REGULARITY OF C^1 SURFACES WITH PRESCRIBED MEAN CURVATURE IN THREE-DIMENSIONAL CONTACT SUB-RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper we consider surfaces of class C^1 with continuous prescribed mean curvature in a three-dimensional contact sub-Riemannian manifold and prove that their characteristic curves are of class C^2. This regularity result also holds for critical points of the sub-Riemannian perimeter under a volume constraint. All results are valid in the first Heisenberg group $\mathbb{H}^1$.

1. INTRODUCTION

Recently Cheng, Hwang and Yang [9], [10], have considered the functional

$$(1.1) \qquad \mathscr{F}(u) = \int_\Omega |\nabla u + \vec{F}| + \int_\Omega f u,$$

on a domain $\Omega \subset \mathbb{R}^{2n}$, where $\vec{F}$ is a vector field and $f \in L^\infty(\Omega)$. In case $\vec{F}(x,y) = (-y, x)$, the integral $\int_\Omega |\nabla u + \vec{F}|$ is the sub-Riemannian area of the horizontal graph of the function u in the Heisenberg group $\mathbb{H}^n$. Among several interesting results, they proved in [10, Thm. A] that, in case $n = 1$, $u \in C^1(\Omega)$ is an stationary point of $\mathscr{F}$ and $f \in C^0(\Omega)$, the integral curves of the vector field $((\nabla u + \vec{F})/|\nabla u + \vec{F}|)^\perp$, defined in the set $|\nabla u + \vec{F}| \neq 0$, are of class C^2. The geometric meaning of their result is that the projection of the characteristic curves of the graph of u are of class C^2. A stationary point u of $\mathscr{F}$ satisfies weakly the prescribed mean curvature equation

$$(1.2) \qquad \operatorname{div}\left(\frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|} \right) = f.$$

Theorem A in [10] is well-known for C^2 minimizers and generalizes a previous result by Pauls [21, Lemma 3.3] for H-minimal surfaces with components of the horizontal Gauss map in the class $W^{1,1}$. For lipschitz continuous vanishing viscosity minimal graphs, it was proven by Capogna, Citti and Manfredini [4, Cor. 1.6].

In order to extend this result to arbitrary surfaces, it is natural to replace $\mathscr{F}$ by the sub-Riemannian prescribed mean curvature functional

$$(1.3) \qquad \mathscr{J}(E, B) = P(E, B) + \int_{E \cap B} f,$$

where E is a set of locally finite sub-Riemannian perimeter in Ω, $P(E, B)$ is the relative sub-Riemannian perimeter of E in a bounded open set $B \subset \Omega$, and $f \in L^\infty(\Omega)$. If $E \subset \mathbb{H}^n$ is the subgraph of a function $t = u(x, y)$ in the Heisenberg group $\mathbb{H}^n$, then $\mathscr{J}(E)$ coincides with (1.1)

taking $\vec{F}(x, y) = (-y, x)$. The notion of sub-Riemannian perimeter used in sub-Riemannian geometry was first introduced by Capogna, Danielli and Garofalo [5] for Carnot-Carathéodory spaces. General properties and existence of sets with minimum perimeter were proved later by Garofalo and Nhieu [17]. A rather complete theory of finite perimeter sets in the Heisenberg group $\mathbb{H}^n$ following De Giorgi's original arguments was developed by Franchi, Serapioni and Serra-Cassano [11], and later extended to step 2 Carnot groups [12] by the same authors. The recent monograph [6] provides a quite complete survey on recent progress on the subject.

We have defined the prescribed mean curvature functional following Massari [19], who considered minimizers of $\mathscr{J}$ for the Euclidean relative perimeter. He obtained existence and regularity results for this problem and observed that, in case E is the subgraph of a Lipschitz function u defined on an open bounded set $D \subset \mathbb{R}^{n-1}$, the function u satisfies weakly the prescribed mean curvature equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)(x) = f(x, u(x))$$

for $x \in D$. In case $\partial E \cap \Omega$ is a hypersurface of class C^2 then the mean curvature of ∂E at a point $p \in \partial E$ equals $g(p)$. See also Maggi [18, pp. 139–140].

The aim of this paper is to extend Cheng, Hwang and Yang's regularity result for characteristic curves [10, Thm. A] from C^1 horizontal graphs satisfying weakly the mean curvature equation in the first Heisenberg group $\mathbb{H}^1$ to surfaces of class C^1 with prescribed mean curvature in arbitrary three-dimensional contact sub-Riemannian manifolds.

In this setting, the Euclidean perimeter is replaced by the sub-Riemannian one and the integral of the function f is computed using Popp's measure [20, § 10.6], [2]. The minimizing condition will be replaced by a stationary one. Our ambient space will be a three-dimensional contact manifold with a sub-Riemannian metric defined on its horizontal distribution. In particular, no assumption on the existence of a pseudo-hermitian structure is made. We shall prove in Theorem 4.1

> Let $E \subset \Omega$ be a set with C^1 boundary and prescribed mean curvature $f \in C^0(\Omega)$ in a domain $\Omega \subset M$ of a three-dimensional contact sub-Riemannian manifold. Then characteristic curves in ∂E are of class C^2.

We remark that [10, Thm. A] states that the projection of characteristic curves to the plane $t = 0$ is of class C^2, but together with [8, (2.22)] this implies that the characteristic curves themselves are C^2. We thank J.-H. Cheng for pointing out this fact.

