The Dirichlet energy of mappings from B^3 into a manifold: density results and gap phenomenon

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Abstract. *Weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded Dirichlet integral give rise to elements of the space $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$. We assume that the 2-homology group of $\mathcal{Y}$ has no torsion and that the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective. Then, in dimension $n = 3$, we prove that every element T in $\mathrm{cart}^{2,1}(B^3 \times \mathcal{Y})$, which has no singular vertical part, can be approximated weakly in the sense of currents by a sequence of smooth graphs $\{u_k\}$ with Dirichlet energies converging to the energy of T. We also show that the injectivity hypothesis on the Hurewicz map cannot be removed. We finally show that a similar topological obstruction on the target manifold holds for the approximation problem of the area functional.*

Let B^n be the unit ball in $\mathbb{R}^n$ and let $\mathcal{Y}$ be a smooth oriented Riemannian manifold of dimension $M \geq 2$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 3$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary and, moreover, that its integral 2-homology group $H_2(\mathcal{Y}) := H_2(\mathcal{Y}, \mathbb{Z})$ has no torsion, so that $H_2(\mathcal{Y}, X) = H_2(\mathcal{Y}) \otimes X$ for $X = \mathbb{R}, \mathbb{Q}$. Note that the last condition automatically holds if $M = 2$. In these circumstances, see e.g. [6], Vol. I, Sec. 5.4.1, there are generators $[\gamma_1], \dots, [\gamma_{\overline{s}}]$, i.e. integral cycles in $\mathcal{Z}_2(\mathcal{Y})$, such that

$$H_2(\mathcal{Y}) = \left\{ \sum_{s=1}^{\overline{s}} n_s \, [\gamma_s] \mid n_s \in \mathbb{Z} \right\}.$$

Definition 0.1 *We say that an integral 2-cycle $C \in \mathcal{Z}_2(\mathcal{Y})$ is* of spherical type *if its homology class contains a Lipschitz image of the 2-sphere S^2. More precisely, if there exist $Z \in \mathcal{Z}_2(\mathcal{Y})$, $R \in \mathcal{R}_3(\mathcal{Y})$ and a Lipschitz function $\phi : S^2 \to \mathcal{Y}$ such that*

$$C - Z = \partial R , \qquad \phi_{\#}[\![S^2]\!] = Z .$$

Denoting then

$$H_2^{sph}(\mathcal{Y}) := \{[\gamma] \in H_2(\mathcal{Y}) \mid \text{there exists } \phi \in \mathrm{Lip}(S^2, \mathcal{Y}) \text{ such that } \phi_{\#}[\![S^2]\!] \in [\gamma]\} ,$$

we also may and do choose the γ_i's in such a way that $[\gamma_1], \dots, [\gamma_{\widetilde{s}}]$ generate the spherical homology classes in $H_2^{sph}(\mathcal{Y})$ for some $\widetilde{s} \leq \overline{s}$. By the De Rham theorem the second real homology group is in duality with the second cohomology group $H^2_{dR}(\mathcal{Y})$, the duality being given by the natural pairing

$$< [\gamma], [\omega] > := \gamma(\omega) = \int_\gamma \omega , \qquad [\gamma]_{\mathbb{R}} \in H_2(\mathcal{Y}, \mathbb{R}) , \quad [\omega] \in H^2_{dR}(\mathcal{Y}) .$$

We will then denote by $[\omega^1], \dots, [\omega^{\overline{s}}]$ a dual basis in $H^2_{dR}(\mathcal{Y})$ so that, δ_{sr} being the Kronecker symbols,

$$\gamma_s(\omega^r) = \delta_{sr} .$$

Also, we may and do assume that ω^s is the harmonic form in its cohomology class.

In this paper we consider sequences of smooth maps $u_k : B^n \to \mathcal{Y}$ with

$$\sup_k \mathbf{D}(u_k) < +\infty ,$$

where $\mathbf{D}(u)$ is the Dirichlet integral

$$\mathbf{D}(u) := \frac{1}{2} \int_{B^n} |Du|^2 \, dx .$$

Modulo passing to a subsequence the $(n,2)$-currents G_{u_k}, integration over the graphs of u_k of n-forms with at most two vertical differentials, converge to a current $T \in \text{cart}^{2,1}(B^n \times \mathcal{Y})$, cf. [6] [7] and Sec. 1 below. Those currents have the form

$$T = G_{u_T} + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q + S_{T,sing}\,,$$

where u_T is the weak $W^{1,2}$ limit of the u_k's, $\mathbb{L}_q$ is an integer multiplicity rectifiable current of dimension $n-2$ in B^n, R_q is a 2-cycle in q and $S_{T,sing}$ is nonzero only on forms ω with exactly two vertical differentials and such that $d_y\omega \neq 0$. Apart from being completely vertical and homologically trivial, otherwise the current $S_{T,sing}$ can be very wild and essentially any measure in $B^n \times \mathcal{Y}$, compare [5]. For every $T \in \text{cart}^{2,1}(B^n \times \mathcal{Y})$, the Dirichlet integral of T, $\mathbf{D}(T)$, turns out to be well defined; in particular $\mathbf{D}(G_u) = \mathbf{D}(u)$ if $T = G_u$.

We proved in [8] that, if $n = 2$, then every $T \in \text{cart}^{2,1}(B^2 \times \mathcal{Y})$ is the weak limit of a sequence of graphs of smooth maps $u_k : B^2 \to \mathcal{Y}$, i.e., $G_{u_k} \rightharpoonup T$, moreover

$$\mathbf{D}(T) = \inf\{ \liminf_{k\to+\infty} \mathbf{D}(u_k) \mid \{u_k\} \subset \text{Lip}(B^2, \mathcal{Y})\,, \quad G_{u_k} \rightharpoonup T\}$$

and actually there is a sequence of smooth maps $u_k : B^2 \to \mathcal{Y}$ such that

$$G_{u_k} \rightharpoonup T \qquad \text{and} \qquad \frac{1}{2}\int_{B^2} |Du_k|^2\,dx \to \mathbf{D}(T)\,.$$

Assume now that the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective. We observe that by the Hurewicz theorem [9], if in particular $\mathcal{Y}$ is 1-connected, i.e., $\pi_1(\mathcal{Y}) = 0$, since $H_2(\mathcal{Y})$ is torsion free, this condition automatically holds.

In this paper we extend the previous density result to dimension $n = 3$, provided, however, that the current $T \in \text{cart}^{2,1}(B^3 \times \mathcal{Y})$ has the form

$$T = G_{u_T} + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q\,.$$

Condition $S_{T,sing} = 0$ is "satisfied" by minimizers T for the Dirichlet integral, compare Proposition 1.5.

We also show that the injectivity assumption of the Hurewicz map cannot be avoided. If the Hurewicz map is not injective, in fact, we show a current T in $\text{cart}^{2,1}(B^3 \times \mathcal{Y})$ of the type $T = G_u$ for which there is a positive constant C such that

$$\mathbf{D}(u) + C \leq \liminf_{k\to+\infty} \mathbf{D}(u_k)$$

for any sequence of smooth maps u_k with $G_{u_k} \rightharpoonup G_u$. Finally, we show that for the area functional, instead of the Dirichlet integral, the same gap phenomenon arises even in dimension 2, due to the non triviality of the first homotopy group $\pi_1(\mathcal{Y})$.

1 Notation and preliminary results

In this section we recall some facts from the theory of Cartesian currents with finite Dirichlet energy. We refer to [6] and [7] for proofs and details.

$\mathcal{D}_{n,2}$-currents. For $k = 2, \ldots, n$, every differential k-form $\omega \in \mathcal{D}^k(B^n \times \mathcal{Y})$ splits as a sum $\omega = \sum_{j=0}^{\underline{j}} \omega^{(j)}$, $\underline{j} := \min(k, M)$, where the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical $\mathcal{Y}$ variables. We denote by $\mathcal{D}^{k,2}(B^n \times \mathcal{Y})$ the subspace of $\mathcal{D}^k(B^n \times \mathcal{Y})$ of k-forms of the type $\omega = \sum_{j=0}^{2} \omega^{(j)}$, and by $\mathcal{D}_{k,2}(B^n \times \mathcal{Y})$ the dual space of $\mathcal{D}^{k,2}(B^n \times \mathcal{Y})$. Every $(k,2)$-current $T \in \mathcal{D}_{k,2}(B^n \times \mathcal{Y})$ splits as $T = \sum_{j=0}^{2} T_{(j)}$, where $T_{(j)}(\omega) = T(\omega^{(j)})$. For example, if $u \in W^{1,2}(B^n, \mathcal{Y})$, i.e., $u \in W^{1,2}(B^n, \mathbb{R}^N)$ with $u(x) \in \mathcal{Y}$ for a.e. $x \in B^n$, then G_u is an $(n,2)$-current in $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$, where in an approximate sense $G_u := (Id \bowtie u)_\# [\![B^n]\!]$, $(Id \bowtie u)(x) := (x, u(x))$, compare [6].

D-norm. For $\omega \in \mathcal{D}^{n,2}(B^n \times \mathcal{Y})$ we set

$$\|\omega\|_{\mathbf{D}} := \max\left\{\sup_{x,y} \frac{|\omega^{(0)}(x,y)|}{1+|y|^2}\,, \int_{B^n} \sup_y |\omega^{(1)}(x,y)|^2\,dx\,, \int_{B^n} \sup_y |\omega^{(2)}(x,y)|\,dx\right\}\,,$$
$$\|T\|_{\mathbf{D}} := \sup\left\{T(\omega) \mid \omega \in \mathcal{D}^{n,2}(B^n \times \mathcal{Y})\,,\ \|\omega\|_{\mathbf{D}} \le 1\right\}.$$

It is not difficult to show that $\|T\|_{\mathbf{D}}$ is a norm on $\{T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y}) \mid \|T\|_{\mathbf{D}} < +\infty\}$.

Weak $\mathcal{D}_{n,2}$-convergence. If $\{T_k\} \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$, we say that $\{T_k\}$ converges weakly in $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^{n,2}(B^n \times \mathcal{Y})$. The class $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ is closed under weak convergence and $\|\cdot\|_{\mathbf{D}}$ is weakly lower semicontinuous. Moreover, if $\sup_k \|T_k\|_{\mathbf{D}} < +\infty$, then there is a subsequence which weakly converges to some $T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ with $\|T\|_{\mathbf{D}} < +\infty$.

Boundaries. The exterior differential d splits into a horizontal and a vertical differential $d = d_x + d_y$. Of course $\partial_x T(\omega) := T(d_x\omega)$ defines a boundary operator $\partial_x : \mathcal{D}_{n,2}(B^n \times \mathcal{Y}) \to \mathcal{D}_{n-1,2}(B^n \times \mathcal{Y})$. Now, for any $\omega \in \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})$, $d_y\omega$ belongs to $\mathcal{D}^{n,2}(B^n \times \mathcal{Y})$ if and only if $d_y\omega^{(2)} = 0$. Then $\partial_y T$ makes sense only as an element of the dual space of $\mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y})$, where

$$\mathcal{Z}^{k,2}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{k,2}(B^n \times \mathcal{Y}) \mid d_y\omega^{(2)} = 0\}\,.$$

D-graphs. The study of weak limits of sequences of maps with equibounded Dirichlet energy, minimization problems and concentration phenomena, see [6], draw the authors of [5] to introduce the subclass **D**-graph$(B^n \times \mathcal{Y})$ given by the $(n,2)$-currents $T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ with $\|T\|_{\mathbf{D}} < +\infty$ and such that

$$T = G_{u_T} + S_T \tag{1.1}$$

for some function $u_T \in W^{1,2}(B^n, \mathcal{Y})$ and some $S_T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ with $S_{T(0)} = S_{T(1)} = 0$, i.e. S_T completely vertical, so that

$$\partial_x T = 0 \quad \text{on } \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})\,, \qquad \partial_y T = 0 \quad \text{on } \mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y})\,.$$

They also showed that:

(i) the decomposition (1.1) is unique;

(ii) weak limits in $\mathcal{D}_{n,2}$ of sequences of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$, with equibounded Dirichlet energy, belong to **D**-graph$(B^n \times \mathcal{Y})$;

(iii) if $T \in$ **D**-graph$(B^n \times \mathcal{Y})$, in general $\partial G_{u_T} \neq 0$. However

$$\partial G_{u_T} = 0 \quad \text{on } \mathcal{D}^{n-1,1}(B^n \times \mathcal{Y})\,, \qquad \partial G_{u_T}(\omega^{(2)}) = 0 \quad \text{if } \omega^{(2)} = d\eta \quad \text{and} \quad \operatorname{spt}\eta \subset B^n \times \mathcal{Y}$$

and

$$\partial_y G_{u_T} = 0 \quad \text{on } \mathcal{Z}^{n-1,1}(B^n \times \mathcal{Y})\,, \qquad \partial G_{u_T} = \partial_x G_{u_T} \quad \text{on } \mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y})\,;$$

in particular

$$\partial_y S_T(\omega^{(2)}) = 0 \quad \text{if } \omega^{(2)} = d\eta \quad \text{and} \quad \operatorname{spt}\eta \subset B^n \times \mathcal{Y}\,, \qquad \partial_x S_T = 0 \quad \text{on } \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})\,;$$

(iv) $\|G_{u_T}\|_{\mathbf{D}} = \|u_T\|_{W^{1,2}} \le \|T\|_{\mathbf{D}}$, consequently $\|S_T\|_{\mathbf{D}} \le 2\,\|T\|_{\mathbf{D}}$;

(v) **D**-graph$(B^n \times \mathcal{Y})$ is closed under weak convergence in $\mathcal{D}_{n,2}$ with equibounded **D**-norm.

