An epiperimetric inequality for the thin obstacle problem

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Abstract. We prove an epiperimetric inequality for the thin obstacle problem, thus extending the pioneering results by Weiss on the classical obstacle problem (*Invent. Math.*, 138 (1999), no. 1, 23–50). This inequality provides the means to study the rate of converge of the rescaled solutions to their limits, as well as the regularity properties of the free boundary.

Keywords: Epiperimetric inequality, thin obstacle problem, regularity of the free boundary.

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1. Introduction

In this paper we develop a homogeneity improvement approach to the thin obstacle problem following the pioneering work by Weiss [20] for the classical obstacle problem. For the sake of simplicity we have restricted ourselves to the simplest case of the *Signorini problem* consisting in minimizing the Dirichlet energy among functions with positive traces on a given hyperplane. Loosely speaking we show that around suitable free boundary points the solutions quantitatively improve the degree of their homogeneity. This fact implies a number of consequences for the study of the regularity property of the free boundary itself as recalled in what follows.

In order to explain the main results of the paper we recall some of the basic known facts on the thin obstacle problem that are most relevant for our purposes. We consider the minimizers of the Dirichlet energy

$$\mathscr{E}(u) := \int_{B_1^+} |\nabla u|^2 dx$$

in the class of admissible functions

$$\mathscr{A}_w := \left\{ u \in H^1(B_1^+) : \, u \geq 0 \;\text{ on } B_1', \, u = w \;\text{ on } (\partial B_1)^+ \right\}, \tag{1.1}$$

where for any subset $A \subseteq \mathbb{R}^n$ we shall indicate by A^+ the set $A \cap \{x \in \mathbb{R}^n : x_n > 0\}$, $B_1' := \partial B_1^+ \cap \{x_n = 0\}$. The function $w \in H^1(B_1^+)$ above prescribes the boundary conditions on $(\partial B_1)^+$ and satisfies the obvious compatibility condition $w \geq 0$ on B_1' (in the usual sense of traces). For the sake of convenience in what follows we shall automatically extend every function in $\mathscr{A}_w$ by even symmetry. For $u \in \mathrm{argmin}_{\mathscr{A}_w} \mathscr{E}$ we denote by $\Lambda(u)$ its coincidence set, i.e. the set where the solution touches the obstacle

$$\Lambda(u) := \left\{ (\hat{x}, 0) \in B_1' \, : u(\hat{x}, 0) = 0 \right\},$$

and by $\Gamma(u)$ the free boundary, namely the topological boundary of $\Lambda(u)$ in the relative topology of B_1'.

Points in the free boundary of u can be classified according to their frequencies. Indeed, Athanasopoulos, Caffarelli and Salsa have established in [3, Lemma 1] that in every free boundary point x_0 Almgren's frequency function

$$(0,1-|x_0|)\ni r\mapsto N^{x_0}(r,u):=\frac{r\int_{B_r(x_0)}|\nabla u|^2\,dx}{\int_{\partial B_r(x_0)}u^2\,d\mathcal{H}^{n-1}}$$

is nondecreasing and has a finite limit as $r\downarrow 0$ satisfying $N^{x_0}(0^+,u)\in[3/2,\infty)$. Clearly, $N^{x_0}(r,u)$ is well-defined if $u|_{\partial B_r(x_0)}\not\equiv 0$, otherwise one can prove that actually $u\equiv 0$ in $B_r(x_0)$.

Following the original works by Weiss [19, 20], Garofalo and Petrosyan [10] have then introduced a family of monotonicity formulas exploiting a parametrized family of *boundary adjusted energies à la* Weiss: for $x_0\in\Gamma(u)$, $\lambda>0$ and $r\in(0,1-|x_0|)$

$$W_\lambda^{x_0}(r,u):=\frac{1}{r^{n+1}}\int_{B_r(x_0)}|\nabla u|^2\,dx-\frac{\lambda}{r^{n+2}}\int_{\partial B_r(x_0)}u^2\,d\mathcal{H}^{n-1}.$$

[10, Theorem 1.4.1] shows that the boundary adjusted energy $W_\lambda^{x_0}$ corresponding to $\lambda=N^{x_0}(0^+,u)$ is monotone non-decreasing. More precisely,

$$\frac{d}{dr}W_\lambda^{x_0}(r,u)=\frac{2}{r^{n+2\lambda}}\int_{\partial B_r(x_0)}(\nabla u\cdot x-\lambda u)^2\,d\mathcal{H}^{n-1}. \tag{1.2}$$

Note that the right-hand side of (1.2) measures the distance of u from a λ-homogeneous function, and essentially explains why suitable rescalings of u converge to homogeneous functions.

In this paper we show that, analogously to the case of the classical obstacle problem as discovered by Weiss [20], there are classes of points x_0 of the free boundary of u where the monotonicity of $W_\lambda^{x_0}$ can be explicitly quantified, meaning that there exist constants $\gamma, r_0, C>0$ such that

$$W_\lambda^{x_0}(r,u)\leq C\,r^\gamma\quad\forall\, r\in(0,r_0), \tag{1.3}$$

thus leading to the above mentioned homogeneity improvement of the solutions. Rather than explaining the important consequences of (1.3) (for which we give only a small essay in § 4), we focus in this paper on the way (1.3) is proven, i.e. by means of what Weiss called *epiperimetric inequality* in homage to Reifenberg's famous result [15] on minimal surfaces. In order to give an idea of the topic, let us discuss here the case of lowest frequency $\lambda=3/2$: roughly speaking, in this case the epiperimetric inequality asserts that, for dimensional constants κ, $\delta>0$ if $c\in H^1(B_1)$ is a $3/2$-homogeneous function with positive trace on B_1' that is δ-close to the cone of $3/2$-homogeneous global solutions, then there exists a function v with the same boundary values of c such that

$$W^0_{3/2}(1,v)\leq(1-\kappa)\,W^0_{3/2}(1,c). \tag{1.4}$$

The derivation of (1.3) from the epiperimetric inequality (1.4) is then done via simple algebraic relations on the boundary adjusted energy $W^0_{3/2}$, linking its derivative with the energy of the $3/2$-homogeneous extension of the boundary values of the solution. Comparing the energy of the latter with that of the solution itself leads to a differential inequality finally implying (1.3) (cp. § 4 for the details).

The main focus of the present note is however the method of proving (1.4). There are indeed only few examples of problems in geometric analysis where such kind of inequality has been established, with a number of far-reaching applications and consequences. The first instance is the remarkable work by Reifenberg [15] on minimal surfaces, then successfully extended in various directions: by Taylor [18] for what concerns soap-films and soap-bubbles minimal surfaces, by White [21] for tangent cones to two-dimensional area minimizing integral currents, by Chang [5] for the analysis of branch points in two-dimensional area minimizing integral currents, and by De Lellis and Spadaro [7] for two-dimensional multiple valued functions minimizing the generalized Dirichlet energy. In all of these instances the proof of the epiperimetric inequality is constructive, i.e. it is performed via an explicit computation of the energy of a suitable comparison solution, most of the time allowing to give an explicit bound on the constant κ.

On the other hand, the proof given by Weiss for the classical obstacle problem [20] is indirect, it exploits an infinite-dimensional version of a simple stability argument: namely, if $\phi \in C^2(\mathbb{R}^n)$ satisfies $\nabla\phi(y_0) = 0$ and $D^2\phi(y_0)$ is positive definite, then $y_0 \in \mathbb{R}^n$ is an attractive point for the dynamical system $\dot{y} = -\nabla\phi(y)$. In regard to this, it is worth mentioning that the infinite-dimensional extension of the quoted stability argument is in general subtle. For what concerns the present paper, the energy involved in the obstacle problem is not regular (because of the constraint given by the obstacle), and in addition, its second variation is not positive definite since there are entire directions where the functional is constant. This point of view shares some similarities with the approach by Allard and Almgren [1] and by Simon [16] for the analysis of the asymptotic of minimal cones with isolated singularities.

Our proof of the epiperimetric inequality for the thin obstacle problem is inspired by the fundamental paper by Weiss [20]. For instance, following Weiss [20] the method of proof is a contradiction argument. However, rather than faithfully reproducing the whole proof in [20], we underline two variational principles at the heart of it, that are more likely to be generalized to other contexts. In the contradiction argument we note that the failure of the epiperimetric inequality leads to a quasi-minimality condition for a sequence of auxiliary functionals related to the second variation of the original energy. The goal is then to understand the asymptotic behavior of such new energies. Indeed, the minimizers of their Γ-limits characterize the directions along which the epiperimetric inequality may fail. To exclude this occurrence, another variational argument leads to an orthogonality condition between the minimizers of the mentioned Γ-limits and a suitable tangent cone to the spaces of blowups, thus giving a contradiction.

We think that this scheme based on two competing variational principles can be applied in many other problems. For this reason, in order to make it as transparent as possible, we have detailed the proof of our main results in several steps, hoping that this effort could be useful for the reader.

We are able to prove the epiperimetric inequality for free boundary points in the following two classes:

(1) for points with lowest frequency $3/2$ (cp. Theorem 3.1);
(2) for *isolated* points of the free boundary with frequency $2m$, $m \in \mathbb{N} \setminus \{0\}$ (cp. Theorem 3.2).

The condition in (2) for the free boundary points being isolated can equivalently be rephrased in terms of the properties of the blowup functions. As explained in § 2, they correspond to the points where the blowups are everywhere positive except at the origin. We remark that it is still an open problem (except for dimension $n = 2$) to classify all the possible limiting values of the frequency: apart from the quoted lower bound $N^{x_0}(0^+, u) \geq 3/2$ for every $x_0 \in \Gamma(u)$, there are examples of free boundary points with limiting frequency equal to $(2m+1)/2$ and $2m$ for every $m \in \mathbb{N} \setminus \{0\}$ (in dimension $n = 2$ these are the unique possible values).

For even frequencies, the fact that our Theorem 3.2 covers only the case of isolated points is a consequence of the indirect approach we use. Indeed, the same restriction holds for the classical obstacle problem, as already explained by Weiss [20], and it is related to the topology considered in the asymptotic analysis of Theorems 3.1 and 3.2. Indeed these results are based on the Γ-convergence with respect to the weak H^1 topology of a family of functionals under a unilateral obstacle conditions, which does not pass to the limit in the case the obstacle constraint is imposed on a set of dimension less than or equal to $n-2$. This restriction on the class of points where Theorem 3.2 applies can be removed, if one could find a direct argument to the epiperimetric inequality for the obstacle problem by exhibiting comparisons, as in the case of minimal surfaces.

After the completion of this paper, we discovered that our results have a big overlap with those contained in a very recent preprint by Garofalo, Petrosyan and Smit Vega Garcia [11]. In this paper the Authors investigate the regularity of the points with least frequency of the free boundary of the Signorini problem with variable Lipschitz coefficients as a consequence of the epiperimetric inequality, as previously done by the Authors of the present note and Gelli [9] for the classical obstacle problem with Lipschitz regular coefficients, using the fundamental energetic approach pioneered by Weiss in [20]. In particular, in [11] the epiperimetric inequality in case (1) above is proved and the results on the regularity of the free boundary cover the ones in § 4. Despite this, we think that the remaining cases of the epiperimetric inequality in (2) are interesting, and furthermore we believe that the variational approach to the epiperimetric inequality we have developed can be generalized to other contexts and therefore that it is worth being shared with the community.

The paper is organized in the following way. In § 2 we give the necessary preliminaries in order to state the epiperimetric inequality. Then in § 3 we provide the proof of the main results. We state two versions of the epiperimetric inequality covering the case (1) and (2) above separately, in Theorem 3.1 and Theorem 3.2 respectively. The proofs of these two results are divided into a sequence of steps, that as explained above are meant to give a clear overview on the structure of the proof. Since the two proofs are very much similar, we provide all the details for the first case and for what concerns the second one, where actually several simplifications occur, we only point out the main differences. Finally in § 4 we prove the regularity of the free boundary near points of least frequency as a consequence of the epiperimetric inequality.

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2. Notation and preliminaries

The open ball in the Euclidean space $\mathbb{R}^n$ centered in x and with radius r is denoted by $B_r(x)$, and by B_r if the center is the origin. We recall that, given any subset $A \subseteq \mathbb{R}^n$, we shall indicate by A^+ the set $A \cap \{x \in \mathbb{R}^n : x_n > 0\}$. Moreover we set

$$B_1' := \partial B_1^+ \cap \{x_n = 0\} \quad \text{and} \quad B_1'^{,-} := B_1' \cap \{x_{n-1} \leq 0\}.$$

The Euclidean scalar product in $\mathbb{R}^n$ among the vectors ξ_1 and ξ_2 shall be denoted by $\xi_1 \cdot \xi_2$. The scalar product in the Sobolev space $H^1(B_1)$ shall be denoted by $\langle \cdot, \cdot \rangle$, and the corresponding norm by $\| \cdot \|_{H^1}$.

We introduce a parametrized family of *boundary adjusted energies à la* Weiss [20]: namely, given $\lambda > 0$ for every $u \in H^1(B_1)$, we set

$$\mathscr{G}_\lambda(u) := \int_{B_1} |\nabla u|^2 dx - \lambda \int_{\partial B_1} u^2 d\mathcal{H}^{n-1}, \tag{2.1}$$

and note that for all u_1, $u_2 \in \mathscr{A}_w$ in the admissible class 1.1 it holds

$$\mathscr{G}_\lambda(u_1) - \mathscr{G}_\lambda(u_2) = 2\big(\mathscr{E}(u_1) - \mathscr{E}(u_2)\big).$$

Throughout the whole paper we shall be interested only in range of values $\lambda \in \{3/2\} \cup \{2m\}_{m\in\mathbb{N}}$.

2.1. **$3/2$-homogeneous solutions.** Next, let $x = (\hat{x}, x_n) \in \mathbb{R}^n$, and adopt the notation $\mathbb{S}^{n-2} = \mathbb{S}^{n-1} \cap \{x_n = 0\}$. Define for $e \in \mathbb{S}^{n-2}$

$$h_e(x) := \Big(2\,(\hat{x}\cdot e) - \sqrt{(\hat{x}\cdot e)^2 + x_n^2}\Big)\sqrt{\sqrt{(\hat{x}\cdot e)^2 + x_n^2} + \hat{x}\cdot e}\,.$$

It is easy to check that

$$h_e(x) = \sqrt{2}\,\mathrm{Re}\big[(\hat{x}\cdot e + i\,x_n)^{3/2}\big],$$

where one chooses the determination of the complex root satisfying $h_e \geq 0$ on $\{x_n = 0\}$. Moreover the following properties of h_e hold true:

(1) h_e is even w.r.t. $\{x_n = 0\}$, i.e.

$$h_e(\hat{x}, -x_n) = h_e(\hat{x}, x_n) \quad \forall\,(\hat{x}, x_n) \in \mathbb{R}^n;$$

(2) $h_e \geq 0$ on $\{x_n = 0\}$ and $h_e = 0$ on $\{x_n = 0,\ x\cdot e \leq 0\}$;
(3) h_e is harmonic on $B_1^+ \cup B_1^-$;
(4) $h_e|_{\{x_n \geq 0\}}$ is $C^{1,1/2}$, i.e. there exists $H_e \in C^{1,1/2}(\mathbb{R}^n)$ such that

$$H_e = h_e \quad \text{in } \{x_n \geq 0\};$$

(5) for $x_n > 0$ we have

$$\begin{aligned}\frac{\partial h_e}{\partial x_n}(\hat{x}, x_n) = &- \frac{x_n}{\sqrt{(\hat{x}\cdot e)^2 + x_n^2}}\sqrt{\sqrt{(\hat{x}\cdot e)^2 + x_n^2} + \hat{x}\cdot \nu} \\ &+ \frac{2\,(\hat{x}\cdot e) - \sqrt{(\hat{x}\cdot e)^2 + x_n^2}}{2\sqrt{\sqrt{(\hat{x}\cdot e)^2 + x_n^2} + \hat{x}\cdot e}}\,\frac{x_n}{\sqrt{(\hat{x}\cdot e)^2 + x_n^2}}.\end{aligned} \tag{2.2}$$

In particular,

$$\frac{\partial h_e}{\partial x_n}(\hat{x}, 0^+) := \lim_{x_n \downarrow 0} \frac{\partial h_e}{\partial x_n}(\hat{x}, x_n) = \begin{cases} -\frac{3}{\sqrt{2}}|\hat{x}\cdot e|^{1/2} & \text{on } \{x_n = 0,\ x\cdot e < 0\} \\ 0 & \text{on } \{x_n = 0,\ x \cdot e \geq 0\}, \end{cases} \tag{2.3}$$

and by (2)

$$h_e(\hat{x}, 0)\,\frac{\partial h_e}{\partial x_n}(\hat{x}, 0^+) = 0 \quad \text{ on } \{x_n = 0\}. \tag{2.4}$$

We introduce next the set of blow-ups at the *regular points* of the free boundary:

$$\mathscr{H}_{3/2} := \big\{\lambda\, h_e : \, e \in \mathbb{S}^{n-2},\ \lambda \in [0, \infty)\big\} \subset H^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Note that $\mathscr{H}_{3/2}$ is a cone in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$, the restrictions

$$\mathscr{H}_{3/2}|_{B_1} := \big\{f|_{B_1} \, : f \in \mathscr{H}_{3/2}\big\} \subset H^1(B_1)$$

is a closed set and $\mathscr{H}_{3/2} \setminus \{0\}$ is parametrized by an $(n-1)$-dimensional manifold by the map

$$\mathbb{S}^{n-2} \times (0, +\infty) \ni (e, \lambda) \quad \stackrel{\Phi}{\longmapsto} \quad \lambda\, h_e \in \mathscr{H}_{3/2} \setminus \{0\}.$$

Note that Φ is an embedding, so that we can then introduce the tangent space to $\mathscr{H}_{3/2}$ at any point $\lambda_0 h_{e_0}$ as

$$T_{\lambda_0\, h_{e_0}} \mathscr{H}_{3/2} := \big\{d_{(\lambda_0, e_0)}\Phi(\xi, \alpha) : \ \xi \cdot e_n = \xi \cdot e_0 = 0,\ \alpha \in \mathbb{R}\big\}\,, \tag{2.5}$$

and notice that

$$T_{\lambda_0\, h_{e_0}} \mathscr{H}_{3/2} = \{\alpha\, h_{e_0} + v_{e_0,\xi} : \ \xi \cdot e_n = \xi \cdot e_0 = 0,\ \alpha \in \mathbb{R}\}\,, \tag{2.6}$$

where we have set

$$v_{e,\xi}(x) := (\hat{x}\cdot \xi)\, \sqrt{\sqrt{(\hat{x}\cdot e)^2 + x_n^2} + \hat{x}\cdot e} \tag{2.7}$$

Note moreover that

$$v_{e,\xi}(x) = \sqrt{2}\,(\hat{x}\cdot\xi)\, \mathrm{Re}\big[(\hat{x}\cdot e + i\, x_n)^{1/2}\big],$$

where the determination of the complex square root is chosen in such a way that $v_{e,\xi} \geq 0$ in $\{x_n = 0\}$.

