ENERGY RELEASE RATE AND QUASI-STATIC EVOLUTION VIA VANISHING VISCOSITY IN A FRACTURE MODEL DEPENDING ON THE CRACK OPENING

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Abstract. In the setting of planar linearized elasticity, we study a fracture model depending on the crack opening. Assuming that the crack path is known a priori and sufficiently smooth, we prove that the energy release rate is well defined. Then, we consider the problem of quasi-static evolution for our model. Thanks to a vanishing viscosity approach, we show the existence of such an evolution satisfying a weak Griffith's criterion.

Keywords: variational models, free-discontinuity problems, energy release rate, energy derivative, cohesive fracture, crack propagation, quasi-static evolution, local minimizers, Griffith's criterion, vanishing viscosity.

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1. Introduction

Griffith's criterion is a well-established principle which predicts in a quasi-static setting whether or not a pre-existing crack in an elastic body grows for a given external force, [15]. If we assume that the fracture evolves only along a prescribed smooth path Σ, so that it can be parametrized by the arc-length s, we are able to state the Griffith's criterion in terms of the energy release rate, which is the negative of the right derivative of the deformation energy with respect to the crack extension, i.e., the parameter s: If the energy release rate is less than a certain constant related to the toughness of the material, then the crack is stable, otherwise it will grow. This principle has been studied in several papers, see e.g. [18, 19, 20, 26, 28] for the case of prescribed crack path, and [21, 22] for a more general setting in linearized antiplane elasticity. The cited works tackle the problem of existence of a quasi-static evolution in brittle fracture satisfying a weak form of the Griffith's criterion.

In this paper, we are interested in the application of the Griffith's criterion to a problem of quasi-static cohesive crack growth in the setting of planar linearized elasticity. We consider a linearly elastic body $\overline{\Omega}$, where $\Omega \subseteq \mathbb{R}^2$ is an open, bounded, connected set with Lipschitz boundary $\partial\Omega$, and a simple C^3-curve Σ which represents the prescribed crack path. Let $L := \mathcal{H}^1(\Sigma)$ and $\gamma\colon [0, L] \to \Sigma$ be its arc-length parametrization. The admissible fractures are of the form

$$\Gamma_s := \{\gamma(\sigma) : 0 \le \sigma \le s\}$$

for $s \in [0, L]$. We set also $\Omega_s := \Omega \setminus \Gamma_s$.

The main feature of the Barenblatt's cohesive model, see e.g [3, 4], is the presence of the so-called cohesive forces acting on the fracture lips. In the mathematical model, the density of the energy spent by the cohesive forces is represented by a function $\varphi\colon [0, +\infty) \to [0, +\infty)$

which depends, in its simplest form, only on the modulus of the jump of the displacement across Σ. In general, φ satisfies:

$$\begin{gathered}\varphi \text{ concave},\\ \varphi(0)=0, \quad \varphi'(0)=\mu<+\infty\,,\\ \lim_{\zeta\to+\infty}\varphi(\zeta)=\kappa<+\infty\,,\\ \varphi(\zeta)\le\kappa\,.\end{gathered} \tag{1.1}$$

In our model, we do not need all these hypotheses. Indeed, given $T>0$, we consider a C^1 function $\varphi\colon[0,T]\times\mathbb{R}^2\to\mathbb{R}$ such that $\varphi(t,0)=0$ and $\varphi(t,\xi)\le c\,(1+|\xi|^p)$ for some $c>0$ and some $p\in(1,+\infty)$, see Section 2 for the precise assumptions. In particular, φ could be time dependent and negative. Thus, with our model we are able to discuss also the case of an external time dependent force $h:[0,T]\to\mathbb{R}^2$ acting on both the fracture lips. In this case, $\varphi(t,\xi):=-h(t)\cdot\xi$.

Different from the Barenblatt's model, we assume, as in [5], that the energy spent by the cohesive forces is completely reversible. Moreover, we introduce a dissipative surface term proportional to the crack length, namely $\mathrm{G}_0 s$, where G_0 is a positive constant related to the physical properties of the material. This additional contibution can be interpreted as an activation threshold, i.e., as the energy required to break the inter-atomic bonds along the fracture. For simplicity, we will set $\mathrm{G}_0:=1$.

We stress that the coexistence of a cohesive term and of an activation threshold has been noticed in several papers related to fracture mechanics: in [12] in the approximation of fracture models via Γ-convergence of Ambrosio-Tortorelli type functionals, in [2, 10] in the study of the asymptotic behavior of composite materials through a homogenization procedure, and in [7, 17] in the framework of fracture models as Γ-limits of damage models.

We are now ready to introduce the total energy of the system. Let $f\colon[0,T]\to L^2(\Omega;\mathbb{R}^2)$ and $w\colon[0,T]\to H^1(\Omega;\mathbb{R}^2)$ denote the volume forces and the Dirichlet boundary datum, respectively. For every $t\in[0,T]$, every $s\in[0,L]$, and every displacement $u\in H^1(\Omega_s;\mathbb{R}^2)$, we define

$$\mathcal{E}(t,s,u):=\frac{1}{2}\int_{\Omega_s}\mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x-\int_{\Omega_s}f(t)\cdot u\,\mathrm{d}x+\int_{\Gamma_s}\varphi(t,[u])\,\mathrm{d}\mathcal{H}^1+s\,, \tag{1.2}$$

where $\mathbb{C}$ is the usual elasticity tensor, $\mathrm{E}u$ stands for the symmetric part of the gradient of u, and $[u]$ denotes the jump of u across Σ.

Hence, the total energy (1.2) is the sum of four terms. The first two volume contributions are the *stored elastic energy* and the power spent by the body forces acting on $\overline{\Omega}$, respectively. The third integral in (1.2) represents the energy spent by the forces acting on the fracture lips, and the last term is the activation threshold of which we have already discussed.

We now describe the main features of the evolutive problem. For $t\in[0,T]$ and $s\in[0,L]$, we define the reduced energy:

$$\mathcal{E}_{min}(t,s):=\min\{\mathcal{E}(t,s,u):\,u\in H^1(\Omega_s,\mathbb{R}^2),\,u=w(t)\text{ on }\partial\Omega\}\,. \tag{1.3}$$

In order to give a definition of quasi-static evolution for our cohesive fracture model via Griffith's criterion, we first have to study the differentiability of $\mathcal{E}_{min}$ with respect to the crack length s. To this end, we notice that because of the non-convexity of $\varphi(t,\cdot)$, the solution to the minimum problem (1.3) is not unique. This will affect the computation of the derivative of the reduced energy $\mathcal{E}_{min}$ with respect to s. Indeed, in Section 3 we show that in general $\mathcal{E}_{min}$ is not differentiable in s. However, we can still compute its right and left derivatives $\partial_s^+\mathcal{E}_{min}$ and $\partial_s^-\mathcal{E}_{min}$, see Theorems 3.1 and 3.2. In particular, we are in a situation different from [19, 28], where the reduced energy is differentiable and has a continuous derivative, and similar in this aspect to [18, 20], where finite-strain elasticity in brittle fracture is considered. In Proposition 3.11 we prove that the two derivatives $\partial_s^+\mathcal{E}_{min}$ and $\partial_s^-\mathcal{E}_{min}$ satisfy a semicontinuity property which will play a central role in the proof of existence of a quasi-static evolution for the cohesive crack growth problem, see Definition 4.7 and the proof of Theorem 4.9.

Let us emphasize the main difference between a quasi-static evolution via global stability, proposed in [13] for the fracture growth, and an evolution via Griffith's principle. Roughly speaking, the former says that an evolution $s(t)$ has to be globally stable, that is, has to satisfy

$$\mathcal{E}_{min}(t, s(t)) \leq \mathcal{E}_{min}(t, s) \qquad \text{for every } s \geq s(t) \text{ and every } t \in [0, T]\,. \tag{1.4}$$

In particular, condition (1.4) is derivative free. Therefore, it allows for the presence of jump discontinuities: for instance, $s(t)$ could jump instantaneously from a stable configuration to another passing through an energetic barrier. This is a typical situation because the function

$$s \mapsto \mathcal{E}_{min}(t, s)$$

is non-convex. On the contrary, the definition of quasi-static evolution via Griffith's principle imposes a condition on the energy release rate. In our setting, we will have some requirements on $\partial_s^+ \mathcal{E}_{min}$ and $\partial_s^- \mathcal{E}_{min}$ (see Definition 4.7). Since $\partial_s^\pm \mathcal{E}_{min}$ are the right and left derivatives of the reduced energy with respect to the crack length s, the Griffith's criterion represents a sort of differential condition on the evolution $s(t)$. Therefore, we should obtain a more regular solution or, at least, a more physical one, i.e., an evolution which jumps later than a globally stable one.

In order to get a quasi-static evolution satisfying a weak version of the Griffith's principle, in Sections 4-7 we tackle the evolutive problem by means of vanishing viscosity. This procedure has been studied for instance in [1, 11, 24, 25] in an abstract setting. It consists in the perturbation of minimum problems with a viscosity term driven by a small positive parameter ε, enforcing a local minimality of the solutions. Let us briefly discuss how we exploit this technique. Given a subdivision $\{t_i^k\}_{i=0}^k$ of the time interval $[0, T]$, we consider, for $i \geq 1$, the incremental minimum problem

$$\min \left\{ \mathcal{E}_{min}(t_i^k, s) + \frac{\varepsilon}{2} \, \frac{(s - s_\varepsilon^{k,i-1})^2}{t_i^k - t_{i-1}^k} \, : \, s \geq s_\varepsilon^{k,i-1} \right\}, \tag{1.5}$$

where $s_\varepsilon^{k,i-1}$ is a solution of (1.5) at time t_{i-1}^k and $s_\varepsilon^{k,0} := s_0$, the initial condition. In (1.5), we are penalizing the distance between the new and the previous cracks with the viscosity term driven by $\varepsilon > 0$. Having constructed the discrete time solutions for every $\varepsilon > 0$, the scheme is to pass to the limit as $k \to +\infty$, in order to find the so-called *viscous evolution* s_ε (Theorem 4.6), and, finally, let ε tend to zero. In this way, we will obtain a *quasi-static evolution* for the cohesive fracture problem (Theorem 4.9).

We notice that this kind of vanishing viscosity approach to the cohesive fracture is a novelty. Indeed, the cohesive crack growth problem, without activation cost, has been investigated in previous works, see e.g. [5, 6, 9]. In [6, 9], the notion of quasi-static evolution is based on global stability and the proof of existence is addressed via the time discretization process introduced by Francfort and Marigo in the field of fracture mechanics [13], and frequently used in the study of rate-independent processes [23]. In [5], following the ideas of [8], the vanishing viscosity approach is applied with an L^2-penalization of the displacement.

The plan of the paper is the following: in Section 2 we discuss the setting of the problem and the notation which will be used throughout the paper. In Section 3 we compute the right and left derivatives of the reduced energy $\mathcal{E}_{min}$ with respect to the crack length s, see Theorems 3.1 and 3.2. In Section 4 we give the definitions of viscous and quasi-static evolutions for the cohesive crack growth problem, see Definitions 4.5 and 4.7, and state the results of existence of such evolutions in Theorems 4.6 and 4.9. These Theorems will be proved in Sections 5-7.

Finally, in Sections 8-9, we generalize the previous results to the case of many non-interacting cracks, in the spirit of [22]. In order to get the same properties of Definition 4.7, we will use the notion of parametrized solution introduced in [25].

2. Setting of the problem

In this section, we introduce the notation which will be used later on and describe the main features of the problem we will discuss in the following sections.

We consider a model in planar linearized elasticity. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, connected, open set with Lipschitz boundary $\partial\Omega$. The reference configuration is its closure $\overline{\Omega}$, which represents a linearly elastic body at rest.

The prescribed crack path is given by a simple C^3-curve $\Sigma \subseteq \overline{\Omega}$ with $\mathcal{H}^1(\Sigma) =: L$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Let $\gamma \in C^3([0,L];\Sigma)$ be its arc-length parametrization and ν, τ be its unit normal and unit tangent vectors, respectively. We make the following assumptions on the geometry of the model:

- $\partial\Omega \cap \Sigma = \{\gamma(0), \gamma(L)\}$;
- $\Omega \setminus \Sigma = \Omega^+ \cup \Omega^-$, where Ω^+, Ω^- are two connected open subsets of $\mathbb{R}^2$ with Lipschitz boundary, defined according to the orientation of the normal vector ν, with $\Omega^+ \cap \Omega^- = \emptyset$.

By the regularity assumptions on Ω, Ω^+, and Ω^-, the trace operators $tr\colon H^1(\Omega;\mathbb{R}^2) \to H^{1/2}(\partial\Omega;\mathbb{R}^2)$, $tr_\pm\colon H^1(\Omega^\pm,\mathbb{R}^2) \to H^{1/2}(\partial\Omega^\pm;\mathbb{R}^2)$ are well defined and continuous. In particular, for every $v \in H^1(\Omega \setminus \Sigma;\mathbb{R}^2)$ we can define its jump across Σ by

$$[v] := tr_+(v)|_\Sigma - tr_-(v)|_\Sigma \ \in H^{1/2}(\Sigma;\mathbb{R}^2).$$

We recall that the embeddings $H^{1/2}(\Sigma;\mathbb{R}^2) \hookrightarrow L^p(\Sigma;\mathbb{R}^2)$ are compact for every $p \in [1,+\infty)$.

For simplicity, we assume that the family of admissible fractures is given by the set

$$\{\Gamma_s : s \in [0,L]\}, \tag{2.1}$$

where, for every $s \in [0,L]$, we define

$$\Gamma_s := \{\gamma(\sigma) : 0 \leq \sigma \leq s\}.$$

This means that the set in (2.1) can be parametrized by the arc-length $s \in [0,L]$. Moreover, with this choice of admissible cracks, we are assuming that all possible fractures are closed and connected subsets of Σ, with a common starting point $\gamma(0) \in \partial\Omega$. In particular, a crack Γ_s may extend only from its end point $\gamma(s)$.

For $s \in [0,L]$ we define $\Omega_s := \Omega \setminus \Gamma_s$ and denote by $H^1(\Omega_s;\mathbb{R}^2)$ the set

$$\{u \in H^1(\Omega \setminus \Sigma;\mathbb{R}^2) : [u] = 0 \ \mathcal{H}^1\text{-a.e. on } \Sigma \setminus \Gamma_s\}.$$

From now on, we will drop the $\mathbb{R}^2$ in the definition of the function spaces, when it is clear that we are dealing with vector-valued functions.

The body outside the crack is supposed to be linearly elastic, with elasticity tensor $\mathbb{C}$. In general, $\mathbb{C}$ is a function of the space variable $x \in \Omega \setminus \Sigma$. For technical reasons, it is assumed to be C^1 with bounded derivative. In particular, the linear function $\mathbb{C}(x)\colon \mathbb{M}^2_{sym} \to \mathbb{M}^2_{sym}$ is defined for every $x \in \Omega \setminus \Sigma$, where $\mathbb{M}^2_{sym}$ is the space of 2×2 symmetric matrices with real coefficients. As usual, we suppose that $\mathbb{C}$ is positive definite, uniformly with respect to $x \in \Omega \setminus \Sigma$, i.e., there exist $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha|\mathrm{F}|^2 \leq \mathbb{C}(x)\mathrm{F}\cdot\mathrm{F} \leq \beta|\mathrm{F}|^2 \qquad \text{for every } \mathrm{F} \in \mathbb{M}^2_{sym} \text{ and every } x \in \Omega \setminus \Sigma, \tag{2.2}$$

where the dot denotes the scalar product between matrices. We notice that we can also think to $\mathbb{C}(x)$ as a tensor acting on the whole $\mathbb{M}^2$, the space of 2×2 matrices with real coefficients. Thanks to the symmetries of $\mathbb{C}$, see e.g. [16], we have

$$\mathbb{C}(x)\mathrm{F} = 0_{\mathbb{M}^2}$$

for every $\mathrm{F} \in \mathbb{M}^2$ skew-symmetric and every $x \in \Omega \setminus \Sigma$. For simplicity of notation, from now on we will not specify the dependence on $x \in \Omega$ of the elasticity tensor.

Given $T > 0$, we consider a function $g\colon [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ with the following properties:

- $t \mapsto g(t,x,\xi)$ is continuous for every $\xi \in \mathbb{R}^2$ and a.e. $x \in \Omega$;
- $x \mapsto g(t,x,\xi)$ is measurable for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$;
- $\xi \mapsto g(t,x,\xi)$ is $C^1(\mathbb{R}^2)$ for every $t \in [0,T]$ and a.e. $x \in \Omega$;
- $t \mapsto D_\xi g(t,x,\xi)$ is continuous for every $\xi \in \mathbb{R}^2$ and a.e. $x \in \Omega$;
- $x \mapsto D_\xi g(t,x,\xi)$ is measurable for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$;
- for every $\varepsilon > 0$, there exists $a_\varepsilon > 0$ such that

$$|g(t,x,\xi)| \leq a_\varepsilon + \varepsilon |\xi|^2 \tag{2.3}$$

  for a.e. $x \in \Omega$, every $t \in [0,T]$, and every $\xi \in \mathbb{R}^2$.
- there exists $a_1 > 0$ such that

$$|D_\xi g(t,x,\xi)| \leq a_1(1 + |\xi|) \tag{2.4}$$

  for a.e. $x \in \Omega$, every $t \in [0,T]$, and every $\xi \in \mathbb{R}^2$.

Remark 2.1. We point out that the function g is a nonlinear generalization of the power spent by the volume forces. Indeed, in Section 4 we will set

$$g(t,x,\xi) := f(t,x) \cdot \xi\,, \tag{2.5}$$

where $f \in AC([0,T]; L^2(\Omega))$, the space of absolutely continuous functions from $[0,T]$ with values in $L^2(\Omega)$. The function f will represent the body forces applied on Ω. In particular, g as in (2.5) satisfies all the properties previously listed.

Finally, we introduce a function $\varphi\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ such that:

- $t \mapsto \varphi(t,\xi)$ is continuous for every $\xi \in \mathbb{R}^2$;
- $\xi \mapsto \varphi(t,\xi)$ is $C^1(\mathbb{R}^2)$ for every $t \in [0,T]$;
- $\varphi(t,0) = 0$ for every $t \in [0,T]$;
- there exist $p \in (1,+\infty)$ and $a_2 > 0$ such that

$$\begin{aligned} &\varphi(t,\xi) \leq a_2(1 + |\xi|^p)\,, \\ &|D_\xi \varphi(t,\xi)| \leq a_2(1 + |\xi|^{p-1}) \end{aligned} \tag{2.6}$$

  for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$;
- for every $\varepsilon > 0$, there exists $b_\varepsilon > 0$ such that

$$\varphi(t,\xi) \geq -b_\varepsilon - \varepsilon |\xi|^2 \tag{2.7}$$

  for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$.

Remark 2.2. At t fixed, the function $\varphi(t,\cdot)$ will represent the density of the energy spent by the inter-atomic forces on the crack lips. It will be concentrated on Σ and depend only on the jump of the displacement across Σ. This is typical in the model of cohesive fracture, see e.g. [4, Section 2.6]. Actually, in the cohesive model the energy density $\varphi\colon \mathbb{R} \to [0,+\infty)$ should depend only on the modulus of the jump of the displacement across Σ. Moreover, it should be monotone increasing, concave, bounded by a constant $\kappa > 0$, and satisfy

$$\begin{aligned} &\varphi(0) = 0\,, \\ &\varphi'(0) = \mu < +\infty\,, \\ &\lim_{|\xi| \to +\infty} \varphi(|\xi|) = \kappa\,. \end{aligned}$$

We notice that, for our purposes, these further hypotheses on φ are not needed.

We stress that in our model the function φ could be time dependent and negative, see (2.7). This means that we are able to discuss also the case of a given force $h\colon [0,T] \to \mathbb{R}^2$ acting on both the fracture lips, namely $\varphi(t,\xi) := -h(t) \cdot \xi$. Moreover, we anticipate that our results can be generalized with minor changes to the case of two different forces h^+ and h^- acting on the two faces of the crack.

We are now ready to define the total energy of the system which will be considered in the computation of the energy release rate and, with g as in (2.5), in the problem of quasi-static evolution as limit of viscous evolution for our cohesive model:
fixed $t \in [0,T]$, $s \in [0,L]$, and $u \in H^1(\Omega_s)$, we set

$$\mathcal{E}(t,s,u) := \frac{1}{2}\int_{\Omega_s} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \,\mathrm{d}x - \int_{\Omega_s} g(t,x,u)\,\mathrm{d}x + \int_{\Gamma_s} \varphi(t,[u])\,\mathrm{d}\mathcal{H}^1 + \mathrm{G}_0 s\,. \tag{2.8}$$

Hence, the energy is the sum of the *stored elastic energy*, a term which generalizes the power spent by the volume forces, a surface term which can be interpreted as the energy spent by the cohesive forces on the fracture Γ_s, and an activation threshold $\mathrm{G}_0 s$ proportional to the crack length which represents the energy dissipated by the process of fracture growth. We notice that, as in [5], we assume the cohesive part of the energy to be reversible. For simplicity, we will set $\mathrm{G}_0 := 1$.