While the proof of [10, Thm. A] was based on the integral formula [10, (2.3)], see also (3.7) in [16, Remark 3.4], the proof of Theorem 4.1 is purely variational and follows by localizing the first variation of perimeter along a characteristic curve. A much weaker version of Theorem 4.1 was given in [16, Thm. 3.5], where it was proven that the regular part of an area-stationary surface of class C^1 in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ is foliated by horizontal geodesics. Theorem 4.1 provides a new result even for the case of the first Heisenberg group $\mathbb{H}^1$.

The regularity of characteristic curves proven in Theorem 4.1 allows us to define in Section 5 a mean curvature function H in the regular part of ∂E, that coincides with f. As a consequence of the definition of the mean curvature, we shall prove in Proposition 5.3 that characteristic curves are of class C^{k+2} in case f is of class C^k when restricted to a characteristic direction. This holds, e.g., when $f \in C^k(\Omega)$ of $f \in C^k_{\mathbb{H}}(\Omega)$, the space of functions with continuous horizontal derivatives of order k, $k \geqslant 1$. This class contains $C^1(\Omega)$ when $k \geqslant 2$. Critical points of the perimeter, eventually under a volume constraint, and C^1 boundary, have constant prescribed mean curvature as

shown in Section 3. Hence Theorem 4.1 applies to these sets and implies that the regular parts of their boundaries are foliated by C^∞ characteristic curves, see Proposition 5.4.

We have organized this paper into several sections. In the second one we provide the necessary background on contact sub-Riemannian manifolds and sets of finite perimeter, and we recall the first variation formula for C^1 surfaces following [13]. In Section 3 we introduce the definition of set of locally finite perimeter with prescribed mean curvature and prove that a set with C^1 boundary and area-stationary under a volume constraint has constant prescribed mean curvature. The main result, Theorem 4.1, is proven in Section 4. The consequences on the mean curvature and higher regularity for characteristic curves will appear in Section 5.

2. PRELIMINARIES

2.1. **Contact sub-Riemannian manifolds.** In this paper we shall consider a 3-dimensional C^∞ manifold M with contact form ω and a sub-Riemannian metric $g_{\mathcal{H}}$ defined on its *horizontal distribution* $\mathcal{H} := \ker(\omega)$. By definition, $d\omega|_{\mathcal{H}}$ is non-degenerate. We shall refer to $(M, \omega, g_{\mathcal{H}})$ as a 3-*dimensional contact sub-Riemannian manifold*. It is well-known that $\omega \wedge d\omega$ is an orientation form in M. Since

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]),$$

the horizontal distribution $\mathcal{H}$ is completely non-integrable. The *Reeb vector field T* in M is the only one satisfying

$$(2.1) \qquad\qquad \omega(T) = 1, \qquad \mathscr{L}_T \omega = 0,$$

where $\mathscr{L}$ is the Lie derivative in M.

A canonical contact structure in Euclidean 3-space $\mathbb{R}^3$ with coordinates (x, y, t) is given by the contact one-form $\omega_0 := dt + x\,dy - y\,dx$. The associated contact manifold is the Heisenberg group $\mathbb{H}^1$. Darboux's Theorem [3, Thm. 3.1] (see also [15]) implies that, given a point $p \in M$, there exists an open neighborhood U of p and a diffeomorphism ϕ_p from U into an open set of $\mathbb{R}^3$ satisfying $\phi_p^* \omega_0 = \omega$. Such a local chart will be called a *Darboux chart*. Composing the map ϕ_p with a contact transformation of $\mathbb{H}^1$ also provides a Darboux chart. This implies we can prescribe the image of a point $p \in U$ and the image of a horizontal direction in $T_p M$.

The metric $g_{\mathcal{H}}$ can be extended to a Riemannian metric g on M by requiring T to be a unit vector orthogonal to $\mathcal{H}$. The Levi-Civita connection associated to g will be denoted by D. The integral curves of the Reeb vector field T are *geodesics* of the metric g. This property can be easily checked since condition $\mathscr{L}_T \omega = 0$ in (2.1) implies $\omega([T, X]) = 0$ for any $X \in \mathcal{H}$. Hence, for any horizontal vector field X, we have

$$g(X, D_T T) = -g(D_T X, T) = -g(D_X T, T) = 0.$$

We trivially have $g(T, D_T T) = 0$, and so we get $D_T T = 0$, as claimed.

The Riemannian volume element in (M, g) will be denoted by dM. It coincides with Popp's measure [20, § 10.6], [2]. The volume of a set $E \subset M$ with respect to the Riemannian metric g will be denoted by $|E|$.

2.2. **Torsion and the sub-Riemannian connection.** The following is taken from [13, § 3.1.2]. In a contact sub-Riemannian manifold, we can decompose the endomorphism $X \in TM \to D_X T$ into its antisymmetric and symmetric parts, which we will denoted by J and τ, respectively,

$$(2.2) \qquad \begin{aligned} 2g(J(X), Y) &= g(D_X T, Y) - g(D_Y T, X), \\ 2g(\tau(X), Y) &= g(D_X T, Y) + g(D_Y T, X). \end{aligned}$$

Observe that $J(X), \tau(X) \in \mathcal{H}$ for any vector field X, and that $J(T) = \tau(T) = 0$. Also note that

$$(2.3) \qquad 2g(J(X), Y) = -g([X, Y], T), \qquad X, Y \in \mathcal{H}.$$

We will call τ the *(contact) sub-Riemannian torsion*. We note that our J differs from the one defined in [14, (2.4)] by the constant $g([X, Y], T)$, but plays the same geometric role and can be easily generalized to higher dimensions, [13, § 3.1.2].