The 2-dimensional case. If $n = 2$, obviously $\mathcal{D}_{n,2}(B^n \times \mathcal{Y}) = \mathcal{D}_2(B^2 \times \mathcal{Y})$ and ∂T is the usual boundary of currents, whereas $\mathbf{M}(T) \le c\,(\|T\|_{\mathbf{D}} + 1)$ for some absolute constant. Consequently, weak limits of smooth graphs with equibounded Dirichlet energy are integer multiplicity (shortly i.m.) rectifiable currents in $\mathcal{R}_2(B^2 \times \mathcal{Y})$, and $\mathbf{D}$-graph$(B^2 \times \mathcal{Y}) \cap \mathcal{R}_2(B^2 \times \mathcal{Y})$ is closed under weak convergence with equibounded $\mathbf{D}$-norm.

As proved in [5] and [6], every T in $\mathbf{D}$-graph$(B^2 \times \mathcal{Y}) \cap \mathcal{R}_2(B^2 \times \mathcal{Y})$ decomposes as

$$T = G_{u_T} + S_T\,, \qquad S_T = \sum_{i=1}^{I} \delta_{x_i} \times C_i + S_{T,sing}\,, \tag{1.2}$$

where δ_x is the Dirac mass in x, $x_i \in B^2$, $C_i \in \mathcal{Z}_2(\mathcal{Y})$ are integral cycles with non trivial homology and $S_{T,sing}$ is a completely vertical, homologically trivial, i.m. rectifiable current supported on a set not containing $\{x_i\} \times \mathcal{Y}$, $i = 1, \ldots, I$. More precisely, $S_{T,sing}(\omega) \neq 0$ only on forms $\omega \in \mathcal{D}^2(B^2 \times \mathcal{Y})$ such that $d_y \omega^{(2)} \neq 0$. Moreover, see [5] [6], if T is in the sequential weak closure of smooth graphs with equibounded Dirichlet energies, then every C_i is of spherical type. This facts lead to the following

Definition 1.1 *If $n = 2$, we denote by $\mathrm{cart}^{2,1}(B^2 \times \mathcal{Y})$ the class of i.m. rectifiable currents T in* $\mathbf{D}$-graph$(B^2 \times \mathcal{Y})$ *which decompose as in* (1.2)*, where the C_i's are of spherical type.*

It turns out, see [5] [7], that $\mathrm{cart}^{2,1}(B^2 \times \mathcal{Y})$ is closed under weak convergence, with equibounded $\mathbf{D}$-norm, and contains the weak limits of sequences of smooth graphs with equibounded $\mathbf{D}$-norm.

The n-dimensional case. Denote by $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ the orthogonal projection onto the first factor. Let P be an oriented 2-plane in $\mathbb{R}^n$, and $P_t := P + \sum_{i=1}^{n-2} t_i \nu_i$ the family of oriented 2-planes parallel to P, $t = (t_1, \ldots, t_{n-2}) \in \mathbb{R}^{n-2}$, $\mathrm{span}(\nu_1, \ldots, \nu_{n-2})$ being the orthogonal subspace to P. Similar to the case of normal currents, for every $T \in \mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ with $\|T\|_{\mathbf{D}} < +\infty$ and for $\mathcal{H}^{n-2}$-a.e. t the *slice* $T \llcorner \pi^{-1}(P_t)$ of T over $\pi^{-1}(P_t)$ is a well defined current in $\mathcal{D}_2((B^n \cap P_t) \times \mathcal{Y})$ with finite $\mathbf{D}$-norm. Moreover, if $T_k \rightharpoonup T$ with equibounded $\mathbf{D}$-norm, for $\mathcal{H}^{n-2}$-a.e. t, passing to a subsequence we have $T_k \llcorner \pi^{-1}(P_t) \rightharpoonup T \llcorner \pi^{-1}(P_t)$ with equibounded $\mathbf{D}$-norm. Finally, if $T \in \mathbf{D}$-graph$(B^n \times \mathcal{Y})$, for $\mathcal{H}^{n-2}$-a.e. t we have $T \llcorner \pi^{-1}(P_t) \in \mathbf{D}$-graph$((B^n \cap P_t) \times \mathcal{Y})$. Therefore in any dimension n it was introduced in [7] the following

Definition 1.2 *We say that T is in $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ if $T \in$* $\mathbf{D}$-graph$(B^n \times \mathcal{Y})$ *and for any 2-plane P and for $\mathcal{H}^{n-2}$-a.e. t the 2-dimensional current $T \llcorner \pi^{-1}(P_t)$ belongs to $\mathrm{cart}^{2,1}((B^n \cap P_t) \times \mathcal{Y})$.*

It turns out that the class $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ is closed under weak convergence with equibounded $\mathbf{D}$-norm.

Definition 1.3 *We say that a Sobolev map $u \in W^{1,2}(B^n, \mathcal{Y})$ is in $\mathrm{cart}^{2,1}(B^n, \mathcal{Y})$ if the current G_u associated to its graph is in $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$.*

Therefore, a $W^{1,2}$ map u is in $\mathrm{cart}^{2,1}(B^n, \mathcal{Y})$ if its graph has no inner boundary, i.e.,

$$\partial_x G_u = 0 \quad \text{on } \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})\,, \qquad \partial_y G_u = 0 \quad \text{on } \mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y})\,. \tag{1.3}$$

Remark 1.4 If $u : B^n \to \mathcal{Y}$ is a continuous map in $W^{1,2}(B^n, \mathcal{Y})$, by a standard convolution and projection argument it can be approximated in $W^{1,2}$-strong sense by a smooth sequence in $C^\infty(B^n, \mathcal{Y})$. This yields in particular that $u \in \mathrm{cart}^{2,1}(B^n, \mathcal{Y})$.

We let $\widetilde{B}^n$ denote a bounded domain in $\mathbb{R}^n$ such that $B^n \subset\subset \widetilde{B}^n$, e.g. $\widetilde{B}^n := B^n(0,2)$, and we let $\varphi : \widetilde{B}^n \to \mathcal{Y}$ be a given smooth $W^{1,2}$ function. In the sequel we will denote

$$\begin{aligned}
W^{1,2}_\varphi(\widetilde{B}^n, \mathcal{Y}) &:= \{u \in W^{1,2}(\widetilde{B}^n, \mathcal{Y}) \mid u = \varphi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\} \\
\mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n, \mathcal{Y}) &:= \{u \in \mathrm{cart}^{2,1}(\widetilde{B}^n, \mathcal{Y}) \mid u = \varphi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\} \\
\mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y}) &:= \{T \in \mathrm{cart}^{2,1}(\widetilde{B}^n \times \mathcal{Y}) \mid (T - G_\varphi) \llcorner (\widetilde{B}^n \setminus \overline{B}^n) \times \mathbb{R}^N = 0\}\,.
\end{aligned}$$

Structure of $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$. Denote by $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ the orthogonal projection onto the second factor. If $T \in \mathbf{D}$-graph$(B^n \times \mathcal{Y})$, for any closed 2-form $\omega \in \mathcal{Z}^2(\mathcal{Y})$ define the $(n-2)$-currents $\mathbb{D}(T; \omega)$ and $\mathbb{L}(T; \omega)$ in B^n by setting

$$\mathbb{D}(T;\omega)(\phi) := G_{u_T}(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\omega)\,, \qquad \mathbb{L}(T;\omega)(\phi) := S_T(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\omega)\,, \qquad \phi \in \mathcal{D}^{n-2}(B^n)\,.$$

Then, compare [5] and [6], $\mathbb{D}(T;\omega)$ and $\mathbb{L}(T;\omega)$ have finite mass, $\partial(\mathbb{D}(T;\omega)+\mathbb{L}(T;\omega))=0$ in $\mathcal{D}^{n-3}(B^n)$ and $\mathbb{L}(T;\omega)$ only depends on the cohomology class of ω. Finally, setting for $s=1,\ldots,\overline{s}$

$$\mathbb{D}_s(T):=\mathbb{D}(T;\omega^s)\,,\qquad \mathbb{L}_s(T):=\mathbb{L}(T;\omega^s)\,,$$

we have

$$T=G_{u_T}+\sum_{s=1}^{\overline{s}}\mathbb{L}_s(T)\times\gamma_s\qquad\text{on } \mathcal{Z}^{n,2}(B^n\times\mathcal{Y}) \tag{1.4}$$

for some function $u_T\in W^{1,2}(B^n,\mathcal{Y})$. Moreover, if $T\in\text{cart}^{2,1}(B^n\times\mathcal{Y})$, it is proved in [7] that the $\mathbb{L}_s(T)$'s are i.m. rectifiable currents in $\mathcal{R}_{n-2}(B^n)$, with $\mathbb{L}_s(T)=0$ for $s=\widetilde{s}+1,\ldots,\overline{s}$. As a consequence, every $T\in\text{cart}^{2,1}(B^n\times\mathcal{Y})$ decomposes as

$$T=G_{u_T}+\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbb{L}_q\times R_q\qquad\text{on}\quad \mathcal{Z}^{n,2}(B^n\times\mathcal{Y})\,, \tag{1.5}$$

where $\mathbb{L}_q$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(B^n)$ and $R_q\in\mathcal{Z}^2(\mathcal{Y})$ is an integral 2-cycle of spherical type in the homology class q. More precisely, if $q=\sum_{s=1}^{\widetilde{s}}n_s[\gamma_s]$, $n_s\in\mathbb{Z}$, then

$$R_q=\sum_{s=1}^{\widetilde{s}}n_s\gamma_s\qquad\text{and}\qquad \mathbb{L}_q=\tau(\mathcal{L}_q,1,\vec{\mathcal{L}})\,,$$

where $\vec{\mathcal{L}}$ is a unit simple orienting $(n-2)$-vector field and the rectifiable sets $\mathcal{L}_q$ are pairwise disjoint.

We finally remark that if

$$S_{T,sing}:=T-G_{u_T}+\sum_{s=1}^{\widetilde{s}}\mathbb{L}_s(T)\times\gamma_s\,,$$

though completely vertical and homologically trivial, i.e. $S_{T,sing}(\omega)=0$ if $\omega^{(2)}=0$ or $\omega\in\mathcal{Z}^{n,2}(B^n\times\mathcal{Y})$, the singular part $S_{T,sing}$ is in general non zero on forms $\omega\in\mathcal{D}^{n,2}(B^n\times\mathcal{Y})$ such that $d_y\omega^{(2)}\neq 0$. However, as the following result of [7] shows in any dimension n, solutions of minimum problems "have" no singular parts.

Proposition 1.5 *Let $T\in\text{cart}^{2,1}_\varphi(\widetilde{B}^n\times\mathcal{Y})$ be a minimizer, or a minimizer in its homology class*

$$[T]:=\{R\in\text{cart}^{2,1}_\varphi(\widetilde{B}^n\times\mathcal{Y})\mid R=T\quad\text{on } \mathcal{Z}^{n,2}(B^n\times\mathcal{Y})\}$$

for the Dirichlet integral. Write T as in (1.5)*, where $R_q\in q$, and set*

$$T^H:=G_{u_T}+\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbb{L}_q\times S_q\,,$$

$S_q\in\mathcal{Z}^2(\mathcal{Y})$ being the least mass integral cycle in q. Then $T^H\in\text{cart}^{2,1}_\varphi(\widetilde{B}^n\times\mathcal{Y})$, $T^H\in[T]$ and $\mathbf{D}(T^H)=\mathbf{D}(T)$; moreover

$$\mathbf{D}(T)=\frac{1}{2}\int|Du_T|^2\,dx+\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbf{M}(\mathbb{L}_q)\,\mathbf{M}(S_q)\,.$$

The 3-dimensional case. Let $R_2^\infty(B^3,\mathcal{Y})$ be the set of all maps $u\in W^{1,2}(B^3,\mathcal{Y})$ which are smooth except on a singular set $\Sigma(u)$ which is discrete. We also say that $u\in R^\infty_{2,\varphi}(\widetilde{B}^3,\mathcal{Y})$ if $u\in R_2^\infty(\widetilde{B}^3,\mathcal{Y})$ and u takes the value φ on $\widetilde{B}^3\setminus\overline{B}^3$. The following density result was proved in [1].

