a duality between forces and velocities in time and space (see [7] for a similar approach, but in a different setting). To avoid occurrence of second time derivatives in the weak formulation, we also need to integrate by parts with respect to time the second order term u_{tt} (cf. (2.31)). Actually, these modifications will permit us to solve our original problem on the whole time interval $(0,T)$, but also to write "pointwise" the duality relation in any subinterval $(0,t)$, with a physically consistent interpretation of the corresponding constraint. Finally, an energy inequality is proved to hold on (almost) every subinterval of $[0,T]$. We end observing that the behavior of weak solutions (at least in the homogeneous Neumann case) may be clarified by considering a spatially homogeneous setting. For instance, if $f(u) = \partial I_{[-1,1]}(u)$ and $g \equiv 0$, (1.1) reduces to the prototype ODE $u_{tt} + \partial I_{[-1,1]}(u) \ni 0$ whose solutions can be easily described, especially in relation with the jumps of u_t (cf. Remark 2.4 for more details).

The remainder of the paper is organized as follows: in Section 2 we introduce some amount of preliminary material mainly related to maximal monotone operators and duality methods; moreover we present the notion of weak solution and state the related existence result. Then, the proof is detailed in Section 3, where we also give a number of remarks illustrating our results at the light of simple finite-dimensional examples.

2 Preliminary notions and main result

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain (with $N \geq 1$) of boundary Γ and let us consider the interval $[0, T]$, for some fixed final time $T > 0$. Let us set $H := L^2(\Omega)$ and use the notation $(\cdot, \cdot)$ for the scalar product both in H and in H^N. Let also the symbol $\| \cdot \|$ denote the corresponding norms. In our analysis, we will consider either Dirichlet or Neumann boundary conditions for (1.1); hence we introduce a notation suitable for addressing both cases in a unified way. So, we put $V := H^1(\Omega)$ in the Neumann case, and $V := H_0^1(\Omega)$ in the Dirichlet case. In both cases, V will be endowed with the standard (Sobolev) norm, indicated by $\| \cdot \|_V$. Moreover, we will denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V. In general, we will indicate by $\| \cdot \|_X$ the norm in some Banach space X (or in X^N).

We let A stand for the weak form of (minus) the Laplace operator seen as an unbounded linear operator on H whose domain $D(A)$ depends on the boundary conditions. Namely, in the Neumann case, we set

$$Av := -\Delta v, \quad D(A) := H^2_{\boldsymbol{n}}(\Omega), \tag{2.1}$$

where $H^2_{\boldsymbol{n}}(\Omega)$ denotes the space of the H^2-functions having zero normal derivative (in the sense of traces) on $\partial\Omega$. Correspondingly, in the Dirichlet case, we set

$$Av := -\Delta v, \quad D(A) := H^2(\Omega) \cap V = H^2(\Omega) \cap H^1_0(\Omega). \tag{2.2}$$

In both cases, A is a positive operator (strictly positive for Dirichlet conditions). Morever, A can be extended to the space V by setting (for both choices of boundary conditions)

$$\langle Av, z \rangle = \int_\Omega \nabla v \cdot \nabla z \, \mathrm{d}x. \tag{2.3}$$

This extension, which turns out to be linear and bounded from V to V', will be identically noted as A; indeed, we believe that no danger of confusion exists at this stage.

Next, we specify our assumptions on the semilinear term $f(u)$. First, we suppose that f may be decomposed as

$$f(u) = \beta(u) - \lambda u, \tag{2.4}$$

where $\lambda \geq 0$ and β is a *maximal monotone graph* in $\mathbb{R} \times \mathbb{R}$ such that

$$\overline{D(\beta)} = [-1, 1], \quad 0 \in \beta(0). \tag{2.5}$$

Indeed, just for simplicity and with no loss of generality, we require the closure of the *domain* of β to be the interval $[-1, 1]$. In addition, it is not restrictive to assume the normalization $0 \in \beta(0)$, which turns out to be useful especially in the Dirichlet case.

Referring the reader to [3, 10] for a complete survey on the theory of maximal monotone operators in Banach and Hilbert spaces, we just observe here that, thanks to (2.5), there exists a convex and lower semicontinuous function $j : \mathbb{R} \to [0, +\infty]$ such that $\beta = \partial j$, $\overline{D(j)} = [-1, 1]$, and $j(0) = \min j = 0$. Here, $D(j)$ denotes the *domain* of the convex function j, i.e., the set where j takes finite values.

It is well known that the graph β induces maximal monotone operators (identically noted as β for simplicity) both in H and in $L^2(Q)$, where $Q := (0, T) \times \Omega$. For instance, one has $\xi \in \beta(u)$ in the H-sense if and only if $u, \xi \in H$ and $\xi(x) \in \beta(u(x))$ for a.e. $x \in \Omega$. Moreover, let us define the *convex* functional

$$J : H \to [0, +\infty], \quad J(u) := \int_\Omega j(u) \, \mathrm{d}x, \tag{2.6}$$

where the integral may well be $+\infty$ in the case when $j(u) \notin L^1(\Omega)$ (i.e., when $u \notin D(J)$). Then, $\beta = \partial J$ in H, namely the operator induced by β on H coincides with the H-subdifferential of the convex functional J. As is customary when dealing with multivalued operators, we shall often identify maximal monotone operators with their graphs (cf., e.g., [3, 10]). With the above notation, equation (1.1), where the coefficients ε and δ have been set to 1 for simplicity, becomes

$$u_{tt} + Au_t + Au + \beta(u) - \lambda u \ni g \quad \text{in } (0, T) \times \Omega. \tag{2.7}$$

Note the occurrence of the inclusion sign, motivated by the fact that β may be multi-valued.

In view of (2.1) (or of (2.2)), (2.7) can be read as a relation holding in $L^2(0,T;H)$ (and thus interpreted as a pointwise inclusion almost everywhere in Q). Indeed, (2.7) looks as the most natural and appropriate weak formulation of the strongly damped wave equation in the case when β is a smooth monotone function defined on the whole real line. On the other hand, though (2.7) is still perfectly meaningful from the mathematical viewpoint under our assumptions (2.4)-(2.5), proving existence of solution in the current setting seems to be out of reach (see Remark 2.4 below for a counterexample in the spatially homogeneous case). Mainly, what seems to fail is the possibility to interpret point-by-point the equation, and in particular the constraint β.

Hence, we need to construct a furtherly relaxed formulation of the equation, for which one might be able to get existence. In performing this program, we would like our new concept of solution to be still somehow physically consistent. Namely, weak solutions should comply with thermodynamical principles (like the energy inequality), satisfy a proper form of the constraint, and be obtained as limit points of families of functions solving physically sound regularizations of the equation. To start with this program, we set

$$\mathcal{V} := H^1(0,T;H) \cap L^2(0,T;V). \tag{2.8}$$

Note that, in view of standard results on vector-valued functions, the above space coincides with $H^1(Q)$ in the Neumann case. The duality pairing between $\mathcal{V}'$ and $\mathcal{V}$ will be noted by $\langle\!\langle \cdot,\cdot \rangle\!\rangle$. We also consider the space $\mathcal{H} := L^2(Q) = L^2(0,T;H)$ endowed with the natural scalar product, noted here as $(\!(\cdot,\cdot)\!)$. Thanks to standard results on Sobolev spaces, the inclusions $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ hold continuously and densely provided $\mathcal{H}$ is identified with its dual by means of the above scalar product. Actually, the weak formulation of our problem will strongly rely on the *parabolic Hilbert triplet* $(\mathcal{V},\mathcal{H},\mathcal{V}')$.

We also need similar concepts in the case when the time interval $(0,T)$ is replaced by $(0,t)$ for $0 < t \le T$. Namely, we set $Q_t := (0,t)\times\Omega$, and, correspondingly, we note by $(\!(v,z)\!)_{(0,t)}$ the (standard) scalar product in $\mathcal{H}_t := L^2(Q_t)$ and by $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{(0,t)}$ the duality between $\mathcal{V}_t := H^1(0,t;H) \cap L^2(0,t;V)$ and its dual. We also set

$$\mathcal{V}_{t,0} := \big\{ v \in \mathcal{V}_t : \ v \equiv 0 \text{ on } \{t\}\times\Omega \big\}, \tag{2.9}$$

where relation $v \equiv 0$ is intended in the sense of traces (in time). Clearly, $\mathcal{V}_{t,0}$ is a closed subspace of $\mathcal{V}_t$. Then, if $\varphi \in \mathcal{V}_{t,0}$, extending it by 0 for times larger than t, we obtain an element of $\mathcal{V}$, noted in the following as $\widetilde{\varphi}$. Correspondingly, if $\eta \in \mathcal{V}'$, we can naturally define its *restriction* η^t to the time interval $(0,t)$ by setting, for $\varphi \in \mathcal{V}_{t,0}$,

$$\langle\!\langle \eta^t, \varphi \rangle\!\rangle_{(0,t)} := \langle\!\langle \eta, \widetilde{\varphi} \rangle\!\rangle. \tag{2.10}$$

Actually, it is readily checked that $\eta^t \in \mathcal{V}_{t,0}'$. Moreover, the restriction operator $\eta \mapsto \eta^t$ is linear and continuous from $\mathcal{V}'$ to $\mathcal{V}_{t,0}'$.

With the above notation at disposal, we extend the functional J to time-dependent functions by setting (see (2.6))

$$\mathcal{J} : \mathcal{H} \to [0,+\infty], \quad \mathcal{J}(u) := \int_0^T J(u)\,\mathrm{d}t = \int_0^T \int_\Omega j(u)\,\mathrm{d}x\,\mathrm{d}t, \tag{2.11}$$

where, as before, the integral may also take the value $+\infty$. Analogously, for $t \in (0,T]$, we put

$$\mathcal{J}_{(t)} : \mathcal{H}_t \to [0,+\infty], \quad \mathcal{J}_{(t)}(u) := \int_0^t J(u)\,\mathrm{d}s = \int_0^t \int_\Omega j(u)\,\mathrm{d}x\,\mathrm{d}s. \tag{2.12}$$

As noted above, the $\mathcal{H}$-subdifferential $\partial\mathcal{J}(u)$ (or the analogue for $\mathcal{J}_{(t)}(u)$) can be still interpreted in the "pointwise" form $\beta(u)$.

