VARIATIONAL ANALYSIS OF A REDUCED ALLEN–CAHN ACTION FUNCTIONAL

ANNIBALE MAGNI

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We consider systems in a bounded domain $\Omega \subset \mathbf{R}^n$ having two stable states and admitting a mean field description in terms of a phase function $u : \Omega \to \mathbf{R}$ and with a free energy of the form

$$\mathcal{F}(u) = \int_\Omega \Big(\frac{\epsilon}{2}|\nabla u|^2 + W(u)\Big) dx\,, \tag{1}$$

where $W : \mathbf{R} \to \mathbf{R}$ is a suitable double well potential, and $\epsilon > 0$.
While the *zero–temperature* dynamics of such a system is given by the gradient flow of (1), the evolution at a temperature $\gamma > 0$ is described by means of the Allen-Cahn equation perturbed by an additive white noise η, i.e.

$$\epsilon\partial_t u = \epsilon\Delta u - \frac{1}{\epsilon}W'(u) + \sqrt{2\gamma}\,\eta\,. \tag{2}$$

Large deviation principles for stochastically perturbed Allen–Cahn equations have been considered among others by [1],[2],[3] and the corresponding action functional for a time $T > 0$ has been computed to be

$$\mathcal{S}_\epsilon(u) = \int_0^T \int_\Omega \Big(\epsilon(\partial_t u)^2 + \frac{1}{\epsilon}(-\epsilon\Delta u + \frac{1}{\epsilon}W'(u))^2\Big)\, dx\, dt\,. \tag{3}$$

For given initial and final states of the system, an action minimizer represents a most likely connecting path between the two states and the value of the minimum of the action is related to the probability that the transition between the two states takes place in the given time T.
In [4] we have studied the minimization of the sharp interface limit of $\mathcal{S}_\epsilon$ for $\epsilon \to 0$, with prescribed initial and final states. The limit functional, also called *reduced* action functional has been computed in [3], [6] and reads

$$\mathcal{S}_0(\Sigma) = \int_0^T \int_{\Sigma_t} \Big(|\vec{v}(x,t)|^2 + |\vec{\mathrm{H}}(x,t)|^2\Big) d\mathcal{H}^n(x)\, dt + 4\sum_{i\in J} \mathcal{H}^n(\Sigma^i)\,, \tag{4}$$

where $\Sigma := (\Sigma_t)_{t\in[0,T]}$ is a family of smoothly evolving smooth hypersurfaces (with normal speed $\vec{v}$ and mean curvature $\vec{\mathrm{H}}$) out of a discrete set of times $J \subset [0,T]$ at which new components can be nucleated.
The straightforward application of the direct method of the Calculus of Variations to the functional (4) in the class of evolving integral varifolds with speed and mean curvature in L^2 fails. This failure is due to the impossibility of keeping track of the initial and final data along a minimizing sequence, which in turn is a consequence of the fact that a bound on (4) ensures only a control in BV((0,T)) for the total area of the evolving varifolds. To overcome this problem, we complemented the evolution of varifolds with a phase evolution according to the following definitions

Definition 1. Consider a family $\boldsymbol{\mu} := (\mu_t)_{t\in(0,T)}$ of Radon measures on $\mathbf{R}^{n+1}$ and set $\mu := \mu_t \otimes \mathcal{L}^1$. We call $\boldsymbol{\mu}$ an L^2-flow if (for almost all $t \in (0,T)$) μ_t is an integral $n-$varifold with mean curvature $\vec{\mathrm{H}} \in L^2(\mu_t; \mathbf{R}^{n+1})$, $\boldsymbol{\mu}$ has generalized normal speed $\vec{v} \in L^2(\mu; \mathbf{R}^{n+1})$ (see [5], Definition 3.1) and $\sup_{0<t<T} \mu_t(\mathbf{R}^{n+1}) < \infty$.

Date: January 12, 2015.

Definition 2. Given $T > 0$ and two open sets $\Omega(0)$, $\Omega(T) \subset \mathbf{R}^{n+1}$ with finite perimeter, let $\mathcal{M} = \mathcal{M}(T, \Omega(0), \Omega(T))$ be the class of pairs $(\boldsymbol{\mu}, \boldsymbol{u})$, with $\boldsymbol{\mu} := (\mu_t)_{t\in(0,T)}$ and $\boldsymbol{u} := (u_t)_{t\in(0,T)}$, such that the evolution $\boldsymbol{\mu}$ is an L^2-flow, for almost all $t \in (0, T)$

$$u(\cdot,t) \in BV(\mathbf{R}^{n+1}, \{0,1\}), \quad \text{with} \quad |\nabla u(\cdot,t)| \leq \mu_t \tag{5}$$

hold, the evolution of phases $\boldsymbol{u}$ satisfies $u \in C^{\frac{1}{2}}([0,T]; L^1(\mathbf{R}^{n+1}))$, $\boldsymbol{u}$ attains the initial and final data

$$u(\cdot,0) = \mathcal{X}_{\Omega(0)}, \quad u(\cdot,T) = \mathcal{X}_{\Omega(T)}, \tag{6}$$

and

$$\int_{\mathbf{R}^{n+1}\times(0,T)} \partial_t\eta(x,t)u(x,t)\,dx\,dt = \int_{\mathbf{R}^{n+1}\times(0,T)} \eta(x,t)\vec{v}(x,t)\cdot\nu(x,t)\,d|\nabla u(\cdot,t)|\,dt \tag{7}$$

holds for all $\eta \in C^1_c(\mathbf{R}^{n+1} \times (0,T))$, where $\vec{v}$ is the generalized velocity of $\boldsymbol{\mu}$ and where $\nu(\cdot,t)$ denotes the generalized inner normal on $\partial^*\{u(\cdot,t) = 1\}$.