Theorem 1.6 (Bethuel) *$R_2^\infty(B^3,\mathcal{Y})$ (respectively $R^\infty_{2,\varphi}(\widetilde{B}^3,\mathcal{Y})$) is dense in $W^{1,2}(B^3,\mathcal{Y})$ (respectively in $W^{1,2}_\varphi(\widetilde{B}^3,\mathcal{Y})$).*

Let now $u \in W^{1,2}(B^3,\mathcal{Y})$. We have that $\partial G_u(\omega)=0$ if $\omega \in \mathcal{D}^{2,2}(B^3\times\mathcal{Y})$ with $\omega^{(2)}=0$ or $d_y\omega=0$. Setting then

$$\mathcal{B}^{k,2}(B^3\times\mathcal{Y}) := \{\omega \in \mathcal{D}^{k,2}(B^3\times\mathcal{Y}) \mid \exists \eta \in \mathcal{D}^{k-1,1}(B^3\times\mathcal{Y}) \ : \ \omega = d_y\eta\}$$

and

$$\mathcal{H}^{k,2}(B^3\times\mathcal{Y}) := \frac{\mathcal{Z}^{k,2}(B^3\times\mathcal{Y})}{\mathcal{B}^{k,2}(B^3\times\mathcal{Y})}\,,$$

we have $\partial G_u = 0$ on $\mathcal{B}^{2,2}(B^3\times\mathcal{Y})$ and $\partial_y\partial G_u=0$, whence $\partial G_u(\omega)$ depends only on the cohomology class of $\omega\in\mathcal{Z}^{2,2}(B^3\times\mathcal{Y})$. As a consequence ∂G_u induces a functional $(\partial G_u)_\star$ on $\mathcal{H}^{2,2}(B^3\times\mathcal{Y})$ given by

$$(\partial G_u)_\star(\omega+\mathcal{B}^{2,2}) := \partial G_u(\omega+\mathcal{B}^{2,2}) = \partial G_u(\omega)\,, \qquad \omega\in\mathcal{Z}^{2,2}\,,$$

compare [6], Vol. II, Sec. 5.4.1. Therefore, since

$$\mathcal{H}^{k,2}(B^3\times\mathcal{Y}) \simeq \mathcal{D}^{k-2}(B^3)\otimes H^2_{dR}(\mathcal{Y})\,,$$

the homology map $(\partial G_u)_\star$ is uniquely represented as an element of $\mathcal{D}_0(B^3;H_2(\mathcal{Y},\mathbb{R}))$. More explicitly, if $\phi\in\mathcal{D}^0(B^3)$, we have $[(\partial G_u)_\star(\phi)]\in H_2(\mathcal{Y},\mathbb{R})$ and for $s=1,\dots,\overline{s}$

$$< (\partial G_u)_\star(\phi),[\omega^s] > = \partial G_u(\pi^\#\phi\wedge\widetilde{\pi}^\#\omega^s)\,,$$

$<,>$ denoting the de Rham duality between $H_2(\mathcal{Y},\mathbb{R})$ and $H^2_{dR}(\mathcal{Y})$.

Following [6], Vol. II, Sec. 5.4.2, we now set

$$\mathbb{P}(u) := (\partial G_u)_\star \in \mathcal{D}_0(B^3;H_2(\mathcal{Y};\mathbb{R}))$$

and, for each $\omega\in[\omega]\in H^2_{dR}(\mathcal{Y})$, we define the current $\mathbb{P}(u;\omega)\in\mathcal{D}_0(B^3)$ by $\mathbb{P}(u;\omega) := \partial\pi_\#(G_u \llcorner \widehat{\pi}^\#\omega)$, so that

$$\mathbb{P}(u;\omega)(\phi) = \partial G_u(\pi^\#\phi\wedge\widehat{\pi}^\#\omega) \qquad \forall\,\phi\in\mathcal{D}^0(B^3)\,,$$

and for every $\omega\in\mathcal{Z}^2(\mathcal{Y})$ the current $\mathbb{D}(u;\omega)\in\mathcal{D}_1(B^3)$ by $\mathbb{D}(u;\omega) := \pi_\#(G_u\llcorner\widehat{\pi}^\#\omega)$, so that

$$\mathbb{D}(u;\omega)(\phi) = G_u(\pi^\#\phi\wedge\widehat{\pi}^\#\omega) \qquad \forall\,\phi\in\mathcal{D}^1(B^3)\,.$$

The following facts hold:

(i) for $s=1,\dots,\widetilde{s}$

$$\mathbb{P}(u;\omega^s)(\phi) = < \mathbb{P}(u)(\phi),[\omega^s] >\,,$$

i.e., $\mathbb{P}(u;\omega^s)$ does not depend on the representative in the cohomology class $[\omega^s]$;

(ii) $\partial\,\mathbb{P}(u)=0$ and $\mathbb{P}(u)=\displaystyle\sum_{s=1}^{\widetilde{s}}\mathbb{P}(u;\omega^s)\otimes[\gamma_s]$, hence it does not depend on the choice of $[\gamma_1],\dots,[\gamma_{\widetilde{s}}]$;

(iii) $\partial\,\mathbb{D}(u;\omega)(\phi) = < \mathbb{P}(u)(\phi),[\omega] >$ and hence $\partial\,\mathbb{D}(u;\omega^s)=\mathbb{P}(u;\omega^s)$ for each representative ω^s in $[\omega^s]$.

Therefore we can set

$$\mathbb{D}_s(u) := \mathbb{D}(u;\omega^s)\,, \qquad \mathbb{P}_s(u) := \mathbb{P}(u;\omega^s) = \partial\,\mathbb{D}_s(u)\llcorner B^3\,, \qquad s=1,\dots,\widetilde{s}\,.$$

Note that if $T=G_{u_T}+S_T\in \operatorname{cart}^{2,1}(B^3\times\mathcal{Y})$, then $\mathbb{D}_s(u_T)=\mathbb{D}_s(T)$ and

$$\mathbb{P}_s(u_T) = -\partial\,\mathbb{L}_s(T)\llcorner B^3 \qquad \forall\,s=1,\dots,\widetilde{s}\,. \tag{1.6}$$

Moreover, if $T=G_{u_T}+S_T\in \operatorname{cart}^{2,1}_\varphi(\widetilde{B}^3\times\mathcal{Y})$, then

$$\mathbb{P}_s(u_T) = -\partial\,\mathbb{L}_s(T) \qquad \forall\,s=1,\dots,\widetilde{s}\,. \tag{1.7}$$

Clearly $\mathbb{P}(u)=0$ if u is smooth, say Lipschitz, or at least in $W^{1,3}(B^3,\mathcal{Y})$. Moreover, taking account of Theorem 1.6 it is proved in [6] the following

Proposition 1.7 *Let* $u \in W^{1,2}_{\varphi}(\widetilde{B}^3, \mathcal{Y})$ *and let* $\{u_k\} \subset R^{\infty}_{2,\varphi}(\widetilde{B}^3, \mathcal{Y})$ *converge strongly in* $W^{1,2}$ *to* u*. Then* $\mathbb{P}(u)$ *is a* 0*-dimensional real flat chain, actually is the real flat limit of currents* $\mathbb{P}(u_k)$ *in* $\mathcal{R}_0(\widetilde{B}^3; H_2^{sph}(\mathcal{Y}))$ *and*

$$\mathbb{P}(u)(\phi) = \sum_{s=1}^{\widetilde{s}} \mathbb{P}_s(u)(\phi)\,[\gamma^s] \in H_2^{sph}(\mathcal{Y}, \mathbb{R}) \qquad \forall\, \phi \in \mathcal{D}^0(\widetilde{B}^3)\,, \tag{1.8}$$

where $H_2^{sph}(\mathcal{Y}, \mathbb{R}) := H_2^{sph}(\mathcal{Y}) \otimes \mathbb{R}$*. Moreover*

(i) $\mathbb{P}(u_k)$ *is a finite combination, with integer coefficients* $d_{i,s} \in \mathbb{Z}$*, of Dirac measures at points* $a_i \in \overline{B}^3$*,*

$$\mathbb{P}(u_k) = \sum_{s=1}^{\widetilde{s}} d_{i,s}\delta_{a_i} \otimes [\gamma_s]\,;$$

(ii) *if* $\chi \in C^{\infty}_c(\widetilde{B}^3)$ *with* $\chi = 1$ *in a neighborhood of* $\overline{B}^3$*, then for every* $s = 1, \ldots, \widetilde{s}$

$$\mathbb{P}_s(u_k)(\chi) = 0 \qquad \text{and} \qquad \mathbb{P}_s(u)(\chi) = 0\,,$$

i.e. the s-degree *of* u_k *and* u *is zero.*

By applying Federer's results in [6], Vol. II, Sec. 4.2.5 and Sec. 5.4.2, it then follows that $\mathbb{P}_s(u) = \partial\,\mathbb{D}_s(u)$ is an integral flat chain.

Proposition 1.8 *Let* $u \in W^{1,2}_{\varphi}(\widetilde{B}^3, \mathcal{Y})$ *and* $\{u_k\} \subset R^{\infty}_{2,\varphi}(\widetilde{B}^3, \mathcal{Y})$ *converge strongly in* $W^{1,2}$ *to* u*. Then*

(i) *there exists* $L \in \mathcal{R}_1(\widetilde{B}^3; H_2^{sph}(\mathcal{Y}))$ *such that* $\mathbb{P}(u) = \partial L$*;*

(ii) *for each* $s = 1, \ldots, \widetilde{s}$ *there exist points* $a_i, b_i \in \overline{B}^3$ *such that*

$$\mathbb{P}_s(u) = \sum_{i=1}^{+\infty}(\delta_{a_i} - \delta_{b_i})\,, \qquad \sum_{i=1}^{+\infty} |a_i - b_i| < +\infty\,;$$

(iii) $\mathbf{M}(\mathbb{D}_s(u_k) - \mathbb{D}_s(u)) \to 0$ *as* $k \to +\infty$*;*

(iv) *if* $L^s_{u_k,u}$ *denotes a* 1*-dimensional i.m. rectifiable current of least mass with support in* $\overline{B}^3$ *such that*

$$\partial L^s_{u_k,u} = \mathbb{P}_s(u) - \mathbb{P}_s(u_k)\,, \tag{1.9}$$

then

$$\mathbf{M}(L^s_{u_k,u}) \to 0 \qquad as \quad k \to +\infty\,.$$

In a similar way it can be proved the following

Proposition 1.9 *Let* $u \in W^{1,2}(B^3, \mathcal{Y})$ *and let* $\{u_k\} \subset R^{\infty}_2(B^3, \mathcal{Y})$ *converge strongly in* $W^{1,2}$ *to* u*. Then* $\mathbb{P}(u)$ *is a* 0*-dimensional real flat chain, actually is the real flat limit of currents* $\mathbb{P}(u_k)$ *in* $\mathcal{R}_0(B^3; H_2^{sph}(\mathcal{Y}))$ *and* (1.8) *holds for every* $\phi \in \mathcal{D}^0(B^3)$*. Also,* (i) *in Proposition* 1.7 *holds again. Moreover,* (i), (ii) *and* (iii) *in Proposition* 1.8 *hold again, with* $L \in \mathcal{R}_1(B^3; H_2^{sph}(\mathcal{Y}))$ *in* (i)*; finally*

(iv) *if* $L^s_{u_k,u}$ *denotes a* 1*-dimensional i.m. rectifiable current of least mass in* B^3 *with support*

$$\operatorname{spt}(\partial L^s_{u_k,u} - \mathbb{P}_s(u) + \mathbb{P}_s(u_k)) \subset \partial B^3\,, \tag{1.10}$$

then

$$\mathbf{M}_{B^3}(L^s_{u_k,u}) := \mathbf{M}(L^s_{u_k,u} \llcorner B^3) \to 0 \qquad as \quad k \to +\infty\,.$$

The Dirichlet energy in $\text{cart}^{2,1}$**.** Denote by $\bigwedge_n \mathbb{R}^{n+M}$ the space of n-vectors in $\mathbb{R}^{n+M}$. If $G : \mathbb{R}^n \to \mathbb{R}^M$ is a linear transformation, and with the same notation $G := (G_i^j)_{i,j=1}^{n,M}$ is the $(M \times n)$-matrix associated to it, we let

$$M(G) := (e_1 + Ge_1) \wedge \cdots \wedge (e_n + Ge_n) \in \bigwedge_n \mathbb{R}^{n+M} ,$$

$(e_i)_{i=1}^n$ being the canonical basis in $\mathbb{R}^n$. Then $M(G)$ identifies the plane graph of G in $\mathbb{R}^{n+M}$, and in fact orients such an n-plane. If $T \in \mathbf{D}\text{-graph}(B^n \times \mathcal{Y})$, we define the Dirichlet density as the function of $y \in \mathcal{Y}$, $\xi \in \bigwedge_n \mathbb{R}^{n+M}$ given by

$$F(y,\xi) := \sup\Big\{ \phi(\xi) \mid \phi : \textstyle\bigwedge_n \mathbb{R}^{n+M} \to \mathbb{R} \quad \text{linear, } \phi(M(G)) \leq \dfrac{1}{2}|G|^2 \\ \text{for all linear maps } G : \mathbb{R}^n \to T_y\mathcal{Y} \Big\} ,$$

