# BV Minimizers of Variational Integrals: Existence, Uniqueness, Regularity

## Habilitationsschrift

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# Chapter 1

# Introduction

This treatise summarizes and contextualizes the main results of the author's publications [P1, P2, P3, P4, P5], among which [P1, P3, P5] are joint works with L. Beck. In order to provide a treatment within a unified framework, several results are presented in a slightly modified form, and, particularly in Chapter 4, some minor refinements are accomplished. We emphasize, however, that in many cases the exposition at hand *does not contain complete proofs* of the stated results. Instead, the aim is to sketch and illustrate the essential ideas and to point out connections between the relevant methods and statements, while we do not enter into all the technical details of the proofs, which can be found in the original articles.

The publications match the subsequent chapters as follows. Chapter 2, which discusses the basic existence theory for BV minimizers, is not based on a specific reference. Chapter 3 describes the duality theory developed in [P3]. Chapter 4 presents the regularity and uniqueness theorems obtained in [P1, P5], and Chapter 5 is concerned with the local and partial regularity results of [P2]. Finally, the approximation results of [P4] are already utilized in the other chapters, but their detailed discussion, which points into a slightly different direction, is postponed to Appendix A.

Primarily, the present exposition is concerned with minimization problems for first-order variational integrals

$$F[w] := \int_\Omega f(\,\cdot\,,\nabla w)\,\mathrm{d}x \qquad \text{among admissible functions } w\colon \Omega \to \mathbb{R}^N\,. \tag{1.1}$$

Here, the dimension $n$ and the codimension $N$ are arbitrary positive integers, $\Omega$ denotes an open subset of $\mathbb{R}^n$, and $\mathrm{d}x$ stands for the integration with respect to the $n$-dimensional Lebesgue measure $\mathcal{L}^n$. Moreover, $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is a given Borel integrand which satisfies the linear growth condition

$$|f(x,z)| \le \Psi(x) + \Gamma|z| \qquad \text{for all } (x,z) \in \Omega \times \mathbb{R}^{N\times n} \tag{1.2}$$

with a fixed ($\mathrm{L}^1$ or $\mathrm{L}^\infty$) function $\Psi$ on $\Omega$ and a fixed non-negative constant $\Gamma < \infty$.

In what follows, $\mathrm{C}^2$ functions $w$ satisfying a suitable boundary condition will always be admissible, and we record that, in this case, the necessary first-order criterion closely relates the minimization problem to a partial differential equation: If $f$ is $\mathrm{C}^2$ in the $z$-variable, every

$C^2$ minimizer $u$ of $F$ satisfies the Euler(-Lagrange) equation

$$\operatorname{div}\big[\nabla_z f(\,\cdot\,,\nabla u)\big] \equiv 0 \qquad \text{on } \Omega\,, \tag{1.3}$$

where the $\mathbb{R}^N$-valued divergence of the matrix-valued function $\nabla_z f(\,\cdot\,,\nabla u)$ is computed row-wise. For $n \geq 2$, the equation (1.3) is a second-order quasilinear PDE, and for $N \geq 2$ it is actually meaningful to regard (1.3) not as a single PDE for a single function, but rather as a system of $N$ PDEs for the $N$ component functions of $u$. In the sequel, we never explicitly rely on (1.3), but we prefer to work with its more convenient weak reformulation. This reformulation makes sense whenever $\nabla_z f$ exists as a bounded continuous function and asserts, even for all $W^{1,1}$ minimizers $u$ of $F$, that we have[1]

$$\int_\Omega \nabla_z f(\,\cdot\,,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in W^{1,1}_0(\Omega,\mathbb{R}^N)\,. \tag{1.4}$$

A model class of convex integrals of the above type — which we use in this introduction to illustrate some results of [P1, P2, P5] — is given by

$$M_p[w] := \int_\Omega (1+|\nabla w|^p)^{\frac{1}{p}}\,\mathrm{d}x \qquad \text{with a parameter } p \in [1,\infty)\,. \tag{1.5}$$

Specifically, in the case $N = 1$, $p = 2$ and for sufficiently smooth $w$, the quantity $M_2[w]$ equals the $n$-dimensional measure $\mathcal{H}^n(\operatorname{Graph} w)$ of the graph of $w$, and $M_2$ is known as the non-parametric area functional (in the hypersurface case). This functional has been extensively studied in classical literature, and its well-developed theory, described in the monograph [73], for instance, has widely inspired the approach to more general functionals of the type (1.1)–(1.2) or (1.5).

For general $N$, instead, the $n$-dimensional area $\mathcal{H}^n(\operatorname{Graph} w)$ of a smooth graph is given by the integral

$$\int_\Omega \sqrt{1+|\mathcal{M}(\nabla w)|^2}\,\mathrm{d}x\,. \tag{1.6}$$

Here, $\mathcal{M}(z) \in \mathbb{R}^{\tau(n,N)}$ denotes the vector of all minors of $z \in \mathbb{R}^{N\times n}$ of all orders from 1 up to $\min\{n,N\}$, and $\tau(n,N)$ stands for the total number of such minors. Thus, in higher codimension $N \geq 2$, the non-parametric area functional (1.6) is non-convex with a non-standard growth behavior. This functional does not coincide, anymore, with $M_2$ (except in the trivial case $n = 1$) and is much more difficult to handle. In fact, we are not aware of a developed theory of (non-smooth minimizers of) the higher-codimension non-parametric area, and though we will not pursue this in the sequel, we hope that some of the arguments employed in order to treat $M_2$ in the higher-codimension case might eventually be useful to make a tiny step towards a better understanding of the functional (1.6).

Another specific integral, which coincides with $M_1$ from the model class (1.5) (up to the irrelevant additive constant $\mathcal{L}^n(\Omega)$), is the total variation, given by

$$\int_\Omega |\nabla w|\,\mathrm{d}x\,. \tag{1.7}$$

[1] We write $z\cdot\widetilde{z}$ for the Hilbert-Schmidt product of two matrices $z,\widetilde{z} \in \mathbb{R}^{N\times n}$, that is the inner product obtained by identifying $\mathbb{R}^{N\times n}$ with $\mathbb{R}^{Nn}$. The corresponding (Hilbert-Schmidt or Frobenius) norm of $z$ is simply denoted by $|z|$.

Minimizers of the total variation are known as functions of least gradient, and in the case $N = 1$ they have been studied, for instance, in [100, 36, 110, 109, 124] — partially out of intrinsic interest and partially motivated by the geometric fact that their level sets are parametric minimal hypersurfaces. Moreover, the term (1.7) is frequently used in a variety of proposed minimization approaches for image restoration, since it has a specific mild regularization effect on minimizers. In this treatise, $\mathrm{M}_1$ and thus the total variation will be included in the considerations of Chapters 2 and 3. However, since $\mathrm{M}_1$ is merely convex, but not strictly convex or differentiable, it will be excluded in most of Chapters 4 and 5, where positivity of the second derivative of the integrand is a crucial assumption (nevertheless, the estimates of Section 4.1 cover the case of $\mathrm{M}_1$, and Section 5.3 allows some non-strict convexity and includes regularizations of $\mathrm{M}_1$, which are partially used in image restoration).

Next we discuss the solvability of minimization problems for the integrals $\mathrm{M}_p[w]$ from (1.5). As a first attempt, it seems natural to study such problems in the Sobolev space $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$, on which the functionals $\mathrm{M}_p$ are finite and strongly continuous. Unfortunately, since $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ is neither a reflexive space nor a dual space, it has bad compactness properties, and minimizers in $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ need not exist in general; and even if they exist, their existence is often hard to prove. For this reason, one is naturally led to consider the integrals $\mathrm{M}_p$ on the space $\mathrm{BV}(\Omega, \mathbb{R}^N)$ of functions of bounded variation, that is, the space of functions $w \in \mathrm{L}^1(\Omega, \mathbb{R}^N)$ whose distributional derivative is represented by a finite $\mathbb{R}^{N\times n}$-valued Radon measure $\mathrm{D}w$ on $\Omega$. Since $\mathrm{BV}(\Omega, \mathbb{R}^N)$ can be seen as a dual space, it exhibits good (weak-$*$) compactness properties, but in exchange one faces the problem to reasonably define $\mathrm{M}_p[w]$ in the case that the gradient of $w$ is merely a measure. However, this problem can be overcome by working with functionals of measures (or BV functions) in the sense of Goffman & Serrin [75] (or Giaquinta & Modica & Souček [71]); see Section 2.1 for a detailed account. In the case considered here, one approach to this concept amounts to defining BV extensions $\overline{\mathrm{M}}_p$ of the functionals $\mathrm{M}_p$ as follows. For every Borel set $A$ in $\mathbb{R}^n$ and every $w \in \mathrm{BV}_{\mathrm{loc}}(U, \mathbb{R}^N)$, defined on an open neighborhood $U$ of $A$, we set

$$\overline{\mathrm{M}}_p[w; A] := \int_A (1 + |\nabla w|^p)^{\frac{1}{p}}\,\mathrm{d}x + |\mathrm{D}^{\mathrm{s}}w|(A) \in [0, \infty]\,,$$

where we have adopted the notation $\mathrm{D}w = (\nabla w)\mathcal{L}^n + \mathrm{D}^{\mathrm{s}}w$ for the Lebesgue decomposition of the gradient measure $\mathrm{D}w$ into its $\mathcal{L}^n$-absolutely continuous part with density $\nabla w \in \mathrm{L}^1_{\mathrm{loc}}(U, \mathbb{R}^{N\times n})$ and its $\mathcal{L}^n$-singular part $\mathrm{D}^{\mathrm{s}}w$ (notice that, in this context, $\nabla w$ is not a gradient of its own). Once these extensions are at hand, one can deal with minimization problems for $\overline{\mathrm{M}}_p$ in $\mathrm{BV}(\Omega, \mathbb{R}^N)$, and it is well known that one can generally solve a BV version of the Dirichlet minimization problem, which is the natural counterpart of the corresponding $\mathrm{W}^{1,1}$ version; see Sections 2.3 and 2.4.

Taking the existence of BV minimizers for granted, we next turn to their regularity properties. In the scalar case $N = 1$, this issue can be approached by classical methods based on the usage of global gradient estimates in the spirit of Serrin [121] and Trudinger [128], and, in particular, Giaquinta & Modica & Souček [72] adapted a method of Gerhardt [69] in order to establish interior Lipschitz regularity for BV minimizers of a class of functionals of the type (1.1)–(1.2). Moreover, under suitable assumptions on the boundary data, Tausch [125] has shown the existence of regular minimizers also for *some* functionals with explicit $w$-dependence and for the model integrals (1.5) with $2 \neq p > 1$. The treatment of BV minimizers of $\overline{\mathrm{M}}_p$ in arbitrary codimension $N \in \mathbb{N}$, however, seems to require different methods, and the situation is less completely understood. The known results from [21, 29, 31, P1, P2, P5] are indeed cumulated in the following statement, which we formulate in terms of local minimizers and without explicit reference to a boundary condition. We stress, however, that the conclusions apply, in particular, to minimizers of the Dirichlet problem, for which also the general existence result in the later Theorem 2.8 is available.

**Theorem 1.1** (gradient regularity for local BV minimizers of $\mathrm{M}_p$, $1<p<\infty$)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ and an exponent $p \in (1,\infty)$. If $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local minimizer of $\overline{\mathrm{M}}_p$ on $\Omega$, that is*

$$\overline{\mathrm{M}}_p[u;\mathrm{B}_r(x_0)] \le \overline{\mathrm{M}}_p[u{+}\varphi;\mathrm{B}_r(x_0)]$$

*for all balls $\mathrm{B}_r(x_0) \Subset \Omega$ and all $\varphi \in \mathrm{BV}(\Omega,\mathbb{R}^N)$ with $\operatorname{spt}\varphi \subset \mathrm{B}_r(x_0)$, then*

**(A)** *in the **case $\boldsymbol{1<p<2}$**, we have*

$$u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \text{ for some } \alpha(n,N,p) \in (0,1]\,,$$

**(B)** *in the **case $\boldsymbol{p=2}$**, we have*

$$u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \qquad \text{and} \qquad |\nabla u|\log(1{+}|\nabla u|) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)\,,$$

**(C)** ***generally**, there exists an open subset $\Omega_0$ of $\Omega$ such that we have*

$$u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0,\mathbb{R}^N) \text{ for some } \alpha(n,N,p) \in (0,1] \qquad \text{and} \qquad \mathcal{L}^n(\Omega\setminus\Omega_0)=0\,.$$

*Here the exponent $\alpha$ in parts* (A) *and* (C) *can be made explicit and is essentially*[2] *the optimal Hölder exponent for the gradient of $\mathbb{R}^N$-valued $p$-harmonic functions in $n$ variables. In particular, in the case $p=2$, the statement in* (C) *holds true with $\alpha=1$ and in fact even with $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0,\mathbb{R}^N)$ replaced by $\mathrm{C}^\infty(\Omega_0,\mathbb{R}^N)$.*

The chronology of Theorem 1.1 is as follows. First, Anzellotti & Giaquinta [21] established part (C) for $p=2$ and, with the weaker conclusion that the singular set $\Omega\setminus\Omega_0$ is merely nowhere dense, even for all $p\in(1,\infty)$. Then, Bildhauer [29, 31] proved part (B) for specific BV minimizers $u$, and a subsequent result of [P1] yields (B) even for arbitrary BV minimizers. Finally, the full claim of (C) has been established in [P2], while the result in part (A) has been obtained in the recent preprint [P5]. Extensions of parts (A), (B), and (C) of Theorem 1.1 to

[2] More precisely, [P2] makes the following assertions about the Hölder exponent $\alpha$ in Theorem 1.1. Whenever the excess estimates leading to interior $\mathrm{C}^{1,\alpha_*}$ regularity of $\mathbb{R}^N$-valued $p$-harmonic functions in $n$ variables generally hold for some $\alpha_* \in (0,1]$, then the claims of Theorem 1.1 hold for every $\alpha \in (0,\alpha_*)$. The relevant excess estimates are known to hold for all $n,N\in\mathbb{N}$ and $p\in(1,\infty)$ with some positive $\alpha_*(n,N,p)$, but in most cases the optimal value of $\alpha_*$ is not known.

more general variational integrals are discussed in Sections 4.3, 4.4, and 5.1–5.3 of this treatise, respectively, and these sections also provide some information on the proofs.

At this stage, we intend to explain that the case distinction above can actually be understood by looking at the second derivatives $\nabla^2 \mathrm{m}_p$ of the integrand $\mathrm{m}_p$ given by

$$\mathrm{m}_p(z) := (1+|z|^p)^{\frac{1}{p}} \qquad \text{for } z \in \mathbb{R}^{N\times n}\,.$$

Indeed, also for the general functional $F$ in (1.1), the weak formulation (1.4) of the Euler equation, involves the first $z$-derivatives $\nabla_z f$ of the integrand $f$. Thus, whenever one linearizes or differentiates the Euler equation (which is an ingredient in essentially all[3] known methods for establishing gradient regularity), then the second derivatives $\nabla_z^2 f$ come into play. Specifically, for the integrands $\mathrm{m}_p$ with $p > 1$, explicit computations show that, for $z, \xi \in \mathbb{R}^{N\times n}$, we have

$$\begin{aligned} C_p^{-1}|z|^{p-2}|\xi|^2 &\le \nabla^2\mathrm{m}_p(z)(\xi,\xi) \le C_p|z|^{p-2}|\xi|^2 \qquad && \text{if } |z| \le 1\,,\\ C_p^{-1}|z|^{-1-p}|\xi|^2 &\le \nabla^2\mathrm{m}_p(z)(\xi,\xi) \le C_p|z|^{-1}|\xi|^2 \qquad && \text{if } |z| \ge 1 \end{aligned}$$

with a constant $C_p \in (0,\infty)$. Here, from the first line of inequalities one reads off that $\nabla^2\mathrm{m}_p(z)$ vanishes at $z = 0$ in the case $p > 2$, while it becomes singular at $z = 0$ in the case $p < 2$. Thus, for $p \neq 2$ one is confronted with a singular/degenerate behavior near points $x \in \Omega$ where $\nabla u(x)$ vanishes. This phenomenon occurs also, in essentially the same manner, in the theory of vector-valued $p$-harmonic functions; it has been well-understood by now, and the relevant methods carry over to the integrals $\overline{\mathrm{M}}_p$ at the cost of some technical effort, which is partially described in Sections 4.2, 4.3, 5.2. But then another — actually more serious and more interesting — degeneration phenomenon manifests in the second line of inequalities above: The dispersion ratio of $\nabla^2\mathrm{m}_p(z)$, that is the ratio between the largest and the smallest eigenvalue of the bilinear form $\nabla^2\mathrm{m}_p(z)$, may (and does) blow-up for $|z| \to \infty$ at the rate of $|z|^p$. It is the rate of this blow-up which is truly responsible for the distinction between parts (A), (B), and (C) of Theorem 1.1. In the case $p < 2$, the blow-up is subquadratic, it can be handled by known techniques developed for the treatment of non-standard growth conditions, and one obtains the $\mathrm{C}^{1,\alpha}$ regularity in (A). The limit case $p = 2$ of a quadratic blow-up is more difficult, but, to some extent, similar methods still apply and yield the weaker regularity gain in (B). For general $p > 1$, finally, it is an open problem whether one can obtain any form of *everywhere* gradient regularity, but the localization technique of Anzellotti & Giaquinta [21] allows to establish — independent of the speed of the blow-up — the partial regularity in (C).

Next we focus on the Dirichlet problem with boundary values given by a globally defined function $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$. The BV formulation of this problem corresponds to the minimization of the functional $\overline{\mathrm{M}}_p\big[\,\cdot\,;\overline{\Omega}\big]$ in the admissible class

$$\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big) := \{w \in \mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N) \,:\, w = u_0 \text{ holds } \mathcal{L}^n\text{-a.e. on } \mathbb{R}^n \setminus \Omega\}\,.$$

We emphasize that this BV Dirichlet problem incorporates the boundary values $u_0$ in a generalized sense. In fact, the admissible functions $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ may jump across $\partial\Omega$ and need

[3]The only exception from this rule is the Gehring improvement which yields, for instance for variational problems with standard superlinear growth conditions, a small gain of gradient integrability. To the author's knowledge, it has remained an open problem whether there is any analogue of this improvement for the linear growth problems considered here.

not coincide with $u_0$ on $\partial\Omega$ in the sense of (interior) trace, but rather the extension of $w$ by the values of $u_0$ outside of $\Omega$ influences the measure $\mathrm{D}w$ on $\partial\Omega$ and consequently the functional $\overline{\mathrm{M}}_p\big[w;\overline{\Omega}\big]$. This amounts to a very natural penalization of functions with wrong boundary values, and this procedure is basically inevitable in the existence theory for BV minimizers. We postpone a more detailed discussion to Sections 2.3 and 2.4, but record already that parts (A) and (B) of Theorem 1.1 imply, in a very simple way, the following uniqueness statement.

**Corollary 1.2** (uniqueness modulo constants for BV minimizers of $\overline{\mathrm{M}}_p$, $1 < p \le 2$)**.** *Consider a connected open set $\Omega$ in $\mathbb{R}^n$ with $\mathcal{L}^n\big(\overline{\Omega}\big) < \infty$, an exponent $p \in (1,2]$, and a function $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$. If $u,v \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ are such that*

$$\overline{\mathrm{M}}_p\big[u;\overline{\Omega}\big] \le \overline{\mathrm{M}}_p\big[w;\overline{\Omega}\big] \qquad \text{and} \qquad \overline{\mathrm{M}}_p\big[v;\overline{\Omega}\big] \le \overline{\mathrm{M}}_p\big[w;\overline{\Omega}\big]$$

*hold for all $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$, then, for some constant $y \in \mathbb{R}^N$, we have*

$$u = v + y \qquad \mathcal{L}^n\text{-a.e. on } \Omega\,.$$

Extensions of Corollary 1.2 to more general variational integrals are later provided in Corollaries 4.5 and 4.7.

*Proof of Corollary* 1.2. If we assume that $\nabla u$ differs from $\nabla v$ on a subset of $\overline{\Omega}$ with positive $\mathcal{L}^n$-measure, then, relying on the strict convexity of the mapping $z \mapsto (1+|z|^p)^{\frac{1}{p}}$, we can compute

$$\begin{aligned}
\overline{\mathrm{M}}_p\big[\tfrac{1}{2}(u{+}v);\overline{\Omega}\big] &= \int_{\overline{\Omega}} \big(1+\big|\tfrac{1}{2}(\nabla u{+}\nabla v)\big|^p\big)^{\frac{1}{p}}\,\mathrm{d}x + \big|\tfrac{1}{2}(\mathrm{D}^{\mathrm{s}}u{+}\mathrm{D}^{\mathrm{s}}v)\big|\big(\overline{\Omega}\big)\\
&< \frac{1}{2}\bigg(\int_{\overline{\Omega}}(1+|\nabla u|^p)^{\frac{1}{p}}\,\mathrm{d}x + |\mathrm{D}^{\mathrm{s}}u|\big(\overline{\Omega}\big) + \int_{\overline{\Omega}}(1+|\nabla v|^p)^{\frac{1}{p}}\,\mathrm{d}x + |\mathrm{D}^{\mathrm{s}}v|\big(\overline{\Omega}\big)\bigg)\\
&= \frac{1}{2}\big(\overline{\mathrm{M}}_p\big[u;\overline{\Omega}\big] + \overline{\mathrm{M}}_p\big[v;\overline{\Omega}\big]\big) = \inf_{\mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)} \overline{\mathrm{M}}_p\big[\,\cdot\,;\overline{\Omega}\big]\,.
\end{aligned}$$

In view of $\frac{1}{2}(u{+}v) \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ this yields a contradiction, and we have shown that

$$\nabla u = \nabla v \qquad \text{holds } \mathcal{L}^n\text{-a.e. on } \overline{\Omega}\,.$$

Now we involve Theorem 1.1. Since $u$ and $v$ are local minimizers of $\overline{\mathrm{M}}_p$ with $p \in (1,2]$ on $\Omega$, the theorem guarantees, in particular, $\mathrm{D}^{\mathrm{s}}u = \mathrm{D}^{\mathrm{s}}v \equiv 0$ on $\Omega$. Consequently, the derivatives of $u$ and $v$ coincide on $\Omega$, and the constancy theorem yields the claim. □

We point out that the de facto proof of the uniqueness statement in the subsequent sections proceeds by the above argument only in the case $p = 2$. In the case $1 < p < 2$, instead, we proceed in a different manner: We first establish the regularity in part (A) of Theorem 1.1 only for specific, but not yet for all BV minimizers, then we exploit a duality trick to deduce the uniqueness statement of Corollary 1.2, and only eventually we obtain the regularity claim for all BV minimizers as stated in Theorem 1.1. Thus, it is not only true that everywhere regularity implies uniqueness, but also uniqueness can serve as a tool in obtaining regularity, and, all in all, in the BV framework these two topics are more bidirectionally connected than the above argument may suggest.

We also stress that the uniqueness modulo constants which is asserted in Corollary 1.2 cannot be improved to full uniqueness of BV minimizers. This effect is due to the generalized understanding of the boundary values, which enter through the non-strictly convex term $|\mathrm{D}^{\mathrm{s}}w|\big(\overline{\Omega}\big)$, and a concrete occurrence of non-uniqueness is demonstrated, already in the two-dimensional scalar case $n = 2$, $N = 1$, by a classical example of Santi [115, Section 2] for the non-parametric area functional $\overline{\mathrm{M}}_2$. Indeed, Santi's example works on a specific, symmetric bounded Lipschitz domain $\Omega$ in $\mathbb{R}^2$ with boundary values equal to a large constant $M$ on one half of the boundary and equal to $-M$ on the other half; see Figure 1. It is then shown that a specific minimizer $\overline{u} \in \mathrm{C}^\infty(\Omega)$ of $\overline{\mathrm{M}}_2\big[\,\cdot\,,\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega}\big)$ is bounded on $\Omega$ independently of $M$. For sufficiently large $M$, it thus follows that the (interior) trace of $\overline{u}$ is at distance at least $M/2$ from $u_0$ on $\partial\Omega$, and also $\overline{u}+\mathbb{1}_\Omega y$ with $y \in [-M/2, M/2]$ minimizes $\overline{\mathrm{M}}_2\big[\,\cdot\,,\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega}\big)$.

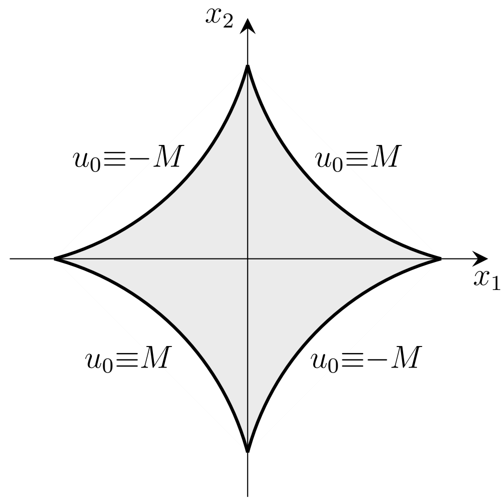


**Figure 1:** Santi's domain

Santi's example actually builds on the comparison with explicit solutions known from a previous example of Finn [62]. Finn's example also concerns the area functional $\overline{\mathrm{M}}_2$ in the case $n = 2$, $N = 1$, but now on the annulus $\mathrm{B}_2 \setminus \overline{\mathrm{B}_1} \subset \mathbb{R}^2$ with boundary values equal to $M$ on $\partial\mathrm{B}_1$ and zero boundary values on $\partial\mathrm{B}_2$; see Figure 2. In this case, the minimizer $u$ smoothly attains the boundary values on the boundary portion $\partial\mathrm{B}_2$ and is thus unique, and indeed, by exploiting the rotational symmetry of the problem, $u$ can be made explicit by solving an ODE. The graph of $u$ is a part of a catenoid, and $u$ is found to be $M$-independently bounded. Thus, this more basic example already shows that the minimizer $u$ deviates, for large $M$, from the given boundary values on the boundary portion $\partial\mathrm{B}_1$. In fact, Finn's example can be further understood by observing that the *parametric* Plateau problem with the given boundary values is always solved by a part of a catenoid, but for large $M$ the parametric solution does not stay in the cylinder over the non-convex domain $\mathrm{B}_2 \setminus \overline{\mathrm{B}_1}$ and is not admissible for the *non-parametric* problem considered here. Instead, the graph of the minimizer $u$ is a part of another catenoid whose bottleneck of diameter 2 lies far below the boundary curve $\partial\mathrm{B}_1{\times}\{M\}$; compare [91, 108, 82] for further related examples.

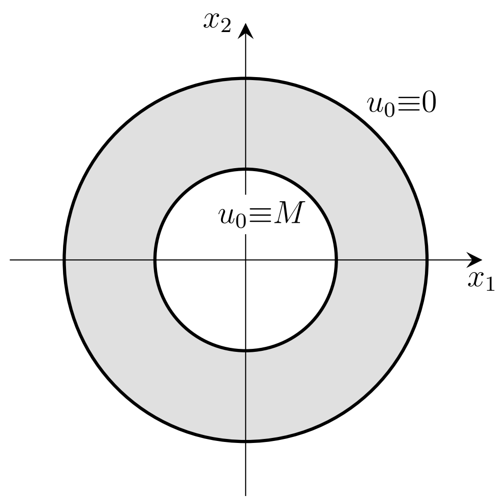


**Figure 2:** Finn's domain

We also mention a noteworthy, complementary criterion of Miranda [101, 102] for the non-occurrence of deviations on the boundary (in case of arbitrary $n \in \mathbb{N}$ and for $N = 1$): If $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$ such that $\partial\Omega$ has non-negative generalized mean curvature near a point $x_0 \in \partial\Omega$ and if $u_0$ (restricted to $\mathbb{R}^n \setminus \Omega$) is continuous at $x_0$, then every minimizer of $\overline{\mathrm{M}}_2\big[\,\cdot\,,\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega}\big)$ continuously attains the boundary value $u_0(x_0)$ at $x_0$. In particular, whenever this applies for a single point $x_0 \in \partial\Omega$, the BV minimizer is unique. Miranda's criterion is in accordance with the solvability results for the classical Dirichlet problem for the minimal surface equation [83], and comparison with the examples shows that the criterion is indeed quite sharp. In Finn's example, the mean curvature of $\partial\Omega$ is negative at the inner boundary $\partial\mathrm{B}_1$, in Santi's example, the mean curvature of $\partial\Omega$ is negative except for four corners, at which continuity fails; and, finally, an example of Baldo & Modica [23] shows that non-uniqueness may even happen if $\Omega$ is a two-dimensional ball (so that $\partial\Omega$ has positive curvature, of course), but the boundary values are everywhere discontinuous on $\partial\Omega$. Consequently, one may not hope to extend Miranda's criterion to points $x_0 \in \partial\Omega$ where either $u_0$ is discontinuous or $\partial\Omega$ has negative mean-curvature, but still some interesting results on the boundary regularity and the regularity of the trace of minimizers of $\overline{\mathrm{M}}_2\big[\,\cdot\,,\overline{\Omega}\big]$ in such points have been obtained; see [122, 132, 90, 133, 92], for instance.

As already said above, the described examples indicate that the conclusion of Corollary 1.2 cannot be strengthened, without making further assumptions, to full uniqueness of BV minimizers. However, returning to the case of arbitrary codimension $N \in \mathbb{N}$, the corollary leaves open the possibility that, for fixed $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$, the minimizers may differ by an *$N$-parameter family* of constants $y \in \mathbb{R}^N$, and this aspect can still be improved. Indeed, an observation of [P1] guarantees that, for every $p \in (1,2]$ and $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$, the functional $\overline{\mathrm{M}}_p\big[\,\cdot\,,\overline{\Omega}\big]$ possesses at most a *1-parameter family* of minimizers in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$. This improvement follows from Corollary 1.2 by an elementary convexity argument, and we refer to Sections 4.5 and 4.6 for a further discussion — and also for some related vector-valued refinements, devised in [P1], of the examples of Finn and Santi. In addition, we believe that it would be very interesting to obtain analogues of Miranda's criterion (or other attainment results for the boundary values) also for general integrands $f$ and in codimension $N \geq 2$. However, so far, this has remained an open issue.

In the subsequent chapters, we mostly discuss various extensions of the above-described existence, uniqueness, and regularity results to general variational integrals of the type (1.1)–(1.2). Additionally, we are also concerned with a corresponding duality theory for BV minimizers, which serves as a tool in some regularity and uniqueness proofs, but may also be of independent interest. We usually strive to state the results under sharp assumptions, and particularly we will be able to include the case of non-smooth, unbounded $\Omega$ and $f$ in some of our considerations. Despite these attempts to reach a large generality, however, we emphasize that our motivation stems mostly from the case of the model integrals in (1.5) and that the model statements of Theorem 1.1 and Corollary 1.2 may be seen as the most relevant outcome of our work.

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# Some terminology and conventions regarding measures

Most notation used in this treatise is either quite standard or explained at the first occurrence. However, since we extensively work with various types of measures, we explicitly set out some related conventions.

We only consider measures on locally compact and separable metric spaces $X$, in fact only on locally compact subsets $X$ of $\mathbb{R}^n$. Non-negative Borel (or Radon) measures $\mu$ are then understood in the usual way as (locally finite) $[0,\infty]$-valued, $\sigma$-additive[4] set functions on the Borel $\sigma$-algebra of $X$; they are occasionally completed by extension to the larger $\sigma$-algebra of $\mu$-measurable sets.

For signed measures, the need to allow merely (one-sided) locally finite ones requires somewhat peculiar conventions in the spirit of [9, Definition 1.40]. Indeed, by a signed Borel measure $\nu$ on $X$, we mean a $(-\infty,\infty]$-valued, $\sigma$-additive set function, which is defined, at least, on all *relatively compact* Borel subsets of $X$. Such a $\nu$ has unique Jordan decomposition $\nu=\nu_+-\nu_-$ into a non-negative Borel measure $\nu_+$ on $X$ and a non-negative Radon measure $\nu_-$ on $X$ such that $\nu_+$ and $\nu_-$ are mutually singular (where, in particular, $\nu_+$ and $\nu_-$ can be defined on the whole Borel $\sigma$-algebra thanks to the fact that $X$ is $\sigma$-compact). Moreover, the variation measure $|\nu|=\nu_++\nu_-$ makes sense as a non-negative Borel measure on $X$, and $\nu$ can be extended in unique way as a $(-\infty,\infty]$-valued, $\sigma$-additive set function to a maximal system of Borel subsets of $X$, namely to the system of $\nu_-$-finite Borel subsets of $X$. We call $\nu$ a Radon measure on $X$ if $|\nu|$ (or, equivalently, $\nu_+$) is a Radon measure on $X$. Moreover, we say that $\nu$ is finite on $X$ if $|\nu|$ is finite on $X$, and we record that, in this case, $\nu$ can be extended to the whole Borel $\sigma$-algebra of $X$.

Similarly, every $\mathbb{R}^m$-valued, $\sigma$-additive set function $\nu$, which is defined, at least, on all relatively compact Borel subsets of $X$, is termed an $\mathbb{R}^m$-valued Radon measure on $X$. This makes perfect sense, since, in this case, the components of $\nu$ are always signed Radon measures on $X$ and the variation measure $|\nu|$ is a non-negative Radon measure on $X$. The space of $\mathbb{R}^m$-valued Radon measures on $X$ is called $\mathrm{RM}_{\mathrm{loc}}(X,\mathbb{R}^m)$, and we observe that every $\nu\in\mathrm{RM}_{\mathrm{loc}}(X,\mathbb{R}^m)$ can be extended in a unique way to the system of $|\nu|$-finite Borel subsets of $X$. We call $\nu$ finite if $|\nu|$ is finite (or, equivalently, whenever $\nu$ can be extended to the whole Borel $\sigma$-algebra of $X$), and write $\mathrm{RM}(X,\mathbb{R}^m)$ for the space of finite $\mathbb{R}^m$-valued Radon measures on $X$. In the case $m=1$, this terminology is consistent with the one for signed measures, and we omit the target space in all the abbreviations.

By local weak-$*$ convergence in either $\mathrm{RM}_{\mathrm{loc}}(X,\mathbb{R}^m)$ or $\mathrm{RM}(X,\mathbb{R}^m)$, we mean the weak convergence of $\mathbb{R}^m$-valued measures which arises by duality with $\mathrm{C}^0_{\mathrm{cpt}}(X,\mathbb{R}^m)$ test functions. If, instead, we admit even test functions in the completion of $\mathrm{C}^0_{\mathrm{cpt}}(X,\mathbb{R}^m)$ with respect to the sup-norm, we refer to the corresponding concept as (global) weak-$*$ convergence in $\mathrm{RM}(X,\mathbb{R}^m)$.