We are now ready to introduce the weak form of the constraint β. We shall present most of the construction by working on the time interval $(0,T)$. The adaptation to subintervals $(0,t)$ is straighforward and we mostly leave it to the reader because we do not want to overburden the notation. That said, we start by setting $\mathcal{J}_{\mathcal{V}} := \mathcal{J}|_{\mathcal{V}}$. It is readily proved that $\mathcal{J}_{\mathcal{V}}$ is convex and lower semicontinuous on $\mathcal{V}$. Hence, we may take its subdifferential with respect to the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$. Namely, for $\xi \in \mathcal{V}'$ and $u \in \mathcal{V}$, we put

$$\xi \in \beta_w(u) \overset{\text{def}}{\Longleftrightarrow} u \in \mathcal{V}, \ \ \xi \in \mathcal{V}', \ \ \text{and} \ \langle\!\langle \xi, v-u \rangle\!\rangle + \mathcal{J}_{\mathcal{V}}(u) \le \mathcal{J}_{\mathcal{V}}(v) \ \text{ for all } \ v \in \mathcal{V}. \tag{2.13}$$

The idea of "relaxing" β in this way is not new; for instance, the same method has been applied in [6, 14, 30] in other contexts. It is worth noting from the very beginning that $u \in D(\beta_w)$ still implies $u \in [-1,1]$ almost everywhere; in other words, the weak operator β_w still forces u to assume only "physically meaningful" values. Note that an alternative, but essentially equivalent, approach based on variational inequalities has been devised in [27] for the Cahn-Hilliard equation with dynamic boundary conditions. The novelty occurring in our case is related to the use of "parabolic" (Sobolev-Bochner) spaces. Indeed, this choice seems particularly appropriate for the present problem as far as it permits us to overcome some issues related with the (expected) low regularity of weak solutions.

Let us now characterize a bit more precisely the operator β_w. We follow here the lines of [11, 18] (see also [6]). Firstly, we observe that (see, e.g., [6, Prop. 2.3]), if $u \in \mathcal{V}$, $\xi \in \mathcal{H}$, and $\xi \in \beta(u)$ a.e. in Q, then $\xi \in \beta_w(u)$. Namely, if $\beta|_{\mathcal{V}}$ denotes the restriction to $\mathcal{V}$ of the "pointwise" operator β, then β_w extends $\beta|_{\mathcal{V}}$. In other words, the "strong" constraint implies the "weak" one. Moreover (cf. [6, Prop. 2.5]),

$$\text{if } u \in \mathcal{V} \text{ and } \xi \in \beta_w(u) \cap \mathcal{H}, \quad \text{then } \xi \in \beta(u) \text{ a.e. in } Q. \tag{2.14}$$

In general, however, the elements $\xi \in \beta_w(u)$ (which lie, by definition, in the space $\mathcal{V}'$) need not belong to $\mathcal{H}$. Hence, the graph inclusion $\beta|_{\mathcal{V}} \subset \beta_w$ is generally a proper one. Nevertheless, if $\xi \in \beta_w(u)$, then ξ "automatically" gains some more regularity.

In order to explain this phenomenon, we proceed along the lines of [30, Sec. 2]. Namely, for $t \in (0,T]$, we set

$$\mathcal{X}_t := C^0(\overline{Q_t}), \quad \text{for Neumann boundary conditions,} \tag{2.15a}$$

$$\mathcal{X}_t := \{u \in C^0(\overline{Q_t}) : \ u \equiv 0 \text{ on } [0,t] \times \Gamma\}, \quad \text{for Dirichlet boundary conditions.} \tag{2.15b}$$

For $t = T$ we simply write $\mathcal{X} = \mathcal{X}_T$. We also set, in both cases,

$$\mathcal{X}_{t,0} := \{v \in \mathcal{X}_t : \ v \equiv 0 \text{ on } \{t\} \times \Omega\}. \tag{2.16}$$

The space $\mathcal{X}_t$ (hence its closed subspace $\mathcal{X}_{t,0}$) is naturally endowed with the supremum norm $\|\cdot\|_\infty$. Moreover, also thanks to the smoothness of Ω in the Neumann case, $\mathcal{X}_t \cap \mathcal{V}_t$ is dense both in $\mathcal{X}_t$ and in $\mathcal{V}_t$. Let now $\xi \in \mathcal{V}'$ (the analogue applies with straighforward modifications to $\xi \in \mathcal{V}_t'$) and let us suppose that ξ, if restricted to the functions $\varphi \in \mathcal{X} \cap \mathcal{V}$, is continuous with respect to the $\mathcal{X}$-norm, i.e., there exists $C > 0$ such that

$$|\langle\!\langle \xi, z \rangle\!\rangle| \le C\|z\|_\infty \quad \text{for any } z \in \mathcal{X} \cap \mathcal{V}. \tag{2.17}$$

In that case, by density, ξ extends *in a unique way* to a bounded linear functional on $\mathcal{X}$. Namely, there exists a unique $\mathcal{T} \in \mathcal{X}'$, which can be seen as a Borel measure on $\overline{Q}$ in view of Riesz' representation theorem, such that

$$\langle\!\langle \xi, z \rangle\!\rangle = \iint_{\overline{Q}} z \, \mathrm{d}\mathcal{T} \qquad \text{for any } z \in \mathcal{X} \cap \mathcal{V}. \tag{2.18}$$

In this situation we say that the measure $\mathcal{T}$ *represents* ξ on $\mathcal{X}$. Actually this situation automatically occurs when ξ is an element of a weak constraint. Indeed, by an easy adaptation of [30, Prop. 2.1] (which, in turn, is based on the results of [11]), one can see that, up to some adjustment related to the boundary conditions, any $\xi \in \beta_w(u)$, when restricted to continuous functions, is represented by a measure $\mathcal{T}$ defined on the parabolic cylinder Q. Such a measure, in turn, is related to the original operator β in the following way (cf. [11, Thm. 3] for further details): noting as $\mathcal{T} = \mathcal{T}_a + \mathcal{T}_s$ the Radon-Nikodym decomposition of $\mathcal{T}$, with $\mathcal{T}_a$ ($\mathcal{T}_s$, respectively) standing for the absolutely continuous (singular, respectively) part, we then have

$$\mathcal{T}_a u \in L^1(Q), \tag{2.19}$$

$$\mathcal{T}_a(t,x) \in \beta(u(t,x)) \quad \text{for a.e. } (t,x) \in Q, \tag{2.20}$$

$$\langle\!\langle \xi, u \rangle\!\rangle - \int_0^T \int_\Omega \mathcal{T}_a u \, \mathrm{d}x \, \mathrm{d}t = \sup \Big\{ \iint_{\overline{Q}} z \, \mathrm{d}\mathcal{T}_s, \ z \in \mathcal{X}, \ z(\overline{Q}) \subset [-1,1] \Big\}. \tag{2.21}$$

Hence, the continuous part $\mathcal{T}_a$ of the measure $\mathcal{T}$ satisfies the constraint pointwise (in view of (2.20)), whereas the singular part $\mathcal{T}_s$ is characterized by (2.21).

In particular, we expect that condition (2.21) could be made more precise. Namely, noting as $\mathcal{T}_s = \rho|\mathcal{T}_s|$ the *polar decomposition* of $\mathcal{T}_s$, where $|\mathcal{T}_s|$ is the total variation of $\mathcal{T}_s$, proceeding along the lines of [18, Thm. 3] one may prove that

$$\rho \in \partial I_{[-1,1]}(u) \quad |\mathcal{T}_s|\text{-a.e. in } \overline{Q}. \tag{2.22}$$

In other words, we expect the singular part of $\mathcal{T}$ to be supported on the set where $|u| = 1$ and that $\rho = 1$ where $u = 1$, $\rho = -1$ where $u = -1$. In this sense, also the singular part of $\mathcal{T}$ is, at least partially, reminiscent of the expression of the graph β.

Actually, the characterization (2.22) is proved in [18] in the case when $\mathcal{V} = H_0^1(\Omega)$, Ω a bounded domain of $\mathbb{R}^N$, and may be likely extended to the present situation. However, a detailed proof may involve some technicalities particularly related to the facts that we are working in the parabolic cylinder and should distinguish between the Dirichlet and Neumann cases. For this reason, we omit details here. We note, however, that (2.22) is straighforward whenever we additionally know that $u \in \mathcal{X} \cap \mathcal{V}$ (i.e., u, beyond lying in $\mathcal{V}$, is continuous). Indeed, in that case, from (2.21) there follows

$$\begin{aligned}\iint_{\overline{Q}} \rho u \,\mathrm{d}|\mathcal{T}_s| &= \iint_{\overline{Q}} u \,\mathrm{d}\mathcal{T}_s = \langle\!\langle \xi, u \rangle\!\rangle - \int_0^T \int_\Omega \mathcal{T}_a u \,\mathrm{d}x \,\mathrm{d}t \\ &= \sup\left\{ \iint_{\overline{Q}} z \,\mathrm{d}\mathcal{T}_s,\ z \in \mathcal{X},\ z(\overline{Q}) \subset [-1,1] \right\} = |\mathcal{T}_s|(\overline{Q}),\end{aligned} \tag{2.23}$$

the latter term denoting the total variation of the measure $\mathcal{T}_s$. Comparing terms, we then deduce $\rho u = 1$ $|\mathcal{T}_s|$-a.e. in $\overline{Q}$, as desired.

Remark 2.1. It is worth observing that, in the Neumann case, the singular component $\mathcal{T}_s$ of the measure $\mathcal{T}$ representing $\xi \in \beta_w(u)$ may be, at least partially, supported on the boundary of Q. Let us see this by a simple one-dimensional example. Let $Q = (-1,1)$, $u(x) = x$, $j = I_{[-1,1]}$. Then, if $\xi = \alpha\delta_1$ for some $\alpha > 0$, where δ_1 is the Dirac delta concentrated in 1, it is clear that, for any $v \in H^1(-1,1)$ such that $v \in D(J)$ (i.e., such that $-1 \le v(x) \le 1$ for all $x \in Q$), there holds

$$\langle \xi, v - u \rangle = \alpha(v(1) - u(1)) = \alpha(v(1) - 1) \le 0 = J(v) - J(u). \tag{2.24}$$

Hence, $\xi \in \beta_w(u)$ by definition of subdifferential.