Let us now briefly discuss the equilibrium condition of the system. Fix $t \in [0,T]$, $s \in [0,L]$, and the Dirichlet boundary datum $w \in H^1(\Omega)$ on $\partial\Omega$. According to the variational principles of linear elasticity, the body, with energy given by (2.8), is in equilibrium with an assigned crack Γ_s if the displacement u is a solution of the minimum problem

$$\min_{u \in A(s,w)} \mathcal{E}(t,s,u)\,, \tag{2.9}$$

where

$$A(s,w) := \{u \in H^1(\Omega_s) : \, [u]\cdot\nu \geq 0,\, u = w \text{ on } \partial\Omega\} \tag{2.10}$$

is the set of all admissible displacements associated to the crack Γ_s and the Dirichlet boundary datum w. In the previous formula, the inequality $[u]\cdot\nu \geq 0$, which is assumed to be satisfied $\mathcal{H}^1$-almost everywhere on Σ, takes into account the non-interpenetration condition, while the equality $u = w$ has to be intended in the trace sense on $\partial\Omega$.

We now state a general lemma which proves the lower semicontinuity of $\mathcal{E}$ and will be useful also in next sections.

Lemma 2.3. *Let $t_k, t \in [0,T]$, $s_k, s \in [0,L]$, $w_k, w \in H^1(\Omega)$, $u_k \in A(s_k,w_k)$ for every k, and $u \in A(s,w)$. Assume that $t_k \to t$, $s_k \to s$, $w_k \to w$ in $H^1(\Omega)$, and $u_k \rightharpoonup u$ weakly in $H^1(\Omega\setminus\Sigma)$. Then*

$$\begin{aligned} &\mathcal{E}(t,s,u) \leq \liminf_k \mathcal{E}(t_k,s_k,u_k)\,,\\ &\int_{\Gamma_s} \varphi(t,[u])\,\mathrm{d}\mathcal{H}^1 = \lim_k \int_{\Gamma_{s_k}} \varphi(t_k,[u_k])\,\mathrm{d}\mathcal{H}^1\,,\\ &\int_{\Omega_s} g(t,x,u)\,\mathrm{d}x = \lim_k \int_{\Omega_{s_k}} g(t_k,x,u_k)\,\mathrm{d}x\,. \end{aligned} \tag{2.11}$$

If, in addition, we assume that

$$\mathcal{E}(t,s,u) = \lim_k \mathcal{E}(t_k,s_k,u_k)\,, \tag{2.12}$$

then $u_k \to u$ strongly in $H^1(\Omega\setminus\Sigma)$.

Proof. By compactness, we have that $u_k \to u$ strongly in $L^p(\Omega)$ and in $L^p(\Sigma)$ for every $p \in [1,+\infty)$. Up to a subsequence, we can assume that $u_k \to u$ pointwise in Ω and on Σ.

By the continuity properties of φ and g, we have the pointwise convergences

$$\varphi(t_k,[u_k]) \to \varphi(t,[u]) \qquad \text{and} \qquad g(t_k,x,u_k) \to g(t,x,u)\,.$$

Thanks to the hypotheses (2.3), (2.6), and (2.7), applying the dominated convergence theorem we get the two equalities in (2.11). Since the stored elastic energy is lower semicontinuous, we obtain also the first inequality in (2.11).

If we assume (2.12), then, by (2.11), we deduce that

$$\int_{\Omega_s} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \,\mathrm{d}x = \lim_k \int_{\Omega_{s_k}} \mathbb{C}\mathrm{E}u_k \cdot \mathrm{E}u_k \,\mathrm{d}x\,.$$

Hence, we have that $u_k \to u$ strongly in $H^1(\Omega \setminus \Sigma)$. □

Thanks to Lemma 2.3, to the hypotheses (2.2)-(2.7), and to the application of Korn's inequality in $\Omega^\pm$, the minimum problem (2.9) admits a solution $u \in A(s,w)$. We notice that, by the lack of convexity of $\varphi(t,\cdot)$ and $g(t,x,\cdot)$, the solution to (2.9) is not unique. For simplicity of notation, we introduce the reduced energy

$$\mathcal{E}_{min}(t,s,w) := \min_{u\in A(s,w)} \mathcal{E}(t,s,u)\,. \tag{2.13}$$

The aim of Section 3 is to compute the derivative of the function $s \mapsto \mathcal{E}_{min}(t,s,w)$, for $t \in [0,T]$ and $w \in H^1(\Omega)$ fixed, in order to find the so-called *energy release rate*. We will see that, in general, this derivative does not exist. This is due to the non-uniqueness of solution to the minimum problem (2.9). However, we will find formulas for the right and left derivatives of the reduced energy $\mathcal{E}_{min}$ with respect to the crack length s, see Theorems 3.1 and 3.2. In Sections 4-7 we will see that these two derivatives will play a central role in the definition of viscous and quasi-static evolution via Griffith's criterion.

In order to do our computations, we need to slightly move the crack tip along the prescribed curve Σ. Hence, fixed $t \in [0,T]$, $s \in (0,L)$, and δ such that $s+\delta \in [0,L]$, we construct a C^3-diffeomorphism $F_{s,\delta}$ such that $F_{s,\delta}(\Omega_s) = \Omega_{s+\delta}$, and $F_{s,\delta}|_{\partial\Omega} = \mathrm{Id}|_{\partial\Omega}$. Indeed, by our regularity assumption, in a neighborhood of the crack tip $\gamma(s)$ the curve Σ can be seen, up to a rotation, as the graph of a C^3 function, i.e., there exists $\eta > 0$ and $\psi_s \in C^3((\gamma_1(s)-\eta, \gamma_1(s)+\eta))$ such that

$$\Sigma = \{(x_1, \psi_s(x_1)) : \, x_1 \in (\gamma_1(s)-\eta, \gamma_1(s)+\eta)\}\,,$$

where x_1 and γ_1 are the first components of $x = (x_1,x_2) \in \mathbb{R}^2$ and of the arc-length parametrization $\gamma = (\gamma_1,\gamma_2)$, respectively.

Choose a cut-off function $\vartheta \in C^\infty_c(\mathrm{B}_{\eta/2}(0))$ with $\vartheta = 1$ on $\overline{\mathrm{B}}_{\eta/3}(0)$. We define $F_{s,\delta}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_{s,\delta}(x) := x + \begin{pmatrix} (\gamma_1(s+\delta)-\gamma_1(s))\vartheta(\gamma(s)-x) \\ \psi_s(x_1 + (\gamma_1(s+\delta)-\gamma_1(s))\vartheta(\gamma(s)-x)) - \psi_s(x_1) \end{pmatrix} \tag{2.14}$$

if $x \in \mathrm{B}_{\eta/2}(\gamma(s))$, while $F_{s,\delta}(x) := x$ if $x \in \mathbb{R}^2 \setminus \mathrm{B}_{\eta/2}(\gamma(s))$.

In the following lemma, we give some properties of $F_{s,\delta}$ (see e.g. [19]).

Lemma 2.4. *For every $s \in (0,L)$, there exists $\delta_0 > 0$ such that:*

(*a*) *$F_{s,\cdot} \in C^3((-\delta_0,\delta_0)\times\mathbb{R}^2;\mathbb{R}^2)$ and, for every $|\delta| < \delta_0$, the map $F_{s,\delta}$ is a C^3-diffeomorphism. Moreover, $F_{s,\delta}(\Omega_s) = \Omega_{s+\delta}$, $F_{s,\delta}(\gamma(s)) = \gamma(s+\delta)$, and $F_{s,\delta}(\Gamma_s) = \Gamma_{s+\delta}$;*

(*b*) *the norms $\|F_{s,\delta}\|_{C^3}$ and $\|F^{-1}_{s,\delta}\|_{C^3}$ are uniformly bounded with respect to δ and there exists $c_1, c_2 > 0$ such that, for every $|\delta| < \delta_0$ and every $x \in \mathbb{R}^2$, we have $c_1 \le \det \nabla F_{s,\delta}(x) \le c_2$;*

(*c*) *$\|\mathrm{Id} - F_{s,\delta}\|_{C^2} \to 0$ as $\delta \to 0$;*

(*d*) *some derivatives:*

$$\begin{gathered} \rho_s(x) := \partial_\delta(F_{s,\delta}(x))|_{\delta=0} = \gamma_1'(s)\vartheta(\gamma(s)-x)\begin{pmatrix} 1 \\ \psi_s'(x_1)\end{pmatrix}\,, \\ \partial_\delta(\det\nabla F_{s,\delta}(x))|_{\delta=0} = \mathrm{div}\rho_s(x)\,, \\ \partial_\delta(\nabla F_{s,\delta}(x))|_{\delta=0} = -\partial_\delta(\nabla F_{s,\delta}(x))^{-1}|_{\delta=0} = \nabla\rho_s(x)\,, \\ \partial_\delta(\mathrm{cof}\,\nabla F_{s,\delta})^T|_{\delta=0} = -\partial_\delta(\mathrm{cof}\,\nabla F_{s,\delta})^{-T}|_{\delta=0} = \mathrm{div}\rho_s\,1_{\mathbb{M}^2} - \nabla\rho_s\,, \end{gathered} \tag{2.15}$$

where, for every $\mathrm{G} \in \mathbb{M}^2$, $\mathrm{cof}\,\mathrm{G}$ *stands for the cofactor matrix of* G.

Proof. See [14] for the proof of (a), (b), and (d) in the case of C^∞ maps. The same arguments are applicable with the C^3 regularity of $F_{s,\delta}$. Property (c) follows immediately from the definition (2.14) of $F_{s,\delta}$. □

Formulas (2.15) will appear in the expressions of the right and left derivatives of the reduced energy $\mathcal{E}_{min}$ with respect to s, see (3.1), (3.5)-(3.8).

3. Energy release rate

The purpose of this section is to give precise formulas for the derivative of the energy with respect to the crack length s. First of all, let us fix some notation. In what follows, for every $t \in [0,T]$, every $s \in [0,L]$, and every $w \in H^1(\Omega)$, we will denote by u_s a solution to the minimum problem (2.9) in $A(s,w)$.

Let $t \in [0,T]$, $s \in (0,L)$, $u \in H^1(\Omega_s)$, and let ϑ be a cut-off function as in (2.14). We set

$$
\begin{aligned}
G(t,u,\vartheta) := & -\frac{1}{2}\int_{\Omega_s} (D\mathbb{C}\,\rho_s)\nabla u \cdot \nabla u \,\mathrm{d}x - \int_{\Omega_s} \mathbb{C}\nabla\big((\nabla\rho_s - \mathrm{div}\rho_s\,1_{\mathbb{M}^2})u\big)\cdot\nabla u\,\mathrm{d}x \\
& + \int_{\Omega_s} \mathbb{C}(\nabla u \nabla\rho_s)\cdot\nabla u\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_s}\mathbb{C}\nabla u\cdot\nabla u\,\mathrm{div}\rho_s\,\mathrm{d}x \\
& + \int_{\Omega_s} D_\xi g(t,x,u)\cdot\big[(\nabla\rho_s - \mathrm{div}\rho_s\,1_{\mathbb{M}^2})u - \nabla u\,\rho_s\big]\,\mathrm{d}x \\
& - \int_{\Gamma_s} D_\xi\varphi(t,[u])\cdot\big((\nabla\rho_s - \mathrm{div}\rho_s\,1_{\mathbb{M}^2})u\big)\,\mathrm{d}\mathcal{H}^1 \\
& - \int_{\Gamma_s}\varphi(t,[u])\,\nu\otimes\tau\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\nabla\rho_s\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_s}\varphi(t,[u])\,\mathrm{div}\rho_s\,\mathrm{d}\mathcal{H}^1\,,
\end{aligned}
\tag{3.1}
$$

where ρ_s has been introduced in Lemma 2.4, ν and τ are the unit normal and unit tangent vectors to Σ, respectively, and $D\mathbb{C}\,\rho_s$ is a fourth order tensor given by

$$
(D\mathbb{C}\,\rho_s)_{ijkl} := \sum_{m=1}^{2}\frac{\partial\mathbb{C}_{ijkl}}{\partial x_m}\,\rho_{s,m}\,, \qquad \rho_s = (\rho_{s,1},\rho_{s,2})\,. \tag{3.2}
$$

In particular, we notice that G depends on ϑ through the definition of ρ_s, see (2.15).

We introduce the right and left derivatives of $\mathcal{E}_{min}$ with respect to the arc-length of the crack s: for every $t \in [0,T]$ and every $w \in H^1(\Omega)$ we define

$$
\partial_s^+\mathcal{E}_{min}(t,s,w) := \lim_{\delta\searrow 0}\frac{\mathcal{E}_{min}(t,s+\delta,w) - \mathcal{E}_{min}(t,s,w)}{\delta} \qquad \text{for every } s \in [0,L)\,, \tag{3.3}
$$

and

$$
\partial_s^-\mathcal{E}_{min}(t,s,w) := \lim_{\delta\nearrow 0}\frac{\mathcal{E}_{min}(t,s+\delta,w) - \mathcal{E}_{min}(t,s,w)}{\delta} \qquad \text{for every } s \in (0,L]\,, \tag{3.4}
$$

if the two limits exist.

We are now ready to state the main results of this section.

Theorem 3.1. *For every* $t \in [0,T]$, *every* $s \in (0,L)$, *and every* $w \in H^1(\Omega)$, *the limit in* (3.3) *exists and*

$$
\partial_s^+\mathcal{E}_{min}(t,s,w) = 1 - \mathcal{G}^+(t,s,w)\,, \tag{3.5}
$$

where we have set

$$
\mathcal{G}^+(t,s,w) := \max\{G(t,u_s,\vartheta) : u_s \in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\,, \tag{3.6}
$$

for a given cut-off function ϑ *as in* (2.14).

Moreover, $\mathcal{G}^+(t,s,w)$ *does not depend on the choice of* ϑ.

Theorem 3.2. *For every* $t \in [0,T]$, *every* $s \in (0,L)$, *and every* $w \in H^1(\Omega)$, *the limit in* (3.4) *exists and*

$$\partial_s^- \mathcal{E}_{min}(t,s,w) = 1 - \mathcal{G}^-(t,s,w)\,, \tag{3.7}$$

where we have set

$$\mathcal{G}^-(t,s,w) := \min\{G(t,u_s,\vartheta) : u_s \in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\,, \tag{3.8}$$

for a given cut-off function ϑ *as in* (2.14).

Moreover, $\mathcal{G}^-(t,s,w)$ *does not depend on the choice of* ϑ.

Remark 3.3. We notice that formulas (3.5)-(3.8) say that the function $s \mapsto \mathcal{E}_{min}(t,s,w)$ is not differentiable in the interval $(0,L)$. This is due to the lack of uniqueness of solution to (2.9) and, more in general, to the fact that a minimizer of $\mathcal{E}(t,s,\cdot)$ might not be approximated by minima of $\mathcal{E}(t,s+\delta,\cdot)$ as $\delta \to 0$. The consequences of this "non-approximability" will be clear in the proofs of Theorems 3.1 and 3.2, and will be stressed in Remark 3.9.

Let us anticipate, as stated in Proposition 3.11 below, that we can not expect to have the continuity of $\partial_s^+\mathcal{E}_{min}$ and $\partial_s^-\mathcal{E}_{min}$ as functions of t, s, and w, thus the arguments used in [19, 22] have to be modified as in [20] in order to find a quasi-static evolution as limit of viscous solutions, see Sections 4-7.

We finally notice that the terms $\mathcal{G}^+$ and $\mathcal{G}^-$ appearing in (3.5) and (3.7) are the generalization of the energy release rate, see e.g. [18, 21]. To be consistent with the existent literature dealing with Griffith's criterion, the definitions of viscous and quasi-static evolutions will involve $\mathcal{G}^+$ and $\mathcal{G}^-$, see Definitions 4.5 and 4.7.

Remark 3.4. We point out that, to prove Theorems 3.1 and 3.2, we can not apply the abstract results in [19, 20], since we can not ensure that property (E2) of [19, 20], that is,

$$|\partial_\delta \mathcal{E}(t,s+\delta,u)| \le c_1(c_2 + \mathcal{E}(t,s+\delta,u)) \quad \text{for } |\delta| \text{ small, } u \in A(s+\delta,w)\,,$$

holds in our framework. Indeed, we are able to prove that

$$\begin{aligned} &|\partial_\delta \mathcal{E}(t,s+\delta,u)| \le c(\|u\|_{H^1}^2 + \|u\|_{H^1}^p)\,,\\ &\mathcal{E}(t,s+\delta,u) \ge \tilde{c}\|u\|_{H^1}^2\,, \end{aligned} \tag{3.9}$$

where $p \in (1,+\infty)$ has been fixed in (2.6). However, (3.9) is not sufficient to get (E2) if $p > 2$.

Moreover, we notice that, with our method, we do not need to assume g to be differentiable with respect to the space variable $x \in \Omega$, as it has been done in [20].

In order to compute $\partial^\pm \mathcal{E}_{min}$, for every $s \in (0,L)$ and $\delta \in (-\delta_0,\delta_0)$ (see Lemma 2.4) we need to introduce the Piola transformation $P_{s,\delta}$ associated to $F_{s,\delta}$:

$$P_{s,\delta}\, u := (\mathrm{cof}\, \nabla F_{s,\delta})^T u \circ F_{s,\delta} \qquad \text{for every } u \in A(s+\delta,w)\,. \tag{3.10}$$

We notice that $P_{s,\delta}$ is an isomorphism between $A(s+\delta,w)$ and $A(s,w)$ whose inverse is given by

$$P_{s,\delta}^{-1}\, u := ((\mathrm{cof}\, \nabla F_{s,\delta})^{-T} u) \circ F_{s,\delta}^{-1} \qquad \text{for every } u \in A(s,w)\,. \tag{3.11}$$

For simplicity of notation, we also set

$$u^\delta := (\mathrm{cof} \nabla F_{s,\delta})^{-T} u = (P_{s,\delta}^{-1}\, u) \circ F_{s,\delta}\,. \tag{3.12}$$

Before starting the proofs of Theorems 3.1 and 3.2, we show some properties concerning the behavior of u_s and $\mathcal{E}_{min}$ with respect to time t, the parameter s, and the Dirichlet boundary datum w. In the next lemmas, we prove the continuity of the energy $\mathcal{E}_{min}$ in $[0,T] \times (0,L) \times H^1(\Omega)$.

Lemma 3.5. *Let* $s \in (0,L)$ *and let* $u_\delta \in H^1(\Omega\backslash\Sigma)$. *Assume that there exists* $u_0 \in H^1(\Omega\backslash\Sigma)$ *such that* $u_\delta \to u_0$ *in* $H^1(\Omega\backslash\Sigma)$ *as* $\delta \to 0$. *Then the sequences* $u_\delta \circ F_{s,\delta}$, $u_\delta \circ F_{s,\delta}^{-1}$, $P_{s,\delta}\, u_\delta$, *and* $P_{s,\delta}^{-1}\, u_\delta$ *converge to* u_0 *strongly in* $H^1(\Omega \setminus \Sigma)$ *as* $\delta \to 0$.

Proof. Thanks to the properties stated in Lemma 2.4, the lemma can be easily proved by using the changes of coordinates $x = F^{-1}_{s,\delta}(y)$ and $x = F_{s,\delta}(y)$. □

Lemma 3.6. *The reduced energy $\mathcal{E}_{min} : [0,T]\times[0,L]\times H^1(\Omega) \to \mathbb{R}$ is lower semicontinuous.*

Proof. Let $t_k, t \in [0,T]$, $s_k, s \in [0,L]$, $w_k, w \in H^1(\Omega)$ be such that $t_k \to t$, $s_k \to s$, and $w_k \to w$ in $H^1(\Omega)$ as $k \to +\infty$. For every k, let us fix $u_k \in A(s_k, w_k)$ minimizer of $\mathcal{E}(t_k, s_k, \cdot)$. Then, by Korn's inequality and by the hypotheses (2.2), (2.3), (2.6), and (2.7), we have, for some $\varepsilon > 0$ small enough and some $c_1, c_2 > 0$,

$$c_1\|u_k\|^2_{H^1} - a_\varepsilon - b_\varepsilon \le \mathcal{E}(t_k, s_k, u_k) \le \mathcal{E}(t_k, s_k, w_k) \le c_2\|w_k\|^2_{H^1} + a_\varepsilon + L\,.$$

The previous inequality and the convergence $w_k \to w$ in $H^1(\Omega)$ imply that the sequence u_k is bounded in $H^1(\Omega\setminus\Sigma)$. Therefore, there exists $u \in H^1(\Omega\setminus\Sigma)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^1(\Omega\setminus\Sigma)$. By the compactness of the trace operator, we deduce that $u \in A(s,w)$. Moreover, (2.11) holds. Hence

$$\mathcal{E}_{min}(t,s,w) \le \mathcal{E}(t,s,u) \le \liminf_k \mathcal{E}(t_k, s_k, u_k) = \liminf_k \mathcal{E}_{min}(t_k, s_k, w_k)\,,$$

and this concludes the proof. □

Lemma 3.7. *Let $t_k, t \in [0,T]$, $s_k, s \in (0,L)$, $w_k, w \in H^1(\Omega)$ be such that $t_k \to t$, $s_k \to s$, and $w_k \to w$ in $H^1(\Omega)$ as $k \to +\infty$. Let $u_k \in A(s_k, w_k)$ be a sequence of minimizers of $\mathcal{E}(t_k, s_k, \cdot)$. Then, there exists $u \in A(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ such that, up to a subsequence, $u_k \to u$ in $H^1(\Omega\setminus\Sigma)$.*

In particular, the reduced energy $\mathcal{E}_{min}$ is continuous on $[0,T]\times(0,L)\times H^1(\Omega)$.