Finally, we extend all conventions to $\mathbb{R}^{N\times n}$-valued measures, simply by identifying $\mathbb{R}^{N\times n}$ with $\mathbb{R}^{Nn}$.

[4]We adopt the convention that $\sigma$-additivity includes the requirement that the empty set has measure zero.

# Chapter 2

# Generalized Dirichlet problem and existence of BV minimizers

This chapter is concerned with functionals of measures and the corresponding existence theory for BV minimizers, and the main outcome are the BV framework of Section 2.3 and the existence and consistency theorems of Section 2.4. The statements are based on the by-now-classical results of [75, 112, 71] and a recent refinement in [88], and the exposition at hand is loosely related to parts of [P1, P2, P3].

## 2.1 Functionals of measures

In this section we explain how a measure can be plugged into a non-linear function(al). As a motivation, we start by discussing the basic case that the function(al) is positively 1-homogeneous. Then we pass to the general case of non-homogeneous function(al)s, first studied by Goffman & Serrin [75].

Indeed, we first consider an open subset $\Omega$ of $\mathbb{R}^n$ and a lower semicontinuous function $h\colon \Omega \times \mathbb{R}^m \to (-\infty, \infty]$, which is positively 1-homogeneous in the second variable in the sense of $h(x,0) = 0$ and $h(x,tz) = th(x,z)$ for all $t \in (0,\infty)$ and $(x,z) \in \Omega \times \mathbb{R}^m$. Assuming that $h$ satisfies

$$h(x,z) \geq -\Gamma|z| \qquad \text{for all } (x,z) \in \Omega \times \mathbb{R}^m \tag{2.1}$$

with a constant $\Gamma < \infty$ and that $\nu \in \mathrm{RM}_{\mathrm{loc}}(X,\mathbb{R}^m)$ is a Radon measure on a locally compact subset $X$ of $\Omega$, there is a very plausible way to define a signed Borel measure $f(\,\cdot\,,\nu)$ on $X$: Choosing any non-negative Radon measure $\mu$ on $X$ such that $|\nu| \ll \mu$ (i.e. $|\nu|$ is absolutely continuous with respect to $\mu$), it makes sense to set

$$\int_A h(\,\cdot\,,\nu) := \int_A h\bigg(\,\cdot\,,\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\bigg)\mathrm{d}\mu \qquad \text{for every } |\nu|\text{-finite Borel set } A \subset X\,, \tag{2.2}$$

where $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ stands for the Radon-Nikodým density of $\nu$ with respect to $\mu$. Relying on the 1-homogeneity of $h$, it can be checked easily that this definition does not depend on the choice of $\mu$, and in fact one can always take $\mu = |\nu|$.

Now we turn to the case that $\Omega$ is open in $\mathbb{R}^n$ and that $f\colon \Omega \times \mathbb{R}^m \to (-\infty,\infty]$ is a (possibly non-homogeneous) Borel function. Then, following the approach in [71], we introduce the homogenized function

$$\bar{f}\colon \big(\Omega \times (0,\infty) \times \mathbb{R}^m\big) \cup \big(\overline{\Omega} \times \{0\} \times \mathbb{R}^m\big) \to [-\infty,\infty]$$

by setting first

$$\bar{f}(x,t,z) := tf(x,z/t) \qquad \text{for } (x,t,z) \in \Omega \times (0,\infty) \times \mathbb{R}^m,$$

and then extending the definition to the case $t=0$ via $\bar{f}(x,0,0) := 0$ for $x \in \overline{\Omega}$ and

$$\bar{f}(x,0,z) := \liminf_{\substack{(\tilde{x},\tilde{z})\to(x,z)\\ t\searrow 0}} \bar{f}(\tilde{x},t,\tilde{z}) \qquad \text{for } (x,z) \in \overline{\Omega} \times (\mathbb{R}^m \setminus \{0\})\,. \tag{2.3}$$

The recession function

$$f^\infty\colon \overline{\Omega} \times \mathbb{R}^m \to [-\infty,\infty]$$

of $f$ is simply given by

$$f^\infty(x,z) := \bar{f}(x,0,z) \qquad \text{for } (x,z) \in \overline{\Omega} \times \mathbb{R}^m,$$

and with this terminology both $f$ and $f^\infty$ are restrictions of $\bar{f}$, namely

$$\bar{f}(\,\cdot\,,1,\,\cdot\,) = f \qquad \text{and} \qquad \bar{f}(\,\cdot\,,0,\,\cdot\,) = f^\infty\,. \tag{2.4}$$

Moreover, $\bar{f}$ is positively 1-homogeneous in $(t,z)$, and $f^\infty$ is positively 1-homogeneous in $z$.

Now we assume that $f$ satisfies

$$f(x,z) \geq -\Psi(x) - \Gamma|z| \qquad \text{for all } (x,z) \in \Omega \times \mathbb{R}^m \tag{2.5}$$

with some $\Psi\colon \Omega \to [0,\infty)$ and a constant $\Gamma < \infty$. Then we have $\bar{f}(x,t,z) \geq -t\Psi(x) - \Gamma|z|$ whenever $t>0$. However, in order to guarantee that the value $-\infty$ does not occur and this estimate carries over to the case $t=0$, or in other words to $f^\infty$, we need to assume slightly more than just (2.5). Indeed, if we know, in addition, either that $\Psi$ is bounded on bounded subsets of $\Omega$ or that the $\liminf$ in (2.3) is always a $\lim$, then we have

$$f^\infty(x,z) \geq -\Gamma|z| \qquad \text{for all } (x,z) \in \overline{\Omega} \times \mathbb{R}^m, \tag{2.6}$$

and $\bar{f}$ is lower semicontinuous at every point of $\overline{\Omega} \times \{0\} \times \mathbb{R}^m$ so that, in particular, $f^\infty$ is lower semicontinuous on $\overline{\Omega} \times \mathbb{R}^m$. Finally, we record that, if $f$ is convex in $z$, then $\bar{f}$ is convex in $(t,z)$ and $f^\infty$ is convex in $z$.

Relying on the preceding specifications and observations, we now apply the basic idea in (2.2) to the homogenized integrand $\bar{f}$ as follows.

**Definition 2.1** (functionals of measures)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ and a Borel function $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that* (2.5) *and* (2.6) *are satisfied. If $\nu \in \mathrm{RM}_{\mathrm{loc}}(X,\mathbb{R}^m)$ is a Radon measure on a locally compact subset $X$ of $\overline{\Omega}$ with $\mathcal{L}^n(X \cap \partial\Omega) = 0$ and $\Psi \in \mathrm{L}^1_{\mathrm{loc}}(X)$,* *and if $\mu$ is a non-negative Radon measure on $X$ such that $\mathcal{L}^n \llcorner X + |\nu| \ll \mu$, then we introduce a signed Borel measure $f(\,\cdot\,,\nu)$ on $X$ by setting*

$$\int_A f(\,\cdot\,,\nu) := \int_A \bar{f}\bigg(\,\cdot\,,\frac{\mathrm{d}(\mathcal{L}^n \llcorner X)}{\mathrm{d}\mu},\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\bigg)\mathrm{d}\mu \qquad \textit{for every } (\Psi\mathcal{L}^n{+}|\nu|)\textit{-finite Borel set } A \subset X$$

(*where the integrand on the right-hand side is well-defined, as the assumption $\mathcal{L}^n(X \cap \partial\Omega) = 0$ implies that $\frac{\mathrm{d}(\mathcal{L}^n \llcorner X)}{\mathrm{d}\mu}$ vanishes $\mu$-a.e. on $X \cap \partial\Omega$*).

We observe that if $f$ is lower semicontinuous and positively 1-homogeneous in $z$, then we have $f(\,\cdot\,,t,\,\cdot\,) = f$ on $\Omega \times \mathbb{R}^m$ for all $t \geq 0$. Thus, in this specific case, the measure introduced in Definition 2.1 coincides on $\Omega$ with the one given by (2.2), and in particular we have the compatibility $\int_A |\nu| = |\nu|(A)$ with the notation $|\nu|$ for the variation measure. Furthermore, back in the general case, it follows from the 1-homogeneity of $\bar{f}$ that the definition does not depend on the choice of $\mu$, and, in fact, we can always take $\mu = \mathcal{L}^n \llcorner X + |\nu^{\mathrm{s}}|$, where $\nu = \nu^{\mathrm{a}} + \nu^{\mathrm{s}}$ denotes the Lebesgue decomposition of $\nu$ with respect to $\mathcal{L}^n \llcorner X$. Exploiting this particular choice of $\mu$ and (2.4), it turns out that $f(\,\cdot\,,\nu)$ is also characterized by the well-known formula

$$\int_A f(\,\cdot\,,\nu) = \int_{A\cap\Omega} f\bigg(\,\cdot\,,\frac{\mathrm{d}\nu^{\mathrm{a}}}{\mathrm{d}\mathcal{L}^n}\bigg)\mathrm{d}x + \int_A f^\infty\bigg(\,\cdot\,,\frac{\mathrm{d}\nu^{\mathrm{s}}}{\mathrm{d}|\nu^{\mathrm{s}}|}\bigg)\mathrm{d}|\nu^{\mathrm{s}}| \tag{2.7}$$

for all Borel sets $A \subset X$ with $\|\Psi\|_{\mathrm{L}^1(A)}{+}|\nu|(A) < \infty$.

We stress that we primarily utilize the concept of Definition 2.1 in two specific cases. On one hand, mainly in connection with local regularity results, we often assume $\Psi \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and consider Radon measures $\nu \in \mathrm{RM}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ on the open set $X = \Omega$. Then, if also a complementary upper bound for $f$ is in force, i.e. if we require $|f(x,z)| \leq \Psi(x) + \Gamma|z|$ instead of merely (2.5), also $f(\,\cdot\,,\nu) \in \mathrm{RM}_{\mathrm{loc}}(\Omega)$ is a Radon measure. On the other hand, in connection with the BV Dirichlet problem, we usually assume $\mathcal{L}^n(\partial\Omega) = 0$ and $\Psi \in \mathrm{L}^1(\Omega)$ and consider *finite* Radon measures $\nu \in \mathrm{RM}\big(\overline{\Omega},\mathbb{R}^m\big)$ on the *closure* $X = \overline{\Omega}$ of the open set $\Omega$. If, in this global case, $|f(x,z)| \leq \Psi(x) + \Gamma|z|$ is assumed, then also $f(\,\cdot\,,\nu) \in \mathrm{RM}\big(\overline{\Omega}\big)$ is a *finite* Radon measure.

To close this section, we provide a side remark on the specific case of the measures $|\mathrm{D}w|$ and $\sqrt{1+|\mathrm{D}w|^2}$ with $w \in \mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$. Clearly, these measures, which essentially correspond to the functionals $\mathrm{M}_1$ and $\mathrm{M}_2$ from (1.5) in the introduction, are defined in the sense of Definition 2.1, but they can also be characterized in a simpler fashion via a well-known partial integration trick. Indeed, it can be shown that, for every open subset $U$ of $\mathbb{R}^n$, one has

$$\int_U |\mathrm{D}w| = \sup\bigg\{-\int_U w\cdot\operatorname{div}\tau\,\mathrm{d}x \,:\, \tau \in \mathrm{C}^\infty_{\mathrm{cpt}}(U,\mathbb{R}^{N\times n}),\, \sup_U|\tau| \leq 1\bigg\}\,,$$

$$\int_U \sqrt{1+|\mathrm{D}w|^2} = \sup\bigg\{\int_U \big(\tau_0 - w\cdot\operatorname{div}\tau\big)\,\mathrm{d}x \,:\, (\tau_0,\tau) \in \mathrm{C}^\infty_{\mathrm{cpt}}(U,\mathbb{R}\times\mathbb{R}^{N\times n}),\, \sup_U|(\tau_0,\tau)| \leq 1\bigg\},$$

where the $\mathbb{R}^N$-valued divergence of the $\mathbb{R}^{N\times n}$-matrix field $\tau$ is computed row-wise. However, it seems that these formulas, which could serve as alternative definitions, carry over only to functionals with a very similar structure to the above two and not to more general ones. Thus, we do not further pursue this issue.

## 2.2 Reshetnyak (semi)continuity

Next we discuss the classical semicontinuity and continuity theorems of Reshetnyak [112] and a recent refinement, due to Kristensen & Rindler [88], of the continuity result.

We start by restating Reshetnyak's semicontinuity theorem for 1-homogeneous, convex functionals of measures, in a version which is close to the original one of [112], and which is proved in [9, Theorem 2.38] with the help of a slicing result for measures; compare also [123] for an alternative proof.

**Theorem 2.2** (Reshetnyak semicontinuity, homogeneous version)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ and a lower semicontinuous function $h\colon \Omega \times \mathbb{R}^m \to [0,\infty]$ which is positively 1-homogeneous and convex in the second argument. If $\nu_k$ locally weakly-$*$ converges to $\nu$ in $\mathrm{RM}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$, then we have*

$$\liminf_{k\to\infty} \int_\Omega h(\,\cdot\,,\nu_k) \ge \int_\Omega h(\,\cdot\,,\nu)\,.$$

After the preparations of Section 2.1, it is straightforward to generalize Theorem 2.2 to (possibly) non-homogeneous functionals. Indeed, applying Theorem 2.2 with (extensions of) $\bar{f}$ in place of $h$ and $(\mathcal{L}^n,\nu_k) \in \mathrm{RM}_{\mathrm{loc}}\big(\overline{\Omega},\mathbb{R}^{m+1}\big)$ in place of $\nu_k$, one obtains the following statement; see [P3, Appendix B] for details on the deduction.

**Corollary 2.3** (Reshetnyak semicontinuity, non-homogeneous version)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ with $\mathcal{L}^n(\partial\Omega)=0$ and a lower semicontinuous function $f : \Omega\times\mathbb{R}^m \to [0,\infty]$, which is convex in the second argument. If $\nu_k$ locally weakly-$*$ converges to $\nu$ in $\mathrm{RM}_{\mathrm{loc}}\big(\overline{\Omega},\mathbb{R}^m\big)$, then we have*

$$\liminf_{k\to\infty} \int_{\overline{\Omega}} f(\,\cdot\,,\nu_k) \ge \int_{\overline{\Omega}} f(\,\cdot\,,\nu)\,.$$

In contrast to the semicontinuity results, the following continuity results do not necessitate any convexity hypothesis. In exchange, however, they rely on the following stronger notion of convergence.

**Definition 2.4** (strict convergence in RM)**.** *Consider a locally compact subset $X$ of $\mathbb{R}^n$. For a non-negative function $\Psi \in \mathrm{L}^1(X)$, we say that $\nu_k$ converges $\Psi$-strictly in $\mathrm{RM}(X,\mathbb{R}^m)$ to $\nu$ if $\nu_k$ weakly-$*$ converges to $\nu$ in $\mathrm{RM}(X,\mathbb{R}^m)$ and if there holds*

$$\lim_{k\to\infty} \int_X \sqrt{\Psi^2+|\nu_k|^2} = \int_X \sqrt{\Psi^2+|\nu|^2}\,.$$

We start also the discussion of the continuity results with a version for 1-homogeneous functionals, which is slightly less general than the original statement of [112]. We refer to [9, Theorem 2.39], once more, for a proof via slicing.

**Theorem 2.5** (Reshetnyak continuity, homogeneous version)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ and a continuous function $h\colon \Omega\times\mathbb{R}^m \to \mathbb{R}$, which is positively 1-homogeneous in the second argument and satisfies*

$$|h(x,z)| \le \Gamma|z| \qquad \text{for all } (x,z)\in\Omega\times\mathbb{R}^m\,,$$

*with a constant* $\Gamma < \infty$*. If* $\nu_k$ *converges* 0*-strictly in* $\mathrm{RM}(\Omega, \mathbb{R}^m)$ *to* $\nu$*, then we have*

$$\lim_{k\to\infty} \int_\Omega h(\,\cdot\,, \nu_k) = \int_\Omega h(\,\cdot\,, \nu)\,.$$

We also refer to [52] for a refined discussion of both the homogeneous semicontinuity and the homogeneous continuity theorem and the relation between them.

In order to pass on to the non-homogeneous case, we need an assumption which guarantees that $\bar{f}$ is continuous in all points of $\overline{\Omega} \times \{0\} \times \mathbb{R}^m$. This is indeed achieved by requiring that the $\liminf$ in (2.3) is always a $\lim$, i.e. that[1]

$$\lim_{\substack{(\tilde{x},\tilde{z})\to(x,z)\\ t\searrow 0}} tf(\tilde{x}, \tilde{z}/t) \text{ exists in } \mathbb{R} \text{ for all } (x, z) \in \overline{\Omega} \times (\mathbb{R}^m \setminus \{0\})\,. \tag{2.8}$$

If we also impose the linear growth assumption

$$|f(x, z)| \le \Psi(x) + \Gamma|z| \qquad \text{for all } (x, z) \in \Omega \times \mathbb{R}^m \tag{2.9}$$

(with an $\mathbb{R}$-valued $\Psi$ and a constant $\Gamma < \infty$), then it follows from (2.8) that $f^\infty = \bar{f}(\,\cdot\,, 0, \,\cdot\,)$ is continuous and that we get the corresponding bound for $f^\infty$, that is $|f^\infty(x, z)| \le \Gamma|z|$ for all $(x, z) \in \overline{\Omega} \times \mathbb{R}^m$. Moreover, we emphasize that (2.8) is usually not difficult to check in concrete cases. In particular, if $f$ is convex in $z$, it suffices to verify the existence of the limit for fixed $\widetilde{z} = z$, and if, additionally, (2.9) with finite-valued $\Psi$ is satisfied, (2.8) is equivalent with continuity of $f$ in $x$ for $|z| \to \infty$ in the following sense. For all $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x, \tilde{x} \in \Omega$ and $z \in \mathbb{R}^m$, we have

$$|x - x_0| + |\tilde{x} - x_0| + |z|^{-1} < \delta \implies |f(\tilde{x}, z) - f(x, z)| < \varepsilon|z|\,.$$

In particular, one can read off *that* (2.8) *is generally valid for* $x$*-independent, convex integrands* $f$ *of linear growth.*

At this stage, the next statement follows readily, by applying Theorem 2.5 to the homogenized integrand $\bar{f}$ and the measures $(\mathcal{L}^n, \nu_k) \in \mathrm{RM}(\Omega, \mathbb{R}^{m+1})$.

**Corollary 2.6** (Reshetnyak continuity, first non-homogeneous version)**.** *Consider an open subset* $\Omega$ *of* $\mathbb{R}^n$ *with* $\mathcal{L}^n(\Omega) < \infty$ *and a continuous function* $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$*. Suppose that we have*

$$|f(x, z)| \le \Gamma + \Gamma|z| \qquad \textit{for all } (x, z) \in \Omega \times \mathbb{R}^m, \tag{2.10}$$

*with a constant* $\Gamma < \infty$*, and that* $f$ *satisfies* (2.8)*. If* $\nu_k$ *converges* 1*-strictly in* $\mathrm{RM}(\Omega, \mathbb{R}^m)$ *to* $\nu$*, then we have*

$$\lim_{k\to\infty} \int_\Omega f(\,\cdot\,, \nu_k) = \int_\Omega f(\,\cdot\,, \nu)\,.$$

[1]The existence of $\lim_{\substack{\tilde{x}\to x\\ t\searrow 0}} tf(\tilde{x}, 0) \in \mathbb{R}$ implies that $f(\,\cdot\,, 0)$ remains bounded near $x$. Therefore, in order to include unbounded integrands $f$ in the sequel, we postulate (2.8) only for $z \neq 0$.

However, Corollary 2.6 still has some unsatisfactory features, and in particular the continuity assumption for $f$ in the joint variable $(x, z)$ appears restrictive. It has been shown only recently, by Kristensen & Rindler [88], that this assumption can be weakened and that it suffices to require merely measurability of $f$ in $x$ and continuity in $z$. This weaker hypothesis is known as the Carathéodory property and seems indeed more natural at this point, as it also occurs in connection with (semi)continuity results in $\mathrm{L}^p$ spaces. Moreover, the dropping of the joint-continuity assumption also allows to work, in an elegant way, with measures on $\overline{\Omega}$ instead of $\Omega$. However, as explained above, also (2.8) comprises *some* continuity of $f$ in $x$, and we stress that *this* continuity requirement is inevitable (already to ensure continuity along a strictly convergent sequence of Dirac measures).

Next we restate the described result of [88, Proposition 2] in the slightly more general form of [P3, Theorem 3.10], which also replaces the assumptions (2.10) and $\mathcal{L}^n(\Omega) < \infty$ by (2.9) with $\Psi \in \mathrm{L}^1(\Omega)$ and thus improves on another small technical drawback of Corollary 2.6.

**Theorem 2.7** (Reshetnyak continuity, refined non-homogeneous version)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ with $\mathcal{L}^n(\partial\Omega) = 0$ and a Carathéodory function $f \colon \Omega \times \mathbb{R}^m \to \mathbb{R}$, which satisfies* (2.9)*, for some positive $\Psi \in \mathrm{L}^1(\Omega)$ which is bounded away from $0$ on every bounded subset of $\Omega$. Moreover, suppose that* (2.8) *holds. If $\nu_k$ converges $\Psi$-strictly in* $\mathrm{RM}\left(\overline{\Omega}, \mathbb{R}^m\right)$ *to $\nu$, then we have*

$$\lim_{k\to\infty} \int_\Omega f(\,\cdot\,, \nu_k) = \int_\Omega f(\,\cdot\,, \nu)\,.$$

To close this section, we remark that the preceding statements, together with standard compactness and approximation results, also identify functionals of measures as natural continuations of functionals on $\mathrm{L}^1$. More precisely — under suitable assumptions, for instance the ones of both Corollary 2.3 and Theorem 2.7 — the functional $\nu \mapsto \int_{\overline{\Omega}} f(\,\cdot\,, \nu)$ on $\mathrm{RM}\left(\overline{\Omega}, \mathbb{R}^m\right)$ is the Lebesgue-Serrin extension (that is, in fact, the sequentially weakly-$*$ lower semicontinuous envelope) of the functional $\varphi \mapsto \int_\Omega f(\,\cdot\,, \varphi)\,\mathrm{d}x$ on $\mathrm{L}^1(\Omega, \mathbb{R}^m)$.

## 2.3 Generalized Dirichlet problem

In this section, we explain the passage from a $\mathrm{W}^{1,1}$-formulation to a BV formulation of the Dirichlet problem for a variational integral of the type

$$F[w] := \int_\Omega f(\,\cdot\,, \nabla w)\,\mathrm{d}x\,. \tag{2.11}$$

To this purpose, we assume, for the remainder of this section, that the Borel integrand $f \colon \Omega \times \mathbb{R}^{N\times n} \to (-\infty, \infty]$ satisfies

$$f(x, z) \geq -\Psi(x) - \Gamma|z| \qquad \text{for all } (x, z) \in \Omega \times \mathbb{R}^{N\times n}\,,$$

with $\Psi \in \mathrm{L}^1(\Omega)$ and a constant $\Gamma < \infty$.

In order to first specify the $\mathrm{W}^{1,1}$-formulation, we work with an arbitrary open subset $\Omega$ of $\mathbb{R}^n$ and a function $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})$. We then abbreviate

$$\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N) := u_0 + \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N)$$

and investigate the

$$\text{minimization of } F \text{ in } \mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)\,, \tag{2.12}$$

where the boundary values are prescribed with the help of $u_0$.

A BV reformulation of this problem needs to overcome two technical problems. The first one is the need to compose the integrand $f$ with the gradient measure of a BV function. However, the solution to this problem and the right way of implementing the composition have already been introduced in Definition 2.1. The second problem consists in the specification of the boundary condition and the admissible class for the minimization problem. Indeed, it is not easy to say, in general, what it means for two BV functions to have the same boundary datum. Moreover, even when one restricts oneself to Lipschitz domains $\Omega$ and defines the admissible class as the set of functions in $\mathrm{BV}(\Omega,\mathbb{R}^N)$ with a prescribed trace, this does not overcome the problem, since the admissible class will not be weakly-$*$ closed in $\mathrm{BV}(\Omega,\mathbb{R}^N)$. Thus, we use a different approach, which is based on an extension procedure, works on quite general domains, and reduces to some extent the second problem to the first.

Concretely, we still consider an open subset $\Omega$ of $\mathbb{R}^n$, but we assume that

$$u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N) \text{ with } \nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n}) \tag{2.13}$$

is defined on the whole $\mathbb{R}^n$. Then we work with the admissible class

$$\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right) := \{w \in u_0 + \mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N) \,:\, w = u_0 \text{ holds } \mathcal{L}^n\text{-a.e. on } \mathbb{R}^n \setminus \Omega\}\,,$$

and we sometimes identify $w \in \mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$ with its restriction to $\Omega$ (which clearly contains the same information). To single out the significance of this class and the reason for involving $\overline{\Omega}$ in the notation, we observe that the $\mathcal{L}^n$-a.e. equality $w = u_0$ on $\mathbb{R}^n \setminus \Omega$ implies

$$\mathrm{D}w \llcorner \left(\mathbb{R}^n \setminus \overline{\Omega}\right) = (\nabla u_0)\mathcal{L}^n \llcorner \left(\mathbb{R}^n \setminus \overline{\Omega}\right) \qquad \text{for all } w \in \mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right).$$

Therefore, the relevant, non-prescribed part of $\mathrm{D}w$ is $\mathrm{D}w \llcorner \overline{\Omega}$, and we often consider $\mathrm{D}w$ as a measure on $\overline{\Omega}$ instead of $\mathbb{R}^n$. We also observe that a function $w \in \mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$ generally satisfies $|\mathrm{D}(w-u_0)|\left(\mathbb{R}^n \setminus \overline{\Omega}\right) = 0$, but not necessarily $|\mathrm{D}(w-u_0)|(\partial\Omega) = 0$. Just for illustration purposes, we also introduce a notation for the class of functions for which the last stronger condition holds, namely we set

$$\mathrm{BV}_{u_0}(\Omega,\mathbb{R}^N) := \left\{w \in \mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right) \,:\, |\mathrm{D}(w-u_0)|(\mathbb{R}^n \setminus \Omega) = 0\right\}.$$

Then we have $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N) \subset \mathrm{BV}_{u_0}(\Omega,\mathbb{R}^N) \subset \mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right) \subset u_0 + \mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$, all these classes are non-empty, and most importantly it is easy to check that $\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$ is closed[2]

[2] One should imagine that $\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$ is even the *closure* of $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ and thus also of $\mathrm{BV}_{u_0}(\Omega,\mathbb{R}^N)$ in the reasonable BV topologies. Indeed, under mild assumptions on $\Omega$, Theorem A.2 guarantees that $\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$ is the closure of $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ under strict convergence in $u_0+\mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$. Moreover, even if $\Omega$ is merely open with countably $\mathcal{H}^{n-1}$-rectifiable and locally $\mathcal{H}^{n-1}$-finite boundary $\partial\Omega$, (a simplified version of) the arguments in [P4, Section 4] and [P3, Section 3.3] can be adapted to show that $\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$ is still the sequential weak-$*$ closure of $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ in $u_0+\mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$. We do not provide a proof of the last assertion, which is not relevant for the subsequent considerations.

under weak-$*$ convergence in $u_0 + \mathrm{BV}(\mathbb{R}^n, \mathbb{R}^N)$, while $\mathrm{BV}_{u_0}(\Omega, \mathbb{R}^N)$ need not have the last closedness property. Moreover, since we assumed only (2.13), for arbitrary $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$, the measure $\mathrm{D}w \in \mathrm{RM}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N\times n})$ is not necessarily finite on all of $\mathbb{R}^n$. However, imposing the assumption

$$\mathcal{L}^n(\partial\Omega) = 0$$

from now on, at least $\mathrm{D}w \llcorner \overline{\Omega} \in \mathrm{RM}(\mathbb{R}^n, \mathbb{R}^{N\times n})$ is always globally finite.

With these conventions, it is straightforward to rephrase the Dirichlet problem in BV as follows. We define

$$\overline{F}[w] := \int_{\overline{\Omega}} f(\,\cdot\,, \mathrm{D}w) \qquad \text{for } w \in \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)\,, \tag{2.14}$$

where the right-hand side is understood as a functional of measures in the sense of Definition 2.1. Then, as the BV version of the Dirichlet problem (2.12), we study the

$$\text{minimization of } \overline{F} \text{ in } \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)\,. \tag{2.15}$$

It follows from (2.7) that $\overline{F}$ coincides with $F$ on $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$, and thus the minimization problems (2.12) and (2.15) are always related by

$$\inf_{\mathrm{BV}_{u_0}(\overline{\Omega}, \mathbb{R}^N)} \overline{F} \le \inf_{\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)} F\,. \tag{2.16}$$

In order to unravel the true meaning of (2.15), let us return, for a moment to the specific case of a bounded Lipschitz domain $\Omega$. In this case, $\mathrm{BV}_{u_0}(\Omega, \mathbb{R}^N)$ is the collection of all functions $w \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ whose interior trace $w^{\mathrm{int}}_{\partial\Omega}$ on $\partial\Omega$ equals the trace of $u_0$ on $\partial\Omega$. However, since BV functions may feature jumps along good hypersurfaces, the class $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ is actually the whole space $\mathrm{BV}(\Omega, \mathbb{R}^N)$, and for $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ the boundary parts of $\mathrm{D}w$ and $\overline{F}$ need not vanish, but rather take the forms

$$\mathrm{D}w \llcorner \partial\Omega = \big(\big(u_0 - w^{\mathrm{int}}_{\partial\Omega}\big) \otimes \mathrm{n}_\Omega\big)\mathcal{H}^{n-1} \llcorner \partial\Omega\,, \tag{2.17}$$

$$\int_{\partial\Omega} f(\,\cdot\,, \mathrm{D}w) = \int_{\partial\Omega} f^\infty\big(\,\cdot\,, \big(u_0 - w^{\mathrm{int}}_{\partial\Omega}\big) \otimes \mathrm{n}_\Omega\big)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{2.18}$$

where $\mathrm{n}_\Omega$ denotes the ($\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$ defined) outward unit normal to $\Omega$. Thus, in this case, the admissible class $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ is independent of the boundary datum $u_0$, and instead $u_0$ affects, through the way in which we understand $\mathrm{D}w$ on $\partial\Omega$, the functional $\overline{F}$. This can indeed be seen as a penalization procedure, which does not entirely rule out functions $w \in \mathrm{BV}\big(\overline{\Omega}, \mathbb{R}^N\big) \setminus \mathrm{BV}(\Omega, \mathbb{R}^N)$ with 'wrong' boundary values, but merely penalizes them, in comparison with functions $w \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ having 'right' boundary values, through the occurrence of the (usually positive) extra term (2.18). The examples discussed after Corollary 1.2 in the introduction show that this procedure is essentially unavoidable (as long as one does not impose more restrictive assumptions on $\Omega$ and $u_0$).

In case of a rough open $\Omega$, the boundary values $u_0$ affect, in addition to the functional $\overline{F}$, also the class $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$. For instance, if $\Omega$ has sharp external cusps, one can show by examples that the class $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ can be strictly smaller than $\mathrm{BV}(\Omega, \mathbb{R}^N)$ and can also depend on $u_0$. However, even then $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ is closed under weak-$*$ convergence in $u_0 + \mathrm{BV}(\mathbb{R}^n, \mathbb{R}^N)$, and it should be viewed as a technically very useful BV counterpart of the Dirichlet class $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$.

## 2.4 Existence of BV minimizers

Relying on the results and conventions of the previous sections, we can now prove the existence of minimizers for the problem (2.15). We come out with the following existence result which is essentially a revision of [71, Theorem 1.3]. The technical details of the statement resemble the ones of [P3, Theorem B.2].

**Theorem 2.8** (existence of BV minimizers)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ such that we have $\mathcal{L}^n(\partial\Omega)=0$ and such that $\overline{\Omega}$ supports the* $\mathrm{BV}_0$*-Poincaré inequality*

$$\int_\Omega |w|\,\mathrm{d}x \le C|\mathrm{D}w|\big(\overline{\Omega}\big) \qquad \textit{for all } w\in \mathrm{BV}_0\left(\overline{\Omega}\right), \tag{2.19}$$

*with a constant $C<\infty$. Moreover, consider some $u_0\in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0\in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$ and a lower semicontinuous function $f\colon \Omega\times\mathbb{R}^{N\times n}\to(-\infty,\infty]$, which is convex in the second argument and satisfies the linear coercivity condition*

$$f(x,z)\ge \varepsilon|z|-\ell(z) \qquad \textit{for all } (x,z)\in\Omega\times\mathbb{R}^{N\times n},$$

*with a constant $\varepsilon>0$ and a linear function $\ell\colon\mathbb{R}^{N\times n}\to\mathbb{R}$. Under these assumptions, whenever the functional $\overline{F}$ from* (2.14) *is finite on some function from $\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$, then there exists also a minimizer of $\overline{F}$ in $\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$.*

In connection with the $\mathrm{BV}_0$-Poincare inequality (2.19), we point out that it is generally valid for bounded $\Omega$ and that even for unbounded $\Omega$ there are comparably simple and sharp criteria for its availability. Indeed, it is sufficient that, for some $\delta>0$, the $\delta$-neighborhood of $\Omega$ does not contain balls with arbitrarily large radius, while it is necessary that $\overline{\Omega}$ itself does not contain balls with arbitrarily large radius; see [P3, Lemma 3.1] for the very simple proof.