Having clarified the nature of the weak constraint β_w, we can now observe that equation (2.7) admits a natural energy functional

$$\mathcal{E}(u, u_t) = \int_\Omega \Big(\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + j(u) - \frac{\lambda}{2}u^2 \Big) \,\mathrm{d}x = \frac{1}{2}\big(\|u_t\|^2 + \|\nabla u\|^2 + 2J(u) - \lambda\|u\|^2 \big). \tag{2.25}$$

Indeed, testing (2.7) by u_t and integrating in time and space, one can get that the value of $\mathcal{E}$ at any time $t > 0$ is bounded by the initial value $\mathcal{E}(u_0, u_1)$ plus the power of the external applied forces g (see (2.41) below). Actually, as will be explained later on, the low regularity of solutions does not allow us to perform this estimate directly for weak solutions, but only for a suitable approximation of the problem. This is basically the reason for which we will only be able to prove an energy *inequality* for weak solutions, cf. Theorem 2.5 below.

We can now introduce our assumption on the source term and on the initial data, the latter corresponding exactly to the finiteness of the "initial energy":

$$g \in L^2(0,T;H), \tag{2.26}$$

$$u_0 \in V, \quad u_1 \in H, \quad J(u_0) < \infty. \tag{2.27}$$

Then, we can make precise our concept of weak solution (to be precise, we shall speak of "parabolic duality weak solution" or something similar, but we will rather use "weak solution" just for simplicity):

Definition 2.2. *A couple (u,ξ) is called* a weak solution *to the initial-boundary value problem for the strongly damped wave equation with constraint whenever the following conditions hold:*

(a) There hold the regularity properties

$$u \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V), \tag{2.28}$$

$$\xi \in \mathcal{V}' \cap \mathcal{X}'; \tag{2.29}$$

more precisely, by (2.29) we may require that there exists a measure $\mathcal{T} \in \mathcal{X}'$ representing ξ over $\mathcal{X}$ in the sense of (2.18).

(b1) There holds the following weak version of (2.7):

$$\begin{aligned} &- (\!(u_t,\varphi_t)\!) + (u_t(T),\varphi(T)) + (\!(\nabla u_t,\nabla\varphi)\!) + (\!(\nabla u,\nabla\varphi)\!) \\ &\quad + \langle\!\langle \xi,\varphi\rangle\!\rangle - \lambda(\!(u,\varphi)\!) = (u_1,\varphi(0)) + (\!(g,\varphi)\!), \quad \forall\,\varphi\in\mathcal{V}. \end{aligned} \tag{2.30}$$

(b2) An analogue of (2.30) holds also on subintervals, in the following sense: for any $t \in (0,T]$ there exists a functional $\xi_{(t)} \in \mathcal{V}_t'$ such that

$$\begin{aligned} &- (\!(u_t,\varphi_t)\!)_{(0,t)} + (u_t(t),\varphi(t)) + (\!(\nabla u_t,\nabla\varphi)\!)_{(0,t)} + (\!(\nabla u,\nabla\varphi)\!)_{(0,t)} \\ &\quad + \langle\!\langle \xi_{(t)},\varphi\rangle\!\rangle_{(0,t)} - \lambda(\!(u,\varphi)\!)_{(0,t)} = (u_1,\varphi(0)) + (\!(g,\varphi)\!)_{(0,t)}, \quad \forall\,\varphi\in\mathcal{V}_t. \end{aligned} \tag{2.31}$$

Moreover, $\xi_{(t)}$ lies in $\mathcal{V}_t' \cap \mathcal{X}_t'$; hence it is (uniquely) represented over $\mathcal{X}_t$ by a measure $\mathcal{T}_{(t)}$. In addition to that, for every $t \in (0,T]$, the functionals ξ and $\xi_{(t)}$ are compatible, *namely, for every $\varphi \in \mathcal{V}_{t,0}$ we have ($\widetilde{\varphi}$ denoting the trivial extension of φ)*

$$\langle\!\langle \xi_{(t)},\varphi\rangle\!\rangle_{(0,t)} = \langle\!\langle \xi,\widetilde{\varphi}\rangle\!\rangle. \tag{2.32}$$

In other words, the functional $\xi_{(t)}$, when computed on the elements of $\mathcal{V}_{t,0}$, coincides with the canonical restriction ξ^t of ξ (cf. (2.10)).

(c) There holds the inclusion

$$\xi \in \beta_w(u) \quad \text{in } \mathcal{V}'. \tag{2.33}$$

More generally, for every $t \in (0,T]$, $\xi_{(t)} \in \beta_{w,(t)}(u)$ in $\mathcal{V}_t'$. Here $\beta_{w,(t)}$ represents the weak version of β in the interval $(0,t)$; namely, $\beta_{w,(t)}$ is the subdifferential of the restriction of $\mathcal{J}_{(t)}$ to $\mathcal{V}_t$ with respect to the duality product between $\mathcal{V}_t$ and $\mathcal{V}_t'$.

(d) There holds the Cauchy condition

$$u|_{t=0} = u_0. \tag{2.34}$$

(e) For every $s,t \in [0,T]$, the couple (u,ξ) satisfies the equality

$$\begin{aligned} &- \|u_t\|^2_{L^2(s,t;H)} + (u_t(t),u(t)) + \frac{1}{2}\|\nabla u(t)\|^2 + \|\nabla u\|^2_{L^2(s,t;H)} + \langle\!\langle \xi_{(t)},u\rangle\!\rangle_{(0,t)} - \langle\!\langle \xi_{(s)},u\rangle\!\rangle_{(0,s)} \\ &\quad - \lambda\|u\|^2_{L^2(s,t;H)} = (u_t(s),u(s)) + \frac{1}{2}\|\nabla u(s)\|^2 + \int_s^t\int_\Omega gu \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{2.35}$$

It is worth discussing a bit how the above formulation has been obtained from (2.7). First of all, β has been replaced with its "relaxed" form β_w. Correspondingly, (2.7) has been restated in the "parabolic" dual space $\mathcal{V}'$ by using the test function $\varphi \in \mathcal{V}$ and performing suitable integrations by parts. In particular, a key point stands in the integration in time of the "hyperbolic" term u_{tt}. Indeed, no second time derivatives of u appear in (2.30) (or in (2.31)). In addition to that, the Cauchy condition for u_t is now "embedded" into (2.30) and (2.31).

Remark 2.3. We need to explain in some detail the "meaning" of (2.31), especially in relation with the constraint term. Actually, if $\xi \in \mathcal{V}'$, there is no canonical way of restricting ξ to obtain an element of $\mathcal{V}_t'$. The best we can do is restricting ξ as explained in (2.10) to obtain a functional $\xi^t \in \mathcal{V}_{t,0}'$. However, writing (2.31) as a relation in $\mathcal{V}_{t,0}'$ (i.e., considering only test functions $\varphi \in \mathcal{V}_{t,0}$) would give rise to some information loss. Namely, it may happen that the singular part $\mathcal{T}_s$ of $\mathcal{T}$ is, at least partially, supported on some set of the form $\{t\}\times\overline{\Omega}$ (or, correspondingly, $\langle\!\langle \xi_{(t)},\varphi\rangle\!\rangle_{(0,t)}$ may also depend on the trace of φ on $\{t\}\times\overline{\Omega}$).

Remark 2.4. It is worth noting that, according to the above definition, u_t need not be continuous with respect to time, independently of the target topology. This fact is a distinctive feature of this problem and there seems to be no hope of avoiding jumps of u_t, at least for a *general* constraint β. Here is a simple example where a jump occurs. Let us consider the case of spatially homogeneous solutions to the Neumann problem with $\lambda = 0$ and $g = 0$. In other words, we reduce our problem to the "toy model" represented by the ODE

$$u_{tt} + \beta(u) \ni 0, \tag{2.36}$$

a weak solution u to which exists according to our theory. Let us also choose $\beta = \partial I_{[-1,1]}$. Then, if we take, for instance, $u_0 = 0$ and $u_1 = 1$, we get that $u(t) = t$ at least for $t \in [0, 1)$. As t gets to 1, u_t *must* develop a discontinuity, otherwise, $u(t)$ would become *strictly* larger than 1 for $t > 1$, and the equation would no longer make sense. Hence, the only possibility for the trajectory $u(t)$ is to jump instantaneously in such a way that, in a right neighbourhood $(1, 1 + \epsilon)$ of $t = 1$,

$$u_t(t) = \ell, \quad u(t) = 1 + \ell(t - 1), \quad \ell \leq 0. \tag{2.37}$$

The trajectory, at least in principle, may "choose" at which level ℓ the time derivative u_t "decides" to jump (hence we have no uniqueness). If it jumps to $\ell < 0$, then $u(t)$ starts to decrease from the value 1 at a constant velocity ℓ until it reaches the value -1 (where a new jump of u_t must occur). On the contrary, if u_t jumps to $\ell = 0$, then it will be either $u(t) = 1$ and $u_t(t) = 0$ forever, or after some time u_t may make a further jump to some $\ell < 0$, starting from which u begins to decrease as specified above. More precisely, we can notice that, for (2.36), the weak formulation over $(0, T)$ (cf. (2.30)) reads

$$-\int_0^T u_t \varphi_t \, \mathrm{d}t + \varphi(T) u_t(T) + \langle\langle \xi, \varphi \rangle\rangle = \varphi(0) u_1 \quad \forall \varphi \in H^1(0, T). \tag{2.38}$$

Hence, it is easy to check that, for all $\ell \leq 0$, the function u described above solves (2.38) on a suitable interval $[0, T]$ with $T > 1$ chosen sufficiently small so that no other jumps of u_t occur. Note in particular that different choices of ℓ correspond to different "values" of $\xi \in \beta_w(u)$. Indeed, from (2.38) we get

$$-\int_0^1 \varphi_t \, \mathrm{d}t - \int_1^T \ell \varphi_t \, \mathrm{d}t + \ell \varphi(T) + \langle\langle \xi, \varphi \rangle\rangle = \varphi(0), \tag{2.39}$$

whence

$$\langle\langle \xi, \varphi \rangle\rangle = (1 - \ell)\varphi(1), \tag{2.40}$$

or, in other words, $\xi = (1 - \ell)\delta_{t=1}$ (δ standing for the Dirac delta) and we can notice that this is consistent with the above characterization of β_w. Indeed, at least for $\ell < 0$, $t = 1$ is the only time at which u takes the value 1 and ξ may have, and in fact has, a "singular" part. However, we will see in the sequel (cf. Remarks 3.7 and 3.8 below) that not every jump of u_t (or, in the current example, every value of $\ell > 0$) is "physically" admissible.