Proof. As in the proof of Lemma 3.6, we can find $u \in A(s,w)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^1(\Omega\setminus\Sigma)$.

Let us prove that u is a minimizer of $\mathcal{E}(t,s,\cdot)$ in $A(s,w)$. Fix $u_s \in A(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$. Then, by Lemma 2.4 and by the properties of the Piola transformation (3.10), for k large enough we have $P^{-1}_{s,s_k-s}\, u_s + w_k - w \in A(s_k, w_k)$. Thanks to Lemma 3.5, $P^{-1}_{s,s_k-s}\, u_s \to u_s$ in $H^1(\Omega\setminus\Sigma)$ as $k \to +\infty$. Thus, by (2.11) and by the minimality of u_k, we obtain

$$\begin{aligned}\mathcal{E}_{min}(t,s,w) &\le \mathcal{E}(t,s,u) \le \liminf_{k\to+\infty} \mathcal{E}(t_k,s_k,u_k) \le \limsup_{k\to+\infty} \mathcal{E}(t_k,s_k,u_k)\\ &\le \lim_{k\to+\infty} \mathcal{E}(t_k, s_k, P^{-1}_{s,s_k-s}\, u_s + w_k - w) = \mathcal{E}(t,s,u_s) = \mathcal{E}_{min}(t,s,w)\,.\end{aligned} \tag{3.13}$$

From (3.13) we deduce that u is a minimizer of $\mathcal{E}(t,s,\cdot)$ in $A(s,w)$ and

$$\mathcal{E}(t,s,u) = \mathcal{E}_{min}(t,s,w) = \lim_{k\to+\infty} \mathcal{E}_{min}(t_k,s_k,w_k) = \lim_{k\to+\infty} \mathcal{E}(t_k,s_k,u_k)\,. \tag{3.14}$$

Therefore, by Lemma 2.3 we get that $u_k \to u$ strongly in $H^1(\Omega\setminus\Sigma)$. Moreover, (3.14) implies that $\mathcal{E}_{min}$ is continuous on $[0,T]\times(0,L)\times H^1(\Omega)$. □

In the proof of Theorem 3.1 we will need the following lemma.

Lemma 3.8. *Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, and connected set with Lipschitz boundary. Let $\vartheta \in C^\infty_c(\Omega)$ and $\delta_0 > 0$ be fixed as in (2.14) and in Lemma 2.4. Then the following facts hold true:*

(*a*) *there exists $c = c(\vartheta) > 0$ such that for every $u \in H^1(\Omega)$:*

$$\left\|\delta^{-1}\big(u\circ F^{-1}_{s,\delta} - u\big)\right\|_{L^2} \le c(\vartheta)\|\nabla u\|_{L^2}\,. \tag{3.15}$$

Moreover, $\delta^{-1}(u\circ F^{-1}_{s,\delta} - u) \to -\nabla u\,\rho_s$ in $L^2(\Omega)$ as $\delta \to 0$;

(*b*) *assume that there exist $\delta_k \to 0$, $|\delta_k| < \delta_0$, and $u_{\delta_k}, u \in H^1(\Omega)$ such that $u_{\delta_k} \rightharpoonup u$ weakly in $H^1(\Omega)$ as $k \to +\infty$. Then $\delta_k^{-1}(u_{\delta_k} - u_{\delta_k}\circ F_{s,\delta_k}) \rightharpoonup -\nabla u\,\rho_s$ weakly in $L^2(\Omega)$ as $k \to +\infty$.*

Proof. We adapt the proof of [18, Lemma 4.1] to the case of a curved prescribed crack path Σ.

Let us fix $u \in H^1(\Omega)$. For $|\delta| < \delta_0$ we define $L_\delta(u) := \delta^{-1}(u \circ F_{s,\delta}^{-1} - u)$ and $L_0(u) := -\nabla u\, \rho_s$. The function $L_\delta \colon H^1(\Omega) \to L^2(\Omega)$ is a linear operator for every $|\delta| < \delta_0$. We want to prove that they are uniformly bounded.

To this end, for $|\delta| < \delta_0$ and $h \in \mathbb{R}$ small enough, we set $x_h := F_{s,\delta+h}^{-1}(y)$ and $x := F_{s,\delta}^{-1}(y)$ for $y \in \Omega$. We compute

$$\lim_{h\to 0} \frac{x_h - x}{h}\,.$$

By definition of $F_{s,\cdot}$, we have

$$0 = \frac{1}{h}(F_{s,\delta+h}(x_h) - F_{s,\delta}(x)) = \frac{1}{h}(F_{s,\delta+h}(x_h) - F_{s,\delta+h}(x)) + \frac{1}{h}(F_{s,\delta+h}(x) - F_{s,\delta}(x))\,. \tag{3.16}$$

By the mean value theorem, there exists $t_h \in (0,1)$ such that

$$F_{s,\delta+h}(x_h) - F_{s,\delta+h}(x) = \nabla F_{s,\delta+h}(x_{t_h})(x_h - x)\,,$$

where $x_{t_h} := x + t_h(x_h - x)$. Since $F_{s,\delta+h}$ is a C^3-diffeomorphism, for every h there exists $(\nabla F_{s,\delta+h}(x_{t_h}))^{-1}$. Hence, (3.16) becomes

$$0 = \frac{x_h - x}{h} + (\nabla F_{s,\delta+h}(x_{t_h}))^{-1}\,\frac{F_{s,\delta+h}(x) - F_{s,\delta}(x)}{h}\,. \tag{3.17}$$

Passing to the limit in (3.17) as $h \to 0$, since $x_{t_h} \to x$ we get

$$\rho_{s,\delta}(x) := \lim_{h\to 0}\frac{x_h - x}{h} = -(\nabla F_{s,\delta}(x))^{-1}\partial_\delta F_{s,\delta}(x)\,. \tag{3.18}$$

Let now $u \in C^\infty(\overline{\Omega})$ be fixed. For every $y \in \Omega$, by (3.18) we have

$$L_\delta(u)(y) = \frac{1}{\delta}\int_0^1 \frac{d}{dh}u(F_{s,h\delta}^{-1}(y))\,\mathrm{d}h = \int_0^1 \nabla u(F_{s,h\delta}^{-1}(y))\,\rho_{s,h\delta}(F_{s,h\delta}^{-1}(y))\,\mathrm{d}h\,. \tag{3.19}$$

Taking the L^2 norm of $L_\delta(u)$ in (3.19) and applying Hölder's inequality and the change of coordinates $y = F_{s,h\delta}(x)$, we obtain

$$\|L_\delta(u)\|_{L^2}^2 \le \int_0^1\int_\Omega |\nabla u\,\rho_{s,h\delta}|^2 \det\nabla F_{s,h\delta}\,\mathrm{d}x \le c(\vartheta)\|\nabla u\|_{L^2}^2\,, \tag{3.20}$$

for some constant $c(\vartheta) > 0$ independent of δ. Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, we deduce that (3.20) holds for every $u \in H^1(\Omega)$, which is exactly (3.15).

Moreover, thanks to (3.19), for every $u \in C^\infty(\overline{\Omega})$ we have

$$\|L_\delta(u) - L_0(u)\|_{L^2}^2 \le \int_0^1\int_\Omega |\nabla u(F_{s,h\delta}^{-1}(y))\rho_{s,h\delta}(F_{s,h\delta}^{-1}(y)) + \nabla u\,\rho_s(y)|^2\,\mathrm{d}y\,\mathrm{d}h\,. \tag{3.21}$$

For $(h,y) \in [0,1]\times\Omega$ fixed, the integrand in (3.21) converges to 0 pointwise as $\delta \to 0$, thus, by the dominated convergence theorem, we get that $L_\delta(u) \to L_0(u)$ strongly in $L^2(\Omega)$ for every $u \in C^\infty(\overline{\Omega})$. By (3.20) and a density argument, the same is true for $u \in H^1(\Omega)$. This concludes the proof of point (a).

Let us now prove (b). For every $v \in C_c^\infty(\Omega)$, it holds

$$\begin{aligned}
&\int_\Omega \delta_k^{-1}(u_{\delta_k} - u_{\delta_k}\circ F_{s,\delta_k})\cdot v\,\mathrm{d}x\\
&\qquad = -\int_\Omega u_{\delta_k}\cdot L_{\delta_k}(v)\,\mathrm{d}x + \delta_k^{-1}\int_\Omega u_{\delta_k}\cdot(v\circ F_{s,\delta_k}^{-1})(1 - \det\nabla F_{s,\delta_k}^{-1})\,\mathrm{d}x\\
&\qquad = -\int_\Omega u_{\delta_k}\cdot L_{\delta_k}(v)\,\mathrm{d}x + \delta_k^{-1}\int_\Omega u_{\delta_k}\cdot(v\circ F_{s,\delta_k}^{-1})\,\frac{\det\nabla F_{s,\delta_k}(F_{s,\delta_k}^{-1}(x)) - 1}{\det\nabla F_{s,\delta_k}(F_{s,\delta_k}^{-1}(x))}\,\mathrm{d}x\,.
\end{aligned} \tag{3.22}$$

In the last integral of (3.22) we perform the change of coordinates $x = F_{s,\delta}(y)$, thus we obtain

$$\begin{aligned}\int_\Omega \delta_k^{-1}(u_{\delta_k} - u_{\delta_k}\circ F_{s,\delta_k})\cdot v\,\mathrm{d}x \\ = -\int_\Omega u_{\delta_k}\cdot L_{\delta_k}(v)\,\mathrm{d}x + \int_\Omega (u_{\delta_k}\circ F_{s,\delta_k})\cdot v\,\frac{\det\nabla F_{s,\delta_k}-1}{\delta_k}\,\mathrm{d}x\,.\end{aligned}\tag{3.23}$$

Passing to the limit in (3.22) as $k\to+\infty$, taking into account point (a), Lemma 2.4, and the weak convergence $u_{\delta_k}\circ F_{s,\delta_k}\rightharpoonup u$ in $H^1(\Omega)$, we get

$$\begin{aligned}\lim_k \int_\Omega \delta_k^{-1}(u_{\delta_k} - u_{\delta_k}\circ F_{s,\delta_k})\cdot v\,\mathrm{d}x &= \int_\Omega u\cdot\nabla v\,\rho_s\,\mathrm{d}x + \int_\Omega u\cdot v\,\mathrm{div}\rho_s\,\mathrm{d}x \\ &= \int_\Omega u\cdot \mathrm{div}(v\otimes\rho_s)\,\mathrm{d}x = -\int_\Omega v\cdot\nabla u\,\rho_s\,\mathrm{d}x\,,\end{aligned}\tag{3.24}$$

where, in the last equality, we have used the divergence theorem.

Since

$$\delta_k^{-1}(u_{\delta_k} - u_{\delta_k}\circ F_{s,\delta_k}) = L_{\delta_k}(u_{\delta_k}\circ F_{s,\delta_k})\,,$$

estimate (3.15) and the weak convergence of u_{δ_k} imply that there exists $C>0$ such that for every k

$$\|\delta_k^{-1}(u_{\delta_k} - u_{\delta_k}\circ F_{s,\delta_k})\|_{L^2}\le C\,.$$

Therefore, taking into account the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$, we deduce that (3.24) holds for every $v\in L^2(\Omega)$, hence $\delta_k^{-1}(u_{\delta_k} - u_{\delta_k}\circ F_{s,\delta_k})\rightharpoonup -\nabla u\,\rho_s$ weakly in $L^2(\Omega)$ as $k\to+\infty$, and the proof of the lemma is thus concluded. □

We are now ready to prove Theorem 3.1. Here, we follow the steps of [18, Theorem 3.3]. Before starting the proof, we notice that, with the notation introduced in (3.10) and (3.11), for $\delta>0$ these inequalities hold:

$$\begin{aligned}\frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta} &\le \frac{\mathcal{E}(t,s+\delta,P_{s,\delta}^{-1}\,u_s)-\mathcal{E}(t,s,u_s)}{\delta}\,, \\ \frac{\mathcal{E}(t,s+\delta,u_{s+\delta})-\mathcal{E}(t,s,P_{s,\delta}\,u_{s+\delta})}{\delta} &\le \frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta}\,,\end{aligned}\tag{3.25}$$

for every $u_s\in A(s,w)$ and every $u_{s+\delta}\in A(s+\delta,w)$ minimizers of $\mathcal{E}(t,s,\cdot)$ and $\mathcal{E}(t,s+\delta,\cdot)$, respectively. Estimates (3.25) will be the key point of the following proofs.

Proof of Theorem 3.1. Fix $t\in[0,T]$, $s\in(0,L)$, and $w\in H^1(\Omega)$. Let us consider the first inequality in (3.25). Recalling the notation introduced in (3.10), (3.11), and (3.12), for $0<\delta<\delta_0$ we have, by the change of variables $x=F_{s,\delta}^{-1}(y)$,

$$\begin{aligned}&\frac{\mathcal{E}(t,s+\delta,P_{s,\delta}^{-1}\,u_s)-\mathcal{E}(t,s,u_s)}{\delta} \\ &\quad= \frac{1}{2\delta}\bigg(\int_{\Omega_s}\mathbb{C}(F_{s,\delta}(x))\nabla u_s^\delta\,(\nabla F_{s,\delta})^{-1}\cdot\nabla u_s^\delta\,(\nabla F_{s,\delta})^{-1}\,\det\nabla F_{s,\delta}\,\mathrm{d}x \\ &\quad-\int_{\Omega_s}\mathbb{C}\mathrm{E}u_s\cdot\mathrm{E}u_s\,\mathrm{d}x\bigg) - \frac{1}{\delta}\bigg(\int_{\Omega_{s+\delta}} g(t,x,P_{s,\delta}^{-1}\,u_s)\,\mathrm{d}x - \int_{\Omega_s} g(t,x,u_s)\,\mathrm{d}x\bigg) \\ &\quad+\frac{1}{\delta}\bigg(\int_{\Gamma_s}\varphi(t,[u_s^\delta])\,\frac{\sqrt{1+(\psi_s'\circ F_{s,\delta})^2}}{\sqrt{1+\psi_s'^2}}\,\det\nabla F_{s,\delta}\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_s}\varphi(t,[u_s])\,\mathrm{d}\mathcal{H}^1\bigg) + 1 \\ &\quad= \frac{1}{\delta}I_1 - \frac{1}{\delta}I_2 + \frac{1}{\delta}I_3 + 1\,.\end{aligned}\tag{3.26}$$

Thanks to the properties of $F_{s,\delta}$ stated in Lemma 2.4 and to the regularity of the elasticity tensor $\mathbb{C}$, applying the dominated convergence theorem we easily get that

$$
\begin{aligned}
\lim_{\delta\searrow 0}\frac{1}{\delta}\,I_1=\frac{1}{2}\int_{\Omega_s}(D\mathbb{C}\,\rho_s)\nabla u_s\cdot\nabla u_s\,\mathrm{d}x+\int_{\Omega_s}\mathbb{C}\nabla\big((\nabla\rho_s-\mathrm{div}\rho_s\,1_{\mathbb{M}^2})u_s\big)\cdot\nabla u_s\,\mathrm{d}x\\
-\int_{\Omega_s}\mathbb{C}(\nabla u_s\nabla\rho_s)\cdot\nabla u_s\,\mathrm{d}x+\frac{1}{2}\int_{\Omega_s}\mathbb{C}\nabla u_s\cdot\nabla u_s\,\mathrm{div}\rho_s\,\mathrm{d}x\,.
\end{aligned}\tag{3.27}
$$

We now deal with the term I_2 of (3.26). In view of the regularity properties of g, we can apply the mean value theorem: for a.e. $x\in\Omega$ there exists $\zeta_\delta(x)\in(0,1)$ such that

$$
\begin{aligned}
&g(t,x,P^{-1}_{s,\delta}\,u_s(x))-g(t,x,u_s(x))\\
&\quad=D_\xi g(t,x,P^{-1}_{s,\delta}\,u_s(x)+\zeta_\delta(x)(P^{-1}_{s,\delta}\,u_s(x)-u_s(x)))\cdot(P^{-1}_{s,\delta}\,u_s(x)-u_s(x))\,.
\end{aligned}\tag{3.28}
$$

Let us set $\bar u_\delta:=P^{-1}_{s,\delta}\,u_s+\zeta_\delta(P^{-1}_{s,\delta}\,u_s-u_s)$, where ζ_δ is as in (3.28). We can continue in (3.28), obtaining

$$
\begin{aligned}
&g(t,x,P^{-1}_{s,\delta}\,u_s(x))-g(t,x,u_s(x))\\
&\quad=D_\xi g(t,x,\bar u_\delta(x))\cdot\big[(P^{-1}_{s,\delta}\,u_s-u_s\circ F^{-1}_{s,\delta})+(u_s\circ F^{-1}_{s,\delta}-u_s)\big]\,.
\end{aligned}\tag{3.29}
$$

By Lemma 3.5, $u_s\circ F^{-1}_{s,\delta}$ and $P^{-1}_{s,\delta}\,u_s$ converge to u_s in $H^1(\Omega\setminus\Sigma)$ as $\delta\searrow 0$. Hence, we also have, up to a subsequence, $\bar u_\delta\to u_s$ pointwise. Thanks to Lemmas 2.4 and 3.8, to condition (2.4) on g, and to the dominated convergence theorem, we get

$$
\lim_{\delta\searrow 0}\frac{1}{\delta}\,I_2=\int_{\Omega_s}D_\xi g(t,x,u_s)\cdot\big[(\nabla\rho_s-\mathrm{div}\rho_s\,1_{\mathbb{M}^2})\,u_s-\nabla u_s\,\rho_s\big]\,\mathrm{d}x\,.\tag{3.30}
$$

We now consider the term I_3 in (3.26). We can write it as

$$
\begin{aligned}
I_3=&\int_{\Gamma_s}\varphi(t,[u^\delta_s])\frac{\sqrt{1+(\psi'_s\circ F_{s,\delta})^2}}{\sqrt{1+\psi'^2_s}}(\det\nabla F_{s,\delta}-1)\,\mathrm{d}\mathcal{H}^1\\
&+\int_{\Gamma_s}\varphi(t,[u^\delta_s])\Big(\frac{\sqrt{1+(\psi'_s\circ F_{s,\delta})^2}}{\sqrt{1+\psi'^2_s}}-1\Big)\,\mathrm{d}\mathcal{H}^1+\int_{\Gamma_s}(\varphi(t,[u^\delta_s])-\varphi(t,[u_s]))\,\mathrm{d}\mathcal{H}^1\\
=&\,I_{1,3}+I_{2,3}+I_{3,3}\,.
\end{aligned}\tag{3.31}
$$

For the first two terms in (3.31) it is easy to see that

$$
\begin{aligned}
\lim_{\delta\searrow 0}&\frac{1}{\delta}\,I_{1,3}+\frac{1}{\delta}\,I_{2,3}\\
&=\int_{\Gamma_s}\varphi(t,[u_s])\,\mathrm{div}\rho_s\,\mathrm{d}\mathcal{H}^1+\int_{\Gamma_s}\varphi(t,[u_s])\,\frac{\psi'_s\psi''_s}{1+\psi'^{\,2}_s}\,\gamma'_1(s)\vartheta(\gamma(s)-x)\,\mathrm{d}\mathcal{H}^1\\
&=\int_{\Gamma_s}\varphi(t,[u_s])\,\mathrm{div}\rho_s\,\mathrm{d}\mathcal{H}^1+\int_{\Gamma_s}\varphi(t,[u_s])\,\nu\otimes\tau\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\nabla\rho_s\,\mathrm{d}\mathcal{H}^1\,.
\end{aligned}\tag{3.32}
$$

For the last term in (3.31), we exploit again the mean value theorem: for every $x\in\Gamma_s$ there exists $\zeta_\delta(x)\in(0,1)$ such that

$$
\varphi(t,[u^\delta_s](x))-\varphi(t,[u_s](x))=D_\xi\varphi(t,[u^\delta_s](x)+\zeta_\delta(x)([u^\delta_s](x)-[u_s](x)))\cdot([u^\delta_s](x)-[u_s](x))
$$

Arguing as in (3.30) and taking into account hypothesis (2.6) on φ, we get

$$
\lim_{\delta\searrow 0}\frac{1}{\delta}\,I_{3,3}=\int_{\Gamma_s}D_\xi\varphi(t,[u_s])\cdot\big((\nabla\rho_s-\mathrm{div}\rho_s\,1_{\mathbb{M}^2})u_s\big)\,\mathrm{d}\mathcal{H}^1\,.\tag{3.33}
$$

Collecting (3.25)-(3.27) and (3.30)-(3.33) we deduce

$$\begin{aligned}\limsup_{\delta\searrow 0}\,&\frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta}\\ &\leq \lim_{\delta\searrow 0}\frac{\mathcal{E}(t,s+\delta,u_s^\delta)-\mathcal{E}(t,s,u_s)}{\delta}=1-G(t,u_s,\vartheta)\,.\end{aligned}\tag{3.34}$$

Since we can repeat the previous argument for every $u_s\in A(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$, taking the infimum in the right-hand side of (3.34) we get

$$\begin{aligned}\limsup_{\delta\searrow 0}\,&\frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta}\\ &\leq 1-\sup\{G(t,u_s,\vartheta):\,u_s\in A(s,w)\text{ is a minimizer of }\mathcal{E}(t,s,\cdot)\}\,.\end{aligned}\tag{3.35}$$

In particular, since the set of minimizers $\{u_s\}$ is bounded in $H^1(\Omega_s)$ for every $s\in(0,L)$, the supremum in (3.35) is finite.

