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REGULARITY THEORY FOR MASS-MINIMIZING CURRENTS (AFTER ALMGREN-DE LELLIS-SPADARO)

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Contents

INTRODUCTION

The development of Geometric Measure Theory has been mostly motivated by the attempts to solve Plateau's problem, that we can state here as follows.

Plateau's problem. Let M be a $(m+n)$-dimensional Riemannian manifold and let $\Gamma \subset M$ be a compact $(m-1)$-dimensional oriented embedded submanifold without boundary. Find an m-dimensional oriented embedded submanifold Σ with boundary Γ such that

$$\mathrm{vol}_m(\Sigma) \leq \mathrm{vol}_m(\Sigma'),$$

for all oriented submanifolds $\Sigma' \subset M$ such that $\partial \Sigma' = \Gamma$.

As a matter of fact, Plateau's problem (here stated in classical terms and for embedded submanifolds) can be very sensitive to the choice of the dimension m, the codimension n and to the class of admissible surfaces. For instance, in the case $m = 2$ and for boundaries Γ parametrized on the boundary of the unit disk D of $\mathbb{R}^2$, J. Douglas [25] and T. Radó [45] provided existence of solutions, using the fact that the so-called conformal parametrizations lead to good compactness properties of minimizing sequences. However, for general dimension and codimension, parametric methods fail.

It is a well-known fact that, in the formulation I gave, the solution of the Plateau's does not always exist. For example, consider $M = \mathbb{R}^4$, $n = m = 2$ and Γ the smooth Jordan curve parametrized in the following way:

$$\Gamma = \left\{ (\zeta^2, \zeta^3) : \zeta \in \mathbb{C},\ |\zeta| = 1 \right\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4,$$

where we use the usual identification between $\mathbb{C}^2$ and $\mathbb{R}^4$, and we choose the orientation of Γ induced by the anti-clockwise orientation of the unit circle $|\zeta| = 1$ in $\mathbb{C}$. It can be shown

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by a calibration technique (see the next sections) that there exists no smooth solution to the Plateau problem for such fixed boundary, and the *(singular) immersed* 2-dimensional disk

$$S = \{(z,w) \,:\, z^3 = w^2,\ |z| \le 1\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4,$$

oriented in such a way that $\partial S = \Gamma$, satisfies

$$\mathcal{H}^2(S) < \mathcal{H}^2(\Sigma),$$

for all smooth, oriented 2-dimensional submanifolds $\Sigma \subset \mathbb{R}^4$ with $\partial\Sigma = \Gamma$. Here and in the following we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, which for $k \in \mathbb{N}$ corresponds to the ordinary k-volume on smooth k-dimensional submanifolds.

This fact motivates the introduction of *weak solutions* to the Plateau problem, including at least immersed submanifolds, and the main questions about their existence and regularity. In this text I will focus on the line of thought that originated from E. De Giorgi's work for oriented hypersurfaces, thought as boundaries of sets (the so-called theory of sets of finite perimeter, closely related also to the theory of BV functions), which eventually led H. Federer and W.H. Fleming [27] to the very succesful theory of currents, which provides weak solutions with no restriction on dimension and codimension (I will not discuss here the topological point of view and the formulation of Plateau's problem adopted by E.R. Reifenberg [46, 47], which is more appropriate for non-oriented surfaces).

In a parallel way, also the regularity theory has been first developed in codimension 1, essentially thanks to the work of E. De Giorgi. The ideas introduced by E. De Giorgi in [17] had an impact also in other fields, as I will illustrate, and they could be almost immediately applied and adapted also to the higher codimension regularity theory (by F. Almgren [6] for currents, by B. Allard [2] for varifolds) to provide regularity *in a dense open subset* of the support. A major open problem, already pointed out in H. Federer's monograph [26], was then the achievement of an *almost everywhere* regularity theory, possibly with an estimate on the codimension of the singular set $\mathrm{Sing}(T)$ inside the support $\mathrm{spt}\,(T) \setminus \mathrm{spt}\,(\partial T)$ out of the boundary.

It took many years to F. Almgren to develop an innovative and monumental program for the almost everywhere regularity and, at the same time, for the optimal estimate of the dimension of the singular set in arbitrary dimension and higher codimension. Announced in the early '80, circulated in preprint form and published posthumous in [7], his work provides the interior partial regularity up to a (relatively) closed set of dimension at most $(m-2)$ inside $\mathrm{spt}\,(T) \setminus \mathrm{spt}\,(\partial T)$:

Theorem 0.1 (F. Almgren). *Let T be an m-dimensional area minimizing integer rectifiable current in a C^5 Riemannian manifold M. Then, there exists a closed set* $\mathrm{Sing}(T)$ *of Hausdorff dimension at most $(m-2)$ such that in $M \setminus (\mathrm{spt}\,(\partial T) \cup \mathrm{Sing}(T))$ the current T is induced by the integration over a smooth oriented submanifold of M.*

Almgren's result can be recovered in the case $m = 2$ (proving the existence of a discrete set of singular points) with simpler proofs for some classes of calibrated currents, see [12, 51, 52], which are indeed area-minimizing. Although some parts of F. Almgren's work, for instance the theory of Q-valued functions and the use of frequency function, were known and used by the specialists (see also the regularity results in the 2-dimensional case [14, 44]), I think it is fair to say that some parts of his technical *tour de force* had not been completely reviewed. In any case, the necessity to have this whole program streamlined and improved was widely felt, as a necessary step towards the advancement of the field and the analysis of the many problems still open in the regularity theory (see the last section). In more recent times, C. De Lellis and his former PhD student E. Spadaro undertook this very ambitious task in a series of papers [20, 23, 24, 21, 22] (actually [23], dealing with the theory of Q-valued functions, originated from his PhD thesis, see also [59]) in the last 5 years.

In this text I will give an overview of these development of the regularity theory for mass-minimizing currents, starting from the codimension 1 case and then moving to the higher codimension. My goal is to illustrate the key ideas of F. Almgren's proof, the new difficulties due to the higher codimension and, to some extent, some of the technical improvements introduced by C. De Lellis and E. Spadaro. Some of these are related to new ideas which spread in the literature only more recently, as higher integrability estimates via reverse Sobolev inequalities [33], the intrinsic point of view in the theory of Dir-minimizing functions, the R. Jerrard-M. Soner BV estimates on jacobians [39] and their applications to the theory of currents [9]. Other important technical improvements and simplifications regard the construction of the so-called center manifold and the error estimates, which measure the deviation of the "sheets" of the current from it; this is for sure the most involved and less expored part of F. Almgren's program, see [7, Chapter 4]. A byproduct of this optimization in the construction of the center manifold is the extension of Theorem 0.1 to $C^{3,\alpha}$ ambient manifolds.

Although the C. De Lellis-E. Spadaro's proof is considerably simpler and in some aspects different from F. Almgren's original one, it remains quite involved. A curse of the regularity theory, particularly of regularity in Geometric Measure Theory, is the complexity of proofs: very often arguments that can be heuristically explained in a few words need very lengthy arguments to be checked, and this involves iterations, change of scales, passage from cubes to balls, from flat to curvilinear systems of coordinates, etc. Moreover, it happens often that nontrivial technical improvements and new estimates appear precisely at this level, and so it is not easy to explain them in a reasonably short text. For this reason I will focus on the most essential aspects of the strategy of proof, leaving out of the discussion other important technical issues discussed in [21, 22] as "splitting before tilting", "persistence of Q-points" and "intervals of flattening".

In the preparation of this text I relied mostly on the Lecture Notes [60] and on E. Spadaro's Phd thesis, which are both recommended to the reader interested to enter more into the (somehow unavoidable) technical details.

1. Background: the H. Federer-W. Fleming theory of currents

One of the most successful theories of oriented generalized submanifolds is the one by H. Federer and W. Fleming in [27] on integer rectifiable currents (see also [15, 16] for the special case of codimension one generalized submanifolds, the so-called sets of finite perimeter). I illustrate the basic definitions in Euclidean spaces, but all definitions and results can be immediately adapted to the case when the ambient manifold is Riemannian. See also [9] for a far reaching extension of the theory to metric spaces, involving a suitable notion of "Lipschitz differential forms".