$T_y\mathcal{Y}$ being the tangent M-space to $\mathcal{Y}$ in y. The Dirichlet integral then extends to $\mathbf{D}$-graphs T, compare [6], as

$$\mathbf{D}(T) := \int F(y, \vec{T})\, d\|T\|_{\mathbf{D}} ,$$

$\vec{T}$ being the Radon-Nikodym derivative $dT/d\|T\|_{\mathbf{D}}$ and, if (1.1) holds, one has

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^n} |Du_T|^2\, dx + \int_{B^n \times \mathcal{Y}} F(y, \vec{S_T})\, d\|S_T\|_{\mathbf{D}} . \tag{1.11}$$

In particular we have

$$\|T\|_{\mathbf{D}} \leq c\, \mathbf{D}(T) \tag{1.12}$$

for some absolute constant $c = c(n)$. Finally, if $A \subset B^n$ is a Borel set we will denote

$$\mathbf{D}(T, A \times \mathcal{Y}) := \mathbf{D}(T \llcorner A \times \mathcal{Y})$$

and if $u \in W^{1,2}(B^n, \mathcal{Y})$

$$\mathbf{D}(u, A) := \frac{1}{2}\int_A |Du|^2\, dx = \mathbf{D}(G_u, A \times \mathcal{Y}) , \qquad \mathbf{D}(u) := \mathbf{D}(u, B^n) .$$

In case $n = 2$, if (1.2) holds we have

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^2} |Du_T|^2\, dx + \sum_{i=1}^{I} \mathbf{M}(C_i) + \mathbf{M}(S_{T,sing}) .$$

In case $n \geq 3$, a part from the case of energy minimizing currents, see [7], we do not have an explicit formula for the second term on the right hand side of (1.11). However, if $n = 3$ and

$$T = G_{u_T} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T) \times \gamma_s ,$$

i.e. $S_{T,sing} = 0$, then we have

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^3} |Du_T|^2\, dx + \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T))\, \mathbf{M}(\gamma_s) \tag{1.13}$$

and, writing T as $T = G_{u_T} + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q$,

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^3} |Du_T|^2\, dx + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\, \mathbf{M}(R_q) . \tag{1.14}$$

As a consequence of the previous facts, for future use we explicitly state the following approximation results.

Proposition 1.10 *Let* $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^3 \times \mathcal{Y})$ *and* $S_{T,sing} = 0$. *Then there exists a sequence* $\{u_k\} \subset R^\infty_{2,\varphi}(\widetilde{B}^3, \mathcal{Y})$, *strongly converging to* u *in* $W^{1,2}$, *such that if* $L^s_{u_k,u}$ *is given by* (1.9), *then we have*

$$T_k := G_{u_k} + \sum_{s=1}^{\widetilde{s}} (L^s_{u_k,u} + \mathbb{L}_s(T)) \times \gamma_s \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^3 \times \mathcal{Y}) ,$$

$\mathbf{M}(\partial(L^s_{u_k,u} + \mathbb{L}_s(T))) < +\infty$ *for every* s *and* k, $T_k \rightharpoonup T$ *weakly in* $\mathcal{D}_{3,2}(\widetilde{B}^3 \times \mathcal{Y})$ *and* $\mathbf{D}(T_k) \to \mathbf{D}(T)$ *as* $k \to +\infty$.

PROOF: By (1.7) and (1.9) we have $\partial(L^s_{u_k,u} + \mathbb{L}_s(T)) = -\,\mathbb{P}_s(u_k)$, which is a finite sum of Dirac masses, see Proposition 1.7. Moreover

$$\partial G_{u_k} \llcorner \widetilde{B}^3 \times \mathbb{R}^N = \sum_{s=1}^{\widetilde{s}} \mathbb{P}_s(u_k) \times \gamma_s , \tag{1.15}$$

whence $T_k \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^3 \times \mathcal{Y})$. The other assertions follow from Theorem 1.6 and Proposition 1.8. □

Proposition 1.11 *Let* $T \in \mathrm{cart}^{2,1}(B^3 \times \mathcal{Y})$ *and* $S_{T,sing} = 0$. *Then there exists a sequence* $\{u_k\} \subset R^\infty_2(B^3, \mathcal{Y})$, *strongly converging to* u *in* $W^{1,2}$, *such that if* $L^s_{u_k,u}$ *is given by* (1.10), *then we have*

$$T_k := G_{u_k} + \sum\nolimits_{s=1}^{\widetilde{s}} (L^s_{u_k,u} + \mathbb{L}_s(T)) \times \gamma_s \in \mathrm{cart}^{2,1}(B^3 \times \mathcal{Y}) ,$$

$\mathbf{M}_{B^3}(\partial(L^s_{u_k,u} + \mathbb{L}_s(T))) := \mathbf{M}(\partial(L^s_{u_k,u} + \mathbb{L}_s(T)) \llcorner B^3) < +\infty$ *for every* s *and* k, $T_k \rightharpoonup T$ *weakly in* $\mathcal{D}_{3,2}(B^3 \times \mathcal{Y})$ *and* $\mathbf{D}(T_k) \to \mathbf{D}(T)$ *as* $k \to +\infty$.

PROOF: By (1.6) and (1.10) we have $\partial(L^s_{u_k,u} + \mathbb{L}_s(T)) \llcorner B^3 = -\,\mathbb{P}_s(u_k)$, which is a finite sum of Dirac masses, see Proposition 1.9. Moreover (1.15) holds again, with B^3 instead of $\widetilde{B}^3$, whence $T_k \in \mathrm{cart}^{2,1}(B^3 \times \mathcal{Y})$. The other assertions follow from Theorem 1.6 and Proposition 1.9. □

Removing point singularities. We finally show how to remove the homologically trivial singularities of a map in R^∞_2, compare [1]. To this aim we will assume that the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective, where $\pi_2(\mathcal{Y})$ is the second free homotopy group of $\mathcal{Y}$, compare Remark 3.3 below.

Proposition 1.12 *Let* $\mathcal{Y}$ *be such that the Hurewicz homomorphism* $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ *is injective. Let* $u \in R^\infty_2(B^3, \mathcal{Y})$ *be in* $\mathrm{cart}^{2,1}(B^3, \mathcal{Y})$, *so that* (1.3) *holds, with* $n = 3$. *Then there exists a sequence of smooth maps* $u_k : B^3 \to \mathcal{Y}$ *which converges to* u *strongly in* $W^{1,2}$. *Moreover, if* $u \in R^\infty_{2,\varphi}(\widetilde{B}^3, \mathcal{Y})$, *we can define the sequence in such a way that* $u_k = \varphi$ *on* $\widetilde{B}^3 \setminus \overline{B}^3$.

PROOF: Since we use a local argument, we may assume that u has only one singularity at the origin, i.e., $u \in C^1(B^3 \setminus \{0\}, \mathcal{Y})$. Moreover, arguing as in [1], we can strongly approximate u by $W^{1,2}$ functions v, smooth outside the origin, such that

$$v(x) = u\left(r \, \frac{x}{|x|}\right)$$

for some small radius $r > 0$. Since the homological condition (1.3) is preserved by strong (and weak) convergence, we have that v satisfies (1.3) too. To remove the singularity of v, we have to show that

$$\{w \in W^{1,2}(B^3_r, \mathbb{R}^N) \cap C^0(B^3_r, \mathcal{Y}) \mid w_{|\partial B^3_r} = v_{|\partial B^3_r}\} \neq \emptyset , \tag{1.16}$$

i.e., $v_{|\partial B^3_r}$ is homotopic to a constant map in $\mathcal{Y}$. Since by assumption the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective, it suffices to show that for every closed 2-form ω in $\mathcal{Y}$, or in a basis $\{\omega^s\}_{s=1}^{\overline{s}}$ of $\mathcal{Z}^2(\mathcal{Y})$, we have $d\, v_{|\partial B^3_r}{}^\#\omega = 0$. This follows from condition (1.3) for $u := v_{|B^3_r}$: in fact

$$\int_{\partial B^3_r} v_{|\partial B^3_r}{}^\# \omega^s = G_{v_{|\partial B^3_r}}(\widehat{\pi}^\# \omega^s) = \partial G_{v_{|B^3_r}}(\widehat{\pi}^\# \omega^s) = G_{v_{|B^3_r}}(d\widehat{\pi}^\# \omega^s) = G_{v_{|B^3_r}}(\widehat{\pi}^\# d\omega^s) = 0 .$$

We then choose a continuous extension $w \in W^{1,2}(B_r^3, \mathcal{Y})$ of $v_{|\partial B_r^3}$ and, for $0 < \delta < r$, define

$$u_{r,\delta}(x) := \begin{cases} v(x) & \text{if} \quad |x| \geq r \\ v\left(r\,\dfrac{x}{|x|}\right) & \text{if} \quad \delta \leq |x| < r \\ w\left(\dfrac{r}{\delta}\,x\right) & \text{if} \quad |x| < \delta\,. \end{cases}$$

We have $\mathbf{D}(u_{r,\delta}, B_\delta^3) \leq (\delta/r)\,\mathbf{D}(w, B_r^3)$. Moreover

$$\mathbf{D}(u_{r,\delta}, B_r^3 \setminus B_\delta^3) \leq r \int_{\partial B_r^3} |D_\tau u|^2 \, d\mathcal{H}^2 = O(r)\,,$$

where $D_\tau u$ is the tangential derivative of u and $O(r_j) \to 0$ for some sequence $r_j \searrow 0$. In fact, if $F(r) := \int_{\partial B_r^3} |D_\tau u|^2 \, d\mathcal{H}^2$, by the coarea formula [3] we have $\int_0^1 rF(r)\,dr \leq 2\,\mathbf{D}(u, B^3) < +\infty$, so that F is a non-negative function in $L^1(0,1)$ and hence $\liminf_{r\to 0^+} r\,F(r) = 0$. Taking first δ small with respect to r, and letting then $r \to 0^+$ along a suitable sequence, we obtain the first assertion by Remark 1.4. The second assertion trivially follows. $\square$

2 The dipole construction

In this section we extend the classical dipole construction, of [2], for maps with values into S^2 to the wider class of target manifolds $\mathcal{Y}$ described at the beginning of this paper.

To fix the notation, let $a_+, a_- \in \mathbb{R}^3$ and $L \in \mathcal{R}_1(\mathbb{R}^3)$ be the 1-dimensional current given by integration over the segment joining a_- to a_+, oriented so that the boundary $\partial L = \delta_{a_+} - \delta_{a_-}$ and with mass $\mathbf{M}(L) = l := |a_+ - a_-|$. Without loss of generality we may and will assume

$$a_+ := (l, 0, 0)\,, \qquad a_- := (0, 0, 0)\,.$$

Also, if $P \in \mathcal{Y}$ we let G_P denote the current carried by the graph of the map constantly equal to P. Finally, in all this section we let $C \in \mathcal{Z}_2(\mathcal{Y})$ denote a nontrivial integral 2-cycle of spherical type, see Definition 0.1.

We will readapt some results from [4], compare [6], Vol. II, Sec. 4.2.3, where it is discussed the case $\mathcal{Y} = S^2$. In the sequel we will also make use of the following density result for spherical cycles, which was proved in [8].

Proposition 2.1 *Let $C \in \mathcal{Z}_2(\mathcal{Y})$ be an integral 2-cycle of spherical type and $P \in \mathcal{Y}$ be a given point. Then there exists a family of Lipschitz functions $f_\varepsilon : B^2 \to \mathcal{Y}$ such that $f_{\varepsilon|\partial B^2} \equiv P$, $f_{\varepsilon\#}[\![\, B^2 \,]\!] \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$ as $\varepsilon \to 0$ and*

$$\mathbf{D}(f_\varepsilon, B^2) \leq \mathbf{M}(C) + \frac{\varepsilon}{2}\,.$$

Finally, the 2-cycle $C_\varepsilon := f_{\varepsilon\#}[\![\, B^2 \,]\!]$ does not depend on the choice of $P \in \mathcal{Y}$.

We first prove existence of functions u_ε from $\mathbb{R}^3$ into $\mathcal{Y}$ which are smooth outside the points $a_\pm$, are constantly equal to P outside small neighborhoods of the segment connecting the points $a_\pm$, neighborhoods which are mapped, with orientation $\pm$, onto a 2-cycle C_ε which converges weakly with the mass to C, so that the graphs G_{u_ε} converge weakly and in energy to the dipole $G_P + L \times C$.