To prove the converse inequality for the $\liminf$, we argue in a similar way on the second inequality of (3.25), taking into account Lemmas 2.4, 3.5, 3.7, and point (b) of Lemma 3.8. Indeed, for every $\delta>0$ we fix $u_{s+\delta}\in A(s+\delta,w)$ minimizer of $\mathcal{E}(t,s+\delta,\cdot)$. By Lemma 3.7, we deduce that there exist a subsequence $\delta_k\searrow 0$ and $u_s\in A(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ such that $u_{s+\delta_k}\to u_s$ in $H^1(\Omega\setminus\Sigma)$. Lemma 3.5 implies that $u_{s+\delta_k}\circ F_{s,\delta_k}\to u_s$ in $H^1(\Omega_s)$. For simplicity, we set $U_{s,\delta_k}:=u_{s+\delta_k}\circ F_{s,\delta_k}$ and notice that $P_{s,\delta_k}u_{s+\delta_k}=(\mathrm{cof}\nabla F_{s,\delta_k})^T U_{s,\delta_k}$.

We can write

$$\begin{aligned}&\frac{\mathcal{E}(t,s+\delta_k,u_{s+\delta_k})-\mathcal{E}(t,s,P_{s,\delta_k}\,u_{s+\delta_k})}{\delta_k}\\ &=\frac{1}{2\delta_k}\bigg(\int_{\Omega_s}\mathbb{C}(F_{s,\delta_k}(x))\nabla U_{s,\delta_k}(\nabla F_{s,\delta_k})^{-1}\cdot\nabla U_{s,\delta_k}(\nabla F_{s,\delta_k})^{-1}\det\nabla F_{s,\delta_k}\,\mathrm{d}x\\ &\quad-\int_{\Omega_s}\mathbb{C}\nabla(P_{s,\delta_k}\,u_{s+\delta_k})\cdot\nabla(P_{s,\delta_k}\,u_{s+\delta_k})\,\mathrm{d}x\bigg)\\ &\quad-\frac{1}{\delta_k}\bigg(\int_{\Omega_s}g(t,x,u_{s+\delta_k})\,\mathrm{d}x-\int_{\Omega_s}g(t,x,P_{s,\delta_k}u_{s+\delta_k})\,\mathrm{d}x\bigg)\\ &\quad+\frac{1}{\delta_k}\bigg(\int_{\Gamma_s}\varphi(t,[U_{s,\delta_k}])\,\frac{\sqrt{1+(\psi_s'\circ F_{s,\delta_k})^2}}{\sqrt{1+\psi_s'^2}}\det\nabla F_{s,\delta_k}\,\mathrm{d}\mathcal{H}^1\\ &\quad-\int_{\Gamma_s}\varphi(t,[P_{s,\delta_k}\,u_{s+\delta_k}])\,\mathrm{d}\mathcal{H}^1\bigg)+1\end{aligned}\tag{3.36}$$

Following step by step the proof of (3.34), in view of Lemma 2.4, of point (b) of Lemma 3.8, and of the previous observations, we can pass to the limit as $k\to+\infty$ in (3.36) getting

$$\begin{aligned}\lim_{k}\,&\frac{\mathcal{E}(t,s+\delta_k,u_{s+\delta_k})-\mathcal{E}(t,s,P_{s,\delta_k}\,u_{s+\delta_k})}{\delta_k}=1-G(t,u_s,\vartheta)\\ &\geq 1-\sup\{G(t,u_s,\vartheta):\,u_s\in A(s,w)\text{ is a minimizer of }\mathcal{E}(t,s,\cdot)\}\,.\end{aligned}\tag{3.37}$$

By a contradiction argument, from inequality (3.37) it follows that

$$\begin{aligned}\liminf_{\delta\searrow 0}\,&\frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta}\\ &\geq 1-\sup\{G(t,u_s,\vartheta):\,u_s\in A(s,w)\text{ is a minimizer of }\mathcal{E}(t,s,\cdot)\}\,.\end{aligned}\tag{3.38}$$

Thus, collecting inequalities (3.35) and (3.38), we get that the limit in (3.3) exists and

$$\partial_s^+\mathcal{E}_{min}(t,s,w)=1-\sup\{G(t,u_s,\vartheta):\,u_s\in A(s,w)\text{ is a minimizer of }\mathcal{E}(t,s,\cdot)\}\,.\tag{3.39}$$

It remains to prove that the supremum in (3.39) is attained. Let us consider a sequence of minimizers u_s^n of $\mathcal{E}(t,s,\cdot)$ in $A(s,w)$ such that

$$\lim_n G(t,u_s^n,\vartheta) = \sup\{G(t,u_s,\vartheta) : u_s \in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}.$$

Since Lemma 3.7 holds, there exist a subsequence, not relabeled, and a minimizer $u \in A(s,w)$ of $\mathcal{E}(t,s,\cdot)$ such that $u_s^n \to u$ in $H^1(\Omega_s)$. Since G is continuous with respect to the strong convergence in $H^1(\Omega\setminus\Sigma)$, we have

$$\lim_n G(t,u_s^n,\vartheta) = G(t,u,\vartheta) = \sup\{G(t,u_s,\vartheta) : u_s \in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}.$$

This concludes the proof of (3.5).

Finally, in view of the definition (3.3) of $\partial_s^+\mathcal{E}_{min}$, we notice that $\mathcal{G}^+$ does not depend on the cut-off function ϑ. □

Exploiting the arguments of Theorem 3.1, we can also prove Theorem 3.2.

Proof of Theorem 3.2. We just have to follow step by step the proof of Theorem 3.1.

In this case, since we are dealing with $\delta < 0$, estimates (3.25) are replaced by

$$\begin{aligned} &\frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta} \le \frac{\mathcal{E}(t,s+\delta,u_{s+\delta})-\mathcal{E}(t,s,P_{s,\delta}\,u_{s+\delta})}{\delta}, \\ &\frac{\mathcal{E}(t,s+\delta,P_{s,\delta}^{-1}\,u_s)-\mathcal{E}(t,s,u_s)}{\delta} \le \frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta}, \end{aligned} \tag{3.40}$$

for every $u_s \in A(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ and every $u_{s+\delta} \in A(s+\delta,w)$ minimizer of $\mathcal{E}(t,s+\delta,\cdot)$.

The second inequality in (3.40) can be treated as the corresponding one in the first part of the proof of Theorem 3.1. This time, it leads us to

$$\liminf_{\delta\nearrow 0} \frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta} \ge 1 - G(t,u_s,\vartheta). \tag{3.41}$$

Since (3.41) holds for every $u_s \in A(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$, taking the supremum we obtain

$$\begin{aligned} &\liminf_{\delta\nearrow 0} \frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta} \\ &\qquad \ge 1 - \inf\{G(t,u_s,\vartheta) : u_s \in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}. \end{aligned} \tag{3.42}$$

For the first inequality in (3.40), we argue again as in the proof of (3.38). In this case, we get

$$\begin{aligned} &\limsup_{\delta\nearrow 0} \frac{\mathcal{E}_{min}(t,s+\delta,w)-\mathcal{E}_{min}(t,s,w)}{\delta} \\ &\qquad \le 1 - \inf\{G(t,u_s,\vartheta) : u_s \in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}. \end{aligned} \tag{3.43}$$

Collecting the inequalities (3.42) and (3.43), we have that the limit in (3.4) exists. Moreover, recalling the definition (3.4), we have

$$\partial_s^-\mathcal{E}_{min}(t,s,w) = 1 - \inf\{G(t,u_s,\vartheta) : u_s \in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}. \tag{3.44}$$

As in the proof of Theorem 3.1, the infimum in (3.44) is actually a minimum, thus (3.7) is proved. Finally, $\mathcal{G}^-$ does not depend on the cut-off function ϑ. This concludes the proof of Theorem 3.2. □

Remark 3.9. As we have already noticed in Remark 3.3, the general non-existence of the derivative of $\mathcal{E}_{min}$ with respect to the crack-length s is due to the lack of approximability of the minimizers $u_s \in A(s,w)$ of $\mathcal{E}(t,s,\cdot)$, that is, it is not true that for every u_s and every $\delta > 0$ there exist $u_{s+\delta} \in A(s+\delta,w)$ minimizer of $\mathcal{E}(t,s+\delta,\cdot)$ and $u_{s-\delta} \in A(s-\delta,w)$ minimizer of $\mathcal{E}(t,s-\delta,\cdot)$ such that $u_{s+\delta}, u_{s-\delta} \to u_s$ in $H^1(\Omega\setminus\Sigma)$ as $\delta \searrow 0$. If this approximation property were true, then, in the inequalities (3.35), (3.38), (3.42), and (3.43), we could take both the infimum and the supremum. As a consequence, it would be $\partial_s^+\mathcal{E}_{min} = \partial_s^-\mathcal{E}_{min}$ and the reduced energy would be differentiable with respect to $s \in (0, L)$. For instance, this is true if the functions $\xi \mapsto \varphi(t, \xi)$ and $\xi \mapsto g(t, x, \xi)$ are convex. Indeed, in this case the minimum problem (2.9) has a unique solution $u_s \in A(s, w)$ and the function $s \mapsto u_s$ is continuous.

Remark 3.10. We briefly notice that if we drop the non-interpenetration condition in the definition (2.10) of the admissible displacements $A(s, w)$, Theorems 3.1 and 3.2 hold with a simpler formula for G, namely

$$\begin{aligned} G(t, u, \vartheta) := & -\frac{1}{2}\int_{\Omega_s} (D\mathbb{C}\,\rho_s)\nabla u \cdot \nabla u \,\mathrm{d}x + \int_{\Omega_s} \mathbb{C}(\nabla u \nabla \rho_s)\cdot \nabla u \,\mathrm{d}x \\ & -\frac{1}{2}\int_{\Omega_s} \mathbb{C}\nabla u \cdot \nabla u \,\mathrm{div}\rho_s \,\mathrm{d}x - \int_{\Omega_s} D_\xi g(t, x, u)\cdot \nabla u\, \rho_s \,\mathrm{d}x \\ & - \int_{\Gamma_s} \varphi(t, [u])\, \nu\otimes\tau \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \nabla\rho_s \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_s} \varphi(t, [u])\,\mathrm{div}\rho_s \,\mathrm{d}\mathcal{H}^1 . \end{aligned}$$

The ideas of the proofs present minor changes due to the fact that we do not need the Piola transformation $P_{s,\delta}$ anymore. Indeed, $u \circ F_{s,\delta} \in A(s, w)$ for every $u \in A(s + \delta, w)$ in this case.

Moreover, we stress that a C^2-regularity of the curve Σ is enough, and that we do not need the differentiability hypothesis on φ.

Thanks to Theorems 3.1 and 3.2, we are allowed to define the functions

$$\mathcal{G}^+, \mathcal{G}^- : [0, T] \times (0, L) \times H^1(\Omega) \to \mathbb{R} ,$$

whose expressions are given by (3.6) and (3.8), respectively.

We now state a property of semicontinuity of $\mathcal{G}^+$ and $\mathcal{G}^-$ which will be useful in the next sections.

Proposition 3.11. *The following facts hold:*

(*a*) *for every $t \in [0, T]$, every $s \in (0, L)$, and every $w \in H^1(\Omega)$*

$$\mathcal{G}^+(t, s, w) \geq \mathcal{G}^-(t, s, w) \geq 0 ;$$

(*b*) *the function $\mathcal{G}^+$ is upper semicontinuous with respect to the strong topology of $\mathbb{R} \times \mathbb{R} \times H^1(\Omega)$;*

(*c*) *the function $\mathcal{G}^-$ is lower semicontinuous with respect to the strong topology of $\mathbb{R} \times \mathbb{R} \times H^1(\Omega)$.*

Proof. To prove property (a), we just notice that $\mathcal{G}^+(t, s, w)$ and $\mathcal{G}^-(t, s, w)$ are the negative of the right and left derivatives of the function

$$s \mapsto \mathcal{E}_{min}(t, s, w) - s .$$

Since this function is monotone non-increasing and Theorems 3.1, 3.2 hold, we get (a).

Let us prove (b). We consider a sequence $(t_k, s_k, w_k) \to (t, s, w)$ in $[0, T] \times (0, L) \times H^1(\Omega)$ and ϑ a cut-off function defined as in (2.14). By Theorem 3.1, for every $k \in \mathbb{N}$ there exists $u_{s_k} \in A(s_k, w_k)$ minimizer of $\mathcal{E}(t_k, s_k, \cdot)$ such that $\mathcal{G}^+(t_k, s_k, w_k) = G(t_k, u_{s_k}, \vartheta)$. By Lemma 3.7, there exists $u_s \in A(s, w)$ minimizer of $\mathcal{E}(t, s, \cdot)$ such that, up to a subsequence, $u_{s_k} \to u_s$ in $H^1(\Omega \backslash \Sigma)$. Formula (3.1), together with the hypotheses on g and on φ, implies that

$$G(t, u_s, \vartheta) = \lim_k G(t_k, u_{s_k}, \vartheta) .$$

By (3.6), $G(t, u_s, \vartheta) \leq \mathcal{G}^+(t, s, w)$, thus we deduce the upper semicontinuity of $\mathcal{G}^+$.

In the same way, taking into account (3.8), we obtain the lower semicontinuity of $\mathcal{G}^-$, and this concludes the proof. $\square$

We conclude this section with a proposition which helps us to give an interpretation to G defined in (3.1). Let $t \in [0,T]$, $s \in (0,L)$, $w \in H^1(\Omega)$, $u \in H^1(\Omega \setminus \Sigma)$, and $\eta > 0$. We define

$$\mathcal{E}^{\eta}_{loc}(t,s,u) := \inf\left\{\mathcal{E}(t,s,v) : v \in A(s,w),\ \|v-u\|_{H^1} \le \eta\right\}. \tag{3.45}$$

By the direct method of the calculus of variations, we can prove that the infimum in (3.45) is attained.

Proposition 3.12. *Let $t \in [0,T]$, $s \in (0,L)$, $w \in H^1(\Omega)$, $u_s \in A(s,w)$ a minimizer of $\mathcal{E}(t,s,\cdot)$, and let ϑ be a cut-off function as in (2.14). Then*

$$\begin{aligned} G(t,u_s,\vartheta) - 1 &= \lim_{\eta\searrow 0}\ \liminf_{\delta\searrow 0} \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta,u_s)}{\delta} \\ &= \lim_{\eta\searrow 0}\ \limsup_{\delta\searrow 0} \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta,u_s)}{\delta}. \end{aligned} \tag{3.46}$$

In particular, $G(t,u_s,\vartheta) =: G(t,u_s)$ does not depend on ϑ.

Proof. Let t, s, w, and u_s be as in the statement of the proposition. Let $\eta > 0$ be fixed. With the notation introduced in Lemma 3.8, for $\delta > 0$ small enough we have $P^{-1}_{s,\delta}\, u_s \in A(s+\delta,w)$ and, by Lemma 3.5, $\|P^{-1}_{s,\delta}\, u_s - u_s\|_{H^1} \le \eta$. Thus, the following estimate from below holds:

$$\frac{\mathcal{E}(t,s,u_s) - \mathcal{E}(t,s+\delta,P^{-1}_{s,\delta}\, u_s)}{\delta} \le \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta,u_s)}{\delta}. \tag{3.47}$$

Therefore, as in the proof of Theorem 3.1, passing to the $\liminf$ as $\delta \searrow 0$ in (3.47) we get

$$G(t,u_s,\vartheta) - 1 \le \liminf_{\delta\searrow 0} \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta,u_s)}{\delta}. \tag{3.48}$$

We now prove that

$$\begin{aligned} &\limsup_{\delta\searrow 0} \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta,u_s)}{\delta} \\ &\qquad \le \sup\left\{G(t,u_\eta,\vartheta) : u_\eta \in A(s,w) \text{ is a minimizer of } \mathcal{E}^{\eta}_{loc}(t,s,u_s)\right\} - 1. \end{aligned} \tag{3.49}$$

Let us fix a sequence $\delta_k \searrow 0$. Since, for every k, $\mathcal{E}^{\eta+1/k}_{loc}(t,s+\delta_k,u_s) \le \mathcal{E}^{\eta}_{loc}(t,s+\delta_k,u_s)$, the following chain of inequalities holds:

$$\begin{aligned} \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta_k,u_s)}{\delta_k} &\le \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta+1/k}_{loc}(t,s+\delta_k,u_s)}{\delta_k} \\ &= \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}(t,s+\delta_k,u^k_\eta)}{\delta_k}, \end{aligned} \tag{3.50}$$

where we denote by $u^k_\eta \in A(s+\delta_k,w)$ a minimizer of $\mathcal{E}^{\eta+1/k}_{loc}(t,s+\delta_k,u_s)$. Since $\mathcal{E}(t,s,u_s) = \mathcal{E}_{min}(t,s,w)$ and $P_{s,\delta_k}\, u^k_\eta \in A(s,w)$, we can continue in (3.50) getting

$$\frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta_k,u_s)}{\delta_k} \le \frac{\mathcal{E}(t,s,P_{s,\delta_k}\, u^k_\eta) - \mathcal{E}(t,s+\delta_k,u^k_\eta)}{\delta_k}. \tag{3.51}$$

Up to a subsequence, we can assume that

$$\limsup_{k} \frac{\mathcal{E}(t,s,P_{s,\delta_k}\, u^k_\eta) - \mathcal{E}(t,s+\delta_k,u^k_\eta)}{\delta_k} = \lim_{k} \frac{\mathcal{E}(t,s,P_{s,\delta_k}\, u^k_\eta) - \mathcal{E}(t,s+\delta_k,u^k_\eta)}{\delta_k}.$$

By construction, we have that u^k_η is bounded in $H^1(\Omega \setminus \Sigma)$. Thus, we may assume that, up to a subsequence, $u^k_\eta \rightharpoonup u$ weakly in $H^1(\Omega\setminus\Sigma)$ as $k \to +\infty$ for some $u \in H^1(\Omega\setminus\Sigma)$. By the compactness of the trace operator and by the lower semicontinuity of the H^1-norm, we have $u \in A(s,w)$ and $\|u - u_s\|_{H^1} \le \eta$.

Let us prove that u is a minimizer of $\mathcal{E}^{\eta}_{loc}(t,s,u_s)$: given $v_\eta \in A(s,w)$ a minimum of $\mathcal{E}^{\eta}_{loc}(t,s,u_s)$, thanks to Lemma 3.5 we can find a sequence ε_k such that $0<\varepsilon_k<\delta_k$, $\varepsilon_{k+1}<\varepsilon_k$, and $\|P^{-1}_{s,\varepsilon_k} v_\eta - v_\eta\|_{H^1} \leq 1/k$ for every $k\in\mathbb{N}$. Therefore, by the triangle inequality we get

$$\|P^{-1}_{s,\varepsilon_k} v_\eta - u_s\|_{H^1} \leq \eta + 1/k\,.$$

Moreover, by our choice of ε_k, $P^{-1}_{s,\varepsilon_k} v_\eta \in A(s+\varepsilon_k, w) \subseteq A(s+\delta_k,w)$. Hence, in view of (2.11) in Lemma 2.3 and of the definition of v_η, we obtain

$$\begin{aligned}\mathcal{E}(t,s,v_\eta) = \mathcal{E}^{\eta}_{loc}(t,s,u_s) \leq \mathcal{E}(t,s,u) \leq \liminf_k \, \mathcal{E}(t,s+\delta_k,u^k_\eta)\\ \leq \limsup_k \, \mathcal{E}(t,s+\delta_k,u^k_\eta) \leq \lim_k \, \mathcal{E}(t,s+\delta_k,P^{-1}_{s,\varepsilon_k} v_\eta) = \mathcal{E}(t,s,v_\eta)\,.\end{aligned} \tag{3.52}$$

where, in the last equality, we have used the strong convergence of $P^{-1}_{s,\varepsilon_k} v_\eta$ to v_η in $H^1(\Omega\backslash\Sigma)$ as $k\to+\infty$. The chain of inequalities (3.52) implies that $u\in A(s,w)$ is a minimizer of $\mathcal{E}^{\eta}_{loc}(t,s,u_s)$ and that

$$\mathcal{E}(t,s,u) = \lim_k \, \mathcal{E}(t,s+\delta_k,u^k_\eta)\,.$$

Thus, by Lemma 2.3 we get that $u^k_\eta \to u$ strongly in $H^1(\Omega\backslash\Sigma)$ as $k\to+\infty$. By Lemma 3.5, we also have $P_{s,\delta_k} u^k_\eta \to u$ in $H^1(\Omega\setminus\Sigma)$.