Now we define the *(contact) sub-Riemannian connection* ∇ as the unique metric connection, [7, eq. (I.5.3)], with torsion tensor $\mathrm{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ given by

$$(2.4) \qquad \mathrm{Tor}(X, Y) := g(X, T) \tau(Y) - g(Y, T) \tau(X) + 2g(J(X), Y) T.$$

From (2.4) and Koszul formula for the connection ∇ it follows that T is a parallel vector field for the sub-Riemannian connection. In particular, their integral curves are geodesics for the connection ∇.

If $X \in \mathcal{H}$, $p \in M$, and $X_p \neq 0$, then $J(X_p) \neq 0$: as $d\omega|_{\mathcal{H}}$ is non-degenerate, there exists $Y \in \mathcal{H}$ such that $d\omega_p(X_p, Y_p) \neq 0$. From (2.2) we have $2g(J(X_p), Y_p) = -g([X, Y]_p, T_p)$, different from 0 since $\omega_p([X, Y]_p) = -d\omega(X_p, Y_p) \neq 0$.

The standard orientation of M is given by the 3-form $\omega \wedge d\omega$. If X_p is horizontal, then the basis $\{X_p, J(X_p), T_p\}$ is positively oriented. To check this, observe first that the sign of $(\omega \wedge d\omega)(X, J(X), T)$ equals the sign of $d\omega(X, J(X))$, and we have

$$d\omega(X, J(X)) = -\omega([X, J(X)]) = -g([X, J(X)], T) = g(\mathrm{Tor}(X, J(X)), T) = 2g(J(X), J(X)) > 0.$$

2.3. Perimeter and C^1 surfaces.

A set $E \subset M$ has *locally finite perimeter* if, for any bounded open set $B \subset M$, we have

$$P(E, B) := \sup \left\{ \int_{E \cap B} \mathrm{div}\, U \, dM : U \text{ horizontal}, \mathrm{supp}(U) \subset B, ||U||_\infty \leqslant 1 \right\} < +\infty.$$

The quantity $P(E, B)$ is the *relative perimeter* of E in B.

Assuming $\Sigma = \partial E$ is a surface of class C^1, the relative perimeter of E in a bounded open set $B \subset M$ coincides with the sub-Riemannian area of $\Sigma \cap B$, given by

$$(2.5) \qquad A(\Sigma \cap B) = \int_{\Sigma \cap B} |N_h| \, d\Sigma.$$

Here N is the Riemannian unit normal to Σ, N_h is the horizontal projection of N to the horizontal distribution, and $d\Sigma$ is the Riemannian area measure, all computed with respect the Riemannian metric g, see [6]. The quantity $|N_h|$ vanishes in the *singular set* $\Sigma_0 \subset \Sigma$ of points $p \in \Sigma$ where the tangent space $T_p\Sigma$ coincides with the horizontal distribution $\mathcal{H}_p$. The *horizontal unit normal* at $p \in \Sigma \setminus \Sigma_0$ is defined by $(\nu_h)_p := (N_h)_p / |(N_h)_p|$. At every point $p \in \Sigma \setminus \Sigma_0$, the intersection $\mathcal{H}_p \cap T_p\Sigma$ is one-dimensional and generated by the characteristic vector field $Z := J(\nu_h)/|J(\nu_h)|$. The vector S_p is defined for $p \in \Sigma \setminus \Sigma_0$ by $S_p := g(N_p, T_p)(\nu_h)_p - |(N_h)_p| T_p$. The tangent space $T_p\Sigma$, $p \in \Sigma \setminus \Sigma_0$, is generated by $\{Z_p, S_p\}$.

2.4. The first variation of the sub-Riemannian perimeter for C^1 surfaces.

Given a set E with C^1 boundary, we can use the flow $\{\varphi_s\}_{s \in \mathbb{R}}$ of a vector field U with compact support in B to produce a variation of $\Sigma \cap B$. The Riemannian area formula gives the following expression of the sub-Riemannian area of $\Sigma_s := \varphi_s(\Sigma \cap B)$,

$$A(\Sigma_s) = \int_\Sigma |N_h^s| \, \mathrm{Jac}(\varphi_s) \, d\Sigma,$$

where N^s is a unit normal to Σ_s. Fix $p \in \Sigma \setminus \Sigma_0$ and the orthonormal basis $\{e_1, e_2\} = \{Z_p, S_p\}$ in $T_p\Sigma$. We consider extensions E_1, E_2 of Z_p, S_p, respectively, along the integral curve of U passing through p. The vector fields $E_1(s), E_2(s)$ are invariant under the flow of U and generate the tangent plane to Σ_s at the point $\varphi_s(p)$. The vector $(E_1 \times E_2)/|(E_1 \times E_2)|$ is normal to Σ_s. Here $\times$ denotes the cross product with respect to a volume form η for the metric g inducing the same orientation as $\omega \wedge d\omega$, i.e. $g(w, u \times v) = \eta(w, u, v)$. It is easy to check that $|(E_1 \times E_2)|(s) = \mathrm{Jac}(\varphi_s)(p)$, and that

$$V(p, s) := (E_1 \times E_2)_h(s) = \big(g(E_1, T)(T \times E_2) - g(E_2, T)(E_1 \times T)\big)(s).$$