We can now introduce the statement of our main result:

Theorem 2.5. *Let us assume* (2.4), (2.5), (2.26), *and* (2.27). *Then, there exists at least one weak solution* (u, ξ), *in the sense of* Definition 2.2, *to the initial-boundary value problem for the strongly damped wave equation with constraint. Moreover* $u_t \in BV(0, T; X)$ *for any Banach space* X *such that* $L^1(\Omega)$ *and* V' *are compactly embedded in* X.

In addition, for almost every $s \in [0, T)$ *(surely including* $s = 0$*) and every* $t \in (s, T]$*, the following version of the energy inequality holds:*

$$\mathcal{E}(u(t), u_t(t)) + \|\nabla u_t\|^2_{L^2(s,t;H)} \leq \mathcal{E}(u(s), u_t(s)) + \int_s^t (g, u_t) \, \mathrm{d}\tau, \tag{2.41}$$

where $\mathcal{E}$ *is defined in* (2.25).

Finally, in the case when we additionally have

$$u_0 \in D(A), \tag{2.42}$$

then u enjoys the additional regularity property

$$u \in C_w([0,T]; D(A)). \tag{2.43}$$

Namely, $u(t)$ belongs to $D(A)$ for every $t \in [0,T]$ and $t \mapsto u(t)$ is continuous when the target space is endowed with the weak topology.

3 Proof of Theorem 2.5

Step 1. Approximation. We start by introducing a natural regularization of (the strong form of) equation (2.7) depending on an approximation parameter ε (which will then be let go to 0). To this aim, for $\varepsilon \in (0,1)$, we let $j^\varepsilon : \mathbb{R} \to [0,\infty)$ denote the *Moreau-Yosida regularization* of j (cf., e.g., [10] for details). In particular, j^ε turns out to be convex and lower semicontinuous. Moreover, its derivative $\beta^\varepsilon := \partial j^\varepsilon$ corresponds to the *Yosida approximation* of $\beta = \partial j$. Under our assumptions β^ε is monotone and globally Lipschitz continuous on the whole real line and it satisfies $\beta^\varepsilon(0) = 0$. We also set

$$J^\varepsilon(u) := \int_\Omega j^\varepsilon(u)\,\mathrm{d}x, \quad \mathcal{J}^\varepsilon(u) := \int_0^T \int_\Omega j^\varepsilon(u)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.1}$$

Moreover, we regularize the initial data by taking, for $\varepsilon \in (0,1)$, u_0^ε and u_1^ε satisfying

$$u_0^\varepsilon \in D(A), \quad u_1^\varepsilon \in V, \quad J^\varepsilon(u_0^\varepsilon) \le J(u_0), \tag{3.2}$$

$$u_0^\varepsilon \to u_0 \ \text{ in } V, \qquad u_1^\varepsilon \to u_1 \ \text{ in } H. \tag{3.3}$$

The construction of approximate initial data complying with (3.2)-(3.3) is standard. For instance, one may take u_0^ε as the solution to the elliptic singular perturbation problem

$$u_0^\varepsilon \in D(A), \qquad u_0^\varepsilon + \varepsilon A u_0^\varepsilon = u_0 \ \text{ in } H. \tag{3.4}$$

In particular, the last of (3.2) can be shown by testing the equation in (3.4) by $\beta^\varepsilon(u_0^\varepsilon)$ and noting that

$$(\beta^\varepsilon(u_0^\varepsilon), u_0^\varepsilon - u_0) \ge J^\varepsilon(u_0^\varepsilon) - J^\varepsilon(u_0) \ge J^\varepsilon(u_0^\varepsilon) - J(u_0), \tag{3.5}$$

the latter inequality following from the monotonicity of the Moreau-Yosida regularization J^ε with respect to ε.

We are now ready to introduce our approximated equation

$$u_{tt}^\varepsilon + A u_t^\varepsilon + A u^\varepsilon + \beta^\varepsilon(u^\varepsilon) - \lambda u^\varepsilon = g \quad \text{in } (0,T) \times \Omega. \tag{3.6}$$

Correspondingly, we have the following well-posedness and regularity result:

Theorem 3.1. *Let us assume (2.4), (2.5), (2.26), and (2.27). For $\varepsilon \in (0,1)$, let u_0^ε, u_1^ε, and β^ε be as detailed above. Then, there exists a unique solution*

$$u^\varepsilon \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;D(A)) \tag{3.7}$$

to equation (3.6), complemented with the initial conditions $u^\varepsilon|_{t=0} = u_0^\varepsilon$ and $u_t^\varepsilon|_{t=0} = u_1^\varepsilon$. Moreover, for all $t, s \in [0,T]$, the following energy equality *holds:*

$$\begin{aligned} &\frac{1}{2}\|u_t^\varepsilon(t)\|^2 + \int_\Omega j^\varepsilon(u^\varepsilon(t))\,\mathrm{d}x - \frac{\lambda}{2}\|u^\varepsilon(t)\|^2 + \frac{1}{2}\|\nabla u^\varepsilon(t)\|^2 + \|\nabla u_t^\varepsilon\|^2_{L^2(s,t;H)} \\ &\quad = \frac{1}{2}\|u_t^\varepsilon(s)\|^2 + \int_\Omega j^\varepsilon(u^\varepsilon(s))\,\mathrm{d}x - \frac{\lambda}{2}\|u^\varepsilon(s)\|^2 + \frac{1}{2}\|\nabla u^\varepsilon(s)\|^2 + \int_s^t (g, u_t^\varepsilon)\,\mathrm{d}\tau. \end{aligned} \tag{3.8}$$

The proof of Theorem 3.1 is fairly standard and could be carried out, e.g., by following the lines of [23]. Here it is just worth noting that the regularity conditions stated in (3.7) are compatible with the assumptions (3.2)-(3.3) on the regularized initial data. Moreover, one could easily check that (3.7) can be (at least formally) obtained testing (3.6) by $u_{tt}^\varepsilon + A u_t^\varepsilon$ and performing integrations by parts. In particular, the term $\beta^\varepsilon(u^\varepsilon)$ can be managed thanks to the Lipschitz continuity of Yosida approximations. It is also worth noting that, in this regularity setting, equation (3.6) makes sense pointwise; indeed, all its single terms belong to the space $L^2(0,T;H)$ (in particular, we do not need to regularize the source term g: condition (2.26) is enough). Hence, testing the equation by u_t^ε is allowed: this gives relation (3.8) by means of well-known chain rule formulas.

Step 2. A priori estimates. We now derive a number of bounds, uniform with respect to the regularization parameter ε, for the solutions u^ε given by Theorem 3.1. First of all, testing (3.6) by $\varphi \in \mathcal{V}_t$, integrating over Q_t, and performing suitable integrations by parts (both in space and in time), we deduce the integrated (weak) formulation

$$\begin{aligned} &-(\!(u_t^\varepsilon, \varphi_t)\!)_{(0,t)} + (u_t^\varepsilon(t), \varphi(t)) + (\!(\nabla u_t^\varepsilon, \nabla\varphi)\!)_{(0,t)} + (\!(\nabla u^\varepsilon, \nabla\varphi)\!)_{(0,t)} \\ &\quad + (\!(\beta^\varepsilon(u^\varepsilon), \varphi)\!)_{(0,t)} - \lambda(\!(u^\varepsilon, \varphi)\!)_{(0,t)} = (u_1^\varepsilon, \varphi(0)) + (\!(g, \varphi)\!)_{(0,t)}, \quad \forall\, \varphi \in \mathcal{V}_t. \end{aligned} \tag{3.9}$$

Of course, (3.9) holds in particular for $t = T$ and $\varphi \in \mathcal{V}$. Next, setting $s = 0$ in (3.8), or, in other words, testing (3.6) by u_t^ε and integrating over Q_t, $t \in (0,T]$, we find (by Young's inequality)

$$\begin{aligned} &\frac{1}{2}\|u_t^\varepsilon(t)\|^2 + \int_\Omega j^\varepsilon(u^\varepsilon(t))\,\mathrm{d}x - \frac{\lambda}{2}\|u^\varepsilon(t)\|^2 + \frac{1}{2}\|\nabla u^\varepsilon(t)\|^2 + \|\nabla u_t^\varepsilon\|^2_{L^2(0,t;H)} = M_1(\varepsilon) + (\!(g, u_t^\varepsilon)\!)_{(0,t)} \\ &\quad \le M_1(\varepsilon) + \|g\|^2_{L^2(0,t;H)} + \frac{1}{4}\|u_t^\varepsilon\|^2_{L^2(0,t;H)} =: M_2(\varepsilon) + \frac{1}{4}\|u_t^\varepsilon\|^2_{L^2(0,t;H)}, \end{aligned} \tag{3.10}$$

where we have set

$$M_1(\varepsilon) := \frac{1}{2}\|u_1^\varepsilon\|^2 + J^\varepsilon(u_0^\varepsilon) - \frac{\lambda}{2}\|u_0^\varepsilon\|^2 + \frac{1}{2}\|\nabla u_0^\varepsilon\|^2, \quad M_2(\varepsilon) := M_1(\varepsilon) + \|g\|^2_{L^2(0,t;H)}, \tag{3.11}$$

and we may notice that actually M_1 and M_2 are bounded uniformly in ε due to (3.2)-(3.3), (2.27), and (2.26). Let us also observe that, thanks to the properties of the Yosida approximation (cf. [10]), there exists a constant $c \ge 0$ such that $j^\varepsilon(r) - \lambda r^2 \ge -c$ for all $r \in \mathbb{R}$ and all $\varepsilon \in (0,1)$. Hence, applying Gronwall's lemma to (3.10), we obtain

$$\|u_t^\varepsilon(t)\| \le M \quad \text{for all } t \in [0,T], \tag{3.12a}$$

and for all $\varepsilon \in (0,1)$. Here and below M denotes a positive constant, possibily different from line to line, depending on the problem data, but independent of ε. From (3.10) we also get

$$\|u^\varepsilon\|_{H^1(0,T;V)} \le M, \tag{3.12b}$$

$$\int_\Omega j^\varepsilon(u^\varepsilon(t))\,\mathrm{d}x \le M \quad \text{for all } t \in [0,T], \tag{3.12c}$$

for all $\varepsilon \in (0,1)$. Setting now $\varphi = u^\varepsilon$ in (3.9) and taking $t = T$, we deduce

$$\begin{aligned} &\int_0^T \int_\Omega \beta^\varepsilon(u^\varepsilon)u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \|\nabla u^\varepsilon\|^2_{L^2(0,T;H)} + \frac{1}{2}\|\nabla u^\varepsilon(T)\|^2 = \|u_t^\varepsilon\|^2_{L^2(0,T;H)} - (u_t^\varepsilon(T), u^\varepsilon(T)) \\ &\quad + \frac{1}{2}\|\nabla u_0^\varepsilon\|^2 + \lambda\|u^\varepsilon\|^2_{L^2(0,T;H)} + (u_1^\varepsilon, u_0^\varepsilon) + (\!(g, u^\varepsilon)\!). \end{aligned} \tag{3.13}$$