Passing to the lim sup in (3.51) as $k\to+\infty$ and taking into account the previous convergences, we get, as in the proofs of Theorems 3.1 and 3.2,

$$\begin{aligned}\limsup_k \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta_k,u_s)}{\delta_k}\\ \leq \lim_k \, \frac{\mathcal{E}(t,s,P_{s,\delta_k} u^k_\eta) - \mathcal{E}(t,s+\delta_k,u^k_\eta)}{\delta_k} = G(t,u,\vartheta) - 1\,.\end{aligned} \tag{3.53}$$

Taking the supremum in (3.53) among all the functions u minimizer of $\mathcal{E}^{\eta}_{loc}(t,s,u_s)$, we deduce that

$$\begin{aligned}\limsup_k \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^{\eta}_{loc}(t,s+\delta_k,u_s)}{\delta_k}\\ \leq \sup\{G(t,u_\eta,\vartheta) : u_\eta\in A(s,w) \text{ is a minimizer of } \mathcal{E}^{\eta}_{loc}(t,s,u_s)\} - 1\,.\end{aligned} \tag{3.54}$$

By a contradiction argument, (3.54) implies (3.49). It is easy to see that, as in Theorem 3.1, the supremum in (3.49) is actually a maximum.

Finally, passing to the limit in inequalities (3.48) and (3.49) as $\eta\searrow 0$, we get (3.46), and the proof is thus concluded. □

Remark 3.13. In view of Proposition 3.12, we can interpret $G(t,u_s)$ as a "local" energy release rate, in the sense that it takes into account only deformations which are close to u_s in the H^1-norm, while $\mathcal{G}^{\pm}$ are "global" energy release rates.

Since we have explicit formulas for the right and left derivatives of the reduced energy $\mathcal{E}_{min}$ in terms of the generalized energy release rates $\mathcal{G}^+$ and $\mathcal{G}^-$, we are now in a position to study the problem of existence of a quasi-static solution of our cohesive fracture model with an activation threshold. Following the ideas of [20], we look for an evolution satisfying a weak form of Griffith's criterion.

4. Quasi-static evolution

We provide a notion of quasi-static evolution based on the technique of vanishing viscosity. The solution is defined through a process of time discretization: we first solve some incremental problems and then pass to the limit as the time step vanishes. This is a typical procedure in the study of fracture mechanics, see e.g. [13], and of other rate-independent processes [23]. In order to enforce local minimality, the incremental problems are perturbed with a viscous parameter $\varepsilon>0$ which tends to zero more slowly than the time step. This approach was employed in [1, 11, 24, 25] in an abstract setting and in [19, 20, 22, 28] for the problem of crack growth.

First of all, let us fix some notation which will be used from now on: the reference configuration is described by $\overline{\Omega}$, where $\Omega \subseteq \mathbb{R}^2$ is an open, bounded, connected set with Lipschitz boundary. The crack path is given by the C^3-curve $\Sigma \subseteq \overline{\Omega}$. See Section 2 for the properties of Ω and Σ and (2.1) for the definition of admissible cracks. Given $T > 0$, we consider

$$w \in AC([0,T]; H^1(\Omega)) \qquad \text{and} \qquad f \in AC([0,T]; L^2(\Omega)) \tag{4.1}$$

which represent the Dirichlet boundary datum and the volume forces applied to Ω, respectively. In particular, $f(t,x) \cdot \xi$ will substitute the function $g(t,x,\xi)$ defined in Section 2. For simplicity of notation, we will not indicate the dependence of f and w on the space variable x.

Finally, we assume that the function $\varphi\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a further property of differentiability: we suppose that $\varphi(\cdot, \xi) \in AC([0,T]; \mathbb{R})$ for every $\xi \in \mathbb{R}^2$ and that there exist $p \in [1, +\infty)$ and $a_3 \in L^1([0,T])$ with $a_3 \geq 0$ such that

$$|D_t \varphi(t,\xi)| \leq a_3(t)(1 + |\xi|^p) \qquad \text{for a.e. } t \in [0,T] \text{ and every } \xi \in \mathbb{R}^2 . \tag{4.2}$$

Fixed $s \in [0,L]$ and $t \in [0,T]$, the energy of the system is, similar to (2.8),

$$\mathcal{E}(t,s,u) := \frac{1}{2} \int_{\Omega_s} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \, \mathrm{d}x - \int_{\Omega_s} f(t) \cdot u \, \mathrm{d}x + \int_{\Gamma_s} \varphi(t, [u]) \, \mathrm{d}\mathcal{H}^1 + s , \tag{4.3}$$

for every $u \in A(s, w(t))$, the set of admissible displacements at time t, defined as in (2.10). We recall that, different from the Barenblatt's model, we assume the cohesive part of the energy to be completely reversible, while the dissipative term of the energy is given by the length of the crack s.

Since the boundary datum is a function of $t \in [0,T]$, we slightly change the notation for the reduced energy $\mathcal{E}_{min}$ and for the energy release rates: for every $s \in [0,L]$ and every $t \in [0,T]$, we define, similar to (2.13),

$$\mathcal{E}_{min}(t,s) := \min_{u \in A(s,w(t))} \mathcal{E}(t,s,u) . \tag{4.4}$$

Remark 4.1. By (4.1), all the results about $\mathcal{E}_{min}$ proved in Section 3 hold: by Lemmas 3.6 and 3.7 the reduced energy $\mathcal{E}_{min}$ is lower semicontinuous on $[0,T] \times [0,L]$ and continuous on $[0,T] \times (0,L)$. By Theorems 3.1 and 3.2, it has right and left derivatives with respect to the crack length s which are now denoted by $\partial_s^+ \mathcal{E}_{min}(t,s)$ and $\partial_s^- \mathcal{E}_{min}(t,s)$ for every $(t,s) \in [0,T] \times (0,L)$. Moreover,

$$\partial_s^+ \mathcal{E}_{min}(t,s) = 1 - \mathcal{G}^+(t,s,w(t)) ,$$
$$\partial_s^- \mathcal{E}_{min}(t,s) = 1 - \mathcal{G}^-(t,s,w(t)) ,$$

where $\mathcal{G}^\pm$ are defined as in (3.6) and in (3.8).

With an abuse of notation, we now set

$$\mathcal{G}^\pm(t,s) := \mathcal{G}^\pm(t,s,w(t)) ,$$

where, in the formulas (3.1), (3.6), and (3.8) for $\mathcal{G}^\pm(t,s,w(t))$, the function $g(t,x,u)$ is replaced by $f(t,x) \cdot u$ for an admissible displacement u.

Remark 4.2. Since w and f are continuous in time, a simple application of Proposition 3.11 shows that $\mathcal{G}^+$ is upper semicontinuous and $\mathcal{G}^-$ is lower semicontinuous on $[0,T] \times (0,L)$.

We now discuss briefly the time incremental minimum problems and then give our definitions of *viscous* and *quasi-static evolutions*.

For every $k \in \mathbb{N}$ we fix a subdivision $\{t_i^k\}_{i=0}^k$ of the time interval $[0,T]$ with $t_i^k := i\tau_k$ and $\tau_k := T/k$. Given $\varepsilon > 0$, we define recursively the solution $s_\varepsilon^{k,i}$ to incremental minimum problems: let $s_\varepsilon^{k,0} := s_0$, where $s_0 \in (0, L)$ is the initial condition, and, for $i \geq 1$, let $s_\varepsilon^{k,i}$ be a solution to

$$\min \left\{ \mathcal{E}_{min}(t_i^k, s) + \frac{\varepsilon}{2} \frac{(s - s_\varepsilon^{k,i-1})^2}{\tau_k} \, : \, s \in [s_\varepsilon^{k,i-1}, L] \right\}. \tag{4.5}$$

We postpone the proof of existence of a solution to (4.5) to the next section, see Proposition 5.1, to comment briefly on the function which appears in (4.5). This function is the sum of two terms: the reduced energy $\mathcal{E}_{min}$ defined by (4.4), which represents the energy of the system at the equilibrium for a fixed $s \in [0, L]$, and a perturbation term driven by $\varepsilon > 0$ which enforces a local minimization of the energy with respect to s. This kind of approximation should guarantee that the evolution in the limit follows "local minimizers" of the energy (see [8, 11, 20, 22, 24, 25, 27] for further discussions and applications).

The passage to the limit will be performed in two steps: we let first $k \to +\infty$ and find a viscous evolution for every $\varepsilon > 0$, and, finally, we obtain a quasi-static evolution as the parameter ε tends to zero.

We introduce the concept of *failure time* and of *jump set*, important from now on.

Definition 4.3. Let $a, b > 0$ and let $s \colon [0, a] \to [0, b]$ be a monotone non-decreasing function. We define

- the *failure time* $\mathcal{T}(s)$ of s
$$\mathcal{T}(s) := \sup \{ t \in [0, a] : \, s(t) < b \} ;$$
- the *jump set* $J(s)$ of s
$$J(s) := \{ t \in [0, a] : \, s \text{ is discontinuous at } t \} .$$

Remark 4.4. We notice that $\mathcal{T}$ is lower semicontinuous with respect to the pointwise convergence, that is, if $s_k \to s$ pointwise, then

$$\mathcal{T}(s) \leq \liminf_k \, \mathcal{T}(s_k) .$$

Of course, from now on we will consider only monotone non-decreasing functions from $[0, T]$ with values in $[0, L]$.

We now give a definition of *viscous evolution* and *quasi-static evolution* for the cohesive crack growth problem.

Definition 4.5. Let $\varepsilon > 0$ and $s_0 \in (0, L)$. We say that a monotone non-decreasing function $s_\varepsilon \in H^1([0, T])$ is a *viscous evolution for the cohesive crack growth problem* with $s_\varepsilon(0) = s_0$ if it satisfies the following rate-dependent Griffith's criterion:
for a.e. $t \in [0, \mathcal{T}(s_\varepsilon))$

(1) $\dot{s}_\varepsilon(t) \geq 0$;
(2) $\mathcal{G}^-(t, s_\varepsilon(t)) - 1 - \varepsilon \dot{s}_\varepsilon(t) \leq 0$;
(3) $(\mathcal{G}^+(t, s_\varepsilon(t)) - 1 - \varepsilon \dot{s}_\varepsilon(t)) \, \dot{s}_\varepsilon(t) \geq 0$.

In Section 6 we will prove the following existence theorem.

Theorem 4.6. *Let $\varepsilon > 0$, $f \in AC([0, T]; L^2(\Omega))$, and $w \in AC([0, T]; H^1(\Omega))$. Then, for every $s_0 \in (0, L)$ there exists a viscous evolution $s_\varepsilon \in H^1([0, T])$ for the cohesive crack growth problem with $s_\varepsilon(0) = s_0$.*

Definition 4.7. Let $s_0 \in (0, L)$. We say that a monotone non-decreasing function $s \in BV([0, T])$ is a *quasi-static evolution for the cohesive crack growth problem* with $s(0) = s_0$ if it satisfies:

(1) for every $t \in [0, \mathcal{T}(s)) \setminus J(s)$:
$$\mathcal{G}^-(t, s(t)) \leq 1 ;$$

(2) for every $t \in [0, \mathcal{T}(s)) \cap J(s)$:

$$\mathcal{G}^+(t,\sigma) \geq 1 \qquad \text{for every } \sigma \in [s(t^-), s(t^+)];$$

(3) if $t \in [0, \mathcal{T}(s))$ and $\mathcal{G}^+(t, s(t)) < 1$, then s is differentiable at t and $\dot{s}(t) = 0$.

Remark 4.8. We notice that in [20, Definition 2.1], the notion of evolution given in Definition 4.7 is called *local energetic solution.* It generalizes the definition of local energetic solution in [19, Definition 2.3] to the case of a non-differentiable reduced energy $\mathcal{E}_{min}$.

We can now state the main theorem of this paper.

Theorem 4.9. *Let $f \in AC([0,T]; L^2(\Omega))$ and $w \in AC([0,T]; H^1(\Omega))$. Then, for every $s_0 \in (0, L)$ there exists a quasi-static evolution $s \in BV([0,T])$ for the cohesive crack growth problem with $s(0) = s_0$.*

Remark 4.10. Theorem 4.9 will be proved in Section 7. Its proof will follow the ideas of [20, Theorem 4.2]. The main difference will be that, starting from the discrete solutions to (4.5), we first construct a viscous evolution as the parameter k tends to $+\infty$ (see Theorem 4.6) and then, passing to the limit as $\varepsilon \searrow 0$, we obtain a quasi-static evolution according to Definition 4.7, while in [20] these steps are carried out simultaneously working with a parameter $k = k(\varepsilon)$.

Finally, we remark that in the proof of Theorem 4.9 we will also show that if $\{s_\varepsilon\}_{\varepsilon>0}$ is a sequence of viscous evolutions for the cohesive crack growth problem with $s_\varepsilon(0) = s_0$, then, up to a subsequence, s_ε converges pointwise to a quasi-static evolution $s \in BV([0,T])$.

5. The discrete-time problems

We now discuss the properties of the discrete-time solutions $s_\varepsilon^{k,i}$ introduced in Section 4. First of all, we have to prove that they are well defined.

Proposition 5.1. *For every $\varepsilon > 0$, $k \in \mathbb{N}$, and $i = 1, \dots, k$, there exists a solution to* (4.5).

Proof. We exploit the direct method of the calculus of variations. Let $\varepsilon > 0$, $k \in \mathbb{N}$, and $i = 1, \dots, k$ be fixed. Let $s_j \in [s_\varepsilon^{k,i-1}, L]$ be a minimizing sequence for the minimum problem (4.5). Up to a subsequence, we may assume that there exists $s \in [s_\varepsilon^{k,i-1}, L]$ such that $s_j \to s$. Taking into account Lemma 3.6, we have that

$$\mathcal{E}_{min}(t_i^k, s) \leq \liminf_j \mathcal{E}_{min}(t_i^k, s_j),$$

hence s is a solution to (4.5). □

We now provide some a priori bounds on the incremental solutions. In what follows, $w_i^k := w(t_i^k)$ and $f_i^k := f(t_i^k)$.

Proposition 5.2. *There exists $C > 0$ such that, for every $k \in \mathbb{N}$ and every $\varepsilon > 0$, the following inequality holds*

$$\frac{\varepsilon}{2} \sum_{j=1}^{k} \frac{(s_\varepsilon^{k,j} - s_\varepsilon^{k,j-1})^2}{\tau_k} \leq C. \tag{5.1}$$

Proof. During the proof of this proposition, we will denote by u_i^k a minimizer of $\mathcal{E}(t_i^k, s_\varepsilon^{k,i}, \cdot)$ in $A(s_\varepsilon^{k,i}, w_i^k)$ and by Ω_i^k, Γ_i^k the sets $\Omega_{s_\varepsilon^{k,i}}$, $\Gamma_{s_\varepsilon^{k,i}}$, respectively.

First, let us prove that the minimizers u_i^k are bounded in $H^1(\Omega \setminus \Sigma)$ uniformly with respect to $k \in \mathbb{N}$, $i = 1, \dots, k$, and $\varepsilon > 0$. Indeed, $w_i^k \in A(s_\varepsilon^{k,i}, w_i^k)$ and, by (2.2), (2.6), the hypothesis $\varphi(t_i^k, 0) = 0$, and Hölder's inequality, we get

$$\mathcal{E}(t_i^k, s_\varepsilon^{k,i}, u_i^k) = \mathcal{E}_{min}(t_i^k, s_\varepsilon^{k,i}) \leq \mathcal{E}(t_i^k, s_\varepsilon^{k,i}, w_i^k) \leq \frac{\beta}{2} \|w_i^k\|_{H^1}^2 + \|f_i^k\|_{L^2} \|w_i^k\|_{H^1} + L. \tag{5.2}$$

From (4.1) and (5.2), we deduce that, for some $c > 0$,

$$\mathcal{E}_{min}(t^k_i, s^{k,i}_\varepsilon) = \mathcal{E}(t^k_i, s^{k,i}_\varepsilon, u^k_i) \leq c\,. \tag{5.3}$$

Therefore, since (2.2) holds and φ satisfies (2.7) uniformly in t, applying Hölder's and Korn's inequalities to (5.3) we obtain

$$c_1\|u^k_i\|^2_{H^1} - \|f^k_i\|_{L^2}\|u^k_i\|_{H^1} - c_2 \leq \mathcal{E}_{min}(t^k_i, s^{k,i}_\varepsilon) \leq c \tag{5.4}$$

for some c_1, $c_2 > 0$. By the absolute continuity of f and by Young's inequality, from (5.4) it follows that there exists $M > 0$ such that for every k, every $i = 1, \ldots, k$, and every $\varepsilon > 0$:

$$\|u^k_i\|_{H^1} \leq M \qquad \text{and} \qquad \mathcal{E}_{min}(t^k_i, s^{k,i}_\varepsilon) \geq -M\,. \tag{5.5}$$

Let $k \in \mathbb{N}$, $i = 1, \ldots, k$, and $\varepsilon > 0$ be fixed. Since $u^k_{i-1} + w^k_i - w^k_{i-1} \in A(s^{k,i-1}_\varepsilon, w^k_i)$, we have, by definition of $s^{k,i}_\varepsilon$ and of the reduced energy $\mathcal{E}_{min}$,

$$\begin{aligned}
\mathcal{E}_{min}(t^k_i, s^{k,i}_\varepsilon) + \frac{\varepsilon}{2}\,\frac{(s^{k,i}_\varepsilon - s^{k,i-1}_\varepsilon)^2}{\tau_k} &\leq \mathcal{E}_{min}(t^k_i, s^{k,i-1}_\varepsilon) \\
&\leq \mathcal{E}(t^k_i, s^{k,i-1}_\varepsilon, u^k_{i-1} + w^k_i - w^k_{i-1}) \\
&= \mathcal{E}_{min}(t^k_{i-1}, s^{k,i-1}_\varepsilon) + \int_{\Omega^k_{i-1}} \mathbb{C}\mathrm{E}u^k_{i-1} \cdot \mathrm{E}(w^k_i - w^k_{i-1})\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \mathbb{C}\mathrm{E}(w^k_i - w^k_{i-1}) \cdot \mathrm{E}(w^k_i - w^k_{i-1})\,\mathrm{d}x - \int_{\Omega^k_{i-1}} (f^k_i - f^k_{i-1}) \cdot u^k_{i-1}\,\mathrm{d}x \\
&\quad - \int_\Omega f^k_i \cdot (w^k_i - w^k_{i-1})\,\mathrm{d}x + \int_{t^k_{i-1}}^{t^k_i} \int_{\Gamma^k_{i-1}} D_t\varphi(\tau, [u^k_{i-1}])\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}\tau\,.
\end{aligned} \tag{5.6}$$

Thanks to (4.1), (4.2), (5.5), to Hölder's inequality, and to the continuity of the trace operator, (5.6) becomes

$$\begin{aligned}
&\mathcal{E}_{min}(t^k_i, s^{k,i}_\varepsilon) + \frac{\varepsilon}{2}\,\frac{(s^{k,i}_\varepsilon - s^{k,i-1}_\varepsilon)^2}{\tau_k} \\
&\quad \leq \mathcal{E}_{min}(t^k_{i-1}, s^{k,i-1}_\varepsilon) + \beta M \int_{t^k_{i-1}}^{t^k_i} \|\dot{w}(\tau)\|_{H^1}\,\mathrm{d}\tau + \beta W_k \int_{t^k_{i-1}}^{t^k_i} \|\dot{w}(\tau)\|_{H^1}\,\mathrm{d}\tau \\
&\quad\quad + M \int_{t^k_{i-1}}^{t^k_i} \|\dot{f}(\tau)\|_{L^2}\,\mathrm{d}\tau + F \int_{t^k_{i-1}}^{t^k_i} \|\dot{w}(\tau)\|_{H^1}\,\mathrm{d}\tau + (L + CM^p) \int_{t^k_{i-1}}^{t^k_i} a_3(\tau)\,\mathrm{d}\tau\,,
\end{aligned} \tag{5.7}$$

where $L = \mathcal{H}^1(\Sigma)$, C is a positive constant independent of k, and

$$\begin{aligned}
W_k &:= \frac{1}{2} \sup_{j=1,\ldots,k} \|w^k_j - w^k_{j-1}\|_{H^1}\,, \\
F &:= \sup_{t\in[0,T]} \|f(t)\|_{L^2}\,.
\end{aligned}$$

Adding to both sides of (5.7) the term $\frac{\varepsilon}{2}\frac{(s^{k,i-1}_\varepsilon - s^{k,i-2}_\varepsilon)^2}{\tau_k}$ and iterating the previous argument, we get

$$\begin{aligned}
&\mathcal{E}_{min}(t^k_i, s^{k,i}_\varepsilon) + \frac{\varepsilon}{2}\sum_{j=1}^{i} \frac{(s^{k,j}_\varepsilon - s^{k,j-1}_\varepsilon)^2}{\tau_k} \\
&\quad \leq \mathcal{E}_{min}(0, s_0) + (\beta M + \beta W_k + F) \int_0^T \|\dot{w}(t)\|_{H^1}\,\mathrm{d}t \\
&\quad\quad + M \int_0^T \|\dot{f}(t)\|_{L^2}\,\mathrm{d}t + (L + CM^p) \int_0^T a_3(t)\,\mathrm{d}t\,.
\end{aligned} \tag{5.8}$$

By (4.1), $F < +\infty$ and $W_k \to 0$ as $k \to +\infty$, so (5.5) and (5.8) imply (5.1), and the proof is thus concluded. □

For every k and every $\varepsilon > 0$, let us define the piecewise constant interpolations $\bar{t}_k(t) := t_i^k$ and $\bar{s}_\varepsilon^k(t) := s_i^k$ for $t \in (t_{i-1}^k, t_i^k]$, and the piecewise affine interpolation function

$$s_\varepsilon^k(t) := s_\varepsilon^{k,i-1} + \frac{s_\varepsilon^{k,i} - s_\varepsilon^{k,i-1}}{\tau_k}(t - t_{i-1}^k) \qquad \text{for } t \in (t_{i-1}^k, t_i^k]\,.$$

The next proposition is the equivalent of the Griffith's criterion in the discrete setting.