The *proof of Theorem* 2.8 is based on the direct method of the calculus of variations. Indeed, since the passage from $f$ to the new integrand $(x,z)\mapsto f(x,z)+\ell(z)$ changes the functional $\overline{F}$ at most by a constant, we can assume $\ell\equiv 0$. Then we consider a minimizing sequence $(w_k)_{k\in\mathbb{N}}$ for $\overline{F}$ in $\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$. By the coercivity assumption, $(\mathrm{D}w_k)_{k\in\mathbb{N}}$ is bounded in $\mathrm{RM}\left(\overline{\Omega},\mathbb{R}^{N\times n}\right)$, and by the $\mathrm{BV}_0$-Poincaré inequality, $(w_k-u_0)_{k\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(\mathbb{R}^n,\mathbb{R}^N)$. Thus, using weak-$*$ compactness and Rellich's theorem, we can pass to a subsequence such that $w_{k_\ell}$ converges $\mathcal{L}^n$-a.e. on $\mathbb{R}^n$ to $u\in\mathrm{BV}_{u_0}\left(\overline{\Omega},\mathbb{R}^N\right)$ and such that $\mathrm{D}w_{k_\ell}$ weakly-$*$ converges to $\mathrm{D}u$ in $\mathrm{RM}\left(\overline{\Omega},\mathbb{R}^{N\times n}\right)$. At this stage, Corollary 2.3 implies

$$\overline{F}[u]\le\liminf_{\ell\to\infty}\overline{F}[w_{k_\ell}]=\inf_{\mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)}\overline{F}\,,$$

and the proof is complete. $\square$

In addition to the preceding existence theorem, it is clearly desirable to know that a minimizer in the $\mathrm{W}^{1,1}$ framework — if it exists, which is indeed only known in specific situations — is always a BV minimizer as well. In other words, we would like to ensure that the class of $\mathrm{W}^{1,1}$ minimizers is contained in the class of BV minimizers. This inclusion follows once we show that the inequality (2.16) is actually an equality, and this coincidence, in turn, can be proven if we impose on $\partial\Omega$ the mild regularity assumption

$$\mathbb{1}_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } |\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega\,. \tag{2.20}$$

Indeed, (2.20) is only relevant in order to have the approximations of Theorem A.2 at hand. Thus, we postpone a more detailed discussion of this hypothesis and its necessity to Appendix A, and for the moment we just record the following facts. The assumption (2.20) is much weaker than the requirement that $\Omega$ has a Lipschitz boundary, but slightly stronger than the condition that $\Omega$ has merely locally finite perimeter. Moreover, (2.20) implies that $\partial\Omega$ is $\mathcal{H}^{n-1}$-$\sigma$-finite and in particular that we have $\mathcal{L}^n(\partial\Omega) = 0$.

Utilizing (2.20) and Theorem A.2, we can establish the following addendum to Theorem 2.8. The precise statement is close in spirit to [71, Section 2], and its first part has been recorded in [P3, Section 2.2].

**Theorem 2.9** (consistency of the BV framework). *Consider an open subset $\Omega$ of $\mathbb{R}^n$ with* (2.20) *and a function $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$. Moreover, suppose that a Carathéodory function $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (*with $\mathbb{R}^{N\times n}$ in place of $\mathbb{R}^m$*) *the growth condition* (2.9) *with $\Psi \in \mathrm{L}^1(\Omega)$ and the continuity assumption* (2.8)*. Then, for the functionals $F$ and $\overline{F}$ in* (2.11) *and* (2.14)*, we have*

$$\inf_{\mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)} \overline{F} = \inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F\,. \tag{2.21}$$

*If, in addition, $\Psi$ is positive and bounded away from $0$ on every bounded subset of $\Omega$, then $u$ minimizes $\overline{F}$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ if and only if there exists a minimizing sequence $(w_k)_{k\in\mathbb{N}}$ for $F$ in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ such that $\|w_k - u\|_{\mathrm{L}^1(\Omega,\mathbb{R}^N)}$ converges to $0$ and such that $\mathrm{D}w_k$ converges $\Psi$-strictly in $\mathrm{RM}\big(\overline{\Omega},\mathbb{R}^{N\times n}\big)$ to $\mathrm{D}u$. Finally, the preceding characterization remains valid if we add the requirement that $\nabla w_k$ converges $\mathcal{L}^n$-a.e. to $\nabla u$.*

We find it worth pointing out that, in contrast to Theorem 2.8, Theorem 2.9 does not require a convexity assumption on $f$.

*Proof of Theorem* 2.9. Theorem A.2 yields, for every $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$, a sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ such that $\|w_k - w\|_{\mathrm{L}^1(\Omega,\mathbb{R}^N)}$ converges to $0$ and $\mathrm{D}w_k$ converges $\Psi$-strictly in $\mathrm{RM}\big(\overline{\Omega},\mathbb{R}^{N\times n}\big)$ to $\mathrm{D}w$. Moreover, under the positivity assumption for $\Psi$, Theorem 2.7 implies that, for such a sequence, we always have

$$\lim_{k\to\infty} F[w_k] = \overline{F}[w]\,.$$

All the claims of Theorem 2.9 follow easily from the preceding observations (where in order to verify (2.21), we can always enlarge $\Psi$ slightly and assume, without loss of generality, that the positivity assumption is satisfied). □

Finally, we remark that the proofs of Theorem 2.8 and Theorem 2.9 show slightly more, namely they provide, in case of the Dirichlet problem, a first-order analogue of the last remark in Section 2.2: Under the hypotheses of both theorems (apart from the Poincaré assumption), the functional $\overline{F}$ on $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ is the Lebesgue-Serrin extension (that is, the sequentially weakly-$*$ lower semicontinuous envelope) of the functional $F$ on $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$.

# Chapter 3

# Convex duality in the BV setting

This chapter is concerned with the duality theory for BV minimizers from [P3]. The main result is the characterization of extremal pairs in Theorem 3.8, which crucially relies on Anzellotti's concept [13, 14] of a pairing between gradient measures of BV functions and divergence-free $\mathrm{L}^\infty$ vector fields. In addition, our considerations implicate the uniqueness criterion of Corollary 3.14, which is particularly useful in the later Section 4.3. The theory presented here partially generalizes previous considerations of Ekeland & Temam [60], Temam & Strang [126], Kohn & Temam [85, 86], Anzellotti & Giaquinta [19, 20], Hardt & Kinderlehrer [80], Anzellotti [15, 17], Bouchitté & Dal Maso [37], Seregin [118, 119, 120], Bildhauer & Fuchs [32, 33], and Bildhauer [27, 28, 30].

## 3.1 Dual problem and duality formula

The concept of convex duality relies, first of all, on the following notion of duality for integrands, which is known as Legendre transformation, Fenchel transformation, convex conjugation, or convex duality.

**Definition 3.1** (convex conjugate)**.** *For a set $X$ and a function $f\colon X \times \mathbb{R}^m \to [-\infty,\infty]$, the (convex) conjugate $f^*\colon X \times \mathbb{R}^m \to [-\infty,\infty]$ of $f$ (with respect to the second variable $z$) is defined by*

$$f^*(x,z^*) := \sup_{z\in\mathbb{R}^{N\times n}} \big[z^*\cdot z - f(x,z)\big] \qquad \text{for } (x,z^*) \in X \times \mathbb{R}^m\,,$$

*where $z^*\cdot z$ stands for the Euclidean inner product of $z^*$ and $z$ in $\mathbb{R}^m$. Moreover, the function $f^{**} := (f^*)^*$ is called the bi-conjugate of $f$ (with respect to the $z$-variable).*

We record the following elementary properties of $f^*$ and $f^{**}$. If we have $f(x,\,\cdot\,) \not\equiv \infty$, then $f^*(x,\,\cdot\,)$ is lower semicontinuous and convex with values in $(-\infty,\infty]$ (and otherwise $f^*(x,\,\cdot\,) \equiv -\infty$). Moreover, if such a function exists at all, then $f^{**}(x,\,\cdot\,)$ is the largest lower semicontinuous, convex function $\mathbb{R}^m \to (-\infty,\infty]$ which is nowhere larger than $f(x,\,\cdot\,)$ (and otherwise $f^{**}(x,\,\cdot\,) \equiv -\infty$). The semicontinuity requirement in the characterization of $f^{**}$ can clearly be omitted in case of finite-valued $f$, and hence, in this case, we call $f^{**}$ the convexification of $f$ with respect to the $z$-variable. Finally, combining the preceding properties, one deduces the general validity of the equality $f^{***} = f^*$.

For an open subset $\Omega$ of $\mathbb{R}^n$ and $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$, we consider again the minimization problem for the functional $F$, defined by

$$F[w] := \int_\Omega f(\,\cdot\,,\nabla w)\,\mathrm{d}x \qquad \text{for } w \in \mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)\,, \tag{3.1}$$

where the Borel integrand $f\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies

$$|f(x,z)| \le \Psi(x)+\Gamma|z| \qquad \text{for } (x,z)\in\Omega\times\mathbb{R}^{N\times n} \tag{3.2}$$

with $\Psi\in\mathrm{L}^1(\Omega)$ and $\Gamma<\infty$.

The minimization problem is closely related to a dual problem in the space

$$\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n}) := \{\tau\in\mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times n}) \,:\, \operatorname{div}\tau\equiv 0 \text{ in } \mathcal{D}'(\Omega,\mathbb{R}^N)\}\,,$$

where the (row-wise) distributional divergence, acting on $\mathbb{R}^{N\times n}$-matrix fields $\tau$, is nothing but the formal adjoint of the gradient operator on $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$. Under (3.2), the dual integrand satisfies $f^*(x,z^*)=\infty$ whenever we have $|z^*|>\Gamma$ and $\Psi(x)<\infty$, and setting

$$R_{u_0}[\tau] := \int_\Omega \big[\tau\cdot\nabla u_0 - f^*(\,\cdot\,,\tau)\big]\mathrm{d}x \qquad \text{for } \tau\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})\,, \tag{3.3}$$

the dual problem then consists in the

$$\text{maximization of } R_{u_0} \text{ in } \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})\,. \tag{3.4}$$

We remark that the setup and the investigation of (3.4) are to a large extent motivated by applications of variational calculus in plasticity theory where the dual maximizers are interpreted as fields of stresses; compare [60, 126, 85, 86, 19, 20, 80, 15, 118, 120], for instance. However, we here intend to explain the relevance of (3.4) on a purely mathematical level. To this end, observe that, for all $w\in\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ and all $\tau\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, the definition of $f^*$ gives

$$f(\,\cdot\,,\nabla w) \ge \tau\cdot\nabla w - f^*(\,\cdot\,,\tau) \qquad \mathcal{L}^n\text{-a.e. on } \Omega\,, \tag{3.5}$$

while integration by parts yields

$$\int_\Omega \big[\tau\cdot\nabla w - f^*(\,\cdot\,,\tau)\big]\mathrm{d}x = \int_\Omega \big[\tau\cdot\nabla u_0 - f^*(\,\cdot\,,\tau)\big]\mathrm{d}x = R_{u_0}[\tau]\,. \tag{3.6}$$

Combining (3.5) and (3.6), we find

$$\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F \ge \sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})} R_{u_0}\,. \tag{3.7}$$

As the first main insight of the convex duality theory, if $f$ is convex in $z$, then (3.7) turns out to be an equality; this statement and similar ones for various classes of variational problems are extensively discussed in the monograph [60]. For our particular case, one can read off the following theorem.

**Theorem 3.2** (duality formula and existence of a dual maximizer)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$ and that the Borel integrand $f\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}$ is convex in $z$ and satisfies the linear growth condition* (3.2) *with $\Psi\in\mathrm{L}^1(\Omega)$. Moreover, consider a function $u_0\in\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ with $\nabla u_0\in\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$. Then, one has*

$$\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F = \sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})} R_{u_0} \in [-\infty,\infty)\,, \tag{3.8}$$

*and the supremum on the right-hand side is, in fact, a maximum.*

One possible proof of Theorem 3.2 is illustrated in Section 3.3.

Clearly, a benefit of (3.8) lies in the identification of the generally solvable variational problem on the right-hand side, which shares the target value of the often unsolvable problem on the left-hand side. Beyond that, we now explain that solutions of the two problems are even coupled by pointwise extremality relations, which generalize the Euler equation (1.3) and remain valid even for non-differentiable integrands $f$ such as the total variation integrand. To this end, we make use of the following basic concept of convex analysis.

**Definition 3.3** (subdifferential)**.** *For a set $X$ and a function $f\colon X\times\mathbb{R}^m\to[-\infty,\infty]$, the subdifferential $\partial_z f(x,z)$ of $f$ (with respect to the $z$-variable) at a point $(x,z)\in X\times\mathbb{R}^m$ is defined as*

$$\partial_z f(x,z) := \{z^*\in\mathbb{R}^m \,:\, f(x,\xi)\ge f(x,z)+z^*\cdot(\xi-z) \text{ for all } \xi\in\mathbb{R}^m\}\,.$$

*We also use the abbreviation $|\partial_z f|(x,z) := \sup\{|z^*| \,:\, z^*\in\partial_z f(x,z)\}$ (and if $\partial_z f(x,z)$ is the empty set, we understand, by convention, $|\partial_z f|(x,z)=0$). Later on, when considering $x$-independent functions $f\colon\mathbb{R}^m\to[-\infty,\infty]$, we omit the subscript $z$ and understand $\partial f$ and $|\partial f|$ in the evident way.*

Once more, we put some elementary properties on record. If $f(x,\,\cdot\,)$ is convex on $\mathbb{R}^m$, then $\partial_z f(x,z)$ is non-empty for all $z\in\mathbb{R}^m$, and if $f(x,\,\cdot\,)$ is differentiable at $z\in\mathbb{R}^m$, then one has $\partial_z f(x,z)\subset\{\nabla_z f(x,z)\}$. In particular, if $f$ is both $\mathrm{C}^1$ and convex in $z$, then $\partial_z f(x,z)$ equals the singleton $\{\nabla_z f(x,z)\}$ for all $(x,z)\in X\times\mathbb{R}^m$. Furthermore, from the definitions of the convex conjugate and the subdifferential, one reads off the equivalence

$$z^*\in\partial_z f(x,z) \iff f(x,z)+f^*(x,z^*)=z^*\cdot z \tag{3.9}$$

for $x\in X$ and $z,z^*\in\mathbb{R}^m$. Finally, if (3.2) is in force, then the subdifferential is evidently bounded by

$$|\partial_z f|(x,z)\le\Gamma \qquad \text{whenever } (x,z)\in X\times\mathbb{R}^m \text{ with } \Psi(x)<\infty\,. \tag{3.10}$$

At this stage, we are ready to establish the following pointwise characterization of minimizer-maximizer pairs.

**Corollary 3.4** (extremality relations in $\mathrm{W}^{1,1}_{u_0}$)**.** *Under the assumptions of Theorem 3.2, for $u\in\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ and $\sigma\in\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, one has*

$$\left.\begin{array}{r}u \text{ minimizes } F \text{ in } \mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)\,,\\ \sigma \text{ maximizes } R_{u_0} \text{ in } \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})\end{array}\right\} \iff \sigma\in\partial_z f(\,\cdot\,,\nabla u) \text{ holds } \mathcal{L}^n\text{-a.e. on } \Omega\,. \tag{3.11}$$

*Proof of Corollary* 3.4. In view of (3.6) and (3.8), $u$ minimizes $F$ and $\sigma$ minimizes $R_{u_0}$ if and only if there holds

$$\int_\Omega f(\,\cdot\,,\nabla u)\,\mathrm{d}x = \int_\Omega \big[\sigma\cdot\nabla u - f^*(\,\cdot\,,\sigma)\big]\,\mathrm{d}x\,.$$

Since (3.5) guarantees a pointwise *in*equality between the two integrands, the last formula holds, in turn, if and only if one has equality

$$f(\,\cdot\,,\nabla u) = \sigma\cdot\nabla u - f^*(\,\cdot\,,\sigma) \qquad \mathcal{L}^n\text{-a.e. on } \Omega\,. \tag{3.12}$$

Finally, taking into account (3.9), the equality (3.12) means precisely that $\sigma \in \partial_z f(\,\cdot\,,\nabla u)$ holds $\mathcal{L}^n$-a.e. on $\Omega$. □

We remark that minimizer-maximizer pairs can in fact be characterized by pointwise extremality relations in (at least) three basic and equivalent formulations: The first one is given by the right-hand side of (3.11), the second one is recorded in (3.12), and the third formulation reads as

$$\nabla u \in \partial_{z^*} f^*(\,\cdot\,,\sigma) \qquad \mathcal{L}^n\text{-a.e. on } \Omega\,. \tag{3.13}$$

Here, (3.13) comes in as an equivalent rephrasing of (3.12), which results from the fact that $f^{**} = f$ holds (by convexity of $f$ in $z$) and from an application of (3.9) with $f^*$ in place of $f$.

Since minimizers in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ need not exist, in general, we next return to the BV framework and the minimization problem for the functional $\overline{F}$, defined by

$$\overline{F}[w] := \int_{\overline{\Omega}} f(\,\cdot\,,\mathrm{D}w) \qquad \text{for } w \in \mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)\,. \tag{3.14}$$

Specifically, we study the relation between the minimization problem for $\overline{F}$ and the dual problem in (3.4), and we aim at finding analogues of the duality formula (3.8) and the extremality relation (3.11). Indeed, the analogue of (3.8) is immediate at this stage. Under the respective assumptions, we can just combine (2.21) and (3.8) to obtain the BV duality formula

$$\inf_{\mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)} \overline{F} = \sup_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})} R_{u_0} \in [-\infty,\infty)\,. \tag{3.15}$$

Obtaining the BV version of the extremality relation (3.11), however, is more elaborate and will only be accomplished, after the preparations of the subsequent section, in Section 3.3.

## 3.2 Anzellotti's pairing of gradient measures and divergence-free $\mathrm{L}^\infty$ vector fields

This section deals with the multiplication of a divergence-free $\mathrm{L}^\infty$ vector field $\tau \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$ and the gradient measure $\mathrm{D}w \llcorner \overline{\Omega} \in \mathrm{RM}(\mathbb{R}^n,\mathbb{R}^{N\times n})$ of a function $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$. The product $\tau\cdot\mathrm{D}w\llcorner\overline{\Omega} \in \mathrm{RM}(\mathbb{R}^n)$ of these two objects is only defined in an obvious way if either $\tau$ is continuous on $\overline{\Omega}$ or $\mathrm{D}w\llcorner\overline{\Omega}$ is absolutely continuous with respect to $\mathcal{L}^n\llcorner\Omega$. Indeed, the common feature of these two cases is that $\tau$ is $|\mathrm{D}w|$-a.e. on $\overline{\Omega}$ (well-)defined, while in general this property is not at hand. However, taking an integration-by-parts formula as a motivation, one can define a generalized product even without any extra assumption on $\tau$ and $w$. This is achieved by the following up-to-the-boundary version, specified in [P3, Definition 5.1], of Anzellotti's pairing from [13, Definition 1.4].

**Definition 3.5** (up-to-the-boundary pairing of $\tau$ and $\mathrm{D}w$)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$, and consider $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^N)$, $\tau \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, and $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$. Then the pairing of $\tau$ and $\mathrm{D}w$ is the distribution $[\![\tau,\mathrm{D}w]\!] \in \mathcal{D}'(\mathbb{R}^n)$, defined by*

$$[\![\tau,\mathrm{D}w]\!](\varphi) := \int_\Omega \varphi\tau\cdot\nabla u_0\,\mathrm{d}x - \int_\Omega \tau\cdot((w-u_0)\otimes\nabla\varphi)\,\mathrm{d}x \qquad \text{for } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,. \tag{3.16}$$

We emphasize that, at this stage, the up-to-the boundary feature lies in the fact that we work with a distribution on $\mathbb{R}^n$, *not* on $\Omega$ and thus require $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$, but *not* $\operatorname{spt}\varphi \subset \Omega$.

Under the mild regularity hypothesis

$$\mathbb{1}_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } |\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega\,. \tag{3.17}$$

of Appendix A, the pairing $[\![\tau,\mathrm{D}w]\!]$ exhibits reasonable properties, which we record in the following. Most basically, the pairing is a measure supported in $\overline{\Omega}$.

**Lemma 3.6** (the pairing is a measure)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$ with (3.17), and consider $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$, $\tau \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, and $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$. Then the pairing $[\![\tau,\mathrm{D}w]\!]$ is represented by a finite signed Radon measure on $\mathbb{R}^n$, which is also denoted by $[\![\tau,\mathrm{D}w]\!] \in \mathrm{RM}(\mathbb{R}^n)$. Furthermore, we have $|[\![\tau,\mathrm{D}w]\!]|\big(\mathbb{R}^n\setminus\overline{\Omega}\big) = 0$ and*

$$|[\![\tau,\mathrm{D}w]\!]| \leq \|\tau\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times n})}|\mathrm{D}w| \tag{3.18}$$

*as an inequality of measures.*

*Proof.* Working with the approximations $w_k \in u_0 + \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ of Theorem A.2 (applied for $\Psi \equiv 0$), and using integration by parts, we deduce from Definition 3.5 the estimate

$$|[\![\tau,\mathrm{D}w]\!](\varphi)| \leq \|\tau\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times n})}|\mathrm{D}w|\big(\overline{\Omega}\big)\sup_{\Omega}|\varphi| \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,. \tag{3.19}$$

Consequently, $[\![\tau,\mathrm{D}w]\!]$ extends to a continuous linear functional on $\mathrm{C}^0_{\mathrm{cpt}}(\mathbb{R}^n)$ with the sup-norm, and the measure representation follows from the Riesz representation theorem. The other claims are now immediate consequences of (3.19). $\square$

It is easily seen from the constancy theorem and Definition 3.5 that the pairing is a local operation. Precisely, this means that, for an open set $U$ in $\mathbb{R}^n$, the equalities $\tau_1 = \tau_2$ on $U\cap\Omega$ and $\mathrm{D}w_1 = \mathrm{D}w_2$ on $U\cap\overline{\Omega}$ imply $[\![\tau_1,\mathrm{D}w_1]\!](\varphi) = [\![\tau_2,\mathrm{D}w_2]\!](\varphi)$ for $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(U)$ and equivalently, in case of (3.17), $[\![\tau_1,\mathrm{D}w_1]\!] = [\![\tau_2,\mathrm{D}w_2]\!]$ as measures on $U$. From the locality property, we deduce, in particular, the very plausible assertion that the restriction of the pairing to $\Omega$ does not depend on the boundary values $u_0$.

In the two simpler cases mentioned at the beginning of the section, the pairing simplifies as expected. Indeed, integrating by parts in (3.16), it follows directly that we have

$$[\![\tau,\mathrm{D}w]\!] = (\tau\cdot\nabla w)\mathcal{L}^n \llcorner \Omega \qquad \text{for } w \in \mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)\,. \tag{3.20}$$

Moreover, if (3.17) holds and $\tau$ is continuous(ly extended) on $\overline{\Omega}$, we conclude $[\![\tau, \mathrm{D}w]\!] = \tau \cdot \mathrm{D}w \llcorner \overline{\Omega}$ essentially from a localized version of (3.18); compare with Proposition 3.10 and the explanation following it.

Further convenient properties of the pairing are summarized in the next statement.

**Lemma 3.7.** *Suppose that $\Omega$, $u_0$, $\tau$, and $w$ are as in Lemma 3.6. Then, for the measure $[\![\tau, \mathrm{D}w]\!] \in \mathrm{RM}(\mathbb{R}^n)$, we have*

$$[\![\tau, \mathrm{D}w]\!](\overline{\Omega}) = \int_\Omega \tau \cdot \nabla u_0 \,\mathrm{d}x\,, \tag{3.21}$$

$$[\![\tau, \mathrm{D}w]\!]^{\mathrm{a}} = (\tau \cdot \nabla w)\mathcal{L}^n \llcorner \Omega\,. \tag{3.22}$$

*Here, $[\![\tau, \mathrm{D}w]\!]^{\mathrm{a}}$ denotes the absolutely continuous part of $[\![\tau, \mathrm{D}w]\!]$ with respect to $\mathcal{L}^n$.*

*Proof.* In order to establish (3.21), it suffices to insert in (3.16) test functions $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ with $\mathbb{1}_{\mathrm{B}_R} \le \varphi \le 1$, $|\nabla\varphi| \le 1$ on $\mathbb{R}^n$ and then send $R \to \infty$.

Turning to (3.22), we fix a common Lebesgue point $x_0 \in \Omega$ of $\tau$ and $\mathrm{D}w$ (for the base measure $\mathcal{L}^n$). Here the $\mathrm{D}w$-Lebesgue-point property corresponds, by definition, to the existence of a Lebesgue value $\nabla w(x_0) \in \mathbb{R}^{N\times n}$ such that we have

$$\lim_{\varrho\searrow 0} \frac{|\mathrm{D}w - \nabla w(x_0)\mathcal{L}^n|(\mathrm{B}_\varrho(x_0))}{\mathcal{L}^n(\mathrm{B}_\varrho(x_0))} = 0\,,$$

and the $\tau$-Lebesgue-point property is defined similarly via the existence of a Lebesgue value $\tau(x_0) \in \mathbb{R}^{N\times n}$. Now we take into account that $[\![\tau, \mathrm{D}w]\!]$ is supported in $\overline{\Omega}$, that $\mathcal{L}^n(\partial\Omega) = 0$ holds by (3.17), and that $\mathcal{L}^n$-a.e. $x_0 \in \Omega$ satisfies the above requirements. Consequently, (3.22) follows once we show

$$\lim_{\varrho\searrow 0} \frac{[\![\tau, \mathrm{D}w]\!](\mathrm{B}_\varrho(x_0))}{\mathcal{L}^n(\mathrm{B}_\varrho(x_0))} = \tau(x_0) \cdot \nabla w(x_0)\,. \tag{3.23}$$

Turning to the verification of (3.23), we first introduce the abbreviation $a(x) \coloneqq \nabla w(x_0)\cdot(x{-}x_0)$, for which we observe $\mathrm{D}(w{-}a) = \mathrm{D}w - \nabla w(x_0)\mathcal{L}^n$ and, relying on the locality of the pairing and (3.20),

$$[\![\tau, \mathrm{D}(w{-}a)]\!] = [\![\tau, \mathrm{D}w]\!] - [\![\tau, \mathrm{D}a]\!] = [\![\tau, \mathrm{D}w]\!] - (\tau \cdot \nabla w(x_0))\mathcal{L}^n \qquad \text{on every ball } \mathrm{B}_\varrho(x_0) \subset \Omega\,.$$

With the help of these observations and (3.18), we then estimate, for $\mathrm{B}_\varrho(x_0) \subset \Omega$,

$$\begin{aligned}
&\left|\frac{[\![\tau, \mathrm{D}w]\!](\mathrm{B}_\varrho(x_0))}{\mathcal{L}^n(\mathrm{B}_\varrho(x_0))} - \tau(x_0)\cdot\nabla w(x_0)\right| \\
&\qquad\le \left|\frac{[\![\tau, \mathrm{D}(w{-}a)]\!](\mathrm{B}_\varrho(x_0))}{\mathcal{L}^n(\mathrm{B}_\varrho(x_0))}\right| + \left|\fint_{\mathrm{B}_\varrho(x_0)} \big[\tau - \tau(x_0)\big]\,\mathrm{d}x \cdot \nabla w(x_0)\right| \\
&\qquad\le \|\tau\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times n})} \frac{|\mathrm{D}w - \nabla w(x_0)\mathcal{L}^n|(\mathrm{B}_\varrho(x_0))}{\mathcal{L}^n(\mathrm{B}_\varrho(x_0))} + \fint_{\mathrm{B}_\varrho(x_0)} |\tau - \tau(x_0)|\,\mathrm{d}x\,|\nabla w(x_0)|\,.
\end{aligned}$$

By the choice of $x_0$, the right-hand side of the last estimate vanishes in the limit $r \searrow 0$. Thus we arrive at (3.23), and the proof is complete. □

We finally remark that Anzellotti [13, Definition 1.4] has also considered pairing operations for $\mathrm{L}^\infty$ vector fields whose divergence does not vanish, but is merely in some $\mathrm{L}^q$. Though these ideas are not relevant for the purposes of the present treatise, we briefly mention that a forthcoming joint work of C. Scheven and the author develops an even more general pairing, which incorporates even $\mathrm{L}^\infty$ divergence-measure fields. We plan to apply this concept in connection with a duality theory for irregular-obstacle total variation minimization.

## 3.3 Extremality relations for BV minimizers

In this section we arrive at the announced BV version of the extremality characterization in (3.11). The statement imposes a continuity hypothesis on the integrand $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ for $|z| \to \infty$. It requires that

$$\lim_{\substack{(\tilde{x},\tilde{z})\to(x,z)\\ t\searrow 0}} t f(\tilde{x},\tilde{z}/t) \text{ exists in } \mathbb{R} \text{ for all } (x,z) \in \overline{\Omega} \times (\mathbb{R}^{N\times n}\setminus\{0\})\,. \tag{3.24}$$

We do not further discuss (3.24) here, but instead we refer the reader to Section 2.2, where a completely analogous condition has already been introduced in (2.8), and we directly proceed to the restatement of [P3, Theorem 2.2].

**Theorem 3.8** (extremality relations in $\mathrm{BV}_{u_0}$)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$ with* (3.17) *and that the Borel integrand $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is convex in $z$ and satisfies the linear growth condition* (3.2) *with $\Psi \in \mathrm{L}^1(\Omega)$ and the continuity condition* (3.24)*. Furthermore, consider a function $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$. Then, for $u \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ and $\sigma \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, we have*

$$\left.\begin{array}{r} u \text{ minimizes } \overline{F}\,,\\ \sigma \text{ maximizes } R_{u_0} \end{array}\right\} \iff \left\{\begin{array}{c} \sigma \in \partial_z f(\,\cdot\,,\nabla u) \text{ holds } \mathcal{L}^n\text{-a.e. on } \Omega\,,\\ f^\infty\!\left(\,\cdot\,,\dfrac{\mathrm{d}\mathrm{D}^{\mathrm{s}}u}{\mathrm{d}|\mathrm{D}^{\mathrm{s}}u|}\right) = \dfrac{\mathrm{d}[\![\sigma,\mathrm{D}u]\!]^{\mathrm{s}}}{\mathrm{d}|\mathrm{D}^{\mathrm{s}}u|} \text{ holds } |\mathrm{D}^{\mathrm{s}}u|\text{-a.e. on } \overline{\Omega} \end{array}\right. \tag{3.25}$$

(*where clearly minimality and maximality are understood within the classes* $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ *and* $\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$*, respectively*)*.*

Though the first condition on the right-hand side of (3.25) resembles the one in (3.11), we emphasize that in the present BV framework it receives a different meaning, since $\nabla u$ *does not denote the full gradient of $u$, but only the density of the absolutely continuous part of* $\mathrm{D}u$. With this interpretation, the validity of the relation $\sigma \in \partial_z f(\,\cdot\,,\nabla u)$ for extremals $u$ and $\sigma$ has been observed, previously to [P3], by Seregin [118, 119, 120] and Bildhauer & Fuchs [32, 28, 30], for instance. However, the corresponding statements in these papers impose various strong assumptions on $f$ (such as $x$-independence, strict convexity, and $\mathrm{C}^2$-regularity with a bound for $\nabla^2 f$) and make use of regularity estimates to obtain the extremality relation for only those BV minimizers which arise as limits of certain minimizing sequences. In contrast, the proof of [P3] does not involve any extra regularity, yields the extremality relation for arbitrary BV minimizers, and thus answers a problem posed in [30, Remark 2.30].

The second relation on the right-hand side of (3.25), the one involving $\mathrm{D}^{\mathrm{s}}u$, is inspired by ideas of Anzellotti [13, 14, 17]. Indeed, Anzellotti developed (a local version of) the pairing $[\![\sigma, \mathrm{D}u]\!]$ in [13] and the unpublished manuscript [14] and arrived at formulas similar to the above $\mathrm{D}^{\mathrm{s}}u$-relation in [17, Theorem 1.3]. Though his formulas are stated in a somewhat different framework (in fact without mention of a dual problem), for sufficiently smooth $\Omega$ and $f$, it is possible to deduce the validity of the second relation in (3.25) for extremals $u$ and $\sigma$ from [17, Theorem 1.3] and the validity of the first relation. In addition, still for sufficiently smooth $\Omega$ and $f$, Anzellotti [16] has also shown that BV minimizers $u$ are characterized by a suitably adapted Euler equation (which has subsequently inspired the notion of BV solutions of elliptic equations in [18, 76]). A common feature of (3.25) and the equation in [16, Section 3] is that both imply, for differentiable $f$, the vanishing of the distributional divergence of $\nabla_z f(\,\cdot\,, \nabla u)$ on $\Omega$. However, the information on $\mathrm{D}^{\mathrm{s}}u$ does not (or not for an obvious reason) coincide, and indeed the statements in [16] do neither involve any Anzellotti type pairing nor do they identify $\nabla_z f(\,\cdot\,, \nabla u)$ as a dual solution.

All in all, even in view of all these previous results, it seems that Theorem 3.8 is new insofar that a simple and systematic treatment of both extremality relations and a full characterization of extremal pairs have not been achieved before.

To finalize the illustration of Theorem 3.8, we point out that several reformulations of the relations on the right-hand side of (3.25) are possible. So, the $\nabla u$-relation is equivalent to (3.12) and also to (3.13) (with the present understanding of $\nabla u$). Moreover, in the terminology of Section 2.1, the $\mathrm{D}^{\mathrm{s}}u$-relation expresses nothing but the equality $f^\infty(\,\cdot\,, \mathrm{D}^{\mathrm{s}}u) = [\![\sigma, \mathrm{D}u]\!]^{\mathrm{s}}$ of measures on $\overline{\Omega}$, and one may also unify both relations in the single equality

$$f(\,\cdot\,, \mathrm{D}u) = [\![\sigma, \mathrm{D}u]\!] - f^*(\,\cdot\,, \sigma)\mathcal{L}^n \llcorner \Omega \qquad \text{on } \overline{\Omega}\,. \tag{3.26}$$

Finally, if $f$ is, in addition to the above assumptions, positively 1-homogeneous in $z$, then we have

$$f^*(x, z^*) = \begin{cases} 0 & \text{if } z^* \in \partial_z f(x, 0) \\ \infty & \text{otherwise} \end{cases} \qquad \text{for } (x, z^*) \in \Omega \times \mathbb{R}^{N\times n}\,.$$

In this case, the extremality relations thus simplify to the equality $f(\,\cdot\,, \mathrm{D}u) = [\![\sigma, \mathrm{D}u]\!]$, combined with the requirement that $f^*(\,\cdot\,, \sigma) < \infty$ holds $\mathcal{L}^n$-a.e. on $\Omega$. The simplification applies, in particular, in case of the total variation, and then the relations read simply as $|\mathrm{D}u| = [\![\sigma, \mathrm{D}u]\!]$ together with the $\mathcal{L}^n$-a.e. inequality $|\sigma| \leq 1$ on $\Omega$. This last formulation (with some divergence-free $\sigma$, of course) has been rediscovered and is called the 1-Laplace equation in some more recent literature, but the corresponding characterization of extremals has been known for quite some time; compare with the introduction of [86], for instance.

As a corollary of Theorem 3.8 we recover, under more general assumptions, Bildhauer's uniqueness result [27, Theorem 7] for the dual problem.