Let us now highlight some additional properties enjoyed by functions $\psi \in \mathscr{H}_{3/2}$. For any given $\varphi \in H^1(B_1)$, a simple integration by parts yields

$$\begin{aligned}\int_{B_1^+} \nabla\psi\cdot\nabla\varphi\, dx &= \int_{B_1^+} \mathrm{div}(\varphi\nabla\psi)dx \\ &= \int_{(\partial B_1)^+} \varphi\frac{\partial\psi}{\partial\nu} d\mathcal{H}^{n-1} - \int_{B_1'} \varphi\, \frac{\partial \psi}{\partial x_n}(\hat{x}, 0^+) d\mathcal{H}^{n-1} \\ &= \frac{3}{2}\int_{(\partial B_1)^+} \varphi\,\psi\, d\mathcal{H}^{n-1} - \int_{B_1'} \varphi\, \frac{\partial \psi}{\partial x_n}(\hat{x}, 0^+) d\mathcal{H}^{n-1},\end{aligned}$$

where $\nu = \frac{x}{|x|}$ and we used that ψ is $3/2$-homogeneous and $\Delta\psi = 0$ in B_1^+. Therefore, by the even symmetry of ψ we conclude

$$\int_{B_1} \nabla\psi\cdot\nabla\varphi\, dx = \frac{3}{2}\int_{\partial B_1} \varphi\,\psi\, d\mathcal{H}^{n-1} - 2\int_{B_1'} \varphi\, \frac{\partial \psi}{\partial x_n}(\hat{x}, 0^+) d\mathcal{H}^{n-1}. \tag{2.8}$$

In particular, (2.8) yields that the first variation of $\mathscr{G}_{3/2}$ at $\psi \in \mathscr{H}_{3/2}$ in the direction $\varphi \in H^1(B_1)$, formally defined as

$$\delta\mathscr{G}_{3/2}(\psi)[\varphi] := 2\int_{B_1} \nabla\psi \cdot \nabla\varphi \, dx - 3\int_{\partial B_1} \psi\,\varphi \, d\mathcal{H}^{n-1},$$

satisfies

$$\delta\mathscr{G}_{3/2}(\psi)[\varphi] = -4\int_{B_1'} \varphi \, \frac{\partial\psi}{\partial x_n}(\hat{x}, 0^+) d\mathcal{H}^{n-1}. \tag{2.9}$$

Furthermore, by taking into account (2.4) and (2.8) applied to $\varphi = \psi$, we get

$$\mathscr{G}_{3/2}(\psi) = 0 \qquad \text{for all } \psi \in \mathscr{H}_{3/2}. \tag{2.10}$$

2.2. **$2m$-homogeneous solutions.** We introduce the even homogeneous solutions representing the lowest stratum in the singular part of the free-boundary according to [10, Theorem 1.3.8].

More precisely, given $m \in \mathbb{N} \setminus \{0\}$ consider the closed convex cone of $H^1(B_1)$

$$\mathscr{H}_{2m} := \Big\{\psi : \psi \text{ } 2m\text{-homogeneous}, \ \triangle\psi = 0 \,\text{in}\, \mathbb{R}^n, \ \psi = \psi(\hat{x}, |x_n|), \ \psi(\hat{x}, 0) \geq 0\Big\}.$$

Note that all the functions in $\mathscr{H}_{2m}$ are actually harmonic polynomials on $\mathbb{R}^n$ satisfying

$$\frac{\partial\psi}{\partial x_n} = 0 \ \text{ on } B_1'. \tag{2.11}$$

Given $\psi \in \mathscr{H}_{2m}$ and $\varphi \in H^1(B_1)$, an integration by parts leads to

$$\int_{B_1} \nabla\psi \cdot \nabla\varphi \, dx = 2m\int_{\partial B_1} \varphi\,\psi \, d\mathcal{H}^{n-1}. \tag{2.12}$$

Therefore, setting

$$\delta\mathscr{G}_{2m}(\psi)[\varphi] := 2\int_{B_1} \nabla\psi \cdot \nabla\varphi \, dx - 4m\int_{\partial B_1} \varphi\,\psi \, d\mathcal{H}^{n-1},$$

it is immediate to check that

$$\delta\mathscr{G}_{2m}(\psi)[\varphi] = 0 \qquad \text{for all } \psi \in \mathscr{H}_{2m} \text{ and all } \varphi \in H^1(B_1), \tag{2.13}$$

and

$$\mathscr{G}_{2m}(\psi) = 0 \qquad \text{for all } \psi \in \mathscr{H}_{2m}. \tag{2.14}$$

We can group the functions in $\mathscr{H}_{2m}$ according to the dimension of their invariant subspace as follows: for $\psi \in \mathscr{H}_{2m}$ consider the subspace

$$\Pi_\psi = \{\xi \in \mathbb{R}^{n-1} : \psi(\hat{x}, 0) = \psi(\hat{x} + \xi, 0) \ \ \forall \hat{x} \in \mathbb{R}^{n-1}\},$$

and, for $d \in \{0, \ldots, n-2\}$ define

$$\mathscr{H}_{2m}^{(d)} := \{\psi \in \mathscr{H}_{2m} : \dim \Pi_\psi = d\}\,.$$

In what follows we shall be dealing only with the lowest stratum $\mathscr{H}_{2m}^{(0)}$ for which we provide the ensuing alternative characterization.

2.3. Proposition. *$\psi \in \mathscr{H}_{2m}^{(0)}$ if and only if $\psi(\hat{x}, 0) > 0$ for every $\hat{x} \neq 0$.*

Proof. First note that by the $2m$-homogeneity the condition $\psi|_{B_1'} \geq 0$ implies that actually $\psi(\cdot, 0)$ is even w.r.t. x_i, for $i \in \{1, \ldots, n-1\}$.

Suppose then by contradiction that $\psi(\hat{y}, 0) = 0$ for some $\hat{y} \neq 0$, and without loss of generality assume that $\hat{y}/|\hat{y}| = e_1$. Hence, $\psi(x_1, 0, \ldots, 0) = 0$ for all $x_1 \in \mathbb{R}$, and we may find a $2(m-\ell)$-homogeneous polynomial q, $1 \leq \ell < m$, such that $\psi(x) = x_1^{2\ell} q(x)$ and $q(0, x_2, \ldots, x_n)$ is not identically zero. Computing the Laplacian of ψ we get

$$0 = \triangle\psi = 2\ell(2\ell-1)\, x_1^{2(\ell-1)} q + 4\ell x_1^{2\ell-1} \frac{\partial q}{\partial x_1} + x_1^{2\ell} \triangle q,$$

and thus for all $x_1 \neq 0$ we have

$$2\ell(2\ell-1)\, q + 4\ell x_1 \frac{\partial q}{\partial x_1} + x_1^2 \triangle q = 0.$$

In turn, by letting $x_1 \to 0$ we conclude that $q(0, x_2, \ldots, x_n) \equiv 0$, a contradiction. □

In view of the latter result, it is easy to check that $\mathscr{H}_{2m}^{(0)}$ is an open subset (with respect to the H^1_{loc} topology) of the linear space

$$\widehat{\mathscr{H}_{2m}} := \Big\{ p : \, p \ 2m\text{-homogeneous}, \ \triangle p = 0 \text{ in } \ \mathbb{R}^n, \ p = p(\hat{x}, |x_n|) \Big\}. \tag{2.15}$$

In particular, $\widehat{\mathscr{H}_{2m}}$ is a supporting tangent plane at every $\psi \in \mathscr{H}_{2m}^{(0)}$ to the convex cone $\mathscr{H}_{2m}$ in the following sense: for every sequence of functions $\{\psi_l\}_{l \in \mathbb{N}} \in \mathscr{H}_{2m}$ such that $\psi_l \to \psi$ in $H^1(B_1)$, there exists a subsequence $\{l'\} \subset \mathbb{N}$ such that the limit $\zeta := \lim_{l' \to +\infty} \frac{\psi_{l'} - \psi}{\|\psi_{l'} - \psi\|_{H^1(B_1)}}$ exists in $H^1(B_1)$ and ζ coincides with the restriction to B_1 of a function in $\widehat{\mathscr{H}_{2m}}$. Note that, in particular, $\widehat{\mathscr{H}_{2m}} \supset \mathscr{H}_{2m}$.

3. The epiperimetric inequality

In this section we establish an epiperimetric inequality *à la* Weiss for the thin obstacle problem. In the rest of the section we agree that a function $c \in H^1(B_1)$ is λ-homogeneous, $\lambda \in \{3/2\} \cup \{2m\}_{m \in \mathbb{N} \setminus \{0\}}$, if there exists $f \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ which is λ-homogeneous and satisfying $c|_{B_1} = f$. Moreover, to avoid cumbersome notation, we shall use, without any risk of ambiguity, the symbol $\mathscr{H}_\lambda$ also to denote the restrictions of the blowup maps to the unit ball (which in the previous section we denoted by $\mathscr{H}_\lambda|_{B_1}$). Moreover, given $\mathscr{K} \subset H^1(B_1)$ closed set, we define

$$\mathrm{dist}_{H^1}(c, \mathscr{K}) := \min \left\{ \|c - \varphi\|_{H^1(B_1)}, \ \varphi \in \mathscr{K} \right\}.$$

We are now ready to state the main results of the paper.

3.1. Theorem. *There exist dimensional constants $\kappa \in (0,1)$ and $\delta > 0$ such that if $c \in H^1(B_1)$ is a $3/2$-homogeneous function with $c \geq 0$ on B_1' and*

$$\mathrm{dist}_{H^1}\big(c, \mathscr{H}_{3/2}\big) \leq \delta, \tag{3.1}$$

then

$$\inf_{v \in \mathscr{A}_c} \mathscr{G}_{3/2}(v) \leq (1-\kappa) \mathscr{G}_{3/2}(c). \tag{3.2}$$

An analogous result for the lowest stratum of the singular set holds.

3.2. Theorem. *There exist dimensional constants $\kappa \in (0,1)$ and $\delta > 0$ such that if $c \in H^1(B_1)$ is a $2m$-homogeneous function with $c \geq 0$ on B_1' and*

$$\mathrm{dist}_{H^1}\big(c, \mathscr{H}_{2m}\big) \leq \delta \text{ and } P(c) \in \mathscr{H}_{2m}^{(0)} \tag{3.3}$$

where $P : H^1(B_1) \to \mathscr{H}_{2m}$ is the projection operator, then

$$\inf_{v \in \mathscr{A}_c} \mathscr{G}_{2m}(v) \leq (1-\kappa)\mathscr{G}_{2m}(c). \tag{3.4}$$

3.3. **The lowest frequency.** Here we prove the epiperimetric inequality for those points of the free boundary with frequency $3/2$. To simplify the notation in the proof below we shall denote $\mathscr{G}_{3/2}$ only by $\mathscr{G}$.

PROOF OF THEOREM 3.1. We argue by contradiction. Therefore we start off assuming the existence of numbers κ_j, $\delta_j \downarrow 0$ and of functions $c_j \in H^1(B_1)$ that are $3/2$-homogeneous, $c_j \geq 0$ on B_1' and such that

$$\mathrm{dist}_{H^1}\big(c_j, \mathscr{H}_{3/2}\big) = \delta_j, \tag{3.5}$$

and

$$(1-\kappa_j)\mathscr{G}(c_j) \leq \inf_{v \in \mathscr{A}_{c_j}} \mathscr{G}(v). \tag{3.6}$$

In particular, setting $h := h_{e_{n-1}}$, up to a change of coordinates depending on j, we may assume that there exists $\lambda_j \geq 0$ such that

$$\psi_j := \lambda_j \, h$$

is a point of minimum distance of c_j from $\mathscr{H}_{3/2}$, i.e.

$$\|c_j - \psi_j\|_{H^1} = \mathrm{dist}_{H^1}\big(c_j, \mathscr{H}_{3/2}\big) = \delta_j \qquad \text{for all } j \in \mathbb{N}. \tag{3.7}$$

We divide the rest of the proof in some intermediate steps.

Step 1: Introduction of a family of auxiliary functionals. We rewrite inequality (3.6) conveniently and interpret it as an almost minimality condition for a sequence of new functionals.

For fixed j, let $v \in \mathscr{A}_{c_j}$ and use (2.9) (applied twice to ψ_j with test functions $\varphi = c_j - \psi_j$ and $\varphi = v - \psi_j$) and (2.10), in order to rewrite (3.6) in the following form

$$(1-\kappa_j)\Big(\mathscr{G}(c_j) - \mathscr{G}(\psi_j) - \delta\mathscr{G}(\psi_j)[c_j - \psi_j] - 4\int_{B_1'} (c_j - \psi_j)\, \frac{\partial \psi_j}{\partial x_n}(\hat{x}, 0^+)\, d\mathcal{H}^{n-1}\Big)$$
$$\leq \mathscr{G}(v) - \mathscr{G}(\psi_j) - \delta\mathscr{G}(\psi_j)[v - \psi_j] - 4\int_{B_1'} (v - \psi_j)\, \frac{\partial \psi_j}{\partial x_n}(\hat{x}, 0^+)\, d\mathcal{H}^{n-1}.$$

Simple algebraic manipulations then lead to

$$(1-\kappa_j)\Big(\mathscr{G}(c_j - \psi_j) - 4\int_{B_1'} (c_j - \psi_j)\frac{\partial \psi_j}{\partial x_n}(\hat{x}, 0^+)\, d\mathcal{H}^{n-1}\Big)$$
$$\leq \mathscr{G}(v - \psi_j) - 4\int_{B_1'} (v - \psi_j)\frac{\partial \psi_j}{\partial x_n}(\hat{x}, 0^+)\, d\mathcal{H}^{n-1}, \tag{3.8}$$

for all $v \in \mathscr{A}_{c_j}$.

Next we introduce the following notation. We set

$$z_j := \frac{c_j - \psi_j}{\delta_j}, \tag{3.9}$$

and, recalling that $\psi_j = \lambda_j h$, we set

$$\vartheta_j := \frac{\lambda_j}{\delta_j}$$

and

$$\mathscr{B}_j := \left\{ z \in z_j + H_0^1(B_1) : (z + \vartheta_j h)|_{B_1'} \geq 0 \right\}. \tag{3.10}$$

Then we define the functionals $\mathscr{G}_j : L^2(B_1) \to (-\infty, +\infty]$ given by

$$\mathscr{G}_j(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - \frac{3}{2} \int_{\partial B_1} z_j^2 \, d\mathcal{H}^{n-1} - 4\,\vartheta_j \int_{B_1'} z \, \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) \, d\mathcal{H}^{n-1} & \text{if } z \in \mathscr{B}_j, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.11}$$

Note that the second term in the formula does not depend on z but only on the boundary conditions $z_j|_{\partial B_1}$.

Therefore, (3.8) reduces to

$$(1 - \kappa_j)\mathscr{G}_j(z_j) \leq \mathscr{G}_j(z) \quad \text{for all } z \in \mathscr{B}_j. \tag{3.12}$$

Moreover, note that by (3.7) and (3.9)

$$\|z_j\|_{H^1(B_1)} = 1. \tag{3.13}$$

This implies that we can extract a subsequence (not relabeled) such that

(a) $(z_j)_{j \in \mathbb{N}}$ converges weakly in $H^1(B_1)$ to some z_∞;
(b) the corresponding traces $(z_j|_{B_1'})_{j \in \mathbb{N}}$ converge strongly in $L^2(B_1') \cup L^2(\partial B_1)$;
(c) $(\vartheta_j)_{j \in \mathbb{N}}$ has a limit $\vartheta \in [0, \infty]$.