Proposition 5.3. *For every $k \in \mathbb{N}$, every $\varepsilon > 0$, and every $t \in [0, \mathcal{T}(\bar{s}_\varepsilon^k))$ we have:*

(*a*) $\dot{s}_\varepsilon^k(t) \geq 0$;
(*b*) $\mathcal{G}^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t)) - 1 - \varepsilon \dot{s}_\varepsilon^k(t) \leq 0$;
(*c*) $(\mathcal{G}^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t)) - 1 - \varepsilon \dot{s}_\varepsilon^k(t))\, \dot{s}_\varepsilon^k(t) = 0$.

Proof. Property (a) follows immediately from the definition of s_ε^k.

Let us prove (b). Fix $t \in (t_{i-1}^k, t_i^k]$ such that $t < \mathcal{T}(\bar{s}_\varepsilon^k)$. By construction, for every $\sigma \geq s_\varepsilon^{k,i-1}$ we have

$$\mathcal{E}_{min}(t_i^k, s_\varepsilon^{k,i}) + \frac{\varepsilon}{2}\frac{(s_\varepsilon^{k,i} - s_\varepsilon^{k,i-1})^2}{\tau_k} \leq \mathcal{E}_{min}(t_i^k, \sigma) + \frac{\varepsilon}{2}\frac{(\sigma - s_\varepsilon^{k,i-1})^2}{\tau_k}\,. \tag{5.9}$$

If $\sigma > s_\varepsilon^{k,i}$, dividing (5.9) by $\sigma - s_\varepsilon^{k,i}$, we obtain

$$\frac{\mathcal{E}_{min}(t_i^k, s_\varepsilon^{k,i}) - \mathcal{E}_{min}(t_i^k, \sigma)}{\sigma - s_\varepsilon^{k,i}} - \frac{\varepsilon}{2\tau_k}\,\frac{(\sigma - s_\varepsilon^{k,i-1})^2 - (s_\varepsilon^{k,i} - s_\varepsilon^{k,i-1})^2}{\sigma - s_\varepsilon^{k,i}} \leq 0\,,$$

so, passing to the limit as $\sigma \searrow s_\varepsilon^{k,i}$ and taking into account Theorem 3.1, we get (b).

If $\dot{s}_\varepsilon^k(t) = 0$, then (c) is clearly satisfied. Otherwise, $s_\varepsilon^{k,i} > s_\varepsilon^{k,i-1}$, hence we can consider (5.9) with $\sigma \in (s_\varepsilon^{k,i-1}, s_\varepsilon^{k,i})$. Dividing by $\sigma - s_\varepsilon^{k,i}$ and passing to the limit as $\sigma \nearrow s_\varepsilon^{k,i}$, from Theorem 3.2 it follows that

$$\mathcal{G}^-(t_i^k, \bar{s}_\varepsilon^k(t)) - 1 - \varepsilon \dot{s}_\varepsilon^k(t) \geq 0\,. \tag{5.10}$$

Thanks to point (a) of Proposition 3.11 and to the previous step, we deduce that

$$\mathcal{G}^+(t_i^k, \bar{s}_\varepsilon^k(t)) = \mathcal{G}^-(t_i^k, \bar{s}_\varepsilon^k(t))\,,$$

hence (c) holds. □

6. Viscous evolution

This section is devoted to the proof of Theorem 4.6. For every $\varepsilon > 0$, we pass to the limit as $k \to +\infty$, in order to find a viscous evolution.

Let us prove the following compactness result.

Proposition 6.1. *For every $\varepsilon > 0$, there exists $s_\varepsilon \in H^1([0,T])$ such that*

(*a*) *up to a subsequence, $s_\varepsilon^k \rightharpoonup s_\varepsilon$ weakly in $H^1([0,T])$ and $s_\varepsilon^k, \bar{s}_\varepsilon^k \to s_\varepsilon$ uniformly in $[0,T]$;*
(*b*) *s_ε is monotone non-decreasing;*
(*c*) *$s_\varepsilon(0) = s_0$;*
(*d*) *$\varepsilon\|\dot{s}_\varepsilon\|_{L^2}^2$ is uniformly bounded with respect to $\varepsilon > 0$.*

Proof. Proposition 5.2 implies that $\varepsilon\|\dot{s}_\varepsilon^k\|_{L^2}^2$ is uniformly bounded with respect to $k \in \mathbb{N}$ and $\varepsilon > 0$, thus the sequence $(s_\varepsilon^k)_k$ is bounded in $H^1([0,T])$. Therefore, for every $\varepsilon > 0$ there exists $s_\varepsilon \in H^1([0,T])$ such that, up to a subsequence, $s_\varepsilon^k \rightharpoonup s_\varepsilon$ weakly in $H^1([0,T])$. In particular, by (5.1) and by the lower semicontinuity of the L^2-norm, property (d) holds.

Applying the Ascoli-Arzelà theorem, up to a further subsequence we can assume that $s^k_\varepsilon \to s_\varepsilon$ uniformly in $[0,T]$ as $k \to +\infty$. Since, by (5.1),

$$|s^k_\varepsilon(t) - \bar{s}^k_\varepsilon(t)| \leq \left| \frac{s^{k,i}_\varepsilon - s^{k,i-1}_\varepsilon}{\tau_k} (t - t^k_{i-1}) \right| + |s^{k,i}_\varepsilon - s^{k,i-1}_\varepsilon| \leq C\sqrt{\tau_k}$$

for some $C > 0$, we deduce that $\bar{s}^k_\varepsilon \to s_\varepsilon$ uniformly in $[0,T]$, hence (a) is proved.

Since, by construction, $s^k_\varepsilon(0) = s_0$ for every k, it follows that $s_\varepsilon(0) = s_0$. Finally, from the monotonicity of $\bar{s}^k_\varepsilon$ and the uniform convergence proved in (a), we deduce that s_ε is monotone non-decreasing. □

We are now ready to prove Theorem 4.6

Proof of Theorem 4.6. Fix $\varepsilon > 0$. Let us prove that $s_\varepsilon \in H^1([0,T])$ found in Proposition 6.1 is a viscous evolution for the cohesive crack growth with $s_\varepsilon(0) = s_0$.

Since $s_\varepsilon \in H^1([0,T])$, its derivative $\dot{s}_\varepsilon$ exists a.e. in $[0,T]$ and is nonnegative by monotonicity (see (b) of Proposition 6.1).

To prove properties (2) and (3) of Definition 4.5, in view of Remark 4.4 we have to distinguish between two possibilities:

$$\mathcal{T}(s_\varepsilon) = \lim_k \mathcal{T}(\bar{s}^k_\varepsilon) \qquad \text{or} \qquad \mathcal{T}(s_\varepsilon) < \limsup_k \mathcal{T}(\bar{s}^k_\varepsilon)\,. \tag{6.1}$$

Let us consider the first case. By properties (a) of Proposition 3.11 and (b) of Proposition 5.3, for every $\psi \in L^2([0,T])$ with $\psi \geq 0$ we have

$$\int_0^{\mathcal{T}(\bar{s}^k_\varepsilon)} (\varepsilon \dot{s}^k_\varepsilon(t) + 1 - \mathcal{G}^-(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)))\, \psi(t)\, \mathrm{d}t \geq 0\,. \tag{6.2}$$

By the weak convergence $s^k_\varepsilon \rightharpoonup s_\varepsilon$ in $H^1([0,T])$, taking the lim sup as $k \to +\infty$ in (6.2) we get

$$\int_0^{\mathcal{T}(s_\varepsilon)} (\varepsilon \dot{s}_\varepsilon(t) + 1)\, \psi(t)\, \mathrm{d}t - \liminf_k \int_0^T \mathcal{G}^-(\bar{t}_k(t), \bar{s}^k_\varepsilon(t))\, \psi(t)\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}^k_\varepsilon))}(t)\, \mathrm{d}t \geq 0\,, \tag{6.3}$$

where we denote by $\mathbf{1}_E$ the characteristic function of the set E. By Proposition 3.11,

$$\mathcal{G}^-(\bar{t}_k(t), \bar{s}^k_\varepsilon(t))\, \psi(t)\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}^k_\varepsilon))}(t) \geq 0 \qquad \text{for a.e. } t \in [0,T]\,.$$

Therefore, applying Fatou's lemma to the last term in (6.3), taking into account (a) of Proposition 6.1, the convergence $\bar{t}_k(t) \to t$ for every $t \in [0,T]$, and the lower semicontinuity of $\mathcal{G}^-$, we deduce that

$$\int_0^{\mathcal{T}(s_\varepsilon)} (\varepsilon \dot{s}_\varepsilon(t) + 1 - \mathcal{G}^-(t, s_\varepsilon(t)))\, \psi(t)\, \mathrm{d}t \geq 0\,. \tag{6.4}$$

Inequality (6.4) holds for every $\psi \in L^2([0,T])$, $\psi \geq 0$, hence we have proved property (2) of Definition 4.5.

In order to prove condition (3), we first notice that, thanks to the bound (5.5), to the definition of $\mathcal{G}^+$ (see (3.1) and (3.6)), and to the hypotheses (2.2), (2.6), and (4.1), there exists $C > 0$ such that

$$\mathcal{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) \leq C \tag{6.5}$$

uniformly with respect to $k \in \mathbb{N}$, $\varepsilon > 0$, and $t \in [0, \mathcal{T}(\bar{s}^k_\varepsilon))$.

Integrating (c) of Proposition 5.3 over the interval $[0, \mathcal{T}(\bar{s}^k_\varepsilon))$, we obtain

$$\int_0^{\mathcal{T}(\bar{s}^k_\varepsilon)} (\mathcal{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) - 1 - \varepsilon \dot{s}^k_\varepsilon(t))\, \dot{s}^k_\varepsilon(t)\, \mathrm{d}t = 0\,. \tag{6.6}$$

Passing to the lim sup in (6.6) as $k \to +\infty$, by Proposition 6.1 and the lower semicontinuity of the L^2-norm, we get

$$
\begin{aligned}
0 &= \limsup_k \int_0^{\mathcal{T}(\bar{s}_\varepsilon^k)} (\mathcal{G}^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t)) - 1 - \varepsilon \dot{s}_\varepsilon^k(t))\, \dot{s}_\varepsilon^k(t)\,\mathrm{d}t \\
&\le \limsup_k \int_0^{\mathcal{T}(\bar{s}_\varepsilon^k)} \mathcal{G}^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t))\, \dot{s}_\varepsilon^k(t)\,\mathrm{d}t - \int_0^{\mathcal{T}(s_\varepsilon)} \dot{s}_\varepsilon(t)\,\mathrm{d}t - \varepsilon \liminf_k \|\dot{s}_\varepsilon^k \mathbf{1}_{[0,\mathcal{T}(\bar{s}_\varepsilon^k))}\|_{L^2}^2 \\
&\le \limsup_k \int_0^T \mathcal{G}^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t))\, \dot{s}_\varepsilon^k(t)\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}_\varepsilon^k))}(t)\,\mathrm{d}t - \int_0^{T(s_\varepsilon)} (1+\varepsilon\dot{s}_\varepsilon(t))\,\dot{s}_\varepsilon(t)\,\mathrm{d}t\,.
\end{aligned}
\tag{6.7}
$$

By property (a) of Proposition 5.3, we can continue the chain of inequalities (6.7), obtaining

$$
\begin{aligned}
0 &\le \limsup_k \int_0^T \Big(\sup_{h\ge k} \mathcal{G}^+(\bar{t}_h(t), \bar{s}_\varepsilon^h(t))\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}_\varepsilon^h))}(t)\Big)\, \dot{s}_\varepsilon^k(t)\,\mathrm{d}t - \int_0^{T(s_\varepsilon)} (1+\varepsilon\dot{s}_\varepsilon(t))\,\dot{s}_\varepsilon(t)\,\mathrm{d}t \\
&= \limsup_k \int_0^T F_k(t)\, \dot{s}_\varepsilon^k(t)\,\mathrm{d}t - \int_0^{T(s_\varepsilon)} (1+\varepsilon\dot{s}_\varepsilon(t))\,\dot{s}_\varepsilon(t)\,\mathrm{d}t\,,
\end{aligned}
\tag{6.8}
$$

where we have set

$$
F_k(t) := \sup_{h\ge k} \mathcal{G}^+(\bar{t}_h(t), \bar{s}_\varepsilon^h(t))\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}_\varepsilon^h))}(t)
$$

for every $t \in [0,T]$ and every $k \in \mathbb{N}$.

By definition, $F_k(t)$ converges pointwise to

$$
F(t) := \limsup_k \mathcal{G}^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t))\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}_\varepsilon^k))}(t) = \limsup_k \mathcal{G}^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t))\, \mathbf{1}_{[0,\mathcal{T}(s_\varepsilon))}(t)\,.
$$

By estimate (6.5) and the dominated convergence theorem, $F_k \to F$ strongly in $L^2([0,T])$. Therefore, by Proposition 6.1, (6.8) becomes

$$
\int_0^{\mathcal{T}(s_\varepsilon)} (F(t) - 1 - \varepsilon\dot{s}_\varepsilon(t))\,\dot{s}_\varepsilon(t)\,\mathrm{d}t \ge 0\,.
$$

Finally, by Proposition 3.11, we deduce that $F(t) \le \mathcal{G}^+(t, s_\varepsilon(t))\,\mathbf{1}_{[0,\mathcal{T}(s_\varepsilon))}(t)$, hence, thanks to the nonnegativity of $\dot{s}_\varepsilon$, we obtain

$$
\int_0^{\mathcal{T}(s_\varepsilon)} (\mathcal{G}^+(t, s_\varepsilon(t)) - 1 - \varepsilon\dot{s}_\varepsilon(t))\,\dot{s}_\varepsilon(t)\,\mathrm{d}t \ge 0\,.
\tag{6.9}
$$

With the same argument, we can prove that (6.9) holds on every $I \subseteq [0, \mathcal{T}(s_\varepsilon))$ measurable. This implies property (3) of Definition 4.5.

For the second case in (6.1), we can assume, up to a further subsequence, that $\mathcal{T}(s_\varepsilon) < \mathcal{T}(\bar{s}_\varepsilon^k)$ for every k. Therefore, we just have to replace $\mathcal{T}(\bar{s}_\varepsilon^k)$ with $\mathcal{T}(s_\varepsilon)$ in (6.2) and (6.6) and repeat the previous arguments. This concludes the proof of the theorem. □

7. The quasi-static evolution

We now pass to the limit as the parameter ε tends to zero. This allows us to prove the existence of a quasi-static evolution of the cohesive crack growth problem in the sense of Definition 4.7.

In order to prove the properties of Definition 4.7, we need the following technical lemma.

Lemma 7.1. *Let $z, z_k : [0,T] \to \mathbb{R}$ be non-decreasing monotone functions such that $z_k(t) \to z(t)$ for every $t \in [0,T]$. Let z be continuous at $\hat{t} \in [0,T]$. Then, for every $t_k \to \hat{t}$ in $[0,T]$ it is $z_k(t_k) \to z(\hat{t})$.*

Proof. Fix $\eta > 0$. By continuity, there exists $\delta > 0$ such that $|z(\hat{t}) - z(t)| < \eta$ for every $|t - \hat{t}| < 2\delta$, $t \in [0, T]$.

Since $t_k \to \hat{t}$, there exists $\bar{k} \in \mathbb{N}$ such that $|t_k - \hat{t}| < \delta$ for every $k \geq \bar{k}$, so that

$$|z(t_k) - z(\hat{t})| < \eta$$

for every $k \geq \bar{k}$. By monotonicity, $z(\hat{t} - \delta) \leq z(t_k) \leq z(\hat{t} + \delta)$ for every $k \geq \bar{k}$.

Pointwise convergence implies that, up to a redefinition of $\bar{k}$,

$$|z_k(\hat{t} - \delta) - z(\hat{t} - \delta)| < \eta \qquad \text{and} \qquad |z_k(\hat{t} + \delta) - z(\hat{t} + \delta)| < \eta$$

for every $k \geq \bar{k}$.

By continuity of z and the choice of δ, we have $|z(\hat{t}) - z(\hat{t} \pm \delta)| < \eta$. Then, by monotonicity and the above inequalities, we get

$$z(\hat{t}) - 2\eta < z(\hat{t} - \delta) - \eta < z_k(\hat{t} - \delta) \leq z_k(t_k) \leq z_k(\hat{t} + \delta) < z(\hat{t} + \delta) + \eta < z(\hat{t}) + 2\eta$$

for $k \geq \bar{k}$. Being $\eta > 0$ arbitrary, the thesis follows. □

We are now ready to prove Theorem 4.9.

Proof of Theorem 4.9. Let $\varepsilon_k \searrow 0$ and let s_{ε_k} be a sequence of viscous evolutions for the cohesive crack growth problem. Since s_{ε_k} are monotone non-decreasing and uniformly bounded in time, by Helly's theorem there exists $s \in BV([0, T])$ monotone non-decreasing such that, up to a subsequence, $s_{\varepsilon_k} \to s$ pointwise in $[0, T]$. Let us prove that s is a quasi-static evolution of the cohesive crack growth problem with $s(0) = s_0$.

Since $s_{\varepsilon_k}(0) = s_0$, of course $s(0) = s_0$. We already know that s is monotone non-decreasing, thus it remains to prove that s satisfies the weak Griffith's principle, that is, properties (1), (2), and (3) of Definition 4.7.

Let us prove condition (1). We argue as in the proof of Theorem 4.6. By Remark 4.4, we distinguish between the two possibilities

$$\mathcal{T}(s) = \lim_k \mathcal{T}(s_{\varepsilon_k}) \qquad \text{or} \qquad \mathcal{T}(s) < \limsup_k \mathcal{T}(s_{\varepsilon_k}) . \tag{7.1}$$

In the first case, by property (2) of Definition 4.5 we have, for every $\psi \in L^2([0, T])$ with $\psi \geq 0$,

$$\int_0^{\mathcal{T}(s_{\varepsilon_k})} (1 + \varepsilon_k \dot{s}_{\varepsilon_k}(t) - \mathcal{G}^-(t, s_{\varepsilon_k}(t)))\, \psi(t)\, \mathrm{d}t \geq 0 . \tag{7.2}$$

Thanks to (d) of Proposition 6.1, we deduce that $\varepsilon_k \dot{s}_{\varepsilon_k} \to 0$ in $L^2([0, T])$ as $k \to +\infty$. Therefore, passing to the lim sup as $k \to +\infty$ in (7.2), we get

$$\begin{aligned} 0 &\leq \limsup_k \int_0^{\mathcal{T}(s_{\varepsilon_k})} (1 + \varepsilon_k \dot{s}_{\varepsilon_k}(t) - \mathcal{G}^-(t, s_{\varepsilon_k}(t)))\, \psi(t)\, \mathrm{d}t \\ &= \int_0^{\mathcal{T}(s)} \psi(t)\, \mathrm{d}t - \liminf_k \int_0^T \mathcal{G}^-(t, s_{\varepsilon_k}(t))\, \psi(t)\, \mathbf{1}_{[0, \mathcal{T}(s_{\varepsilon_k}))}(t)\, \mathrm{d}t . \end{aligned} \tag{7.3}$$

Applying Fatou's lemma to (7.3), taking into account the lower semicontinuity of $\mathcal{G}^-$ and the convergence $\mathcal{T}(s_{\varepsilon_k}) \to \mathcal{T}(s)$, we obtain

$$\int_0^{\mathcal{T}(s)} (1 - \mathcal{G}^-(t, s(t)))\, \psi(t)\, \mathrm{d}t \geq 0$$

for every $\psi \in L^2([0, T])$ with $\psi \geq 0$, hence

$$\mathcal{G}^-(t, s(t)) \leq 1 \qquad \text{for a.e. } t \in [0, \mathcal{T}(s)) . \tag{7.4}$$

In particular, (7.4) is true for every $t \in [0, \mathcal{T}(s)) \backslash J(s)$.

For the second case of (7.1), we may assume, up to a subsequence, that $\mathcal{T}(s) < \mathcal{T}(s_{\varepsilon_k})$ for every k. Then, we have to replace $\mathcal{T}(s_{\varepsilon_k})$ with $\mathcal{T}(s)$ in (7.2) and repeat the previous argument. Thus, property (1) of Definition 4.7 holds.