**Corollary 3.9** (uniqueness of the dual solution)**.** *Assume that $\Omega$, $f$, and $u_0$ are as in Theorem* 3.8 *and that moreover $f$ is* $\mathrm{C}^1$ *in $z$. If a minimizer of $\overline{F}$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ exists, then $R_{u_0}$ has a unique maximizer in $\mathrm{L}^\infty_{\mathrm{div}}(\Omega, \mathbb{R}^{N\times n})$.*

We remark that the $\mathrm{C}^1$ assumption on $f$ is indeed very natural in order to obtain this uniqueness result, since this assumption is equivalent with a strict convexity property of $f^*$

and of the dual problem. Moreover, we stress that the existence of a BV minimizer is also a reasonable hypothesis, since Theorem 2.8 yields a sufficient criterion for existence in terms of mild requirements on $\Omega$ and $f$.

*Proof of Corollary* 3.9. The existence of a maximizer has already been recorded in Theorem 3.2, and in view of $\partial_z f(x,z) = \{\nabla_z f(x,z)\}$ it is uniquely determined by the first relation on the right-hand side of (3.25). □

In connection with the proof of Theorem 3.8, two different lines of argument are supplied in [P3]. The first reasoning works with minimizing sequences in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ and widely avoids all BV and pairing issues. Correspondingly, this reasoning only shows that all pairs of extremals satisfy the $\nabla u$-relation on the right-hand side of (3.25), but it also comes along with a proof of the duality formula in Theorem 3.2. The second reasoning crucially works with BV minimizers and the pairing from Definition 3.5, uses Theorem 3.2 as a prerequisite, and establishes the full characterization in (3.25). In the following we briefly illustrate both these strategies.

*Duality proof. First strategy.* We first explain the proof of the duality formula from Theorem 3.2, momentarily only when $\Omega$ is bounded and $f$ is $\mathrm{C}^1$ in $z$. In particular, (3.10) then yields the bound $|\nabla_z f(x,z)| \le \Gamma$ for $z \in \mathbb{R}^{N\times n}$ and $\mathcal{L}^n$-a.e. $x \in \Omega$. In view of (3.7), it suffices to deal with the case that $\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F$ is finite, and we start our reasoning with a minimizing sequence for $F$ in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$, that is a sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ such that

$$\lim_{k\to\infty} F[w_k] = \inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F\,.$$

We fix a corresponding null sequence $(\delta_k)_{k\in\mathbb{N}}$ in $(0,\infty)$ with

$$\lim_{k\to\infty} \delta_k \|\nabla w_k - \nabla u_0\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})} = 0\,. \tag{3.27}$$

By Ekeland's variational principle [58, 59], applied for each $k \in \mathbb{N}$ in the complete metric space $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ with the $\delta_k$-multiple of the $\mathrm{L}^1$ gradient distance as the metric, we can then find an improved minimizing sequence $(v_k)_{k\in\mathbb{N}}$ for $F$ in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ with the following properties. There holds

$$\lim_{k\to\infty} \delta_k \|\nabla v_k - \nabla w_k\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})} = 0\,, \tag{3.28}$$

and additionally we have, for every $k \in \mathbb{N}$, the perturbed minimality property

$$\int_\Omega f(\,\cdot\,,\nabla v_k)\,\mathrm{d}x \le \int_\Omega f(\,\cdot\,,\nabla v_k + \nabla\varphi)\,\mathrm{d}x + \delta_k\|\nabla\varphi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})} \qquad \text{for all } \varphi \in \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)\,.$$

As a consequence, we obtain the perturbed Euler equation

$$\left|\int_\Omega \nabla_z f(\,\cdot\,,\nabla v_k)\cdot\nabla\varphi\,\mathrm{d}x\right| \le \delta_k\|\nabla\varphi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})} \qquad \text{for all } \varphi \in \mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)\,. \tag{3.29}$$

We next choose $\tau_k := \nabla_z f(\,\cdot\,,\nabla v_k) \in \mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times n})$ and use (3.9) and integration by parts to find

$$F[v_k] = \int_\Omega f(\,\cdot\,,\nabla v_k)\,\mathrm{d}x = \int_\Omega \big[\tau_k\cdot\nabla v_k - f^*(\,\cdot\,,\tau_k)\big]\,\mathrm{d}x\,.$$

Via (3.29), applied with $\varphi = v_k - u_0 \in \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^N)$, we then arrive at

$$F[v_k] \leq \int_\Omega \big[\tau_k \cdot \nabla u_0 - f^*(\,\cdot\,, \tau_k)\big] \,\mathrm{d}x + \delta_k \|\nabla v_k - \nabla u_0\|_{\mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})} \,. \tag{3.30}$$

Taking into account the boundedness of $|\nabla_z f|$ and possibly passing to a subsequence, we can assume that $\tau_k$ weakly-$*$ converges in $\mathrm{L}^\infty(\Omega, \mathbb{R}^{N\times n})$ to a vectorfield $\sigma$, and thanks to (3.29) we infer that $\sigma \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega, \mathbb{R}^{N\times n})$ is actually divergence-free. Now we take limits in (3.30). Involving the weak-$*$ upper semicontinuity of the concave functional $\tau \mapsto \int_\Omega \big[\tau \cdot \nabla u_0 - f^*(\,\cdot\,, \tau)\big] \,\mathrm{d}x$ and (3.27)–(3.28), we then end up with

$$\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)} F \leq \int_\Omega \big[\sigma \cdot \nabla u_0 - f^*(\,\cdot\,, \sigma)\big] \,\mathrm{d}x = R_{u_0}[\sigma] \,.$$

Recalling (3.7), we arrive at the claim of Theorem 3.2.

The $\nabla u$-relation in (3.25) can be obtained, still when $\Omega$ is bounded and $f$ is $\mathrm{C}^1$ in $z$, by a closely related argument. Given a pair $(u, \sigma)$ of extremals, this argument works with the zero-order functional $F_\sigma$, given by

$$\begin{aligned} f_\sigma(x, z) &:= f(x, z) - \sigma(x) \cdot z \qquad \text{for } (x, z) \in \Omega \times \mathbb{R}^{N\times n} \,, \\ F_\sigma[\Theta] &:= \int_\Omega f_\sigma(\,\cdot\,, \Theta) \,\mathrm{d}x \qquad \text{for } \Theta \in \mathrm{L}^1(\Omega, \mathbb{R}^{N\times n}) \,. \end{aligned}$$

Via the estimate $f(x, z) - z^* \cdot z \geq -f^*(x, z^*)$, Theorem 3.2, and integration by parts, we obtain the crucial chain of (in)equalities

$$\begin{aligned} \inf_{\mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})} F_\sigma &\geq \int_\Omega \big[-f^*(\,\cdot\,, \sigma)\big] \,\mathrm{d}x = R_{u_0}[\sigma] - \int_\Omega \sigma \cdot \nabla u_0 \,\mathrm{d}x \\ &= \inf_{w \in \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)} \left( F[w] - \int_\Omega \sigma \cdot \nabla u_0 \,\mathrm{d}x \right) = \inf_{w \in \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)} F_\sigma[\nabla w] \,. \end{aligned}$$

By Theorem 2.9, there exists a minimizing sequence $(w_k)_{k\in\mathbb{N}}$ for $F$ in $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$ such that $\nabla w_k$ converges $\mathcal{L}^n$-a.e. to $\nabla u$, and, taking into account the preceding considerations, $(\nabla w_k)_{k\in\mathbb{N}}$ is a minimizing sequence for the zero-order functional $F_\sigma$ in $\mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})$. Using Ekeland's principle once more, we arrive at a refined minimizing sequence $(V_k)_{k\in\mathbb{N}}$ for $F_\sigma$ in $\mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})$ such that $V_k$ still converges $\mathcal{L}^n$-a.e. to $\nabla u$ and such that, for some null sequence $\delta_k$ in $(0, \infty)$, we have

$$\left| \int_\Omega \nabla_z f_\sigma(\,\cdot\,, V_k) \cdot \Phi \,\mathrm{d}x \right| \leq \delta_k \|\Phi\|_{\mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})} \qquad \text{for all } \Phi \in \mathrm{L}^1(\Omega, \mathbb{R}^{N\times n}) \,. \tag{3.31}$$

It follows by continuity of $\nabla_z f_\sigma$ in $z$ that $\tau_k := \nabla_z f_\sigma(\,\cdot\,, V_k)$ converges $\mathcal{L}^n$-a.e. to $\nabla_z f_\sigma(\,\cdot\,, \nabla u)$. By the boundedness of $|\nabla_z f_\sigma|$, the same convergence is at hand in $\mathrm{L}^p(\Omega, \mathbb{R}^{N\times n})$, for every $p < \infty$, and then (3.31) forces the limit to be 0. Thus, we have shown that $\nabla_z f_\sigma(\,\cdot\,, \nabla u) \equiv 0$ and, equivalently, the targeted extremality relation $\sigma = \nabla_z f(\,\cdot\,, \nabla u)$ hold $\mathcal{L}^n$-a.e. on $\Omega$. We stress that the $\mathcal{L}^n$-a.e. convergence of $\nabla w_k$, which originally results from Theorem A.2 and comes in implicitly through Theorem 2.9, plays a crucial role in the preceding reasoning.

In order to treat non-differentiable integrands $f$, [P3, Section 4] provides refinements of the preceding arguments. Though these refinements apply analogously to the above proof of the duality formula, we here describe only how they affect the reasoning around (3.31). A main difficulty in the non-differentiable case is evidently that $\tau_k := \nabla_z f_\sigma(\,\cdot\,, V_k)$ is not a meaningful choice anymore and that the inclusion $\tau_k \in \partial_z f_\sigma(\,\cdot\,, V_k)$ determines $\tau_k$ only partially, so that one actually needs an additional criterion in order to implement a suitable choice. This is done in [P3] by exploiting the elementary fact that the convex functions $f(x,\,\cdot\,)$ have one-sided directional derivatives, which are realized (in their respective direction) by subgradients. Specifically, this allows to find a null sequence $(\delta_k)_{k\in\mathbb{N}}$ and, for every given field of directions $\Phi \in \mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})$, a field $\tau_k \in \partial_z f_\sigma(\,\cdot\,, V_k)$ of subgradients such that, as a replacement for (3.31), there holds

$$\int_\Omega \tau_k \cdot \Phi \,\mathrm{d}x \leq \delta_k \|\Phi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})} \,. \tag{3.32}$$

However, the finding of $\tau_k$ brings up two further technical issues, which we discuss only very briefly; for more details see [P3, Lemma 4.1, Proposition 4.2].

One issue is that, while the product $\tau_k \cdot \Phi$ is obtained as a directional derivative and is thus uniquely determined and measurable, there is still some freedom in the choice of $\tau_k$ itself and it is not obvious from the arguments supplied so far that $\tau_k$ can be chosen as a measurable function. This measurability problem is overcome in [P3] by reformulating the requirements on $\tau_k$ in the language of measurable multifunctions, as explained in [113], and then applying the measurable selection principle [113, Theorem 1.C].

The second issue is that the above explanations yield, for every $\Phi \in \mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})$, a corresponding $\tau_k$, but in order to conclude the proof one needs to find, possibly for smaller $\delta_k$, a single $\tau_k$ which works for all $\Phi \in \mathrm{L}^1(\Omega, \mathbb{R}^{N\times n})$. Restricting the considerations from $\mathrm{L}^1$ vector fields to $\mathrm{L}^p$ ones with $p > 1$, it is shown in [P3] that the relevant exchange of quantifiers boils down to a (weak-$*$) separation problem for a convex set and a point in $\mathrm{L}^{\frac{p}{p-1}}$. The separation problem, in turn, is then solved via the Hahn-Banach-Mazur theorem and the reflexivity of $\mathrm{L}^p$.

Once we have found $\tau_k \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{N\times n})$ such that $\tau_k \in \partial_z f_\sigma(\,\cdot\,, V_k)$ holds $\mathcal{L}^n$-a.e. on $\Omega$ and such that (3.32) holds for all $\Phi \in \mathrm{L}^p(\Omega, \mathbb{R}^{N\times n})$, the proof of the extremality relation continues essentially as in the differentiable case. Indeed, $\tau_k$ still converges to 0, and a continuity property of subgradients ensures the $\mathcal{L}^n$-a.e. inclusion $0 \in \partial_z f_\sigma(\,\cdot\,, \nabla u)$ on $\Omega$ in the limit, so that we arrive at the $\mathcal{L}^n$-a.e. relation $\sigma \in \partial_z f(\,\cdot\,, \nabla u)$ on $\Omega$.

Finally, the adaption of the above methods to unbounded domains $\Omega$ with possibly infinite $\mathcal{L}^n$-measure causes some, but comparably minor technical problems, which we do not discuss here. For all further details we refer once more to [P3, Section 4]. □

*Duality proof. Second strategy.* Next we describe the proof of the full characterization of extremality in (3.25). The main difficulty in this regard is to understand the $|\mathrm{D}w|$-a.e. behavior of the pairing $[\![\tau, \mathrm{D}w]\!]$ and obtain an analogue of the basic inequality (3.5) for the singular parts. Once this is achieved, the remaining arguments are, in principle, analogous to the simple proof of Corollary 3.4.

Indeed, for Lipschitz domains $\Omega$ in the scalar case $N = 1$, the $|\mathrm{D}w|$-a.e. behavior of $[\![\tau, \mathrm{D}w]\!]$ is very accurately described by the representation formula [14, Theorem 3.6] in an unpublished manuscript of Anzellotti, and we could base our reasoning on a suitable adaption of this formula.

However, in order to avoid several related technicalities, we prefer to use only the weaker information in the following proposition, which is a restatement of [P3, Theorem 5.2] and still suffices for our purposes.

**Proposition 3.10** ($|\mathrm{D}w|$-a.e. density control on $[\![\tau,\mathrm{D}w]\!]$)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$ with (3.17), and consider $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$, $\tau \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$, and a common Lebesgue point $x_0 \in \overline{\Omega}$ of $\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}$ and $\frac{\mathrm{d}[\![\tau,\mathrm{D}w]\!]}{\mathrm{d}|\mathrm{D}w|}$ (for the base measure $|\mathrm{D}w|$), where all measures are understood as measures on $\overline{\Omega}$. If, for some closed convex set $K$ in $\mathbb{R}^{N\times n}$, there holds*

$$\tau \in K \qquad \mathcal{L}^n\text{-a.e. on a neighborhood of } x_0 \text{ in } \Omega\,,$$

*then there exists a*

$$z_0^* \in K$$

*such that we have*

$$\frac{\mathrm{d}[\![\tau,\mathrm{D}w]\!]}{\mathrm{d}|\mathrm{D}w|}(x_0) = z_0^* \cdot \frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}(x_0)\,.$$

Basically, we have already seen that Proposition 3.10 holds in the case that $K$ is a ball in $\mathbb{R}^{N\times n}$: After reducing to the case that the center of the ball is 0, the statement follows from a localized version of the previous estimate (3.18), namely from $|[\![\tau,\mathrm{D}w]\!]|\big(U\cap\overline{\Omega}\big) \le \|\tau\|_{\mathrm{L}^\infty(U\cap\Omega,\mathbb{R}^{N\times n})}|\mathrm{D}w|\big(U\cap\overline{\Omega}\big)$ for open neighborhoods $U$ in $\mathbb{R}^n$. In the general case, Proposition 3.10 is established by using the smooth approximations $w_k$ of Theorem A.2, by suitably decomposing the products $\tau\cdot\nabla w_k$ in the directions parallel and orthogonal to $\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}(x_0)$, and by some elementary estimations. The implementation of this reasoning has been carried out in [P3, Section 5] and is not detailed here.

Imposing the assumptions of Theorem 3.8 on $\Omega$, $f$, and $u_0$, but still working with arbitrary competitors $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ and $\tau \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, we next claim that Proposition 3.10 induces the following decisive consequence: Whenever $f^*(\,\cdot\,,\tau) < \infty$ holds $\mathcal{L}^n$-a.e. on $\Omega$, then we have the inequality of measures

$$f^\infty(\,\cdot\,,\mathrm{D}w) \ge [\![\tau,\mathrm{D}w]\!] \qquad \text{on } \overline{\Omega}\,. \tag{3.33}$$

Once (3.33) is at hand, we keep only the information about the singular parts

$$f^\infty(\,\cdot\,,\mathrm{D}^{\mathrm{s}}w) \ge [\![\tau,\mathrm{D}w]\!]^{\mathrm{s}} \qquad \text{on } \overline{\Omega}\,,$$

and, bearing (3.22) in mind, we recombine this with (3.5) to

$$f(\,\cdot\,,\mathrm{D}w) \ge [\![\tau,\mathrm{D}w]\!] - f^*(\,\cdot\,,\sigma)\mathcal{L}^n \llcorner \Omega \qquad \text{on } \overline{\Omega}\,. \tag{3.34}$$

In order to explain now how (3.33) follows from Proposition 3.10, we first consider the simpler case that $f$ does not depend of $x$. It then suffices to reason as follows. By (3.5), for $\mathcal{L}^n$-a.e. $x\in\Omega$ and all $(t,z)\in(0,\infty)\times\mathbb{R}^{N\times n}$, we have $tf(z/t) \ge \tau(x)\cdot z - tf^*(\tau(x))$. Sending $t \searrow 0$ and exploiting the finiteness requirement for $f^*(\tau)$, we conclude

$$\tau(x) \in \partial_z f^\infty(0) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x\in\Omega\,.$$

Fixing a common Lebesgue point $x_0 \in \overline{\Omega}$ of $\frac{\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}$ and $\frac{\mathrm{d}[\![\tau,\mathrm{D}w]\!]}{\mathrm{d}|\mathrm{D}w|}$, we now apply Proposition 3.10 with the closed convex set $K := \partial_z f^\infty(0) \subset \mathbb{R}^{N\times n}$. We thus find some $z_0^* \in \partial_z f^\infty(0)$ with

$$\frac{\mathrm{d}[\![\tau,\mathrm{D}w]\!]}{\mathrm{d}|\mathrm{D}w|}(x_0) = z_0^* \cdot \frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}(x_0) \le f^\infty\bigg(\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}(x_0)\bigg)\,.$$

Exploiting the validity of the last inequality for $|\mathrm{D}w|$-a.e. point $x_0$ in $\Omega$ and recalling (2.2), we infer

$$[\![\tau,\mathrm{D}w]\!] \le f^\infty\bigg(\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}\bigg)|\mathrm{D}w| = f^\infty(\mathrm{D}w)\,,$$

and we have established (3.33) in the simpler, $x$-independent case.

For an $x$-dependent integrand $f$, the derivation of (3.33) is slightly more complicated, since also the closed convex sets $K_x := \partial_z f^\infty(x,0) \subset \mathbb{R}^{N\times n}$ vary with $x$ and the situation of Proposition 3.10 with a fixed $K$ is not directly at hand. However, to overcome this point, it suffices to observe that, by a short and elementary argument [P3, Lemma 5.4], $K_x$ is set-upper-semicontinuous in $x$. Relying on this observation and applying Proposition 3.10 to neighborhoods of $K_{x_0}$, it is straightforward to establish (3.33) also for the general case.

Once (3.33) and thus also (3.34) are established, we proceed to the final proof of the equivalence (3.25) for $\sigma \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$ and $u \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$. It follows from (3.15) that $(u,\sigma)$ is a minimizer-maximizer pair if and only if we have $\overline{F}[u] = R_{u_0}[\sigma]$. The last equality, in turn, is now rephrased with the help of (3.21); it holds if and only if we have

$$\int_{\overline{\Omega}} f(\,\cdot\,,\mathrm{D}u) = [\![\sigma,\mathrm{D}u]\!]\big(\overline{\Omega}\big) - \int_\Omega f^*(\,\cdot\,,\sigma)\,\mathrm{d}x\,. \tag{3.35}$$

Since (3.35) implies $\mathcal{L}^n$-a.e. finiteness of $f^*(\,\cdot\,,\sigma)$, the inequality (3.34) is at hand for the terms in (3.35), and in view of this pointwise *inequality*, (3.35) is finally equivalent with the pointwise *equality*

$$f(\,\cdot\,,\mathrm{D}u) = [\![\sigma,\mathrm{D}u]\!] - f^*(\,\cdot\,,\sigma)\mathcal{L}^n \llcorner \Omega \qquad \text{on } \overline{\Omega}\,.$$

Since we already observed in (3.26) that the formula in the last line is just a reformulation of those on the right-hand side of (3.25), this finishes the proof of Theorem 3.8. □

## 3.4 Uniqueness of BV minimizers via a duality criterion

In this section we first discuss criteria, which allow to read off uniqueness and regularity results for BV minimizers from properties of the dual maximizers. As the final — and most interesting — outcome of these considerations we obtain the uniqueness criterion in Corollary 3.14, *which does not, anymore, involve the dual problem.*

In order to proceed to the statements, we first introduce some convenient terminology. For an open subset $\Omega$ of $\mathbb{R}^n$, a point $x_0 \in \overline{\Omega}$, and a function $\eta\colon \Omega \to \mathbb{R}^m$, we write $\limsup_{x\to x_0}\{\eta(x)\}$ for the (necessarily closed) set of all cluster points of sequences of the form $(\eta(x_k))_{k\in\mathbb{N}}$ where $(x_k)_{k\in\mathbb{N}}$ is a sequence in $\Omega$ with limit $x_0$. If $\tau$ is a Lebesgue class of functions from $\Omega$ to $\mathbb{R}^m$, we understand $\limsup_{x\to x_0}\{\tau(x)\}$ as the (still closed) intersection of all sets $\limsup_{x\to x_0}\{\eta(x)\}$

where $\eta$ is a representative of $\tau$. Moreover, for a set $X$, a function $f\colon X \times \mathbb{R}^m \to [-\infty,\infty]$, and $x_0 \in X$, we abbreviate

$$\operatorname{Im}\partial_z f(x_0,\,\cdot\,) := \{\partial_z f(x_0,z) \,:\, z \in \mathbb{R}^m\} \subset \mathbb{R}^m\,. \tag{3.36}$$

With these conventions, we now restate [P3, Theorem 2.4].

**Theorem 3.11** (duality criterion for the absence of singular parts)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$ with* (3.17) *and that the Borel integrand $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is convex in $z$ and satisfies the linear growth condition* (3.2) *with $\Psi \in \mathrm{L}^1(\Omega)$ and the continuity condition* (3.24)*. Furthermore, consider $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$, and assume that $u$ minimizes $\overline{F}$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ and that $\sigma$ minimizes $R_{u_0}$ in $\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$. If $A$ is a Borel subset of $\overline{\Omega}$ and $\limsup_{x\to x_0}\{\sigma(x)\}$ is contained in the interior of $\operatorname{Im}\partial_z f(x_0,\,\cdot\,)$ for $|\mathrm{D}^{\mathrm{s}}u|$-a.e. $x_0 \in A$, then we necessarily have $|\mathrm{D}^{\mathrm{s}}u|(A) = 0$.*

The *proof of Theorem* 3.11 is closely related to the second strategy from the previous section. In particular, it makes use of the following statement, which reproduces [P3, Lemma 5.6] and complements the crucial inequality (3.33) in the previous reasoning.

**Lemma 3.12.** *Suppose that $\Omega$, $f$, $u_0$ are as in Theorem* 3.11*, and consider $\tau \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$, $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$, and a common Lebesgue point $x_0 \in \overline{\Omega}$ of $\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}$ and $\frac{\mathrm{d}[\![\tau,\mathrm{D}w]\!]}{\mathrm{d}|\mathrm{D}w|}$ (for the base measure $|\mathrm{D}w|$), where once more all measures are regarded as measures on $\overline{\Omega}$. If $\limsup_{x\to x_0}\{\tau(x)\}$ is contained in the interior of $\operatorname{Im}\partial_z f(x_0,\,\cdot\,)$, then we have the strict inequality*

$$f^\infty\bigg(x_0,\frac{\mathrm{d}\mathrm{D}w}{\mathrm{d}|\mathrm{D}w|}(x_0)\bigg) > \frac{\mathrm{d}[\![\tau,\mathrm{D}w]\!]}{\mathrm{d}|\mathrm{D}w|}(x_0)\,.$$

The deduction of Lemma 3.12 from Proposition 3.10 is implemented in a similar spirit as the one of (3.33). We omit the details, which can be found in [P3, Section 5].

The point of Lemma 3.12 is that its strict inequality is, as far as the singular parts are concerned, incompatible with the extremality relations. Indeed, under the assumption of Theorem 3.11, the lemma implies, for the minimizer-maximizer pair $(u,\sigma)$, the strict inequality

$$f^\infty\bigg(\,\cdot\,,\frac{\mathrm{d}\mathrm{D}^{\mathrm{s}}u}{\mathrm{d}|\mathrm{D}^{\mathrm{s}}u|}\bigg) > \frac{\mathrm{d}[\![\sigma,\mathrm{D}u]\!]^{\mathrm{s}}}{\mathrm{d}|\mathrm{D}^{\mathrm{s}}u|} \qquad |\mathrm{D}^{\mathrm{s}}u|\text{-a.e. on } A\,,$$

and this is only compatible with the $\mathrm{D}^{\mathrm{s}}u$-relation in (3.25) if we have $|\mathrm{D}^{\mathrm{s}}u|(A) = 0$. Thus, we arrive at the claim of Theorem 3.11. □

As a corollary of Theorem 3.11 we recover, under slightly different assumptions, a result of Bildhauer [30, Theorem A.9, Remark A.10], which has previously been established by a different strategy based on variations for the dual problem.

**Corollary 3.13** (continuity of $\sigma$ implies uniqueness of $\mathrm{D}u$)**.** *Suppose that $\Omega$, $f$, $\Psi$, $u_0$ are as in Theorem* 3.11 *and that, additionally, $f(x,\,\cdot\,)\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a strictly convex function for all $x \in \Omega$. If $R_{u_0}$ has a* continuous *minimizer $\sigma$ in $\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})$ such that $\sigma(x)$ lies in the interior of $\operatorname{Im}\partial_z f(x,\,\cdot\,)$ for all $x \in \Omega$, then all minimizers $u$ and $v$ of $\overline{F}$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ are necessarily in $\mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and $\nabla u = \nabla v$ holds $\mathcal{L}^n$-a.e. on $\Omega$.*

*Proof.* Theorem 3.11 gives $|\mathrm{D}^{\mathrm{s}}u|(\Omega) = 0$, and uniqueness of $\nabla u$ follows straightforwardly from the strict convexity of $f$. Finally, the remaining claim $\nabla u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ follows from the validity, under the present assumptions, of the following convex analysis facts (compare [P3, Proposition 3.8, Lemma 3.9] for elementary proofs): The set

$$G_f := \{(x, z^*) \in \Omega \times \mathbb{R}^{N\times n} \,:\, z^* \in \operatorname{Im} \partial_z f(x, \,\cdot\,)\}$$

is open in $\Omega \times \mathbb{R}^{N\times n}$, and $\nabla_{z^*} f^*$ exists and is locally bounded on $G_f$. With these assertions and the extremality relation $\nabla u = \nabla_{z^*} f^*(\,\cdot\,, \sigma)$ from (3.13) at hand, local boundedness of $\nabla u$ follows by taking into account the continuity of the mapping $\Omega \to G_f, x \mapsto (x, \sigma(x))$. □

We point out that, in Corollary 3.13, continuity of $\sigma$ is assumed only on $\Omega$, but *not* up to the boundary. Correspondingly, the conclusion gives $|\mathrm{D}^{\mathrm{s}}u|(\Omega) = 0$ and $u \in u_0 + \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$, but *not* $|\mathrm{D}^{\mathrm{s}}u|(\partial\Omega) = 0$ or $u \in \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$.

The most important aspect of Theorem 3.11 and Corollary 3.13 is the following uniqueness criterion, which is formulated only in terms of BV minimizers and does not involve the dual solution $\sigma$ anymore. This criterion is conveniently applied in the more concrete situation of the eventual Corollary 4.5.

**Corollary 3.14** (a $\mathrm{C}^1$ minimizer is unique modulo constants in $\mathrm{BV}_{u_0}$)**.** *Suppose that $\Omega$, $f$, $\Psi$, $u_0$ are as in Theorem* 3.11 *and, additionally, that $\Omega$ is connected, that $f(x, \,\cdot\,) \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is a strictly convex $\mathrm{C}^1$ function for all $x \in \Omega$, and that $\nabla_z f$ is continuous on $\Omega \times \mathbb{R}^{N\times n}$. If $\overline{F}$ has a minimizer in $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$, which is $\mathrm{C}^1$ on $\Omega$, then, for each two minimizers $u$ and $v$ of $\overline{F}$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$, there exists a constant $y \in \mathbb{R}^N$ such that $u = v + y$ holds $\mathcal{L}^n$-a.e. on $\Omega$.*

*Proof.* In view of Theorem 3.8[1] and Corollary 3.9, the existence of the $\mathrm{C}^1$ minimizer and the continuity of $\nabla_z f$ show that the unique maximizer $\sigma := \nabla_z f(\,\cdot\,, \nabla u)$ of $R_{u_0}$ in $\mathrm{L}^\infty_{\mathrm{div}}(\Omega, \mathbb{R}^{N\times n})$ is continuous. Therefore, the claim follows from Corollary 3.13 and the constancy theorem. □

## 3.5 Duality for some non-convex cases

In this section, we point out that the duality results of Sections 3.1 and 3.3 remain valid in *some* non-convex cases. While we believe that this feature is actually worth recording, we also emphasize that its practicability is limited, since *there are no general existence theorems for these cases at hand.*

In what follows, we restrict[2] ourselves to the case that the domain $\Omega$ and the function $\Psi$ in (3.2) are bounded. We utilize the concept of the quasiconvexification, which is also known as the quasiconvex envelope or the quasiconvex hull and is defined as follows. For a Carathéodory integrand $f \colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$, its quasiconvexification $\mathrm{Q}f \colon \Omega \times \mathbb{R}^{N\times n} \to [-\infty, \infty)$ (with respect to the $z$-variable) is given by

$$\mathrm{Q}f(x, z) := \sup\{g(z) \,:\, g \colon \mathbb{R}^{N\times n} \to \mathbb{R} \text{ is quasiconvex with } g \le f(x, \,\cdot\,) \text{ on } \mathbb{R}^{N\times n}\}$$

[1] We find it worth pointing out that the simpler extremality relation of Corollary 3.4 cannot be used at this point, since we do not assume that the $\mathrm{C}^1$ minimizer realizes the boundary values given by $u_0$.

[2] The reason for this slight restriction is that the relaxation results on which we rely are only available in the literature for bounded $\Omega$ and $\Psi$. It seems very likely that these boundedness assumptions can actually be dropped, but we do not want to enter into a technical discussion of this aspect.

for $(x,z) \in \Omega \times \mathbb{R}^{N\times n}$. Here we adopt the convention $\sup\emptyset = -\infty$, and we say that a locally bounded Borel function $g\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex if

$$\fint_{\operatorname{spt}\varphi} g(z+\nabla\varphi(x))\,\mathrm{d}x \geq g(z) \qquad \text{holds for all } z \in \mathbb{R}^{N\times n} \text{ and } 0 \not\equiv \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n,\mathbb{R}^N)\,.$$

By Jensen's inequality, every convex function $\mathbb{R}^{N\times n} \to \mathbb{R}$ is also quasiconvex, evidently we have $\mathrm{Q}f \leq f$ on $\Omega \times \mathbb{R}^{N\times n}$, and if $\mathrm{Q}f > -\infty$ holds on $\Omega \times \mathbb{R}^{N\times n}$, then it follows that $\mathrm{Q}f(x,\,\cdot\,)$ is quasiconvex for all $x \in \Omega$ and that also $\mathrm{Q}f$ has the Carathéodory property.

With this terminology, if $\Omega$ is a bounded open set in $\mathbb{R}^n$, if $u_0$ is a function in $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$, and if the Carathéodory function $f$ satisfies

$$-\Gamma - \Gamma|z| \leq \mathrm{Q}f(x,z) \leq f(x,z) \leq \Gamma + \Gamma|z| \qquad \text{for } (x,z) \in \Omega \times \mathbb{R}^{N\times n} \tag{3.37}$$

with a constant $\Gamma < \infty$, then the linear growth version of Dacorogna's relaxation formula [48] asserts

$$\inf_{w\in \mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} \int_\Omega \mathrm{Q}f(\,\cdot\,,\nabla w)\,\mathrm{d}x = \inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F\,. \tag{3.38}$$

This equality (which can be inferred from [49, Theorem 9.8], for instance) and the observation $\mathrm{Q}f \leq f$ imply that every minimizer of the right-hand problem in (3.38) is also a minimizer of the left-hand problem. Thus, one may take the position that the passage from the integrand $f$ to the integrand $\mathrm{Q}f$ is not only a (quasi)convexification procedure, but also the passage to a functional whose minimizers are more likely to exist. This is a very reasonable point of view, since quasiconvexity has been identified — in quite general situations; see [103, 99, 67, 48, 96, 1, 25, 7, 63, 64, 6, 87, 89], for instance — as the proper assumption in the existence theory for minimizers, and since the quasiconvex integral on the left-hand side of (3.38) can indeed be minimized in a suitable BV framework [7, 89]. Nevertheless, for our purposes it is not necessary to enter further into the quasiconvex analysis, since the duality theory needs, anyway, *some* true convexity hypothesis. Indeed, we will assume that at least

$$\text{the quasiconvexification } \mathrm{Q}f \text{ is convex in } z\,. \tag{3.39}$$

We emphasize that in the scalar case $N = 1$ (and also in the 1-dimensional case $n = 1$), where quasiconvexity is nothing but convexity, this assumption is tautologically satisfied, so that *in the scalar case we actually manage to work without any convexity assumption*. Back to the general case, (3.39) is equivalent to the requirement that $\mathrm{Q}f$ equals the convexification $f^{**}$ of $f$, and [49, Theorem 6.30] provides a criterion for rotationally symmetric integrands satisfying $\mathrm{Q}f = f^{**}$, but still (3.39) may, for $N > 1$, be considered as a somewhat artificial assumption.

Clearly, under the hypothesis (3.39), our duality results apply to the convex functional on the left-hand side of (3.38). However, we next observe that eventually these results carry over even to the (now possibly non-convex) functionals $F$ and $\overline{F}$. This is recorded in the following corollaries of Theorems 3.2 and 3.8, which are taken from [P3, Appendix A].