Now, the right hand side is bounded uniformly in ε due to (3.12), (2.26)-(2.27) and (3.2)-(3.3). Moreover, it is easy to check (cf. also [27, Appendix]) that there exist constants $c_1 > 0$, $c_2 \ge 0$ independent of $\varepsilon \in (0,1)$ such that $c_1|\beta^\varepsilon(r)| \le \beta^\varepsilon(r)r + c_2$ for all $r \in \mathbb{R}$. Hence, (3.13) entails

$$\|\beta^\varepsilon(u^\varepsilon)\|_{L^1(0,T;L^1(\Omega))} \le M, \tag{3.14a}$$

for all $\varepsilon \in (0,1)$. Then, using once more (3.12) and comparing terms in (3.6), we also find

$$\|u_{tt}^\varepsilon\|_{L^1(0,T;X)} \le M, \tag{3.14b}$$

for any Banach space X such that $L^1(\Omega) \subset X$ and and $V' \subset X$ with continuous and compact embeddings. In particular, it is not restrictive to assume X be the dual of a reflexive and separable space (for instance, in dimension $N = 3$, one may take $X = H^{-2}(\Omega)$). Hence, we have

$$\|u_t^\varepsilon\|_{W^{1,1}(0,T;X)} \leq M, \tag{3.14c}$$

for all $\varepsilon \in (0, 1)$.

Step 3. Passage to the limit. Now we aim at letting $\varepsilon \searrow 0$. From (3.12a)-(3.12b) we deduce that there exists a function u of the regularity specified in (2.28) such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } \; H^1(0,T;V), \tag{3.15a}$$

$$u^\varepsilon \rightharpoonup u \quad \text{weakly star in } \; W^{1,\infty}(0,T;H), \tag{3.15b}$$

and in particular

$$u^\varepsilon(t) \to u(t) \quad \text{strongly in } \; H \; \text{ for all } t \in [0,T], \tag{3.15c}$$

$$u^\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } \; V \; \text{ for all } t \in [0,T]. \tag{3.15d}$$

It is worth stressing that the above convergence relations, as well as the ones that will follow, are intended to hold up to extraction of suitable (nonrelabelled) subsequences of $\varepsilon \searrow 0$.

Since V is compactly embedded into H, in view of conditions (3.12b) and (3.14c) we can apply [31, Corollary 4] with the three spaces $V \subset\subset H \subset X$ and $p = 2$ in order to obtain that

$$u_t^\varepsilon \to u_t \quad \text{strongly in } \; L^2(0,T;H). \tag{3.15e}$$

Moreover, condition (3.14c) implies that the functions u_t^ε are uniformly bounded in $BV(0,T;X)$ (for the properties of vector-valued BV-spaces one can refer, e.g., to [10, Appendix]). In view of the fact that we may assume X be the dual of a reflexive and separable space, we can employ a generalization of Helly's theorem (cf., e.g., [26, Thm. 3.1] or [13, Lemma 7.2]), providing a function $v \in BV(0,T;X)$ such that

$$u_t^\varepsilon(t) \rightharpoonup v(t) \quad \text{weakly star in } X \; \text{ for all } t \in [0,T]. \tag{3.15f}$$

It is easily seen that v coincides with u_t almost everywhere. Hence, up to changing the representative of u_t, we may assume $v = u_t$ everywhere on $[0,T]$. Moreover, combining (3.15f) with (3.12a), we obtain

$$u_t^\varepsilon(t) \rightharpoonup u_t(t) \quad \text{weakly in } H \; \text{ for all } t \in [0,T]. \tag{3.15g}$$

Moreover we get

$$u_t^\varepsilon \rightharpoonup u_t \quad \text{weakly star in } \; BV(0,T;X). \tag{3.15h}$$

Let us now show that the functions $\beta^\varepsilon(u^\varepsilon)$ are uniformly bounded (with respect to ε) in $\mathcal{V}'$. Actually, writing (3.9) for $t = T$, and using Holder's inequality, (3.3), (2.26) and the estimates (3.12a) and (3.12b), we find

$$\begin{aligned} \big|\langle\!\langle \beta^\varepsilon(u^\varepsilon), \varphi \rangle\!\rangle\big| &\leq \|u_t^\varepsilon\|_{L^2(0,T;H)} \|\varphi_t\|_{L^2(0,T;H)} + \|u_t^\varepsilon(T)\| \|\varphi(T)\| \\ &\quad + \|\nabla u_t^\varepsilon\|_{L^2(0,T;H)} \|\nabla\varphi\|_{L^2(0,T;H)} + \|\nabla u^\varepsilon\|_{L^2(0,T;H)} \|\nabla \varphi\|_{L^2(0,T;H)} \\ &\quad + \lambda \|u^\varepsilon\|_{L^2(0,T;H)} \|\varphi\|_{L^2(0,T;H)} + \|u_1^\varepsilon\| \|\varphi(0)\| + \|g\|_{L^2(0,T;H)} \|\varphi\|_{L^2(0,T;H)} \leq C \|\varphi\|_{\mathcal{V}}, \end{aligned} \tag{3.15i}$$

for all $\varphi \in \mathcal{V}$, where $C > 0$ is independent of ε. Therefore, we can infer that there exists $\xi \in \mathcal{V}'$ such that

$$\beta^\varepsilon(u^\varepsilon) \rightharpoonup \xi \quad \text{weakly in } \; \mathcal{V}'. \tag{3.15j}$$

Next, recalling the definition (2.15) of $\mathcal{X}$, from (3.14a) we obtain that there exists a measure $\mathcal{T} \in \mathcal{X}'$ such that

$$\beta^\varepsilon(u^\varepsilon) \rightharpoonup \mathcal{T} \quad \text{weakly star in } \; \mathcal{X}'. \tag{3.15k}$$

In view of the density of $\mathcal{X} \cap \mathcal{V}$ both in $\mathcal{X}$ and in $\mathcal{V}$, the measure $\mathcal{T}$ represents ξ on $\mathcal{X}$, i.e. (2.18) holds.

Let us now go back to (3.9), now rewritten for general $t \in (0,T]$ and $\varphi \in \mathcal{V}_t$. Then, rearranging terms, and using the above convergence relations (3.15), we obtain that there exists the limit

$$\begin{aligned}\lim_{\varepsilon\searrow 0} \langle\!\langle \beta^\varepsilon(u^\varepsilon), \varphi \rangle\!\rangle_{(0,t)} &= \lim_{\varepsilon\searrow 0} \Big[(\!(u_t^\varepsilon, \varphi_t)\!)_{(0,t)} - (u_t^\varepsilon(t), \varphi(t)) - (\!(\nabla u_t^\varepsilon, \nabla\varphi)\!)_{(0,t)} - (\!(\nabla u^\varepsilon, \nabla\varphi)\!)_{(0,t)} \\ &\qquad + \lambda (\!(u^\varepsilon, \varphi)\!)_{(0,t)} + (u_1^\varepsilon, \varphi(0)) + (\!(g, \varphi)\!)_{(0,t)} \Big] \\ &= (\!(u_t, \varphi_t)\!)_{(0,t)} - (u_t(t), \varphi(t)) - (\!(\nabla u_t, \nabla\varphi)\!)_{(0,t)} - (\!(\nabla u, \nabla\varphi)\!)_{(0,t)} \\ &\qquad + \lambda (\!(u, \varphi)\!)_{(0,t)} + (u_1, \varphi(0)) + (\!(g, \varphi)\!)_{(0,t)}. \end{aligned} \tag{3.16}$$

A crucial point in our argument is that the left hand side tends, *with no need of extracting a further subsequence*, to a linear and continuous functional on $\mathcal{V}_t$ that acts on φ as specified by the right hand side. Noting as $\xi_{(t)}$ such a functional, we have in other words

$$\beta^\varepsilon(u^\varepsilon) \rightharpoonup \xi_{(t)} \quad \text{weakly in } \mathcal{V}_t'. \tag{3.17}$$

Moreover, (3.16) can be restated as

$$\begin{aligned} &- (\!(u_t, \varphi_t)\!)_{(0,t)} + (u(t), \varphi(t)) + (\!(\nabla u_t, \nabla\varphi)\!)_{(0,t)} + (\!(\nabla u, \nabla\varphi)\!)_{(0,t)} \\ &\quad + \langle\!\langle \xi_{(t)}, \varphi \rangle\!\rangle_{(0,t)} - \lambda (\!(u, \varphi)\!)_{(0,t)} = (u_1, \varphi(0)) + (\!(g, \varphi)\!)_{(0,t)}. \end{aligned} \tag{3.18}$$

Hence, (2.31) and (2.30), which is a particular case of it, are proved. Note now that, from (3.14a), it also follows

$$\beta^\varepsilon(u^\varepsilon) \rightharpoonup \mathcal{T}_{(t)} \quad \text{weakly star in } \mathcal{X}_t' \tag{3.19}$$

and also this convergence holds with no need of extracting further subsequences. Indeed, the limit of the whole (sub)sequence is already identified as $\xi_{(t)}$ on the dense subspace $\mathcal{X}_t \cap \mathcal{V}_t$. This also implies that the measure $\mathcal{T}_{(t)}$ represents $\xi_{(t)}$ on $\mathcal{X}_t$ in the sense of (2.18). Using the fact that for any $\varphi \in \mathcal{V}_{t,0}$ the extension $\widetilde{\varphi}$ lies in $\mathcal{V}$, it is easy to check that the functionals ξ and $\xi_{(t)}$ are "compatible". Hence, we have checked points *(a)* and *(b1)-(b2)* of Definition 2.2 of weak solution.