Theorem 2.2 (Approximate dipoles) *There exists a family of maps $\{u_\varepsilon\} \subset C^1(\mathbb{R}^3 \setminus \{a_-, a_+\}, \mathcal{Y})$ such that*

$$G_{u_\varepsilon} \rightharpoonup G_P + L \times C \qquad \text{weakly in } \mathcal{D}_{3,2}(\mathbb{R}^3 \times \mathcal{Y}) \tag{2.1}$$

as $\varepsilon \to 0$ and

$$\mathbf{D}(u_\varepsilon, \mathbb{R}^3) \to l \cdot \mathbf{M}(C)\,, \qquad l := |a_+ - a_-|\,.$$

PROOF: First define $w_\varepsilon : \mathbb{R}^2 \times (0,l) \to \mathcal{Y}$ by

$$w_\varepsilon(x_1,x_2,x_3) := \begin{cases} f_\varepsilon(x_1,x_2) & \text{if} \quad (x_1,x_2) \in B^2 \\ P & \text{if} \quad (x_1,x_2) \notin B^2 , \end{cases}$$

where f_ε is given by Proposition 2.1, so that w_ε is a smooth function. Moreover, let $\mathbb{R}^2_{x_3}$ denote the 2-dimensional plane through $(0,0,x_3)$ parallel to the coordinate plane $x_3 = 0$. By means of an homothety on $\mathbb{R}^2_{x_3}$ depending on x_3, we shall shrink the set where w_ε is different from P near the segment connecting the points a_- and a_+. More precisely, we apply the following result, which is proved as in [6], Vol. II, Sec. 4.2.3.

Lemma 2.3 *Let* $v : B^2_R \times (0,l) \to \mathcal{Y}$ *be a* $W^{1,2}$ *function and for every* $0 < \delta < 1$ *and* $0 < m \le 1$ *let*

$$\varphi^m_\delta(t) := \min\{mt, m(l-t), \delta\} , \qquad 0 \le t \le l .$$

Consider the map $\phi^m_\delta : B^2_R \times (0,l) \to \mathbb{R}^3$

$$\phi^m_\delta(x_1,x_2,x_3) := (\varphi^m_\delta(x_3)x_1, \varphi^m_\delta(x_3)x_2, x_3)$$

and define

$$v^m_\delta(x) := v \circ (\phi^m_\delta)^{-1}(x) , \qquad x \in \phi^m_\delta(B^2_R \times (0,l)) .$$

Then we have

$$\begin{aligned} \int_{\phi^m_\delta(B^2_R \times (0,l))} |Dv^m_\delta|^2 \, dx \quad &\le \quad \int_{B^2_R \times (0,l)} |D_{(x_1,x_2)} v|^2 \, dx \\ &+ \quad c\, m^2 R^2 \int_{B^2_R \times ((0,\delta/m) \cup (l-\delta/m,l))} |D_{(x_1,x_2)} v|^2 \, dx + c\,\delta^2 \int_{B^2_R \times (0,l)} |D_{x_3} v|^2 \, dx , \end{aligned} \tag{2.2}$$

where c *is an absolute constant.*

Applying Lemma 2.3 with $v := w_\varepsilon$, $R = 1$, $m = 1$ and δ sufficiently small in dependence of ε and of the Lipschitz constant of w_ε, we obtain a map $w_{\varepsilon,\delta} := w_\varepsilon \circ (\phi^1_\delta)^{-1}$. Finally, setting

$$u_\varepsilon(x) := \begin{cases} w_{\varepsilon,\delta}(x) & \text{if} \quad x \in \phi^1_\delta(B^2 \times (0,l)) \\ P & \text{otherwise} , \end{cases}$$

u_ε belongs to $C^1(\mathbb{R}^3 \setminus \{a_-, a_+\}, \mathcal{Y})$, is equal to P outside

$$\Omega_\delta := \phi^1_\delta(B^2 \times (0,l)) = \{x \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 < (\varphi^1_\delta(x_3))^2 , \quad 0 < x_3 < l\}$$

and proves the assertion. $\square$

For future use, we now consider the following variant of Theorem 2.2.

Lemma 2.4 *Let* $\Gamma(t)$ *be a smooth function from* $\mathbb{R}$ *to* $\mathcal{Y}$ *which is constant for* $t \le 0$ *and for* $t \ge l$*. Let* $C \in \mathcal{Z}_2(\mathcal{Y})$ *be an integral 2-cycle of spherical type. Then for all positive* ε *and for* δ *sufficiently small there exists a smooth function* $u_\varepsilon : \mathbb{R}^3 \setminus \{a_-, a_+\} \to \mathcal{Y}$ *such that* $u_\varepsilon(x) = \Gamma(x_3)$ *outside* Ω_δ*,*

$$\partial G_{u_\varepsilon} = (\delta_{a_+} - \delta_{a_-}) \times C_\varepsilon , \tag{2.3}$$

where $C_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ *is such that* $C_\varepsilon \rightharpoonup C$ *with* $\mathbf{M}(C_\varepsilon) \to \mathbf{M}(C)$*, and finally*

$$\mathbf{D}(u_\varepsilon, \Omega_\delta) \le l \cdot \mathbf{M}(C) + \varepsilon .$$

PROOF: We readapt an argument by [4]. For $0 \le z \le l$ let

$$\mathcal{L}(z) := \int_0^z |\Gamma'(t)|\, dt\,,$$

so that $\mathcal{L}(l)$ is the total length of Γ, and let $\gamma_z : [0, \mathcal{L}(z)] \to \mathcal{Y}$ be the arc length parametrization of the curve described by $\Gamma(t)$ as $0 \le t \le z$, i.e., such that $|\gamma_z'(t)| = 1$ for a.e. t, with $\gamma_z(0) = \Gamma(0)$ and $\gamma_z(\mathcal{L}(z)) = \Gamma(z)$. For $0 < r \ll 1$ let

$$\Psi_r(\rho) := \frac{\log r - \log \rho}{\log r}\,, \qquad r < \rho < 1\,.$$

Finally, we set $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$, $x_3 = z$ and denote $\widetilde{w}(\rho,\theta,z) := w(\rho\cos\theta, \rho\sin\theta, z)$ a function w.r.t. cylindrical coordinates. First define $w_\varepsilon : B^2 \times (0,l) \to \mathcal{Y}$ by

$$\widetilde{w}_\varepsilon(\rho,\theta,z) := \begin{cases} \widetilde{f}_\varepsilon(\rho/r,\theta) & \text{if} \quad 0 \le \rho < r \\ \gamma_z(\mathcal{L}(z)\,\Psi_r(\rho)) & \text{if} \quad r \le \rho < 1 \end{cases} \qquad \theta \in [0,2\pi]\,, \quad z \in (0,l)\,, \tag{2.4}$$

where f_ε is given by Proposition 2.1 in correspondence of $P = \Gamma(0) \in \mathcal{Y}$. Trivially

$$\mathbf{D}(w_\varepsilon, B_r^2 \times (0,l)) = \frac{1}{2}\int_{B_r^2 \times (0,l)} |D_{(x_1,x_2)} w_\varepsilon|^2\, dx = l \cdot \mathbf{D}(f_\varepsilon, B^2)\,.$$

Moreover, for some absolute constant $c > 0$ we estimate

$$\int_{(B^2 \setminus B_r^2)\times(0,l)} |D_{(x_1,x_2)} w_\varepsilon|^2\, dx \le c\, \mathcal{L}(l)^2\, \frac{l}{-\log r} \le \varepsilon\,,$$

if r is chosen sufficiently small, and

$$\int_{(B^2 \setminus B_r^2)\times(0,l)} |D_{x_3} w_\varepsilon|^2\, dx \le \mathcal{H}^2(B^2 \setminus B_r^2) \int_0^l |\mathcal{L}'(z)|^2\, dz \le C \int_0^l |\Gamma'(z)|^2\, dz\,.$$

Applying Lemma 2.3 with $v := w_\varepsilon$, $R = 1$ and $m = 1$, we obtain a map $u_\varepsilon := w_\varepsilon \circ (\phi_\delta^1)^{-1}$, which is extended as $u_\varepsilon(x) = \Gamma(x_3)$ outside Ω_δ. By (2.2), by Proposition 2.1 and by the previous estimates

$$\mathbf{D}(u_\varepsilon, \phi_\delta^1(B^2 \times (0,l))) \le l \cdot \mathbf{M}(C) + \varepsilon$$

if we choose δ sufficiently small in dependence of ε, of l, of the total length $\mathcal{L}(l)$ of Γ and of the L^2 norm of Γ'. Finally (2.3) holds with $C_\varepsilon := f_{\varepsilon\#}[\![B^2]\!]$. $\square$

We now wish to extend Lemma 2.4 to the following result, Proposition 2.6, in which we consider a map u from a neighborhood U of the segment joining a_- to a_+ into $\mathcal{Y}$. We will suppose u smooth outside a_+ and a_-. There are two main difficulties: the first one is to connect the function $\Gamma(x_3) := u(0,0,x_3)$ with the restriction of u to $\partial B_R^2 \times \{x_3\}$, for a small radius R: this will be done by using local charts on $\mathcal{Y}$. The second one is to remove point singularities which are produced by the construction. To this aim we will assume that the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective and we will extend Proposition 1.12. We first make the following

Remark 2.5 Similarly to [1], for any given point p in U we can choose small balls of radius r around p and replace u there by the radial maps

$$u_r(x) := u\left(p + r\frac{x-p}{|x-p|}\right) \tag{2.5}$$

so that

$$\mathbf{D}(u_r, B_r^3(p)) = \frac{r}{2}\int_{\partial B_r^3(p)} |D_\tau u|^2\, d\mathcal{H}^2 = O(r)\,,$$

where $O(r_j) \to 0$ for some sequence $r_j \searrow 0$, compare the proof of Proposition 1.12. In particular, if

$$\partial G_u \llcorner B^3_R(p) \times \mathbb{R}^N = \delta_p \times C$$

for $R > 0$ small and for some $C \in \mathcal{Z}_2(\mathcal{Y})$, then for every $0 < r < R$

$$\partial G_{u_r} \llcorner B^3_r(p) \times \mathbb{R}^N = \delta_p \times C_r \,,$$

where $C_r \in \mathcal{Z}_2(\mathcal{Y})$ is in the homology class of C, i.e., $C_r - C = \partial R_r$ for some $R_r \in \mathcal{R}_3(\mathcal{Y})$.

For $\sigma > 0$ and $0 < m \leq 1$ we now set

$$\widetilde{\Omega}^m_\sigma := \{x \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 < 4(\varphi^m_\sigma(x_3))^2 \,, \quad 0 < x_3 < l\} \,.$$

In the proof of Theorems 3.1 and 3.2 below, we will make use of the following

Proposition 2.6 *Let U be a neighborhood of the segment joining a_- and a_+. Let $u : U \to \mathcal{Y}$ be a $W^{1,2}$ map which is smooth except at a_- to a_+, with*

$$u(x) = u\left(a_\pm + r_\pm \frac{x - a_\pm}{|x - a_\pm|}\right) \qquad \text{if} \quad 0 < |x - a_\pm| < r_\pm \tag{2.6}$$

for some small radius $0 < r_\pm \ll l$. Moreover, suppose that

$$\partial G_u \llcorner U \times \mathbb{R}^N = -\delta_{a_+} \times C^+ + \delta_{a_-} \times C^- \,, \tag{2.7}$$

where $C^\pm \in \mathcal{Z}_2(\mathcal{Y})$. Finally, let $C \in \mathcal{Z}_2(\mathcal{Y})$ be a 2-cycle of spherical type. Then for every positive ε there exists a smooth map $u_\varepsilon : U \setminus \{a_-, a_+\} \to \mathcal{Y}$ such that

$$\partial G_{u_\varepsilon} \llcorner U \times \mathbb{R}^N = \delta_{a_+} \times (C_\varepsilon - C^+_\varepsilon) - \delta_{a_-} \times (C_\varepsilon - C^-_\varepsilon) \,, \tag{2.8}$$

where $C^\pm_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ is in the same homology class of $C^\pm$ and $C_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ is such that $C_\varepsilon \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$, with $\mathbf{M}(C_\varepsilon) \to \mathbf{M}(C)$ as $\varepsilon \to 0$. Moreover $u_\varepsilon(x) = u(x)$ in $U \setminus \widetilde{\Omega}^m_\sigma$, for some arbitrarily small $\sigma, m > 0$, with $\widetilde{\Omega}^m_\sigma \subset U$. Finally $G_{u_\varepsilon} \rightharpoonup G_u + L \times C$ weakly in $\mathcal{D}_{3,2}(U \times \mathcal{Y})$ and

$$\mathbf{D}(u_\varepsilon, \widetilde{\Omega}^m_\sigma) \leq l \cdot \mathbf{M}(C) + \varepsilon \,. \tag{2.9}$$

To prove Proposition 2.6, we will make use of the following extension of Proposition 1.12 to remove point singularities.