**Corollary 3.15** ($\mathrm{W}^{1,1}$ duality in some non-convex cases)**.** *Consider a bounded open set $\Omega$ in $\mathbb{R}^n$, a function $u_0 \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$, and a Carathéodory integrand $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ such that* (3.37) *and* (3.39) *hold. Then, for the functionals in* (3.1) *and* (3.3)*, we have the equality*

$$\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F = \max_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})} R_{u_0} \in [-\infty,\infty)\,.$$

*Moreover, for* $u \in \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$ *and* $\sigma \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega, \mathbb{R}^{N\times n})$*, we have*

$$\left.\begin{array}{r} u \textit{ minimizes } F \textit{ in } \mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)\,, \\ \sigma \textit{ maximizes } R_{u_0} \textit{ in } \mathrm{L}^\infty_{\mathrm{div}}(\Omega, \mathbb{R}^{N\times n}) \end{array}\right\} \iff \sigma \in \partial_z f(\,\cdot\,, \nabla u) \textit{ holds } \mathcal{L}^n\textit{-a.e. on } \Omega\,.$$

*Proof.* Applying Theorem 3.2 to the convex function $\mathrm{Q}f$ with $(\mathrm{Q}f)^* = f^{***} = f^*$, we find

$$\inf_{w \in \mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} \int_\Omega \mathrm{Q}f(\,\cdot\,, \nabla w)\,\mathrm{d}x = \max_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})} R_{u_0} \in [-\infty, \infty)\,.$$

Taking into account (3.38), we arrive at the claimed equality, and then the characterization of the minimizer-maximizer pair follows exactly as explained in the proof of Corollary 3.4. □

**Corollary 3.16** (BV duality in some non-convex cases)**.** *Consider a bounded open set* $\Omega$ *in* $\mathbb{R}^n$ *with* (3.17)*, a function* $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$*, and a Carathéodory integrand* $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ *such that* (3.37) *and* (3.39) *hold and such that the continuity condition* (3.24) *holds for* $\mathrm{Q}f$ *in place of* $f$*. Then, for the functionals in* (3.3) *and* (3.14)*, we have the equality*

$$\inf_{\mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)} \overline{F} = \max_{\mathrm{L}^\infty_{\mathrm{div}}(\Omega,\mathbb{R}^{N\times n})} R_{u_0} \in [-\infty, \infty)\,.$$

*Moreover, for* $u \in \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ *and* $\sigma \in \mathrm{L}^\infty_{\mathrm{div}}(\Omega, \mathbb{R}^{N\times n})$*, we have*

$$\left.\begin{array}{r} u \textit{ minimizes } \overline{F} \textit{ in } \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)\,, \\ \sigma \textit{ maximizes } R_{u_0} \textit{ in } \mathrm{L}^\infty_{\mathrm{div}}(\Omega, \mathbb{R}^{N\times n}) \end{array}\right\} \iff f(\,\cdot\,, \mathrm{D}u) = [\![\sigma, \mathrm{D}u]\!] - f^*(\,\cdot\,, \sigma)\mathcal{L}^n \llcorner \Omega \textit{ on } \overline{\Omega}\,.$$

In the formulation of Corollary 3.16, we have replaced the two separate extremality relations of Theorem 3.8 with the single equivalent relation for measures, which has been introduced in (3.26). Apart from conciseness there is, however, no specific reason for doing so, and indeed all the other forms of the extremality relations which have been discussed in connection with Corollary 3.4 and Theorem 3.8 can still be employed in the non-convex case with (3.39). The only small subtlety in this regard concerns the reformulated extremality relation $\nabla u \in \partial_{z^*} f^*(\,\cdot\,, \sigma)$, which is still necessary but not anymore sufficient for $F$- or $\overline{F}$-minimality of $u$. This results from the fact that we do not anymore have $f^{**} = f$ on $\Omega \times \mathbb{R}^{N\times n}$, but that at least $F$- or $\overline{F}$-minimality of $u$ and (3.39) imply the $\mathcal{L}^n$-a.e. equality $f^{**}(\,\cdot\,, \nabla u) = f(\,\cdot\,, \nabla u)$ on $\Omega$.

*Proof of Corollary* 3.16*.* Applying, in the first step, Theorem 2.9 with $\mathrm{Q}f$ in place of $f$, we obtain

$$\inf_{w \in \mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} \int_\Omega \mathrm{Q}f(\,\cdot\,, \nabla w)\,\mathrm{d}x = \inf_{w \in \mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)} \int_{\overline{\Omega}} \mathrm{Q}f(\,\cdot\,, \mathrm{D}w) \le \inf_{\mathrm{BV}_{u_0}(\overline{\Omega},\mathbb{R}^N)} \overline{F} \le \inf_{\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)} F\,.$$

In view of (3.38) it follows that all the infima in the preceding line are in fact equal. Particularly, we deduce the coincidence of the last two infima, which implies by Corollary 3.15 the claimed equality.

In addition, we also get the coincidence of the two BV infima, which we now exploit to prove the forwards implication of the claimed equivalence. Indeed, if $(u, \sigma)$ is a minimizer-maximizer pair for $\overline{F}$ and $R_{u_0}$, then, by the preceding observation and the general inequality $\mathrm{Q}f \leq f$, it is also a minimizer-maximizer pair for $w \mapsto \int_{\overline{\Omega}} \mathrm{Q}f(\,\cdot\,,\mathrm{D}w)$ and $R_{u_0}$, and $u$ satisfies the equality of measures $\mathrm{Q}f(\,\cdot\,,\mathrm{D}u) = f(\,\cdot\,,\mathrm{D}u)$ on $\overline{\Omega}$. In view of the assumption (3.39) and the equality $(\mathrm{Q}f)^* = f^*$, we can then apply Theorem 3.8 with $\mathrm{Q}f$ in place of $f$, and, formulating the conclusion in the style of (3.26), we obtain $\mathrm{Q}f(\,\cdot\,,\mathrm{D}u) = [\![\sigma,\mathrm{D}u]\!] - f^*(\,\cdot\,,\sigma)\mathcal{L}^n \llcorner \Omega$ on $\overline{\Omega}$. All in all, we thus arrive at $f(\,\cdot\,,\mathrm{D}u) = [\![\sigma,\mathrm{D}u]\!] - f^*(\,\cdot\,,\sigma)\mathcal{L}^n \llcorner \Omega$ on $\overline{\Omega}$.

To show the backwards implication, we first observe that, involving (3.34) with $\mathrm{Q}f$ in place of $f$, we have the general inequalities $f(\,\cdot\,,\mathrm{D}u) \geq \mathrm{Q}f(\,\cdot\,,\mathrm{D}u) \geq [\![\sigma,\mathrm{D}u]\!] - f^*(\,\cdot\,,\sigma)\mathcal{L}^n \llcorner \Omega$ on $\overline{\Omega}$. If we assume $f(\,\cdot\,,\mathrm{D}u) = [\![\sigma,\mathrm{D}u]\!] - f^*(\,\cdot\,,\sigma)\mathcal{L}^n \llcorner \Omega$ on $\overline{\Omega}$, we can thus deduce that all three measures in the preceding chain of inequalities coincide on $\overline{\Omega}$. Applying, once more, Theorem 3.8 with $\mathrm{Q}f$ in place of $f$, we find that $(u,\sigma)$ is a minimizer-maximizer pair for $w \mapsto \int_{\overline{\Omega}} \mathrm{Q}f(\,\cdot\,,\mathrm{D}w)$ and $R_{u_0}$. Since we have already reasoned that $\mathrm{Q}f(\,\cdot\,,\mathrm{D}u)$ equals $f(\,\cdot\,,\mathrm{D}u)$ on $\overline{\Omega}$, we can finally conclude that $(u,\sigma)$ is a minimizer-maximizer pair for $F$ and $R_{u_0}$. This completes the proof of Corollary 3.16. □

Finally, we remark that also Corollary 3.9 and Theorem 3.11 can be extended to the case of non-convex integrands $f$ with (3.39). Indeed, the uniqueness assertion for $\sigma$ from Corollary 3.9 remains valid whenever, in addition to the hypotheses of the preceding Corollary 3.16, $\mathrm{Q}f = f^{**}$ is $\mathrm{C}^1$ in $z$, and criteria for this differentiability property of the (quasi)convexification can in turn be found in [77, 26, 24, 84]. Moreover, the conclusion of Theorem 3.11 remains valid under precisely the assumptions of the preceding Corollary 3.16.

# Chapter 4

# Everywhere regularity and uniqueness of BV minimizers

This chapter revisits the everywhere regularity and uniqueness results of [P1, P5], which generalize previous, closely related considerations of Bildhauer & Fuchs [33], Bildhauer [29, 31, 30], and Marcellini & Papi [95]. The results presented here have been adapted, whenever possible, to the setting of the previous chapters with possibly non-smooth domains and integrands, and thus some statements have not been recorded in the same form before.

In contrast to the other chapters, however, we here restrict the exposition to (local) BV minimizers of functionals *without explicit $x$-dependence*. Indeed, this restriction could partially be avoided, since regularity results for BV minimizers in *some* $x$-dependent framework have been obtained in [35, Section 3]; see also [30, Section 4.2.2.2] and [P1, Appendix C]. However, up to now it has not been possible to avoid the unsatisfactory hypothesis[1] that the $x$-dependence is Lipschitz or even better. We thus prefer not to enter into the technical details of the $x$-dependent case.

Concretely, the integrands in this chapter are continuous functions $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ which satisfy

$$\limsup_{|z|\to\infty} \frac{|f(z)|}{|z|} < \infty\,, \tag{4.1}$$

The linear growth condition (4.1) clearly implies boundedness on $\mathbb{R}^{N\times n}$ of the function $|\partial f|$ from Definition 3.3, and we can then introduce the oscillation $\Gamma_f$ of the subdifferential $\partial f$ as the finite diameter of the set $\operatorname{Im}\partial f$ in (3.36), for short

$$\Gamma_f := \operatorname{diam}(\operatorname{Im}\partial f) < \infty\,. \tag{4.2}$$

For such an integrand $f$ and for a function $w \in \mathrm{BV}_{\mathrm{loc}}(U,\mathbb{R}^N)$, defined on an open neighborhood $U$ of a Borel set $A$ in $\mathbb{R}^n$, our functionals are now given by

$$\overline{F}[w;A] := \int_A f(\mathrm{D}w) \in \mathbb{R} \qquad \text{whenever } \mathcal{L}^n(A) + |\mathrm{D}w|(A) < \infty\,. \tag{4.3}$$

[1]The Lipschitz hypothesis is relevant in order to obtain second derivative estimates by differentiation of the Euler equation. The case of a merely Hölder continuous dependence, instead, does not allow to proceed in this way and has, due to a lack of suitable freezing techniques, remained inaccessible.

Here the integral is understood in the sense of Definition 2.1, and we record, as a side remark for later use, that we can admit $\mathcal{L}^n(A) = \infty$ in (4.3) in case that $f$ is convex with $f(0) = 0$, because we then have $|f(z)| \le \widetilde{\Gamma}|z|$ for all $z \in \mathbb{R}^{N\times n}$, with the constant $\widetilde{\Gamma} := |\partial f|(0) + \Gamma_f < \infty$.

Since most of this chapter is concerned with local properties in the interior of an open set $\Omega \subset \mathbb{R}^n$, it is convenient to formulate the statements independent of boundary values. This is achieved by employing the following notion of local BV minimizers of $\overline{F}$.

**Definition 4.1** (local BV minimizers). *Suppose that* $\Omega$ *is open in* $\mathbb{R}^n$ *and that* $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ *is continuous with* (4.1). *Then we call* $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *a local minimizer of the functional* $\overline{F}$ *in* (4.3) *on* $\Omega$ *if there holds*

$$\overline{F}[u; \mathrm{B}_r(x_0)] \le \overline{F}[u+\varphi; \mathrm{B}_r(x_0)]$$

*for all balls* $\mathrm{B}_r(x_0) \Subset \Omega$ *and all* $\varphi \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ *with* $\operatorname{spt}\varphi \subset \mathrm{B}_r(x_0)$.

## 4.1 L$^\infty$ estimates (allowing for non-differentiable cases)

In this section we deal with the first (and simplest) regularity property of (local) BV minimizers, namely with interior L$^\infty$ estimates. To this end we impose the following additional hypothesis on the integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$. We assume the existence of some $z_0^* \in \mathbb{R}^{N\times n}$ such that we have[2]

$$\gamma_f := \liminf_{|z|\to\infty} \frac{f(z) - z_0^* \cdot z}{|z|} > 0\,, \tag{4.4}$$

$$\lambda_f := \sup\{\xi^{\mathrm{T}}(z_0^* - z^*) z^{\mathrm{T}}\xi \,:\, z \in \mathbb{R}^{N\times n},\, z^* \in \partial f(z),\, \xi \in \mathbb{R}^N,\, |\xi| = 1\} < \infty\,. \tag{4.5}$$

Here, the linear coercivity condition (4.4) resembles the one from the existence result in Theorem 2.8, and is, for lower semicontinuous $f$, equivalent with the requirement that $z_0^*$ is an interior point of $\operatorname{Im}\partial f$. The choice of $\gamma_f$ means that $z_0^*$ is at distance $\gamma_f$ from the boundary of $\operatorname{Im}\partial f$, and thus, when also (4.1) is in force, we clearly have $\gamma_f \le \frac{1}{2}\Gamma_f < \infty$. Moreover, since $f^*$ is continuous and $[-f(0), \infty)$-valued in the interior of $\operatorname{Im}\partial f$, we can set

$$\kappa_f := f(0) + \sup_{\mathrm{B}_{\gamma_f/2}(z_0^*)} f^* \in [0, \infty)\,.$$

We record that $\Gamma_f$, $\gamma_f$, $\lambda_f$, and $\kappa_f$ are invariant under addition of an affine function to $f$ (provided that, in case of $\gamma_f$, $\lambda_f$, $\kappa_f$, we correspondingly modify $z_0^*$). This makes these quantities very convenient in order to state our estimates, since also (local) minimality is invariant under addition of an affine function to the integrand.

Before commenting in more detail on the above assumptions and specifically on (4.5), we next state the L$^\infty$ estimates, which extend [P1, Theorem 1.11] to a class of non-differentiable integrands $f$ including the total variation integrand $\mathrm{m}_1$ (for which every $z_0^* \in \mathrm{B}_1$ is admissible with $\Gamma_{\mathrm{m}_1} = 2$, $\gamma_{\mathrm{m}_1} = 1 - |z_0^*|$, $\lambda_{\mathrm{m}_1} = \kappa_{\mathrm{m}_1} = 0$). For $N = 1$ and the regularized total variation integrand $f = \mathrm{m}_{1,2}$ in the later formula (5.9), similar L$^\infty$ estimates for BV minimizers have previously been obtained by Hardt & Kinderlehrer [81, Theorem 2.2]. The method of proof in [81], however, differs from the one described below.

[2] We use $(\,\cdot\,)^{\mathrm{T}}$ for the transposition of vectors and matrices.

**Theorem 4.2** (interior $\mathrm{L}^\infty$ estimates for BV minimizers)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$, a convex integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$, and some $z_0 \in \mathbb{R}^{N\times n}$ such that (4.1), (4.4), and (4.5) hold with $\Gamma_f \le \Gamma < \infty$ and $\gamma_f \ge \gamma > 0$. Then every local minimizer $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ of $\overline{F}$ on $\Omega$ satisfies*

$$u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\,,$$

*and, for all concentric balls $\mathrm{B}_r(x_0) \Subset \mathrm{B}_R(x_0) \Subset \Omega$, we have the estimate*[3]

$$\sup_{\mathrm{B}_r(x_0)} |u| \overset{n,\gamma,\Gamma}{\lesssim} \frac{1}{(R{-}r)^n}\int_{\mathrm{B}_R(x_0)} |u|\,\mathrm{d}x + \frac{R^n}{(R{-}r)^{n-1}}(\lambda_f{+}\kappa_f)\,. \tag{4.6}$$

In order to illustrate the assumptions of Theorem 4.2 and to prepare its proof, we next show that (4.4) induces the quantitative lower bounds

$$f(z) - f(0) - z_0^* \cdot z \ge \frac{\gamma_f}{2}|z| - \kappa_f \qquad \text{for all } z \in \mathbb{R}^{N\times n}\,, \tag{4.7}$$

$$(z^*{-}z_0^*)\cdot z \ge \frac{\gamma_f}{2}|z| - \kappa_f \qquad \text{for all } z \in \mathbb{R}^{N\times n} \text{ and } z^* \in \partial f(z)\,. \tag{4.8}$$

Indeed (4.7) and (4.8) are quickly verified by employing the definitions of $\partial f$, $f^*$, and $\kappa_f$ in the estimate $z^*\cdot z \ge f(z) - f(0) \ge \big(z_0^* + \frac{\gamma_f}{2}\frac{z}{|z|}\big)\cdot z - f^*\big(z_0^* + \frac{\gamma_f}{2}\frac{z}{|z|}\big) - f(0) \ge z_0^*\cdot z + \frac{\gamma_f}{2}|z| - \kappa_f$ for $z \ne 0$.

In particular, (4.8) shows that, in the scalar case $N=1$, assumption (4.5) with $\lambda_f \le \kappa_f$ follows automatically from (4.4) and can therefore be dropped. We thus recognize (4.5) as a structure condition which is specific for the vectorial case $N \ge 2$. Indeed, by a well-known series of counterexamples [51, 74, 107, 79, 130], regularity results in the case $n \ge 3$, $N \ge 2$ cannot hold without some additional structural hypothesis, and the most recent work [131] of Šverák & Yan demonstrates the necessity of such an hypothesis even for $\mathrm{L}^\infty$ estimates. Reversely, a first sufficient hypothesis in this connection has been identified by Uhlenbeck [129], who exploited the rotationally symmetric structure $f(z) = \tilde{f}(|z|)$ in case of the vectorial $p$-energy and established interior $\mathrm{C}^{1,\alpha}$ regularity for the minimizers. Also Theorem 4.2 includes rotationally symmetric integrands $f$ if they are only non-constant and convex with (4.1), as in this case (4.4) and (4.5) are automatically satisfied with $z_0^* = 0$ and $\lambda_f = 0$. However, since we are only interested in $\mathrm{L}^\infty$ instead of $\mathrm{C}^{1,\alpha}$ estimates, we can weaken rotational symmetry to the one-sided condition (4.5), which is inspired by Meier's indicator function from [98] and is also satisfied with $z_0^* = 0$ and $\lambda_f = 0$ if a convex $f$ takes the form $f(z) = \tilde{f}(|z_1|,|z_2|,\ldots,|z_n|)$ with the columns $z_i \in \mathbb{R}^N$ of $z \in \mathbb{R}^{N\times n}$. In the sequel, (4.5) is used in form of the estimate

$$\xi^{\mathrm{T}}(z^*{-}z_0^*)z^{\mathrm{T}}\xi \ge -\lambda_f|\xi|^2 \qquad \text{for all } z \in \mathbb{R}^{N\times n},\ z^* \in \partial f(z), \text{ and } \xi \in \mathbb{R}^N\,. \tag{4.9}$$

The *proof of Theorem* 4.2 follows essentially the lines of [P1, Section 4] and is briefly sketched now. Since a local minimizer of $\overline{F}$ on $\Omega$ solves a Dirichlet problem[4] on every ball $\mathrm{B}_r(x_0) \Subset \Omega$, we

[3] The symbol $\overset{n,\gamma,\Gamma}{\lesssim}$ indicates an estimate up to a multiplicative positive constant which depends only on $n$, $\gamma$, and $\Gamma$. An analogous notation is widely used in the sequel.

[4] More precisely, a local minimizer of $\overline{F}$ on $\Omega$ minimizes also $\overline{F}\big[\,\cdot\,;\mathrm{B}_r(x_0)\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\mathrm{B}_r(x_0)},\mathbb{R}^N\big)$ whenever the trace of $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$ and the exterior trace of the local minimizer coincide on the boundary of the ball $\mathrm{B}_r(x_0) \Subset \Omega$. Gagliardo's characterization of traces [68, Teorema 1.II] guarantees that a suitable $u_0$ exists.

can assume in the following that $\Omega$ is smooth and that $u$ minimizes $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ for some $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$. Then we can approximate $u$ by the sequence $(w_k)_{k\in\mathbb{N}}$ of Theorem 2.9. Arguing via Ekeland's principle as for (3.29) and handling the non-differentiability as explained around (3.32), it is then possible to improve the situation as follows. We can find a minimizing sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$, a null sequence $(\delta_k)_{k\in\mathbb{N}}$ in $(0,\infty)$, and a sequence $(\tau_k)_{k\in\mathbb{N}}$ in $\mathrm{L}^\infty(\Omega,\mathbb{R}^{N\times n})$ such that $v_k$ converges to $u$ in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$, such that $\tau_k \in \partial_z f(\,\cdot\,,\nabla v_k)$ holds $\mathcal{L}^n$-a.e. on $\Omega$, and such that we have the perturbed Euler equation

$$\bigg|\int_\Omega \tau_k\cdot\nabla\varphi\,\mathrm{d}x\bigg| \le \delta_k\|\nabla\varphi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})} \qquad \text{for all } \varphi\in\mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)\,. \tag{4.10}$$

We exploit this information in order to implement, in a more or less standard fashion, a version of Moser's iteration scheme [104, 105] as follows. As test functions $\varphi$ we insert (suitable truncations of) $\eta^s|v_k|^{t-1}v_k$ with parameters $s,t\ge 1$ and non-negative cut-off functions $\eta\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$. We then rely on the lower bounds (4.8), (4.9) and the upper bound $|\partial_z f|\le\Gamma_f$ in explicit computations and estimations, which are completely analogous to the ones carried out for differentiable $f$ in [P3, Lemma 4.1, Lemma 4.2] and which we do not repeat here. Involving also the Sobolev embedding and assuming that the perturbation term is controlled by $\delta_k\le\frac{\gamma_f}{2t}$, the outcome are estimates for the $\mathrm{L}^{\frac{n}{n-1}t}$-norm of $v_k$ on a ball in terms of its $\mathrm{L}^t$-norm on an enlarged ball. However, the requirement $\delta_k\le\frac{\gamma_f}{2t}$ is always valid for $k\ge k_0(t)$, and thus, after finite iteration and passage to the limit $k\to\infty$, we can estimate the $\mathrm{L}^q$-norm of $u$ on $\mathrm{B}_r(x_0)$, for all $q<\infty$, in terms of its $\mathrm{L}^1$-norm on $\mathrm{B}_R(x_0)$. Finally, it can be checked that the constants in these estimates remain bounded in the limit $q\to\infty$, and we can deduce the claimed control on the $\mathrm{L}^\infty$-norm of $u$ on $\mathrm{B}_r(x_0)$. □

Next we comment on variants of both Theorem 4.2 and its proof.

First of all we point out that once more, at the cost of loosing the corresponding existence result, the convexity assumption on $f$ can be avoided. Indeed, the following generalization in the spirit of Section 3.5 is a corollary to Theorem 4.2: If $f$ is just continuous with (4.1), (4.4), (4.5) and its quasiconvexification $\mathrm{Q}f$ is convex, then the estimate of Theorem 4.2 still holds for local minimizers of $\overline{F}$. This statement simply follows by checking that (4.1), (4.4), (4.5) hold also for $\mathrm{Q}f$ in place of $f$, by applying Theorem 4.2 with $\mathrm{Q}f$ in place of $f$, and by observing finally that local minimizers of $\overline{F}$ are also local minimizers of $[w;A]\mapsto\int_A \mathrm{Q}f(\mathrm{D}w)$. However, as already pointed out in [P1, Remark 1.12], for differentiable integrands $f\in\mathrm{C}^1(\mathbb{R}^{N\times n})$, one can even go beyond this, and also convexity of $\mathrm{Q}f$ is dispensable. In fact, we have the following modified statement (only relevant for $N\ge 2$, of course): As soon as the continuity hypothesis (2.8) (which was automatic in the convex case) is ensured and suitable adaptions of (4.2), (4.8), (4.9) (essentially with $\{\nabla f\}$ in place of $\partial f$) are assumed, the conclusion of Theorem 4.2 remains valid under only these assumptions on $f\in\mathrm{C}^1(\mathbb{R}^{N\times n})$.

An alternative way to approach the $\mathrm{L}^\infty$ estimates of Theorem 4.2, say, for simplicity, for a bounded Lipschitz domain $\Omega$ and $u_0\in\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$, is to choose the $v_k$ as minimizers of suitable regularized functionals like

$$u_0+\mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)\to\mathbb{R}\,,\; w\mapsto\int_\Omega\big[f(\nabla w)+k^{-1}|\nabla w_k|^2\big]\mathrm{d}x\,. \tag{4.11}$$

On one hand, an advantage of this approach over the Ekeland one is that each $v_k$ satisfies an Euler equation without a perturbation term like the one on the right-hand side of (4.10), but rather with an additional 'good' term with prefactor $k^{-1}$ coming from the quadratic part $w \mapsto k^{-1} \int_\Omega |\nabla w_k|^2 \,\mathrm{d}x$ of the functional. In contrast to the above reasoning, when one also tracks the additional $k^{-1}$ terms throughout the iteration procedure, one can then prove that the $v_k$ are uniformly bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and indeed we will make use of this fact in the following Section 4.2. On the other hand, the usage of the regularizations in (4.11) brings also a disadvantage: This approach does work with an *a priori given* minimizer $u$, but rather one gets *some* $v_k$ and then *some* BV minimizer $u$ as their weak-$*$ limit in $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$. Therefore, this strategy establishes, in (quasi)convex cases, the existence of *one* $\mathrm{L}^\infty_{\mathrm{loc}}$ minimizer corresponding to every given boundary datum $u_0$, but, without an uniqueness result at hand, it does *not* yield $\mathrm{L}^\infty_{\mathrm{loc}}$ estimates for *all* minimizers. Eventually, in Section 4.3 below, we overcome this point by combining the Ekeland and the regularization approach in order to choose approximations $v_k$ with the advantages of both strategies.

Last but not least, we mention that, in many regards, maximum (or convex hull) principles can serve as a replacement for interior $\mathrm{L}^\infty$ estimates. Like $\mathrm{L}^\infty$ estimates, also maximum principles extend to the vectorial case $N \geq 2$ only under additional structural assumptions; compare with [54, 34], for instance. In a BV framework, the usage of maximum principles has been discussed in [P1, Appendix D] with the tentative conclusion that they require slightly stronger hypotheses than our condition (4.5). However, more importantly, an advantage of maximum principles is that they are often simpler to establish than $\mathrm{L}^\infty$ estimates, while the main disadvantage is the imposition of an additional boundedness assumption on $u_0$.

## 4.2 $\mathrm{W}^{1,\infty}$ estimates for asymptotically $\mu$-elliptic problems

In this section we turn to gradient estimates for BV minimizers. The achievement of such estimates generally requires stronger assumptions on the integrand $f$, namely strict positivity hypotheses and bounds for the second derivatives $\nabla^2 f$. Concretely, our assumptions are similar to the $\mu$-ellipticity condition of Bildhauer & Fuchs [33][5]

$$\frac{\gamma\, \mathrm{I}_{N\times n}}{(1+|z|)^\mu} \leq \nabla^2 f(z) \leq \frac{\Gamma\, \mathrm{I}_{N\times n}}{1+|z|} \qquad \text{for all } z \in \mathbb{R}^{N\times n}\,, \tag{4.12}$$

with fixed constants $0 < \gamma \leq \Gamma < \infty$ and $1 < \mu \leq 3$. A peculiarity of the present setting lies in the fact that one cannot take $\mu = 1$ in (4.12), as there exists[6] no integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ which satisfies (4.1) and (4.12) with $\mu = 1$ and $\gamma > 0$. Therefore, in the linear growth case one is generally faced with a degeneration of the ellipticity ratio for $|z| \to \infty$ — as already illustrated by means of the model integrands $\mathrm{m}_p$ in the introduction. In the case $\mu < 1 + 2/n$,

[5] Here, we understand $\nabla^2 f(z)$ as a bilinear form on $\mathbb{R}^{N\times n}$, the symbol $\mathrm{I}_{N\times n}$ denotes the Euclidean inner product on $\mathbb{R}^{N\times n}$, and an inequality $\mathcal{B}_1 \leq \mathcal{B}_2$ between bilinear forms $\mathcal{B}_1$ and $\mathcal{B}_2$ on $\mathbb{R}^{N\times n}$ evidently means $\mathcal{B}_1(\xi,\xi) \leq \mathcal{B}_2(\xi,\xi)$ for all $\xi \in \mathbb{R}^{N\times n}$.

[6] Indeed, one can see by integration that the lower bound in (4.12) with $\mu = 1$ requires $f(z)$ to grow at least as fast as $|z| \log |z|$.

Bildhauer & Fuchs [33] were able to control such degeneration effects via second derivative estimates, and they obtained interior $\mathrm{C}^{1,\alpha}$ regularity for BV minimizers under the assumptions (4.1) and (4.12) on the integrand. In the two-dimensional case $n = 2$ and in cases with $\mathrm{L}^\infty$ estimates for minimizers (for instance, the ones of the previous section suffice), Bildhauer [29, 31] has eventually reached the improved bound $\mu < 3$, which corresponds to a subquadratic blow-up of the ellipticity ratio and seems very natural. Actually, it has been shown in [72, Section 3] and [30, Section 4.4] that the assumption $\mu < 3$ has some optimality property, and moreover it is also strongly reminiscent of classical bounds for the Bernstein genre, which occur in Serrin's extensive treatise [121] on quasilinear elliptic equations. Unfortunately, the results of [33, 29, 31] do not directly apply to the integrands $\mathrm{m}_p$ with $p \neq 2$, since the degenerate or singular behavior of $\nabla^2\mathrm{m}_p(z)$ for $|z| \to 0$ is not included in (4.12), but a way to incorporate $\mathrm{m}_p$ consists in imposing (4.12) only for large values of $|z|$. This ties in with the known fact that such asymptotic requirements on $f$ suffice in many cases to guarantee Lipschitz regularity of minimizers; see [46, 70, 93, 111, 66] and several more recent papers. Specifically, among the asymptotic regularity results in the vectorial case $N \geq 2$ we mention the one of Cupini & Guidorzi & Mascolo [47, Theorem 1.1] for integrands of superlinear $(p, q)$-growth and the one of Marcellini & Papi [95, Theorem B], which includes linear growth cases with $\mu < 1 + 2/n$ but considers only minimizers which are a priori in $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$. In order to deal with BV minimizers of the model integrals $\overline{\mathrm{M}}_p$, we now report on an analogue of these results which hypothesizes the following asymptotic version of (4.12), with positive parameters $\mu$, $R$, $\gamma$, and $\Gamma$:

$$f \in \mathrm{W}^{2,\infty}_{\mathrm{loc}}\left(\mathbb{R}^{N\times n} \setminus \overline{\mathrm{B}_R}\right), \tag{4.13}$$

$$f \text{ is rotationally symmetric on } \mathbb{R}^{N\times n} \setminus \mathrm{B}_R\,, \tag{4.14}$$

$$\frac{\gamma\, \mathrm{I}_{N\times n}}{|z|^\mu} \leq \nabla^2 f(z) \leq \frac{\Gamma\, \mathrm{I}_{N\times n}}{|z|} \text{ holds for } \mathcal{L}^{N\times n}\text{-a.e. } z \in \mathbb{R}^{N\times n} \setminus \mathrm{B}_R\,. \tag{4.15}$$

We remark that these assumption imply, in particular, the validity of (4.4) and (4.5) with $z_0^* = 0$ and $\lambda_f \leq \Gamma_f R$, and we proceed to a restatement of [P5, Theorem 1.2] in a slightly modified framework. To this end, we rely once more on the mild regularity hypothesis

$$\mathbb{1}_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } |\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega\,. \tag{4.16}$$

of Appendix A.

**Theorem 4.3** (interior $\mathrm{W}^{1,\infty}$ estimates for one BV minimizer)**.** *Suppose that $\Omega$ is a bounded open set in $\mathbb{R}^n$ with (4.16), that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is convex, and that (4.1), (4.13), (4.14), and (4.15) hold for a positive radius $R$, an exponent*

$$1 < \mu < 3\,,$$

*and constants $0 < \gamma \leq \gamma_f \leq \Gamma_f \leq \Gamma < \infty$. Then, for every $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n, \mathbb{R}^N)$, there exists a minimizer $u$ of $\overline{F}\big[\,\cdot\,; \overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ such that we have*

$$u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$$

*and*

$$|\nabla u(x_0)| \overset{n,\mu,R,\gamma,\Gamma}{\lesssim} \left(1+\operatorname{dist}(x_0,\partial\Omega)^{-n}\int_\Omega |\nabla u_0|\,\mathrm{d}x\right)^{1+\frac{3(\mu-1)}{2(3-\mu)}} \qquad \textit{for } \mathcal{L}^n\textit{-a.e. } x_0\in\Omega\,.$$

By explicit computations, one verifies that the integrands $\mathrm{m}_p$ with $p>1$ satisfy the assumptions (4.13), (4.14), (4.15) with $\mu=1+p$. Therefore, the model integrals $\overline{\mathrm{M}}_p$ with $1<p<2$ are included in Theorem 4.3.

Next we sketch the *proof of Theorem* 4.3. To this end, we follow the reasoning of [P5, Section 4], which adapts the approach of Bildhauer & Fuchs [33] to the present setting.