Definition 1.1 (Integer rectifiable currents). *An integer rectifiable current T of dimension m in $\mathbb{R}^{m+n}$ is a triple $T = (R, \tau, \theta)$ such that:*

(i) *R is a rectifiable set, i.e. $R = \bigcup_{i\in\mathbb{N}} C_i$ with $\mathcal{H}^m(R_0) = 0$ and $C_i \subset M_i$ for every $i \in \mathbb{N} \setminus \{0\}$, where M_i are m-dimensional oriented C^1 submanifolds of $\mathbb{R}^{m+n}$;*

(ii) *$\tau : R \to \Lambda_m$ is a measurable map, called orientation, taking values in the space of simple unit m-vectors and more precisely such that, for $\mathcal{H}^m$-a.e. $x \in C_i$, $\tau(x) = v_1 \wedge \cdots \wedge v_m$ with $\{v_1, \ldots, v_m\}$ an oriented orthonormal basis of T_xM_i;*

(iii) *$\theta : R \to \mathbb{Z}$ is a measurable function, called multiplicity, which is integrable with respect to $\mathcal{H}^m$.*

An integer rectifiable current $T = (R, \tau, \theta)$ induces a continuous linear functional (with respect to the natural Fréchet topology) on smooth, compactly supported m-dimensional differential forms ω (i.e. smooth compactly supported m-covector fields), denoted by $\mathscr{D}^m$, acting as follows

$$T(\omega) = \int_R \theta \, \langle \omega, \tau \rangle \, d\mathcal{H}^m.$$

Remark 1.2. *The continuous linear functionals defined in the Fréchet space $\mathscr{D}^m$ are a more general class (basically vector-valued L. Schwarz's distributions taking their values in the space of m-vectors in $\mathbb{R}^{m+n}$), called m-dimensional currents. Note that the submanifold M_i in Definition 1.1 are only C^1 regular, and that they could even be taken to be Lipschitz regular, providing an equivalent definition. This low regularity requirement is crucial for the H. Federer-W. Fleming closure and compactness theorem, discussed below. The price to pay, of course, is that regularity theory is hard, because of the very low initial regularity level.*

For an integer rectifiable current T, one can define the analog of the boundary and the volume for smooth submanifolds.

Definition 1.3 (Boundary and mass)**.** *Let T be a m-dimensional current in $\mathbb{R}^{m+n}$. The boundary of T is defined as the $(m-1)$-dimensional currents acting as follows*

$$\partial T(\omega) := T(d\omega) \quad \forall\, \omega \in \mathscr{D}^{m-1}.$$

If T is integer rectifiable, the mass of T is defined as the quantity

$$\mathbf{M}(T) := \int_R |\theta| \, d\mathcal{H}^m.$$

More generally, one can define by duality the boundary and the mass even for a general m-dimensional current. The general definition of mass involves the choice of a suitable norm on the space of m-covectors, but we will not need this definition in the sequel.

Note that, in the case $T = (\Sigma, \tau_\Sigma, 1)$ is the current induced by an oriented submanifold Σ, with τ_Σ a continuous orienting vector for Σ, then by Stokes' Theorem one has

$$\partial T = (\partial\Sigma, \tau_{\partial\Sigma}, 1),$$

where $\tau_{\partial\Sigma}$ is the induced orientation. Moreover, $\mathbf{M}(T) = \mathrm{vol}_m(\Sigma)$.

Finally, we recall that the space of currents is usually endowed with the weak* topology (often called in this context *weak* topology).

Definition 1.4 (Weak topology)**.** *We say that a sequence of currents $(T_\ell)_{\ell\in\mathbb{N}}$ weakly converges to some current T, and we write $T_\ell \rightharpoonup T$, if*

$$T_\ell(\omega) \to T(\omega) \quad \forall\, \omega \in \mathscr{D}^m.$$

Using the dual definition of mass, it is not hard to show that $T \mapsto \mathbf{M}(T)$ is (sequentially) weakly lower semicontinuous even in the class of general m-dimensional currents.

The Plateau problem has now a straightforward generalization to the context of integer rectifiable currents.

Generalized Plateau problem. Let Γ be a compactly supported $(m-1)$-dimensional integer rectifiable current in $\mathbb{R}^{m+n}$ with $\partial\Gamma = 0$. Find an m-dimensional integer rectifiable current T such that $\partial T = \Gamma$ and

$$\mathbf{M}(T) \le \mathbf{M}(S), \tag{1}$$

for every S integer rectifiable with $\partial S = \Gamma$.

It is not hard to see, by a classical cone construction, that the class of admissible currents T in (1) is not empty. In addition, the existence of a minimizing sequence (T_ℓ) weakly convergent to a m-dimensional current T is not hard to prove. The key technical point, on which ultimately the success of the theory of integer rectifiable currents relies, is the following closure theorem by H. Federer and W. Fleming, proven in their pioneering paper [27]. It states that T is *still* an integer rectifiable current. Then, the lower semicontinuity of mass under weak convergence provides existence of solutions to the generalized Plateau problem.

Theorem 1.5 (H. Federer and W. Fleming [27])**.** *Let $(T_\ell)_{\ell\in\mathbb{N}}$ be m-dimensional integer rectifiable currents in $\mathbb{R}^{m+n}$ with*

$$\sup_{\ell\in\mathbb{N}}\big(\mathbf{M}(T_\ell)+\mathbf{M}(\partial T_\ell)\big)<\infty,$$

and assume that $T_\ell\rightharpoonup T$. Then, T is an integer rectifiable current.

It is then natural to ask about the regularity properties of the solutions to the generalized Plateau problem, called in the sequel *area-minimizing* integer rectifiable currents.

Another important operator in the theory of currents, widely used in arguments involving induction on the dimension and cut&paste procedures, is the slicing operator. If $p\le n$ and $f:\mathbb{R}^{m+n}\to\mathbb{R}^p$ is a Lipschitz map, then there exists a unique (up to $\mathcal{H}^p$-negligible sets in $\mathbb{R}^p$) family of $(m-p)$-dimensional currents

$$\langle T,f,z\rangle \qquad z\in\mathbb{R}^p$$

concentrated on the fiber $f^{-1}(z)$ and satisfying, for all bounded Borel $g:\mathbb{R}^p\to\mathbb{R}$,

$$T\llcorner(g\circ f)df^1\wedge\ldots\wedge df^p=\int g(z)\langle T,f,z\rangle\,dz,$$

the equality being undersdood in the sense of superposition of currents, i.e.

$$T((g\circ f)hdf\wedge dq)=\int_{\mathbb{R}^p}g(z)\langle T,f,z\rangle(hdq)\,dz \qquad \text{for all } hdq\in\mathscr{D}^{m-p}.$$

Notice that this is a geometric counterpart of the existence conditional probability measures in Probability. In the case $p=1$ the slice operator can be obtained as a kind of commutator between boundary and restriction:

$$\langle T,f,z\rangle=(\partial T)\llcorner\{f<z\}-\partial\big(T\llcorner\{f<z\}\big).$$

In general, the slice operator can be built by iterating this procedure. The slice operator preserves the property of being (integer) rectifiable.

We close this section providing some additional notation that will consistently be used in this text. Given an m-dimensional integer rectifiable current $T=(R,\tau,\theta)$, we set:

$$\|T\|:=|\theta|\,\mathcal{H}^m\llcorner R,\quad \vec{T}:=\tau\quad\text{and}\quad \operatorname{spt}(T):=\operatorname{spt}(\|T\|).$$

The regular and the singular part of a current are naturally defined as follows.

$$\operatorname{Reg}(T):=\big\{x\in\operatorname{spt}(T)\ :\ \operatorname{spt}(T)\cap B_r(x)\text{ is induced by a smooth submanifold for some }r>0\big\},$$

$$\operatorname{Sing}(T):=\operatorname{spt}(T)\setminus\big(\operatorname{spt}(\partial T)\cup\operatorname{Reg}(T)\big).$$

The definition of $\operatorname{Sing}(T)$ is motivated by the fact that we will deal only with "interior" regularity, since in general codimension the boundary regularity is still open.
We shall also denote by ω_m the Lebesgue measure of the unit ball in $\mathbb{R}^m$.

2. Codimension 1 regularity theory

The codimension 1 case has been studied first, starting from E. De Giorgi's work [17] (see also E. Giusti's monograph [34] and also the more recent one by F. Maggi [43]) and more refined results can be proven, compared to the higher codimension case (see also [29, 54, 56, 58, 46] for the interior regularity and [3, 37] for the boundary regularity).

In this section I will illustrate the key technical tools needed to attack the regularity problem in the original context of [17], namely sets E of finite perimeter. In the language of currents, a set of finite perimeter corresponds to a $(m+1)$-dimensional current T_E associated to the integration on E, i.e.

$$T_E(f dx^1 \wedge \ldots \wedge dx^{m+1}) = \int_E f\, dx$$

having the property that ∂T_E has finite mass. It turns out (and this corresponds to the so-called boundary rectifiability theorem of the theory of currents) that

$$\partial T_E = (\partial^* E, \tau_E, 1)$$

is an integer rectifiable current with multiplicity 1. The countably $\mathcal{H}^m$-rectifiable set $\partial^* E$ is the so-called *essential boundary*, and the perimeter is precisely $\mathcal{H}^m(\partial^* E)$. In codimension 1 it is also customary to write ν_E for $*\tau_E$ (here $*$ is the canonical operator mapping m-vectors to 1-vectors in $\mathbb{R}^{m+1}$), the so-called *approximate unit normal*.

2.1. **Excess and ϵ-regularity theorem.** We define the *excess* $\mathbf{E}(\partial T_E, B_r(x))$ of ∂T_E as follows:

$$\begin{aligned} \mathbf{E}(\partial T_E, B_r(x)) := & \frac{1}{r^m}\big(\mathcal{H}^m(B_r(x)\cap\partial^* E) - \big|\int_{B_r(x)\cap\partial^* E} \nu_E\, d\mathcal{H}^m\big|\big) \\ = & \frac{1}{2r^m}\int_{B_r(x)\cap\partial^* E} |\nu_E(y) - \overline{\nu_E(x,r)}|^2\, d\mathcal{H}^m(y), \end{aligned} \tag{2}$$

where

$$\overline{\nu_E(x,r)} = \frac{1}{\big|\int_{B_r(x)\cap\partial^* E}\nu_E(y)\, d\mathcal{H}^m(y)\big|}\int_{B_r(x)\cap\partial^* E}\nu_E(y)\, d\mathcal{H}^m(y). \tag{3}$$

If we write the excess in terms of ∂T_E, instead, we get

$$\mathbf{E}(\partial T_E, B_r(x)) = \frac{1}{r^m}\big(\|{*\partial T_E}\|(B_r(x)) - |{*\partial T_E}(B_r(x))|\big), \tag{4}$$

where $*\partial T_E$ is the $\mathbb{R}^{m+1}$-valued measure canonically associated to ∂T_E by letting ∂T_E act on forms $\phi\widehat{dx_i}$ (indeed, the distributional derivative of the characteristic function of E).