We now prove property (2). Let $t \in [0, \mathcal{T}(s)) \cap J(s)$ be a jump point of s. Since $s_{\varepsilon_k} \to s$ pointwise, we may suppose that $t < \mathcal{T}(s_{\varepsilon_k})$. By the monotonicity of s, $s(t^-) < s(t^+)$. For every $s(t^-) \leq a < b \leq s(t^+)$, there exist two sequences $t_k^a, t_k^b \to t$ such that $s_{\varepsilon_k}(t_k^a) = a$ and $s_{\varepsilon_k}(t_k^b) = b$ for every $k \in \mathbb{N}$. For every $\psi \in L^2([s_0, L])$ with $\psi \geq 0$, we have, by (3) of Definition 4.5,

$$\int_{t_k^a}^{t_k^b} (\mathcal{G}^+(\tau, s_{\varepsilon_k}(\tau)) - 1 - \varepsilon_k \dot{s}_{\varepsilon_k}(\tau))\, \psi(s_{\varepsilon_k}(\tau))\, \dot{s}_{\varepsilon_k}(\tau)\, \mathrm{d}\tau \geq 0\,. \tag{7.5}$$

Since $\dot{s}_{\varepsilon_k} \geq 0$ a.e. in $[0, T]$, from (7.5) we deduce that

$$\int_{t_k^a}^{t_k^b} (\mathcal{G}^+(\tau, s_{\varepsilon_k}(\tau)) - 1)\, \psi(s_{\varepsilon_k}(\tau))\, \dot{s}_{\varepsilon_k}(\tau)\, \mathrm{d}\tau \geq 0\,. \tag{7.6}$$

We perform a change of variable setting $\sigma := s_{\varepsilon_k}(\tau)$ and

$$\hat{t}_k(\sigma) := \min\{\tau \in [t_k^a, t_k^b] : s_{\varepsilon_k}(\tau) = \sigma\}\,,$$

so that (7.6) becomes

$$\int_a^b (\mathcal{G}^+(\hat{t}_k(\sigma), \sigma) - 1)\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0\,. \tag{7.7}$$

Passing to the lim sup in (7.7) as $k \to +\infty$, applying Fatou's lemma and recalling Proposition 3.11, we get

$$\int_a^b (\mathcal{G}^+(t, \sigma) - 1)\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0\,. \tag{7.8}$$

Since (7.8) holds for every $\psi \in L^2([s_0, L])$, $\psi \geq 0$, and every $a < b$ in $[s(t^-), s(t^+)]$, then

$$\mathcal{G}^+(t, \sigma) \geq 1 \qquad \text{for every } \sigma \in [s(t^-), s(t^+)]\,.$$

It remains to prove property (3) of Definition 4.7. Let $t \in [0, \mathcal{T}(s))$ be such that $\mathcal{G}^+(t, s(t)) < 1$. By the previous step, $t \notin J(s)$. Let us prove that s is constant in a neighborhood of t. To this end, we first prove that there exists $\delta > 0$ such that, for k large enough,

$$\mathcal{G}^+(\tau, s_{\varepsilon_k}(\tau)) < 1 \qquad \text{for every } \tau \in (t - \delta, t + \delta)\,. \tag{7.9}$$

Assume by contradiction that this is not the case. From the pointwise convergence $s_{\varepsilon_k} \to s$, we deduce that, for k large enough, $t \in [0, \mathcal{T}(s_{\varepsilon_k}))$. Therefore, we may assume that there exist a subsequence $\varepsilon_{k_h} \searrow 0$ and a sequence $\delta_h \searrow 0$ such that (7.9) is not satisfied in the interval $(t - \delta_h, t + \delta_h)$, i.e., we can find $t_h \in (t - \delta_h, t + \delta_h)$ such that, for every h,

$$\mathcal{G}^+(t_h, s_{\varepsilon_{k_h}}(t_h)) \geq 1\,. \tag{7.10}$$

Since $t_h \to t$ and $t \notin J(s)$, by Lemma 7.1 we have $s_{\varepsilon_{k_h}}(t_h) \to s(t)$ as $h \to +\infty$. By the upper semicontinuity of $\mathcal{G}^+$ we get, passing to the lim sup in (7.10) as $h \to +\infty$, $\mathcal{G}^+(t, s(t)) \geq 1$, which is a contradiction.

Combining (7.9) and properties (1) and (3) of Definition 4.5, we deduce that, for k large enough, $\dot{s}_{\varepsilon_k}(\tau) = 0$ for every $\tau \in (t - \delta, t + \delta)$, thus s_{ε_k} is constant in this interval. Since $s_{\varepsilon_k} \to s$ pointwise in $[0, T]$ as $k \to +\infty$, we get that s is constant in the same interval. Therefore, s is differentiable in t and $\dot{s}(t) = 0$. This concludes the proof of the theorem. □

We conclude this section with a remark on the energy balance.

Remark 7.2. At this stage, we do not have any energy balance. This is due to the fact that we can not ensure that along a quasi-static evolution $s \in BV([0, T])$ the generalized energy release rates $\mathcal{G}^+$ and $\mathcal{G}^-$ coincide.

We give the hypotheses on the energy functional (4.3) which guarantee, applying the abstract results in [20], the existence of a *special* quasi-static evolution satisfying an energy balance and a more restrictive Griffith's criterion. Let $\mathbb{C}$ be $C^{1,1}$, Σ be a simple $C^{3,1}$ curve, and let $\varphi \in C^{1,1}([0,T]\times\Sigma;\mathbb{R})$ be such that (2.6) and (4.2) hold with $p=2$. Moreover, let $f \in C^{1,1}([0,T]\times\overline{\Omega};\mathbb{R}^2)$ and $w \in C^{1,1}([0,T];H^1(\Omega))$. Then, with the arguments used is [20, Sections 3.1, 3.2], it is possible to show that for every $t \in (0,T)$ and every $s \in (0,L)$ there exists the left derivative $\partial_t^- \mathcal{E}_{min}$ of the reduced energy with respect to time. In particular,

$$\partial_t^- \mathcal{E}_{min}(t,s) = \min\{H(t,s,u) : \, u \in A(t,s) \text{ is a minimizer of } \mathcal{E}(t,s,w(t))\}\,,$$

where we have set

$$H(t,s,u) := \int_\Omega \mathbb{C}\mathrm{E}u \cdot \mathrm{E}\dot{w}(t)\,\mathrm{d}x - \int_\Omega \dot{f}(t)\cdot u\,\mathrm{d}x - \int_\Omega f(t)\cdot \dot{w}(t)\,\mathrm{d}x + \int_{\Gamma_s} D_t\varphi(t,[u])\,\mathrm{d}\mathcal{H}^1\,.$$

Applying the results in [20, Section 5.2], we can also prove that for every $s_0 \in (0,L)$ there exists a quasi-static evolution $s \in BV([0,T])$ for the cohesive crack growth problem with $s(0)=s_0$, which satisfies a refined Griffith's criterion: condition (1) in Definition 4.7 is replaced by

(1') for every $t \in [0,\mathcal{T}(s))\setminus J(s)$:

$$\mathcal{G}^+(t,s(t)) \le 1\,.$$

Moreover, we have the following energy balance:
for every $t \in (0,\mathcal{T}(s))$

$$\begin{aligned}\mathcal{E}_{min}(t,s(t)) + s(t^-) - s(0^+) &+ \int_{s_0}^{s(0^+)} \mathcal{G}^+(0,\sigma)\,\mathrm{d}\sigma + \int_{s(t^-)}^{s(t)} \mathcal{G}^+(t,\sigma)\,\mathrm{d}\sigma \\ &+ \sum_{\tau\in(0,t)\cap J(s)} \left(s(\tau^+) - s(\tau^-) + \int_{s(\tau^-)}^{s(\tau^+)} \mathcal{G}^+(\tau,\sigma)\,\mathrm{d}\sigma \right) \\ &= \mathcal{E}_{min}(0,s_0) + \int_0^t \partial_t^- \mathcal{E}_{min}(\tau,s(\tau))\,\mathrm{d}\tau\,.\end{aligned}$$

In [20], such an evolution is called *special local energetic solution.*

8. The case of many curves

In this section we address the study of the evolution of multiple non-interacting cracks.

We assume that the fractures grow along a prescribed number of pairwise disjoint simple C^3-curves $\Sigma_1,\dots,\Sigma_M$ with $\mathcal{H}^1(\Sigma_m) =: L_m$. The assumptions on every Σ_m are the same of Section 2. For $m=1,\dots,M$, we denote by $\gamma_m\colon [0,L_m]\to\mathbb{R}^2$ the arc-length parametrization of the m-th curve Σ_m and by ν_m, τ_m the unit normal and unit tangent vectors to Σ_m, respectively.

We define $\Lambda := [0,L_1]\times\ldots\times[0,L_M] \subseteq \mathbb{R}^M$. For every $s=(s_1,\dots,s_M)\in\Lambda$, we set

$$\Gamma_s := \Gamma^1_{s_1}\cup\ldots\cup\Gamma^M_{s_M} \qquad \text{and} \qquad \Omega_s := \Omega\setminus\Gamma_s\,,$$

where $\Gamma^m_{s_m} \subseteq \Sigma_m$ is as in (2.1). Then, the set of admissible fractures is given by

$$\{\Gamma_s : \, s\in\Lambda\}\,. \tag{8.1}$$

In this setting, we generalize the activation threshold considered in the energy (2.8) with the norm defined by

$$|s|_1 := \sum_{m=1}^M |s_m| \qquad \text{for every } s\in\mathbb{R}^M\,.$$

Therefore, for every $t\in[0,T]$, $s\in\Lambda$, and $u\in H^1(\Omega_s)$, the total energy of the system is

$$\mathcal{E}(t,s,u) := \frac{1}{2}\int_{\Omega_s} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x - \int_{\Omega_s} g(t,x,u)\,\mathrm{d}x + \int_{\Gamma_s}\varphi(t,[u])\,\mathrm{d}\mathcal{H}^1 + |s|_1\,,$$

where $\mathbb{C}$, φ, and g have the usual hypotheses stated in Section 2 and 4. Given the Dirichlet boundary datum $w \in H^1(\Omega)$, we define $A(s,w)$ and the reduced energy $\mathcal{E}_{min}(t,s,w)$ as in (2.10) and in (2.13), respectively.

We now show how to extend the results of Section 3 to this setting. In particular, we are interested in the analogous of the energy release rates. For $m = 1, \dots, M$, let us define

$$\Lambda_m := [0, L_1] \times \ldots \times [0, L_{m-1}] \times (0, L_m) \times [0, L_{m+1}] \times \ldots \times [0, L_M] \,.$$

Let $m = 1, \dots, M$ and $s \in \Lambda_m$ be fixed. By hypothesis, there exists $\eta > 0$ such that the curve Σ_m is the graph of a C^3 function ψ_s^m on $(\gamma_m^1(s_m) - \eta, \gamma_m^1(s_m) + \eta)$, where γ_m^1 is the first component of $\gamma_m = (\gamma_m^1, \gamma_m^2)$. We may also assume that $d(\gamma_m(s_m), \Sigma_l) \geq 2\eta$ for every $l \neq m$, where $d(\cdot, E)$ denotes the distance function from a set E. Given $\delta \in \mathbb{R}$ such that $s_m + \delta \in [0, L_m]$ and a cut-off function $\vartheta \in C_c^\infty(\mathrm{B}_{\eta/2}(0))$ with $\vartheta = 1$ in $\overline{\mathrm{B}}_{\eta/3}(0)$, we define, as in (2.14), $F_{s,\delta}^m \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_{s,\delta}^m(x) := x + \begin{pmatrix} (\gamma_m^1(s_m + \delta) - \gamma_m^1(s_m))\vartheta(\gamma_m(s_m) - x) \\ \psi_s^m(x_1 + (\gamma_m^1(s_m + \delta) - \gamma_m^1(s_m))\vartheta(\gamma_m(s_m) - x)) - \psi_s^m(x_1) \end{pmatrix} \tag{8.2}$$

if $x = (x_1, x_2) \in \mathrm{B}_{\eta/2}(\gamma_m(s_m))$, while $F_{s,\delta}^m(x) := x$ for $x \in \mathbb{R}^2 \setminus \mathrm{B}_{\eta/2}(\gamma_m(s_m))$.

The equivalent to Lemma 2.4 holds in this setting.

Lemma 8.1. *There exists $\delta_0 > 0$ such that the following facts hold:*

- (a) $F_{s,\cdot}^m \in C^3((-\delta_0, \delta_0) \times \mathbb{R}^2; \mathbb{R}^2)$ *and, for every $|\delta| < \delta_0$, the map $F_{s,\delta}^m$ is a C^3-diffeomorphism. Moreover, $F_{s,\delta}^m(\gamma_m(s_m)) = \gamma_m(s_m + \delta)$, $F_{s,\delta}^m(\Gamma_{s_m}^m) = \Gamma_{s_m+\delta}^m$, and $F_{s,\delta}^m(\Gamma_{s_l}^l) = \Gamma_{s_l}^l$ for $l \neq m$;*
- (b) *the norms $\|F_{s,\delta}^m\|_{C^3}$ and $\|(F_{s,\delta}^m)^{-1}\|_{C^3}$ are uniformly bounded with respect to δ and there exists $c_1, c_2 > 0$ such that, for every $|\delta| < \delta_0$ and every $x \in \mathbb{R}^2$, we have $c_1 \leq \det \nabla F_{s,\delta}^m(x) \leq c_2$;*
- (c) $\|\mathrm{Id} - F_{s,\delta}^m\|_{C^2} \to 0$ *as $\delta \to 0$;*
- (d) *some derivatives*

$$\rho_s(x) := \partial_\delta(F_{s,\delta}^m(x))|_{\delta=0} \,, \qquad \partial_\delta(\det \nabla F_{s,\delta}^m(x))|_{\delta=0} = \mathrm{div}\rho_s^m(x) \,,$$
$$\partial_\delta(\nabla F_{s,\delta}^m(x))|_{\delta=0} = -\partial_\delta(\nabla F_{s,\delta}^m(x))^{-1}|_{\delta=0} = \nabla \rho_s^m(x) \,,$$
$$\partial_\delta(\mathrm{cof}\, \nabla F_{s,\delta}^m)^T|_{\delta=0} = -\partial_\delta(\mathrm{cof}\, \nabla F_{s,\delta}^m)^{-T}|_{\delta=0} = \mathrm{div}\rho_s^m \, 1_{\mathbb{M}^2} - \nabla \rho_s^m \,,$$

Similar to (3.1), for $m = 1, \dots, M$, $t \in [0,T]$, $s \in \Lambda_m$, $w \in H^1(\Omega)$, and $u \in A(s,w)$, we set

$$\begin{aligned} G_m(t,u,\vartheta) := &-\frac{1}{2}\int_{\Omega_s} (D\mathbb{C}\,\rho_s^m)\nabla u \cdot \nabla u \,\mathrm{d}x - \int_{\Omega_s} \mathbb{C}\nabla\big((\nabla\rho_s^m - \mathrm{div}\rho_s^m \,1_{\mathbb{M}^2})u\big)\cdot \nabla u \,\mathrm{d}x \\ &+ \int_{\Omega_s} \mathbb{C}(\nabla u \nabla \rho_s^m)\cdot \nabla u \,\mathrm{d}x - \frac{1}{2}\int_{\Omega_s} \mathbb{C}\nabla u \cdot \nabla u \,\mathrm{div}\rho_s^m \,\mathrm{d}x \\ &+ \int_{\Omega_s} D_\xi g(t,x,u)\cdot\big[(\nabla\rho_s^m - \mathrm{div}\rho_s^m\,1_{\mathbb{M}^2})u - \nabla u\,\rho_s^m\big]\,\mathrm{d}x \\ &- \int_{\Gamma_s} D_\xi\varphi(t,[u])\cdot\big((\nabla\rho_s^m - \mathrm{div}\rho_s^m\,1_{\mathbb{M}^2})u\big)\,\mathrm{d}\mathcal{H}^1 \\ &- \int_{\Gamma_s} \varphi(t,[u])\,\nu\otimes\tau \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \nabla\rho_s^m \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_s}\varphi(t,[u])\,\mathrm{div}\rho_s^m\,\mathrm{d}\mathcal{H}^1 \,, \end{aligned} \tag{8.3}$$

where ϑ is as in (8.2) and $D\mathbb{C}\,\rho_s^m$ is as in (3.2).

Moreover, we define

$$\partial_{s,m}^+ \mathcal{E}_{min}(t,s,w) := \lim_{\delta \searrow 0} \frac{\mathcal{E}_{min}(t, s + \delta e_m, w) - \mathcal{E}_{min}(t,s,w)}{\delta} \,, \tag{8.4}$$

$$\partial^-_{s,m}\mathcal{E}_{min}(t,s,w) := \lim_{\delta\nearrow 0}\frac{\mathcal{E}_{min}(t,s+\delta e_m,w)-\mathcal{E}_{min}(t,s,w)}{\delta}\,, \tag{8.5}$$

where $\{e_1,\dots,e_M\}$ is the canonical basis of $\mathbb{R}^M$. With the same techniques used in Theorems 3.1, 3.2, and in Proposition 3.11, we can prove that the limits in (8.4) and (8.5) exist and have explicit formulas similar to (3.5) and (3.7).

Theorem 8.2. *For every $t\in[0,T]$, every $m=1,\dots,M$, every $s\in\Lambda_m$, and every $w\in H^1(\Omega)$, the limits in (8.4) and (8.5) exist and*

$$\begin{aligned}\partial^+_{s,m}\mathcal{E}_{min}(t,s,w) &= 1-\mathcal{G}^+_m(t,s,w)\,,\\ \partial^-_{s,m}\mathcal{E}_{min}(t,s,w) &= 1-\mathcal{G}^-_m(t,s,w)\,,\end{aligned} \tag{8.6}$$

where we have set

$$\begin{aligned}\mathcal{G}^+_m(t,s,w) &:= \max\{G_m(t,u_s,\vartheta) : u_s\in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\,,\\ \mathcal{G}^-_m(t,s,w) &:= \min\{G_m(t,u_s,\vartheta) : u_s\in A(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\end{aligned} \tag{8.7}$$

for a given cut-off function ϑ as in (8.2). In particular, $\mathcal{G}^+_m$ and $\mathcal{G}^-_m$ do not depend on the choice of ϑ.

Moreover, $\mathcal{G}^+_m,\mathcal{G}^-_m\colon[0,T]\times\Lambda_m\times H^1(\Omega)\to[0,+\infty)$ are upper and lower semicontinuous on $[0,T]\times\mathring{\Lambda}\times H^1(\Omega)$, respectively.

Remark 8.3. The functions $\mathcal{G}^+_m$ and $\mathcal{G}^-_m$ introduced in Theorem 8.2 can be interpreted as partial energy release rates, in the sense that they characterize the partial derivatives with respect to the variable $s_m\in[0,L_m]$ of the reduced energy $\mathcal{E}_{min}$.

Also in this setting, the notion of quasi-static evolution will be related to the properties of $\mathcal{G}^\pm_m$, see Theorems 8.7 and 9.1.

We now deal with the construction of a quasi-static evolution. As in Section 4, we replace g with the power spent by the body forces $f\in AC([0,T];L^2(\Omega))$. Given a boundary datum $w\in AC([0,T];H^1(\Omega))$, we redefine the reduced energy $\mathcal{E}_{min}\colon[0,T]\times\Lambda\to\mathbb{R}$ and the energy release rates $\mathcal{G}^\pm_m\colon[0,T]\times\Lambda_m\to[0,+\infty)$ by

$$\mathcal{E}_{min}(t,s) := \mathcal{E}_{min}(t,s,w(t)) \qquad\text{and}\qquad \mathcal{G}^\pm_m(t,s) := \mathcal{G}^\pm_m(t,s,w(t))\,.$$

We notice again that $\mathcal{E}_{min}$ is continuous on $[0,T]\times\mathring{\Lambda}$, while, for every $m=1,\dots,M$, $\mathcal{G}^+_m$ and $\mathcal{G}^-_m$ are upper and lower semicontinuous, respectively.

For every $k\in\mathbb{N}$, we consider a time discretization $\{t^k_i\}_{i=0}^k$ of the form $t^k_i := i\tau_k$, where $\tau_k := T/k$. Fixed $\varepsilon>0$, we define recursively $s^{k,i}_\varepsilon\in\Lambda$: $s^{k,0}_\varepsilon := s_0\in\mathring{\Lambda}$, the initial condition, and, for $i\geq 1$, we set $s^{k,i}_\varepsilon$ to be a solution of the incremental minimum problem

$$\min\left\{\mathcal{E}_{min}(t^k_i,s)+\frac{\varepsilon}{2}\frac{|s-s^{k,i-1}_\varepsilon|_2^2}{\tau_k} : s\in\Lambda,\ s_m\geq(s^{k,i-1}_\varepsilon)_m \text{ for } m=1,\dots,M\right\}, \tag{8.8}$$

where

$$|s|_2 := \left(\sum_{m=1}^M s_m^2\right)^{1/2} \qquad \text{for every } s\in\mathbb{R}^M\,.$$

The proof of existence of solution to (8.8) is similar to the proof of Proposition 5.1.

We introduce the interpolation functions: for every $t\in(t^k_{i-1},t^k_i]$ we set

$$\bar{t}_k(t) := t^k_i\,,$$

$$\bar{s}^k_{\varepsilon,m}(t) := (s^{k,i}_\varepsilon)_m\,,\quad \bar{s}^k_\varepsilon(t) := (\bar{s}^k_{\varepsilon,1}(t),\dots,\bar{s}^k_{\varepsilon,M}(t))\,,$$

$$s^k_{\varepsilon,m}(t) := (s^{k,i-1}_\varepsilon)_m+\frac{(s^{k,i}_\varepsilon)_m-(s^{k,i-1}_\varepsilon)_m}{\tau_k}(t-t^k_{i-1})\,,\quad s^k_\varepsilon(t) := (s^k_{\varepsilon,1}(t),\dots,s^k_{\varepsilon,M}(t))\,.$$

In particular, as in Proposition 5.2, we get

$$\varepsilon\int_0^T|\dot{s}^k_\varepsilon(t)|_2^2\,\mathrm{d}t\leq C \tag{8.9}$$

uniformly in ε and k, where $\dot{s}^k_\varepsilon(t) := (\dot{s}^k_{\varepsilon,1}(t), \dots, \dot{s}^k_{\varepsilon,M}(t))$.