By a mollification procedure, one can reduce to the case that we have $u_0\in\mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)$, and in the following we only deal with this slightly simplified situation. Following essentially the regularization strategy described at the end of Section 4.1, we then work, for $k\in\mathbb{N}$, with the unique minimizers $v_k$ of the strictly convex functionals

$$F_k[w;\Omega]:=\int_\Omega \big[f_{k^{-1}}(\nabla w)+k^{-1}|\nabla w|^2\big]\mathrm{d}x \qquad \text{for } w\in u_0{+}W^{1,2}_0(\Omega,\mathbb{R}^N)\,, \tag{4.17}$$

where $f_{k^{-1}}$ stands for a standard mollification of $f$ with mollification radius $k^{-1}$. It is straightforward to show that $(v_k)_{k\in\mathbb{N}}$ is a minimizing sequence for $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ in $u_0+\mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$, and by density the same is true in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$. Moreover, a subsequence of $(v_k)_{k\in\mathbb{N}}$ weakly-$*$ converges in $\mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$ to a limit $u\in\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$, and via Corollary 2.3 and Theorem 2.9 it then follows that the limit $u$ is a minimizer of $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ even in $\mathrm{BV}\big(\overline{\Omega},\mathbb{R}^N\big)$. As already pointed out, as the main advantage of the regularization approach, we can use the corresponding Euler equation

$$\int_\Omega \big[\nabla f_{k^{-1}}(\nabla v_k)\cdot\nabla\varphi+2k^{-1}\nabla v_k\cdot\nabla\varphi\big]\mathrm{d}x=0 \qquad \text{for all } \varphi\in\mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N) \tag{4.18}$$

to obtain uniform $\mathrm{L}^\infty$ estimates for the whole sequence $(v_k)_{k\in\mathbb{N}}$. Indeed, using the same test functions as in the proof of Theorem 4.2 and implementing an analogous Moser type iteration scheme, we find an analogue of (4.6) for each $v_k$. The only difference lies in an additional $\mathrm{L}^2$ term on the right-hand side, and concretely, after subtraction of means and usage of Poincaré's inequality, these $\mathrm{L}^\infty$ estimates read as[7]

$$\sup_{\mathrm{B}_{2r}(x_0)}\frac{|v_k-(v_k)_{\mathrm{B}_{2r}(x_0)}|}{r} \overset{n,\mu,R,\gamma,\Gamma}{\lesssim} \fint_{\mathrm{B}_{4r}(x_0)}|\nabla v_k|\,\mathrm{d}x+k^{-1}\fint_{\mathrm{B}_{4r}(x_0)}|\nabla v_k|^2\,\mathrm{d}x+1 \tag{4.19}$$

on every ball $\mathrm{B}_{4r}(x_0)\Subset\Omega$. Here, taking into account the minimality property $F_k[v_k;\Omega]\le F_k[u_0;\Omega]$, the right-hand side of (4.19) remains bounded for $k\to\infty$. Next we make use of the fact that, thanks to the quadratic regularization term in (4.17), a standard argument with

[7] One may wonder why the quantity $\kappa_f$, involved in (4.8), occurs in (4.6) but not in (4.19). The reason for this is that, under the present stronger assumptions, $\kappa_f$ can be controlled in terms of $\mu$, $R$, $\gamma$, and $\Gamma$; compare [P5, Lemma 2.4]

difference quotients shows that the $v_k$ posses second derivatives in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$. By differentiation of the Euler equation we then obtain, for every $i \in \{1,2,\ldots,n\}$,

$$\int_\Omega \big[\nabla^2 f_{k^{-1}}(\nabla v_k)(\partial_i \nabla v_k, \nabla\psi) + 2k^{-1}\partial_i \nabla v_k \cdot \nabla\psi\big]\,\mathrm{d}x = 0 \qquad \text{for all } \psi \in \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)\,. \tag{4.20}$$

As test functions $\psi$ in (4.20) we use (truncations of) $T(|\nabla v_k|)\eta^2\partial_i v_k$, where $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ is arbitrary and where $T \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is non-negative and non-decreasing on $[0,\infty)$ and constant on $[0,R]$. Exploiting the structure assumption (4.14) and the growth condition (4.15), we then arrive at the general Caccioppoli type inequality

$$\int_\Omega \eta^2(|\nabla v_k|^{-\mu}+k^{-1})|\nabla^2 v_k|^2 T(|\nabla v_k|)\,\mathrm{d}x \overset{\gamma,\Gamma}{\lesssim} \int_\Omega |\nabla\eta|^2(|\nabla v_k|^{-1}+k^{-1})|\nabla v_k|^2 T(|\nabla v_k|)\,\mathrm{d}x\,. \tag{4.21}$$

In the next step, we return to the Euler equation (4.18) and choose the test function $\varphi = \chi^{\frac{2p}{3-\mu}}(|\nabla v_k|-R-2)^p_+(v_k-(v_k)_{\mathrm{B}_{2r}(x_0)})$, where $R$ and $\mu$ denote the parameters from the assumptions of the theorem, and where we have moreover fixed $p \in (1,\infty)$, a ball $\mathrm{B}_{4r}(x_0) \Subset \Omega$, and a cut-off function $\chi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\mathbb{1}_{\mathrm{B}_r(x_0)} \le \chi \le \mathbb{1}_{\mathrm{B}_{2r}(x_0)}$ and $|\nabla\chi| \le 2/r$ on $\Omega$. The result of the testing can be combined, in a longer series of estimations, with both (4.19) and (4.21), the latter applied with the choices $\eta = \chi^{\frac{p}{3-\mu}+\frac12}$ and $T(\tau) := (\tau-R-2)^{p-1}_+\tau^{\frac{\mu-1}{2}}$. After all, this leads to the $\mathrm{W}^{1,p}$ estimate of [P5, Lemma 4.3], which asserts

$$\fint_{\mathrm{B}_r(x_0)} |\nabla v_k|^p \overset{n,\mu,R,\gamma,\Gamma,p}{\lesssim} \left(\fint_{\mathrm{B}_{4r}(x_0)} |\nabla v_k|\,\mathrm{d}x + k^{-1}\fint_{\mathrm{B}_{4r}(x_0)} |\nabla v_k|^2\,\mathrm{d}x + 1\right)^{1+\frac{2}{3-\mu}(p-1)} \tag{4.22}$$

for every ball $\mathrm{B}_{4r}(x_0) \Subset \Omega$. The last estimate shows, in particular, $\nabla u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$ for all $p \in [1,\infty)$, but, as the constant depends on $p$, we cannot pass to the limit to conclude $\nabla u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$. Indeed, it seems that even somewhat modified estimates (with constants whose dependence on $p$ is carefully tracked) can only be iterated finitely many times and that one cannot implement a Moser type argument here. Instead, it seems necessary to involve a different technique, and the reasoning in [P5] relies in fact on (a slight adaption of) the De Giorgi approach to $\mathrm{L}^\infty$ estimates. More precisely, we make use of the following general lemma, which is a restatement of [P5, Lemma 4.4].

**Lemma 4.4** (De Giorgi type lemma with weights)**.** *Consider non-negative exponents $\theta$ and $\sigma$ with $\theta+\sigma \ge 4$, an open set $\Omega$ in $\mathbb{R}^n$, and a function $w \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^m) \cap \mathrm{L}^p(\Omega,\mathbb{R}^m)$ with $p > (\theta+\sigma)n/2$. If $w$ satisfies, for some constant $C \ge 1$, all levels $\ell \ge \ell_0 \ge 1$, and all $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, the Caccioppoli type inequality on superlevel sets*

$$\int_\Omega \eta^2(|w|-\ell)^2_+|w|^{-\theta}|\nabla w|^2\,\mathrm{d}x \le C\int_\Omega |\nabla\eta|^2(|w|-\ell)^2_+|w|^{\sigma-2}\,\mathrm{d}x\,, \tag{4.23}$$

*then we have*

$$w \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^m)\,,$$

*and, for all balls $\mathrm{B}_r(x_0) \Subset \Omega$, we can estimate*

$$\sup_{\mathrm{B}_{r/2}(x_0)} |w| - \ell_0 \overset{n,\theta,\sigma,p,C}{\lesssim} \left(\fint_{\mathrm{B}_r(x_0)} |w|^p\,\mathrm{d}x\right)^{\frac{\theta+\sigma}{4p}}\,. \tag{4.24}$$

We do not discuss Lemma 4.4 and its proof (which follows essentially the classical strategy of De Giorgi) in detail here. However, even though we use the lemma only for $\theta + \sigma > 4$, we briefly remark that, in the case $\theta+\sigma = 4$, in which (4.23) becomes 'homogeneous', the exponent $\frac{\theta+\sigma}{4p}$ on the right-hand side of (4.24) consistently simplifies to $\frac{1}{p}$.

In order to conclude the proof of Theorem 4.3, we now aim at applying the lemma with the choices $w := \nabla u$, $\theta = \mu$, $\sigma \in \{3, 4\}$, $\ell_0 = R + 2$, and an arbitrary, but now fixed $p = p(n, \mu) > (\mu + 4)n/2$. The hypothesized inequality (4.23) (first only with $\sigma = 4$, but in a second step and for sufficiently large $k$, also with the better exponent $\sigma = 3$) is then available as a consequence of the Caccioppoli type inequality (4.21), applied for $T(\tau) = (\tau - \ell)_+^2$ with $\ell \geq R + 2$. Consequently, the lemma yields (4.24) for the above choice of quantities, and by rewriting this inequality and taking into account the $\mathrm{W}^{1,p}$ bound (4.22), we arrive at

$$\limsup_{k\to\infty} \sup_{\mathrm{B}_{r/2}(x_0)} |\nabla v_k| \overset{n,\mu,R,\gamma,\Gamma}{\lesssim} \left(1 + \limsup_{k\to\infty} \fint_{\mathrm{B}_{4r}(x_0)} |\nabla v_k| \,\mathrm{d}x\right)^{1+\frac{3(\mu-1)}{2(3-\mu)}} \tag{4.25}$$

for all balls $\mathrm{B}_{4r}(x_0) \Subset \Omega$. Finally, we recall that $(v_k)_{k\in\mathbb{N}}$ is a minimizing sequence, which weakly-$*$ converges to the BV minimizer $u$. Relying on these facts it is now straightforward to bound the right-hand term in (4.25) in terms of the boundary values $u_0$ and to deduce the claims of the theorem. $\square$

## 4.3 $\mathbf{C}^{1,\alpha}$ regularity for some singular problems

Our next goal are (Hölder) continuity estimates for the gradient of (local) BV minimizers. To this end, we require the following set of conditions, which is tailored out for the case of the model integrands $\mathrm{m}_p$ with $1 < p < 2$ and incorporates their singular behavior for $|z| \to 0$. Precisely, for positive parameters $\mu$, $q$, $\beta$ and a non-decreasing function $\Xi$, we assume:

$$f \in \mathrm{C}^2(\mathbb{R}^{N\times n} \setminus \{0\}) \cap \mathrm{C}^1(\mathbb{R}^{N\times n}), \tag{4.26}$$

$$f \text{ is rotationally symmetric on } \mathbb{R}^{N\times n}, \tag{4.27}$$

$$\frac{\gamma\, \mathrm{I}_{N\times n}}{|z|^{2-q} + |z|^{\mu}} \leq \nabla^2 f(z) \leq \frac{\Gamma\, \mathrm{I}_{N\times n}}{|z|^{2-q} + |z|} \text{ for all } z \in \mathbb{R}^{N\times n} \setminus \{0\}, \tag{4.28}$$

$$|\nabla^2 f(z_2) - \nabla^2 f(z_1)| \leq \Xi(|z_1|+|z_2|)\mathrm{S}_{q,\beta}(|z_1|, |z_2|)|z_2-z_1|^{\beta} \text{ for all } z_1, z_2 \in \mathbb{R}^{N\times n} \setminus \{0\}. \tag{4.29}$$

Here, the last condition with the scaling factor

$$\mathrm{S}_{q,\beta}(s,t) := \begin{cases} (s+t)^{q-2-\beta} & \text{for } q > 2 \\ 1 & \text{for } q = 2 \\ s^{q-2}t^{q-2}(s+t)^{2-q-\beta} & \text{for } q < 2 \end{cases}$$

expresses a local $\beta$-Hölder continuity requirement on $\nabla^2 f$, which is suitably adapted to a singular (if $q < 2$) or degenerate behavior (if $q > 2$) for $|z| \to 0$. Moreover, we remark that (4.28) reduces to (4.12) in the non-degenerate case $q = 2$, and we record that the preceding conditions comprise the ones of the previous section, and thus imply, in particular, (4.4) and (4.5) with $z_0^* = 0$ and $\lambda_f = 0$.

Taking the assumptions for granted, we have the following result on $\mathrm{C}^{1,\alpha}$ regularity and uniqueness of (local) BV minimizers. The non-degenerate case $q = 2$ of the statement has been established by Bildhauer in [29, Theorem 2.7], while the general case has eventually been treated in [P5, Theorem 1.3].

**Corollary 4.5** (interior $\mathrm{C}^{1,\alpha}$ regularity and uniqueness modulo constants for BV minimizers)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$ and that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies* (4.1), (4.26), (4.27), (4.28), *and* (4.29) *with exponents*

$$1 < \mu < 3\,, \qquad 1 < q < \infty\,, \qquad 0 < \beta \le 1\,,$$

*constants $0 < \gamma \le \gamma_f \le \Gamma_f \le \Gamma < \infty$, and a non-decreasing function $\Xi\colon (0,\infty) \to [0,\infty)$. Then, for every local minimizer $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ of $\overline{F}$ on $\Omega$ and every open subset $U \Subset \Omega$, there exists an $\alpha \in (0,1]$ such that we have*

$$u \in \mathrm{C}^{1,\alpha}(U,\mathbb{R}^N)\,.$$

*If we additionally have that $\Omega$ is connected and that either $\mathcal{L}^n\big(\overline{\Omega}\big) < \infty$ or $f(0) = 0$ holds, and if $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1\big(\overline{\Omega},\mathbb{R}^{N\times n}\big)$ is fixed, then, for each two minimizers $u$ and $v$ of $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$, there exists a constant $y \in \mathbb{R}^N$ such that $u = v + y$ holds $\mathcal{L}^n$-a.e. on $\Omega$.*

For $p > 1$, the integrands $\mathrm{m}_p$ satisfy (4.1), (4.26), (4.27), (4.28), and (4.29) with $\mu = 1 + p$ and $q = p$, so that the model integrals $\overline{\mathrm{M}}_p$ with $1 < p < 2$ are included in Corollary 4.5. However, the statement of the corollary leaves open the possibility that $\alpha$ depends on $U$ and that $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ does not hold for a fixed positive exponent $\alpha$. Therefore, part (A) of Theorem 1.1 in the introduction is not contained in Corollary 4.5, but in fact this result follows only by combining Corollary 4.5 with the later Theorem 5.5.

For comparison, we briefly mention a recent related result on a model PDE which resembles the Euler equation of $\mathrm{M}_p$ in some aspects, but remains non-degenerate. Indeed, in the two-dimensional scalar case $n = 2$, $N = 1$ and under strong assumptions on $\Omega$ and $u_0$, it is shown in [38] that this equation has a unique $\mathrm{C}^1$ solution, which realizes the boundary datum in the $\mathrm{W}^{1,1}_{u_0}$-sense.

The *proof of Corollary* 4.5 is based on three ingredients: the preceding Theorem 4.3, a well-known regularity result for variational integrals with superquadratic $q$-growth, and the uniqueness criterion in Corollary 3.14. These ingredients are combined to prove the regularity claim for BV minimizers of the Dirichlet problem on a smooth and bounded domain $\Omega$. Then, as explained in connection with Theorem 4.2, the regularity claim follows also for local minimizers on arbitrary open $\Omega$, and the uniqueness claim is obtained by the simple reasoning already used in the proof of Corollary 1.2.

To show regularity in case of smooth and bounded $\Omega$, we now fix an arbitrary datum $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$ and the minimizer $u \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ of $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ which was found in Theorem 4.3. We then argue that, with this regularity at hand, the growth of the integrand $f$ becomes irrelevant. Indeed, the fact that $u$ solves the Euler equation on an open $U \Subset \Omega$, depends only on the values of the integrand $f$ on the ball $\overline{\mathrm{B}_M} \subset \mathbb{R}^{N\times n}$ with radius

$M := \sup_U |\nabla u| < \infty$. In particular, it is thus possible to modify $f$ outside $\overline{\mathrm{B}_M}$ and construct, as detailed in [P5, Section 5], a rotationally symmetric $\mathrm{C}^2$-integrand $f_M$ with $q$-growth (and corresponding requirements for $\nabla^2 f_M$) such that $u$ still solves the weak formulation of the Euler equation

$$\int_U \nabla f_M(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = 0 \qquad \text{for all } \varphi\in \mathrm{W}^{1,q}_0(U,\mathbb{R}^N)\,.$$

Carrying out the construction of $f_M$ in a sufficiently careful way, it turns out that the above assumptions on the integrand $f$ induce suitable properties of $f_M$, so that we may apply classical regularity results [127, 70, 2, 78, 94] in the spirit of Uhlenbeck's regularity theorem [129] for the $q$-Laplace system. These results imply $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(U,\mathbb{R}^N)$ for some exponent $\alpha\in(0,1]$ (which depends also on $M$ and thus on $U$). By Corollary 3.14, finally, the regularity gain carries over from one to all BV minimizers. □

## 4.4 $\mathrm{W}^{1,\mathrm{L}\log\mathrm{L}}$ regularity up to a limit case

Next we turn to the case that the $\mu$-ellipticity condition (4.12) is satisfied for $f\in \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^{N\times n})$ *with the limit exponent* $\mu = 3$, which has been excluded in the previous sections. In other words, this means that we choose $\mu = 3$ and $q = 2$ in (4.28), which then reads as the requirement that

$$\frac{\gamma\,\mathrm{I}_{N\times n}}{1+|z|^3} \le \nabla^2 f(z) \le \frac{\Gamma\,\mathrm{I}_{N\times n}}{1+|z|} \text{ holds for } \mathcal{L}^{N\times n}\text{-a.e. } z\in\mathbb{R}^{N\times n}\,. \tag{4.30}$$

The relevance of this assumption lies mainly in two facts. On one hand, it is just the condition satisfied by the model integrand $\mathrm{m}_2$, which has not yet been included in the considerations of Sections 4.2 and 4.3. On the other hand, a specific interest in the case $\mu = 3$ arises from the fact, already pointed out in the introduction and in Section 4.2, that it is a borderline case.

We observe that (4.30) implies (4.4) with some $z_0^*\in\mathbb{R}^{N\times n}$, which we fix for the following restatement of [P1, Theorem 1.10] (in a slightly refined version with a $\mathrm{W}^{2,\infty}_{\mathrm{loc}}$ instead of a $\mathrm{C}^2$ integrand).

**Theorem 4.6** ($\mathrm{W}^{1,\mathrm{L}\log\mathrm{L}}_{\mathrm{loc}}$ estimates and uniqueness modulo constants for $\mathrm{L}^\infty_{\mathrm{loc}}$-BV minimizers)**.** *Suppose that* $\Omega$ *is open in* $\mathbb{R}^n$ *and that* $f\in\mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^{N\times n})$ *satisfies* (4.1) *and* (4.30) *with constants* $0<\gamma\le\gamma_f\le\Gamma_f\le\Gamma<\infty$. *Then, every locally bounded local minimizer* $u\in\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\cap\mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *of* $\overline{F}$ *on* $\Omega$ *satisfies*

$$u\in\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \qquad \textit{and} \qquad |\nabla u|\log(1{+}|\nabla u|)\in\mathrm{L}^1_{\mathrm{loc}}(\Omega) \tag{4.31}$$

*and is controlled, for all balls* $\mathrm{B}_{2r}(x_0)\Subset\Omega$*, by*

$$\int_{\mathrm{B}_r(x_0)}|\nabla u|\log(1{+}|\nabla u|^2)\,\mathrm{d}x \overset{n,\gamma,\Gamma}{\lesssim} \left(\kappa_f+\sup_{\mathrm{B}_{2r}(x_0)}\frac{|u|}{r}\right)\int_{\mathrm{B}_{2r}(x_0)}\big(\kappa_f{+}|\nabla u|\big)\,\mathrm{d}x\,. \tag{4.32}$$

*If we additionally have that* $\Omega$ *is connected and that either* $\mathcal{L}^n\big(\overline{\Omega}\big)<\infty$ *or* $f(0)=0$ *holds, and if* $u_0\in\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ *with* $\nabla u_0\in\mathrm{L}^1\big(\overline{\Omega},\mathbb{R}^{N\times n}\big)$ *is fixed, then, for each two locally bounded minimizers* $u,v\in\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *of* $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ *in* $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$*, there exists a constant* $y\in\mathbb{R}^N$ *such that* $u=v+y$ *holds* $\mathcal{L}^n$*-a.e. on* $\Omega$*.*

We emphasize that the regularity gain in (4.31) has first been obtained by Bildhauer in [29, Theorem 2.5] — under assumptions quite similar to the ones of Theorem 4.6, *but for only one* BV *minimizer*. Precisely, Bildhauer proved that, whenever a *bounded* datum $u_0$ is fixed, then $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ has *one* minimizer $u$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ which satisfies (4.31). However, since the obtained regularity is much weaker than $\mathrm{C}^1$, it is not possible to argue as in the previous section and to pass from only one to every minimizer via Corollary 3.14 or a similar trick. Therefore, Bildhauer's result does not imply the uniqueness statement in Theorem 4.6, which is the principal contribution of [P1] and is basically the outcome of a refined approximation procedure. We discuss these issues in some more detail below, when describing the proof of Theorem 4.6.

However, before coming to the proof, we highlight an interesting point, which is also present in Bildhauer's previous result. Indeed, neither (4.5) nor rotational symmetry of $f$ are assumed in Theorem 4.6, and thus, even the vectorial case $N \geq 2$, the slight integrability gain of the theorem holds true if one imposes just growth conditions, but none of the usual structural hypotheses on $f$. However, once we bring also (4.5) into play, then we can rely on Theorem 4.2 to eliminate the $\mathrm{L}^\infty_{\mathrm{loc}}$ assumption in Theorem 4.6. This accounts for the next statement, which reproduces [P1, Corollary 1.13].

**Corollary 4.7** ($\mathrm{W}^{1,\mathrm{L}\log\mathrm{L}}_{\mathrm{loc}}$ estimates and uniqueness modulo constants for all BV minimizers)**.** *In addition to the hypotheses of Theorem* 4.6*, suppose that* (4.4) *and* (4.5) *are satisfied for some* $z_0^* \in \mathbb{R}^{N\times n}$*. Then the local boundedness assumptions on* $u$ *and* $v$ *in Theorem* 4.6 *are automatically valid and can thus be dropped from the statement. Moreover, for every local minimizer* $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *of* $\overline{F}$ *on* $\Omega$ *and every ball* $\mathrm{B}_{3r}(x_0) \Subset \Omega$*, we then have the estimate*

$$\fint_{\mathrm{B}_r(x_0)} |\nabla u| \log(1{+}|\nabla u|^2)\,\mathrm{d}x \overset{n,\gamma,\Gamma}{\lesssim} \left(\kappa_f + \lambda_f + \fint_{\mathrm{B}_{3r}(x_0)} |\nabla u|\,\mathrm{d}x\right)^2. \tag{4.33}$$

In particular, Corollary 4.7 applies to the model integrands $\mathrm{m}_2$. Thus, the corollary contains Part (B) of Theorem 1.1 in the introduction as a special case.

It has remained an open problem — even for the model integrands $\mathrm{m}_2$ in the vectorial case $N \geq 2$ — whether one can improve the $\mathrm{L}\log\mathrm{L}$ integrability gain to the assertion on the $\mathrm{L}^p$ scale that at least $\nabla u \in \mathrm{L}^{1+\varepsilon}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$ holds for some $\varepsilon > 0$. Indeed, the author has tried to achieve this via an endpoint version of Gehring's lemma, but this approach fails, essentially due to the presence of the exponent 2 on the right-hand side of (4.33). However, a version of the Gehring lemma devised in this connection has eventually found another application in the quite different context of [116], namely in the regularity theory of Alexandrov solutions of the Monge-Ampère equation.

We next outline a *proof of Theorem* 4.6*,* which proceeds by approximation of $u$ with Sobolev functions. To this end, we basically follow the ideas in [P1], which combine the regularization and the Ekeland approximation strategy as already discussed towards the end of Section 4.1. However, we also put forward a slight refinement of the arguments in [P1, Section 5], which is related to the function $h$ below and enables us to bypass some quite technical parts of the reasoning in [P1, Sections 5.2, 5.3].

Now we enter into some of the details. Possibly adding an affine function to the integrand $f$, we assume that $f(0)$ equals 0 and that (4.4) holds with $z_0^* = 0$. Moreover, as in case of

Theorem 4.2 and Corollary 4.5, we can reduce to establishing (4.31) and (4.32) in the situation that $\Omega$ is smooth and bounded and that $u \in \mathrm{L}^\infty(\Omega,\mathbb{R}^N)$ minimizes $\overline{F}[\,\cdot\,;\overline{\Omega}]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ for some $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n,\mathbb{R}^N)$. Setting

$$M := \sup_\Omega |u| < \infty \tag{4.34}$$

and relying on the simple convexity argument of [P1, Lemma 2.9], we can further assume $\sup_{\mathbb{R}^n}|u_0| \le M$. Thus, we confine ourselves to working in the preceding simplified framework, and in order to avoid an additional mollification procedure we also suppose in the sequel that we have $u_0 \in \mathrm{W}^{1,2n}(\mathbb{R}^n,\mathbb{R}^N)$. We then start with the minimizing sequence $(w_k)_{k\in\mathbb{N}}$ for $F[\,\cdot\,;\Omega]$ in $\mathrm{W}^{1,1}_{u_0}(\Omega,\mathbb{R}^N)$ from Theorem 2.9, for which specifically $w_k$ converges in $\mathrm{L}^1(\Omega,\mathbb{R}^N)$ to $u$ and $\mathrm{D}w_k$ converges 1-strictly to $\mathrm{D}u$ in $\mathrm{RM}\big(\overline{\Omega},\mathbb{R}^{N\times n}\big)$. By density, a simple cut-off argument, and Theorem 2.7, we can reduce to the case that we additionally have $w_k \in u_0 + \mathrm{W}^{1,2n}_0(\Omega,\mathbb{R}^N)$, and $\sup_\Omega |w_k| \le M$ for all $k\in\mathbb{N}$.

By setting, for $y\in\mathbb{R}^N$,

$$h(y) := \begin{cases} 0 & \text{for } |y|\le 1 \\ \frac{(|y|-1)^3}{(2-|y|)^{2n}} & \text{for } 1<|y|<2 \\ \infty & \text{for } |y|\ge 2 \end{cases}$$

we next introduce an auxiliary function $h\colon \mathbb{R}^N \to (-\infty,\infty]$. The main properties of this function, besides the obvious facts that it vanishes on $\overline{\mathrm{B}_1}$ and equals $\infty$ outside $\mathrm{B}_2$, are that $h$ is continuous and convex on $\mathbb{R}^N$ and $\mathrm{C}^2$ on $\mathrm{B}_2$. Abbreviating

$$\mathrm{g}_{2n}(z) := (1{+}|z|^2)^n \qquad \text{for } z\in\mathbb{R}^{N\times n}$$

and writing $f_{k^{-1}}$ for the standard mollification with mollification radius $k^{-1}$ of the convex integrand $f$, we then define regularized functionals by setting

$$F_k[w;\Omega] := \int_\Omega \Big[f_{k^{-1}}(\nabla w) + k^{-1}\mathrm{g}_{2n}(\nabla w) + h\Big(\frac{w}{M}\Big)\Big]\,\mathrm{d}x \qquad \text{for } w\in u_0+\mathrm{W}^{1,2n}_0(\Omega,\mathbb{R}^N)\,. \tag{4.35}$$

Here, the introduction of the zero-order term is inspired by a regularization procedure of Carozza & Kristensen & Passarelli di Napoli [39] and a modification of their idea in [P1, Section 5]. However, in contrast to the previous procedures, the present approach via the function $h$ allows to conveniently transform the $\mathrm{L}^\infty$ control (4.34) on $u$ into a *uniform* $\mathrm{L}^\infty$ *control* on its approximations; see (4.38) below.

The functionals $F_k[\,\cdot\,;\Omega]$ are (formally) extended to a larger space, namely the negative Sobolev space $\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)$ by assigning the value $\infty$ outside $u_0+\mathrm{W}^{1,2n}_0(\Omega,\mathbb{R}^N)$. We record that each resulting functional $F_k[\,\cdot\,;\Omega]$ is lower semicontinuous with respect to norm convergence in $\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)$, essentially since $\mathrm{W}^{-1,1}$ convergent sequences with bounded energy are also $\mathrm{W}^{1,2n}$ bounded and thus weakly $\mathrm{W}^{1,2n}$ convergent. In view of the semicontinuity property we can next apply the Ekeland principle in the Banach space $\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)$. From the minimizing sequence $(w_k)_{k\in\mathbb{N}}$ we then obtain an Ekeland improved sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)$ such that, for some null sequence $(\delta_k)_{k\in\mathbb{N}}$ and all $k\in\mathbb{N}$, we have $\|v_k-w_k\|_{\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)}\le\delta_k$ and the perturbed minimality property

$$F_k[v_k;\Omega]\le F_k[w;\Omega]+\delta_k\|w-v_k\|_{\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)}\qquad\text{for all } w\in\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)\,. \tag{4.36}$$

Comparing $(v_k)_{k\in\mathbb{N}}$ with an arbitrary $w\in u_0+\mathrm{W}^{1,2n}_0(\Omega,\mathbb{R}^N)$ such that $\sup_\Omega|w|\le M$, we infer $F_k[v_k;\Omega]<\infty$ for all $k\in\mathbb{N}$ and

$$\limsup_{k\to\infty}F_k[v_k;\Omega]\le\overline{F}[w;\Omega]\,. \tag{4.37}$$

For later usage we next draw a couple of conclusions from these last observations. First, the finiteness of $F_k[v_k;\Omega]$ requires that we have $v_k\in u_0+\mathrm{W}^{1,2n}_0(\Omega,\mathbb{R}^N)$. Second, taking into account the $h$ term in (4.35) and the inclusion $v_k\in\mathrm{W}^{1,2n}(\mathbb{R}^n,\mathbb{R}^N)\subset\mathrm{C}^{0,1/2}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$, which results from the Sobolev embedding, the finiteness of $F_k[v_k;\Omega]$ also enforces[8] the $k$-uniform bound

$$\max_K|v_k|<2M\qquad\text{for every compact subset } K \text{ of } \Omega\,. \tag{4.38}$$

Third, relying on (4.37) and the lower bound in (4.4), we find that every subsequence of $(v_k)_{k\in\mathbb{N}}$ weakly-$*$ converges in $\mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$ (where clearly the $v_k$ are extended on $\mathbb{R}^n\setminus\Omega$ by the values of $u_0$). However, the $\mathrm{W}^{-1,1}$-closeness of $v_k$ and $w_k$ suffices to ensure that the whole sequence weakly-$*$ converges in $\mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$ to the only possible limit $u$. Fourth, inserting the members of the minimizing sequence $(w_k)_{k\in\mathbb{N}}$ for the comparison function $w$ in (4.37), involving (4.2), and recalling that we assumed $f(0)=0$ and $z_0^*=0$, we also get the energy bound

$$\limsup_{k\to\infty}F_k[v_k;\Omega]\le\overline{F}\big[u;\overline{\Omega}\big]\le\Gamma_f|\mathrm{D}u|\big(\overline{\Omega}\big)\,,$$

and in view of (4.7) we infer in particular

$$\limsup_{k\to\infty}\int_\Omega|\nabla v_k|\,\mathrm{d}x\overset{\gamma,\Gamma}{\lesssim}\kappa_f\mathcal{L}^n(\Omega)+|\mathrm{D}u|\big(\overline{\Omega}\big)\,. \tag{4.39}$$

From (4.36) we can now derive, with (4.38) at hand, the perturbed Euler equation

$$\begin{gathered}\left|\int_\Omega\Big[\nabla f_{k^{-1}}(\nabla v_k)\cdot\nabla\varphi+k^{-1}\nabla \mathrm{g}_{2n}(\nabla v_k)\cdot\nabla\varphi+\nabla h\Big(\frac{v_k}{M}\Big)\cdot\frac{\varphi}{M}\Big]\mathrm{d}x\right|\le\delta_k\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)}\\ \text{for all } \varphi\in\mathrm{W}^{1,2n}_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)\,.\end{gathered} \tag{4.40}$$

Indeed, in order to verify (4.36), we use $w=v_k+t\varphi$ with a parameter $t\in\mathbb{R}$ and a (by the Sobolev embedding necessarily continuous) $\varphi\in\mathrm{W}^{1,2n}_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ as a comparison function in (4.36), and then we rely on the first-order necessary criterion for minima. At this point, we crucially exploit that, thanks to (4.38) and for sufficiently small $|t|$, the values of $\frac{v_k+t\varphi}{M}$ remain in the ball $\mathrm{B}_2$ where $h$ is finite and differentiable.

[8] Amendment (October 2019): The author is grateful to Franz Gmeineder for pointing out a flaw in the original version of this thesis, where $h$ above was defined with $2-|y|$ instead of $(2-|y|)^{2n}$. While the original choice does not seem sufficient to deduce the claim (4.38), with the present corrected choice we can justify (4.38) by the following extra argument. If we assume $|v_k(x_0)|\ge 2M$ for some $x_0\in\Omega$, then $\mathrm{C}^{0,1/2}_{\mathrm{loc}}$ regularity gives $|v_k(x)|\ge 2M-C|x-x_0|^{1/2}\ge\frac32 M$ for $x\in\mathrm{B}_r(x_0)\subset\Omega$, with sufficiently small $r>0$ and some $C<\infty$. The choice of $h$ and the last estimate yield $\int_\Omega h\big(\frac{v_k}{M}\big)\,\mathrm{d}x\ge\int_{\mathrm{B}_r(x_0)}\frac{1/8}{(2-|v_k|/M)^{2n}}\,\mathrm{d}x\ge\int_{\mathrm{B}_r(x_0)}\frac{1/8}{(C/M)^{2n}|x-x_0|^n}\,\mathrm{d}x=\infty$. This contradicts the finiteness of $F_k[v_k;\Omega]$, and thus we necessarily have $|v_k|<2M$ on $\Omega$.

Next we perform several estimations in order to establish higher differentiability and integrability properties of the approximations $v_k$. The first such property is obtained from the Euler equation (4.40) and the finiteness condition $F_k[v_k;\Omega]<\infty$ by an application of the difference quotient method, and it reads as

$$v_k \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \qquad \text{and} \qquad (1+|\nabla v_k|)^{2n-2}|\nabla^2 v_k|^2 \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \qquad \text{for all } k\in\mathbb{N}\,. \tag{4.41}$$

However, we emphasize that this gain of second derivatives *comes with estimates which are non-uniform in k*. Indeed, the derivation of (4.41) is detailed in [P1, Lemma 5.1], and instead of repeating the details here we only make some comments on the relevance of the various terms in (4.35) and (4.40). So, we point out that, since at this stage we deal with non-uniform-in-$k$ estimates, we can decisively exploit the good quantitative strict convexity properties of the $\mathrm{g}_{2n}$ regularization term, which are not impaired by the $f_{k^{-1}}$ and $h$ terms, essentially due to the convexity of the integrands $f_{k^{-1}}$ and $h$. Moreover, the perturbation term on the right-hand side of (4.40) is controlled via the simple inequality $\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)}\le\|\varphi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^N)}$, and in the course of the estimation it can be absorbed on the left-hand side (basically since it contains $|\varphi|$ only linearly and not quadratically).