In the class $\mathcal{M}$ of generalized evolutions, we have given a generalized definition for the action functional taking into account also the phase evolution $\boldsymbol{u}$.

Definition 3. For $\boldsymbol{\Sigma} \in \mathcal{M}$, $\boldsymbol{\Sigma} = (\boldsymbol{\mu}, \boldsymbol{u})$ as above, we define

$$\mathcal{S}(\boldsymbol{\Sigma}) := \mathcal{S}_+(\boldsymbol{\Sigma}) + \mathcal{S}_-(\boldsymbol{\Sigma}), \tag{8}$$

$$\mathcal{S}_+(\boldsymbol{\Sigma}) := \sup_\eta \Big[2|\nabla u(\cdot,T)|(\eta(\cdot,T)) - 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) + \int_{\mathbf{R}^{n+1}\times(0,T)} -2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v} - \vec{\mathrm{H}}|^2\,d\mu_t\,dt\Big], \tag{9}$$

$$\mathcal{S}_-(\boldsymbol{\Sigma}) := \sup_\eta \Big[-2|\nabla u(\cdot,T)|(\eta(\cdot,T)) + 2|\nabla u(\cdot,0)|(\eta(\cdot,0)) + \int_{\mathbf{R}^{n+1}\times(0,T)} 2\big(\partial_t\eta + \nabla\eta\cdot\vec{v}\big) + (1-2\eta)_+\frac{1}{2}|\vec{v} + \vec{\mathrm{H}}|^2\,d\mu_t\,dt\Big], \tag{10}$$

where the supremum is taken over all $\eta \in C^1(\mathbf{R}^{n+1} \times [0,T])$ with $0 \leq \eta \leq 1$.

For sufficiently regular evolutions, the generalized action functional agrees with (4), as proven in the following

Theorem 4. *Let $\boldsymbol{\Sigma}$ be given by an evolution $(\Omega(t))_{t\in[0,T]}$ of open sets $\Omega(t) \subset \mathbf{R}^{n+1}$, which means*

$$u(\cdot,t) = \mathcal{X}_{\Omega(t)} \qquad \text{and} \qquad \mu_t := \mathcal{H}^n \llcorner \partial\Omega(t).$$

Assume that $(\partial\Omega(t))_{t\in[0,T]}$ represents, outside of a set (possibly empty) of singular times $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = T$, a smooth evolution of smooth hypersurfaces. Then

$$\mathcal{S}(\boldsymbol{\Sigma}) = \int_0^T \int_{\partial\Omega(t)} (|\vec{v}(\cdot,t)|^2 + |\vec{\mathrm{H}}(\cdot,t)|^2)\,d\mathcal{H}^n\,dt + 2\sum_{j=0}^{k+1} \sup_\psi |\mu_{t_j+}(\psi) - \mu_{t_j-}(\psi)|, \tag{11}$$

where the supremum is taken over all $\psi \in C^1(\mathbf{R}^{n+1})$ with $|\psi| \leq 1$, and where we have set $\mu_t := \mathcal{H}^n \llcorner \partial\Omega(0)$ for $t < 0$ and $\mu_t := \mathcal{H}^n \llcorner \partial\Omega(T)$ for $t > T$.

Within the setting of generalized evolutions, we finally were able to apply the direct method and prove existence of global minimizers for $\mathcal{S}$.

Theorem 5. *Let $T > 0$ and let $\Omega(0) \subset \mathbf{R}^{n+1}$, $\Omega(T) \subset \mathbf{R}^{n+1}$ be two given open bounded sets with finite perimeter. Consider a family of evolutions $(\boldsymbol{\Sigma}_l)_{l\in\mathbb{N}} \subset \mathcal{M}(T, \Omega(0), \Omega(T))$ with*

$$\mathcal{S}(\boldsymbol{\Sigma}_l) \leq \Lambda \quad \text{for all } l \in \mathbb{N}, \tag{12}$$

where $\Lambda > 0$ is a fixed constant.
Then there exists a subsequence $l \to \infty$ (not relabelled) and a limit evolution $\mathbf{\Sigma} \in \mathcal{M}(T, \Omega(0), \Omega(T))$, such that

$$u^l \to u \qquad \text{in } L^1(Q_T) \cap C^0([0,T]; L^1(\mathbf{R}^{n+1})), \tag{13}$$

$$\mu_t^l \to \mu_t \quad \text{for almost all } t \in (0,T) \text{ as integral varifolds on } \mathbf{R}^{n+1}, \tag{14}$$

$$\mu^l \to \mu \quad \text{as Radon measures on } Q_T, \tag{15}$$

and such that $u \in C^{0,1/2}([0,T]; L^1(\mathbf{R}^{n+1}))$ and $\mu \ll \mathcal{H}^{n+1}$.
Moreover it holds

$$\mathcal{S}(\mathbf{\Sigma}) \le \liminf_{l \to \infty} \mathcal{S}(\mathbf{\Sigma}_l)\,. \tag{16}$$

In particular, the minimum of $\mathcal{S}$ in $\mathcal{M}(T, \Omega(0), \Omega(T))$ is attained.

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