As in Proposition 5.3, we have a discrete Griffith's criterion.

Proposition 8.4. *For every $\varepsilon > 0$, every $k \in \mathbb{N}$, every $m = 1, \dots, M$, and every $t \in [0, \mathcal{T}((\bar{s}^k_\varepsilon)_m))$ we have*

- (a) $\dot{s}^k_{\varepsilon,m}(t) \geq 0$;
- (b) $\mathcal{G}^+_m(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) - 1 - \varepsilon\, \dot{s}^k_{\varepsilon,m}(t) \leq 0$;
- (c) $(\mathcal{G}^+_m(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) - 1 - \varepsilon\, \dot{s}^k_{\varepsilon,m}(t))\, \dot{s}^k_{\varepsilon,m}(t) = 0$.

Proof. It is sufficient to repeat the argument of Proposition 5.3 componentwise. □

We define the *failure time* and the *jump set* for a vector valued function whose components are monotone non-decreasing.

Definition 8.5. Let $a, b_1, \dots, b_M > 0$ and let $s_m : [0, a] \to [0, b_m]$ be a monotone non-decreasing function for $m = 1, \dots, M$. Let $s = (s_1, \dots, s_M)$. We define:

- the *failure time* of s as
$$\mathcal{T}(s) := \min_{m=1,\dots,M} \mathcal{T}(s_m),$$
where $\mathcal{T}(s_m)$ is as in Definition 4.3;
- the *jump set* of s as
$$J(s) := \bigcup_{m=1}^{M} J(s_m),$$
where $J(s_m)$ is as in Definition 4.3.

We can now pass to the limit as $k \to +\infty$. As in Proposition 6.1, for fixed $\varepsilon > 0$, we find $s_\varepsilon \in H^1([0,T])$ such that, up to a subsequence, s^k_ε converges to s_ε weakly in $H^1([0,T])$ and uniformly in $[0,T]$. Moreover, $\bar{s}^k_\varepsilon \to s_\varepsilon$ uniformly. By (8.9) and the lower semicontinuity of the L^2-norm, there exists $C > 0$ such that for every $\varepsilon > 0$

$$\varepsilon \int_0^T |\dot{s}_\varepsilon(t)|_2^2 \,\mathrm{d}t \leq C, \tag{8.10}$$

where $\dot{s}_\varepsilon(t) := (\dot{s}^1_\varepsilon(t), \dots, \dot{s}^M_\varepsilon(t))$.

The map $t \mapsto s_\varepsilon(t)$ is a viscous evolution with $s_\varepsilon(0) = s_0$, see Definition 4.5. Indeed, taking into account Proposition 3.11, the following result holds.

Proposition 8.6. *For every $\varepsilon > 0$, every $m = 1, \dots, M$, and a.e. $t \in [0, \mathcal{T}(s_\varepsilon))$:*

- (a) $\dot{s}^m_\varepsilon(t) \geq 0$;
- (b) $\mathcal{G}^-_m(t, s_\varepsilon(t)) - 1 - \varepsilon \dot{s}^m_\varepsilon(t) \leq 0$;
- (c) $(\mathcal{G}^+_m(t, s_\varepsilon(t)) - 1 - \varepsilon \dot{s}^m_\varepsilon(t))\, \dot{s}^m_\varepsilon(t) \geq 0$.

Proof. Argue componentwise as in Theorem 4.6. □

As in the proof of Theorem 4.9, there exist a subsequence $\varepsilon_k \to 0$ and a function $s \in BV([0,T], \Lambda)$ such that $s_{\varepsilon_k} \to s$ pointwise. Moreover, every component s_m is monotone non-decreasing in $[0,T]$.

Repeating componentwise the argument of Theorem 4.9, we can prove a Griffith's criterion in the continuity points of s.

Theorem 8.7. *The following facts hold:*

- (a) *s_m is monotone non-decreasing for every $m = 1, \dots, M$;*
- (b) *for every $m = 1, \dots, M$ and every $t \in [0, \mathcal{T}(s)) \setminus J(s)$, $\mathcal{G}^-_m(t, s(t)) \leq 1$;*
- (c) *if $t \in [0, \mathcal{T}(s)) \setminus J(s)$ and $\mathcal{G}^+_m(t, s(t)) < 1$ for some $m = 1, \dots, M$, then s_m is differentiable in t and $\dot{s}_m(t) = 0$.*

However, in this setting it is difficult to state the properties of $\mathcal{G}_m^\pm$ in the jump points: in particular, we do not have the equivalent to condition (2) of Definition 4.7. Therefore, following the steps of [19, 22, 25], we define a reparametrization that shall give some information on the behavior of the cracks at the jump points.

9. Parametrized solutions

We perform a change of variable which transforms the lengths in absolutely continuous functions. Roughly speaking, this is done by a parametrization of time on the jump points of the viscous solution s_ε.

For $\varepsilon > 0$ and $t \in [0,T]$, we set

$$\sigma_\varepsilon(t) := t + |s_\varepsilon(t)|_1 - |s_0|_1 = t + \sum_{m=1}^{M} (s_\varepsilon^m(t) - s_0^m)\,. \tag{9.1}$$

Thanks to the properties of s_ε, see Proposition 8.6, σ_ε is strictly increasing, continuous, and $\dot\sigma_\varepsilon(t) \geq 1$ for every $\varepsilon > 0$ and a.e. $t \in [0,T]$, hence we can find its inverse $\sigma \mapsto \tilde t_\varepsilon(\sigma)$ for $0 \leq \sigma \leq S_\varepsilon := \sigma_\varepsilon(T)$. We deduce that $\tilde t_\varepsilon$ is strictly increasing, continuous, and $0 < \tilde t_\varepsilon'(\sigma) \leq 1$ for every $\varepsilon > 0$ and a.e. $\sigma \in [0, S_\varepsilon]$ (from now on, the symbol $'$ denotes the derivative with respect to σ).

For $m = 1, \dots, M$ and $\sigma \in [0, S_\varepsilon]$, we set

$$\tilde s_\varepsilon^m(\sigma) := s_\varepsilon^m(\tilde t_\varepsilon(\sigma))\,, \quad \tilde s_\varepsilon(\sigma) := (\tilde s_\varepsilon^1(\sigma), \dots, \tilde s_\varepsilon^M(\sigma))\,,$$
$$\tilde s_\varepsilon'(\sigma) := ((\tilde s_\varepsilon^1)'(\sigma), \dots, (\tilde s_\varepsilon^M)'(\sigma))\,.$$

By (9.1), we have $\sigma = \tilde t_\varepsilon(\sigma) + |\tilde s_\varepsilon(\sigma)|_1 - |s_0|_1$. Deriving this relation, we obtain

$$\tilde t_\varepsilon'(\sigma) + |\tilde s_\varepsilon'(\sigma)|_1 = 1 \tag{9.2}$$

for every $\varepsilon > 0$ and a.e. $\sigma \in [0, S_\varepsilon]$. By (9.2) and the monotonicity of $\tilde s_\varepsilon^m$, we get $0 \leq (\tilde s_\varepsilon^m)'(\sigma) \leq 1$ for every $\varepsilon > 0$, every $m = 1, \dots, M$, and a.e. $\sigma \in [0, S_\varepsilon]$. Moreover, in view of (9.2), $\tilde t_\varepsilon$ and $\tilde s_\varepsilon$ are Lipschitz functions.

We define $\widetilde{\mathcal{G}}_{m,\varepsilon}^\pm(\sigma) := \mathcal{G}_m^\pm(\tilde t_\varepsilon(\sigma), \tilde s_\varepsilon(\sigma))$ for $\sigma \in [0, \mathcal{T}(\tilde s_\varepsilon))$ and $\bar S := \sup_{\varepsilon>0} S_\varepsilon$, which is bounded by a constant depending on T and on the lengths L_m. Since in the limit $\varepsilon \searrow 0$ it will be useful to deal with functions defined on the same interval, we extend the functions $\tilde t_\varepsilon$, $\tilde s_\varepsilon$, $\tilde t_\varepsilon'$, and $\tilde s_\varepsilon'$ on $(S_\varepsilon, \bar S]$ by $\tilde t_\varepsilon(\sigma) := \tilde t_\varepsilon(S_\varepsilon)$, $\tilde s_\varepsilon(\sigma) := \tilde s_\varepsilon(S_\varepsilon)$, $\tilde t_\varepsilon'(\sigma) := 0$, and $\tilde s_\varepsilon'(\sigma) := 0$. In the sequel, we will also need $\widetilde{\mathcal{T}}(\tilde s_\varepsilon) := \min\{S_\varepsilon, \mathcal{T}(\tilde s_\varepsilon)\}$.

Recalling that $\tilde t_\varepsilon'(\sigma) > 0$ on $[0, S_\varepsilon]$, the Griffith's criterion stated in Proposition 8.6 reads in the new variables as

$$(\tilde s_\varepsilon^m)'(\sigma) \geq 0\,, \tag{9.3}$$

$$\widetilde{\mathcal{G}}_{m,\varepsilon}^-(\sigma)\,\tilde t_\varepsilon'(\sigma) - \tilde t_\varepsilon'(\sigma) - \varepsilon(\tilde s_\varepsilon^m)'(\sigma) \leq 0\,, \tag{9.4}$$

$$\big(\widetilde{\mathcal{G}}_{m,\varepsilon}^+(\sigma)\,\tilde t_\varepsilon'(\sigma) - \tilde t_\varepsilon'(\sigma) - \varepsilon(\tilde s_\varepsilon^m)'(\sigma)\big)\,(\tilde s_\varepsilon^m)'(\sigma) \geq 0 \tag{9.5}$$

for every m, every ε, and a.e. $\sigma \in [0, \widetilde{\mathcal{T}}(\tilde s_\varepsilon))$.

We now pass to the limit along a subsequence $\varepsilon_k \searrow 0$. Since $\tilde t_{\varepsilon_k}, \tilde s_{\varepsilon_k}$ are bounded in $W^{1,\infty}([0,\bar S])$, up to a further subsequence we have that $\tilde t_{\varepsilon_k}$ and $\tilde s_{\varepsilon_k}$ converge weakly* in $W^{1,\infty}([0,\bar S])$ to some functions $\tilde t$ and $\tilde s$, respectively. We can also assume that $S_{\varepsilon_k} \to S$ and $\tilde t, \tilde s \in W^{1,\infty}([0,S])$. In particular, writing (9.2) in an integral form and passing to the limit, we deduce that for a.e. $\sigma \in [0,S]$

$$\tilde t'(\sigma) + |\tilde s'(\sigma)|_1 = 1\,. \tag{9.6}$$

We set $\widetilde{\mathcal{T}}(\tilde s) := \min\{S, \mathcal{T}(\tilde s)\}$ and, for $m = 1 \dots, M$ and $\sigma \in [0, \widetilde{\mathcal{T}}(\tilde s))$,

$$\widetilde{\mathcal{G}}_m^\pm(\sigma) := \mathcal{G}_m^\pm(\tilde t(\sigma), \tilde s(\sigma))\,.$$

As in Remark 4.4, we have

$$\widetilde{\mathcal{T}}(\tilde{s}) \leq \liminf_k \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}) . \tag{9.7}$$

Finally, we observe that (8.10) gives

$$\begin{aligned} \varepsilon_k \int_0^{S_{\varepsilon_k}} |\tilde{s}'_{\varepsilon_k}(\sigma)|_2^2 \, \mathrm{d}\sigma &= \varepsilon_k \int_0^{S_{\varepsilon_k}} |\dot{s}_{\varepsilon_k}(\tilde{t}_{\varepsilon_k}(\sigma))|_2^2 (\tilde{t}'_{\varepsilon_k})^2(\sigma) \, \mathrm{d}\sigma \\ &\leq \varepsilon_k \int_0^{S_{\varepsilon_k}} |\dot{s}_{\varepsilon_k}(\tilde{t}_{\varepsilon_k}(\sigma))|_2^2 \, \tilde{t}'_{\varepsilon_k}(\sigma) \, \mathrm{d}\sigma = \varepsilon_k \int_0^T |\dot{s}_{\varepsilon_k}(t)|_2^2 \, \mathrm{d}t \leq C \end{aligned} \tag{9.8}$$

uniformly in k. Therefore, $\varepsilon_k \tilde{s}'_{\varepsilon_k} \mathbf{1}_{[0,S_{\varepsilon_k}]} \to 0$ in $L^2([0,\bar{S}])$.

Passing to the limit as $k \to +\infty$, we are now able to show that the parametrized solution $\tilde{s}$ satisfies a Griffith's criterion involving also the jump points of $\tilde{s}$. This is the aim of the following theorem.

Theorem 9.1. *The Lipschitz continuous functions $\tilde{t}$ and $\tilde{s}$ satisfy for a.e. $\sigma \in [0, \widetilde{\mathcal{T}}(\tilde{s}))$:*

(a) *$\tilde{t}'(\sigma) \geq 0$ and $\tilde{s}'_m(\sigma) \geq 0$ for $m = 1, \dots, M$;*
(b) *if $\tilde{t}'(\sigma) > 0$, then $\widetilde{\mathcal{G}}_m^-(\sigma) \leq 1$ for $m = 1, \dots, M$;*
(c) *if $\tilde{t}'(\sigma) > 0$ and $\tilde{s}'_m(\sigma) > 0$ for some $m \in \{1, \dots, M\}$, then $\widetilde{\mathcal{G}}_m^+(\sigma) \geq 1$;*
(d) *if $\tilde{t}'(\sigma) = 0$, then there exists $m \in \{1, \dots, M\}$ such that $\tilde{s}'_m(\sigma) > 0$. Moreover, $\widetilde{\mathcal{G}}_m^+(\sigma) \geq 1$ for such m.*

Proof. By the monotonicity of $\tilde{t}$ and $\tilde{s}$, we have $\tilde{t}'(\sigma) \geq 0$ and $\tilde{s}'_m(\sigma) \geq 0$ for every m and a.e. $\sigma \in [0, S]$. Moreover, by (9.6) they can not be simultaneously zero.

As in the proofs of Theorems 4.6 and 4.9, we have to distinguish between two possibilities:

$$\widetilde{\mathcal{T}}(\tilde{s}) = \lim_k \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}) \qquad \text{or} \qquad \widetilde{\mathcal{T}}(\tilde{s}) < \limsup_k \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}) . \tag{9.9}$$

Let us consider the first case. Let us fix $m = 1, \dots, M$ and $\psi \in L^2([0, \bar{S}])$ with $\psi \geq 0$. Thanks to (9.4), for every k we have

$$\int_0^{\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})} (\tilde{t}'_{\varepsilon_k}(\sigma) - \widetilde{\mathcal{G}}^-_{m,\varepsilon_k}(\sigma) \tilde{t}'_{\varepsilon_k}(\sigma) + \varepsilon_k (\tilde{s}^m_{\varepsilon_k})'(\sigma)) \, \psi(\sigma) \, \mathrm{d}\sigma \geq 0 , \tag{9.10}$$

where ε_k is the subsequence previously fixed.

Since $\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}) \to \widetilde{\mathcal{T}}(\tilde{s})$, $\tilde{t}'_{\varepsilon_k}$ converges to $\tilde{t}'$ weakly* in $L^\infty([0, \bar{S}])$, and $\varepsilon_k \tilde{s}'_{\varepsilon_k} \mathbf{1}_{[0,S_{\varepsilon_k}]} \to 0$ in $L^2([0, \bar{S}])$, passing to the lim sup in (9.10) as $k \to +\infty$ we get

$$\begin{aligned} 0 &\leq \limsup_k \int_0^{\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})} (\tilde{t}'_{\varepsilon_k}(\sigma) - \widetilde{\mathcal{G}}^-_{m,\varepsilon_k}(\sigma) \tilde{t}'_{\varepsilon_k}(\sigma) + \varepsilon_k (\tilde{s}^m_{\varepsilon_k})'(\sigma)) \, \psi(\sigma) \, \mathrm{d}\sigma \\ &= \int_0^{\widetilde{\mathcal{T}}(\tilde{s})} \tilde{t}'(\sigma) \, \psi(\sigma) \, \mathrm{d}\sigma - \liminf_k \int_0^{\bar{S}} \widetilde{\mathcal{G}}^-_{m,\varepsilon_k}(\sigma) \tilde{t}'_{\varepsilon_k}(\sigma) \, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}))} \, \psi(\sigma) \, \mathrm{d}\sigma . \end{aligned} \tag{9.11}$$

By the monotonicity of $\tilde{t}_\varepsilon$, we can continue the chain of inequalities in (9.11)

$$\begin{aligned} 0 &\leq \int_0^{\widetilde{\mathcal{T}}(\tilde{s})} \tilde{t}'(\sigma) \, \psi(\sigma) \, \mathrm{d}\sigma - \liminf_k \int_0^{\bar{S}} \left(\inf_{h \geq k} \widetilde{\mathcal{G}}^-_{m,\varepsilon_h}(\sigma) \, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_h}))}(\sigma) \right) \tilde{t}'_{\varepsilon_k}(\sigma) \, \psi(\sigma) \, \mathrm{d}\sigma \\ &= \int_0^{\widetilde{\mathcal{T}}(\tilde{s})} \tilde{t}'(\sigma) \, \psi(\sigma) \, \mathrm{d}\sigma - \liminf_k \int_0^{\bar{S}} F_k(\sigma) \, \tilde{t}'_{\varepsilon_k}(\sigma) \, \psi(\sigma) \, \mathrm{d}\sigma , \end{aligned} \tag{9.12}$$

where we have set

$$F_k(\sigma) := \inf_{h \geq k} \widetilde{\mathcal{G}}^-_{m,\varepsilon_h}(\sigma) \, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_h}))}(\sigma) .$$

The sequence F_k is uniformly bounded and converges pointwise to

$$F(\sigma) := \liminf_k \widetilde{\mathcal{G}}^-_{m,\varepsilon_k}(\sigma) \, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}))}(\sigma) = \liminf_k \widetilde{\mathcal{G}}^-_{m,\varepsilon_k}(\sigma) \, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}))}(\sigma) .$$

Therefore, applying the dominated convergence theorem, we get $F_k \to F$ in L^2 and

$$\int_0^{\widetilde{\mathcal{T}}(\tilde{s})} (\tilde{t}'(\sigma) - F(\sigma)\, \tilde{t}'(\sigma))\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0\,. \tag{9.13}$$

By Proposition 3.11, we deduce that $F(\sigma) \geq \widetilde{\mathcal{G}}^-_m(\sigma)$. Hence, in view of property (a), (9.13) becomes

$$\int_0^{\widetilde{\mathcal{T}}(\tilde{s})} (\tilde{t}'(\sigma) - \widetilde{\mathcal{G}}^-_m(\sigma)\, \tilde{t}'(\sigma))\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0\,,$$

which proves (b) by the arbitrariness of ψ.

For the second case of (9.9), we may assume, up to a subsequence, that $\widetilde{\mathcal{T}}(\tilde{s}) < \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})$, hence it is sufficient to replace $\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})$ with $\widetilde{\mathcal{T}}(\tilde{s})$ in (9.10) and repeat the previous argument. Thus property (b) is proved.

We notice that if (a), (b) and (9.6) hold, then (c) and (d) are equivalent to the following property:
if $\widetilde{\mathcal{G}}^+_m(\bar{\sigma}) < 1$ for some m and some $\bar{\sigma} \in [0, \widetilde{\mathcal{T}}(\tilde{s}))$, then $\tilde{s}_m$ is locally constant around $\bar{\sigma}$.
Let us assume that $\widetilde{\mathcal{G}}^+_m(\bar{\sigma}) < 1$. Then, arguing as in the proof of Theorem 4.9, there exist $\bar{k} \in \mathbb{N}$ and $\delta > 0$ such that $\widetilde{\mathcal{G}}^+_{m,\varepsilon_k}(\sigma) < 1$ for every $\sigma \in (\bar{\sigma} - \delta, \bar{\sigma} + \delta)$ and every $k \geq \bar{k}$. From (9.5) we deduce that $\tilde{s}^m_{\varepsilon_k}$ is constant in $(\bar{\sigma} - \delta, \bar{\sigma} + \delta)$. Since $\tilde{s}^m_{\varepsilon_k}$ converges to $\tilde{s}_m$ weakly* in $W^{1,\infty}([0, \bar{S}])$, we get that $\tilde{s}_m$ is locally constant around $\bar{\sigma}$, and this concludes the proof of the theorem. □

Remark 9.2. As usual in these cases, since the reduced energy $\mathcal{E}_{min}$ is continuous only on $[0, T] \times \mathring{\Lambda}$ and, as a consequence, $\mathcal{G}^{\pm}_m$ are not upper and lower semicontinuous on the whole $[0, T] \times \Lambda$, the evolution we described is meaningful up to the failure time $\widetilde{\mathcal{T}}(\tilde{s})$.

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