Let us now show relation (2.35), i.e., point *(e)* of Definition 2.2. To this aim, we write (2.31) with $\varphi = u$ for $s,t \in (0,T]$ and take the difference. Note that the choice $\varphi = u$ is admissible since $u \in \mathcal{V}$. We then infer

$$\begin{aligned} &- \|u_t\|^2_{L^2(s,t;H)} + (u_t(t), u(t)) + \int_s^t (\nabla u_t, \nabla u) \, \mathrm{d}\tau + \|\nabla u\|^2_{L^2(s,t;H)} \\ &\quad + \langle\!\langle \xi_{(t)}, u \rangle\!\rangle_{(0,t)} - \langle\!\langle \xi_{(s)}, u \rangle\!\rangle_{(0,s)} - \lambda \|u\|^2_{L^2(s,t;H)} = (u_t(s), u(s)) + \int_s^t \int_\Omega g u \, \mathrm{d}x \, \mathrm{d}\tau. \end{aligned} \tag{3.20}$$

Then, computing explicitly the integral on the left hand side, (2.35) readily follows.

Step 4. Identification of ξ. To conclude our proof we need to identify ξ (and $\xi_{(t)}$) in the sense of the weak constraint (2.33). This will give *(c)* of Definition 2.2. We start working on ξ, and, to get the identification, we shall implement the so-called Minty's trick in the duality between $\mathcal{V}'$ and $\mathcal{V}$. This corresponds to checking the following two conditions:

(i) There holds the lim sup-inequality

$$\limsup_{\epsilon\searrow 0} \langle\!\langle \beta^\varepsilon(u^\varepsilon), u^\varepsilon \rangle\!\rangle \le \langle\!\langle \xi, u \rangle\!\rangle. \tag{3.21}$$

(ii) The operators β^ε *suitably* converge to β_w, in such a way that (2.33) may follow as a consequence of (3.21).

We start by checking property (i), postponing the discussion regarding the correct notion of convergence for (ii) and its implications. Writing (3.9) for $\varphi = u^\varepsilon$ and $t = T$, we obtain

$$\begin{aligned} \langle\!\langle \beta^\varepsilon(u^\varepsilon), u^\varepsilon \rangle\!\rangle &= \|u_t^\varepsilon\|^2_{L^2(0,T;H)} - (u_t^\varepsilon(T), u^\varepsilon(T)) + (u_1^\varepsilon, u_0^\varepsilon) - \frac{1}{2}\|\nabla u^\varepsilon(T)\|^2 + \frac{1}{2}\|\nabla u_0^\varepsilon\|^2 \\ &\quad - \|\nabla u^\varepsilon\|^2_{L^2(0,T;H)} + \lambda \|u^\varepsilon\|^2_{L^2(0,T;H)} + (\!(g, u^\varepsilon)\!). \end{aligned} \tag{3.22}$$

Now, thanks to (3.15a), (3.15c), (3.15d), (3.15g), and (3.15e), we see that the lim sup (as $\varepsilon \searrow 0$ along a proper subsequence) of the right hand side is less or equal than

$$\|u_t\|^2_{L^2(0,T;H)} - (u_t(T), u(T)) + (u_1, u_0) - \frac{1}{2}\|\nabla u(T)\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \|\nabla u\|^2_{L^2(0,T;H)} + \lambda\|u\|^2_{L^2(0,T;H)} + (\!(g, u)\!). \tag{3.23}$$

Hence, using (2.35) written for $t = T$ and $s = 0$, we see that the above expression is equal to $\langle\!\langle \xi, u\rangle\!\rangle$. Therefore, (3.21) is proved.

Let us now switch to discussing (ii), which requires the introduction of some additional machinery. We present it by following the lines of the book by Attouch [1]. At first, we observe that the restriction to $\mathcal{V}$ of the function β^ε can be seen as a monotone operator from $\mathcal{V}$ to $\mathcal{V}'$ (once one works in the parabolic Hilbert triplet $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$). Indeed, if $v \in \mathcal{V}$, then $\beta^\varepsilon(v) \in \mathcal{H} \subset \mathcal{V}'$ by the Lipschitz continuity of β^ε. Hence, for any $v_1, v_2 \in \mathcal{V}$, we have

$$\langle\!\langle \beta^\varepsilon(v_2) - \beta^\varepsilon(v_1), v_2 - v_1\rangle\!\rangle = (\!(\beta^\varepsilon(v_2) - \beta^\varepsilon(v_1), v_2 - v_1)\!) \geq 0. \tag{3.24}$$

Moreover, if $v, z \in \mathcal{V}$, then, by definition of subdifferential,

$$\langle\!\langle \beta^\varepsilon(v), z - v\rangle\!\rangle = (\!(\beta^\varepsilon(v), z - v)\!) \leq \int_0^T \int_\Omega (j^\varepsilon(z) - j^\varepsilon(v))\,\mathrm{d}x\,\mathrm{d}t = \mathcal{J}^\varepsilon|_\mathcal{V}(z) - \mathcal{J}^\varepsilon|_\mathcal{V}(v). \tag{3.25}$$

In other words, we have the graph inclusion

$$\beta^\varepsilon|_\mathcal{V} \subset \partial_{\mathcal{V},\mathcal{V}'}\mathcal{J}^\varepsilon|_\mathcal{V}, \tag{3.26}$$

where the notation used on the right hand side stands for the subdifferential of $\mathcal{J}^\varepsilon|_\mathcal{V}$ with respect to the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$. By the standard theory of subdifferentials, this is a maximal monotone operator from $\mathcal{V}$ to $2^{\mathcal{V}'}$, which includes (in the sense of graphs) the (monotone, but not necessarily maximal) operator $\beta^\varepsilon|_\mathcal{V}$.

In view of the fact that the family of functionals $\{\mathcal{J}^\varepsilon|_\mathcal{V}\}$ (defined on $\mathcal{V}$ and taking values in $[0, +\infty)$) is increasing as ε decreases to 0, applying [1, Thm. 3.20], we obtain

$$\mathcal{J}^\varepsilon|_\mathcal{V} \to \sup_{\varepsilon\in(0,1)} \mathcal{J}^\varepsilon|_\mathcal{V} \tag{3.27}$$

in the sense of *Mosco convergence* (that is Gamma-convergence both in the strong and in the weak topology of $\mathcal{V}$). Moreover, by the monotone convergence theorem it is readily seen that the functional on the right hand side coincides in fact with $\mathcal{J}|_\mathcal{V}$. Hence, owing to [1, Thm. 3.66], the family of maximal monotone operators $\partial_{\mathcal{V},\mathcal{V}'}\mathcal{J}^\varepsilon|_\mathcal{V}$, identified with the family of their graphs in the product space $\mathcal{V} \times \mathcal{V}'$, converges *in the sense of graphs* (cf. [1, Def. 3.58]) to $\partial_{\mathcal{V},\mathcal{V}'}\mathcal{J}|_\mathcal{V} = \beta_w$. Namely,

$$\forall\,[x; y] \in \partial_{\mathcal{V},\mathcal{V}'}\mathcal{J}|_\mathcal{V}, \quad \exists\,[x^\varepsilon; y^\varepsilon] \in \partial_{\mathcal{V},\mathcal{V}'}\mathcal{J}^\varepsilon|_\mathcal{V} \quad \text{such that} \quad [x^\varepsilon; y^\varepsilon] \to [x; y] \ \text{ strongly in } \mathcal{V} \times \mathcal{V}'. \tag{3.28}$$

Hence, in view of the facts that $[u^\varepsilon; \beta^\varepsilon(u^\varepsilon)] \in \mathcal{J}^\varepsilon|_\mathcal{V}$ (thanks to (3.26)), $[u^\varepsilon; \beta^\varepsilon(u^\varepsilon)] \to [u; \xi]$ weakly in $\mathcal{V} \times \mathcal{V}'$ (thanks to (3.15a) and (3.15j)), and to the lim sup-inequality (3.21), we may apply [1, Prop. 3.59], yielding that $[u; \xi] \in \partial_{\mathcal{V},\mathcal{V}'}\mathcal{J}^\varepsilon|_\mathcal{V} = \beta_w$. Hence, (2.33) is proved.

To conclude this part, we need to prove that $\xi_{(t)} \in \beta_{w,(t)}(u)$. To this aim, it is sufficient to adapt the above argument by working on the subinterval $(0, t)$. Indeed, relations (3.9) and (2.35) (the latter for $s = 0$) hold on any subinterval $(0, t)$. Moreover, we can take advantage of (3.15a) (whose analogue obviously holds also on subintervals) and (3.17).

Step 5. Further properties of solutions. Let us start proving that inequality (3.21) is in fact an equality. Indeed, owing to (3.28), there exist $[x^\varepsilon; y^\varepsilon] \in \partial_{\mathcal{V},\mathcal{V}'}\mathcal{J}^\varepsilon|_\mathcal{V}$ such that $[x^\varepsilon; y^\varepsilon] \to [u; \xi]$ strongly in $\mathcal{V} \times \mathcal{V}'$. Hence, noting that, by monotonicity,

$$0 \leq \langle\!\langle \beta^\varepsilon(u^\varepsilon) - y^\varepsilon, u^\varepsilon - x^\varepsilon\rangle\!\rangle, \tag{3.29}$$

taking the lim inf as $\varepsilon \searrow 0$, and recalling (3.21), we obtain

$$\lim_{\epsilon \searrow 0} \langle\!\langle \beta^\varepsilon(u^\varepsilon), u^\varepsilon \rangle\!\rangle = \langle\!\langle \xi, u \rangle\!\rangle. \tag{3.30}$$

As a consequence, the limit of the right hand side of (3.22) exists and coincides with (3.23). In view of the fact that convergence of most terms of (3.22) is already known from the previous estimates we get in particular that

$$\lim_{\varepsilon \searrow 0} \left(\frac{1}{2} \|\nabla u^\varepsilon(T)\|^2 + \|\nabla u^\varepsilon\|^2_{L^2(0,T;H)} \right) = \frac{1}{2} \|\nabla u(T)\|^2 + \|\nabla u\|^2_{L^2(0,T;H)}. \tag{3.31}$$

As before, this argument can be repeated on any subinterval $(0,t)$. Hence, recalling (3.15a) and (3.15d), we finally arrive at

$$u^\varepsilon \to u \quad \text{strongly in } L^2(0,T;V), \tag{3.32}$$

$$u^\varepsilon(t) \to u(t) \quad \text{strongly in } V \quad \text{for all } t \in [0,T]. \tag{3.33}$$