Proposition 2.7 *Let $\mathcal{Y}$ be such that the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is injective. Let $u : B^3 \to \mathcal{Y}$ be a $W^{1,2}$ function smooth outside the origin and such that*

$$\partial G_u \llcorner B^3 \times \mathbb{R}^N = -\delta_0 \times Z \tag{2.10}$$

for some $Z \in \mathcal{Z}_2(\mathcal{Y})$ which is in the zero homology class. Then there exists a sequence $u_\varepsilon : B^3 \to \mathcal{Y}$ of smooth maps such that $u_\varepsilon = u$ in a neighborhood of the boundary ∂B^3, $G_{u_\varepsilon} \rightharpoonup G_u$ weakly in $\mathcal{D}_{3,2}(B^3 \times \mathcal{Y})$ as $\varepsilon \to 0$ and

$$\mathbf{D}(u_\varepsilon, B^3) \leq \mathbf{D}(u, B^3) + \varepsilon \,.$$

In the next section we will also make use of the following

Proposition 2.8 *Let $\mathcal{Y}$ be as in Proposition 2.7 and let $C \in \mathcal{Z}_2(\mathcal{Y})$. Let $u_\varepsilon : B^3 \to \mathcal{Y}$ be a sequence of $W^{1,2}$ functions smooth outside the origin and such that*

$$\partial G_{u_\varepsilon} \llcorner B^3 \times \mathbb{R}^N = \delta_0 \times (C_\varepsilon - Z_\varepsilon) \tag{2.11}$$

for some $Z_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ which is in the same homology class of C and for some $C_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ such that $C_\varepsilon \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$ with $\mathbf{M}(C_\varepsilon) \to \mathbf{M}(C)$ as $\varepsilon \to 0$. Then there exists a sequence $v_\varepsilon : B^3 \to \mathcal{Y}$ of smooth maps such that $v_\varepsilon = u_\varepsilon$ in a neighborhood of the boundary ∂B^3, $(G_{v_\varepsilon} - G_{u_\varepsilon}) \rightharpoonup 0$ weakly in $\mathcal{D}_{3,2}(B^3 \times \mathcal{Y})$ as $\varepsilon \to 0$ and

$$\mathbf{D}(v_\varepsilon, B^3) \leq \mathbf{D}(u_\varepsilon, B^3) + \varepsilon \,.$$

We postpone the proof of Propositions 2.7 and 2.8 and we first give the

PROOF OF PROPOSITION 2.6: We wish to readapt the argument of Lemma 2.4, to which we refer for the notation, with

$$\Gamma(x_3) := u(0,0,x_3)\,, \qquad 0 \le t \le l\,.$$

To connect the function $\Gamma(x_3)$ with the restriction of u to $\partial B^2_R \times \{x_3\}$, for a suitable radius R, we make use of local charts, hence we first divide the cylinder $B^2_R \times (0,l)$ so that the oscillation of u is small.

By (2.6) we know that u has small oscillation in cones centered at $a_\pm$ and with small angles. More precisely, we may and do fix $0 < m \ll 1$ so that u maps the intersection of $B^3(a_\pm, r_\pm)$ with the lower (upper) half cone of center $a_\pm$, x_3-axis, and with angle $\arctan(2m)$, in exactly one local chart of $\mathcal{Y}$.

Therefore, since u is smooth outside $a_\pm$ and $\mathcal{Y}$ is compact, if ϕ^m_1 is given by Lemma 2.3 with $\delta = 1$, there exists a small radius $0 < R_0 < \min\{r_+, r_-, 1/2\}$, a finite set of numbers $0 = z_0 < z_1 < \cdots < z_n = l$, and local charts (U_i, ψ_i) on $\mathcal{Y}$, such that

$$u \circ \phi^m_1(B^2_{2R} \times (z_{i-1}, z_i)) \subset U_i \qquad \forall\, i = 1, \ldots, n$$

for every $0 < R < R_0$. Also, possibly taking smaller m and R_0, we may and do suppose that $\psi_i(U_i)$ is a convex subset of $\mathbb{R}^M$, where $M = \dim(\mathcal{Y})$. Set now

$$p_i = (0,0,z_i)\,, \qquad i = 0, \ldots, n\,,$$

so that $p_0 = a_-$ and $p_n = a_+$. By applying Remark 2.5 with $r = r(\varepsilon, p_i) \in (0, R_0)$, and $i = 1, \ldots, n-1$, we replace u in small balls $B^3_{r_\varepsilon}(p_i)$, with $r_\varepsilon := \min_{1 \le i \le n-1} r(\varepsilon, p_i)$, so that the new function v_ε is such that

$$\mathbf{D}(v_\varepsilon, U) \le \mathbf{D}(u, U) + \frac{\varepsilon}{3}\,. \tag{2.12}$$

As a consequence, in the sequel we relabel $v_\varepsilon = u$ and we assume that (2.5) holds for $p = p_i$ and for every i, with $r = r_\varepsilon$ previously given. Finally, by Remark 2.5, since u is smooth outside $a_\pm$, we have that

$$\partial G_u \llcorner B^3_{r_\varepsilon}(p_i) \times \mathbb{R}^N = \delta_{p_i} \times Z_i\,, \tag{2.13}$$

where the Z_i's, for $i = 1, \ldots, n-1$, are in the zero homology class, $Z_0 = C^-$ and $Z_n = -C^+$.

As before, we denote $\widetilde{w}(\rho, \theta, z) = w(\rho\cos\theta, \rho\sin\theta, z)$ a function in cylindrical coordinates. For $0 < r \ll 1$ we define $w_\varepsilon : B^2 \times (0,l) \to \mathcal{Y}$ by (2.4). Moreover, for every $i = 1, \ldots, n$ we let

$$\varphi^{m,i}_\delta(t) := \min\{m(t - z_{i-1}), m(z_i - t), \delta\}\,, \qquad z_{i-1} \le t \le z_i\,,$$

consider the map $\phi^{m,i}_\delta : B^2 \times (z_{i-1}, z_i) \to \mathbb{R}^3$

$$\phi^{m,i}_\delta(x_1, x_2, x_3) := (\varphi^{m,i}_\delta(x_3) x_1, \varphi^{m,i}_\delta(x_3) x_2, x_3)$$

and define

$$v^i_\varepsilon(x) := w_\varepsilon \circ (\phi^{m,i}_\delta)^{-1}(x)\,, \qquad x \in \phi^{m,i}_\delta(B^2 \times (z_{i-1}, z_i))\,.$$

As in Lemma 2.4 we estimate

$$\mathbf{D}(v^i_\varepsilon, \phi^{m,i}_\delta(B^2 \times (z_{i-1}, z_i))) \le |z_i - z_{i-1}|\, \mathbf{D}(f_\varepsilon, B^2) + \frac{\varepsilon}{3n} \tag{2.14}$$

if we choose $\delta = \delta(v^i_\varepsilon, \varepsilon, m, n) \le R_0/2$ sufficiently small. We then define

$$\widetilde{u}^i_\varepsilon(\rho, \theta, z) := \begin{cases} \widetilde{v}^i_\varepsilon(\rho, \theta, z) & \text{if} \quad 0 \le \rho < \varphi^{m,i}_\delta(z) \\ \widetilde{\Psi}^{m,i}_\delta(\rho, \theta, z) & \text{if} \quad \varphi^{m,i}_\delta(z) \le \rho < 2\varphi^{m,i}_\delta(z) \end{cases} \qquad \theta \in [0, 2\pi]\,, \quad z \in (z_{i-1}, z_i)\,,$$

where

$$\widetilde{\Psi}^{m,i}_\delta(\rho, \theta, z) := {\psi_i}^{-1}\Big(\Big(\frac{\rho}{\varphi^{m,i}_\delta(z)} - 1\Big) \cdot \psi_i \circ \widetilde{u}(2\varphi^{m,i}_\delta(z), \theta, z) + \Big(2 - \frac{\rho}{\varphi^{m,i}_\delta(z)}\Big) \cdot \psi_i \circ \widetilde{\Gamma}(z)\Big)\,.$$

Setting $\widetilde{u}_\varepsilon(\rho,\theta,z) := \widetilde{u}^i_\varepsilon(\rho,\theta,z)$ if $0 \le \rho < 2\varphi^i_\delta(z)$ and $z \in (z_{i-1}, z_i)$ for some i, and $u_\varepsilon = u$ elsewhere on U, we find that u_ε is smooth outside the set $\{p_i \mid i = 0, \dots, n\}$. By the construction, u_ε satisfies (2.13) for every i, where the 2-cycles Z_i, for $i = 1, \dots, n-1$, are in the zero homology class. Moreover

$$Z_0 = -(C_\varepsilon - C^-_\varepsilon) \qquad \text{and} \qquad Z_n = C_\varepsilon - C^+_\varepsilon\,,$$

where $C^\pm_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ is in the same homology class of $C^\pm$ and $C_\varepsilon := f_{\varepsilon\#}[\![B^2]\!]$, compare Proposition 2.1, so that $C_\varepsilon \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$, with $\mathbf{M}(C_\varepsilon) \to \mathbf{M}(C)$ as $\varepsilon \to 0$.

By applying Proposition 2.7, we may and do remove the singularities of u_ε at the points p_i, for $i = 1, \dots, n-1$, by modifying u_ε in small neighborhoods and by paying a small amount of energy. As a consequence, u_ε is smooth outside $a_\pm$ and satisfies (2.8). Moreover, by Proposition 2.1 we infer that $G_{u_\varepsilon} \rightharpoonup G_u + L \times C$ weakly in $\mathcal{D}_{3,2}(U \times \mathcal{Y})$. Therefore it remains to show that if δ is small, then

$$\mathbf{D}(u_\varepsilon, \{u_\varepsilon \ne u\}) \le l \cdot \mathbf{D}(f_\varepsilon, B^2) + \varepsilon\,, \tag{2.15}$$

so that (2.9) follows from Proposition 2.1.

To this aim, since u satisfies (2.5) near the p_i's, we first choose δ sufficiently small (depending on m and r_ε) so that for every $i = 0, \dots, n$ the map u does not depend on $|x - p_i|$ on the set

$$\{x \,:\, 0 < |x - p_i| < \delta\,, \quad \rho < 2mz \quad \text{and} \quad |z - z_i| < \delta/m\}\,.$$

Note that such a dependence is given by condition

$$\rho^2 + z^2 = \delta^2 \left(\frac{1 + 4m^2}{m^2} \right) < {r_\varepsilon}^2.$$

For fixed $i = 1, \dots, n$, if $z_{i-1} + \delta/m < x_3 < z_i - \delta/m$, then ${\varphi^{m,i}_\delta}(x_3) \equiv \delta$ and we estimate

$$|Du^i_\varepsilon| \le c\, \|D{\psi_i}^{-1}\|_\infty \left(\frac{1}{\delta}\, \Phi^i_{\delta,m} + \|D\psi_i\|_\infty \left(\frac{1}{\rho}\, |\widetilde{u}_\theta(2\delta,\theta,z)| + |\widetilde{u}_z(2\delta,\theta,z)| + |\widetilde{\Gamma}'(z)| \right) \right),$$

where $c > 0$ is an absolute constant and

$$\Phi^i_{\delta,m} := \sup\{ |\psi_i \circ \widetilde{u}(2\delta,\theta,z) - \psi_i \circ \widetilde{\Gamma}(z)| \,:\, \theta \in [0, 2\pi]\,, \quad z \in (z_{i-1} + \delta/m, z_i - \delta/m) \}\,.$$

If $z_{i-1} < x_3 < z_{i-1} + \delta/m$, then ${\varphi^{m,i}_\delta}(x_3) = m(x_3 - z_{i-1})$ and u does not depend on $|x - p_{i-1}|$, in particular $\Gamma'(x_3) = 0$. Then

$$|Du^i_\varepsilon| \le c\, \|D{\psi_i}^{-1}\|_\infty \left(\frac{1}{m}\, \Delta_i + \|D\psi_i\|_\infty\, |D_\tau u| \right),$$

where Δ_i is the diameter of $\psi_i(U_i)$ and $|D_\tau u|$ is the derivative of u in the tangential direction to $\partial B^3_R(p_{i-1})$. Finally, a similar estimate holds if $z_i - \delta/m < x_3 < z_i$. Therefore we obtain

$$\begin{aligned}
\mathbf{D}(u^i_\varepsilon, \{\rho \in ({\varphi^{m,i}_\delta}(z), 2{\varphi^{m,i}_\delta}(z))\,,\ z \in (z_{i-1}, z_i)\,,\ \theta \in [0, 2\pi]\}) \le \\
\le C_i \left({\Phi^i_{\delta,m}}^2 + \delta \int_{\partial B^2_{2\delta} \times (z_{i-1}, z_i)} |Du|^2\, d\mathcal{H}^2 + \delta \int_{z_{i-1}}^{z_i} |\Gamma'(x_3)|^2\, dx_3 \right) \\
+ C_i\, \frac{\delta}{m} \left(\delta^2\, {\Delta_i}^2 + \int_{\partial B^3_{r_\varepsilon}(p_{i-1})} |Du|^2\, d\mathcal{H}^2 + \int_{\partial B^3_{r_\varepsilon}(p_i)} |Du|^2\, d\mathcal{H}^2 \right),
\end{aligned}$$

where $r_\varepsilon > 0$ is previously given and $C_i > 0$ is an absolute constant depending on the Lipschitz constant of ψ_i and of its inverse. Now, $\Phi^i_{\delta,m}$ is small if m and δ are chosen small (independently). Moreover, if $F_i(\rho) := \int_{\partial B^2_\rho \times (z_{i-1}, z_i)} |Du|^2\, d\mathcal{H}^2$, as in the proof of Proposition 1.12 we obtain that $\liminf_{\rho \to 0^+} \rho\, F_i(\rho) = 0$. Therefore, if we first choose m small, and then $\delta = \delta(v^i_\varepsilon, \varepsilon, m, n, r_\varepsilon)$ sufficiently small, then for every i

$$\mathbf{D}(u^i_\varepsilon, \{\rho \in ({\varphi^{m,i}_\delta}(z), 2{\varphi^{m,i}_\delta}(z))\,,\ z \in (z_{i-1}, z_i)\,,\ \theta \in [0, 2\pi]\}) \le \frac{\varepsilon}{3n}\,.$$