Hence

$$A(\Sigma_s) = \int_\Sigma |V(p, s)| \, d\Sigma(p),$$

and we get

$$\frac{d}{ds}\bigg|_{s=0} |V(s, p)| = \frac{g(\nabla_{U_p} V, V_p)}{|V_p|}.$$

Since $\{(\nu_h)_p, Z_p, T_p\}$ is positively oriented, observe that $V_p = |(N_h)_p| (\nu_h)_p$. On the other hand,

$$\nabla_{U_p} V = g(\nabla_{U_p} E_1, T_p)(T_p \times (E_2)_p) - g(\nabla_{U_p} E_2, T_p)((E_1)_p \times T_p)$$
$$- g((E_2)_p, T_p)(\nabla_{U_p} E_1 \times T_p),$$

and so

$$\frac{g(\nabla_{U_p} V, V_p)}{|V_p|} = -g(\nabla_{U_p} E_2, T_p) + |(N_h)_p| \, g(\nabla_{U_p} E_1 \times T_p, (\nu_h)_p).$$

Since

$$\begin{aligned}
g(\nabla_{U_p} E_2, T_p) &= g(\nabla_{(E_2)_p} U + \mathrm{Tor}(U_p, (E_2)_p), T_p) \\
&= S_p(g(U, T)) + g(\mathrm{Tor}(U_p, S_p), T_p) \\
&= S_p(g(U, T)) + 2 \, g(J(U_p), S_p),
\end{aligned}$$

and

$$\begin{aligned}
g(\nabla_{U_p} E_1 \times T_p, (\nu_h)_p) &= g((\nabla_{(E_1)_p} U + \mathrm{Tor}(U_p, (E_1)_p)) \times T_p, (\nu_h)_p) \\
&= \eta((\nu_h)_p, \nabla_{(E_1)_p} U + \mathrm{Tor}(U_p, (E_1)_p), T_p) \\
&= +g(\nabla_{Z_p} U + \mathrm{Tor}(U_p, Z_p), Z_p) \\
&= +g(\nabla_{Z_p} U, Z_p) + g(U_p, T_p) \, g(\tau(Z_p), Z_p),
\end{aligned}$$

we conclude that the first variation of the sub-Riemannian perimeter is given by

$$\begin{aligned}
(2.6) \qquad \frac{d}{ds}\bigg|_{s=0} A(\Sigma_s) = \int_{\Sigma \cap B} \Big\{ &-S(g(U, T)) - 2 \, g(J(U), S) + |N_h| \, g(\nabla_Z U, Z) \\
&+ |N_h| \, g(U, T) \, g(\tau(Z), Z) \Big\} \, d\Sigma.
\end{aligned}$$

This formula was obtained in [14, Lemma 3.4].

3. SETS WITH PRESCRIBED MEAN CURVATURE

The reader is referred to [18, (12.32) and Remark 17.11] for background and references in the Euclidean case. Consider a domain $\Omega \subset M$, and a function $f : \Omega \to \mathbb{R}$. We shall say that a set of locally finite perimeter $E \subset \Omega$ has *prescribed mean curvature f on Ω* if, for any bounded open set $B \subset \Omega$, E is a critical point of the functional

$$(3.1) \qquad P(E, B) - \int_{E \cap B} f,$$

where $P(E,B)$ is the relative perimeter of E in B, and the integral on $E \cap B$ is computed with respect to the canonical Popp's measure on M, see [20] and [2]. The admissible variations for this problem are the flows induced by vector fields with compact support in B.

If $\Sigma = \partial E$ is a surface of class C^1 in Ω, then Σ has prescribed mean curvature f if it is a critical point of the functional

$$(3.2) \qquad A(\Sigma \cap B) - \int_{E \cap B} f,$$

for any bounded open set $B \subset \Omega$.

If E is a critical point of the relative perimeter $P(E,B)$ in any bounded open set $B \subset \Omega$, then E has zero or vanishing prescribed mean curvature.

Assume now that $E \subset \Omega$ is a set of locally finite perimeter with C^1 boundary Σ, and that E is a *critical point of the perimeter under a volume constraint*. This means $(d/ds)_{s=0} A(\varphi_s(\Sigma \cap B)) = 0$ for any flow associated to a vector field with compact support in Ω satisfying $(d/ds)_{s=0}|\varphi_s(E \cap B)| = 0$. If the perimeter of E in Ω is positive, then there exists a (horizontal) vector field U_0 with compact support in Ω so that $\int_{E \cap \Omega} \operatorname{div} U_0 \, dM > 0$. By the Divergence Theorem,

$$\int_{\Sigma \cap \Omega} g(U_0, N) \, d\Sigma \neq 0,$$

where N is the outer normal to E. Let $\{\psi_s\}_{s \in \mathbb{R}}$ be the flow associated to the vector field U_0 and define

$$(3.3) \qquad H_0 := \frac{d/ds\big|_{s=0} A(\psi_s(\Sigma))}{d/ds\big|_{s=0} |\psi_s(E)|}.$$

Let $B \subset \Omega$ be a bounded open subset and W a vector field with compact support in B and associated flow $\{\varphi_s\}_{s \in \mathbb{R}}$. Choose $\lambda \in \mathbb{R}$ so that $W - \lambda U_0$ satisfies

$$\frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)| - \lambda \frac{d}{ds}\bigg|_{s=0} |\psi_s(E)| = \int_\Sigma g(W - \lambda U_0, N) \, d\Sigma = 0.$$

Then the flow associated to $W - \lambda U_0$ preserves the volume of $E \cap (B \cup B_0)$, where $B_0 \subset \Omega$ is a bounded open set containing $\operatorname{supp}(U_0)$. Let $Q(U)$ be the integral expression in (2.6). From our hypothesis and the linearity of (2.6), $Q(W - \lambda U_0) = 0$. Hence $Q(W) = \lambda Q(U_0)$. From (3.3) we get

$$Q(W) = \lambda Q(U_0) = \lambda H_0 \frac{d}{ds}\bigg|_{s=0} |\psi_s(E)| = H_0 \frac{d}{ds}\bigg|_{s=0} |\varphi_s(E)|,$$

and so E has (constant) prescribed mean curvature H_0.