Next, let us show that, under assumption (2.42), the additional regularity (2.43) holds. To this aim, we go back to the approximate problem, and, in the spirit of [28], we test (3.6) by Au^ε. Indeed, u^ε has sufficient smoothness in order for this procedure to be admissible (cf. (3.7)). Integrating by parts, and using the monotonicity of β^ε, we then easily infer

$$\frac{\mathrm{d}}{\mathrm{d}t}(u^\varepsilon_t, Au^\varepsilon) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Au^\varepsilon\|^2 + \|Au^\varepsilon\|^2 = (\lambda u^\varepsilon + g, Au^\varepsilon) + (u^\varepsilon_t, Au^\varepsilon_t). \tag{3.34}$$

By some further integration by parts and using Hölder's and Young's inequalities (and the definition of the operator A), the right hand side can be easily estimated as follows:

$$(\lambda u^\varepsilon + g, Au^\varepsilon) + (u^\varepsilon_t, Au^\varepsilon_t) \le \frac{1}{2}\|Au^\varepsilon\|^2 + C\big(\|g\|^2 + \|u^\varepsilon\|^2\big) + \|\nabla u^\varepsilon_t\|^2. \tag{3.35}$$

Here and below, $C > 0$ is a constant independent of ε. Hence, integrating (3.34) over $(0,t)$ for arbitrary $t \in (0,T]$, using (3.35), and recalling estimate (3.12a), we easily obtain

$$2(u^\varepsilon_t(t), Au^\varepsilon(t)) + \|Au^\varepsilon(t)\|^2 + \int_0^t \|Au^\varepsilon\|^2 \,\mathrm{d}s \le C + \|Au^\varepsilon_0\|^2. \tag{3.36}$$

Now, one can immediately check that, under assumption (2.42), if u^ε_0 is defined as in (3.4), then the right hand side of (3.36) is bounded independently of ε. Hence, noticing that the left hand side is larger or equal than

$$\frac{1}{2}\|Au^\varepsilon(t)\|^2 - C\|u^\varepsilon_t(t)\|^2 + \int_0^t \|Au^\varepsilon\|^2 \,\mathrm{d}s, \tag{3.37}$$

where the second term is uniformly controlled due to (3.12a), we readily arrive at

$$\|u^\varepsilon\|_{L^\infty(0,T;D(A))} \le M. \tag{3.38}$$

Letting $\varepsilon \searrow 0$, we then infer

$$u \in L^\infty(0,T;D(A)) \tag{3.39}$$

thanks to semicontinuity of norms with respect to weak convergence. Finally, (2.43), i.e., weak continuity of u with values in $D(A)$, follows by combining (3.39) with the regularity $u \in C([0,T];V)$ following from (2.28), and applying standard results.

Eventually, we show that weak solutions constructed as limit points of $\{u^\varepsilon\}$ also satisfy a form of the energy *inequality*. We start by proving it on intervals of the form $[0,t]$, $t \in (0,T]$. To this aim, we write relation (3.8) for $s = 0$ and take the lim inf as $\varepsilon \searrow 0$. Then, using standard semicontinuity arguments together with relations (3.2)-(3.3), (3.15a), and (3.15c)-(3.15d), it is not difficult to infer, for every $t \in (0,T]$,

$$\frac{1}{2}\|u_t(t)\|^2 + \int_\Omega j(u(t))\ \mathrm{d}x - \frac{\lambda}{2}\|u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \|\nabla u_t\|^2_{L^2(0,t;H)}$$
$$\leq \frac{1}{2}\|u_1\|^2 + \int_\Omega j(u_0)\ \mathrm{d}x - \frac{\lambda}{2}\|u_0\|^2 + \frac{1}{2}\|\nabla u_0\|^2 + \int_0^t (g, u_t)\ \mathrm{d}\tau. \tag{3.40}$$

Note in particular that relation

$$J(u(t)) = \int_\Omega j(u(t))\ \mathrm{d}x \leq \liminf_{\varepsilon\searrow 0} \int_\Omega j^\varepsilon(u^\varepsilon(t))\ \mathrm{d}x = \liminf_{\varepsilon\searrow 0} J^\varepsilon(u^\varepsilon(t)) \tag{3.41}$$

is a consequence of (3.15c) and of the fact that the functionals J^ε converge to J in the sense of Mosco (cf. [1, Par. 3.3]) in the space H. Recalling (2.25), (3.40) reduces to (2.41) in the case $s = 0$. Let us now consider a generic interval $[s,t]$ for $0 < s < t \leq T$ and let us go back to (3.8) written for this choice of s, t. Let us take once more the lim inf as $\varepsilon \searrow 0$. Then, the left hand side is treated exactly as before. On the other hand, when looking at the right hand side, it is easy to check that

$$-\frac{\lambda}{2}\|u^\varepsilon(s)\|^2 + \frac{1}{2}\|\nabla u^\varepsilon(s)\|^2 + \int_s^t (g, u_t^\varepsilon)\ \mathrm{d}\tau \to -\frac{\lambda}{2}\|u(s)\|^2 + \frac{1}{2}\|\nabla u(s)\|^2 + \int_s^t (g, u_t)\ \mathrm{d}\tau \tag{3.42}$$

thanks in particular to (3.15a), and (3.33). Next, thanks to (3.15e), and up to extracting a further subsequence of $\varepsilon \searrow 0$, we have

$$\frac{1}{2}\|u_t^\varepsilon(s)\|^2 \to \frac{1}{2}\|u_t(s)\|^2 \tag{3.43}$$

for *almost* every choice of $s \in (0,T)$. Next, we need to control the component of the energy related with the constraint. Namely, we would like to prove that, at least for a.e. $s \in (0,T)$,

$$\limsup_{\varepsilon\searrow 0} \int_\Omega j^\varepsilon(u^\varepsilon(s))\ \mathrm{d}x \leq \int_\Omega j(u(s))\ \mathrm{d}x \tag{3.44}$$

(hence, coupling the above with (3.41) written for $t = s$, we would get convergence of that term). We can start noticing that

$$\int_\Omega j^\varepsilon(u^\varepsilon(s))\ \mathrm{d}x = \int_\Omega \big(j^\varepsilon(u^\varepsilon(s)) - j^\varepsilon(u(s))\big)\ \mathrm{d}x + \int_\Omega j^\varepsilon(u(s))\ \mathrm{d}x. \tag{3.45}$$

Moreover,

$$\int_\Omega j^\varepsilon(u(s))\ \mathrm{d}x \to \int_\Omega j(u(s))\ \mathrm{d}x \tag{3.46}$$

by the monotone convergence theorem. Now, by definition of subdifferential, we may write

$$\int_\Omega \big(j^\varepsilon(u^\varepsilon(s)) - j^\varepsilon(u(s))\big)\ \mathrm{d}x \leq \int_\Omega \beta^\varepsilon(u^\varepsilon(s))(u^\varepsilon(s) - u(s))\ \mathrm{d}x =: \mu^\varepsilon(s), \tag{3.47}$$

and we have to discuss the behavior of the functions μ^ε. First, we observe that

$$\int_0^T \mu^\varepsilon(s)\ \mathrm{d}s = \langle\!\langle \beta^\varepsilon(u^\varepsilon), u^\varepsilon - u \rangle\!\rangle \to 0, \tag{3.48}$$

the latter property following from (3.30) and (3.15j). Moreover, thanks to (3.47), we have

$$\mu^\varepsilon(s) \geq J^\varepsilon(u^\varepsilon(s)) - J^\varepsilon(u(s)) \geq J^\varepsilon(u^\varepsilon(s)) - J(u(s)), \tag{3.49}$$

whence $\liminf_{\varepsilon\searrow 0} \mu^\varepsilon(s) \geq 0$ thanks to (3.41). Hence, we have in particular $\lim_{\varepsilon\searrow 0}(\mu^\varepsilon)^-(s) = 0$, $(\cdot)^-$ denoting the negative part. Moreover, from (3.49), (3.41) and (3.12c) we infer

$$\mu^\varepsilon(s) \geq -J(u(s)) \geq -\liminf_{\delta\searrow 0} J^\delta(u^\delta(s)) \geq -M, \tag{3.50}$$

for all $\varepsilon \in (0,1)$ and $s \in (0,T]$. Hence, by the dominated convergence theorem we obtain that $(\mu^\varepsilon)^- \to 0$ in $L^1(0,T)$. Consequently, thanks to (3.48), we conclude that $\mu^\varepsilon \to 0$ in $L^1(0,T)$. Hence, up to a subsequence, $\mu^\varepsilon \to 0$ almost everywhere in $(0,T)$, whence (3.44) follows. This actually implies (2.41) for *almost* every $s \in (0,T)$ and every $t \in (s,T]$, as desired. The proof of Theorem 2.5 is concluded.

Remark 3.2. If the source term g is 0, from (2.41) follows in particular that the energy loss in the time interval (s,t) is *at least* as large as the dissipation term $\mathcal{D}(s,t) := \|\nabla u_t\|^2_{L^2(s,t;H)}$. Of course, as commonly occurs situations characterized by bad regularity, the energy dissipated may be in fact strictly larger than $\mathcal{D}(s,t)$. Indeed, we may observe that proving equality in (3.40) appears out of reach in the present regularity setting.

Remark 3.3. In view of our strategy of proof for Theorem 2.5, we can give some further observation complementing Remark 2.4. Hence, let us go back to the "toy problem" (2.36), for example with $\beta = \partial I_{[-1,1]}$ (but our consideration also apply to different choices of β). Then, implementing our regularization method we get the equation

$$u^\varepsilon_{tt} + \beta^\varepsilon(u^\varepsilon) = 0. \tag{3.51}$$

Setting $v^\varepsilon := u^\varepsilon_t$, the shape of solution trajectories of the 2D ODE system associated to (3.51) in the phase space $(u^\varepsilon, v^\varepsilon)$ can be easily described. In particular, since (in this spatially homogeneous setting) no dissipation occurs, trajectories are periodic. Moreover, we may notice that, for $\varepsilon \searrow 0$, $(u^\varepsilon, v^\varepsilon)$ converges in a suitable way to a couple (u,v), where $v = u_t$ and u solves (2.36). Clearly, (u,v) is also a periodic trajectory and its image in the phase space lies in some level set $\{j(u) + v^2/2 = c\}$, $c \geq 0$, of the "energy" functional. In particular, whenever $D(j) = [-1,1]$ (as happens in the case of the indicator function $j = I_{[-1,1]}$, and also for the "logarithmic potential" mentioned in the introduction), such level sets are (at least for large initial energy, i.e., for large values of c) not connected. Namely, their shape determines the jumps of u_t (which, consequently, cannot occur in an "arbitrary" way). Note also that taking different choices for the approximations β^ε of β does not modify the shape of (u,v). Of course it is clear that, in the case of our equation (1.1), the situation is much more complicated than for (2.36) in view of the infinite-dimensional setting. However, the fact that our weak solutions u are still built as limit points of families u^ε solving a very natural regularization of the equation suggests that the jumps of u_t occurring in the limit may be in some sense "physical", i.e., they are determined by the fact that u^ε_t, as $\varepsilon \searrow 0$, may tend to develop discontinuities. In other words, the occurrence of "spurious" jumps of u_t (as are the somehow "arbitrary" jumps described in Remark 2.4) should be excluded in view of the fact that our weak solutions descend from the approximation scheme.