Step 2: First properties of $(\mathscr{G}_j)_{j \in \mathbb{N}}$. We establish the equi-coercivity and some further properties of the family of the auxiliary functionals $(\mathscr{G}_j)_{j \in \mathbb{N}}$.

Notice that for all $w \in \mathscr{B}_j$, being $w|_{\partial B_1} = z_j|_{\partial B_1}$, it holds that

$$\begin{aligned} -\int_{B_1'} w \, \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) d\mathcal{H}^{n-1} &= \int_{B_1'} -(w + \vartheta_j \, h) \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) d\mathcal{H}^{n-1} \\ &\quad + \vartheta_j \int_{B_1'} h \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) d\mathcal{H}^{n-1} \geq 0, \end{aligned} \tag{3.14}$$

where we used (2.3), (2.4) and $(w + \vartheta_j \, h)|_{B_1'} \geq 0$. Therefore, we deduce from the very definition (3.11) that for all $w \in \mathscr{B}_j$

$$\int_{B_1} |\nabla w|^2 dx - \frac{3}{2} \int_{\partial B_1} z_j^2 \leq \mathscr{G}_j(w), \tag{3.15}$$

thus establishing the equi-coercivity of the sequence $(\mathscr{G}_j)_{j \in \mathbb{N}}$.

By taking into account (3.13), if $\vartheta \in [0, +\infty)$ then

$$\liminf_j \mathscr{G}_j(z_j) \geq 1 - \int_{B_1} z_\infty^2 - \frac{3}{2} \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^{n-1} - 4\vartheta \int_{B_1'} z_\infty \, \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) \, d\mathcal{H}^{n-1}. \tag{3.16}$$

Instead, if $\vartheta = +\infty$ then (3.13) and (3.15) yield

$$\liminf_j \mathscr{G}_j(z_j) \geq 1 - \int_{B_1} z_\infty^2 - \frac{3}{2} \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^{n-1}.$$

Hence in all instances, it is not restrictive (up to passing to a further subsequence which we do not relabel) to assume that $(\mathscr{G}_j(z_j))_{j\in\mathbb{N}}$ has a limit in $(-\infty,+\infty]$. Finally, note that

$$\lim_j \mathscr{G}_j(z_j) = +\infty \quad \Longleftrightarrow \quad \lim_j \vartheta_j \int_{B_1'} z_j \,\frac{\partial h}{\partial x_n}(\hat{x},0^+)\, d\mathcal{H}^{n-1} = -\infty. \tag{3.17}$$

Step 3: Asymptotic analysis of $(\mathscr{G}_j)_{j\in\mathbb{N}}$. Here we prove a Γ-convergence result for the family of energies $\mathscr{G}_j$.

More precisely, we distinguish three cases.

(1) If $\vartheta \in [0,+\infty)$, then

$$(z_\infty + \vartheta\, h)|_{B_1'} \geq 0,$$

and $\Gamma(L^2(B_1))\text{-}\lim_j \mathscr{G}_j = \mathscr{G}^{(1)}_\infty$, where

$$\mathscr{G}^{(1)}_\infty(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - \frac{3}{2}\int_{\partial B_1} z_\infty^2\, d\mathcal{H}^{n-1} - 4\vartheta \int_{B_1'} z\,\frac{\partial h}{\partial x_n}(\hat{x},0^+)\, d\mathcal{H}^{n-1} & \text{if } z \in \mathscr{B}^{(1)}_\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\mathscr{B}^{(1)}_\infty := \Big\{ z \in z_\infty + H^1_0(B_1) : (z+\vartheta\, h)|_{B_1'} \geq 0 \Big\}.$$

(2) If $\vartheta = +\infty$ and $\lim_j \mathscr{G}_j(z_j) < +\infty$, then

$$z_\infty|_{B_1',-} = 0$$

recalling that $B_1'^{,-} = B_1' \cap \{x_{n-1} \leq 0\}$, and $\Gamma(L^2(B_1))\text{-}\lim_j \mathscr{G}_j = \mathscr{G}^{(2)}_\infty$, where

$$\mathscr{G}^{(2)}_\infty(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - \frac{3}{2}\int_{\partial B_1} z_\infty^2\, d\mathcal{H}^{n-1} & \text{if } z \in \mathscr{B}^{(2)}_\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\mathscr{B}^{(2)}_\infty := \Big\{ z \in z_\infty + H^1_0(B_1) : z|_{B_1',-} = 0 \Big\}.$$

(3) if $\vartheta = +\infty$ and $\lim_j \mathscr{G}_j(z_j) = +\infty$, then $\Gamma(L^2(B_1))\text{-}\lim_j \mathscr{G}_j = \mathscr{G}^{(3)}_\infty$, where $\mathscr{G}^{(3)}_\infty \equiv +\infty$ on the whole $L^2(B_1)$.

Equality $\mathscr{G}^{(i)}_\infty = \Gamma(L^2(B_1))\text{-}\lim_j \mathscr{G}_j$ for $i = 1,2,3$ consists, by definition, in showing the following two assertions:

(a) for all $(w_j)_{j\in\mathbb{N}}$ and $w \in L^2(B_1)$ such that $w_j \to w$ in $L^2(B_1)$ we have

$$\liminf_j \mathscr{G}_j(w_j) \geq \mathscr{G}^{(i)}_\infty(w); \tag{3.18}$$

(b) for all $w \in L^2(B_1)$ there is $(w_j)_{j\in\mathbb{N}}$ such that $w_j \to w$ in $L^2(B_1)$ and

$$\limsup_j \mathscr{G}_j(w_j) \leq \mathscr{G}^{(i)}_\infty(w). \tag{3.19}$$

Proof of the Γ*-convergence in case (1).* For what concerns the lim inf inequality (3.18), we can assume without loss of generality (up to passing to a subsequence we do not relabel) that

$$\liminf_j \mathscr{G}_j(w_j) = \lim_j \mathscr{G}_j(w_j) < \infty.$$

In view of (3.15), there exists a subsequence (not relabel) such that $(w_j)_{j\in\mathbb{N}}$ converges to w weakly in $H^1(B_1)$ and thus the corresponding traces strongly in $L^2(\partial B_1)\cup L^2(B_1')$. This implies that $w+\vartheta h \geq 0$ on B_1' and, in particular, taking $w_j = z_j$ and $w = z_\infty$, we deduce that $z_\infty \in \mathscr{B}^{(1)}_\infty$. (3.18) is then a simple consequence of the lower semicontinuity of the Dirichlet energy under the weak convergence in H^1.

For what concerns the lim sup inequality (3.19), we start noticing that it is enough to consider the case $w \in \mathscr{B}^{(1)}_\infty$ with

$$\operatorname{supp}(w - z_\infty) \subseteq B_\rho \quad \text{for some } \rho \in (0,1). \tag{3.20}$$

Indeed, in order to deal with the general case, consider the functions

$$w_t(x) := w\big(x/t\big)\,\chi_{B_1}\big(x/t\big) + z_\infty\big(x/t\big)\,\chi_{B_t\setminus \overline{B}_1}\big(x/t\big) \quad \text{with } t>1.$$

Clearly, $w_t \in H^1(B_1)$, $\operatorname{supp}(w_t - z_\infty) \subseteq B_{1/t}$, and $w_t \to w$ in $H^1(B_1)$ for $t \downarrow 1$. Since the upper bound inequality (3.19) holds for each w_t, a diagonalization argument provides the conclusion.

Moreover, by a simple contradiction argument it also suffices to show the following: given w as in (3.20), for every sequence $j_k \uparrow +\infty$ there exist subsequences $j_{k_l} \uparrow +\infty$ and $w_l \to w$ in $L^2(B_1)$ such that

$$\limsup_l \mathscr{G}_{j_{k_l}}(w_l) \leq \mathscr{G}^{(i)}_\infty(w). \tag{3.21}$$

After these reductions, we first use (3.13) to find a subsequences (not relabeled) such that $(|\nabla z_{j_k}|^2 \mathcal{L}^n \llcorner B_1)_{h\in\mathbb{N}}$ converges weakly* in the sense of measures to some finite Radon measure μ. Fixed $r\in(\rho,1)$, let $R := \frac{1+r}{2}$ and let $\varphi \in C^1_c(B_1)$ be a cut-off function such that

$$\varphi|_{B_r} \equiv 1, \quad \varphi|_{B_1\setminus \overline{B}_R} \equiv 0 \quad \text{and} \quad \|\nabla\varphi\|_{L^\infty} \leq \frac{4}{1-r}.$$

Defining

$$w^r_k := \varphi\big(w + (\vartheta - \vartheta_{j_k})h\big) + (1-\varphi) z_{j_k},$$

we easily infer that $w^r_k \in \mathscr{B}_{j_k}$ since $w \in \mathscr{B}^{(1)}_\infty$, $z_{j_k} \in \mathscr{B}_{j_k}$ and

$$w^r_k + \vartheta_{j_k} h = \varphi(w+\vartheta h) + (1-\varphi)(z_{j_k} + \vartheta_{j_k} h).$$

Moreover, since $\vartheta_{j_k} \to \vartheta \in [0,+\infty)$ we get that $w^r_k \to \varphi\, w + (1-\varphi) z_\infty$ in $L^2(B_1)$. Simple calculations then lead to

$$\int_{B_1} |\nabla w^r_k|^2 dx \leq \int_{B_r} |\nabla w + (\vartheta - \vartheta_{j_k})\nabla h|^2 dx + \underbrace{\int_{B_R\setminus \overline{B}_r} |\nabla w^r_{j_k}|^2 dx}_{=:I_k} + \int_{B_1\setminus \overline{B}_R} |\nabla z_{j_k}|^2 dx \tag{3.22}$$

By taking into account that $r > \rho$, we estimate the term I_k above as follows

$$I_k \leq 2\int_{B_R\setminus \overline{B}_r} |\nabla w + (\vartheta - \vartheta_{j_k})\nabla h|^2 dx + 2\int_{B_R\setminus \overline{B}_r} |\nabla z_{j_k}|^2 dx + 2\int_{B_R\setminus \overline{B}_r} |\nabla\varphi|^2 |z_\infty - z_{j_k} + (\vartheta - \vartheta_{j_k})h|^2 dx. \tag{3.23}$$

Hence, provided $\mu(\partial B_r)=0$, from (3.22) and (3.23) we deduce that

$$\limsup_k \int_{B_1} |\nabla w_k^r|^2 dx \le \int_{B_R} |\nabla w|^2 dx + \int_{B_R\setminus \overline{B}_r} |\nabla w|^2 dx + 3\,\mu(B_1\setminus \overline{B}_r). \tag{3.24}$$

In particular, we may apply the construction above to a sequence $r_l \uparrow 1$ and $R_l := \frac{1+r_l}{2}$, such that $\mu(\partial B_{r_l})=0$ for all $l\in\mathbb{N}$. A diagonalization argument provides a subsequence $j_{k_l}\uparrow\infty$ such that $w_l := w_{k_l}^{r_l} \to w$ in $L^2(B_1)$ and

$$\limsup_l \int_{B_1} |\nabla w_l|^2 dx \le \int_{B_1} |\nabla w|^2 dx,$$

and (3.21) follows at once by considering the strong convergence of traces of $z_{j_{k_l}}$ in $L^2(B_1')$.

Proof of the Γ*-convergence in case (2).* For what concerns the lim inf inequality (3.18), we assume without loss of generality that

$$\liminf_j \mathscr{G}_j(w_j) = \lim_j \mathscr{G}_j(w_j) < \infty.$$

Since $w_j \in \mathscr{B}_j$, the stated convergences yield that $w \ge 0$ on $B_1^{\prime,-}$. Moreover, (3.14) gives

$$\begin{aligned} 0 \le &-\vartheta_j \int_{B_1'} w_j \frac{\partial h}{\partial x_n}(\hat{x},0^+)\, d\mathcal{H}^{n-1} \le \mathscr{G}_j(w_j) + \frac{3}{2}\int_{\partial B_1} z_j^2\, d\mathcal{H}^{n-1} \\ &\le \sup_j \left(\mathscr{G}_j(w_j) + \frac{3}{2}\int_{\partial B_1} z_j^2\, d\mathcal{H}^{n-1}\right) < +\infty. \end{aligned}$$

Therefore the convergence of traces implies

$$\int_{B_1'} w \frac{\partial h}{\partial x_n}(\hat{x},0^+)\, d\mathcal{H}^{n-1} = \lim_j \int_{B_1'} w_j \frac{\partial h}{\partial x_n}(\hat{x},0^+)\, d\mathcal{H}^{n-1} = 0.$$

By (2.3) we deduce that actually $w=0$ on $B_1^{\prime,-}$, i.e. $w\in\mathscr{B}_\infty^{(2)}$. In particular, this holds true for z_∞ by taking into account that $\sup_j \mathscr{G}_j(z_j) < +\infty$. The inequality (3.18) then follows at once.

Let us now deal with the lim sup inequality (3.19). Arguing as in case (1), we need only to consider the case of $w\in\mathscr{B}_\infty^{(2)}$ such that (3.20) holds and, for every $j_k\uparrow+\infty$, we need to find a subsequence $j_{k_l}\uparrow+\infty$ and a sequence $w_l\to w$ in $L^2(B_1)$ such that

$$\limsup_l \mathscr{G}_{j_{k_l}}(w_l) \le \mathscr{G}_\infty^{(i)}(w). \tag{3.25}$$

Introduce the positive Radon measures

$$\nu_k := |\nabla z_{j_k}|^2 \mathcal{L}^n \llcorner B_1 - 4\vartheta_{j_k}(z_{j_k} + \vartheta_{j_k} h)\frac{\partial h}{\partial x_n}(\cdot,0^+)\, \mathcal{H}^{n-1}\llcorner B_1^{\prime,-}.$$

Note that, for k sufficiently large it follows that

$$\nu_k(B_1) = \mathscr{G}_{j_k}(z_{j_k}) + \frac{3}{2}\int_{\partial B_1} z_{j_k}^2\, d\mathcal{H}^{n-1} \le \sup_j \mathscr{G}_j(z_j) + 2\int_{\partial B_1} z_\infty^2\, d\mathcal{H}^{n-1} < \infty.$$

Thus, $(\nu_k)_{k\in\mathbb{N}}$ is equi-bounded in mass and, up to a subsequence that we do not relabel, we may assume that $(\nu_k)_{k\in\mathbb{N}}$ converges weakly* to a finite positive Radon measure ν.

Next we fix two constants $\varepsilon,\delta>0$ sufficiently small and we introduce the sets $K:=\partial B_1\cup B_1^{\prime,-}$ and $G_\varepsilon := \{x\in B_1 : \operatorname{dist}(x,K)>\varepsilon\}$ for every $\varepsilon>0$. In order to find the sequence in (3.25), we modify w in two different steps. First we find $w^{\varepsilon,\delta} \in \mathscr{B}_\infty^{(2)}$ such that

$$w^{\varepsilon,\delta}|_{G_\varepsilon} \in C^\infty(G_\varepsilon) \tag{3.26}$$

$$\|w - w^{\varepsilon,\delta}\|_{H^1}^2 := \int_{B_1} \left(|w - w^{\varepsilon,\delta}|^2 + |\nabla w - \nabla w^{\varepsilon,\delta}|^2\right) dx \leq \delta. \tag{3.27}$$

This modification can be achieved in view of Meyers & Serrin's approximation result that provides a function $v \in C^\infty(B_1)$ with $\|v - w\|_{H^1} \leq \delta'$, for some small δ' to be specified in what follows, and having the same trace as w on ∂B_1. Then set

$$w^{\varepsilon,\delta} := \phi_\varepsilon\, v + (1 - \phi_\varepsilon)\, w,$$

where $\phi_\varepsilon : B_1 \to [0,1]$ is a smooth cut-off function such that $\phi_\varepsilon \equiv 1$ on G_ε and $\phi_\varepsilon \equiv 0$ on $B_1 \setminus G_{\varepsilon/2}$, and $\|\nabla\phi_\varepsilon\|_{L^\infty} \leq {}^4\!/\!_\varepsilon$. Since $w^{\varepsilon,\delta} - w = \phi_\varepsilon\,(v - w)$, it follows that

$$\|w^{\varepsilon,\delta} - w\|_{H^1}^2 \leq \|v - w\|_{H^1}^2 \left(\|\phi_\varepsilon\|_{L^\infty}^2 + \|\nabla\phi_\varepsilon\|_{L^\infty}^2\right) \leq (\delta')^2 \left(1 + \frac{16}{\varepsilon^2}\right) \leq \delta,$$

for δ' suitably chosen and depending only on δ and ε.