4. MAIN RESULT

In this Section we shall prove our main result

Theorem 4.1. *Let M be a 3-dimensional contact sub-Riemannian manifold, $\Omega \subset M$ a domain, and $E \subset \Omega$ a set of prescribed mean curvature $f \in C^0(\Omega)$ with C^1 boundary Σ. Then the characteristic curves in Σ are of class C^2.*

Proof. Given any point $p \in \Sigma \setminus \Sigma_0$, consider a Darboux chart (U_p, ϕ_p) such that $\phi_p(p) = 0$. The metric $g_{\mathcal{H}}$ can be described in this local chart by the matrix of smooth functions

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g(X,X) & g(X,Y) \\ g(Y,X) & g(Y,Y) \end{pmatrix}.$$

After a Euclidean rotation around the t-axis, which is a contact transformation in $\mathbb{H}^1$ [22, p. 640], we may assume there exists an open neighborhood $B \cap \Sigma$ of $p \in \Sigma \setminus \Sigma_0$, where $B \subset \mathbb{H}^1$ is an open set containing p, so that $B \cap \Sigma$ is the intrinsic graph G_u of a C^1 function $u : D \to \mathbb{R}$ defined on a domain D in the vertical plane $y = 0$. We can also assume that $E \cap B$ is the subgraph of u. The graph G_u can be parameterized by the map $f_u : D \to \mathbb{R}^3$ defined by

$$f_u(x, t) := (x, u(x, t), t - xu(x, t)), \qquad (x, t) \in D.$$

The tangent plane to any point in G_u is generated by the vectors

$$\frac{\partial}{\partial x} \mapsto (1, u_x, -u - xu_x) = X + u_x Y - 2uT,$$
$$\frac{\partial}{\partial t} \mapsto (0, u_t, 1 - xu_t) = u_t Y + T,$$

and so the characteristic direction is given by $Z = \widetilde{Z}/|\widetilde{Z}|$, where

$$\widetilde{Z} = X + (u_x + 2uu_t)Y.$$

If $\gamma(s) = (x(s), t(s))$ is a C^1 curve in D, then

$$\Gamma(s) = (x(s), u(x(s), t(s)), t(s) - x(s)u(x(s), t(s))) \subset G_u$$

is also C^1, and so

$$\Gamma'(s) = x'(X + u_x Y - 2uT) + t'(u_t Y + T) = x'X + (x'u_x + t'u_t)Y + (t' - 2ux')T.$$

In particular, horizontal curves in G_u satisfy the ordinary differential equation $t' = 2ux'$. Since $u \in C^1(D)$, we have uniqueness of characteristic curves through any given point in G_u.

A unit normal vector to Σ is given by $\widetilde{N}/|\widetilde{N}|$, where

$$\widetilde{N} = (X + u_x Y - 2uT) \times (u_t Y + T).$$

Here $\times$ is the cross product with respect to the Riemannian metric g and a given volume form η chosen so that $\eta(X, Y, T) > 0$. Hence $g(w, u \times v) = \eta(w, u, v)$. If $\{e_1, e_2, e_3\}$ is an orthonormal basis so that $\eta(e_1, e_2, e_3) = 1$ and A is the matrix whose columns are the coordinates of X, Y, T in the basis $\{e_1, e_2, e_3\}$, then $\eta(X, Y, T) = \det(A)$. On the other hand, as

$$A^t A = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we get $\det(A)^2 = \det(G)$. Since $\det(A) > 0$ we obtain $\det(A) = \det(G)^{1/2}$ and so

$$\eta(X, Y, T) = \det(G)^{1/2}.$$

Let $E_1 = X + u_x Y - 2uT$, $E_2 = u_t Y + T$. We compute the scalar product of $\widetilde{N} = E_1 \times E_2$ with X, Y, T to obtain

$$g(X, E_1 \times E_2) = \eta(X, E_1, E_2) = \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & u_x & u_t \\ 0 & -2u & 1 \end{pmatrix} \eta(X, Y, T) = (u_x + 2uu_t)\det(G)^{1/2}.$$

$$g(Y, E_1 \times E_2) = \eta(Y, E_1, E_2) = \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & u_x & u_t \\ 0 & -2u & 1 \end{pmatrix} \eta(X, Y, T) = -\det(G)^{1/2}.$$

$$g(T, E_1 \times E_2) = \eta(T, E_1, E_2) = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & u_x & u_t \\ 1 & -2u & 1 \end{pmatrix} \eta(X, Y, T) = u_t \det(G)^{1/2}.$$