This scale-invariant quantity measures the quadratic variance of the approximate unit normal and it is the key ingredient of the regularity theory. We state the main result for local minimizers, i.e. we assume that in some open set $\Omega\subset\mathbb{R}^{m+1}$ one has

$$\mathcal{H}^m(B_r(x)\cap\partial^* E) \le \mathcal{H}^m(B_r(x)\cap\partial^* F) \qquad \text{whenever } E\Delta F \Subset B_r(x)\Subset\Omega.$$

Theorem 2.1 (De Giorgi)**.** *There exists a dimensional constant $\epsilon(m)>0$ such that if E is locally perimeter minimizing in Ω, $B_r(x)\subset\Omega$ and $x\in\operatorname{spt}\partial T_E$, then*

$$\mathbf{E}(\partial T_E, B_r(x)) < \epsilon \tag{5}$$

implies that $B_{r/2}(x)\cap\operatorname{spt}\partial T_E$ is the graph of a smooth (actually analytic) function, in a suitable system of coordinates. In particular $\mathcal{H}^m(\Omega\cap\operatorname{Sing}(\partial T_E))=0$.

The final part of the statement follows by the fact that at $\mathcal{H}^m$-a.e. point $x \in \partial^* E$ one always has $\mathbf{E}(\partial T_E, B_r(x)) < \epsilon$ for $r = r(x) > 0$ sufficiently small, roughly speaking this happens at Lebesgue points of the approximate normal.

Theorem 2.1 reveals a deep phenomenon, a kind of good separation between smooth and singular objects: there is a critical treshold, based on the excess, such that we are in the smooth scenario as soon as we are below this critical treshold. After E. De Giorgi's work, and after his discovery that full regularity can not be expected in general for systems of partial differential equations with nonconstant coefficients, the idea was immediately exploited to prove regularity theorems for systems (first in [35], then in many other papers). It is now widely used also in other geometric contexts, as harmonic maps between manifolds, mean curvature flow, etc. and this kind of statements are named *ϵ-regularity theorems*.

The proof of Theorem 2.1 is in turn based on the *excess decay lemma*: for $\alpha = \alpha(m) \in (0, 1)$ sufficiently small one has

$$\mathbf{E}(\partial T_E, B_{\alpha r}(x)) \leq \frac{1}{2}\mathbf{E}(\partial T_E, B_r(x)) \tag{6}$$

as soon as $x \in \operatorname{spt}(\partial T_E)$ and $\mathbf{E}(\partial T_E, B_r(x)) < \epsilon$. At all points x where the excess goes below the critical treshold ϵ, and at all nearby points (since the condition (5) is open) one can then initiate a standard iteration scheme to show that the approximate normals (3) are Hölder continuous in space, uniformly w.r.t. the scale parameter r. This shows that in a neighbourhood of x the set $\partial^* E$ is the graph of a $C^{1,\gamma}$ function ϕ (an optimization of E. De Giorgi's proof actually gives that one can reach any power $\gamma < 1$). Since ϕ solves in the weak sense the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla \phi}{\sqrt{1 + |\nabla \phi|^2}}\right) = 0$$

one eventually gets the smoothness of ϕ by the regularity theory for quasilinear elliptic equations.

2.2. **Lipschitz approximation and comparison with harmonic functions.** The proof of the excess decay is achieved by contradiction (although in more recent times effective proofs that lead to an effective estimate of the decay treshold ϵ have been given, see [13, 53] and [11]) and it relies on a deep intuition of E. De Giorgi. Here is a very rough sketch. In balls where the excess is small, after scalings and rotations we reduce ourselves to a family of sets of finite perimeter E_h in the unit ball B_1 with $0 \in \operatorname{spt}(\partial T_{E_h})$ whose normal deviates very little from a given direction independent of h. Choosing a system of coordinates adapted to this situation, one can expect that E_h can be well approximated by graphs of functions ϕ_h with very small Lipschitz constant, so that by linearizing the area functional one obtains (see also Proposition 3.6 for a more precise expansion of the area of a graph)

$$\text{perimeter of } E_h \text{ in the ball} \quad \sim \quad \omega_m + \frac{1}{2}\int_{B_1^m} |\nabla \phi_h|^2 \, dx. \tag{7}$$

The main point is that one can use this expansion to transfer informations from E_h to ϕ_h in two ways: first, the expansion suggests that because of minimality of E_h the functions ϕ_h, suitably rescaled, should be close (possibly passing to a subsequence) to an harmonic function ϕ. Second, the assumption that the excess of E_h does not decay as required in (6) contradicts, in the limit, the decay

$$\int_{B_\alpha^m} |\nabla \phi|^2 \, dx \leq \alpha^m \int_{B_1^m} |\nabla \phi|^2 \, dx$$

typical of harmonic functions. Notice that, in order to get the contradiction, it is necessary to have sufficiently strong convergence of the rescalings of ϕ_h to ϕ and this depends very much on the accuracy in the expansion (7); this aspect will be even more crucial in the higher codimension case.

I will give later on more precise statements, and say a few words about the proof of the Lipschiz and harmonic approximation in the next sections, when dealing with general codimension currents. In E. De Giorgi's original proof, the functions ϕ_h are obtained by a suitable convolution procedure, on scales given by a suitable power of the excess. In the first extension of E. De Giorgi's approach to currents [6] and varifolds [2], instead, the Lipschitz functions are built by looking at the set of points where the excess is small on sufficiently small scales: on this set we have a "Lipschitz behaviour" and a covering argument then shows that the complement of this set can be estimated with the excess.

2.3. **Monotonicity and tangent cones.** Another crucial tool introduced by E. De Giorgi is the monotonicity formula: if E is locally minimizing in Ω, then

$$\frac{\mathcal{H}^m(B_r(x)\cap\partial^*E)}{r^m}\leq\frac{\mathcal{H}^m(B_s(x)\cap\partial^*E)}{s^m}\qquad\text{whenever } B_r(x)\subset B_s(x)\Subset\Omega.$$

The same result holds, in general codimension, for area-minimizing currents, namely

$$\frac{\|T\|(B_r(x))}{r^m}\leq\frac{\|T\|(B_s(x))}{s^m}\qquad\text{whenever } B_r(x)\subset B_s(x)\Subset\mathbb{R}^{m+n}\setminus\mathrm{spt}\,(\partial T).$$

A careful analysis of the nonnegative term arising in the proof of the monotonicity formula shows that it measures somehow the deviation of ∂T_E from being a cone. This leads to the fact that a blow-up procedure not only preserves the (local) minimality property, but also somehow leads to a simpler object, namely a cone. Since the same result holds for currents, we state it at this more general level (for sets of finite perimeter this corresponds to considering the sets $E_{x,r}=(E-x)/r$).

For any $r>0$ and $x\in\mathbb{R}^{m+n}$, let $\iota_{x,r}$ denote the map

$$\iota_{x,r}: y\mapsto\frac{y-x}{r},\tag{8}$$

and set $T_{x,r}:=(\iota_{x,r})_\sharp T$, where $_\sharp$ is the push-forward operator, namely

$$(\iota_{x,r})_\sharp T(\omega):=T(\iota_{x,r}^*\omega)\qquad\forall\,\omega\in\mathscr{D}^m.$$

Theorem 2.2 (Tangent cones). *If T is area-minimizing and $x\in\mathrm{spt}\,(T)\setminus\mathrm{spt}\,(\partial T)$, any weak limit point S of $T_{x,r}$ as $r\downarrow 0$ is a cone without boundary (i.e. $S_{0,r}=S$ for all $r>0$ and $\partial S=0$) which is locally area minimizing in $\mathbb{R}^{m+n}$, i.e.*

$$\|S\|(B_r(x))\leq\|S'\|(B_r(x))\qquad\textit{whenever } \mathrm{spt}\,(S-S')\Subset B_r(x)\Subset\mathbb{R}^{m+n}.$$

Such a cone S is called, as usual, *a tangent cone to T at x.*

2.4. **Persistence of singularities and dimension reduction.** Theorem 2.1 can be substantially improved, we state it for sets of finite perimeter but the same result holds for codimension 1 currents:

Theorem 2.3. *If E is a locally minimizing set of finite perimeter in Ω, then the singular set is empty if $m\leq 6$, is discrete if $m=7$ and it has Hausdorff dimension $(m-7)$ if $m>7$. More precisely*

$$\mathcal{H}^{m-7+\epsilon}\big(\mathrm{Sing}(\partial T_E)\cap\Omega\big)=0\qquad\forall\epsilon>0.$$