Next we consider the Lipschitz functions $\psi_\varepsilon, \chi_\varepsilon : B_1 \to \mathbb{R}$ defined by

$$\psi_\varepsilon(x) := \begin{cases} 1 & B_{1-2\varepsilon} \\ 1 - \frac{1-|x|}{\varepsilon} & B_{1-\varepsilon} \setminus B_{1-2\varepsilon} \\ 0 & B_1 \setminus B_{1-\varepsilon}, \end{cases}$$

and

$$\chi_\varepsilon(x) := \left(2 - \frac{1}{\varepsilon}\operatorname{dist}(x, \{x_n = x_{n-1} = 0\})\right) \wedge 1 \vee 0.$$

Note that actually $\chi_\varepsilon(x) = \chi_\varepsilon(x_{n-1}, x_n)$ with

$$\chi_\varepsilon(x) = \begin{cases} 1 & x_n^2 + x_{n-1}^2 \leq \varepsilon^2 \\ 0 & x_n^2 + x_{n-1}^2 \geq 4\varepsilon^2. \end{cases}$$

Set $\varphi_\varepsilon := \psi_\varepsilon \wedge (1 - \chi_\varepsilon)$, then $\varphi_\varepsilon \in \operatorname{Lip}(\mathbb{R}^n, [0,1])$ with $\|\nabla\varphi_\varepsilon\|_{L^\infty} \leq \frac{1}{\varepsilon}$, and

$$\{\varphi_\varepsilon = 0\} = \{\psi_\varepsilon = 0\} \cup \{\chi_\varepsilon = 1\} = (B_1 \setminus B_{1-\varepsilon}) \cup \left\{\sqrt{x_n^2 + x_{n-1}^2} \leq \varepsilon\right\} \tag{3.28}$$

$$\{\varphi_\varepsilon = 1\} = \{\psi_\varepsilon = 1\} \cap \{\chi_\varepsilon = 0\} = B_{1-2\varepsilon} \cap \left\{\sqrt{x_n^2 + x_{n-1}^2} \geq 2\varepsilon\right\} \tag{3.29}$$

We finally define

$$w_k^{\varepsilon,\delta} := \varphi_\varepsilon\, w^{\varepsilon,\delta} + \left(1 - \varphi_\varepsilon\right) z_{j_k},$$

where j_k is the sequence considered for (3.25). First, we show that $w_k^{\varepsilon,\delta} \in \mathscr{B}_{j_k}$ for k sufficiently large. By construction $w_k^{\varepsilon,\delta}|_{\partial B_1} = z_{j_k}|_{\partial B_1}$. Moreover, we have that

$$w_k^{\varepsilon,\delta} + \vartheta_{j_k}\, h = \varphi_\varepsilon\,(w^{\varepsilon,\delta} + \vartheta_{j_k}\, h) + \left(1 - \varphi_\varepsilon\right)(z_{j_k} + \vartheta_{j_k} h),$$

and both the two terms above are positive on B_1'. Indeed, for what concerns the latter one, it is enough to recall that $z_{j_k} \in \mathscr{B}_{j_k}$ and that $1 - \varphi_\varepsilon \geq 0$. Instead, for the former addend we notice that $(w^{\varepsilon,\delta} + \vartheta_{j_k}\, h)|_{B_1',-} = 0$ and, due to the fact that

(i) $w^{\varepsilon,\delta}$ is smooth in G_ε by (3.26),
(ii) $h > c_\varepsilon > 0$ on $B_1' \cap G_\varepsilon$ for some positive constant $c_\varepsilon > 0$,
(iii) $\operatorname{supp}\varphi_\varepsilon \cap B_1' = \bar{G}_\varepsilon \cap B_1'$ (cp. (3.28)),
(iv) $\vartheta_{j_k} \uparrow +\infty$

for sufficiently large k it follows that

$$\varphi_\varepsilon\,(w^{\varepsilon,\delta}+\vartheta_{j_k}\,h)\geq\varphi_\varepsilon\,(-\|w^{\varepsilon,\delta}\|_{L^\infty(G_\varepsilon\cap B_1')}+\vartheta_{j_k}\,c_\varepsilon)\geq 0\quad\text{on } B_1'\setminus B_1'^{,-}.$$

We now compute the distance between $w_k^{\varepsilon,\delta}$ and w. By (3.27) we have that

$$\begin{aligned}\|w_k^{\varepsilon,\delta}-w\|_{L^2}&\leq\|w^{\varepsilon,\delta}-w\|_{L^2(B_1)}+\|w-z_{j_k}\|_{L^2(\{\varphi_\varepsilon<1\})}\\&\leq\delta+\|z_\infty-z_{j_k}\|_{L^2(B_1)}+\|z_\infty\|_{L^2(\{\varphi_\varepsilon<1\})}+\|w\|_{L^2(\{\varphi_\varepsilon<1\})}.\end{aligned}\tag{3.30}$$

Furthermore, straightforward computations just like in (3.22) and (3.23) give

$$\begin{aligned}&\int_{B_1}|\nabla w_k^{\varepsilon,\delta}|^2dx\\&=\int_{\{\varphi_\varepsilon=1\}}|\nabla w^{\varepsilon,\delta}|^2dx+\int_{\{0<\varphi_\varepsilon<1\}}|\nabla w_k^{\varepsilon,\delta}|^2dx+\int_{\{\varphi_\varepsilon=0\}}|\nabla z_{j_k}|^2dx\\&\leq\int_{\{\varphi_\varepsilon=1\}}|\nabla w^{\varepsilon,\delta}|^2dx+2\int_{\{0<\varphi_\varepsilon<1\}}|\nabla w^{\varepsilon,\delta}|^2dx+2\int_{\{0<\varphi_\varepsilon<1\}}|\nabla z_{j_k}|^2dx\\&+\frac{2}{\varepsilon^2}\int_{\{0<\varphi_\varepsilon<1\}}|w^{\varepsilon,\delta}-z_{j_k}|^2+\int_{\{\varphi_\varepsilon=0\}}|\nabla z_{j_k}|^2dx.\end{aligned}\tag{3.31}$$

Taking into account (3.27), we obtain from (3.31)

$$\begin{aligned}\int_{B_1}|\nabla w_k^{\varepsilon,\delta}|^2dx\leq\int_{B_1}|\nabla w^{\varepsilon,\delta}|^2dx&+\int_{\{0<\varphi_\varepsilon<1\}}|\nabla w^{\varepsilon,\delta}|^2dx\\&+3\int_{\{\varphi_\varepsilon<1\}}|\nabla z_{j_k}|^2dx+\frac{4}{\varepsilon^2}\int_{\{0<\varphi_\varepsilon<1\}}|w-z_{j_k}|^2+\frac{4\,\delta}{\varepsilon^2}.\end{aligned}\tag{3.32}$$

By choosing $\delta_\varepsilon=\varepsilon^4$, and setting $w_k^\varepsilon:=w_k^{\varepsilon,\delta_\varepsilon}$, we conclude from (3.32) that

$$\begin{aligned}\int_{B_1}|\nabla w_k^\varepsilon|^2dx\leq\int_{B_1}|\nabla w^{\varepsilon,\delta_\varepsilon}|^2dx&+\int_{\{0<\varphi_\varepsilon<1\}}|\nabla w^{\varepsilon,\delta_\varepsilon}|^2dx\\&+3\int_{\{\varphi_\varepsilon<1\}}|\nabla z_{j_k}|^2dx+\frac{4}{\varepsilon^2}\int_{\{0<\varphi_\varepsilon<1\}}|w-z_{j_k}|^2+4\,\varepsilon^2.\end{aligned}\tag{3.33}$$

Next, in view of (2.3), (2.4) and since $(w^{\varepsilon,\delta_\varepsilon}+\vartheta_{j_k}\,h)|_{B_1'^{,-}}=0$, the very definition of w_k^ε gives

$$\begin{aligned}0&\leq-4\vartheta_{j_k}\int_{B_1'}w_k^\varepsilon\frac{\partial h}{\partial x_n}(\hat{x},0^+)\,d\mathcal{H}^{n-1}\\&=-4\vartheta_{j_k}\int_{B_1'^{,-}}\big(1-\varphi_\varepsilon\big)(z_{j_k}+\vartheta_{j_k}h)\frac{\partial h}{\partial x_n}(\hat{x},0^+)\,\mathcal{H}^{n-1}\\&\leq-4\vartheta_{j_k}\int_{B_1'^{,-}\cap\{\varphi_\varepsilon<1\}}(z_{j_k}+\vartheta_{j_k}h)\frac{\partial h}{\partial x_n}(\hat{x},0^+)\,d\mathcal{H}^{n-1},\end{aligned}\tag{3.34}$$

having also used in the last inequality that $z_{j_k}\in\mathscr{B}_{j_k}$.

Therefore, by the very definition of ν and by collecting (3.33) and (3.34) we conclude, provided $\nu(\partial\{\varphi_\varepsilon < 1\}) = 0$, that

$$\limsup_k \mathscr{G}_{j_k}(w_k^\varepsilon) \leq \int_{B_1} |\nabla w^{\varepsilon,\delta_\varepsilon}|^2 dx + \int_{\{0<\varphi_\varepsilon<1\}} |\nabla w^{\varepsilon,\delta_\varepsilon}|^2 dx + 3\,\nu(\{\varphi_\varepsilon < 1\}) + \underbrace{\frac{4}{\varepsilon^2}\int_{B_{1-\varepsilon}\cap\{\varepsilon^2<x_{n-1}^2+x_n^2<4\varepsilon^2\}} |w - z_\infty|^2 dx}_{I_\varepsilon :=} + 4\,\varepsilon^2 - \frac{3}{2}\int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^{n-1}, \tag{3.35}$$

where we have used (3.20), as $\rho > 1-\varepsilon$ for ε small enough, and the equality

$$\{0 < \varphi_\varepsilon < 1\} = \{x \in B_{1-\varepsilon} \setminus B_{1-2\varepsilon} : x_{n-1}^2 + x_n^2 > \varepsilon^2\} \cup \{x \in B_{1-\varepsilon} : \varepsilon^2 < x_{n-1}^2 + x_n^2 < 4\varepsilon^2\},$$

that follows from (3.28) and (3.29). We next claim that

$$\lim_{\varepsilon\downarrow 0} I_\varepsilon = 0. \tag{3.36}$$

To this aim we use Fubini's theorem, a scaling argument and a 2-dimensional Poincarè inequality (recalling that the trace of $w - z_\infty$ is null on $B_1^{\prime,-}$) to deduce that for some positive constant C independent of ε we have

$$I_\varepsilon = \frac{4}{\varepsilon^2}\int_{\{y\in\mathbb{R}^{n-2}:\,|y|\leq 1-\varepsilon\}} dy \int_{\{(s,t)\in\mathbb{R}^2:\,\varepsilon^2<s^2+t^2<4\varepsilon^2\wedge((1-\varepsilon)^2-|y|^2)\}} |(w-z_\infty)(y,s,t)|^2 ds\,dt \leq C\int_{B_{1-\varepsilon}\cap\{\varepsilon^2<x_{n-1}^2+x_n^2<4\varepsilon^2\}} |\nabla(w-z_\infty)|^2 dx,$$

from which (3.36) follows at once.

To provide the recovery sequence we perform the construction above for a sequence $\varepsilon_i \downarrow 0$ such that $\nu(\partial\{\varphi_{\varepsilon_i} < 1\}) = 0$ for all $i \in \mathbb{N}$, with the choice $\delta_i := \varepsilon_i^4$. In view of (3.27), (3.29), (3.30), (3.35) and (3.36) a simple diagonal argument implies the existence of a subsequence $j_{k_i} \uparrow \infty$ such that $w_{k_i}^{\varepsilon_i} \to w$ in $L^2(B_1)$ and

$$\limsup_i \mathscr{G}_{j_{k_i}}(w_{k_i}^{\varepsilon_i}) \leq \mathscr{G}_\infty^{(2)}(w).$$

Proof of the Γ*-convergence in case (3).* The proof of (3.18) and (3.19) in case (3) is immediate: the former follows, indeed, from (3.12) and the fact that $\lim_j \mathscr{G}_j(z) = +\infty$; while the latter is trivial.

Step 4: Improving the convergence of $(z_j)_{j\in\mathbb{N}}$ if $\lim_j \mathscr{G}_j(z_j) < +\infty$. Standing the latter assumption, we show that actually $(z_j)_{j\in\mathbb{N}}$ converges strongly to z_∞ in $H^1(B_1)$.

To this aim, we use some standard results in the theory of Γ-convergence. The equi-coercivity of $(\mathscr{G}_j)_{j\in\mathbb{N}}$ established in (3.15), the Poincarè inequality and the condition $\|z_j\|_{H^1}^2 = 1$ in (3.13) imply the existence of an absolute minimizer ζ_j of $\mathscr{G}_j$ on L^2 with fixed $i \in \{1,2\}$. By [6, Theorem 7.4], for $i = 1,2$ we have that there exists $\zeta_\infty \in H^1(B_1)$ such that

$$\zeta_j \to \zeta_\infty \quad \text{in } L^2(B_1), \tag{3.37}$$

$$\mathscr{G}_j(\zeta_j) \to \mathscr{G}_\infty^{(i)}(\zeta_\infty), \tag{3.38}$$

$$\text{and } \zeta_\infty \text{ is the unique minimizer of } \mathscr{G}_\infty^{(i)}, \tag{3.39}$$

where we have used the strict convexity of $\mathscr{G}^{(i)}_\infty$ to deduce the uniqueness of the minimizer of $\mathscr{G}^{(i)}_\infty$. In addition, using the strong convergence of the traces in $L^2(\partial B_1) \cup L^2(B'_1)$ and the estimate

$$\mathscr{G}_j(\zeta_j) \le \mathscr{G}_j(z_j) \le \sup_j \mathscr{G}_j(z_j) < +\infty, \tag{3.40}$$

we infer that

$$\int_{B_1} |\nabla \zeta_j|^2 dx \to \int_{B_1} |\nabla \zeta_\infty|^2 dx,$$

in turn implying the strong convergence of $(\zeta_j)_{j\in\mathbb{N}}$ to ζ_∞ in $H^1(B_1)$.

Next note that by (3.12) and (3.40) z_j is an almost minimizer of $\mathscr{G}_j$, in the following sense:

$$0 \le \mathscr{G}_j(z_j) - \mathscr{G}_j(\zeta_j) \le \kappa_j\, \mathscr{G}_j(z_j) \le \kappa_j \cdot \sup_j \mathscr{G}_j(z_j).$$

Hence, by taking into account that κ_j is infinitesimal, that $z_j \rightharpoonup z_\infty$ weakly in $H^1(B_1)$ and (3.38), Step 3 yields that

$$\mathscr{G}_\infty(z_\infty) \le \liminf_j \mathscr{G}_j(z_j) = \lim_j \mathscr{G}_j(\zeta_j) = \mathscr{G}^{(i)}_\infty(\zeta_\infty),$$

in both cases $i = 1, 2$. Therefore, by the uniqueness of the absolute minimizer of $\mathscr{G}^{(i)}_\infty$, we conclude that $z_\infty = \zeta_\infty$. Arguing as above, the strong convergence of $(z_j)_{j\in\mathbb{N}}$ to z_∞ in $H^1(B_1)$ follows. In particular, note that by (3.13) we infer

$$\|z_\infty\|_{H^1} = 1. \tag{3.41}$$

The rest of the proof is devoted to find a contradiction to all the instances in Step 3. We start with the easier cases (1) and (3). Instead, to rule out case (2) we shall need to establish more refined properties of the function z_∞.

Step 5: Case (1) cannot occur. We recall what we have achieved so far about z_∞, namely

(i) $\|z_\infty\|_{H^1} = 1$,
(ii) z_∞ is $3/2$-homogeneous and even with respect to $x_n = 0$,
(iii) z_∞ is the unique minimizer of $\mathscr{G}^{(1)}_\infty$ with respect to its own boundary conditions,
(iv) $z_\infty \in \mathscr{B}^{(1)}_\infty$, i.e. $z_\infty + \vartheta\, h \ge 0$ on B'_1.

As an easy consequence of the properties above, we show now that

$$w_\infty := z_\infty + \vartheta h$$

minimizes the Dirichlet energy among all maps w such that $w \in w_\infty + H^1_0(B_1)$ and $w \ge 0$ in B'_1 in the sense of traces. In other words, w_∞ is a solution of the Signorini problem. To show this claim, for every $z \in \mathscr{B}^{(1)}_\infty$ we set $w := z + \vartheta\, h$ and by means of

(2.9) we write

$$\begin{aligned}\mathscr{G}^{(1)}_\infty(z) &= \int_{B_1} |\nabla w|^2 dx - \vartheta^2 \int_{B_1} |\nabla h|^2 dx - \frac{3}{2}\int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^{n-1} \\ &\quad - 2\vartheta \int_{B_1} \nabla z \cdot \nabla h \, dx - 4\vartheta \int_{B_1'} z \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) \, d\mathcal{H}^{n-1} \\ &\overset{(2.9)}{=} \int_{B_1} |\nabla w|^2 dx - \vartheta^2 \int_{B_1} |\nabla h|^2 dx - \frac{3}{2}\int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^{n-1} \\ &\quad - 3\vartheta \int_{\partial B_1} z_\infty \, h \, d\mathcal{H}^{n-1}.\end{aligned}$$

Therefore, since z_∞ is the unique minimizer of $\mathscr{G}^{(1)}_\infty$ and $w_\infty \geq 0$ on B_1', it follows from the previous computation that w_∞ is a solution of the Signorini problem. Using now the $3/2$-homogeneity of w_∞ and the classification of global solutions of the thin obstacle problem with such homogeneity in [3, Theorem 3] (see also [14, Proposition 9.9]), we deduce that

$$w_\infty = \lambda_\infty h_{\nu_\infty} \in \mathscr{H}_{3/2}, \qquad \text{for some } \lambda_\infty \geq 0 \text{ and } \nu_\infty \in \mathbb{S}^{n-2}.$$

Eventually, in view of (3.7), we reach the desired contradiction: by the strong convergence $z_j \to z_\infty$ in $H^1(B_1)$ (cp. Step 4 above) and by (3.9), we deduce that

$$\frac{c_j}{\delta_j} = \vartheta_j h + z_j \to \vartheta h + z_\infty = w_\infty \in \mathscr{H}_{3/2} \quad \text{in } H^1(B_1), \tag{3.42}$$

which implies, for j sufficiently large,

$$\mathrm{dist}_{H^1}(c_j, \mathscr{H}_{3/2}) \leq \|c_j - \delta_j \lambda_\infty h_{\nu_\infty}\|_{H^1(B_1)} \overset{(3.42)}{=} o(\delta_j) < \delta_j = \mathrm{dist}_{H^1}(c_j, \mathscr{H}_{3/2}),$$

having used in the last line that $\delta_j \lambda_\infty h_{\nu_\infty} \in \mathscr{H}_{3/2}$.