Next, with the extra information (4.41) at hand, we differentiate (4.40). Crucially exploiting the basic estimate $\|\partial_i\psi\|_{\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)}\le\|\psi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^N)}$ for the $\mathrm{W}^{-1,1}$-norm, we then obtain

$$\begin{gathered}\left|\int_\Omega\Big[\nabla^2(f_{k^{-1}}+k^{-1}\mathrm{g}_{2n})(\nabla v_k)(\partial_i\nabla v_k,\nabla\psi)+\nabla^2h\Big(\frac{v_k}{M}\Big)\Big(\frac{\partial_i v_k}{M},\frac{\psi}{M}\Big)\Big]\,\mathrm{d}x\right|\le\delta_k\|\psi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^N)}\\ \text{for all } \psi\in\mathrm{W}^{1,2n}_0(\Omega,\mathbb{R}^N)\end{gathered} \tag{4.42}$$

(where we have, for brevity, summarized the $f_{k^{-1}}$ and $\mathrm{g}_{2n}$ terms). We test (4.42) with $\psi=\eta^2\partial_i v_k$, where $\eta\in\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ is an arbitrary test function, in order to obtain second derivative estimates *with k-uniform constants* and more specifically the Caccioppoli type estimate

$$\begin{aligned}\limsup_{k\to\infty}\int_\Omega\eta^2\big[(1+|\nabla v_k|)^{-3}|\nabla^2 v_k|^2+k^{-1}(1+|\nabla v_k|)^{2n-2}|\nabla^2 v_k|^2\big]\,\mathrm{d}x&\\ \overset{n,\gamma,\Gamma}{\lesssim}\sup_\Omega|\nabla\eta|^2\big(\kappa_f\mathcal{L}^n(\Omega)+|\mathrm{D}u|\big(\overline{\Omega}\big)\big)\,.&\end{aligned} \tag{4.43}$$

For the details of the computations leading to (4.43), we refer to [P1, Lemma 5.2], while here we restrict ourselves to pointing out some specific issues. To begin with, we remark that, clearly, the strict positivity of both $\nabla^2 f_{k^{-1}}$ and $\nabla^2\mathrm{g}_{2n}$, in the former case expressed by (4.30), plays a crucial role in the derivation of (4.43). Moreover, the $\nabla^2 h$ term in (4.42) is non-negative (after inserting the test function) and can thus be neglected in the respective estimates. Finally, the perturbation term involves — this is the crucial benefit of applying the Ekeland principle in the negative Sobolev space $\mathrm{W}^{-1,1}$ — only first derivatives of $v_k$ and is thus well-controlled. We also remark that the derivation of (4.43) makes use of (4.39), which is indeed responsible for the form of the term on the right-hand side. Finally, we stress that the second derivative control in (4.43) does not persist in the limit $k\to\infty$ in the sense that it does not (or at least not by an argument of which the author is aware) imply that the BV minimizer $u$ has second derivatives as functions or measures; the reason for this lies mostly in the negative 'homogeneity' of the quantity $(1+|\nabla v_k|)^{-3}|\nabla^2 v_k|^2$ in $v_k$.

The final step of the proof essentially reproduces a reasoning introduced by Bildhauer [29]. The argument proceeds by testing (4.40) once more and combining the resulting estimate with (4.43) in order to obtain the final uniform-in-$k$ integrability gain for the first derivatives of the approximations $v_k$. To illustrate this, we fix a ball $\mathrm{B}_{2r}(x_0) \subset \Omega$ and $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\mathbb{1}_{\mathrm{B}_r(x_0)} \le \eta \le \mathbb{1}_{\mathrm{B}_{2r}(x_0)}$ and $|\nabla\eta| \le 2/r$. Then we plug the logarithmic test function $\varphi = \eta^2 v_k \log(1+|\nabla v_k|^2)$ into (4.40) and rely on another series of estimations, which are essentially carried out in [P1, Lemma 5.3]. As before, these estimations exploit good properties of the integrands $f_{k^{-1}}$ and $\mathrm{g}_{2n}$, in particular the inequality (4.8), while the terms involving the integrand $h$ have the right sign and can be discarded. Moreover, the perturbation term on the right-hand side of (4.40) is simply estimated via $\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega,\mathbb{R}^N)} \le \|\varphi\|_{\mathrm{L}^1(\Omega,\mathbb{R}^N)}$ and can eventually be absorbed, thanks to the small prefactor $\delta_k$, on the left-hand side. Testing and estimating in this way, using Young's inequality, and involving both (4.38) and (4.43), we finally end up with the next estimate. Though we do not enter further into the details of the derivation, we stress that the availability of (4.38) allows to considerably simplify the original proof of [P1, Lemma 5.3]. Anyway, the resulting estimate is not significantly changed and reads as

$$\limsup_{k\to\infty} \int_{\mathrm{B}_r(x_0)} |\nabla v_k| \log(1+|\nabla v_k|^2)\,\mathrm{d}x \overset{n,\gamma,\Gamma}{\lesssim} \left(\kappa_f + \frac{M}{r}\right)\big(\kappa_f \mathcal{L}^n(\Omega) + |\mathrm{D}u|\big(\overline{\Omega}\big)\big)\,.$$

This inequality then carries over to the limit $u$, and via Theorem 2.2 we conclude, in fact,

$$\int_{\mathrm{B}_r(x_0)} |\mathrm{D}u| \log(1+|\mathrm{D}u|^2) \overset{n,\gamma,\Gamma}{\lesssim} \left(\kappa_f + \frac{M}{r}\right)\big(\kappa_f \mathcal{L}^n(\Omega) + |\mathrm{D}u|\big(\overline{\Omega}\big)\big)\,, \tag{4.44}$$

where the left-hand integral is understood according to Definition 2.1, or equivalently in the sense of (2.7). As the recession function of $z \mapsto |z|\log(1+|z|^2)$ equals $\infty$ on $\mathbb{R}^{N\times n} \setminus \{0\}$, the estimate (4.44) implies $|\nabla u|\log(1+|\nabla u|^2) \in \mathrm{L}^1(\mathrm{B}_r(x_0))$ and $|\mathrm{D}^{\mathrm{s}}u|(\mathrm{B}_r(x_0)) = 0$. Since $\mathrm{B}_{2r}(x_0)$ is an arbitrary ball in $\Omega$, we have thus verified (4.31), and the estimate (4.32) follows simply by observing that (4.44) remains true with $\Omega$ replaced by $\mathrm{B}_{2r}(x_0)$. □

## 4.5 Uniqueness modulo a 1-parameter family of constants

In this section we exploit a strict convexity hypothesis on the recession function $f^\infty$ in order to slightly refine the uniqueness assertions in Corollaries 4.5 and 4.7. Since, evidently, the 1-homogeneous function $f^\infty$ is never strictly convex on $\mathbb{R}^{N\times n}$, we work with the following concept. We say that a $[-\infty,\infty]$-valued function $g$ on $\mathbb{R}^m$ has strictly convex sublevel sets, if, for every $t \in \mathbb{R}$, the set $\{y \in \mathbb{R}^m : g(y) < t\}$ is bounded and convex and its boundary does not contain a line segment of positive length. For our purposes, the main benefit of this definition lies in the convexity property of the next elementary lemma; compare with [P1, Lemma 6.1].

**Lemma 4.8.** *Suppose that $g\colon \mathbb{R}^m \to [-\infty,\infty]$ is positively 1-homogeneous and has strictly convex sublevel sets[9]. If $y_1, y_2 \in \mathbb{R}^m$ are not positively collinear, then we have the strict convexity inequality $g(y_1+y_2) < g(y_1) + g(y_2)$.*

[9] Since our notion of strictly convex sublevel sets includes the boundedness of these sublevel sets, the 1-homogeneous function $g$ in Lemma 4.8 is necessarily positive on $\mathbb{R}^m \setminus \{0\}$.

Our concrete hypothesis on $f^\infty$ is now that, for some $z_0^* \in \mathbb{R}^{N\times n}$ and

$$\begin{gathered}\text{for all } \nu \in \mathbb{R}^n \setminus \{0\}\,,\\ \text{the function } y \mapsto f^\infty(y\otimes\nu) - z_0^*\cdot(y\otimes\nu) \text{ on } \mathbb{R}^N \text{ has strictly convex sublevel sets.}\end{gathered} \tag{4.45}$$

We remark that there are good criteria for the validity of this assumption. In particular, it is satisfied whenever $z \mapsto f^\infty(z) - z_0^*\cdot z$ on $\mathbb{R}^{N\times n}$ has strictly convex level sets, and it is always satisfied with $z_0^* = 0$ if $f$ is non-constant, convex, and rotationally symmetric on $\mathbb{R}^{N\times n}$ (or if $f$ is non-constant and convex with the less restrictive structure $f(z) = \tilde{f}(|z_1|,|z_2|,\ldots,|z_n|)$).

Involving (4.45), we now restate [P3, Theorem 1.16] in a more concrete framework.

**Corollary 4.9** (uniqueness of BV minimizers modulo a 1-parameter family of constants)**.** *Suppose that $\Omega$ is open and connected in $\mathbb{R}^n$ and that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ either satisfies the assumptions of Corollary* 4.5 *or satisfies the assumptions of Corollary* 4.7 *and* (4.45) *for some $z_0^* \in \mathbb{R}^{N\times n}$. Moreover, assume that we have either $\mathcal{L}^n\big(\overline{\Omega}\big) < \infty$ or $f(0) = 0$. Then, for every $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1\big(\overline{\Omega},\mathbb{R}^{N\times n}\big)$, the set of minimizers of $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ can be written as the* 1*-parameter family*

$$\{\overline{u} + t\overline{y} \,:\, t \in [-1,1]\}$$

*with a fixed function $\overline{u} \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ satisfying $|\mathrm{D}^{\mathrm{s}}\overline{u}|(\Omega) = 0$ and a fixed vector $\overline{y} \in \mathbb{R}^N$.*

Roughly speaking, Corollary 4.9 asserts that, even in the vector-valued case $N \geq 2$, non-uniqueness does not manifest in a form worse than the 1-parameter family of minimizers in Santi's scalar example [115, Section 2] (which is described after the statement of Corollary 1.2 in the introduction). However, we stress that this is only true if the assumption (4.45) is in force, and that the conclusion of Corollary 4.9 does not hold without this assumption. Indeed, for $n = 2$, for arbitrary $N \in \mathbb{N}$, and for an integrand $f$ which satisfies all assumptions of Corollary 4.7, but violates (4.45), the refinement [P1, Theorem 3.14] of Santi's example shows that even an $N$-parameter family of BV minimizers may occur.

The *proof of Corollary* 4.9 in [P1, Section 6] rests on the elementary Lemma 4.8. To explicate this, we assume $z_0^* = 0$ and we recall that, by Corollaries 4.5 and 4.7, if we fix one minimizer $u$, then we have $|\mathrm{D}^{\mathrm{s}}u|(\Omega) = 0$ and all minimizers of $\overline{F}\big[\,\cdot\,;\overline{\Omega}\big]$ in $\mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ have the form $u{+}y\mathbb{1}_\Omega$ with some $y \in \mathbb{R}^N$. In the case $\mathbb{1}_\Omega \notin \mathrm{BV}(\mathbb{R}^n)$ that $\Omega$ has infinite perimeter, this clearly means $u{+}y\mathbb{1}_\Omega \notin u_0 + \mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N)$ for $y \neq 0$, so that $u$ is the unique minimizer and there is nothing to prove. Otherwise, we get $\mathrm{D}(u + y\mathbb{1}_\Omega) = \mathrm{D}u + y \otimes \mathrm{D}\mathbb{1}_\Omega$ and

$$\overline{F}\big[u+y;\overline{\Omega}\big] = \int_{\overline{\Omega}} f\big(\mathrm{D}u + (u_0 - u^{\mathrm{int}}_{\partial^*\Omega})\otimes \mathrm{D}\mathbb{1}_\Omega\big) + \int_{\partial^*\Omega} f^\infty\big((u^{\mathrm{int}}_{\partial^*\Omega} - u_0 + y)\otimes \mathrm{D}\mathbb{1}_\Omega\big)\,, \tag{4.46}$$

where $u^{\mathrm{int}}_{\partial^*\Omega}$ stands for the interior trace of $u$ on the reduced boundary $\partial^*\Omega$ and $u_0$ is evaluated through its $\mathcal{H}^{n-1}$-a.e. defined representative. We remark that the deduction of last equality exploits the subtlety that $|\mathrm{D}u + (u_0 - u^{\mathrm{int}}_{\partial^*\Omega})\otimes \mathrm{D}1_\Omega|$ still vanishes on $\partial^*\Omega$, though maybe not on $\partial\Omega$, when $\Omega$ has merely finite perimeter; compare [9, Theorem 3.84]. Anyway, it now suffices to show that the minima of the last term in (4.46) as a function of $y \in \mathbb{R}^N$ form a line segment of finite length through the origin. To verify this, we take into account that $f^\infty$ is convex with $\gamma_f|z| \leq f^\infty(z) \leq \Gamma_f|z|$, we observe that consequently the set of minima $y_0 \in \mathbb{R}^N$ is convex, closed, and bounded, and we finally demonstrate that

$$\text{every minimum } y_0 \in \mathbb{R}^N \text{ is collinear to } u^{\mathrm{int}}_{\partial^*\Omega}(x) - u_0(x) \text{ for } |\mathrm{D}\mathbb{1}_\Omega|\text{-a.e. } x \in \partial\Omega\,. \tag{4.47}$$

Indeed, if (4.47) were false, Lemma 4.8 would yield strict convexity of the maps $y \mapsto f^\infty(y \otimes \nu)$ along non-radial line segments $[u^{\mathrm{int}}_{\partial^*\Omega}(x) - u_0(x), u^{\mathrm{int}}_{\partial^*\Omega}(x) - u_0(x) + y_0]$, and we would arrive at a contradiction to the minimality of 0 and $y_0$. This contradiction completes the reasoning. □

## 4.6 Boundary behavior and (non-)uniqueness

Finally, we record that the preceding results entail some restrictions for the boundary behavior of BV minimizers. This has been formulated in [P1, Theorem 1.17] as follows.

**Corollary 4.10** (non-uniqueness comes with 1-dimensional jumps at the boundary)**.** *Under the assumptions of Corollary* 4.9*, consider the* (*uniquely determined*) *function* $\overline{u}$ *and the* (*determined up to change of sign*) *vector* $\overline{y}$ *introduced there. If we are in the case* $\overline{y} \neq 0$ *of non-uniqueness of minimizers, then* $\Omega$ *has necessarily finite perimeter in* $\mathbb{R}^n$*, and we have*

$$\overline{u}^{\mathrm{int}}_{\partial^*\Omega}(x) + J(x)\overline{y} = u_0(x) \qquad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \partial^*\Omega \tag{4.48}$$

*for some function* $J\colon \partial^*\Omega \to \mathbb{R} \setminus (-1,1)$ *which takes values arbitrarily close to* 1 *and* $-1$ *on sets of positive* $\mathcal{H}^{n-1}$*-measure.*

The *proof of Corollary* 4.10 is mostly contained in the arguments already given. Indeed, the finiteness of the perimeter of $\Omega$ has been observed implicitly in the proof of Corollary 4.9, the existence of $J$ follows straightforwardly by applying (4.47) to the minimizers $\overline{u} + t\overline{y}$ with $t \in [-1,1]$ and the vector $\overline{y}$, and the fact that $J$ takes values arbitrarily close to 1 and $-1$ can be verified by related arguments. We omit the details which are worked out in [P1, Section 6]. □

The following restatement of [P1, Theorem 1.5] shows that in the case $\overline{y} = 0$, in which $\overline{u}$ is the unique minimizer, (4.48) need not be at hand, and thus the non-uniqueness assumption $\overline{y} \neq 0$ in Corollary 4.10 cannot be dropped. Viewing this from a different angle, one may also say that uniqueness seems to be the generic case, which can come with a variety of boundary configurations, while non-uniqueness happens only in specific situations.

**Theorem 4.11** (uniqueness may come with more complicated jumps at the boundary)**.** *In the case* $n = N = 2$*, consider the annulus* $\Omega = \mathrm{B}_2 \setminus \overline{\mathrm{B}_1}$ *in* $\mathbb{R}^2$*, a constant* $M \in \mathbb{R}$*, and a function* $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^2, \mathbb{R}^2)$ *with* $u_0(x) = Mx$ *for* $|x| = 1$ *and* $u_0(x) = 0$ *for* $|x| = 2$*. Then* $\overline{\mathrm{M}}_2\big[\,\cdot\,;\overline{\Omega}\big]$ *has a unique minimizer* $\overline{u}$ *in* $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^2\big)$*, which satisfies* $\sup_\Omega |\overline{u}| \leq 2/(1-\log 2)$*. In particular, for* $|M| > 2/(1-\log 2)$ *there is no* $\overline{y} \in \mathbb{R}^N$ *such that* (4.48) *holds with an* $\mathbb{R}$*-valued function* $J$*.*

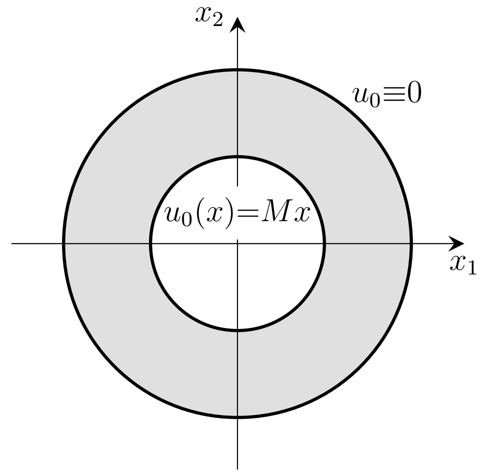


**Figure 3:** The domain of Theorem 4.11

*Idea of proof.* The example of Theorem 4.11 refines Finn's scalar example [62] on the annulus, which is presented after Corollary 1.2 in the introduction. Similar to the reasoning in the scalar case, the proof of Theorem 4.11 proceeds by reduction to an ODE problem. However, since $u_0$ is non-constant on the inner boundary portion $\partial\mathrm{B}_1$, in the situation of Theorem 4.11 the ODE problem takes a more complicated form with an explicit $u$-dependence and cannot be solved explicitly. While uniqueness and some basic properties of the minimizer $\overline{u}$ are still easy to check, it seems non-trivial to verify the sup-bound for $\overline{u}$. Nevertheless, it has been possible to establish this bound by some explicit, but somewhat tricky estimations with a radially shifted competitor. The details are contained in [P1, Section 3.2] and go beyond the scope of the present exposition. □

Finally, we mention that further examples in [P1, Section 6] support the presumption that the above description of the boundary behavior of BV minimizers is quite sharp as far as general domains $\Omega$ and general boundary data $u_0$ are concerned. However, as already emphasized at the end of the introduction, this leaves open the (seemingly more challenging) issue to provide good criteria in terms of $\Omega$ and $u_0$ for the actual attainment $\overline{u}^{\mathrm{int}}_{\partial^*\Omega} = u_0$ of the boundary values on $\partial^*\Omega$ and for full uniqueness of BV minimizers.

# Chapter 5

# Partial regularity of BV minimizers

This chapter deals with the local and partial regularity results of Anzellotti & Giaquinta [21] and their degenerate counterparts from [P2]. The latter results crucially rely on the combination of the localization method of [21] and the $p$-harmonic-comparison strategy of [56].

We first discuss the simpler case of autonomous integrals without lower-order terms, but eventually we also deal with more general functionals. In the simpler case, the integrands are, as in the preceding chapter, continuous functions $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ of linear growth

$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|} < \infty\,, \tag{5.1}$$

and the corresponding BV integrals $\overline{F}$ are given as follows. If $w \in \mathrm{BV}_{\mathrm{loc}}(U,\mathbb{R}^N)$ is defined on an open neighborhood $U$ of a Borel set $A$ in $\mathbb{R}^n$, then, in the terminology of Definition 2.1, we set

$$\overline{F}[w;A] := \int_A f(\mathrm{D}w) \in \mathbb{R} \qquad \text{whenever } \mathcal{L}^n(A) + |\mathrm{D}w|(A) < \infty\,. \tag{5.2}$$

In addition, we continue using the corresponding notion of local minimizers of $\overline{F}$, introduced in Definition 4.1.

## 5.1 The local regularity result of Anzellotti & Giaquinta

The basis for the investigations of this chapter is a result of Anzellotti & Giaquinta on local-in-phase-space gradient regularity [21, Theorem 1.1]. We start by restating the linear growth case of their result, which reads in our setting as follows.

**Theorem 5.1** (local-in-phase-space $\mathrm{C}^{1,\alpha}$ regularity for BV minimizers)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$ and a convex integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ with (5.1) such that $f$ is $\mathrm{C}^2$ near some $z_0 \in \mathbb{R}^{N\times n}$ with $\nabla^2 f(z_0) > 0$. If $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local minimizer of $\overline{F}$ on $\Omega$ and if* $\mathrm{D}u$ *has Lebesgue value $z_0$ at a point $x_0 \in \Omega$ in the sense of*

$$\lim_{\varrho\searrow 0} \frac{|\mathrm{D}u - z_0\mathcal{L}^n|(\mathrm{B}_\varrho(x_0))}{\mathcal{L}^n(\mathrm{B}_\varrho(x_0))} = 0\,,$$

*then*

$$u \text{ is of class } \mathrm{C}^{1,\alpha}\,, \text{ for all } \alpha \in (0,1)\,, \text{ in a neighborhood of } x_0\,.$$

The point of Theorem 5.1 lies in the fact that it guarantees regularity under assumptions which are not only localized near a point $x_0$ in $\Omega$, *but are almost localized near a value* $z_0$ *of the gradient variable.* Here, the formulation 'almost localized' is meant to express that the convexity assumption on $f$ is still imposed globally on $\mathbb{R}^{N\times n}$, but that all other assumptions on $f$ depend only on its restrictions to arbitrarily small neighborhoods of $z_0$. Indeed, it is not clear (to the author) whether the global convexity assumption can be entirely dropped, but relaxation methods in the spirit of Section 3.5 show that it can, at least, be weakened as indicated in the following corollaries: In the case $N = 1$, Theorem 5.1 remains valid under the sole hypotheses on $f$ that (5.1) holds and that the convexification $f^{**}$ is $\mathrm{C}^2$ near $z_0$ with $\nabla^2 f^{**}(z_0) > 0$; and in the case $N > 1$, it suffices to require that (5.1) holds and that the quasiconvexification $\mathrm{Q}f$ is convex on $\mathbb{R}^{N\times n}$ and $\mathrm{C}^2$ near $z_0$ with $\nabla^2\mathrm{Q}f(z_0) > 0$. Anyhow, we do not want to pursue the weakening of the convexity assumption in the sequel, but rather we turn to the case that $\nabla^2 f(z) > 0$ holds for *every* $z \in \mathbb{R}^{N\times n}$, which clearly implies convexity of $f$. In this case, since $\mathcal{L}^n$-a.e. $x_0 \in \Omega$ is a Lebesgue point of $\mathrm{D}u$, Theorem 5.1 implies the following statement, which has been observed in [21, Corollary 1.1].

**Corollary 5.2** (partial $\mathrm{C}^{1,\alpha}$ regularity for BV minimizers)**.** *Consider an open subset* $\Omega$ *of* $\mathbb{R}^n$ *and an integrand* $f \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ *with* (5.1) *such that* $\nabla^2 f(z) > 0$ *holds for all* $z \in \mathbb{R}^{N\times n}$*. If* $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *is a local minimizer of* $\overline{F}$ *on* $\Omega$*, then there exists an open subset* $\Omega_0$ *of* $\Omega$ *such that we have*

$$u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0,\mathbb{R}^N) \textit{ for all } \alpha \in (0,1) \qquad \textit{and} \qquad \mathcal{L}^n(\Omega \setminus \Omega_0) = 0\,.$$

The localized hypotheses in Theorem 5.1 are reflected in Corollary 5.2 in the fact that — in contrast to many other partial regularity results — no growth conditions for $\nabla^2 f$ get involved. This means, in other words, that Corollary 5.2 covers integrands with an arbitrarily fast blow-up of the dispersion ratio of $\nabla^2 f(z)$ for $|z| \to \infty$, and in particular the corollary (or a slight adaption making assumptions only away from $z = 0$) applies to the model integrands $\mathrm{m}_p$ also in the cases $p \geq 2$, which were excluded in the previous chapter. Indeed, the precise outcome in case of the integrals $\mathrm{M}_p$ is as follows.

**Corollary 5.3** (partial $\mathrm{C}^{1,\alpha}$ regularity for BV minimizers of $\mathrm{M}_p$, $1 < p < \infty$)**.** *Consider an open subset* $\Omega$ *of* $\mathbb{R}^n$ *and an exponent* $p \in (1,\infty)$*. If* $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *is a local minimizer of* $\overline{\mathrm{M}}_p$ *on* $\Omega$*, then there exists an open and dense subset* $\Omega_0$ *of* $\Omega$ *such that we have*

$$u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega_0,\mathbb{R}^N) \textit{ for all } \alpha \in (0,1)\,.$$

*In the case* $p = 2$*, one moreover has* $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$*.*

Here, the conclusion in the case $p = 2$ follows directly from Corollary 5.2. In the case $p \neq 2$ instead, one needs to get back to Theorem 5.1, which then applies near all points $x_0 \in \Omega$ where $\mathrm{D}u$ has a Lebesgue value $\neq 0$. However, since $\nabla^2\mathrm{m}_p(0)$ vanishes for $p > 2$ and does not even exist for $p < 2$, the theorem is not applicable in the degeneration points, that are the points $x_0 \in \Omega$ such that $\mathrm{D}u$ has Lebesgue value $= 0$. This is responsible for the fact that we cannot yet assert, in these cases, $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$, but only that $\Omega \setminus \Omega_0$ is nowhere dense. The next section is essentially concerned with the overcoming of this drawback and the closing of the gap from Corollary 5.3 to Theorem 1.1(C) by a careful analysis near the degeneration points.

However, before turning to the degeneration points, we describe the *proof of Theorem* 5.1. Indeed, the reasoning is based on the principle that the local minimizer $u$ should satisfy the Euler equation $\operatorname{div}\big[\nabla f(\mathrm{D}u)\big] \equiv 0$ on $\Omega$ and should thus be close, up to a hopefully small error, to a weak solution $h$ of the linearization

$$\operatorname{div}\big[\nabla^2 f(z_0)(\nabla h{-}z_0,\,\cdot\,)\big] \equiv 0 \qquad \text{on a small ball } \mathrm{B}_r(x_0)\,. \tag{5.3}$$

The prevalent idea in partial regularity proofs is now to exploit the good regularity theory for the solution $h$ of the linear elliptic system (5.3) and to carry over $\mathrm{C}^{1,\alpha}$ estimates to the local minimizer $u$.

Under the assumptions of Theorem 5.1, the rigorous implementation of these heuristic ideas is highly non-trivial for several reasons. First of all, since $\mathrm{D}u$ is not a continuous function, its values need not be close to $z_0$ in small neighborhoods of $x_0$. Consequently, since $f$ need not be differentiable away from $z_0$, the weak formulation of the Euler equation does not make sense (even disregarding the difficulty that $\mathrm{D}u$ is merely a measure). Even more decisively, it is not at all clear that comparison estimates for $u$ and $h$ will not depend on bounds for $\nabla^2 f$ far from $z_0$. These difficulties are overcome in [21] by permanently working with the minimality property instead of the Euler equation and by a delicate localization strategy, which is inspired by a previous regularity proof of Schoen & Simon [117] in the context of geometric measure theory. The localization strategy, which, as a side benefit, avoids all indirect arguments, proceeds in two steps. First one constructs a good $\mathrm{C}^{1,\alpha}$ competitor $w$ with a well-controlled deviation from minimality such that $\nabla w$ is $\mathrm{C}^{0,\alpha}$-close to $z_0$ on $\mathrm{B}_r(x_0)$. The competitor $w$ is obtained in [21] by a mollification procedure, and it is at this point where global convexity of $f$ crucially enters the reasoning. We remark that the mollification step replaces the usage of a classical Lipschitz approximation lemma in the somewhat different framework of [117]. Anyway, in a second step, $w$ is approximated by a solution $h$ of the linear system (5.3), with the Dirichlet boundary condition $h = w$ on $\partial\mathrm{B}_r(x_0)$. It follows, by global Schauder estimates on $\mathrm{B}_r(x_0)$, that $\nabla h$ is still $\mathrm{C}^{0,\alpha}$-close to $z_0$ on the same ball $\mathrm{B}_r(x_0)$. Then, in view of the good control on $\nabla w$ and $\nabla h$, it is possible to work with the integrand $f$ only in a small neighborhood of $z_0$ in order to derive comparison estimates and gain better $\mathrm{C}^{1,\alpha}$-control for $w$ near $x_0$ from the corresponding control on $h$. Finally, exploiting the explicit construction of $w$ as a mollification of $u$, one can suitably carry over some $\mathrm{C}^{1,\alpha}$ estimates to the minimizer $u$.

We do not enter into further details of the described method, which lie beyond the scope of the present exposition. Notably, the accomplishment of the last-mentioned steps is technically demanding and requires a lengthy reasoning, which is detailed in the original article [21]. □

## 5.2 $\mathbf{C}^{1,\alpha}$ regularity near degeneration points

The publication [P2] aims at improving Theorem 5.1 in several regards. A first objective, already foreshadowed above, is the investigation of degeneration points and a corresponding improvement of Corollary 5.3 in the case $p \neq 2$. A second aim is the treatment of lower-order terms, and this is in fact achieved in [P2] by working with a general notion of almost-minimizers. Finally, as a side benefit, [P2] comes up with some technical refinements of the Anzellotti-Giaquinta localization method.

Among these aspects, we first discuss the usage of almost-minimizers. The basic idea in this regard is to weaken the local minimality property, and, suitably adapting the definitions of Anzellotti [12] and Duzaar & Gastel & Grotowski [55], a starting point is to require that a function $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ on an open set $\Omega$ in $\mathbb{R}^n$ satisfies

$$\overline{F}[u; \mathrm{B}_r(x_0)] \le \overline{F}[u{+}\varphi; \mathrm{B}_r(x_0)] + Lr^\beta\big(\mathcal{L}^n{+}|\mathrm{D}u|{+}|\mathrm{D}\varphi|\big)(\mathrm{B}_r(x_0)) \tag{5.4}$$

with a fixed exponent $\beta \in (0,\infty)$ and a fixed constant $L < \infty$, for all balls $\mathrm{B}_r(x_0) \Subset \Omega$ and all $\varphi \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ with $\operatorname{spt}\varphi \subset \mathrm{B}_r(x_0)$. The aim is then to keep working with the simple functionals from (5.2), but to extend Theorem 5.1 to functions $u$ which are not local minimizers, but merely almost-minimizers of $\overline{F}$ in the sense of (5.4). Such an extended result should, in principle, apply to local minimizers of functionals with lower-order terms (and actually also to minimizers of problems with volume or other constraints), since these minimizers turn out to be almost-minimizers of $\overline{F}$. Unfortunately, the notion in (5.4) is still too restrictive in order to include functionals with a genuine zero-order occurrence of the dependent variable $w$, as it will be admitted in the subsequent Section 5.3. Indeed, minimizers of such functionals exhibit only weaker almost-minimality properties with respect to $\overline{F}$, and for this reason we found it necessary to introduce the weaker notion of almost-minimality in the next definition.

**Definition 5.4** ($\mathrm{L}^q$-$\beta$ minimizers)**.** *Suppose that $\Omega$ is open in $\mathbb{R}^n$ and that $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is continuous with* (5.1). *Moreover, fix $q \in [1,\infty]$ and $\beta \in (0,\infty)$. Then we say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is an $\mathrm{L}^q$-$\beta$ minimizer of $\overline{F}$ at $x_0 \in \Omega$ if there exists a function $\omega\colon [0,\infty) \to [0,\infty)$ with the following property: For all balls $\mathrm{B}_r(x_0) \Subset \Omega$, all $\varphi \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ with $\operatorname{spt}\varphi \subset \mathrm{B}_r(x_0)$, and all $M \in [0,\infty)$ with*

$$\big(|\mathrm{D}u|{+}|\mathrm{D}\varphi|\big)(\mathrm{B}_r(x_0)) \le M\mathcal{L}^n(\mathrm{B}_r(x_0))\,, \tag{5.5}$$

$$\|\varphi\|_{\mathrm{L}^q(\mathrm{B}_r(x_0),\mathbb{R}^N)} \le M\Big[r^{1+\frac{n}{q}} + \|u{-}u_{\mathrm{B}_r(x_0)}\|_{\mathrm{L}^q(\mathrm{B}_r(x_0),\mathbb{R}^N)}\Big]\,, \tag{5.6}$$

*there holds*

$$\overline{F}[u; \mathrm{B}_r(x_0)] \le \overline{F}[u{+}\varphi; \mathrm{B}_r(x_0)] + \omega(M)r^\beta\mathcal{L}^n(\mathrm{B}_r(x_0))\,.$$

*Finally, we say that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is an $\mathrm{L}^q$-$\beta$ minimizer of $\overline{F}$ on $\Omega$ if it is an $\mathrm{L}^q$-$\beta$ minimizer of $\overline{F}$ at every $x_0 \in \Omega$ in such a way that $\omega$ can be chosen independent of $x_0$.*

We remark that, in the case $q \le \frac{n}{n-1}$, the inequality (5.6) (possibly with a larger $M$) is an automatic consequence of (5.5) and the Sobolev embedding, so that (5.6) can actually be discarded. Hence, within the range $q \in \big[1, \frac{n}{n-1}\big]$, the notion of $\mathrm{L}^q$-$\beta$ minimizers does actually not depend on $q$. Despite this, there is no doubt that the notion of $\mathrm{L}^q$-$\beta$ minimizers is quite technical. However, its justification lies in the facts that, on one hand, it is weaker than the requirement (5.4) but suffices in order to establish regularity results, and that, on the other hand, in contrast to (5.4) it can be verified in the concrete situations of Section 5.3.

Next we discuss the treatment of degeneration points. Our assumptions in this regard are tailored out for the case of the model integrands $\mathrm{m}_p$, whose second derivatives exhibit at 0 the same degenerate behavior as those of the $p$-energy integrand

$$\mathrm{e}_p(z) := \tfrac{1}{p}|z|^p \qquad \text{for } z \in \mathbb{R}^{N\times n}\,.$$

Since our approach proceeds, to some extent, by reduction to the case of the $p$-energy and vector-valued $p$-harmonic functions, it is natural to include not only the integrands $\mathrm{m}_p$, but also other integrands $f$, which resemble, near some $z_0 \in \mathbb{R}^{N\times n}$, the model integrands $\mathrm{m}_p$ and $\mathrm{e}_p$ near 0. This is achieved by assumption (5.7) below.

Now we are ready to state the variant of Theorem 5.1 which covers almost-minimizers and applies even in degeneration points. The result is taken from [P2, Theorem 2.5, Remark 2.6].