The proof of Theorem 2.3 rests on another powerful heuristic principle, namely the persistence of singularities under blow-up limits and, more generally, weak convergence. In order to understand this, let us first state a stability result for mass minimizing currents, which already plays a role in Theorem 2.2, to show that tangent cones are still mass minimizing. We state the result in local form, saying that T is locally mass-minimizing in an open set $\Omega \subset \mathbb{R}^{m+n}$ if

$$\|T\|(A) \le \|T'\|(A) \qquad \text{whenever } \operatorname{spt}(T - T') \Subset A \Subset \Omega. \tag{9}$$

Theorem 2.4 (Stability of mass-minimizing currents and improved convergence)**.** *If $\Omega \subset \mathbb{R}^{m+n}$ is an open set and T_ℓ are currents locally mass-minimizing in Ω weakly convergent to T, then:*

(i) *T is locally mass-minimizing in Ω;*
(ii) *the mass measures $\|T_\ell\|$ weakly converge to $\|T\|$, in duality with $C_c(\Omega)$.*

The proof of Theorem 2.4 can be achieved with the slicing operator, which allows to perform the standard cut&paste procedure to transfer the minimality property from T_ℓ to T (at the same time, this argument provides the proof of (ii)). Now, if we take into account the expression (4) for the excess, remembering that at singular points the excess has to be larger than ϵ on all scales, in codimension 1 we have the following principle, for sequences of sets E_h of finite perimeter which are locally minimizing in an open set Ω:

$$x_h \in \operatorname{Sing}(\partial T_{E_h}), \quad \lim_{h\to\infty} x_h = x \in \Omega \qquad \Longrightarrow \qquad x \in \operatorname{Sing}(\partial T_E). \tag{10}$$

We can now explain informally the proof of Theorem 2.3: if we are able to rule out the existence of singular minimal cones of dimension smaller than 6 (starting from the easy case $m = 2$, this was actually achieved in progressively higher dimensions thanks to the work of E. De Giorgi, W. Fleming, F. Almgren and culminated in J. Simons' work [58]), then Theorem 2.2 and (10) show that the singular set of any locally mass-minimizing set is empty for $m \le 6$. In dimension $m = 7$ there is indeed a singular minimal cone, the celebrated J. Simon's cone

$$\left\{(y,z) \in \mathbb{R}^4 \times \mathbb{R}^4 : \ |y|^2 < |z|^2\right\}.$$

We can prove that for 7-dimensional boundaries the singular set is always discrete as follows: if, for some E, $x_h \in \operatorname{Sing}(\partial T_E)$ and $x_h \to x$, then $x \in \operatorname{Sing}(\partial T_E)$ and we can blow up at x along the scales $r_h = |x_h - x|$ to find a minimal cone S in $\mathbb{R}^8$ whose singular set contains 0 and another point on the unit sphere: by the cone property, the singular set contains a halfline L. But now we can blow-up at a point different from 0 and on L to find a cone S' which splits, choosing appropriately the coordinates, as $S'' \times \mathbb{R}$, with S'' singular. Since $\partial T_{S''}$ is a 6-dimensional singular minimal boundary in R^7, we have a contradiction.

When $m > 8$ we can somehow repeat this argument, by the so-called dimension reduction argument (first used by H. Federer in [28]) and it is convenient to put it in an abstract form as follows (see [55]) to obtain the proof of Theorem 2.4.

We let $p \ge q \ge 2$ and we consider a collection $\mathcal{F}$ of functions $\phi : \mathbb{R}^p \to \mathbb{R}^k$ with the topology induced by the weak convergence, in the duality with $C_c(\mathbb{R}^p)$, of the corresponding measures $f\mathcal{H}^q$. We assume that:

- [A1] $\mathcal{F}$ is invariant under the transformation $\phi \mapsto \phi \circ \iota_{x,r}^{-1}$, with $\iota_{x,r}$ defined as in (8);
- [A2] for all x the family $\phi \circ \iota_{x,r}^{-1}$ has limit points in $\mathcal{F}$ as $r \downarrow 0$ and any limit point ϕ is a cone, i.e. $\phi_{0,r} = \phi_0$ for all $r > 0$;
- [A3] there exists a map Σ from $\mathcal{F}$ to the closed subsets of $\mathbb{R}^p$ such that $\Sigma(\phi) = \emptyset$ if ϕ is the constant multiple of the characteristic function of a q-dimensional subspace

of $\mathbb{R}^p$, Σ is upper semicontinuous with respect to the (local) Hausforff convergence of closed sets in $\mathbb{R}^p$ and scale-invariant:

$$\Sigma(\phi_{x,r}) = \frac{1}{r}\big(\Sigma(\phi) - x\big).$$

Theorem 2.5 (Federer's dimension reduction argument)**.** *Under the assumptions above, let $d \in [0, q-1]$ be the largest dimension of a subspace $L \subset \mathbb{R}^p$ such that*

$$\phi_{x,r} = \phi \quad \textit{for all } x \in L \textit{ and } r > 0 \textit{ and } \Sigma(\phi) = L, \textit{ for some } \phi \in \mathcal{F}.$$

Then, the Hausdorff dimension of $\Sigma(\phi)$ does not exceed d for all $\phi \in \mathcal{F}$ and, if $d = 0$, $\Sigma(\phi)$ is discrete for all $\phi \in \mathcal{F}$.

In the higher codimension case the dimension reduction argument will indeed play a role in the estimate of the singular set of Dir-minimizing functions, see Theorem 3.3 below.

3. Codimension $n > 1$ regularity theory

The structure of the area minimizing integer rectifiable currents and their regularity theory depend very much on the dimension m of the current and its *codimension* in the ambient space (i.e., with the notation above, if T is an m-dimensional current in $\mathbb{R}^{m+n}$, the codimension is n). We now illustrate the new singular examples arising in codimension higher than 1 (basically due to the appearence of *branch* points), the technical challenges and the new fundamental ideas behind the proof of Theorem 0.1.

3.1. **The basic examples and the new difficulties. Calibrated currents.** The calibration method, going back to [26, 5.4.19], is a powerful tool to prove that an integer rectifiable current T is (locally) mass-minimizing. It is based on the construction of a smooth (although in many cases this requirement can be weakened) closed m-form ω defined in an open set Ω of $\mathbb{R}^{m+n}$ with $|\omega(x)| \leq 1$ for all $x \in \Omega$ and

$$\langle \vec{T}(x), \omega(x) \rangle = 1 \qquad \text{for } \|T\|\text{-a.e. } x \in \Omega.$$

If this happens, it is not hard to show that T is locally mass-minimizing in Ω, according to (9), and we say that T is *calibrated* by ω.

It is not hard to show that the form $\omega = \lambda^k/k!$, where λ is the Kähler form

$$\lambda := \sum_{i=1}^{d} dx_i \wedge dy_i \qquad \mathbb{C}^d \sim (\mathbb{R}^2)^d$$

is a calibration for any complex manifold $S \subset \mathbb{C}^d$ of complex dimension k. This observation provides plenty of examples of locally mass-minimizing currents and shows that the estimate of the singular set in Theorem 0.1 is optimal.

Flat tangent cones do not imply regularity. We have seen in codimension $n = 1$ that regularity is driven by the excess. So, a point x is regular if and only if (some) tangent cone to the current T at x is flat. This is not the case for higher codimension currents, unless one requires some upper bounds on multiplicity (for instance currents with multiplicity 1). In order to illustrate this phenomenon, let us consider the current $T_{\mathscr{V}}$ induced by the complex curve mentioned in the introduction:

$$\mathscr{V} = \big\{(z, w) \;:\; z^3 = w^2,\ |z| \leq 1\big\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4.$$

By the calibration method it is simple to show that $T_{\mathscr{V}}$ is an area minimizing integer rectifiable current (cp. [26, 5.4.19]), which is singular in the origin. Nevertheless, the unique tangent cone to $T_{\mathscr{V}}$ at 0 is the current $S = (\mathbb{R}^2 \times \{0\}, e_1 \wedge e_2, 2)$ which is associated to the integration on the horizontal plane $\mathbb{R}^2 \times \{0\} \simeq \{w = 0\}$ with multiplicity 2. The tangent cone is actually regular, although the origin is a singular point!

Non-homogeneous blow up. One of the main ideas by F. Almgren is then to extend this reasoning to different types of blow ups, by rescaling differently the "horizontal directions", namely those of a flat tangent cone at the point, and the "vertical" ones, which are the orthogonal complement of the former. In this way, in place of preserving the geometric properties of the rectifiable current T, one is led to preserve the *energy* of the associated *multiple-valued function*.

In order to explain this point, let us consider again the current $T_{\mathscr{V}}$. The support of such current, namely the complex curve $\mathscr{V}$, can be viewed as the graph of a function which associates to any $z \in \mathbb{C}$ with $|z| \leq 1$ *two* points in the w-plane:

$$z \mapsto \{w_1(z), w_2(z)\} \quad \text{with } w_i(z)^2 = z^3 \text{ for } i = 1, 2. \tag{11}$$

Then the right rescaling according to F. Almgren is the one producing in the limit a multiple valued *harmonic function* preserving the *Dirichlet energy* (for the definitions see the next sections). In the case of $\mathscr{V}$, the correct rescaling is the one fixing $\mathscr{V}$. For every $\lambda > 0$, we consider $\Phi_\lambda : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\Phi_\lambda(z, w) = (\lambda^2 \, z, \lambda^3 \, w),$$

and note that $(\Phi_\lambda)_\sharp T_{\mathscr{V}} = T_{\mathscr{V}}$ for every $\lambda > 0$. Indeed, in the case of $\mathscr{V}$ the functions w_1 and w_2, being the two determinations of the square root of z^3, are already harmonic functions (at least away from the origin).