Step 6: Case (3) cannot occur. The heuristic idea to rule out case (3) is to correct the scaling of the energies in order to get a non-trivial Γ-limit for the rescaled functionals.

More in details, we start recalling that by (3.17) if $\lim_j \mathscr{G}_j(z_j) = +\infty$, then

$$\gamma_j := -4\vartheta_j \int_{B_1'} z_j \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) \, d\mathcal{H}^{n-1} \uparrow +\infty. \tag{3.43}$$

Further, the convergence $z_j \to z_\infty$ in $L^2(B_1')$ and (3.14) yield

$$\lim_j \frac{\gamma_j}{\vartheta_j} = -4 \lim_j \int_{B_1'} z_j \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) \, d\mathcal{H}^{n-1} = -4 \int_{B_1'} z_\infty \frac{\partial h}{\partial x_n}(\hat{x}, 0^+) \, d\mathcal{H}^{n-1} \in [0, +\infty),$$

so that

$$\vartheta_j \, \gamma_j^{-1/2} \uparrow +\infty. \tag{3.44}$$

It is then immediate to deduce that the right rescaling of the functionals $\mathscr{G}_j$ is obtained by dividing by a factor γ_j^{-1}: namely, for every $z \in \mathscr{B}_j$ we consider $\gamma_j^{-1}\mathscr{G}_j(z)$ and notice that

$$\gamma_j^{-1}\mathscr{G}_j(z) = \widetilde{\mathscr{G}_j}\Big(\gamma_j^{-1/2} z\Big), \tag{3.45}$$

where the functional $\widetilde{\mathscr{G}_j}$ is given by

$$\widetilde{\mathscr{G}_j}(w) := \begin{cases} \displaystyle\int_{B_1} |\nabla w|^2 dx - \frac{3}{2}\int_{\partial B_1} w^2\, d\mathcal{H}^{n-1} - 4\,\frac{\vartheta_j}{\gamma_j^{1/2}}\int_{B_1'} w\,\frac{\partial h}{\partial x_n}(\hat{x},0^+)\, d\mathcal{H}^{n-1} & \text{if } w \in \widetilde{\mathscr{B}_j}, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.46}$$

where

$$\widetilde{\mathscr{B}_j} := \Big\{ w \in \gamma_j^{-1/2} z_j + H_0^1(B_1) :\ \big(w + \vartheta_j\, \gamma_j^{-1/2}\, h\big)|_{B_1'} \geq 0 \Big\}. \tag{3.47}$$

Setting $\widetilde{z_j} := \gamma_j^{-1/2} z_j$, by (3.13) and $\gamma_j \uparrow +\infty$ we get $\widetilde{z_j} \to 0$ in $H^1(B_1)$. In addition, (3.45) and the very definition of γ_j in (3.43) imply that

$$\widetilde{\mathscr{G}_j}(\widetilde{z_j}) = 1 + O(\gamma_j^{-1}). \tag{3.48}$$

Furthermore, (3.12) rewrites as

$$(1-\kappa_j)\widetilde{\mathscr{G}_j}(\widetilde{z_j}) \leq \widetilde{\mathscr{G}_j}(\widetilde{z}) \qquad \text{for all } \widetilde{z} \in \widetilde{\mathscr{B}_j}.$$

In particular, by taking into account (3.44), $\tilde{z}_j \to 0$ in $H^1(B_1)$ and (3.48), namely $\lim_j \widetilde{\mathscr{G}_j}(\widetilde{z_j}) < +\infty$, we can argue exactly as in case (2) of Step 3 to deduce that

$$\Gamma(L^2(B_1))\text{-}\lim_j \widetilde{\mathscr{G}_j} = \widetilde{\mathscr{G}_\infty}$$

with

$$\widetilde{\mathscr{G}_\infty}(\widetilde{z}) := \begin{cases} \displaystyle\int_{B_1} |\nabla \widetilde{z}|^2 dx & \text{if } \widetilde{z} \in \widetilde{\mathscr{B}_\infty}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\widetilde{\mathscr{B}_\infty} := \{\widetilde{z} \in H_0^1(B_1) :\ \widetilde{z}|_{B_1'^{,-}} = 0\}$.

By Step 4 and the convergence $\widetilde{z_j} \to 0$ in $H^1(B_1)$, the null function turns out to be the unique minimizer of $\widetilde{\mathscr{G}_\infty}$ and $\lim_j \widetilde{\mathscr{G}_j}(\widetilde{z_j}) = \widetilde{\mathscr{G}_\infty}(0) = 0$, thus leading to a contradiction to (3.48).

We are then left with excluding case (2) of Step 3 to end the contradiction argument. To this aim, as already pointed out, we need to investigate more closely the properties of the limit z_∞.

From now on we assume that we are in the setting of case (2) of Step 3: i.e. $\vartheta = +\infty$ and $\lim_j \mathscr{G}_j(z_j) < +\infty$.

Step 7: An orthogonality condition. We exploit the fact that ψ_j is a point of minimal distance of c_j from $\mathscr{H}_{3/2}$ to deduce that z_∞ is orthogonal to the tangent space $T_h\mathscr{H}_{3/2}$.

We start noticing that $\vartheta = +\infty$ implies that $\lambda_j > 0$ for all j large enough. Moreover, by the minimal distance condition (3.7) we infer that, for all $\nu \in \mathbb{S}^{n-2}$ and $\lambda \geq 0$,

$$\|c_j - \psi_j\|_{H^1} \leq \|c_j - \lambda h_\nu\|_{H^1},$$

that, by the very definition of z_j in (3.9), can actually be rewritten as

$$\delta_j \|z_j\|_{H^1} \leq \|\psi_j - \lambda h_\nu + \delta_j z_j\|_{H^1},$$

or, equivalently,

$$-\|\psi_j - \lambda h_\nu\|^2_{H^1(B_1)} \leq 2\delta_j \langle z_j, \psi_j - \lambda h_\nu\rangle. \tag{3.49}$$

Therefore, assuming $(\lambda_j, e_{n-1}) \neq (\lambda, \nu)$ and renormalizing (3.49), we get

$$-\|\psi_j - \lambda h_\nu\|_{H^1} \leq 2\delta_j \langle z_j, \frac{\psi_j - \lambda h_\nu}{\|\psi_j - \lambda h_\nu\|_{H^1}} \rangle,$$

and by taking the limit $(\lambda, \nu) \to (\lambda_j, e_{n-1})$ we conclude (recall the definition of the tangent space in (2.5))

$$\langle z_j, \zeta \rangle = 0 \qquad \text{for all } \zeta \in T_{\psi_j} \mathscr{H}_{3/2} = T_h \mathscr{H}_{3/2},$$

where we used that $\lambda_j > 0$ in computing the tangent vectors. Now letting $j \uparrow \infty$ in the equality above we get that

$$\langle z_\infty, \zeta \rangle = 0 \qquad \text{for all } \zeta \in T_h \mathscr{H}_{3/2}. \tag{3.50}$$

Step 8: Identification of z_∞ in case (2) of Step 3. We show that

$$z_\infty(x) = a_0\, h(x) + \left(\sum_{i=1}^{n-2} a_i\, x_i \right) \sqrt{\sqrt{x_{n-1}^2 + x_n^2} + x_{n-1}}, \tag{3.51}$$

for some $a_0, \ldots, a_{n-2} \in \mathbb{R}$, i.e. $z_\infty \in T_h \mathscr{H}_{3/2}$ (cp. (2.6)).

The above claim is consequence of the following facts:

(a) z_∞ solves the boundary value problem

$$\begin{cases} \Delta z_\infty = 0 & \text{in } B_1 \setminus B_1^{\prime,-}, \\ z_\infty = 0 & \text{on } B_1^{\prime,-}; \end{cases} \tag{3.52}$$

(b) $z_\infty(x', x_n) = z_\infty(x', -x_n)$ for every $(x', x_n) \in B_1$;

(c) z_∞ is $3/2$-homogeneous.

The proof consists of three parts:

(I) to show the Hölder regularity of z_∞ and of all its transversal derivatives in the sense of distributions

$$v_\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n-2}} z_\infty}{\partial x_{n-2}^{\alpha_{n-2}}} \quad \text{with } \alpha = (\alpha_1, \ldots, \alpha_{n-2}) \in \mathbb{N}^{n-2};$$

(II) the use of a bidimensional conformal transformation in the variable (x_{n-1}, x_n) to reduce the problem to the upper half ball B_1^+;

(III) the classification of all $3/2$-homogeneous solutions.

As for (I), we start noticing that for every $\alpha \in \mathbb{N}^{n-2}$ (in particular also for $z_\infty = v_{(0,\ldots,0)}$), it follows from (a) and (b) that $v_\alpha|_{B_1^+}$ is a solution to the boundary value problem

$$\begin{cases} \Delta v_\alpha = 0 & \text{in } B_1^+, \\ v_\alpha = 0 & \text{on } B_1^{\prime,-}, \\ \frac{\partial v_\alpha}{\partial x_n} = 0 & \text{on } B_1' \setminus B_1^{\prime,-}. \end{cases} \tag{3.53}$$

It then follows from [17, Theorem 14.5] that the distributions v_α are represented by uniformly Hölder continuous functions in every $\bar{B}_r^+$ with $r < 1$. In particular, by homogeneity, we conclude that v_α is Hölder continuous in the whole $B_1 \subset \mathbb{R}^n$.

Next, for (II) we follow a suggestion by S. Luckhaus [13] (see also the appendix of [8] for a similar procedure) and consider the conformal transformation $\Phi : B_1^+ \to B_1 \setminus B_1^{\prime,-}$ defined by

$$(x_1, \ldots, x_{n-2}, y_1, y_2) \mapsto (x_1, \ldots, x_{n-2}, y_2^2 - y_1^2, -2\, y_1\, y_2), \tag{3.54}$$

and set

$$u_\alpha := v_\alpha \circ \Phi \quad \forall\, \alpha \in \mathbb{N}^{n-2}.$$

We next introduce the following Laplace operators:

$$\Delta' := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-2}^2}$$

$$\Delta'' := \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial x_n^2}$$

$$\Delta_y := \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}.$$

By a simple computation, it follows that (we set for simplicity $x' := (x_1, \ldots, x_{n-2})$)

$$\begin{aligned} \Delta_y u_\alpha(x', y_1, y_2) &= 4\,(y_1^2 + y_2^2)\,\Delta'' v_\alpha(x', y_2^2 - y_1^2, -2\,y_1\,y_2) \\ &\overset{(3.52)}{=} -4\,(y_1^2 + y_2^2)\,\Delta' v_\alpha(x', y_2^2 - y_1^2, -2\,y_1\,y_2) \end{aligned} \tag{3.55}$$

for all $\alpha \in \mathbb{N}^{n-2}$ and for all $(x', y_1, y_2) \in B_1^+$.

Note that the right hand side of (3.55) is Hölder continuous, because by (I) $\Delta' v_\alpha$ is Hölder continuous for every $\alpha \in \mathbb{N}^{n-2}$. Therefore, the usual Schauder theory for the Laplace equation implies that u_α is twice continuously differentiable with Hölder continuous second order partial derivatives.

We can then bootstrap this conclusion and infer that

$$\begin{aligned} \Delta' v_\alpha(x', y_2^2 - y_1^2, -2\,y_1\,y_2) &= \Delta' u_\alpha(x', y_1, y_2) \\ &= \sum_{i=1}^{n-2} u_{\alpha+2\,e_i}(x', y_1, y_2) \end{aligned} \tag{3.56}$$

with $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0)$, $\ldots$, $e_{n-2} = (0, 0, \ldots, 1) \in \mathbb{N}^{n-2}$, and therefore $\Delta'' v_\alpha$ is twice differentiable, thus implying by (3.55) that u_α is $C^{4,\kappa}$ for some $\kappa \in (0,1)$, and so on. In conclusion, it follows from (3.53), (3.55) and (3.56) that $u_0 \in C^\infty(\bar{B}_1^+)$ and

$$u_0(x', y_1, 0) = 0 \quad \forall\, |(x', y_1, 0)| < 1. \tag{3.57}$$

In order to perform the final classification in (III), we consider a Taylor expansion of u_0 up to order three. For the sake of simplicity we write $(w_1, \ldots, w_n) = (x', y_1, y_2)$ and use (3.57) to simplify the expansion: there exist real numbers $b_l, b_{i,j} \in \mathbb{R}$ for $l \in \{0, \ldots, n\}$ and $i, j \in \{1, \ldots, n\}$ such that $b_{i,j} = b_{j,i}$ for every i, j and

$$u_0(w) = w_n \left(b_0 + \sum_{i=1}^n b_i\, w_i + \sum_{i,j=1}^n b_{i,j} w_i\, w_j \right) + g(w),$$

with $g(w) \le C\,|w|^4$ for every $w \in B_1^+$ for some $C > 0$.

Next we perform the change of coordinates Φ^{-1} to deduce an expansion for z_∞. First note that

$$\Phi^{-1} : B_1 \setminus (B_1' \cap \{x_{n-1} \le 0\}) \to B_1^+$$

$$\Phi^{-1}(x_1, \ldots, x_n) = \big(x_1, \ldots, x_{n-2}, f(x_{n-1}, x_n), g(x_{n-1}, x_n)\big).$$

with

$$f(x_{n-1}, x_n) := \frac{\operatorname{sgn}(x_n)}{\sqrt{2}} \sqrt{\sqrt{x_{n-1}^2 + x_n^2} - x_{n-1}} \tag{3.58}$$

$$g(x_{n-1}, x_n) := \frac{1}{\sqrt{2}} \sqrt{\sqrt{x_{n-1}^2 + x_n^2} + x_{n-1}} \tag{3.59}$$

and

$$\mathrm{sgn}(x_n) = \begin{cases} 1 & \text{if } x_n \geq 0 \\ -1 & \text{if } x_n < 0. \end{cases}$$

Therefore, for every $x \in B_1 \setminus B_1^{\prime,-}$ we have

$$z_\infty(x) = u_0 \circ \Phi^{-1}(x) = g(x_{n-1}, x_n) \left(b_0 + \sum_{i=1}^{n-2} b_i\, x_i \right) \tag{3.60}$$

$$+ g(x_{n-1}, x_n)\Big(b_{n-1}\, f(x_{n-1}, x_n) + b_n\, g(x_{n-1}, x_n) + \sum_{i,j=1}^{n-2} b_{i,j} x_i\, x_j \Big) \tag{3.61}$$

$$+ 2\, g(x_{n-1}, x_n) \sum_{i=1}^{n-2} \big(b_{i,n-1}\, x_i\, f(x_{n-1}, x_n) + b_{i,n}\, x_i\, g(x_{n-1}, x_n) \big) \tag{3.62}$$

$$\begin{aligned} &+ b_{n-1,n-1}\, g(x_{n-1}, x_n)\, f^2(x_{n-1}, x_n) + 2\, b_{n-1,n}\, f(x_{n-1}, x_n)\, g^2(x_{n-1}, x_n) \\ &+ b_{n,n}\, g^3(x_{n-1}, x_n) + H(x), \end{aligned} \tag{3.63}$$

with

$$|H(x)| \leq C\, |x|^2 \quad \forall\, x \in B_1 \setminus B_1^{\prime,-}$$

for some $C > 0$.