Since $g(Y, E_1 \times E_2) < 0$ and $E \cap B$ is the subgraph of u, the vector field $E_1 \times E_2$ points into the interior of E. If $\widetilde{N} = E_1 \times E_2 = \alpha X + \beta Y + \gamma T$, then

$$\begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \det(G)^{1/2} \begin{pmatrix} u_x + 2uu_t \\ -1 \\ u_t \end{pmatrix},$$

whence

(4.1)
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \det(G)^{1/2} G^{-1} \begin{pmatrix} u_x + 2uu_t \\ -1 \end{pmatrix},$$

$$\gamma = \det(G)^{1/2} u_t.$$

Let us compute now the sub-Riemmanian area of the intrinsic graph G_u. It is easy to check that $d\Sigma = |E_1 \times E_2| \, dx \, dt$, i.e., that $\mathrm{Jac}(f_u) = |E_1 \times E_2|$. Since $|N_h| = |(E_1 \times E_2)_h|/|E_1 \times E_2|$, then using (4.1) and the explicit expression of the inverse matrix G^{-1} we get

$$|N_h| \, \mathrm{Jac}(f_u) = |(E_1 \times E_2)_h| = \left(\begin{pmatrix} \alpha & \beta \end{pmatrix} G \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right)^{\frac{1}{2}}$$

$$= \left((g_{22} \circ f_u)(u_x + 2uu_t)^2 + 2(g_{12} \circ f_u)(u_x + 2uu_t) + (g_{11} \circ f_u) \right)^{1/2}.$$

Finally, from (2.5) we obtain

(4.2)
$$A(G_u) = \int_D \left(g_{22}(u_x + 2uu_t)^2 + 2g_{12}(u_x + 2uu_t) + g_{11} \right)^{1/2} dx \, dt,$$

where, by abuse of notation, we have written g_{ij} instead of the cumbersome notation $(g_{ij} \circ f_u)$.

Now we consider variations of G_u by graphs of the form $s \mapsto u + sv$, where $v \in C_0^\infty(D)$ and s is a real parameter close to 0. This variation is obtained by applying the flow associated to the vector field $\tilde{v}Y$ to the graph G_u. The function $\tilde{v}$ is obtained by extending v to be constant along the integral curves of the vector field Y, and multiplying by an appropriate function with compact support equal to 1 in a neighborhood of Σ.

When F is a function of (x, y, t), we have

$$\frac{d}{ds}\bigg|_{s=0} (F \circ f_{u+sv})(x, t) = \left(\frac{\partial F}{\partial y} - x \frac{\partial F}{\partial t} \right)_{f_u(x,t)} v(x, t) = Y_{f_u(x,t)}(F) \, v(x, t).$$

So we get

$$\frac{d}{ds}\bigg|_{s=0} A(G_{u+sv}) = \int_D \left(K_1 v + M \left(v_x + 2uv_t + 2vu_t \right) \right) dx \, dt,$$

where the functions K_1 and M are given by

$$K_1 = \frac{1}{2} \frac{Y(g_{22})(u_x + 2uu_t)^2 + 2Y(g_{12})(u_x + 2uu_t) + Y(g_{11})}{(g_{22}(u_x + 2uu_t)^2 + 2g_{12}(u_x + 2uu_t) + g_{11})^{1/2}},$$

and

$$M = \frac{g_{22}(u_x + 2uu_t) + g_{12}}{(g_{22}(u_x + 2uu_t)^2 + 2g_{12}(u_x + 2uu_t) + g_{11})^{1/2}}.$$

Observe that the functions K_1 and M are continuous. Since

$$Z = \frac{X + (u_x + 2uu_t)Y}{(g_{22}(u_x + 2uu_t)^2 + 2g_{12}(u_x + 2uu_t) + g_{11})^{1/2}},$$

the function M coincides with $g(Z, Y) \circ f_u$. A straightforward computation implies

$$1 = |Z|^2 = \det(G)^{-1} \left(g_{22} g(Z, X)^2 - 2g_{12} g(Z, X) g(Z, Y) + g_{11} g(Z, Y)^2 \right)$$

and so

$$(4.3) \qquad g(Z,X) = \frac{g_{12} g(Z,y) \pm (\det(G)(g_{22} - g(Z,Y)^2))^{1/2}}{g_{22}}.$$

By Schwarz's inequality $g(Z,Y)^2 \leqslant g(Y,Y) = g_{22}$. Inequality is strict since otherwise Y and Z would be collinear. Hence $g(Z,X)$ has the same regularity as $g(Z,Y)$ by (4.3).

The subgraph of u can be parameterized by the map $(x,t,s) \to (x,s,t-xs)$. The Jacobian of this map is easily seen to be equal to $\det(G)$. Hence

$$\frac{d}{ds}\bigg|_{s=0} \int_{\text{subgraph } G_{u+sv}} f = \int_D f \, \det(G) \, v \, dx \, dt.$$

If Σ has prescribed mean curvature f, this implies

$$(4.4) \qquad \int_D \big(K v + M (v_x + 2u v_t + 2v u_t)\big) dx \, dt = 0,$$

for any $v \in C_0^\infty(D)$, where the *continuous* function K is given by $K = K_1 - f \det(G)$. By Remark 4.3 below, (4.4) also holds for any $v \in C_0^0(D)$ for which $v_x + 2u v_t$ exists and it is continuous.