Finally, by (2.12), (2.14) and by summation on i we obtain (2.15), as required. $\square$

Proof of Proposition 2.7: As in Proposition 1.12, we first strongly approximate u by $W^{1,2}$ functions v_r, smooth outside the origin, such that $v_r(x) = u(rx/|x|)$ for some small radius $r > 0$. To remove the singularity of v_r, we again show that $v_{r|\partial B^3_r}$ is homotopic to a constant map in $\mathcal{Y}$ and hence, by the assumption on the Hurewicz map $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$, that for every closed 2-form in a basis $\{\omega^s\}_{s=1}^{\overline{s}}$ of $\mathcal{Z}^2(\mathcal{Y})$ we have $d\, v_{r|\partial B^3_r}{}^{\#}\omega^s = 0$. Now we compute

$$\int_{\partial B^3_r} v_{r|\partial B^3_r}{}^{\#}\omega^s = \int_{\partial B^3_r} u_{|\partial B^3_r}{}^{\#}\omega^s = G_{u_{|\partial B^3_r}}(\widehat{\pi}^{\#}\omega^s)\,.$$

By (2.10) we have

$$\partial G_{u|B^3_r} = \partial(G_u \llcorner B^3_r \times \mathbb{R}^N) = G_{u|\partial B^3_r} - \delta_0 \times Z$$

and hence

$$G_{u_{|\partial B^3_r}}(\widehat{\pi}^{\#}\omega^s) = G_{u_{|B^3_r}}(d\widehat{\pi}^{\#}\omega^s) + Z(\omega^s)\,.$$

Since Z is in the zero homology class, i.e., $Z = \partial R$ for some $R \in \mathcal{R}_3(\mathcal{Y})$, we have

$$Z(\omega^s) = R(d\omega^s) = 0$$

whereas

$$G_{u_{|B^3_r}}(d\widehat{\pi}^{\#}\omega^s) = G_{u_{|B^3_r}}(\widehat{\pi}^{\#}d\omega^s) = 0\,,$$

so that $v_{r|\partial B^3_r}$ is homotopically trivial in $\mathcal{Y}$, and therefore we conclude as in Proposition 1.12. $\square$

Proof of Proposition 2.8: As before, we first strongly approximate u_ε by $W^{1,2}$ functions v^ε_r, smooth outside the origin, such that $v^\varepsilon_r(x) = u_\varepsilon(rx/|x|)$ for some small radius $r > 0$. To remove the singularity of v_r, as before we compute

$$\int_{\partial B^3_r} v^\varepsilon_{r|\partial B^3_r}{}^{\#}\omega^s = \int_{\partial B^3_r} u_{\varepsilon|\partial B^3_r}{}^{\#}\omega^s = G_{u_{\varepsilon|\partial B^3_r}}(\widehat{\pi}^{\#}\omega^s)\,.$$

By (2.11) we have

$$\partial G_{u_\varepsilon|B^3_r} = \partial(G_{u_\varepsilon} \llcorner B^3_r \times \mathbb{R}^N) = G_{u_\varepsilon|\partial B^3_r} + \delta_0 \times (C_\varepsilon - Z_\varepsilon)$$

and hence

$$G_{u_{\varepsilon|\partial B^3_r}}(\widehat{\pi}^{\#}\omega^s) = G_{u_{\varepsilon|B^3_r}}(d\widehat{\pi}^{\#}\omega^s) - C_\varepsilon(\omega^s) + Z_\varepsilon(\omega^s)\,.$$

Moreover, since $Z_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ is in the same homology class of C we have that $Z_\varepsilon - C = \partial R_\varepsilon$ for some $R_\varepsilon \in \mathcal{R}_3(\mathcal{Y})$ and hence

$$Z_\varepsilon(\omega^s) = R_\varepsilon(d\omega^s) + C(\omega^s) = C(\omega^s)\,.$$

As a consequence, by strong convergence of C_ε to C

$$C_\varepsilon(\omega^s) - Z_\varepsilon(\omega^s) = C_\varepsilon(\omega^s) - C(\omega^s) \to 0$$

as $\varepsilon \to 0$, whereas

$$G_{u_{\varepsilon|B^3_r}}(d\widehat{\pi}^{\#}\omega^s) = G_{u_{\varepsilon|B^3_r}}(\widehat{\pi}^{\#}d\omega^s) = 0\,.$$

Taking ε small, $r = r(\varepsilon)$ small with ε and $u_\varepsilon := v^\varepsilon_{r(\varepsilon)}$, we infer again that $u_{\varepsilon|\partial B^3_r}$ is homotopically trivial in $\mathcal{Y}$, and therefore we conclude as in Proposition 1.12. $\square$

3 A strong density result

In this section we will prove a strong density result for the Dirichlet energy of maps from 3-dimensional domains into general target manifolds $\mathcal{Y}$ as described in the introduction. We first consider the case with fixed boundary data, Theorem 3.1, and then the case with no boundary data, Theorem 3.2. We will readapt the proof of Theorem 1 in Sec. 4.2.5, and of Theorem 1 in Sec. 4.2.6, respectively, of [6] Vol. II, where these results are proved in the case $\mathcal{Y}=S^2$, see [8] for the case of dimension $n=2$.

Theorem 3.1 (Approximation by smooth graphs with prescribed boundary data) *Let $\mathcal{Y}$ be a smooth, compact, connected, oriented Riemannian manifold of dimension $M\geq 2$, isometrically embedded in $\mathbb{R}^N$, $N\geq 3$. Assume that the integral 2-homology group $H_2(\mathcal{Y},\mathbb{Z})$ has no torsion and that the Hurewicz homomorphism $\pi_2(\mathcal{Y})\to H_2(\mathcal{Y},\mathbb{Q})$ is injective. Finally, let $\varphi:\widetilde{B}^3\to\mathcal{Y}$ be a smooth map and $T\in\mathrm{cart}^{2,1}_\varphi(\widetilde{B}^3\times\mathcal{Y})$ be such that $S_{T,sing}=0$. Then there exists a sequence of smooth maps $u_k:\widetilde{B}^3\to\mathcal{Y}$, with $u_k=\varphi$ in $\widetilde{B}^3\setminus\overline{B}^3$, such that $G_{u_k}\rightharpoonup T$ weakly in $\mathcal{D}_{3,2}(\widetilde{B}^3\times\mathcal{Y})$ and*

$$\lim_{k\to+\infty}\mathbf{D}(u_k,\widetilde{B}^3)=\mathbf{D}(T,\widetilde{B}^3\times\mathcal{Y})\,.$$

Theorem 3.2 (Approximation by smooth graphs) *Let $\mathcal{Y}$ be as in Theorem 3.1 and $T\in\mathrm{cart}^{2,1}(B^3\times\mathcal{Y})$ be such that $S_{T,sing}=0$. Then there exists a sequence of smooth maps $u_k:B^3\to\mathcal{Y}$ such that $G_{u_k}\rightharpoonup T$ weakly in $\mathcal{D}_{3,2}(B^3\times\mathcal{Y})$ and*

$$\lim_{k\to+\infty}\mathbf{D}(u_k,B^3)=\mathbf{D}(T,B^3\times\mathcal{Y})\,.$$

Remark 3.3 In the case that the Hurewicz homomorphism $\pi_2(\mathcal{Y})\to H_2(\mathcal{Y},\mathbb{Q})$ is not injective, it will be shown in Sec. 4 that there are Cartesian maps in $\mathrm{cart}^{2,1}(B^3,\mathcal{Y})$ which are smooth outside the origin, i.e., which have one point singularity, and which cannot be approximated in strong sense by smooth maps. Therefore, such an hypothesis cannot be removed even in Proposition 1.12.

PROOF OF THEOREM 3.1: We may and do suppose that (1.4) and (1.5) hold on the whole class of forms $\mathcal{D}^{3,2}(\widetilde{B}^3\times\mathcal{Y})$. We divide the proof in five steps.

Step 1: Reduction to finite singularities. By Proposition 1.10, for every fixed $\varepsilon>0$ we can find $v_\varepsilon\in R^\infty_{2,\varphi}(\widetilde{B}^3,\mathcal{Y})$ such that v_ε is close to u in the $W^{1,2}$-norm. Moreover, we may and do choose v_ε so that

$$T_\varepsilon:=G_{v_\varepsilon}+\sum\nolimits_{s=1}^{\widetilde{s}}(L^s_{v_\varepsilon,u_T}+\mathbb{L}_s(T))\times\gamma_s\in\mathrm{cart}^{2,1}_\varphi(\widetilde{B}^3\times\mathcal{Y}) \tag{3.1}$$

and

$$\mathbf{D}(v_\varepsilon,\widetilde{B}^3)+\sum\nolimits_{s=1}^{\widetilde{s}}\mathbf{M}(L^s_{v_\varepsilon,u_T}+\mathbb{L}_s(T))\cdot\mathbf{M}(\gamma_s)\leq\mathbf{D}(u_T,\widetilde{B}^3)+\sum\nolimits_{s=1}^{\widetilde{s}}\mathbf{M}(\mathbb{L}_s(T))\cdot\mathbf{M}(\gamma_s)+\varepsilon\,. \tag{3.2}$$

As a consequence, since $\partial(L^s_{v_\varepsilon,u_T}+\mathbb{L}_s(T))=\mathbb{P}_s(v_\varepsilon)$ has finite mass, hence is rectifiable, we can assume that u_T has only a finite number of singularities or, equivalently, that $\mathbb{P}_s(u_T)$ is rectifiable for every $s=1,\dots,\widetilde{s}$. This yields that we may and do suppose

$$T=G_{u_T}+\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbb{L}_q\times R_q \tag{3.3}$$

and by (1.14)

$$\mathbf{D}(T)=\mathbf{D}(u_T,\widetilde{B}^3)+\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbf{M}(\mathbb{L}_q)\,\mathbf{M}(R_q)<+\infty\,, \tag{3.4}$$

where the $\mathbb{L}_q$'s are i.m. rectifiable currents in $\mathcal{R}_1(B^3)$ with pairwise disjoint supports and finite boundary mass

$$\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbf{M}(\partial\,\mathbb{L}_q)<+\infty\,.$$

In particular, by the boundary rectifiability theorem [3], the $\partial\,\mathbb{L}_q$'s are i.m. rectifiable currents.

Step 2: Approximation by polyhedral chains. Since for every $q \in H_2^{sph}(\mathcal{Y})$ the $\mathbb{L}_q$'s are integral currents with pairwise disjoint supports, using Federer's polyhedral approximation theorem [3], for every q we find an integral polyhedral chain P_q^ε with support contained in a small neighborhood of radius $c\,\varepsilon$ of the support of $\mathbb{L}_q$, and a function $u_\varepsilon \in R^\infty_{2,\varphi}(\widetilde{B}^3, \mathcal{Y})$ such that

$$\widetilde{T}_\varepsilon := G_{u_\varepsilon} + \sum_{q\in H_2^{sph}(\mathcal{Y})} P_q^\varepsilon \times R_q \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^3 \times \mathcal{Y})\,,$$

$\widetilde{T}_\varepsilon$ converges weakly in $\mathcal{D}_{3,2}(\widetilde{B}^3 \times \mathcal{Y})$ to T as $\varepsilon \to 0$ and

$$\mathbf{D}(\widetilde{T}_\varepsilon) = \mathbf{D}(u_\varepsilon, \widetilde{B}^3) + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbf{M}(P_q^\varepsilon)\,\mathbf{M}(R_q) \to \mathbf{D}(u_T, \widetilde{B}^3) + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\,\mathbf{M}(R_q)$$

as $\varepsilon \to 0$. Moreover, since the $\mathbb{L}_q$'s have disjoint supports, we may and do choose the P_q^ε's so that for every small $\varepsilon > 0$ they have pairwise disjoint supports.