Due to the $3/2$-homogeneity of z_∞ and the $1/2$-homogeneity of f and g, we deduce that the first term in (3.60), as well as the first two terms of (3.61), (3.62) and the function H in (3.63) have the wrong homogeneity and therefore are identically zero, thus reducing the expansion of z_∞ to the following

$$\begin{aligned} z_\infty(x) = g(x_{n-1}, x_n) \sum_{i=1}^{n-2} b_i\, x_i + b_{n-1,n-1} g(x_{n-1}, x_n)\, f^2(x_{n-1}, x_n) \\ + 2\, b_{n-1,n}\, f(x_{n-1}, x_n)\, g^2(x_{n-1}, x_n) + b_{n,n} g^3(x_{n-1}, x_n). \end{aligned} \tag{3.64}$$

In addition, (3.58) and (3.59) yield

$$f(x_{n-1}, x_n)\, g(x_{n-1}, x_n) = \frac{x_n}{2},$$

and, in turn, plugging the latter identity in (3.64) implies

$$\begin{aligned} z_\infty(x) = g(x_{n-1}, x_n) \cdot \\ \cdot \Big(\sum_{i=1}^{n-2} b_i\, x_i + b_{n-1,n-1}\, f^2(x_{n-1}, x_n) + b_{n-1,n}\, x_n + b_{n,n}\, g^2(x_{n-1}, x_n) \Big). \end{aligned} \tag{3.65}$$

To conclude the proof of Step 8 we need only to check for which choices of the coefficients the right hand side of (3.65) is harmonic. To this aim we notice that g is itself a harmonic function, i.e. $\Delta g = 0$ in $B_1 \setminus B_1^{\prime,-}$. Therefore, we compute Δz_∞ thanks to

(3.58), (3.59) and (3.65) as follows

$$\Delta z_\infty(x) = \frac{(3\,b_{n,n} + b_{n-1,n-1})\sqrt{2\,\sqrt{x_{n-1}^2 + x_n^2} + x_{n-1}}}{\sqrt{x_{n-1}^2 + x_n^2}} + \frac{b_{n-1,n}\,x_n}{\sqrt{x_{n-1}^2 + x_n^2}\,\sqrt{\sqrt{x_{n-1}^2 + x_n^2} + x_{n-1}}}. \tag{3.66}$$

Note that the function on the right hand side of (3.66) is identically zero on $B_1 \setminus B_1^{\prime,-}$ if and only if

$$3\,b_{n,n} + b_{n-1,n-1} = 0 \quad \text{and} \quad b_{n-1,n} = 0.$$

Thus, coming back to (3.65) we conclude that

$$\begin{aligned} z_\infty(x) &= g(x_{n-1}, x_n)\,\Big(\sum_{i=1}^{n-2} b_i\,x_i - 3\,b_{n,n}\,f^2(x_{n-1}, x_n) + b_{n,n}\,g^2(x_{n-1}, x_n)\Big) \\ &\overset{(3.58),\,(3.59)}{=} g(x_{n-1}, x_n)\sum_{i=1}^{n-2} b_i\,x_i + b_{n,n}\,g(x_{n-1}, x_n)\left(2\,x_{n-1} - \sqrt{x_{n-1}^2 + x_n^2}\right), \end{aligned}$$

which is the desired formula (3.51) for $a_0 = b_{n,n}$ and $a_i = b_i$ for $i = 1, \ldots, n-2$.

Step 9: Case (2) of Step 3 cannot occur. We finally reach a contradiction by excluding also case (2) in Step 3.

We use the orthogonality condition derived in Step 7, i.e.,

$$\langle z_\infty, \zeta\rangle = 0 \qquad \text{for all } \zeta \in T_h\mathscr{H}_{3/2}. \tag{3.67}$$

Since z_∞ has the form in (3.51), we can choose h as test function in (3.67) to deduce $a_0 = 0$. Then take $\zeta = v_{e_{n-1},\xi}$ (cp. (2.7)) to deduce $a_1 = \ldots = a_{n-2} = 0$ by the arbitrariness of $\xi \in \mathbb{S}^{n-1}$ with $\xi \cdot e_n = \xi \cdot e_{n-1} = 0$.

Therefore, z_∞ is the null function, contradicting (3.41). In this way we have excluded all the cases of Step 3 and conclude the proof of the theorem. □

3.4. **Even frequencies: the lowest stratum of the singular set.** In this subsection we prove Theorem 3.2. As already remarked the arguments are similar to those of Theorem 3.1, some simplifications are actually occurring. Therefore, we shall only underline the substantial changes. Further, we keep the notation introduced in Theorem 3.1 except for $\mathscr{G}$, that in the ensuing proof stands for $\mathscr{G}_{2m}$.

Proof of Theorem 3.2. We start off as in Theorem 3.1 with a contradiction argument assuming the existence of κ_j, $\delta_j \downarrow 0$, and of $c_j \in H^1(B_1)$ such that c_j is $2m$-homogeneous, $c_j \geq 0$ on B_1',

$$\mathrm{dist}_{H^1}\big(c_j, \mathscr{H}_{2m}\big) = \delta_j, \tag{3.68}$$

$$(1-\kappa_j)\mathscr{G}(c_j) \leq \inf_{v \in \mathscr{A}_{c_j}} \mathscr{G}(v) \tag{3.69}$$

and $\psi_j := P(c_j) \in \mathscr{H}_{2m}^{(0)}$ with

$$\|c_j - \psi_j\|_{H^1} = \mathrm{dist}_{H^1}\big(c_j, \mathscr{H}_{2m}\big) = \delta_j \qquad \text{for all } j \in \mathbb{N}. \tag{3.70}$$

We divide the rest of the proof in some intermediate steps corresponding to those of Theorem 3.1.

Step 1: Introduction of a family of auxiliary functionals. We rewrite inequality (3.69) conveniently and interpret it as an almost minimality condition.

For fixed j, we use (2.13) and algebraic manipulations to rewrite (3.69) for all $v \in \mathscr{A}_{c_j}$ as

$$(1-\kappa_j)\mathscr{G}(c_j-\psi_j) \le \mathscr{G}(v-\psi_j). \tag{3.71}$$

Setting

$$z_j := \frac{c_j-\psi_j}{\delta_j}, \tag{3.72}$$

$$\hat{\psi}_j := \frac{\psi_j}{\|\psi_j\|_{H^1}} \quad \text{and} \quad \vartheta_j := \frac{\|\psi_j\|_{H^1}}{\delta_j},$$

(3.71) reduces to

$$(1-\kappa_j)\mathscr{G}_j(z_j) \le \mathscr{G}_j(z) \quad \text{for all } z \in \mathscr{B}_j, \tag{3.73}$$

where $\mathscr{G}_j : L^2(B_1) \to (-\infty,+\infty]$ is given by

$$\mathscr{G}_j(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - 2m \int_{\partial B_1} z_j^2 \, d\mathcal{H}^{n-1} & \text{if } z \in \mathscr{B}_j, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.74}$$

and

$$\mathscr{B}_j := \left\{ z \in z_j + H^1_0(B_1) : \, (z+\vartheta_j\hat{\psi}_j)|_{B_1'} \ge 0 \right\}. \tag{3.75}$$

Moreover, note that by (3.70) and (3.72)

$$\|z_j\|_{H^1(B_1)} = 1. \tag{3.76}$$

This implies that we can extract a subsequence (not relabeled) such that

(a) $(z_j)_{j\in\mathbb{N}}$ converges weakly in $H^1(B_1)$ to some z_∞;
(b) the corresponding traces $(z_j|_{B_1'})_{j\in\mathbb{N}}$; converge strongly in $L^2(B_1') \cup L^2(\partial B_1)$;
(c) $(\vartheta_j)_{j\in\mathbb{N}}$ has a limit $\vartheta \in [0,\infty]$;
(d) $(\hat{\psi}_j)_{j\in\mathbb{N}}$ converges in $H^1(B_1)$ to some non-trivial $2m$-homogeneous harmonic polynomial ψ_∞ satisfying $\psi_\infty \ge 0$ on B_1'. Actually, being the ψ_j's polynomials the convergence occurs in any $C^\ell(B_1)$ norm.

Step 2: First properties of $(\mathscr{G}_j)_{j\in\mathbb{N}}$. We establish the equi-coercivity and some further properties of the family of the auxiliary functionals $(\mathscr{G}_j)_{j\in\mathbb{N}}$.

In this case equi-coercivity is a straightforward consequence of definition (3.74) and item (b) in Step 1: for j sufficiently large and for all $w \in H^1(B_1)$

$$\int_{B_1} |\nabla w|^2 dx - 3m \int_{\partial B_1} z_\infty^2 \le \mathscr{G}_j(w). \tag{3.77}$$

Moreover notice that (3.76) and item (b) imply

$$\sup_j |\mathscr{G}_j(z_j)| < \infty. \tag{3.78}$$

Step 3: Asymptotic analysis of $(\mathscr{G}_j)_{j\in\mathbb{N}}$. Here we prove a Γ-convergence result for the family of energies $\mathscr{G}_j$.

More precisely, we distinguish two cases.

(1) If $\vartheta \in [0, +\infty)$, then
$$(z_\infty + \vartheta\, \psi_\infty)|_{B_1'} \geq 0,$$
and $\Gamma(L^2(B_1))$- $\lim_j \mathscr{G}_j = \mathscr{G}_\infty^{(1)}$, where
$$\mathscr{G}_\infty^{(1)}(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - 2m \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^{n-1} & \text{if } z \in \mathscr{B}_\infty^{(1)}, \\ +\infty & \text{otherwise,} \end{cases}$$
and $\mathscr{B}_\infty^{(1)} := \{z \in z_\infty + H_0^1(B_1) : (z + \vartheta\, \psi_\infty)|_{B_1'} \geq 0\}$.

(2) If $\vartheta = +\infty$, $\Gamma(L^2(B_1))$- $\lim_j \mathscr{G}_j = \mathscr{G}_\infty^{(2)}$, where
$$\mathscr{G}_\infty^{(2)}(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 dx - 2m \int_{\partial B_1} z_\infty^2 \, d\mathcal{H}^{n-1} & \text{if } z \in \mathscr{B}_\infty^{(2)}, \\ +\infty & \text{otherwise,} \end{cases}$$
and $\mathscr{B}_\infty^{(2)} := z_\infty + H_0^1(B_1)$.

Proof of the Γ*-convergence in case (1).* The fact that $z_\infty \in \mathscr{B}_\infty^{(1)}$ and the $\liminf$ inequality (3.18) can be deduced exactly as in case (1) of Theorem 3.1.

Instead, for what concerns the $\limsup$ inequality (3.19), after performing the reductions in the corresponding step of Theorem 3.1, one has to take
$$w_k^r := \varphi\,(w + \vartheta\, \psi_\infty - \vartheta_{j_k}\, \hat{\psi}_{j_k}) + (1 - \varphi)\, z_{j_k}$$
to infer (3.24). The conclusion then follows by a diagonalization argument.

Proof of the Γ*-convergence in case (2).* The fact that $z_\infty \in \mathscr{B}_\infty^{(2)}$ and the $\liminf$ inequality (3.18) are simple consequences of the lower semicontinuity of the Dirichlet energy under L^2 convergence and of the equi-coercivity of $(\mathscr{G}_j)_{j\in\mathbb{N}}$ in (3.77).

For the proof of the the $\limsup$ inequality (3.19) we need first to observe that the set $\{\psi_\infty = 0\} \cap B_1'$ is contained in a $(n-2)$-dimensional subspace H. In order to prove this claim, we start noticing that ψ_∞ is a non-trivial $2m$-homogeneous harmonic polynomial satisfying $\psi_\infty \geq 0$ on B_1'. Therefore, its zero set is a cone containing the origin. Moreover, $\psi_j|_{B_1'}$ is convex by [2, Theorem 1]: this implies that $\{\psi_\infty = 0\} \cap B_1'$ is actually a convex cone. Since $\psi_\infty|_{B_1'} \geq 0$ we infer from (2.11) that $\{\psi_\infty = 0\} \cap B_1' \subseteq \{\psi_\infty = |\nabla \psi_\infty| = 0\}$. Thus the harmonicity of ψ_∞ finally yields the claim thanks to [4, Theorem 3.1] (see also [12, Lemma 1.9]).

Given this, the construction of the recovery sequence is analogous to that of case (2) in Theorem 3.1. With $K := \partial B_1$, let G_ε and ϕ_ε be defined correspondingly, then $w^{\varepsilon,\delta} := \phi_\varepsilon\, v + (1 - \phi_\varepsilon) w$ satisfies (3.26)-(3.27). Finally, assuming $H \subseteq \{x_{n-1} = x_n = 0\}$, and setting $w_k^{\varepsilon,\delta} := \chi_\varepsilon\, w^{\varepsilon,\delta} + (1 - \chi_\varepsilon) z_{j_k}$ we conclude by choosing $\delta = \varepsilon^2$ and by a suitable diagonalization argument.

Step 4: Improving the convergence of $(z_j)_{j\in\mathbb{N}}$. $(z_j)_{j\in\mathbb{N}}$ converges strongly to z_∞ in $H^1(B_1)$.

This follows thanks to (3.78) and by standard Γ-convergence arguments (cp. the corresponding step of Theorem 3.1). In particular, $\|z_\infty\|_{H^1} = 1$.

To reach the final contradiction we exclude next both the instances in Step 3 above.

Step 5: Case (1) cannot occur. We recall what we have achieved so far about z_∞, namely

(i) $\|z_\infty\|_{H^1} = 1$,

(ii) z_∞ is $2m$-homogeneous and even with respect to $x_n = 0$,

(iii) z_∞ is the unique minimizer of $\mathscr{G}_\infty^{(1)}$ with respect to its own boundary conditions,

(iv) $z_\infty \in \mathscr{B}_\infty^{(1)}$, i.e. $z_\infty + \vartheta\, \psi_\infty \geq 0$ on B_1'.

Arguing as in Step 5 of Theorem 3.1, the properties above and a change of variables for $\mathscr{G}_\infty^{(1)}$ show that $w_\infty := z_\infty + \vartheta\psi_\infty$ minimizes the Dirichlet energy among all maps $w \in w_\infty + H_0^1(B_1)$ such that $w \geq 0$ in B_1' in the sense of traces.

Therefore, w_∞ is a $2m$-homogeneous solution of the Signorini problem, that is $w_\infty \in \mathscr{H}_{2m}$ by [10, Lemma 1.3.3] (see also [14, Proposition 9.11]). We get a contradiction as

$$\frac{c_j}{\delta_j} = \frac{\psi_j}{\delta_j} + z_j = \vartheta_j \hat{\psi}_j + z_j \to \vartheta\, \psi_\infty + z_\infty = w_\infty \in \mathscr{H}_{2m} \quad \text{in } H^1(B_1), \tag{3.79}$$

which implies, for j sufficiently large,

$$\mathrm{dist}_{H^1}(c_j, \mathscr{H}_{2m}) \leq \|c_j - \delta_j w_\infty\|_{H^1(B_1)} \stackrel{(3.79)}{=} o(\delta_j) < \delta_j = \mathrm{dist}_{H^1}(c_j, \mathscr{H}_{2m}),$$

having used in the last line that $\delta_j w_\infty \in \mathscr{H}_{2m}$.

We are then left with excluding case (2) of Step 3 to end the contradiction argument. Since in the present proof the Step 3 has two cases rather than three, for a more clear comparison with Theorem 3.1 we number the next step as 7 rather than 6.

Step 7: An orthogonality condition. We exploit the fact that ψ_j is the point of minimal distance of c_j from $\mathscr{H}_{2m}$ to deduce that z_∞ is orthogonal to $\widehat{\mathscr{H}_{2m}}$.

The very definitions of ψ_j in (3.70) and of z_j in (3.72) imply that for all $\psi \in \mathscr{H}_{2m}$

$$-\|\psi_j - \psi\|^2_{H^1(B_1)} \leq 2\delta_j \langle z_j, \psi_j - \psi\rangle.$$

Therefore, assuming $\psi_j \neq \psi$ and renormalizing, we get

$$-\|\psi_j - \psi\|_{H^1} \leq 2\delta_j \langle z_j, \frac{\psi_j - \psi}{\|\psi_j - \psi\|_{H^1}}\rangle,$$

and by taking the limit $\psi \to \psi_j$ in H^1 we conclude that

$$\langle z_j, \zeta\rangle = 0 \qquad \text{for all } \zeta \in T_{\psi_j}\mathscr{H}_{2m} \stackrel{(2.15)}{=} \widehat{\mathscr{H}_{2m}}.$$

Now letting $j \uparrow \infty$ in the equality above we get that

$$\langle z_\infty, \zeta\rangle = 0 \qquad \text{for all } \zeta \in \widehat{\mathscr{H}_{2m}}. \tag{3.80}$$

Step 8: Identification of z_∞ in case (2) of Step 3. We have that

$$z_\infty \in \widehat{\mathscr{H}_{2m}} \setminus \{0\}. \tag{3.81}$$

Indeed we have already shown that

(i) $\|z_\infty\|_{H^1} = 1$,

(ii) z_∞ is $2m$-homogeneous and even with respect to $x_n = 0$,

(iii) z_∞ is the unique minimizer of $\mathscr{G}_\infty^{(2)}$ with respect to its own boundary conditions, i.e. is harmonic.

Step 9: Case (2) of Step 3 cannot occur. We finally reach a contradiction by excluding also case (2) in Step 3.

Because of (3.81) we can choose z_∞ itself as a test function in (3.80) to deduce that it is actually the null function, thus contradicting (3.41).

In this way we have excluded all the cases of Step 3 and we can conclude the proof of the theorem. □

4. Regularity of the free boundary

In this section we show how to derive the regularity of the free boundary around points of least frequency as a simple consequence of the epiperimetric inequality. To this aim we need to recall some notation and some results from the literature. Since we are going to use the monotonicity formulas proven in [10], we try to follow the notation therein as closely as possible. In what follows $u \in H^1(B_1)$ shall denote a solution to the Signorini problem (even symmetric respect to $\{x_n = 0\}$). We denote with $\Lambda(u)$ the coincidence set, i.e.

$$\Lambda(u) := \big\{(\hat{x}, 0) \in B_1' : u(\hat{x}, 0) = 0\big\}$$

and by $\Gamma(u)$ the free boundary of u, namely the topological boundary of $\Lambda(u)$ in the relative topology of B_1'.