Now we proceed as in the proof of Theorem 3.5 in [16]. Assume the point $p \in G_u$ corresponds to the point (a, b) in the xt-plane. The curve $s \mapsto (s, t(s))$ is (a reparameterization of the projection of) a characteristic curve if and only if the function $t(s)$ satisfies the ordinary differential equation $t'(s) = u(s, t(s))$. For ε small enough, we consider the solution t_ε of equation $t'_\varepsilon(s) = 2u(s, t_\varepsilon(s))$ with initial condition $t_\varepsilon(a) = b + \varepsilon$, and define $\gamma_\varepsilon(s) := (s, t_\varepsilon(s))$, with $\gamma = \gamma_0$. We may assume that, for small enough ε, the functions t_ε are defined in the interval $[a-r, a+r]$ for some $r > 0$. The function $\partial t_\varepsilon / \partial \varepsilon$ satisfies

$$(4.5) \qquad \left(\frac{\partial t_\varepsilon}{\partial \varepsilon}\right)'(s) = 2u_t(s, t_\varepsilon(s))\left(\frac{\partial t_\varepsilon}{\partial \varepsilon}\right)(s), \qquad \frac{\partial t_\varepsilon}{\partial \varepsilon}(a) = 1.$$

where $'$ is the derivative with respect to the parameter s.

We consider the parameterization

$$F(\xi, \varepsilon) := (\xi, t_\varepsilon(\xi)) = (s, t)$$

near the characteristic curve through (a, b). The jacobian of this parameterization is given by

$$\det\begin{pmatrix} 1 & t'_\varepsilon \\ 0 & \partial t_\varepsilon / \partial \varepsilon \end{pmatrix} = \frac{\partial t_\varepsilon}{\partial \varepsilon},$$

which is positive because of the choice of initial condition for t_ε and the fact that the curves $\gamma_\varepsilon(s)$ foliate a neighborhood of (a, b). Any function φ can be considered as a function of the variables (ξ, ε) by making $\tilde\varphi(\xi, \varepsilon) := \varphi(\xi, t_\varepsilon(\xi))$. Changing variables, and assuming the support of φ is contained in a sufficiently small neighborhood of (a, b), we can express the integral (4.2) as

$$\int_I \left\{ \int_{a-r}^{a+r} \left(K\tilde\varphi + M\left(\frac{\partial \tilde\varphi}{\partial \xi} + 2\tilde\varphi \tilde u_t \right) \right) \frac{\partial t_\varepsilon}{\partial \varepsilon} \, d\xi \right\} d\varepsilon,$$

where I is a small interval containing 0. Instead of $\tilde\varphi$, we can consider the function $\tilde\varphi h/(t_{\varepsilon+h} - t_\varepsilon)$, where h is a sufficiently small real parameter. We get that

$$\frac{\partial}{\partial \xi}\left(\frac{h\tilde\varphi}{t_{\varepsilon+h} - t_\varepsilon} \right) = \frac{\partial \tilde\varphi}{\partial \xi} \cdot \frac{h}{t_{\varepsilon+h} - t_\varepsilon} - 2\tilde\varphi \cdot \frac{\tilde u(\xi, \varepsilon+h) - \tilde u(\xi, \varepsilon)}{t_{\varepsilon+h} - t_\varepsilon} \cdot \frac{h}{t_{\varepsilon+h} - t_\varepsilon}$$

tends to

$$\frac{\partial \tilde\varphi / \partial \xi}{\partial t_\varepsilon / \partial \varepsilon} - \frac{2\tilde\varphi \tilde u_t}{\partial t_\varepsilon / \partial \varepsilon},$$

when $h \to 0$. So using that G_u is area-stationary we have that

$$\int_I \left\{ \int_{a-r}^{a+r} \frac{h}{t_{\varepsilon+h} - t_\varepsilon} \left(K \tilde\varphi + M \left(\frac{\partial \tilde\varphi}{\partial \xi} \cdot +2\tilde\varphi \cdot \left(\tilde u_t - \frac{\tilde u(\xi, \varepsilon + h) - \tilde u(\xi, \varepsilon)}{t_{\varepsilon+h} - t_\varepsilon} \right) \right) \right) \frac{\partial t_\varepsilon}{\partial \varepsilon} d\xi \right\} d\varepsilon$$

vanishes. Furthermore, letting $h \to 0$ we conclude

$$\int_I \left\{ \int_{a-r}^{a+r} \left(K \tilde\varphi + M \frac{\partial \tilde\varphi}{\partial \xi} \right) d\xi \right\} d\varepsilon = 0.$$

Let now $\eta : \mathbb{R} \to \mathbb{R}$ be a positive function with compact support in the interval I and consider the family $\eta_\rho(x) := \rho^{-1} \eta(x/\rho)$. Inserting a test function of the form $\eta_\rho(\varepsilon)\psi(\xi)$, where ψ is a C^∞ function with compact support in $(a - r, a + r)$, making $\rho \to 0$, and using that G_u is area-stationary we obtain

$$\int_{a-r}^{a+r} \left(K(0, \xi) \psi(\xi) + M(0, \xi) \psi'(\xi) \right) d\xi = 0$$

for any $\psi \in C_0^\infty((a - r, a + r))$. By Lemma 4.2, the function $M(0, \xi)$, which is the restriction of $g(Z, Y)$ to the characteristic curve, is a C^1 function on the curve. By equation (4.3), the restriction of $g(Z, X)$ to the characteristic curve is also C^1. This proves that horizontal curves are of class C^2. ∎

Lemma 4.2. *Let $I \subset \mathbb{R}$ be an open interval, $k, m \in C^0(I)$, and $K \in C^1(I)$ be a primitive of k. Assume*

$$(4.6) \qquad\qquad \int_I k\psi + m\psi' = 0,$$

for any $\psi \in C_0^\infty(I)$. Then the function $-K + m$ is constant on I. In particular, $m \in C^1(I)$.