Abstracting from the above example, one is led to consider multiple-valued functions from a domain in $\mathbb{R}^m$ which take a fixed number $Q \in \mathbb{N} \setminus \{0\}$ of values in $\mathbb{R}^n$. These functions were called by F. Almgren *Q-valued functions*. What about harmonic Q-valued functions, the natural candidate for the approximation of a current? The definition of harmonic Q-valued function is simple around any "regular point" $x_0 \in \mathbb{R}^m$, for it is enough to consider just the superposition of classical harmonic functions (possibly with a constant integer multiplicity), i.e.

$$\mathbb{R}^m \supset B_r(x_0) \ni x \mapsto \{u_1(x), \ldots, u_Q(x)\} \in (\mathbb{R}^n)^Q, \tag{12}$$

with u_i harmonic and either $u_i = u_j$ or $u_i(x) \neq u_i(x)$ for every $x \in B_r(x_0)$. The issue becomes much more subtle around singular points. In the example (11), in a neighborhood of the origin there is no representation of the map $z \mapsto \{w_1(z), w_2(z)\}$ as in (12). In this case the two values $w_1(z)$ and $w_2(z)$ cannot be ordered in a consistent way (due to the *branch point* at 0), and hence cannot be distinguished one from the other. We are then led to consider a multiple valued function as a map taking Q values in the quotient space $(\mathbb{R}^n)^Q/\sim$ induced by the symmetric group $\mathbf{S}_Q$ of permutations of Q elements and to provide a more intrinsic and "variational" notion of harmonicity for Q-valued functions, see the next sections.

The need of centering and the order of contact. A major geometric and analytic problem has to be addressed in order to find non-trivial blow-up limits. In order to make it apparent, let us discuss another example. Consider the complex curve $\mathscr{W}$ given by

$$\mathscr{W} = \big\{(z, w) \,:\, (w - z^2)^2 = z^5, \; |z| \leq 1\big\} \subset \mathbb{C}^2.$$

As before, $\mathscr{W}$ can be associated to an area minimizing integer rectifiable current $T_{\mathscr{W}}$ in $\mathbb{R}^4$, which is singular at the origin. It is easy to prove that the unique tangent plane to $T_{\mathscr{W}}$ at 0 is the plane $\{w = 0\}$ taken with multiplicity 2. On the other hand, the only nontrivial inhomogeneous blow up in these vertical and horizontal coordinates is given by

$$\Phi_\lambda(z, w) = (\lambda \, z, \lambda^2 \, w),$$

and $(\Phi_\lambda)_\sharp T_{\mathscr{W}}$ converges as $\lambda \to +\infty$ to the current induced by the *smooth* complex curve $\{w = z^2\}$ taken with multiplicity 2. In other words, the inhomogeneous blow up did not produce in the limit any singular current and cannot be used to study or to detect the singularity of $T_{\mathscr{W}}$ at the origin.

For this reason it is essential to "renormalize" $T_{\mathscr{W}}$ by averaging out its regular first expansion, on top of which the singular branching behavior happens. In this case, the regular part of $T_{\mathscr{W}}$ is exactly the smooth complex curve $\{w = z^2\}$, while the singular branching is due to the determination of the square root of z^5. It is then clear that one should look for parametrizations of $\mathscr{W}$ defined in the regular embedded manifold $\{w = z^2\}$, so that the singular map to be considered reduces to

$$z \mapsto \{u_1(z), u_2(z)\} \quad \text{with } u_1(z)^2 = z^5.$$

The regular surface $\{w = z^2\}$ is called *center manifold* by F. Almgren, because it behaves like (and in this case it is exactly) the average of the sheets of the current in a suitable system of coordinates. The construction of the center manifold actually constitutes the most intricate part of the proof of Theorem 0.1.

Having taken care of the geometric problem of the averaging, one has to be sure that the first singular expansion of the current around its regular part does not occur with an infinite order of contact, because in that case the blow up would be by necessity zero. This issue involves one of the most interesting and original ideas of F. Almgren, namely a new monotonicity formula for the so-called *frequency function* (which is a suitable ratio between the energy and a zero degree norm of the function parametrizing the current). This is in fact the right monotone quantity for the inhomogeneous blow ups introduced before, and it leads to a nontrivial limiting current.

In order to introduce the frequency function, we consider the case of a real valued harmonic function $f : B_1 \subset \mathbb{R}^2 \to \mathbb{R}$ with an expansion in polar coordinates

$$f(r, \theta) = a_0 + \sum_{k=1}^{\infty} r^k \big(a_k \cos(k\theta) + b_k \sin(k\theta)\big).$$

It is not difficult to show that the quantity

$$I_f(r) := \frac{r \int_{B_r} |\nabla f|^2}{\int_{\partial B_r} |f|^2} \tag{13}$$

is monotone increasing in r and that its limit as $r \downarrow 0$ gives exactly the smallest non-zero index k in the expansion above. One of the most striking discoveries of F. Almgren is that the frequency function can be defined for Q-valued functions, retaining its monotonicity. This allows to obtain a non-trivial blow up limit: indeed, by monotonicity the frequency is, locally in space, uniformly bounded as $r \to 0$, thus excluding the infinite order of contact. In a PDE context, this idea has been used in [31, 32] to study the regularity of the nodal set of solutions to partial differential equations and the unique continuation property.

3.2. **Q-valued functions.** Let $\mathcal{A}_Q(\mathbb{R}^n) := (\mathbb{R}^n)^Q/\!\sim$ be the set of unordered Q-tuples of points in $\mathbb{R}^n$, where $Q \in \mathbb{N} \setminus \{0\}$ is a fixed number. It can be identified with the class of positive measures of mass Q which are the sum of integer multiplicity Dirac delta:

$$(\mathbb{R}^n)^Q/\!\sim \quad \simeq \quad \mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} [[P_i]] \ : \ P_i \in \mathbb{R}^n \right\},$$

We can then endow $\mathcal{A}_Q(\mathbb{R}^n)$ with one of the canonical distances defined for (probability) measures, the most appropriate and consistent with the case $Q = 1$ is the quadratic

Wasserstein distance: for every $T_1 = \sum_i [[P_i]]$ and $T_2 = \sum_i [[S_i]] \in \mathcal{A}_Q(\mathbb{R}^n)$, we set

$$W_2(T_1, T_2) := \min_{\sigma \in \mathbf{S}_Q} \sqrt{\sum_{i=1}^{Q} \left|P_i - S_{\sigma(i)}\right|^2}, \tag{14}$$

where we recall that $\mathbf{S}_Q$ denotes the permutation group of Q elements.

A Q-valued function is simply a map $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^m$ is an open domain. We can then talk about measurable (with respect to the Borel σ-algebra of $\mathcal{A}_Q(\mathbb{R}^n)$), bounded, uniformly-, Hölder- or Lipschitz-continuous Q-valued functions. However, for the development of a Sobolev space theory, more is needed. In the following remark we describe the original *extrinsic* approach followed by F. Almgren, while in the sequel we describe the *intrinsic* point of view adopted by C. De Lellis and E. Spadaro in the development of the theory of Q-valued functions, based on relatively more recent advances in Analysis in metric spaces.

Remark 3.1 (F. Almgren's extrinsic approach)**.** *A standard procedure to define Sobolev maps with a manifold target $\mathcal{M}$ is, when the manifold is isometrically embedded in some Euclidean space $\mathbb{R}^p$, to define*

$$W^{1,p}(\Omega, \mathcal{M}) := \left\{ f \in W^{1,p}(\Omega, \mathbb{R}^p) : \ f(x) \in \mathcal{M} \text{ for a.e. } x \in \Omega \right\}.$$

With this definition, the theory still works well even under weaker requirements on the target: it suffices to assume that $\mathcal{M}$ is a Lipschitz retract of $\mathbb{R}^p$, since in this case one can use standard convolution arguments in the ambient linear space and use eventually the retraction map to produce $\mathcal{M}$-valued maps. F. Almgren proved the existence of $p = p(n, Q)$ such that $\mathcal{A}_Q(\mathbb{R}^n)$ is bi-Lipschitz equivalent to a Lipschitz retract $\mathcal{M}$ of $\mathbb{R}^p$ and, building on this, he developed the theory of Sobolev and Dir-minimizing functions (actually he proved a bit more, also a kind of local isometry between $\mathcal{A}_Q(\mathbb{R}^n)$ and $\mathcal{M}$, which plays an important role in the theory).

The intrinsic approach is developed following [8] (see also [49, 48], metric theory of harmonic functions developed in [36, 40, 41] and finally the very recent papers [20, 42]).