For $x_0 \in \Gamma(u)$ and $0 < r < 1 - |x_0|$ let $N^{x_0}(r, u)$ be the *frequency function* defined by

$$N^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, dx}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}}$$

provided $u|_{\partial B_r(x_0)} \not\equiv 0$. As proven in [3] the function $(0, \mathrm{dist}(x_0, \partial B_1)) \ni r \mapsto N^{x_0}(r, u)$ is nondecreasing for every $x_0 \in B_1'$. It is then possible to define the limit $N^{x_0}(0^+, u) := \lim_{r \downarrow 0} N^{x_0}(r, u)$ and as shown in [3] it holds $N^{x_0}(0^+, u) \geq 3/2$ for every $x_0 \in \Gamma(u)$. We then denote with $\Gamma_{3/2}$ the points of the free boundary with minimal frequency, also called *regular points* in [10]:

$$\Gamma_{3/2} := \big\{x_0 \in \Gamma(u) : N^{x_0}(0^+, u) = 3/2\big\}.$$

Note that by the monotonicity of the frequency it follows that $\Gamma_{3/2}(u) \subset \Gamma(u)$ is open in the relative topology. We also introduce the shorthand notation

$$D^{x_0}(r) := \int_{B_r(x_0)} |\nabla u|^2 \, dx \quad \text{and} \quad H^{x_0}(r) := \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}$$

and we always omit to write the point x_0 in the notation if $x_0 = 0$. Finally we recall the following simple consequence of the monotonicity of the frequency function, proven in [3, Lemma 1]: the function $(0, 1 - |x_0|) \ni r \mapsto \frac{H^{x_0}(r)}{r^{n+2}}$ is nondecreasing and in particular

$$H^{x_0}(r) \leq \frac{H(1 - |x_0|)}{(1 - |x_0|)^{n+2}} \, r^{n+2} \quad \forall \, 0 < r < 1 - |x_0|, \tag{4.1}$$

and for every $\varepsilon > 0$ there exists $r_0(\varepsilon) > 0$ such that

$$H(r) \geq \frac{H(r_0)}{r_0^{n+2+\varepsilon}} \, r^{n+2+\varepsilon} \quad \forall \, 0 < r < r_0. \tag{4.2}$$

For the readers' convenience we provide a short account of these statements in Appendix A.

4.1. **Decay of the boundary adjusted energy.** The boundary adjusted energy $\mathscr{G}$ considered above is the unscaled version of a member of a family of energies *à la* Weiss introduced in [10], namely the one corresponding to the lowest frequency. Since we also need to consider its rescaled version, we shift to the notation used in [10]: $\mathscr{G}(u) = W_{3/2}(1,u)$ with

$$W^{x_0}_{3/2}(r,u) := \frac{1}{r^{n+1}}\int_{B_r(x_0)} |\nabla u|^2\,dx - \frac{3}{2\,r^{n+2}}\int_{\partial B_r(x_0)} u^2\,d\mathcal{H}^{n-1}.$$

Note that by the monotonicity result in [10] it follows that

$$\frac{d}{dr}W_{3/2}(r,u) = \frac{2}{r}\int_{\partial B_1}\Big(\nabla u_r\cdot\nu - \frac{3}{2}u_r\Big)^2\,d\mathcal{H}^{n-1}. \tag{4.3}$$

and for every $x_0\in\Gamma_{3/2}(u)$ it holds that $W^{x_0}_{3/2}(r,u)\geq 0$ for every $0<r<1-|x_0|$ with

$$\lim_{r\to 0} W^{x_0}_{3/2}(r,u) = 0.$$

In this subsection we show the main consequence of the epiperimetric inequality in Theorem 3.1, namely the decay of the boundary adjusted energy. The next lemma enable us to apply the epiperimetric inequality uniformly on the radius at any free boundary point in $\Gamma_{3/2}(u)$.

4.2. Lemma. *Let $u\in H^1(B_1)$ be a solution to the Signorini problem. Then for every compact set $K\subset\subset B_1'$ and for every $\eta>0$ there exists $r_0>0$ such that*

$$\mathrm{dist}(c^{x_0}_r,\mathscr{H}_{3/2})\leq\eta\quad\forall\,x_0\in\Gamma_{3/2}(u)\cap K,\quad\forall\,0<r<r_0, \tag{4.4}$$

where $c^{x_0}_r\in H^1(B_1)$ is given by

$$c^{x_0}_r(x) := \frac{|x|^{3/2}}{r^{3/2}}u\Big(\frac{rx}{|x|}+x_0\Big).$$

Proof. The proof is done by contradiction: we assume that there exist $\eta>0$ and sequences of points $x_k\in K\cap\Gamma_{3/2}(u)$ and radii $r_k\downarrow 0$ such that

$$\mathrm{dist}(c^{x_k}_{r_k},\mathscr{H}_{3/2})>\eta\quad\forall\,k\in\mathbb{N}. \tag{4.5}$$

Let us introduce the following rescaled function (cp. next subsection for further discussions)

$$u^{x_0}_r(x) := \frac{u(x_0+rx)}{r^{3/2}}\quad\forall\,0<r<1-|x_0|,\quad\forall\,x\in B_{(1-|x_0|)/r}.$$

It follows from (4.1) that

$$\sup_{k\in\mathbb{N}}\|u^{x_k}_{r_k}\|_{L^2(B_1)}<+\infty,$$

from which (by the regularity for the solution of the Signorini problem – cp. Appendix A) we deduce

$$\sup_{k\in\mathbb{N}}\|u^{x_k}_{r_k}\|_{C^{1,1/2}(B_1)}<+\infty. \tag{4.6}$$

We can then conclude that up to passing to a subsequence (not relabeled) there exists a function $w_0\in C^{1,1/2}(B_1)$ such that $\|u^{x_k}_{r_k}-w_0\|_{C^{1,\alpha}(B_1)}\to 0$ for every $\alpha<1/2$. Moreover without loss of generality we can also assume that $x_k\to x_0\in K\cap\Gamma_{3/2}$.

By a simple argument (whose details are left to the reader) the function w_0 is itself a solution to the Signorini problem. Moreover we claim that w_0 is $3/2$-homogeneous. In order to show the claim we start noticing the following: for every $\delta>0$ we can fix

$\rho > 0$ such that $N^{x_0}(\rho, u) \leq 3/2 + \delta$. Therefore, for k sufficiently large we infer that for every $s \in (0, 1)$:

$$\begin{aligned} N(s, u^{x_k}_{r_k}) &= N^{x_k}(s\, r_k, u) \\ &= N^{x_k}(s\, r_k, u) - N^{x_k}(\rho, u) + N^{x_k}(\rho, u) - N^{x_0}(\rho, u) + N^{x_0}(\rho, u) \\ &\leq N^{x_k}(\rho, u) - N^{x_0}(\rho, u) + \frac{3}{2} + \delta \leq \frac{3}{2} + 2\,\delta, \end{aligned} \tag{4.7}$$

where we used the monotonicity of the frequency and the fact that for k large enough $|N^{x_k}(\rho, u) - N^{x_0}(\rho, u)| \to 0$ as $x_k \to x_0$. In particular from the convergence of $u^{x_k}_{r_k}$ to w_0 and the arbitrariness of δ we deduce that $N(s, w_0) \equiv 3/2$, i.e. w_0 is $3/2$-homogeneous.

From the already cited classification result in [3, Theorem 3] we infer that $w_0 \in \mathscr{H}_{3/2}$, thus clearly contradicting (4.5). □

We are now in the position to prove the decay of the boundary adjusted energy. To this aim we recall some elementary formulas, whose verification is left to the reader (details can be also found in [10]):

$$H'(r) := \frac{d}{dr} H(r) = \frac{n-1}{r} H(r) + 2 \int_{\partial B_r} u \nabla u \cdot \nu \, d\mathcal{H}^{n-1}, \tag{4.8}$$

$$D'(r) := \frac{d}{dr} D(r) = \frac{n-2}{r} D(r) + 2 \int_{\partial B_r} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^{n-1}, \tag{4.9}$$

$$D(r) = \int_{\partial B_r} u \nabla u \cdot \nu \, d\mathcal{H}^{n-1}. \tag{4.10}$$

We mention once for all that all the integral quantities $D(r)$, $H(r)$ etc... considered in this section are absolutely continuous functions of the radius and therefore can be differentiated almost everywhere.

4.3. Proposition. *There exists a dimensional constant $\gamma > 0$ with this property. For every compact set $K \subset B_1'$ there exists a constant $C > 0$ such that*

$$W^{x_0}_{3/2}(r, u) \leq C\, r^{\gamma} \quad \forall\, 0 < r < \operatorname{dist}(K, \partial B_1),\ \forall\, x_0 \in \Gamma_{3/2} \cap K. \tag{4.11}$$

PROOF. We start considering the case of $x_0 = 0 \in \Gamma_{3/2}$. We can then compute as follows:

$$\begin{aligned} \frac{d}{dr} W_{3/2}(r, u) &= -\frac{n+1}{r^{n+2}} D(r) + \frac{D'(r)}{r^{n+2}} - \frac{3}{2r^{n+2}} H'(r) + \frac{3(n+2)}{2r^{n+3}} H(r) \\ &= -\frac{n+1}{r} W_{3/2}(r, u) - \frac{3(n+1)}{2r^{n+3}} H(r) \\ &\quad + \underbrace{\frac{D'(r)}{r^{n+2}} + \frac{9}{2r^{n+3}} H(r) - 3 \frac{D(r)}{r^{n+2}}}_{=:I}. \end{aligned} \tag{4.12}$$

In order to treat the terms in I, we introduce the rescaled functions

$$u_r(x) := \frac{u(rx)}{r^{3/2}} \tag{4.13}$$

and deduce by simple computations that

$$I = \frac{1}{r}\int_{\partial B_1}\left(|\nabla u_r|^2 - 3u_r\nabla u_r\cdot\nu + \frac{9}{2}u_r^2\right)d\mathcal{H}^{n-1}$$
$$= \frac{1}{r}\int_{\partial B_1}\left[\left(\nabla u_r\cdot\nu - \frac{3}{2}u_r\right)^2 + |\nabla_\theta u_r|^2 + \frac{9}{4}u_r^2\right]d\mathcal{H}^{n-1}, \qquad (4.14)$$

where we denoted by $\nabla_\theta u_r$ the differential of u_r in the directions tangent to ∂B_1. Let c_r be the $3/2$-homogeneous extension of $u_r|_{\partial B_1}$, i.e.

$$c_r(x) := |x|^{3/2}u_r\Big(\frac{x}{|x|}\Big).$$

It is simple to verify that

$$\int_{\partial B_1}\left(|\nabla_\theta u_r|^2 + \frac{9}{4}u_r^2\right) = (n+1)\int_{B_1}|\nabla c_r|^2\,dx.$$

We then conclude that

$$\frac{d}{dr}W_{3/2}(r,u) = \frac{n+1}{r}\Big(W_{3/2}(1,c_r) - W_{3/2}(1,u_r)\Big)$$
$$+ \frac{1}{r}\int_{\partial B_1}\left(\nabla u_r\cdot\nu - \frac{3}{2}u_r\right)^2 d\mathcal{H}^{n-1}. \qquad (4.15)$$

By (4.4) there exists $r_0 > 0$ such that the epiperimetric inequality in Theorem 3.1 can be applied (recall that $\mathscr{G}(w) = W_{3/2}(1,w)$ for every $w \in H^1(B_1)$). In view of (4.3), we then deduce that

$$\frac{d}{dr}W_{3/2}(r,u) \geq 2\frac{n+1}{r}\frac{\kappa}{1-\kappa}W_{3/2}(1,u_r) \quad \forall\, 0 < r < r_0. \qquad (4.16)$$

Integrating this inequality we get

$$W_{3/2}(r,u) \leq W_{3/2}(1,u)\,r^\gamma \quad \forall\, 0 < r < r_0, \qquad (4.17)$$

with $\gamma := 2(n+1)\kappa/1-\kappa$.

In order to conclude the proof it is enough to observe that for every $x_0 \in K \subset\subset B_1'$ the decay (4.17) can be derived by the same arguments with a constant $C > 0$ which depends only on $D(1)$ and $\mathrm{dist}(K, \partial B_1)$. □

4.4. Remark. Note that (4.15) also shows the monotonicity of the boundary adjusted energy: using the minimizing property for the Dirichlet energy of u_r with respect to functions with the same trace, we deduce indeed the less refined with respect to (4.3) estimate

$$\frac{d}{dr}W_{3/2}(r,u) \geq \frac{1}{r}\int_{\partial B_1}\left(\nabla u_r\cdot\nu - \frac{3}{2}u_r\right)^2 d\mathcal{H}^{n-1} > 0.$$

4.5. **Rescaled profiles.** In order to be able to use the computations above in the study of the property of the free boundary, we need to consider the following limiting profiles of our solution u: under the assumption that $0 \in \Gamma_{3/2}(u)$ we set

$$u_r(x) := \frac{u(rx)}{r^{3/2}} \qquad (4.18)$$

Note that the rescalings in (4.18) are different from those considered in the literature, e.g. in [3].

In order to deduce the existence of nontrivial blowups under the rescaling in (4.18) we need to prove some growth estimates on the solution to the Signorini problem.

A first consequence of (4.1) is that the rescaled profiles u_r have equi-bounded Dirichlet energies:

$$\int_{B_1} |\nabla u_r|^2\,dx = \frac{\int_{B_r} |\nabla u|^2\,dx}{r^{n+1}} = \frac{r\int_{B_r} |\nabla u|^2\,dx}{\int_{\partial B_r} u^2\,d\mathcal{H}^{n-1}}\,\frac{\int_{\partial B_r} u^2\,d\mathcal{H}^{n-1}}{r^{n+2}}$$

$$\stackrel{(4.1)}{\leq} N(r,u)\,H(1) \leq N(1,u)\,H(1). \tag{4.19}$$

Therefore, for every infinitesimal sequence of radii $r_k \downarrow 0$ there exists a subsequence $r_{k'} \downarrow 0$ such that $u_{r_{k'}} \to u_0$ in $L^2(B_1)$.

Note however that (4.2) is not enough to deduce that there exists a limiting function u_0 which is not identically 0. As an application of the epiperimetric inequality and the related decay of the energy in Proposition 4.3 we can deduce that this is actually the case for every such limiting profiles u_0.

4.6. Proposition (Nondegeneracy). *Let $u \in H^1(B_1)$ be a solution to the Signorini problem and assume that $0 \in \Gamma_{3/2}(u)$. Then there exists a constant $H_0 > 0$ such that*

$$H(r) \geq H_0\,r^{n+2} \quad \forall\, 0 < r < 1. \tag{4.20}$$

Proof. The starting point is the computation of the derivative in (A.4):

$$\frac{d}{dr}\left(\log\frac{H(r)}{r^{n+2}}\right) = 2\frac{D(r)}{H(r)} - \frac{3}{r} = \frac{2\,r^{n+1}}{H(r)}W_{3/2}(r,u). \tag{4.21}$$

Let $\gamma > 0$ be the constant in Proposition 4.3 and let $\varepsilon = \gamma/2$ in Lemma A.2. Then by using (4.11) and (4.2) in conjunction we infer from (4.21) that there exists a constant $C = C(\gamma) > 0$ such that

$$\frac{d}{dr}\left(\log\frac{H(r)}{r^{n+2}}\right) \leq C\,r^{\gamma/2-1} \quad \forall\, 0 < r < r_0, \tag{4.22}$$

where $r_0 = r_0(\varepsilon) > 0$ is the constant given by Lemma A.2. Integrating this differential inequality we get that the function

$$\frac{H(r)}{r^{n+2}e^{\frac{2C}{\gamma}r^{\gamma/2}}}$$

is nonincreasing. In particular we conclude that

$$\lim_{r\to 0}\frac{H(r)}{r^{n+2}e^{Cr^{\gamma/2}}} = \lim_{r\to 0}\frac{H(r)}{r^{n+2}} =: H_0 \tag{4.23}$$

and for sufficiently small $r > 0$

$$H_0 \geq \frac{H(r)}{r^{n+2}e^{\frac{2C}{\gamma}r^{\gamma/2}}} \geq \frac{H(r)}{r^{n+2}}(1 - \frac{2C}{\gamma}r^{\gamma/2}) > 0 \tag{4.24}$$

where we used the elementary inequality $e^x \leq \frac{1}{1-x}$ for $x > 0$ sufficiently small. By the monotonicity of the function $\frac{H(r)}{r^{n+2}}$ proven in Lemma A.2 we then conclude. □

Note now that by (4.20) we then deduce that

$$\int_{\partial B_1} u_r^2\,d\mathcal{H}^{n-1} \geq H_0.$$

Therefore, since from (4.19) we also deduce the convergence of the traces of u_r on ∂B_1, we finally get that

$$\int_{\partial B_1} u_0^2 \, d\mathcal{H}^{n-1} \geq H_0 > 0$$

for every limiting profile u_0, thus showing that $u_0 \not\equiv 0$.