**Theorem 5.5** ($\mathrm{C}^{1,\alpha}$ regularity for BV almost-minimizers near degeneration points)**.** *Consider an open subset $\Omega$ of $\mathbb{R}^n$, a convex integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ with* (5.1)*, and some $z_0 \in \mathbb{R}^{N\times n}$ such that $f$ is either $\mathrm{C}^2$ near $z_0$ with $\nabla^2 f(z_0) > 0$ or $\mathrm{C}^2$ in a punctured neighborhood of $z_0$ with*

$$\lim_{z\to 0} \frac{|\nabla^2 f(z_0{+}z) - \theta\nabla^2\mathrm{e}_p(z)|}{|z|^{p-2}} = 0 \tag{5.7}$$

*for some $p \in (1,\infty)$ and some $\theta \in (0,\infty)$. If $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is an $\mathrm{L}^q$-$\beta$ minimizer of $\overline{F}$ on $\Omega$ with $q \in [1,\infty]$ and $\beta \in (0,\infty)$ and if $\mathrm{D}u$ has Lebesgue value $z_0$ at a point $x_0 \in \Omega$, then*

$$u \text{ is of class } \mathrm{C}^{1,\alpha}\,, \text{ for some } \alpha(n,N,p,\beta) \in (0,1]\,, \text{ in a neighborhood of } x_0\,.$$

*Here, the exponent $\alpha$ is* (*in the sense of Footnote* 2 *in the introduction*) *the minimum of $\beta/\max\{p,2\}$ and the optimal Hölder exponent for the gradient of $\mathbb{R}^N$-valued $p$-harmonic functions in $n$ variables.*

In particular, Theorem 5.5 applies to minimizers of the model integrals $\overline{\mathrm{M}}_p$ from the introduction whenever $1 < p < \infty$. Therefore, part (C) of Theorem 1.1 is contained here as a special case.

We remark that the original statement in [P3, Theorem 2.5] actually works with three more precise and more general assumptions in place of (5.7), but we prefer to avoid the corresponding technical details here. We further point out that [55, Example 3] implies, at least in the case $p = 2 > \beta$, the optimality of the Hölder exponent $\alpha = \beta/2$ described in the theorem.

We only comment on some aspects of the *proof of Theorem* 5.5 in [P3, Section 5]. Actually, we here restrict the discussion to the case of true minimizers of $\overline{F}$ and avoid all issues connected to almost-minimizers. Then, since the remaining cases are already covered by Theorem 5.1, we can evidently reduce to the case that (5.7) holds with $p \neq 2$. The underlying idea of proof in this case is to complement the localization method of Anzellotti & Giaquinta [21], described in the previous section, with a $p$-harmonic comparison technique and corresponding iteration scheme, similar to the ones of Esposito & Mingione [61] and Duzaar & Mingione [56, 57].

Particularly, this means that our reasoning relies not only on the comparison with solutions of linear systems like (5.3), but rather proceeds by *comparison with both p-Laplace and linear systems.* Actually, for each ball $\mathrm{B}_r(x_0)$ in an iteration scheme, one considers the excess $\Phi[u;\mathrm{B}_r(x_0)]$, that is an (in the present situation small) integral quantity which measures the deviation of $u$ from being affine on $\mathrm{B}_r(x_0)$, and distinguishes between two cases: If the ratio $\frac{|(\mathrm{D}u)_{\mathrm{B}_r(x_0)}|^p}{\Phi[u;\mathrm{B}_r(x_0)]}$ is below a (typically large) threshold, the situation at hand is *truly degenerate* and one reasons by comparison with solutions of the $p$-Laplace system; if, however, the ratio is above the threshold, the situation is a *non-uniformly non-degenerate* one, in which the comparison with solutions of linear systems is more adequate, but non-uniform factors like $|(\mathrm{D}u)_{\mathrm{B}_r(x_0)}|^{p-2}$ occur and need to be handled with care. The final $\mathrm{C}^{1,\alpha}$ estimates are the outcome of a careful combination of degenerate and non-degenerate estimates in the iteration scheme.

The techniques described so far are already present in [56] or [57], but the framework of Theorem 5.5 requires further ideas. First of all, since (5.7) fixes only the behavior of $\nabla^2 f$ near $z_0$, and also since $u$ is — in contrast to the situations of [61, 56, 57] — not a priori in $\mathrm{W}^{1,p}_{\mathrm{loc}}$, it seems impossible to directly employ the $p$-harmonic harmonic approximation lemma of [57]. Therefore, all the comparison estimates in [P2] for both the non-degenerate and the degenerate regime are instead implemented via the direct localization method of [21]. However, also this method does not directly apply, since the global $\mathrm{C}^{1,\alpha}$ Schauder estimates for linear systems have no analogue for the degenerate $p$-Laplace system with non-zero boundary values, and indeed the validity of these estimates for the $p$-Laplace system is an major unsolved problem. Motivated by this difficulty, the localization method is customized in [P2] such that it works with a Lipschitz (instead of $\mathrm{C}^{1,\alpha}$) competitor $w$ and with $\nabla w$ just $\mathrm{L}^\infty$-close (instead of $\mathrm{C}^{0,\alpha}$-close) to $z_0$ on $\mathrm{B}_r(x_0)$. Relying on global $\mathrm{W}^{1,p+\varepsilon}$ rather than global $\mathrm{C}^{1,\alpha}$ estimates for the $p$-Laplace system, we can then control the $p$-harmonic function $h$ with $h = w$ on $\partial\mathrm{B}_r(x_0)$ at least in $\mathrm{W}^{1,p+\varepsilon}(\mathrm{B}_r(x_0), \mathbb{R}^N)$. A somewhat surprising insight of [P2] is that one actually need not know that $\nabla h$ is $\mathrm{L}^\infty$-close to $z_0$ at this stage, but that the slight extra control on $h$ in $\mathrm{W}^{1,p+\varepsilon}$ (and in some model cases merely the a priori available control in $\mathrm{W}^{1,p}$) already suffices to suitably implement the comparison of $w$ and $h$. The reason for this improvement of the localization approach is technical in nature, but it roughly corresponds to the observation that, though the $\mathrm{W}^{1,p+\varepsilon}$ estimates for $h$ in combination with the $\mathrm{L}^\infty$-smallness of $|\nabla w - z_0|$ do not anymore imply that the set $S := \{x \in \mathrm{B}_r(x_0) : |\nabla h(x) - z_0| \gg 1\}$ is empty, at least these tools yield an improved control on $\mathcal{L}^n(S)$; compare [P2, Proof of Proposition 5.2] for the precise implementation of the argument. We remark that an improvement based on the usage of $\mathrm{W}^{1,s}$ estimates, possibly with larger $s < \infty$, carries over, to some extent, to the non-degenerate case (where it is, however, less crucial) and also to the origins of the method in geometric measure theory; in the latter regard see the joint work [8] of L. Ambrosio, C. De Lellis, and the author.

At this point we refrain from entering into further details of the proof of Theorem 5.5, which can be found in [P2, Section 5]. □

## 5.3 Selective $\mathrm{C}^{1,\alpha}$ regularity for model problems with lower-order terms

Now we come to the treatment of functionals with lower-order terms, for which the notion of $\mathrm{L}^q$-$\beta$ minimizers has been designed, and we roughly describe the results of [P2, Section 3] in this regard. The considerations apply to local minimizers $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ on an open set $\Omega$ in $\mathbb{R}^n$ of splitting-type functionals

$$[w; A] \mapsto \int_A \left[ f(\,\cdot\,, \mathrm{D}w) + g(\,\cdot\,, w)\mathcal{L}^n \right], \tag{5.8}$$

defined for Borel sets $A$ in $\Omega$ and functions $w \in \mathrm{BV}_{\mathrm{loc}}(U, \mathbb{R}^N)$ on an open neighborhood $U$ of $A$. Here, the Carathéodory integrands $f \colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ and $g \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ are subject to fairly mild hypotheses, and indeed it suffices to suppose that $f(x,z)$ is convex and of linear growth in $z$ and suitably Hölder continuous in $x$, while $g(x,y)$ satisfies a Hölder condition in $y$, but only very mild growth assumptions. Under these assumptions, the essential outcome of [P2, Proposition 3.1] is that local minimizers of the functionals in (5.8) can be regarded as $L^q$-$\beta$ minimizers in the sense of Definition 5.4, where $\beta$ is the minimum of the Hölder exponents of $f$ in $x$ and of $g$ in $y$ and $q \in [1,\infty]$ is an exponent related to the growth of $g$. For this reason, (a minor modification of) Theorem 5.5 is general enough to include local minimizers of the functionals in (5.8). We do not intend to restate this regularity result, which is recorded in [P2, Corollary 3.3], with full details, but rather we illustrate its significance by means of an exemplary case. For the regularizations of the total variation integrand $m_1$ given by

$$m_{1,p}(z) := \max\left\{\tfrac{1}{p}|z|^p, |z| - \tfrac{p-1}{p}\right\} \qquad \text{for } z \in \mathbb{R}^{N\times n}\,, \tag{5.9}$$

an instance of the obtained results is contained in the next theorem. While the given statement is taken from [P2, Theorem 1.3], the case $p=2$, $\zeta \le 1$ has already been obtained by Anzellotti & Giaquinta [21, Theorem 6.1], and the scalar, quadratic case $N=1$, $p=\zeta=2$ has been established by Chen & Rao & Tonegawa & Wunderli [45, Theorem 1.2].

**Theorem 5.6** (selective $C^{1,\alpha}$ regularity for BV minimizers of some model problems). *Consider an open subset $\Omega$ of $\mathbb{R}^n$, an exponent $p \in (1,\infty)$, parameters $\lambda \in [0,\infty)$, $\zeta \in (0,\infty)$, and a function $S \in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$. If $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \cap L^\zeta_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ satisfies*

$$\int_{B_r(x_0)} \left[m_{1,p}(Du) + \lambda|u-S|^\zeta \mathcal{L}^n\right] \le \int_{B_r(x_0)} \left[m_{1,p}(D(u+\varphi)) + \lambda|u+\varphi-S|^\zeta \mathcal{L}^n\right]$$

*for all balls $B_r(x_0) \Subset \Omega$ and all $\varphi \in \mathrm{BV}(\Omega,\mathbb{R}^N) \cap L^\zeta(\Omega,\mathbb{R}^N)$ with $\operatorname{spt}\varphi \subset B_r(x_0)$, then the set*

$$\Omega_\sharp := \left\{x \in \Omega \,:\, \lim_{\varrho \searrow 0} \frac{|Du - z\mathcal{L}^n|(B_\varrho(x))}{\mathcal{L}^n(B_\varrho(x))} = 0 \text{ for some } z \in \mathbb{R}^{N\times n} \text{ with } |z| < 1\right\}$$

*is open, and we have*

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_\sharp,\mathbb{R}^N)\,, \text{ for some } \alpha(n,N,p,\zeta) \in (0,1]\,.$$

In order to prove Theorem 5.6, one first establishes, by more or less the arguments described in Section 4.1, interior $L^\infty$ regularity for $u$. Then, involving this regularity, one applies the above-mentioned result of [P2] to conclude that $u$ is an $L^q$-$\beta$ minimizer of $[w;A] \mapsto \int_A m_{1,p}(Dw)$ with $\beta = \min\{1,\zeta\}$ and $q = \max\{(\zeta-1)n, 1\}$. Once this is shown, the claims of Theorem 5.6 follow from Theorem 5.5. For more details on the deduction of Theorem 5.6 and for similar results under partially weakened assumptions on $S$, we refer again to [P2, Section 3].

We point out that Theorem 5.6 and the previous results in [45, 134] are partially motivated by the occurrence of the functionals

$$w \mapsto \int_\Omega \left[|Dw| + \lambda|w-S|^\zeta\right] \tag{5.10}$$

in image restoration; compare, for instance, the work of Chambolle & Lions [42] for a variety of proposed models in this direction. Here, the function $S$ represents a blurred recording of some image, and one attempts to find a reconstruction of the original image as a minimizer of (5.10) among functions $w \in \mathrm{BV}(\Omega, \mathbb{R}^N)$. The functionals in (5.10) seem particularly suitable in this regard. On one hand, the total variation term $|\mathrm{D}w|$ has a mild regularizing effect on minimizers, but also preserves some jump discontinuities which may correspond to edges in the original picture. On the other hand, the fidelity term $\lambda|w-S|^\zeta$, with a suitably chosen, large parameter $\lambda$, forces minimizers to stay close to the recorded image $S$. The parameter $\zeta$ equals 2 in the original celebrated model of Rudin & Osher & Fatemi [114], but a more recent work of Chan & Esedoḡlu [43] has also raised some interest in the choice $\zeta = 1$ (which may, however, bring up non-uniqueness of minimizers). Anyhow, difficulties with the approach based on (5.10) arise from the total variation term, which is difficult to handle in both numerical and analytical regards. Therefore, it is also common to regularize this term by replacing the total variation integrand $\mathrm{m}_1$ with integrands of the type $\mathrm{m}_p$ or $\mathrm{m}_{1,p}$, usually with $p = 2$, however. Indeed, the most promising replacement for $\mathrm{m}_1$ might be the rescalings $\mathrm{m}_{1,p}^\varepsilon(z) := \varepsilon \mathrm{m}_{1,p}(z/\varepsilon)$, which are regularized on $\varepsilon$-balls in $\mathbb{R}^{N\times n}$ and coincide elsewhere (up to an additive constant) with $\mathrm{m}_1$. Needless to say, Theorem 5.6 applies correspondingly with $\mathrm{m}_{1,p}^\varepsilon$ in place of $\mathrm{m}_{1,p}$ and then yields local $\mathrm{C}^{1,\alpha}$ regularity of BV minimizers $u$ on the set

$$\Omega_\sharp^\varepsilon := \left\{ x \in \Omega \,:\, \lim_{\varrho \searrow 0} \frac{|\mathrm{D}u - z\mathcal{L}^n|(\mathrm{B}_\varrho(x))}{\mathcal{L}^n(\mathrm{B}_\varrho(x))} = 0 \text{ for some } z \in \mathbb{R}^{N\times n} \text{ with } |z| < \varepsilon \right\}.$$

Hence, the theorem rigorously confirms that the minimization of the relevant functional has, as desirable in the image restoration context, a smoothing effect near points with small gradients.

Next we briefly discuss the *other* desirable effect in image restoration, namely the preservation of edges or, in other words, the inheritance of jump discontinuous from $S$ to $u$. We believe that would be very interesting to complement Theorem 5.6 with a corresponding result, which specifies conditions on a point $x$ in the approximate jump discontinuity set $\mathrm{J}_S$ of $S$ in order to ensure its membership also in $\mathrm{J}_u$. While, we are not aware of any published contribution in this direction, we briefly mention two converse results of Caselles & Chambolle & Novaga in the case $N = 1$, $\zeta = 2$. For $S \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, they showed in [40, Theorem 3.4] the inclusion $\mathrm{J}_u \subset \mathrm{J}_S$ (up to an $\mathcal{H}^{n-1}$-negligible set) for every BV minimizer $u$ of (5.10). For $n \le 7$ and $S \in \mathrm{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha \in (0,1]$, they proved in [41, Theorem 4.5] that every BV minimizer $u$ of (5.10) satisfies $u \in \mathrm{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$. For a more detailed discussion of BV minimizers of (5.10) (mostly with $\zeta \ge 1$) and in particular for a detailed study of their level sets in the case $N = 1 < n \le 7$, we also refer to Allard's papers [3, 4, 5].

Finally, we briefly comment on functionals of the general type

$$[w; A] \mapsto \int_A f(\,\cdot\,, w, \mathrm{D}w)\,,$$

where the integrand $f$ has still linear growth in the gradient variable, but exhibits also a non-splitting dependence on the zero-order variable. Though such general functionals are not approachable via Definition 2.1, the questions for semicontinuity, relaxation, and existence of BV minimizers have essentially been settled in a suitably adapted framework [50, 10, 22, 65, 63, 11, 64]. Proving any regularity of the BV minimizers, however, has remained an open problem, mainly due to the fact that it has not yet been possible to apply any Gehring type result, to establish any extra gradient integrability, or to implement a suitable freezing method; compare with [21, Section 6].

# Appendix A

# Strict approximation on non-Lipschitz domains

In this chapter we describe two approximation results, which have been obtained in [P4]. The second result is in fact stated in a slightly refined form, which extends the corresponding result of [P4] to possibly unbounded domains and has been worked out in [P3, Section 3.3].

## A.1 Strict interior approximation of a set of finite perimeter

The perimeter $\mathrm{P}(\Omega)$ of an $\mathcal{L}^n$-measurable subset $\Omega$ of $\mathbb{R}^n$ is defined as

$$\mathrm{P}(\Omega) := \sup\left\{ \int_\Omega \operatorname{div}\varphi \,\mathrm{d}x \,:\, \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n,\mathbb{R}^n),\, \sup_{\mathbb{R}^n}|\varphi| \le 1 \right\} \in [0,\infty]\,,$$

and in the case $\mathrm{P}(\Omega) < \infty$ one calls $\Omega$ a set of finite perimeter. For smooth sets $\Omega$, it follows from the divergence theorem that $\mathrm{P}(\Omega)$ coincides with the $(n{-}1)$-dimensional measure $\mathcal{H}^{n-1}(\partial\Omega)$ of the boundary $\partial\Omega$, while, in general, one only has the inequality $\mathrm{P}(\Omega) \le \mathcal{H}^{n-1}(\partial\Omega)$. A basic question about a set $\Omega$ of finite perimeter is whether it can be approximated by smooth sets $\Omega_k$ such that $\mathrm{P}(\Omega_k)$ converges to $\mathrm{P}(\Omega)$. More precisely, one would aim at finding a sequence of open sets $(\Omega_k)_{k\in\mathbb{N}}$ with smooth boundaries in $\mathbb{R}^n$ such that $\Omega_k$ converges in measure and perimeter to $\Omega$, that is, in the sense of

$$\lim_{k\to\infty} \big[\mathcal{L}^n((\Omega_k\setminus\Omega)\cup(\Omega\setminus\Omega_k)) + |\mathrm{P}(\Omega_k)-\mathrm{P}(\Omega)|\big] = 0\,.$$

If one imposes no additional requirements, such approximations $\Omega_k$ can be found by a well-known reasoning, namely by choosing, via the coarea formula, good level sets of mollifications of the indicator function $\mathbb{1}_\Omega$; see [9, Theorem 3.42] for details. However, in case of an open $\Omega$, it is also reasonable to require that the approximations come strictly from within $\Omega$ in the sense that $\Omega_k \Subset \Omega$ holds for all $k \in \mathbb{N}$. Specifically, the latter requirement gets relevant in connection with a boundary condition (see below) or if some functions considered are defined only on $\Omega$. In any case, the problem of finding approximations from within is considerably harder, and seemingly it has been solved in the classical literature [106, 97, 53] only for the case of a bounded Lipschitz domain $\Omega$. In [P4, Theorem 1.1], the previous results are extended to a (possibly) rough $\Omega$ as follows.

**Theorem A.1** (strict interior approximation of a set of finite perimeter)**.** *Suppose that a bounded open subset $\Omega$ of $\mathbb{R}^n$ satisfies*

$$\mathrm{P}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) < \infty\,. \tag{A.1}$$

*Then, there exists a sequence of open sets $\Omega_k \Subset \Omega$ with smooth boundaries such that $\Omega_k$ converges to $\Omega$ in measure and perimeter. Moreover, we can achieve that $\Omega_k$ converges to $\Omega$ and that $\partial\Omega_k$ converges to $\partial\Omega$ also in the Hausdorff distance*[1] $\mathrm{d}_{\mathcal{H}}$.

Before describing a proof of Theorem A.1, we comment on the decisive hypothesis (A.1). To this end, we recall that, for every set $\Omega$ of finite perimeter in $\mathbb{R}^n$, we have $\mathbb{1}_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, and moreover, by De Giorgi's structure theorem, $|\mathrm{D}\mathbb{1}_\Omega|$ coincides with the restriction $\mathcal{H}^{n-1} \llcorner \partial^*\Omega$ of the Hausdorff measure $\mathcal{H}^{n-1}$ to the reduced boundary[2] $\partial^*\Omega$ of $\Omega$; compare [9, Chapter 3.5]. In this light, (A.1) turns out to be equivalent to having $\mathrm{P}(\Omega) < \infty$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$, and thus it expresses the requirement that the topological boundary exceeds the measure-theoretic boundary at most by an $\mathcal{H}^{n-1}$-negligible set. In fact, for $n \geq 2$ this hypothesis (or a similar one) is inevitable in Theorem A.1, as one can see considering the simple example $\Omega = (0,1)^2 \setminus \left[\left\{\frac{1}{2}\right\}\times(0,1)\right] \subset \mathbb{R}^2$ depicted in Figure 4, where the 'missing' line segment $\left\{\frac{1}{2}\right\}\times(0,1)$ does not contribute to the perimeter $\mathrm{P}(\Omega) = 4$, but contributes once to the measure $\mathcal{H}^1(\partial\Omega) = 5$ of the topological boundary and contributes essentially twice to the perimeter of good interior approximations. Beyond this basic example, the necessity of (A.1) is discussed in [P4, Section 5]. There, it is also shown that the assumption (A.1) has a partial optimality property in dimension $n = 2$ (which is based on a reasoning with the path-connected components of $\partial\Omega_k$ and the Golab theorem on the lower semicontinuity of $\mathcal{H}^1$ along $\mathrm{d}_{\mathcal{H}}$-convergent sequences of connected sets).

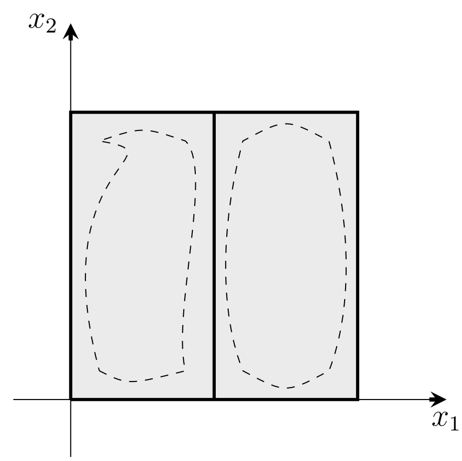


**Figure 4:** The exemplary domain $\Omega$ and an interior approximation with perimeter (dashed curves) not so far from 6.

Next we sketch the *proof of Theorem* A.1*,* given in [P4, Section 3]. We first observe that it suffices to consider a fixed $\varepsilon \in \left(0, \frac{1}{2}\right]$ and to construct an $\mathcal{L}^n$-measurable set $\Omega_\varepsilon \Subset \Omega$ which is close to $\Omega$ in measure and Hausdorff distance and satisfies $\mathrm{P}(\Omega_\varepsilon) \leq \mathrm{P}(\Omega) + \varepsilon$. Indeed, once

[1]The Hausdorff distance $\mathrm{d}_{\mathcal{H}}(A, B)$ of two non-empty, bounded subsets $A, B \subset \mathbb{R}^n$ is defined as $\max\{\sup_{a\in A} \mathrm{dist}(a, B), \sup_{b\in B} \mathrm{dist}(A, b)\}$.

[2]The reduced boundary $\partial^*\Omega$ of $\Omega$ can be defined as the Lebesgue set of the $\mathrm{L}^1$-vector field $\frac{\mathrm{d}\mathrm{D}\mathbb{1}_\Omega}{\mathrm{d}|\mathrm{D}\mathbb{1}_\Omega|}$ with respect to $|\mathrm{D}\mathbb{1}_\Omega|$. Essentially, $\partial^*\Omega$ can be imagined as the set of boundary points $x \in \partial\Omega$ in which $\Omega$ possesses a well-defined inward unit normal, given by the Lebesgue value $\frac{\mathrm{d}\mathrm{D}\mathbb{1}_\Omega}{\mathrm{d}|\mathrm{D}\mathbb{1}_\Omega|}(x)$.

this is achieved, the proof can be completed by the mollification procedure already mentioned above and by using the lower semicontinuity of the perimeter. The construction of $\Omega_\varepsilon$, in turn, exploits arguments from the proof of De Giorgi's structure theorem to find a decomposition of the reduced boundary $\partial^*\Omega$ into countably many pieces $R_1, R_2, R_3, \ldots$ such that each $R_i$ is almost flat in the balls $\mathrm{B}_{1/i}(x)$ around all points $x \in R_i$, in the sense that $R_i \cap \mathrm{B}_{1/i}(x)$ remains in a cone with vertex $x$ and opening angle $\varepsilon$ around the approximate tangent space to $\partial^*\Omega$ at $x$. In the next step, each $R_i$ is covered by countably many $n$-dimensional cylinders $C_{i,j}$ with radii $r_{i,j} \lesssim \min\{\varepsilon, 1/i\}$ and heights $h_{i,j}$ such that the sum $\sum_{j=1}^\infty \omega_{n-1} r_{i,j}^{n-1}$ of the measures of the $(n{-}1)$-dimensional cross sections is close to $\mathcal{H}^{n-1}(R_i)$. The crux of the proof lies then in the achievement of the following additional properties of the cylinders, which are illustrated in Figure 5 below. On one hand, thanks to the choice of the $R_i$, the $C_{i,j}$ can be taken almost flat, in fact $h_{i,j} \approx \varepsilon r_{i,j}$. On the other hand, the $C_{i,j}$ can be arranged such that, up to sets of small measure, only one half of $\partial C_{i,j}$ intersects $\Omega$. In combination, these properties imply that $\sum_{j=1}^\infty \mathcal{H}^{n-1}(\Omega \cap \partial C_{i,j})$ is still close to $\mathcal{H}^{n-1}(R_i)$, and all in all $(C_{i,j})_{i,j\in\mathbb{N}}$ is a cover of $\partial^*\Omega$ with $\sum_{i,j=1}^\infty \mathcal{H}^{n-1}(\Omega \cap \partial C_{i,j}) \le \mathcal{H}^{n-1}(\partial^*\Omega) + \frac{1}{2}\varepsilon$. In addition, by the hypothesis (A.1), the remainder $\partial\Omega \setminus \partial^*\Omega$ of $\partial\Omega$ is $\mathcal{H}^{n-1}$-negligible and can be covered, in a much simpler way, by countably many balls $B_i$ with small radii and total perimeter at most $\frac{1}{2}\varepsilon$. At this point, finally, $\Omega_\varepsilon$ is obtained by removing finite subcovers of both the cylinders $(C_{i,j})_{i,j\in\mathbb{N}}$ and the balls $(B_i)_{i\in\mathbb{N}}$ from $\Omega$. Then, $\Omega_\varepsilon$ is close to $\Omega$ in measure and Hausdorff distance, and most significantly $\partial\Omega_\varepsilon$ is contained in the union of the boundary portions $\Omega \cap \partial C_{i,j}$ and $\partial B_i$, so that we can control $\mathrm{P}(\Omega_\varepsilon) \le \mathcal{H}^{n-1}(\partial^*\Omega) + \varepsilon = \mathrm{P}(\Omega) + \varepsilon$. As indicated in the beginning, this suffices to establish Theorem A.1. □

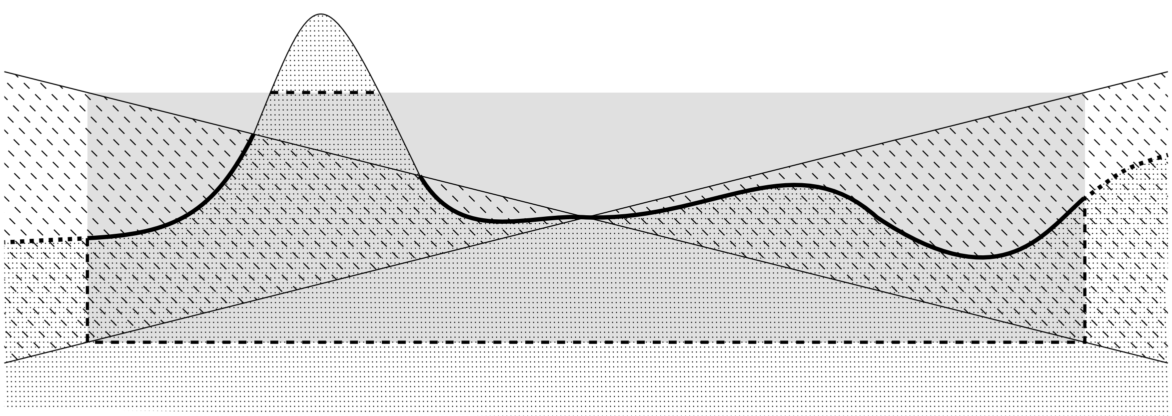

**Figure 5:** The construction in the proof of Theorem A.1:
A portion $R_i$ (thick curves) of the boundary of $\Omega$ (dotted area) is locally contained in a cone (ruled area) around a tangent space and can locally be covered by an almost flat cylinder $C_{i,j}$ (gray shaded area) with boundary portion $\Omega \cap \partial C_{i,j}$ (dashed lines) of controlled $\mathcal{H}^{n-1}$-measure.

We point out that it is actually the convergence in perimeter which is responsible for most difficulties in the above proof. In contrast, it is much simpler to construct interior approximations $\Omega_k \Subset \Omega$ such that $\Omega_k$ converges to $\Omega$ in measure and such that there holds merely

$$\limsup_{k\to\infty} \mathrm{P}(\Omega_k) \le C\mathcal{H}^{n-1}(\partial\Omega)\,, \qquad \text{with } \textit{some} \text{ dimensional constant } C < \infty\,. \tag{A.2}$$

To achieve (A.2), there is no need to involve De Giorgi's structure theorem or ideas from its proof, further no need to cover $\partial\Omega$ by *almost flat* objects, and also no need to care that only *one half* of their boundaries intersects $\Omega$. Instead, a well-known argument based on a simple covering with balls suffices; see [9, Proof of Proposition 3.62]. We remark that the same basic argument has also been used in connection with [44, Proposition 8.1] where, unfortunately, a claim on convergence in perimeter is stated, but only a property of the type (A.2) is established.

## A.2 Strict approximation of a BV function from a Dirichlet class

The publication [P4] also provides a counterpart of Theorem A.1, which concerns the approximation of a given BV function from a prescribed Dirichlet class and which has already been used extensively in the previous chapters. While the original statement [P4, Theorem 1.2] applies on bounded open sets $\Omega \subset \mathbb{R}^n$ with (A.1), we here restate the slight refinement of [P3, Lemma 3.12], which covers even the case of (possibly) unbounded $\Omega$ with (possibly) infinite perimeter. Employing the terminology of Sections 2.1 and 2.3, and imposing on $\partial\Omega$ the mild regularity condition

$$\mathbb{1}_\Omega \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } |\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega\,, \tag{A.3}$$

this refinement reads as follows.

**Theorem A.2** (strict approximation of a BV function from a Dirichlet class)**.** *Suppose that an open subset $\Omega$ of $\mathbb{R}^n$ satisfies* (A.3)*, and fix $u_0 \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\nabla u_0 \in \mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$. Then, for every $w \in \mathrm{BV}_{u_0}\big(\overline{\Omega},\mathbb{R}^N\big)$ and every $\Psi \in \mathrm{L}^1(\Omega)$, there exists a sequence $(w_k)_{k\in\mathbb{N}}$ of approximations in $u_0 + \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^N)$ such that $\|w_k - w\|_{\mathrm{L}^1(\Omega,\mathbb{R}^N)}$ converges to $0$ and such that $\mathrm{D}w_k$ converges $\Psi$-strictly in $\mathrm{RM}\big(\overline{\Omega},\mathbb{R}^{N\times n}\big)$ to $\mathrm{D}w$. In addition, we can achieve that $\nabla w_k$ converges $\mathcal{L}^n$-a.e. on $\Omega$ to $\nabla w$.*

In connection with the hypothesis (A.3), we point out that it is nothing but a local version of (A.1). Indeed, it allows for $\mathrm{P}(\Omega) = \infty$ (in case that $\Omega$ is unbounded), but is equivalent to saying that $\Omega$ is a set of locally finite perimeter in $\mathbb{R}^n$ with $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$. In particular, one can deduce from (A.3) that $\partial\Omega$ is $\mathcal{H}^{n-1}$-$\sigma$-finite and in fact even countably $\mathcal{H}^{n-1}$-rectifiable.

We only describe the *proof of Theorem* A.2 in the slightly simpler case of bounded $w$, $u_0$, and $\Omega$, while we refer to [P4, Section 4] and [P3, Section 3.3] for the cut-off arguments relevant to the general case. In the bounded case, the reasoning follows the proof of [P4, Proposition 4.1] and refines the previously illustrated proof of Theorem A.1. Indeed, the main issue is to construct, for fixed $\varepsilon \in \big(0,\frac{1}{2}\big]$, an open set $\Omega_\varepsilon \Subset \Omega$ such that $\Omega_\varepsilon$ is close to $\Omega$ in measure and such that the interior trace $w^{\mathrm{int}}_{\partial^*\Omega_\varepsilon}$ of $w$ on $\partial^*\Omega_\varepsilon$ is controlled by the up-to-an-$\varepsilon$-error estimate

$$\int_{\partial^*\Omega_\varepsilon} |w^{\mathrm{int}}_{\partial^*\Omega_\varepsilon} - u_0|\,\mathrm{d}\mathcal{H}^{n-1} \le \int_{\partial^*\Omega} |w^{\mathrm{int}}_{\partial^*\Omega} - u_0|\,\mathrm{d}\mathcal{H}^{n-1} + \varepsilon\,.$$

Suitable $\Omega_\varepsilon$ are constructed in [P4] by covering, as in the previous Section A.1, (parts of) $\partial^*\Omega$ with almost flat cylinders $C_{i,j}$ and by eventually removing these cylinders and also some balls from $\Omega$. Here we also need to choose the $C_{i,j}$ such that $w$ is $\varepsilon$-almost constant on each of them, and we additionally rely on some fine properties of traces, but otherwise the reasoning remains close to the previously described one and we omit further details. Once, suitable $\Omega_\varepsilon$ are found, the desired $w_k$ are then obtained as functions of the form $u_0 + (\mathbb{1}_{\Omega_\varepsilon}(w-u_0))_{r(\varepsilon)}$, where the subscript $r(\varepsilon)$ stands for the mollification with a suitably small radius $r(\varepsilon) \le \varepsilon$.

What has been said so far does, however, not yet grant the claim on $\mathcal{L}^n$-a.e. convergence $\nabla w_k \to \nabla w$ to the density $\nabla w$ of the absolutely continuous part of $\mathrm{D}w$, a claim which occurs similarly in [18, Lemma 5.1] and in the present connection in [P3, Lemma 3.12]. Anyway, achieving this convergence is fairly easy once one observes that standard mollifications of $w$ actually converge in the required fashion (since evidently mollifications of $\nabla w$ converge $\mathcal{L}^n$-a.e. to $\nabla w$, while mollifications of $\mathrm{D}^s w$ converge, as functions, $\mathcal{L}^n$-a.e. to 0). Therefore, the required convergence follows either directly from the above-sketched construction or by patching up the just-constructed $w_k$ near $\partial\Omega$ and standard mollifications away from $\partial\Omega$; see [P3, Section 3.3] for further details. $\square$

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