Proof. Since $(K\psi)' = k\psi + K\psi'$, integrating by parts we see that (4.6) is equivalent to

$$\int_I \left(-K + m \right) \psi' = 0,$$

for any $\psi \in C_0^\infty(I)$. This implies that $-K + m$ is a constant function on I. ∎

Remark 4.3. Let us check that (4.4) holds for any $w \in C_0^0(D)$ such that $w_x + 2uw_t$ exists and is continuous. Let us consider a sequence $w_j \in C_0^\infty(D)$, where $w_j = \rho_j * w$, and ρ_j denote the standard mollifiers. We have that w_j converges to w and that $(w_j)_x + 2u(w_j)_t$ converges to $w_x + 2uw_t$ uniformly on compact subsets of D, for $j \to \infty$. We conclude

$$0 = \lim_{j \to \infty} \int_D \left(K(w_j) + M \left((w_j)_x + 2u(w_j)_t + 2w_j u_t \right) \right) dx dt = \int_D \left(Kw + M \left(w_x + 2uw_t + 2wu_t \right) \right) dx dt,$$

thus proving the claim.

Remark 4.4. In case M is the Heisenberg group $\mathbb{H}^1$, G is the identity matrix and the expression for the sub-Riemannian area of the graph G_u given in (4.2) reads

$$A(G_u) = \int_D ((u_x + 2uu_t)^2 + 1)^{1/2} dx dt,$$

a well-known formula obtained in [1].

5. THE MEAN CURVATURE FOR C^1 SURFACES

Given a surface $\Sigma \subset M$ of class C^1 such that the vector fields Z and ν_h are of class C^1 along the characteristic curves in $\Sigma \setminus \Sigma_0$, we define the *mean curvature* of Σ at $p \in \Sigma \setminus \Sigma_0$ by

$$(5.1) \qquad \big(\operatorname{div}_\Sigma^h(\nu_h)\big)(p) := -g(\nabla_Z \nu_h, Z)(p).$$

This is the standard definition of mean curvature for C^2 surfaces, see e.g. [14, (3.8)] and the references there. The mean curvature is usually denoted by H. The mean curvature depends on the choice of ν_h. In case Σ is the boundary of a set E, we shall always choose N as the inner normal and $\nu_h = N_h/|N_h|$.

Using the regularity Theorem 4.1 we get

Proposition 5.1. *Let $\Omega \subset M$ be a domain and $E \subset \Omega$ a set of prescribed mean curvature $f \in C^0(\Omega)$ with C^1 boundary Σ with $H \in L^1_{loc}(\Sigma)$. Then the first variation of the functional (3.2) induced by a vector field $U \in C_0^1(\Omega)$ is given by*

$$(5.2) \qquad \int_\Sigma H\, g(U, N)\, d\Sigma - \int_\Sigma f\, g(U, N)\, d\Sigma.$$

Proof. We first observe that formula

$$\frac{d}{ds}\bigg|_{s=0} A(\Sigma_s) = -\int_\Sigma H\, g(U, N)\, d\Sigma$$

can be proved as in [14, Prop. 6.3], see also [14, Remark 6.4]. On the other hand, it is well-known that

$$\frac{d}{ds}\bigg|_{s=0} \left(\int_{\phi_s(\Omega)} f\right) = -\int_\Sigma f\, g(U, N)\, d\Sigma,$$

see e.g. [18, 17.8]. $\qquad\square$

Then we have that the mean curvature H defined in (5.1) coincides with the prescribed mean curvature f

Corollary 5.2. *Let $E \subset \Omega$ be a set of prescribed mean curvature $f \in C^0(\Omega)$ with C^1 boundary Σ in a domain $\Omega \subset M$. Assume $H \in L^1_{loc}(\Sigma)$. Then $H(p) = f(p)$ for any $p \in \Sigma \setminus \Sigma_0$.*

Finally we can improve the regularity of Theorem 4.1 assuming the mean curvature function is more regular. This result is specially useful if we assume that higher order horizontal derivatives of the function f exist and are continuous.

Proposition 5.3. *Let $E \subset \Omega$ be a set of prescribed mean curvature $f \in C^0(\Omega)$ with C^1 boundary Σ in a domain $\Omega \subset M$. Assume that f is also C^k in the Z-direction, $k \geqslant 1$. Then the characteristic curves of Σ are of class C^{k+2} in $\Sigma \setminus \Sigma_0$.*

Proof. Since we have $\nabla_Z Z = H \nu_h$, we can write

$$\nabla_Z(\nabla_Z Z) = Z(H)\nu_h + H\nabla_Z \nu_h = Z(H)\nu_h - H^2 Z.$$

Iterating the procedure we obtain the statement. $\qquad\square$

In particular, this result holds when $f \in C_{\mathbb{H}}^1(\Omega)$, i.e., when f has horizontal derivatives of class $(k-1)$, see [11].

An important particular case is that of critical points of perimeter, possibly under a volume constraint. Assuming C^1 regularity of the boundary, these sets are known to have constant prescribed mean curvature from the discussion in Section 3. From Proposition 5.3, we immediately obtain

Proposition 5.4. *Let $E \subset \Omega$ be either a critical point of the sub-Riemannian perimeter or a critical point of the sub-Riemannian perimeter under a volume constraint. If E has C^1 boundary, then the regular part of ∂E is foliated by C^∞ characteristic curves.*

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