4.7. **Uniqueness of blowups.** By compactness for every $x_0 \in \Gamma_{3/2}(u)$ and for every infinitesimal sequence $r_k \downarrow 0$ there exists at least a subsequence (in the sequel not relabeled) such that $u_{r_k}^{x_0} \to u_0^{x_0}$ in the weak topology of $H^1(B_1)$ for some $u_0^{x_0} \in H^1(B_1)$ nontrivial function. These limiting functions $u_0^{x_0}$ are called in the sequel *blowups* of u at the point x_0

It is very simple to show that $u_0^{x_0}$ are solutions to the Signorini problem. Moreover $u_0^{x_0}$ is $3/2$-homogeneous, i.e. $x \cdot \nabla u_0^{x_0} - 3u_0^{x_0}/2 \equiv 0$. Therefore, from the classification result in [3, Theorem 3] we infer that $u_0^{x_0} \in \mathscr{H}_{3/2}$.

A key ingredient of the analysis of the free boundary we are going to perform is to show that

(i) the blowup u_0 to a solution of the Signorini problem is actually *unique*, meaning that the *whole* sequence $u_r \to u_0$ in $L^2(B_1)$ as $r \to 0$,

(ii) there is a rate of convergence of the rescaled profiles to unique limiting blowup.

This is now an easy consequence of the epiperimetric inequality and it is shown in the next proposition.

4.8. Proposition. *Let u be a solution to the Signorini problem and $K \subset\subset B_1'$. Then there exist a constant $C > 0$ such that for every $x_0 \in \Gamma_{3/2}(u) \cap K$*

$$\int_{\partial B_1} |u_r^{x_0} - u_0^{x_0}| \, d\mathcal{H}^{n-1} \leq C\, r^{\gamma/2} \quad \text{for all } \; 0 < r < \operatorname{dist}(K, \partial B_1), \tag{4.25}$$

where $\gamma > 0$ is the constant in Proposition 4.3. In particular, the blow-up limit u_0 at x_0 is unique.

PROOF. Arguing as in the proof of Proposition 4.3 it is enough to show (4.25) for $0 \in \Gamma_{3/2}(u)$ and for a constant $C > 0$ which depend only on the L^2 norm of u and on its Dirichlet energy.

Let $0 < s < r < r_0$ be fixed radii with r_0 the constant in (4.4) such that the epiperimetric inequality can be applied. We can then use the formula (4.15) to compute as follows:

$$\begin{aligned}
\int_{\partial B_1} |u_r - u_s| \, d\mathcal{H}^{n-1} &\leq \int_{\partial B_1} \int_s^r t^{-1} \Big| \nabla u_t \cdot \nu - \frac{3}{2} u_t \Big| \, dt \, d\mathcal{H}^{n-1} \\
&\leq \sqrt{n\, \omega_n} \int_s^r t^{-1/2} \left(t^{-1} \int_{\partial B_1} \Big| \nabla u_t \cdot \nu - \frac{3}{2} u_t \Big|^2 d\mathcal{H}^{n-1} \right)^{1/2} dt \\
&\overset{(4.15)}{\leq} \sqrt{n\, \omega_n} \int_s^r t^{-1/2} \Big(\frac{d}{dt} W_{3/2}(t, u) \Big)^{1/2} dt \\
&\leq \sqrt{n\, \omega_n} \, \log\big(\frac{r}{s}\big) \big(W_{3/2}(r, u) - W_{3/2}(s, u) \big)^{1/2}.
\end{aligned} \tag{4.26}$$

By (4.11) and a simple dyadic argument (applying (4.26) to $s = r/2 = 2^{-k}$ for $k \in \mathbb{N}$ sufficiently large) we easily deduce that for every $0 < s < r < r_0$

$$\int_{\partial B_1} |u_r - u_s| \, d\mathcal{H}^{n-1} \leq C\, r^{\gamma/2}$$

for a constant $C > 0$ which in turn depends only on the constants in Proposition 4.3. Sending s to 0 and eventually changing the value of the constant C, we then conclude the proof of (4.25). □

4.9. $C^{1,\alpha}$ **regularity of the free boundary** $\Gamma_{3/2}$. In view of the uniqueness result in Proposition 4.8 we are in the position to give a new proof of the $C^{1,\alpha}$ regularity of the part of the free boundary with least frequency.

4.10. Proposition. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then there exists a dimensional constant $\alpha > 0$ such $\Gamma_{3/2}(u)$ is locally in B_1' a $C^{1,\alpha}$ regular submanifold of dimension $n-2$.*

PROOF. Without loss of generality it is enough to prove that if $0 \in \Gamma_{3/2}(u)$ then $\Gamma_{3/2}(u)$ is a $C^{1,\alpha}$ submanifold in a neighborhood of 0. To this aim we start noticing that by the openness of $\Gamma_{3/2}(u)$ there exists $s > 0$ such that $B_s \cap \Gamma(u) = B_s \cap \Gamma_{3/2}(u)$. Since for every $x_0 \in B_s \cap \Gamma_{3/2}(u)$ the unique blowup of the rescaled functions (4.18) is of the form

$$u^{x_0} = \lambda_{x_0} h_{e(x_0)} \in \mathscr{H}^{3/2}$$

for some $\lambda_{x_0} > 0$ and $e(x_0) \in \mathbb{S}^{n-2}$.

We first prove the Hölder continuity of $x_0 \mapsto \lambda_{x_0}$. To this aim we start observing that, thanks to Proposition 4.3 and Proposition 4.6 we can further estimate (4.21) in the following way

$$\frac{d}{dr}\left(\log \frac{H^{x_0}(r)}{r^{n+2}}\right) = \frac{2\, r^{n+1}}{H^{x_0}(r)} W^{x_0}_{3/2}(r,u) \le C\, r^{\gamma-1} \quad \forall r \in (0,1). \tag{4.27}$$

Notice that by the strong convergence in $L^2(B_1)$ of the rescaled functions it follows that

$$\lambda^2_{x_0} = c_0 \lim_{r\to 0} \frac{H^{x_0}(r)}{r^{n+2}}$$

for some *dimensional* constant $c_0 > 0$. Integrating (4.27) we can then deduce that

$$c_0\, \frac{H^{x_0}(r)}{r^{n+2}} - \lambda^2_{x_0} \le C\, r^{\gamma} \quad \forall\, r \in (0,1). \tag{4.28}$$

Notice moreover that for $x_0, y_0 \in B_s \cap \Gamma_{3/2}(u)$ and $r = |x_0 - y_0|^{1-\theta}$ with $\theta = \gamma/(1+\gamma)$ it holds that

$$\begin{aligned}\int_{\partial B_1} &|u^{x_0}_r - u^{y_0}_r|\, d\mathcal{H}^{n-1} \\ &\le r^{-3/2} \int_{\partial B_1}\int_0^1 \big|\nabla u\big(s(x_0+rx)+(1-s)(y_0+rx)\big)\big|\, |y_0 - x_0|\, ds\, d\mathcal{H}^{n-1}(x) \\ &\le C r^{-1}\, |y_0 - x_0| \le C\, |y_0 - x_0|^{\theta}.\end{aligned} \tag{4.29}$$

Therefore we can conclude that for $r = |x_0 - y_0|^{1-\theta}$ with $\theta = \gamma/(1+\gamma)$ it holds that

$$\begin{aligned}\big|\lambda_{x_0} - \lambda_{y_0}\big| &\le \Big|\lambda_{x_0} - c_0 \frac{H^{x_0}(r)}{r^{n+2}}\Big| + c_0 \Big|\frac{H^{x_0}(r)}{r^{n+2}} - \frac{H^{y_0}(r)}{r^{n+2}}\Big| + \Big|c_0 \frac{H^{y_0}(r)}{r^{n+2}} - \lambda_{y_0}\Big| \\ &\le C\, r^{\gamma} + C \int_{\partial B_1} \big|(u^{x_0}_r)^2 - (u^{y_0}_r)^2\big|\, d\mathcal{H}^{n-1} \\ &\le C\, r^{\gamma} + C \int_{\partial B_1} \big|u^{x_0}_r - u^{y_0}_r\big|\, d\mathcal{H}^{n-1} \le C\, r^{\theta}\end{aligned} \tag{4.30}$$

where we used the uniform L^∞ (actually $C^{1,1/2}$) bound on $u_r^{x_0}$ for every $x_0 \in \Gamma_{3/2}(u) \cap B_s$ (cp. Appendix A for more details).

By Proposition 4.8 and a similar computation we can show that

$$\int_{\partial B_1} |u_0^{x_0} - u_0^{y_0}| \, d\mathcal{H}^{n-1} \leq \int_{\partial B_1} |u_0^{x_0} - u_r^{x_0}| \, d\mathcal{H}^{n-1} + \int_{\partial B_1} |u_r^{x_0} - u_r^{y_0}| \, d\mathcal{H}^{n-1}$$
$$+ \int_{\partial B_1} |u_r^{y_0} - u_0^{y_0}| \, d\mathcal{H}^{n-1}$$
$$\stackrel{(4.25)\,\&\,(4.29)}{\leq} C\, r^{\gamma/2} + C\, |x_0 - y_0|^\theta \leq C\, |x_0 - y_0|^{\gamma\theta/2} \tag{4.31}$$

Note finally that there exists a geometric constant $\bar{C} > 0$ such that

$$|e(x_0) - e(y_0)| \leq \bar{C} \int_{\partial B_1} \left| h_{e(x_0)} - h_{e(y_0)} \right| \, d\mathcal{H}^{n-1}.$$

Therefore from (4.30) and (4.31) we easily deduce that

$$|e(x_0) - e(y_0)| \leq C\, |x_0 - y_0|^{\gamma\theta/2}. \tag{4.32}$$

Next we show that the vectors $e(x_0)$ do actually encode a geometric property of the free boundary. To this aim we introduce the following notation for cones centered at points $x_0 \in \Gamma_{3/2}(u)$: for any $\varepsilon > 0$ we set

$$C^\pm(x_0, \varepsilon) := \left\{ x \in \mathbb{R}^{n-1} \times \{0\} \, : \, \pm\langle x - x_0, e(x_0) \rangle \geq \varepsilon |x - x_0| \right\}.$$

The main claim is then the following: for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x_0 \in \Gamma_{3/2}(u) \cap B_{s/2}$,

$$u > 0 \quad \text{on} \quad C^+(x_0, \varepsilon) \cap B_\delta(x_0), \tag{4.33}$$
$$u = 0 \quad \text{on} \quad C^-(x_0, \varepsilon) \cap B_\delta(x_0). \tag{4.34}$$

For what concerns (4.33) assume by contradiction that there exist $x_j \in \Gamma_{3/2}(u) \cap B_{s/2}$ with $x_j \to x_0 \in \Gamma_{3/2}(u) \cap \bar{B}_{s/2}$, and $y_j \in C^+(x_j, \varepsilon)$ with $y_j - x_j \to 0$ such that $u(y_j) = 0$.

By the $C^{1,1/2}$ regularity of the solution, (4.25) and (4.31), the rescalings $u_{r_j}^{x_j}$ with $r_j := |y_j - x_j|$, converge uniformly to $u_0^{x_0}$. Up to subsequences, by the Hölder continuity of the normals proved in (4.32) we can assume that $r_j^{-1}(y_j - x_j) \to z \in C^+(x_0, \varepsilon) \cap \mathbb{S}^{n-1}$ and by uniform convergence (cp. the Appendix A) $u_0^{x_0}(z) = 0$. This contradicts the fact that $x_0 \in \Gamma_{3/2}(u)$ and $u_0^{x_0} > 0$ on $C^+(x_0, \varepsilon)$. Clearly, the proof of (4.34) is at all analogous and the details are left to the readers.

We can now easily conclude that $\Gamma_{3/2}(u) \cap B_{s/2}$ is the graph of a function g, for a suitably chosen small $\rho_1 > 0$. Without loss of generality assume that $e(0) = e_{n-1}$ and set

$$g(x') := \max \left\{ t \in \mathbb{R} \, : \, (x', t, 0) \in \Lambda(u) \right\}$$

for all points $x' \in \mathbb{R}^{n-2}$ with $|x'| \leq \delta\sqrt{1 - \varepsilon^2}$. Note that by (4.33) and (4.34) this maximum exists and belongs to $[-\varepsilon\delta, \varepsilon\delta]$. Moreover $u(x', t, 0) = 0$ for every $-\varepsilon\,\delta < t < g(x')$ and $u(x', t, 0) > 0$ for every $g(x') < t < \varepsilon\,\delta$. Eventually, by applying (4.33) and (4.34) with respect to arbitrary ε, we deduce that g is differentiable and in view of (4.32) we can conclude that g is $C^{1,\alpha}$ regular for a suitable $\alpha > 0$. □

APPENDIX A.

In this section we recall few known results concerning the solutions to the Signorini problem, which are mainly contained in the references [2, 3, 10, 14].

A.1. **Frequency function.** We recall the definition of frequency function: for $x_0 \in \Gamma(u)$ and $0 < r < 1 - |x_0|$

$$N^{x_0}(r,u) := \frac{r\int_{B_r(x_0)} |\nabla u|^2\,dx}{\int_{\partial B_r(x_0)} u^2\,d\mathcal{H}^{n-1}},$$

if $u|_{\partial B_r(x_0)} \not\equiv 0$ (note that $u|_{\partial B_r(x_0)} \equiv 0$ if and only if $u|_{B_1} \equiv 0$).

As proven in [3] the function $(0, \mathrm{dist}(x_0, \partial B_1)) \ni r \mapsto N^{x_0}(r,u)$ is nondecreasing for every $x_0 \in B_1'$. The proof of this statement is an immediate consequence of (4.8), (4.9) and (4.10): without loss of generality we can assume $x_0 = 0$ and compute

$$\begin{aligned}\frac{N'(r,u)}{N(r,u)} &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\\ &= 2\left(\frac{\int_{\partial B_r}(\nabla u\cdot\nu)^2\,d\mathcal{H}^{n-1}}{\int_{\partial B_r} u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}} - \frac{\int_{\partial B_r} u\nabla u\cdot\nu\,d\mathcal{H}^{n-1}}{\int_{\partial B_r} u^2\,d\mathcal{H}^{n-1}}\right) \geq 0,\end{aligned} \tag{A.1}$$

where the last inequality follows from the Cauchy–Schwartz inequality. Since infimum of a sequence of decreasing functions, the map $x_0 \mapsto N^{x_0}(0^+,u)$ turns then out to be upper-semicontinuous.

Analyzing the case of equality in (A.1) is important for later applications: it is clear from the Cauchy–Schwartz inequality that $N(r,u) \equiv \kappa$ for some $\kappa \in \mathbb{R}$ if and only if u is κ-homogeneous, i.e. $x \cdot \nabla u - \kappa\, u \equiv 0$.

As a consequence of the monotonicity of the frequency function we also can prove the estimates (4.1) and (4.2), that we restate for readers' convenience.

A.2. Lemma. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem and assume that $0 \in \Gamma_{3/2}(u)$. Then the function $(0,1) \ni r \mapsto \frac{H(r)}{r^{n+2}}$ is nondecreasing and in particular*

$$H(r) \leq H(1)\, r^{n+2} \quad \forall\, 0 < r < 1, \tag{A.2}$$

and for every $\varepsilon > 0$ there exists $r_0(\varepsilon) > 0$ such that

$$H(r) \geq \frac{H(r_0)}{r_0^{n+2+\varepsilon}}\, r^{n+2+\varepsilon} \quad \forall\, 0 < r < r_0. \tag{A.3}$$

PROOF. We start computing the following derivative:

$$\frac{d}{dr}\left(\log\frac{H(r)}{r^{n+2}}\right) = 2\frac{D(r)}{H(r)} - \frac{3}{r}. \tag{A.4}$$

As an immediate consequence of the monotonicity of the frequency $N(r,u) \geq N(0^+,u) = 3/2$ we deduce that $\frac{d}{dr}\left(\log\frac{H(r)}{r^{n+2}}\right) \geq 0$ from which (4.1) clearly follows.

Similarly, by the monotonicity of the frequency function, for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) > 0$ such that

$$N(r,u) \leq N(0^+,u) + \frac{\varepsilon}{2} = \frac{3+\varepsilon}{2} \quad \forall\, 0 < r < r_0.$$

Therefore we infer from (A.4) that

$$\frac{d}{dr}\left(\log\frac{H(r)}{r^{n+2}}\right) = \frac{2}{r}\left(N(r,u) - \frac{3}{2}\right) \leq \frac{\varepsilon}{r} \quad \forall\, 0 < r < r_0,$$

and integrating this differential inequality (4.2) follows at once. □

A.3. **Optimal regularity.** Finally we recall the optimal regularity for the solution to the Signorini problem proven in [2].

A.4. Theorem. *Let $u \in H^1(B_1)$ be a solution to the Signorini problem. Then, $u \in C^{1,1/2}(B^+_{1/2})$ and there exists a dimensional constant $C > 0$ such that*

$$\|u\|_{C^{1,1/2}(B_1^+)} \leq C\,\|u\|_{L^2(B_1)}. \tag{A.5}$$

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