Definition 3.2 (Sobolev Q-valued functions)**.** *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set. A measurable function $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is in the Sobolev class $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ if there exist functions $\varphi_j \in L^2(\Omega)$ for $j = 1, \ldots, m$, such that*

- (i) *$x \mapsto W_2(f(x), T) \in W^{1,2}(\Omega)$ for all $T \in \mathcal{A}_Q(\mathbb{R}^n)$;*
- (ii) *$|\partial_j W_2(f, T)| \leq \varphi_j$ almost everywhere in Ω for all $T \in \mathcal{A}_Q(\mathbb{R}^n)$ and for all $j \in \{1, \ldots, m\}$, where $\partial_j W_2(f, T)$ denotes the weak partial derivatives of the functions in (i).*

By simple reasonings, one can infer the existence of minimal functions $|\partial_j f|$ fulfilling (ii), namely $|\partial_j f| \leq \varphi_j$ a.e., for any other φ_j satisfying (ii). We set

$$|Df|^2 := \sum_{j=1}^{m} |\partial_j f|^2, \tag{15}$$

and define the Dirichlet energy of a Q-valued function as (cp. also [40, 41, 42] for alternative definitions)

$$\mathrm{Dir}(f) := \int_\Omega |Df|^2.$$

A Q-valued function f is said *Dir-minimizing* if

$$\int_\Omega |Df|^2 \le \int_\Omega |Dg|^2 \tag{16}$$
$$\text{for all } g \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \text{ with } W_2(f,g)|_{\partial\Omega} = 0,$$

where the last equality is meant in the sense of traces (whose existence can be easily shown, for instance, using condition (i) and appealing to the usual trace theory for Sobolev functions)

The main tool of the theory of Q-valued functions, in the development of the regularity theory for area-minimizing currents, is the following existence and regularity result for Dir-minimizing functions.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open domain with Lipschitz boundary, and let $g \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ be fixed. Then, the following holds.*

- (i) *There exists a Dir-minimizing function f solving the minimization problem* (16).
- (ii) *Every such function f belongs to $C^{0,\kappa}_{\rm loc}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ for a dimensional constant $\kappa = \kappa(m, Q) > 0$ and $|Df| \in L^p_{\rm loc}(\Omega)$ for some dimensional constant $p = p(m, n, Q) > 2$.*
- (iii) *There exists a relatively closed set* $\mathrm{Sing}(u) \subset \Omega$ *of Hausdorff dimension at most $(m-2)$ such that the graph of u outside* $\mathrm{Sing}(u)$, *i.e. the set*
$$\mathrm{graph}\big(u|_{\Omega\setminus \mathrm{Sing}(u)}\big) = \{(x,y) \,:\, x \in \Omega \setminus \Sigma,\ y \in \mathrm{spt}\,(u(x))\}\,,$$
is a smooth embedded m-dimensional submanifold of $\mathbb{R}^{m+n}$.

As we already said, the proof of Theorem 3.3(iii) can be achieved with a dimension reduction argument, with a careful analysis of homogeneous Dir-minimizing functions which arise as blow-up limits, while the proof of statement (ii) relies on a reverse Hölder inequality

$$\left(\frac{1}{\omega_m r^m}\int_{B^m_r(x)} g^\alpha\right)^{1/\alpha} \le C\frac{1}{\omega_m (2r)^m}\int_{B^m_{2r}(x)} g$$

satisfied by $g = |Df|^{2m/(m+2)}$ with $\alpha = (m+2)/m$ and the so-called Gehring's lemma [33].

For the reasons explained in the previous section, a Q-valued function has to be considered as an intrinsic map taking values in the non-smooth space of Q-points $\mathcal{A}_Q$, and cannot be reduced to a "superposition" of Q single-valued functions. Nevertheless, in many situations it is possible to handle Q-valued functions as a superposition. For example, as shown in [20, Proposition 0.4] every measurable function $f : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ can be written (not uniquely) as

$$f(x) = \sum_{i=1}^{Q} [[f_i(x)]] \quad \text{for } \mathcal{H}^m\text{-a.e. } x, \tag{17}$$

with $f_1, \ldots, f_Q : \mathbb{R}^m \to \mathbb{R}^n$ measurable functions.

Similarly, for weakly differentiable functions it is possible to define a notion of pointwise approximate differential (cp. [20, Corollary 2,7])

$$Df = \sum_i [[Df_i]] \in \mathcal{A}_Q(\mathbb{R}^{n\times m}),$$

with the property that at almost every x it holds $Df_i(x) = Df_j(x)$ if $f_i(x) = f_j(x)$. This property ensures that several push forward maps related to f, see for instance (18) below, are well defined.

There is a canonical way to give the structure of integer rectifiable current to the graph of a Lipschitz Q-valued function, in analogy with the classical theory.

By a simple induction argument (cp. [23, Lemma 1.1]), one can prove the existence of a countable partition of M in bounded measurable subsets M_i ($i \in \mathbb{N}$) and Lipschitz functions $f_i^j : M_i \to \mathbb{R}^{m+n}$ ($j \in \{1, \ldots, Q\}$) such that

(a) $F|_{M_i} = \sum_{j=1}^{Q} [[f_i^j]]$ for every $i \in \mathbb{N}$ and $\mathrm{Lip}(f_i^j) \leq \mathrm{Lip}(F)$ $\forall i, j$;
(b) $\forall\, i \in \mathbb{N}$ and $j, j' \in \{1, \ldots, Q\}$, either $f_i^j \equiv f_i^{j'}$ or $f_i^j(x) \neq f_i^{j'}(x)$ $\forall x \in M_i$;
(c) $\forall\, i$ we have $DF(x) = \sum_{j=1}^{Q} [[Df_i^j(x)]]$ for a.e. $x \in M_i$.

In the next definition we consider *proper* Q-valued functions, i.e. measurable functions $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ (where M is any m-dimensional submanifold of $\mathbb{R}^{m+n}$) such that there is a measurable selection $F = \sum_i [[F_i]]$ for which

$$\bigcup_i \overline{(F_i)^{-1}(K)}$$

is compact for every compact $K \subset \mathbb{R}^{m+n}$. This is indeed an intrinsic property: if there exists such a selection, then *every* measurable selection shares the same property.

Definition 3.4 (Q-valued push-forward)**.** *Let M be an oriented submanifold of $\mathbb{R}^{m+n}$ of dimension m and let $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ be a proper Lipschitz map. Then, we define the push-forward $\mathbf{T}_F$ of M through F as the current*

$$\mathbf{T}_F = \sum_{i,j} (f_i^j)_\sharp [[M_i]],$$

where M_i and f_i^j are as above: that is,

$$\mathbf{T}_F(\omega) := \sum_{i \in \mathbb{N}} \sum_{j=1}^{Q} \int_{M_i} \langle \omega(f_i^j(x)), Df_i^j(x)_\sharp \vec{e}(x) \rangle \, d\mathcal{H}^m(x) \qquad \forall\, \omega \in \mathscr{D}^m(\mathbb{R}^n)\,. \tag{18}$$

One can prove that the current $\mathbf{T}_F$ in Definition 3.4 does not depend on the decomposition chosen for M and f and, moreover, it is integer rectifiable (cp. [23, Proposition 1.4]). It is also not hard to see that the boundary operator is coniugated to the restriction operator via the push forward (see [23, Theorem 2.1]), namely if $M \subset \mathbb{R}^{m+n}$ is an m-dimensional submanifold with boundary, $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ is a proper Lipschitz function and $G = F|_{\partial M}$, $\partial \mathbf{T}_F = \mathbf{T}_G$.

Graphs are a special and important class of push-forwards.

Definition 3.5 (Q-graphs)**.** *Let $f = \sum_i [[f_i]] : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz and define the map $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ as $F(x) := \sum_{i=1}^{Q} [[(x, f_i(x))]]$. Then, $\mathbf{T}_F$ is the* current associated to the graph $\mathrm{Gr}(f)$ *and will be denoted by $\mathbf{G}_f$.*

In connection with the energy comparison between a current and its harmonic approximation, the following Taylor expansion of the mass of a graph plays also a fundamental role (cp. [23, Corollary 3.3]).

Proposition 3.6 (Expansion of $\mathbf{M}(\mathbf{G}_f)$)**.** *There exist dimensional constants $\bar{c}, C > 0$ such that, if $\Omega \subset \mathbb{R}^m$ is a bounded open set and $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Lipschitz map with $Lip(f) \leq \bar{c}$, then*

$$\mathbf{M}(\mathbf{G}_f) = Q|\Omega| + \frac{1}{2} \int_\Omega |Df|^2 + \int_\Omega \sum_i \bar{R}_4(Df_i)\,, \tag{19}$$

where $\bar{R}_4 \in C^1(\mathbb{R}^{n \times m})$ satisfies $|\bar{R}_4(D)| = |D|^3 \bar{L}(D)$ for $\bar{L} : \mathbb{R}^{n \times m} \to \mathbb{R}$ Lipschitz with $Lip(\bar{L}) \leq C$ and $\bar{L}(0) = 0$.