Remark 3.4. Let us give some further observation complementing Remark 2.3. Again, we consider, just for simplicity, the "toy" model (2.36); however, our considerations also apply to the original equation (1.1). Actually, from our approximation argument we know that, for any $t \in (0,T]$, (a subsequence of) $\beta^\varepsilon(u^\varepsilon)$ (weakly star) converges to a measure $\mathcal{T}_{(t)}$ on $\overline{Q_t}$ (in particular, we have convergence to some $\mathcal{T}$ on the whole interval). In the toy case, of course, $\overline{Q_t} = [0,t]$; moreover, we are allowed to identify $\mathcal{T}_{(t)} = \xi_{(t)} \in \mathcal{V}'_t$ because Sobolev functions are continuous in 1D. Let us now consider the particular case when $\beta^\varepsilon(u^\varepsilon)$ is supported in some interval of the form $[t^\varepsilon - \varepsilon, t^\varepsilon + \varepsilon]$ and is 0 outside that interval. Then, assuming that t^ε converges to some point $t \in (0,T)$ as $\varepsilon \searrow 0$, and $\beta^\varepsilon(u^\varepsilon)$ "spikes" around t^ε in a proper way, it may happen that $\beta^\varepsilon(u^\varepsilon)$ (weakly star) converges to $\mathcal{T} = \delta_t$ (the Dirac delta concentrated in t) in $\mathcal{X}'$. This kind of behavior may be (possibly) driven for instance by inserting a nonzero forcing term g in the equation. Then, in the case when, for instance, $t^\varepsilon = t - 2\varepsilon$, it turns out that the singularity of $\mathcal{T}$ develops *before* t. Consequently, $\beta^\varepsilon(u^\varepsilon)$ also converges to δ_t in $\mathcal{X}'_t$. In particular, (2.31) holds in $[0,t]$ with $\xi_{(t)} = \delta_t$. On the other hand, if $t^\varepsilon = t + 2\varepsilon$, i.e., the singularity of $\mathcal{T}$ develops *after* t, in that case $\beta^\varepsilon(u^\varepsilon)$ converges to $\mathcal{T}_{(t)} = 0$ in $\mathcal{X}'_t$, whence (2.31) holds in $[0,t]$ with $\xi_{(t)} = 0$. Note that this happens in spite of the fact that the limit measure $\mathcal{T}$ over the whole $[0,T]$ is the same in the two cases. This fact suggests that the formulation (2.31) on the subinterval $(0,t)$ contains some additional information that cannot be simply inferred by restricting the *global* formulation (2.30). This is the reason why we decided to include *(b2)* in our existence theorem.

Remark 3.5. Let t be one of the (at most countably many) jump points of u_t. Then, both the point value of u_t at t and the occurrence of concentration phenomena for the measure $\mathcal{T}_{(t)}$ at the same point also depend on the choice of the approximating problem (i.e., of β^ε; actually our argument works provided that β^ε is smooth and converges to β in the graph sense) and of the selection of converging subsequences via Helly's theorem. This can be seen again by looking at the "toy equation" (2.36) with $\beta = \partial I_{[-1,1]}$ and initial values $u_0 = 0$ and $u_1 = 1$. Then, we know that the (first) jump of u_t occurs at $t = 1$. Let us now consider the approximation (3.51) with the choice

$$\beta^\varepsilon(r) = \begin{cases} 0 & \text{if } |r| \le r^\varepsilon, \\ \varepsilon^{-2}(r - r^\varepsilon) & \text{if } r > r^\varepsilon, \\ \varepsilon^{-2}(r + r^\varepsilon) & \text{if } r < -r^\varepsilon, \end{cases} \tag{3.52}$$

where, for any $\varepsilon \in (0,1)$, one may choose (in an arbitrary way) r^ε in the interval $[1-\varepsilon\pi, 1]$. It is then clear that, whatever are the chosen values of r^ε, β^ε tends to $\beta = \partial I_{[-1,1]}$ in the sense of graphs as $\varepsilon \searrow 0$. Hence, our limit problem is the desired one. Let us notice that, for $r \ge 0$, we have

$$j^\varepsilon(r) = \frac{1}{2\varepsilon^2}\big((r - r^\varepsilon)^+\big)^2. \tag{3.53}$$

In particular, $j^\varepsilon(1) = 0$ if $r^\varepsilon = 1$, whereas

$$j^\varepsilon(1) = \frac{(1-r^\varepsilon)^2}{2\varepsilon^2} \in \Big(0, \frac{\pi^2}{2}\Big] \quad \text{if } r^\varepsilon \in [1-\varepsilon\pi, 1). \tag{3.54}$$

Then, $u^\varepsilon(1) = 1$ if $r^\varepsilon = 1$, whereas in case $r^\varepsilon \in [1-\varepsilon\pi, 1)$ one can easily compute $u^\varepsilon(t) = r^\varepsilon + \varepsilon \sin\big(\frac{t-r^\varepsilon}{\varepsilon}\big)$ for $t \in (r^\varepsilon, r^\varepsilon + \varepsilon\pi]$, whence $u^\varepsilon_t(t) = \cos\big(\frac{t-r^\varepsilon}{\varepsilon}\big)$ and $u^\varepsilon_t(1) = \cos\big(\frac{1-r^\varepsilon}{\varepsilon}\big)$. Hence, choosing appropriately (and somehow "wildly") r^ε in the interval $[1-\varepsilon\pi, 1]$ as ε varies in $(0,1)$, one may obtain the effect that for any number $\ell \in [-1,1]$ there exists a subsequence $\varepsilon_n \searrow 0$ such that $u^{\varepsilon_n}_t(1)$ tends to ℓ. The use of Helly's theorem selects *one* of these subsequences and determines the limit value $u_t(1) = \ell$ (and, in turn, how the limit measures $\mathcal{T}_{(s)}$ concentrate at the jump point $t = 1$).

Remark 3.6. As observed in the previous Remark, $\xi_{(t)}$ is not represented, in general, by the restricion of the measure $\mathcal{T}$ to the set $\overline{Q_t}$. However, we can give a more explicit characterization of this restriction in the following sense. From (3.15h) we have $u_t \in BV(0,T;X)$, where we may assume X be the dual of a separable space. Hence, for all times $t \in [0,T]$ there exists (in the weak star topology of X) the limit

$$u_t(t^+) := w^*\text{–}\lim_{s\to t^+} u_t(s). \tag{3.55}$$

Moreover this value coincides with the weak star limit

$$w^*\text{–}\lim_{s\to t^+} \frac{1}{s-t}\int_t^s u_t(r)\,\mathrm{d}r. \tag{3.56}$$

In particular the limits above must hold with respect to the weak topology of H, since u_t is bounded in H uniformly in time. Let us now write (2.30) with φ replaced by $\varphi h_s \in \mathcal{X}$, with $\varphi \in \mathcal{X} \cap \mathcal{V}$ and $h_s : [0,T] \to [0,1]$ be the function such that $h_s = 1$ on $[0,t]$, $h_s = 0$ on $[s,T]$, and h_s be affine in $[t,s]$. We obtain

$$- (\!(u_t, \varphi_t)\!)_{(0,t)} - (\!(u_t, \varphi_t h_s)\!)_{(t,s)} + \frac{1}{s-t}(\!(u_t, \varphi)\!)_{(t,s)} + (\!(\nabla u_t, \nabla(\varphi h_s))\!) + (\!(\nabla u, \nabla(\varphi h_s))\!)$$
$$+ \iint_{\overline{Q}} \varphi h_s \,\mathrm{d}\mathcal{T} - \lambda(\!(u, \varphi h_s)\!) = (u_1, \varphi(0)) + (\!(g, \varphi h_s)\!).$$

Letting $s \searrow t$, we see that the third term tends to $(u_t(t^+), \varphi(t))$, while the other terms pass to the limit thanks to the dominated convergence theorem and the fact that $\varphi h_s \to \varphi\chi_{[0,t]}$ pointwise, so in particular $\mathcal{T}$-almost everywhere. We then obtain

$$- (\!(u_t, \varphi_t)\!)_{(0,t)} + (u_t(t^+), \varphi(t)) + (\!(\nabla u_t, \nabla\varphi)\!)_{(0,t)} + (\!(\nabla u, \nabla\varphi)\!)_{(0,t)}$$
$$+ \iint_{\overline{Q}} \varphi \,\mathrm{d}(\mathcal{T}\chi_{[0,t]}) - \lambda(\!(u, \varphi)\!)_{(0,t)} = (u_1, \varphi(0)) + (\!(g, \varphi)\!)_{(0,t)}.$$

Comparing with (2.31), we deduce that $\xi_{(t)}$ is represented by the restriction of $\mathcal{T}$ to the closed set $\overline{Q_t}$ whenever the pointwise value $u_t(t)$ coincides with $u_t(t^+)$, which happens in fact in the complementary of a countable set of times. In other words, in that case we have $\mathcal{T}\llcorner_{\overline{Q_t}} = \mathcal{T}_{(t)}$.

Remark 3.7. Relation (2.41) implies in particular that, at least when $g \equiv 0$, the energy functional coincides almost everywhere with a nonincreasing function. In a sense this fact provides an additional criterion for selecting which are the "admissible" jumps of u_t (cf. Remark 3.3). Namely, jumps may occur only in such a way that they do not increase the total energy of the system. For $g \neq 0$ similar considerations hold, up to the fact that g acts somehow as an additional energy source.

Remark 3.8. It is maybe also worth stressing that Theorem 2.5 states the existence of *at least one* weak solution satisfying the properties detailed above. Due to nonuniqueness, there may well exist "spurious" solutions having worse properties. For example they may be constructed in such a way that the time derivative u_t admits somehow "nonphysical" jumps. However our procedure shows that *every* weak solution that is a limit point of our natural regularization scheme is "physical" (for example, in view of (2.41), energy-increasing jumps cannot occur).

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