3.3. **Approximation of area-minimizing currents.** In this section we illustrate some approximation results of currents by means of graphs. We have basically three levels of approximation: on the first level one can approximate general currents by Lipschitz graphs, while on the second level one approximates area-minimizing currents still by Lipschitz graphs, but with a much better degree of approximation. In these first two levels, by approximation we mean that the current and the Lipschitz graph coincide on a large set, with an error controlled by the excess. In the third level (maybe the one closer to E. De Giorgi's original one) one approximates the area-minimizing current by the graph of an harmonic function, but in this case (obviously) the current and the graph may not overlap in a large set and the approximation should be understood in area or, at the functional level, in the sense of Dirichlet energy. This part of F. Almgren's program has been greatly simplified by C De Lellis and E. Spadaro, using the R. Jerrard-M. Soner BV estimates on jacobians [39] and their applications to the theory of currents [9], as I will illustrate.

We first introduce more notation. We consider closed cylinders in $\mathbb{R}^{m+n}$ of the form $C_s(x) := \overline{B}^m_s(x) \times \mathbb{R}^n$ with $x \in \mathbb{R}^m$. One can show that the following setting is not restrictive for the purpose of interior regularity theory: for some cylinder $C_{4r}(x)$ (with $r \leq 1$) and some positive integer Q, the area-minimizing current T has compact support in $C_{4r}(x)$ and satisfies

$$\mathbf{p}_\sharp T = Q[[B^m_{4r}(x)]] \quad \text{and} \quad (\partial T) \llcorner B^m_{4r}(x) \times \mathbb{R}^n = 0, \tag{20}$$

where $\mathbf{p} : \mathbb{R}^{m+n} \to \pi_0 := \mathbb{R}^m \times \{0\}$ is the orthogonal projection and B^m stands for m-dimensional ball.

We introduce next the main regularity parameter for area minimizing currents, namely the *Excess* (notice the analogy with E. De Giorgi's Excess (2)).

Definition 3.7 (Excess and excess measure)**.** *For a current T as above we define the cylindrical excess $\mathbf{E}(T, C_r(x))$ as follows:*

$$\begin{aligned}\mathbf{E}(T, C_r(x)) &:= \frac{\|T\|(C_r(x))}{\omega_m r^m} - Q \\ &= \frac{1}{2\,\omega_m r^m}\int_{C_r(x)} |\vec{T} - \vec{\pi}_0|^2 \, \mathrm{d}\, \|T\|,\end{aligned}$$

where $\vec{\pi}_0$ is the unit simple m-vector orienting π_0. We also define the mass measure μ_T as follows:

$$\mu_T(A) := \mathbf{M}(T \llcorner A \times \mathbb{R}^n) - Q\mathcal{H}^m(A) \tag{21}$$

for $A \subset B^m_r(x)$ Borel.

The first approximation result we state is based on the following idea. Any BV (or Sobolev) function f is "Lipschitz on a large set", more precisely there exist a nonnegative function g in the weak L^1 space $L^1_w(\mathbb{R}^m)$ (in particular finite $\mathcal{H}^m$-a.e.) and a $\mathcal{H}^m$-negligible set N such that

$$|f(x) - f(y)| \leq (g(x) + g(y))|x - y| \qquad \forall x,\, y \in \mathbb{R}^m \setminus N.$$

The function g is, up to a multiplicative dimensional constant, the maximal function of the distributional derivative of f, i.e. $\sup_r |Df|(B_r(x))/r^m$. In the geometric context, the basic observation of [39, 9] is that, for a current T (not necessarily integer rectifiable) with finite mass and boundary with finite mass, the slice operator is BV as a function of the slicing parameter. A particular instance of this statement, relevant for the application to the approximation with graphs is the following: if T is a m-dimensional current in $\mathbb{R}^{m+n}$

with finite mass and boundary with finite mass, and if $P : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the canonical projection on the first m coordinates, then the map

$$x \mapsto \langle T, P, x\rangle$$

is BV as a map from $\mathbb{R}^m$ to the space of 0-dimensional currents in $\mathbb{R}^{m+n}$ (supported in the fiber $\{x\} \times \mathbb{R}^n$), when the latter space is endowed with the so-called flat distance:

$$d_{\mathcal{F}}(S_1, S_2) = \inf\left\{\mathbf{M}(A) + \mathbf{M}(B) :\ S_1 - S_2 = A + \partial B\right\}.$$

For currents without boundary, whose slices have no boundary as well, it is more appropriate to consider this variant of the flat distance:

$$\tilde{d}_{\mathcal{F}}(S_1, S_2) = \inf\left\{\mathbf{M}(B) :\ S_1 - S_2 = \partial B\right\}.$$

and the BV property can be proved in a stronger form, for this larger flat distance. For 0-dimensional currents induced by elements of $\mathcal{A}_Q(\mathbb{R}^{m+n})$ this variant of the flat distance is closely related to the Wasserstein distance W_2 in (14). Indeed, for every $T_1 = \sum_i [[P_i]]$ and $T_2 = \sum_i [[S_i]]$ in $\mathcal{A}_Q(\mathbb{R}^n)$, one has

$$\tilde{d}_{\mathcal{F}}(T_1, T_2) = \sum_{i=1}^{Q} |P_i - S_{\sigma(i)}|$$

for some permutation σ, hence

$$W_2(T_1, T_2) \le \left(\sum_{i=1}^{Q} |P_i - S_{\sigma(i)}|^2\right)^{1/2} \le \sum_{i=1}^{Q} |P_i - S_{\sigma(i)}| = d_{\mathcal{F}}(T_1, T_2).$$

These remarks lead to an elegant proof of the following approximation result, where we use also the notation

$$M_T(x) := \sup_{B^m_s(y)\subset B^m_r(x)} \mathbf{E}(T, C_s(y)).$$

Theorem 3.8. *There exist dimensional constants c, $C > 0$ with the following property. If T is a mass-minimizing current in $C_{4r}(x)$ as in (20), then for all $\eta \in (0, c)$ there exist a compact set $K \subset B^m_{3r}(x)$ and $f \in \mathrm{Lip}(B^m_{3r}(x), \mathcal{A}_Q(\mathbb{R}^n))$ such that* $\mathrm{graph}\big(f|_{K\times\mathbb{R}^n}\big) = T \llcorner K \times \mathbb{R}^n$ *and*

$$\mathcal{H}^m(B^m_{3r}(x)) \setminus K) \le \frac{C}{\eta}\mu_T\big(\{M_T > \frac{\eta}{2}\}\big), \qquad \mathrm{Lip}(f) \le C\eta^{1/2}.$$

The most general approximation result of area minimizing currents is the one due to F. Almgren, and reproved in [24] with more refined techniques and building on Theorem 3.8, which asserts that under suitable smallness condition of the cylindrical excess, an area minimizing current coincides on a big set with the graph of a Lipschitz Q-valued function. Another novel and important technical ingredient introduced by C. De Lellis and E. Spadaro, is the following "higher integrability" of the density δ_T of the mass measure μ_T in (21), namely

$$\int_{B^m_2(x)\cap\{\delta_T\le 1\}} \delta_T^p dy \le C\mathbf{E}^p(T, C_{4\,r}(x)) \quad \text{as soon as} \quad \mathbf{E}(T, C_{4\,r}(x)) < \epsilon,$$

for dimensional constant $p > 1$, $C > 0$ and $\epsilon > 0$. In turn, this result derives from an analogous property proved for Dir-minimizing Q-valued functions, see Theorem 3.3(ii).

The most important improvement of the theorem below with respect to the preexisting approximation results is the small power E^{γ_1} in the three estimates (22) - (24). Indeed, these play a crucial role in the construction of the center manifold. When $Q = 1$ and $n = 1$, this approximation theorem was first proved with different techniques by E. De Giorgi in [17] (cp. also [19, Appendix]).

Theorem 3.9 (F. Almgren's strong approximation)**.** *There exist constants C, γ_1, $\varepsilon_1 > 0$ (depending on m, n, Q) with the following property. Assume that T is area minimizing in the cylinder $C_{4r}(x)$ and assume that*

$$E := \mathbf{E}(T, C_{4\,r}(x)) < \varepsilon_1.$$

Then, there exist a map $f : B_r(x) \to \mathcal{A}_Q(\mathbb{R}^n)$ and a closed set $K \subset \bar{B}_r(x)$ such that the following holds:

$$Lip(f) \leq C E^{\gamma_1}, \tag{22}$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad \text{and} \quad |B_r(x) \setminus K| \leq C\, E^{1+\gamma_1}\, r^m, \tag{23}$$

$$\left| \|T\|(C_r(x)) - Q\,\omega_m\, r^m - \tfrac{1}{2} \int_{B_r(x)} |Df|^2 \right| \leq C\, E^{1+\gamma_1}\, r^m. \tag{24}$$

An important ingredient in the proof of Theorem 3.9 is the so called *harmonic approximation*, which allows us to compare the Lipschitz approximation of Theorem 3.8 with a Dir-minimizing function. Actually, the harmonic approximation could also be seen as a consequence of Theorem 3.9, choosing w as the solution of a suitable Dirichlet problem with f as boundary datum.

Theorem 3.10 (Harmonic approximation)**.** *Then, for every $\bar{\eta} > 0$, there exists a positive constant $\bar{\varepsilon}_1$ with the following property. Assume that T is as in Theorem 3.8,*

$$E := \mathbf{E}(T, C_{4\,r}(x)) < \bar{\varepsilon}_1$$

and let f be the the map provided by Theorem 3.8 with $\eta = E^\alpha$, for some $\alpha \in (0, 1/(4m))$. Then there exists a Dir-minimizing function w in $B^m_{2r}(x)$ such that

$$r^{-2} \int_{B^m_r(x)} W_2(f, w)^2 + \int_{B^m_r(x)} (|Df| - |Dw|)^2 + \int_{B^m_r(x)} |D(\mathsf{b}\circ f) - D(\mathsf{b}\circ w)|^2 \leq \bar{\eta}\, E\, r^m, \tag{25}$$

where $\mathsf{b} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^n$ is the barycenter map, i.e. $\mathsf{b}\left(\sum_i [[P_i]]\right) = \frac{1}{Q} \sum_i P_i$.

3.4. **Center manifold and normal approximation.** The center manifold $\mathcal{M}$ is the graph of a classical function over an m-dimensional plane with respect to which the excess of the minimizing current is sufficiently small. To achieve a suitable accuracy in the approximation of the average of the sheets of the current, it is necessary to define the function at an appropriate scale, which varies locally. Around any given point such scale is morally the first at which the sheets of the current cease to be close. This leads to a Whitney-type decomposition of the reference m-plane, where the refining algorithm is based on a stopping time argument, as in the classical Calderon-Zygmund decomposition. In each cube of the decomposition the center manifold is then a smoothing of the average of the Lipschitz multiple valued approximation of Theorem 3.9, performed in a suitable orthonormal system of coordinates, which changes from cube to cube. Using a kind of discrete Schauder estimates, C. De Lellis and E. Spadaro obtain $C^{3,\alpha}$ estimates for the center manifold. The possibility to get estimates up to the order 3 is deeply related to the expansion in Proposition 3.6, where the error term has order 4. It is interesting to notice that, if the current has multiplicity 1 everywhere (i.e., roughly speaking, it is made of a single sheet), then the center manifold coincides with it and, hence, one can conclude directly a higher regularity than the one given by the usual E.De Giorgi's argument (as I explained, higher regularity in the classical theory codimension 1 theory is obtained by the PDE regularity for the minimal surface equation, at the continuous level). This is already remarked in the introduction of [7] and it has been proved in [19] with a relatively simple and short direct argument.

The normal approximation to the current is then a multivalued map $F : \mathcal{M} \to \mathcal{A}_Q(U)$ of the form

$$F(x) := \sum_{i=1}^{Q} [[x + N_i(x)]], \tag{26}$$

where U is a kind of tubular neighbourhood of $\mathcal{M}$ and $N_i(x) \in [T_x\mathcal{M}]^\perp$.

3.5. **Strategy of proof.** We can give now a sketch of the C.De Lellis-E. Spadaro's proof of Theorem 0.1, referring to [20, 23, 24, 21, 22] for the many more details. The proof is done by contradiction.

Contradiction assumption: there exist numbers $m \geq 2$, $n \geq 1$, $\alpha > 0$ and an area-minimizing m-dimensional integer rectifiable current T in $\mathbb{R}^{m+n}$ such that

$$\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}(T)) > 0.$$

Note that the hypothesis $m \geq 2$ is justified because, for $m = 1$ an area minimizing current is locally the union of finitely many non-intersecting open segments. The aim of the proof is now to show that there exist suitable points of $\mathrm{Sing}(T)$ where we can perform a blow up analysis leading to a Dir-minimizing Q-valued functions with a large set of singular points, thus contradicting Theorem 3.3(iii). This process consists of different steps.

(A). Find a point $x_0 \in \mathrm{Sing}(T)$ and a sequence of radii $(r_k)_k$ with $r_k \downarrow 0$ such that:

- (A_1) the rescaling currents $T_{x_0,r_k} := (\iota_{x_0,r_k})_\sharp T$ converge to a flat mass-minimizing tangent cone;
- (A_2) $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}(T_{x_0,r_k}) \cap B_1) > \eta > 0$ for some $\eta > 0$ and for every $k \in \mathbb{N}$.

(B). Construction of the center manifold $\mathcal{M}$ and of a normal Lipschitz approximation $N : \mathcal{M} \to \mathbb{R}^{m+n}/\!\sim$ as in (26). This is the most technical part of the proof, and most of the conclusions of the subsequent steps intimately depend on the fine details and estimates relative to this construction.

(C). The center manifold that one constructs in step (B) can only be used in general for a finite number of radii r_k of step (A). The reason is that in general its degree of approximation of the average of the minimizing currents T is under control only up to a certain distance from the singular point under consideration. This leads to the definition of the sets where the approximation works, called *intervals of flattening*, and to the construction of an entire *sequence of center manifolds* which will be used in the blow up analysis.

(D). Next one has to take care of the problem of the infinite order of contact. This is done in two steps. For the first one the authors an *almost monotonicity formula* for a geometric variant of Almgren's frequency function, which involves once more the displacement part N of the normal approximation (26), deducing that the order of contact remains finite within each center manifold of the sequence in (C) (so, for scales belonging to the same interval of flattening). In the second step one needs to compare different center manifolds and to show that the order of contact still remains finite. This is done by exploiting a deep consequence of the construction in (C), called *splitting before tilting* (the terminology is borrowed from T. Rivière's paper [50]). Roughly speaking, this is a kind of multivalued version of the so-called "tilt lemma" where the L^2 deviation from a tangent plane can be estimated with the Excess. In a (elliptic) PDE context, this corresponds to the R. Caccioppoli-J. Leray inequality

$$\int_{B_{r/2}} |\nabla f|^2 \leq C r^{-2} \inf_c \int_{B_r} |f - c|^2 + \text{lower order terms}.$$

(E). With this analysis at hand, one can pass to the limit and conclude the convergence of the rescaling of N to the graph of a Dir-minimizing Q-valued function u.

(F). Finally, one can use a the capacitary (quite more delicate, if compared with the usual pointwise arguments of the codimension 1 theory based on the excess) leading to the persistence of the singularities, to show that the function u in (E) needs to have a singular set with positive $\mathcal{H}^{m-2+\alpha}$ measure, thus contradicting the partial regularity estimate for Q-valued harmonic functions.

4. Open problems

I close this survey on the regularity theory for mass-minimizing currents by listing a few open questions. All of them are quite challenging, and therefore brief or simple solutions should not be expected. Nevertheless, as I wrote in the introduction, the long term program undertaken by C. De Lellis and E. Spadaro makes F. Almgren's work readable and exploitable for a larger community of specialists, therefore after several years without essentially new developments we may hope to see some new progress in this field. See also [1, 18] for more open problems in the field.

(1). One of the main, perhaps the most well-known, open problems is the uniqueness of tangent cones to an area-minimizing current, i.e. the uniqueness of the limit $(\iota_{x,r})_\sharp T$ as $r \to 0$ for every $x \in \operatorname{spt}(T)$. The uniqueness is known for 2-dimensional currents (cp. [61]), and there are only partial results in the general case (see [5, 54]). A related question is that of the uniqueness of the inhomogeneous blow up for Dir-minimizing Q-valued functions. Also in this case the uniqueness is known for 2-dimensional domains (cp. [20], following ideas of [14]).

(2). It is unknown whether the singular set of an area-minimizing current has always locally finite $\mathcal{H}^{m-2}$ measure. This is the case for 2-dimensional currents (as proven by S. Chang [14], claiming in his proof a modification of the construction of the center manifold adapted to this purpose); note that in this result the uniqueness of the blow up of a Dir-minimizing map plays a fundamental role.

(3). It is unknown whether the singular set of an area minimizing current has some geometric structure, e.g. if it is rectifiable (i.e., roughly speaking, if it is contained in lower dimensional $(m-2)$-dimensional submanifolds). Once again, the positive answer is known for 2-dimensional currents, where the singularities are known to be locally isolated, and the uniqueness of the tangent map is one of the fundamental steps in the proof, and for codimension 1 currents (in this case the singular set is countably $\mathcal{H}^{m-7}$-rectifiable, [56, 57]). An analogous question can obviously be raised for Q-valued functions and a positive answer is presently known only in the case $m = 2$, [20].

(4). Recent progress on the theory of currents (see [9] and the more recent paper [10]) has provided weak solutions to Plateau's problem even when the ambient space is infinite-dimensional. In [11] the effective approach to regularity theory of [53] has been adapted to this more general situation: a careful analysis of the constructions shows that the constants involved in the proofs do not depend on the codimension, at least when the ambient space is an Hilbert space. This provides regularity *in a dense open set* for mass-minimizing currents (and almost everywhere regularity in the case of multiplicity 1 currents). Thanks to the work of C. De Lellis and E. Spadaro and to their intrinsic approach, also a large part of the theory of Q-valued functions seems to be independent of codimension. A challenging open question is to make also other parts of F. Almgren's theory, as for instance Theorem 3.9, equally "codimension free", with the final goal of getting at least an almost everywhere result for mass-minimizing currents in Hilbert spaces. However, I have to stress that already [24] makes a deep use of F. Almgren's extrinsic approach, which seems to be codimension-dependent.

(5). Finally, I mention the problem of boundary regularity. For higher codimension area-minimizing currents, the only positive known case is when the prescribed boundary is a unit-multiplicity current with support contained in the boundary of a uniformly convex set [4]. See also the recent work [38] for the case of Dir-minimizing Q-valued maps.

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