Step 3: Approximation by non auto-intersecting polyhedral chains. Let

$$T := G_{u_T} + \sum_{q\in H_2^{sph}(\mathcal{Y})} P_q \times R_q \in \mathrm{cart}^{1}_\varphi(\widetilde{B}^3 \times \mathcal{Y})\,, \tag{3.5}$$

where the P_q's are polyhedral, with pairwise disjoint supports $\mathrm{spt}\, P_q \subset \overline{B}^3$, and $u_T : \widetilde{B}^3 \to \mathcal{Y}$ is locally Lipschitz on $\widetilde{B}^3 \setminus \bigcup_q \mathrm{spt}\,\partial P_q$, compare Step 2. Similarly to Step 3 of the cited theorem in [6], we can find for each q a sequence $\{P_q^{(k)}\}_k$ of finite polyhedral lines (which are either closed or start and finish on ∂P_q), without auto-intersections, such that

$$\partial P_q^{(k)} = \partial P_q\,, \qquad P_q^{(k)} \rightharpoonup P_q\,, \qquad \mathbf{M}(P_q^{(k)}) \to \mathbf{M}(P_q)$$

as $k \to +\infty$. Also, since the P_q's have pairwise disjoint supports, we can define the $P_q^{(k)}$'s so that for every k they have pairwise disjoint supports too.

Step 4: Approximating the dipoles. Let T be as in (3.5), where the P_q's are polyhedral chains as in the conclusion of Step 3. Let us fix $\varepsilon > 0$ small. If S is anyone of the segments of the P_q's, by Remark 2.5 we first replace u, on small balls of radius $r = r(\varepsilon, p) > 0$ near each end point p of S, by a function satisfying (2.5). Moreover, if $S := [\![\,(n_i, p_i)\,]\!]$, by a suitable change of coordinates we may and do assume that $n_i = a_-$ and $p_i = a_+$, as in Sec. 2. Therefore (2.7) holds, where $C^\pm$ is in the homology class of R_q, if $a_\pm$ is an end point of P_q, whereas $C^\pm$ is in the zero homology class, if $a_\pm$ is an interior point of P_q. We then apply Proposition 2.6, with $C = R_q$. To this aim, note that we can take m and σ sufficiently small so that the neighborhoods $\widetilde{\Omega}^m_\sigma$ corresponding to different segments S are pairwise disjoint. We then replace u in small neighborhoods of each S by a function $u_\varepsilon \in R^\infty_{2,\varphi}(\widetilde{B}^3, \mathcal{Y})$ satisfying

$$\mathbf{D}(u_\varepsilon, \widetilde{B}^3) \le \mathbf{D}(u_T, \widetilde{B}^3) + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbf{M}(P_q)\,\mathbf{M}(R_q) + \varepsilon\,.$$

Step 5: Elimination of the singularities. The map u_ε constructed in Step 4 is such that there exist pairwise disjoint neighborhoods U_q of the support of the P_q's such that the restriction $u_{\varepsilon|U_q}$ is smooth except on a finite number of points, the end points of the segments $[n_i, p_i]$ in P_q. Now, by Proposition 2.1 the 2-cycle $C_\varepsilon := f_{\varepsilon\#}[\![\,B^2\,]\!]$ does not depend on the choice of the point $P \in \mathcal{Y}$, but only on $C = R_q$ and ε. Therefore, from the proof of Proposition 2.6 it follows that we may and do choose the C_ε's in (2.8) so that they are the same for every segment S of P_q. As a consequence, if $p := p_i = n_{i+1} \notin \partial P_q$, for $r > 0$ small we have

$$\partial G_{u_\varepsilon} \llcorner B^3_r(p) \times \mathbb{R}^N = \delta_p \times Z_p\,,$$

where $Z_p \in \mathcal{Z}_2(\mathcal{Y})$ is in the zero homology class. By Proposition 2.7 we then eliminate the singularity of u_ε in p. If $\partial P_q = \delta_{p^q_+} - \delta_{p^q_-}$, by (2.8) we then have

$$\partial G_{u_\varepsilon} \llcorner U_q \times \mathbb{R}^N = \delta_{p^q_+} \times (C_\varepsilon - C^+_\varepsilon) - \delta_{p^q_-} \times (C_\varepsilon - C^-_\varepsilon)\,,$$

where $C^\pm_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ is in the same homology class of R_q and $C_\varepsilon \in \mathcal{Z}_2(\mathcal{Y})$ is such that $C_\varepsilon \rightharpoonup R_q$ weakly in $\mathcal{D}_2(\mathcal{Y})$ with $\mathbf{M}(C_\varepsilon) \to \mathbf{M}(R_q)$, as $\varepsilon \to 0$. By Proposition 2.8 we then eliminate the singularities of u_ε in the points $p^q_\pm$. Finally, by a diagonal argument we find $u_\varepsilon \in \mathrm{Lip}(\widetilde{B}^3, \mathcal{Y})$ as required. □

In a similar way we now give the

Proof of Theorem 3.2: We may and do suppose that (1.4) and (1.5) hold on the whole class of forms $\mathcal{D}^{3,2}(B^3 \times \mathcal{Y})$. By Proposition 1.11, for every fixed $\varepsilon > 0$ we can find $v_\varepsilon \in R^\infty_2(B^3, \mathcal{Y})$ such that v_ε is close to u in the $W^{1,2}$-norm. Moreover, we may and do choose v_ε so that (3.1) and (3.2) hold, with B^3 instead of $\widetilde{B}^3$. As a consequence, since $\partial(L^s_{v_\varepsilon,u_T} + \mathbb{L}_s(T)) \llcorner B^3 = \mathbb{P}_s(v_\varepsilon)$ has finite mass, we may again and do assume that u_T has only a finite number of singularities or, equivalently, that $\mathbb{P}_s(u_T)$ is rectifiable for every $s = 1, \dots, \widetilde{s}$. This yields that we may and do suppose that (3.3) and (3.4) hold again, with B^3 instead of $\widetilde{B}^3$, where the $\mathbb{L}_q$'s are i.m. rectifiable currents in $\mathcal{R}_1(B^3)$ with pairwise disjoint supports.

If the boundary of the $\mathbb{L}_q$'s has finite mass, then the $\mathbb{L}_q$'s are integral currents and the rest of the proof is carried over similarly to the one of Theorem 3.1.

However, in general the $\mathbb{L}_q$'s do not have finite boundary mass, but only

$$\sum_{q \in H^{sph}_2(\mathcal{Y})} \mathbf{M}((\partial \mathbb{L}_q) \llcorner B^3) < +\infty\,.$$

In this case, we take any Lipschitz function ϕ on $\mathbb{R}^3$ with $\phi > 0$ in B^3, $\phi = 0$ in $\mathbb{R}^3 \setminus B^3$, such that

$$\Omega_\lambda := \{x \in B^3 \mid \phi(x) > \lambda\} \subset\subset B^3 \qquad \forall\, \lambda > 0\,.$$

For a.e. $\lambda > 0$ we have $T_\lambda := T \llcorner \Omega_\lambda \times \mathbb{R}^N \in \mathrm{cart}^{2,1}(\Omega_\lambda \times \mathcal{Y})$ with the mass $\mathbf{M}(\partial T_\lambda) < +\infty$, so that the boundary of any of the $\mathbb{L}_q \llcorner \Omega_\lambda$'s has finite mass. By changing by a small quantity the energy $\mathbf{D}(T, \Omega_\lambda \times \mathcal{Y})$, as in Theorem 3.1 we can now approximate weakly in $\mathcal{D}_{3,2}$ and in energy T_λ by the graph of a new map $v_\lambda : B^3 \to \mathcal{Y}$ which is smooth in Ω_λ, agrees with u_T on $B^3 \setminus \Omega_\lambda$ and is singular in at most a finite number of points on $\partial\Omega_\lambda$. More precisely, we have

$$|\mathbf{D}(v_\lambda, \Omega_\lambda) - \mathbf{D}(T, \Omega_\lambda \times \mathcal{Y})|$$

small, $\mathbb{P}_s(v_\lambda) \llcorner \Omega_\lambda = 0$ and $\mathbb{P}_s(v_\lambda) \llcorner \partial\Omega_\lambda$ has finite mass for every $s = 1, \dots, \widetilde{s}$. We now connect the singularities in $\mathbb{P}_s(v_\lambda) \llcorner \partial\Omega_\lambda$ by means of an i.m rectifiable current $L^s_\lambda \in \mathcal{R}_1(B^3)$ with $\mathrm{spt}(\partial L^s_\lambda - \mathbb{P}_s(v_\lambda)) \subset \partial(B^3 \setminus \Omega_\lambda)$ and with finite boundary mass $\mathbf{M}(\partial L^s_\lambda) < +\infty$. Since by (1.6) we have

$$\mathbb{P}_s(v_\lambda) = -\partial\, \mathbb{L}_s(T) \llcorner (B^3 \setminus \Omega_\lambda) \qquad \forall\, s = 1, \dots, \widetilde{s}\,,$$

one such connection is given by $\mathbb{L}_s(T) \llcorner (B^3 \setminus \Omega_\lambda)$ and therefore we can also assume that the mass $\mathbf{M}(L^s_\lambda)$ is small with λ. As a consequence, since the Cartesian current

$$T_\lambda := G_{v_\lambda} + \sum\nolimits_{s=1}^{\widetilde{s}} L^s_\lambda \times \gamma_s \in \mathrm{cart}^{2,1}(B^3 \times \mathcal{Y})$$

is such that $\sum\nolimits_{s=1}^{\widetilde{s}} \mathbf{M}(\partial L^s_\lambda) < +\infty$, we can argue as in Theorem 3.1 in order to eliminate the dipoles at ∂B^3. The proof is completed by a diagonal argument and by letting $\lambda \searrow 0$. □

A consequence of Theorems 3.1 and 3.2, compare [6], Vol. II, Sec. 4.2.5, is the following

Corollary 3.4 *Let $\mathcal{Y}$ and φ be as in Theorem 3.1. Then every map in $W^{1,2}_\varphi(\widetilde{B}^3, \mathcal{Y})$ (respectively in $W^{1,2}(B^3, \mathcal{Y})$) can be approximated weakly in $W^{1,2}$ by a sequence of maps in $C^1(\widetilde{B}^3, \mathcal{Y}) \cap W^{1,2}_\varphi(\widetilde{B}^3, \mathcal{Y})$ (respectively in $C^1(B^3, \mathcal{Y}) \cap W^{1,2}(B^3, \mathcal{Y})$).*

For a more general result, see [10].

4 Counterexamples

In this section we show that the injectivity hypothesis on the Hurewicz homomorphism is a necessary condition to strong density of smooth maps, compare Theorems 3.1 and 3.2.

We let $\mathcal{Y}$ be a smooth, compact, connected, oriented Riemannian manifold of dimension $M \geq 2$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 3$. We will suppose that the first and second homotopy groups are not trivial, $\pi_1(\mathcal{Y}) \neq 0$ and $\pi_2(\mathcal{Y}) \neq 0$, otherwise the Hurewicz map is automatically injective. We will consider the homogeneous extension $u : B^3 \to \mathcal{Y}$ of Lipschitz continuous maps $\varphi : S^2 \to \mathcal{Y}$, where $S^2 := \partial B^3$ is the unit 2-sphere, i.e.,

$$u(x) := \varphi\left(\frac{x}{|x|}\right). \tag{4.1}$$

Of course $u \in W^{1,2}(B^3, \mathcal{Y})$, whereas

$$\partial G_u \llcorner B^3 \times \mathbb{R}^N = -\delta_0 \times \varphi_\# [\![S^2]\!] ,$$

compare [5]. Therefore $u \in \mathrm{cart}^{2,1}(B^3, \mathcal{Y})$ if and only if $\varphi_\# [\![S^2]\!] = 0$.

Let us introduce for every $u \in \mathrm{cart}^{2,1}(B^3, \mathcal{Y})$ the relaxed functional

$$\overline{\mathbf{D}}(u, B^3) := \inf\{\liminf_{k\to+\infty} \mathbf{D}(u_k, B^3) \mid \{u_k\} \subset \mathrm{Lip}(B^3, \mathcal{Y}) \,,\ G_{u_k} \rightharpoonup G_u \quad \text{weakly in } \mathcal{D}_{3,2}(B^3 \times \mathcal{Y})\} \,.$$

By lower semicontinuity we trivially have

$$\mathbf{D}(u, B^3) \leq \overline{\mathbf{D}}(u, B^3) \,,$$

with equality false, in general. Moreover, equality holds if there exists a smooth sequence $u_k : B^3 \to \mathcal{Y}$ such that $G_{u_k} \rightharpoonup G_u$ weakly in $\mathcal{D}_{3,2}(B^3 \times \mathcal{Y})$ and $\mathbf{D}(u_k, B^3) \to \mathbf{D}(u, B^3)$.

Remark 4.1 Of course, if $u_k \to u$ strongly in $W^{1,2}$, then $\mathbf{D}(u, B^3) = \overline{\mathbf{D}}(u, B^3)$. Moreover, due to uniform convexity, we have that the converse holds true too.

Theorem 4.2 *Suppose that the second homotopy group $\pi_2(\mathcal{Y})$ is non trivial and that the Hurewicz homomorphism $\rho : \pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is not injective. Then there exists a Lipschitz continuous function $\varphi : S^2 \to \mathcal{Y}$ such that its homogeneous extension u is a Cartesian map in* $\mathrm{cart}^{2,1}(B^3, \mathcal{Y})$ *such that for every sequence of smooth maps $\{u_k\} \subset \mathrm{Lip}(B^3, \mathcal{Y})$, with $G_{u_k} \rightharpoonup G_u$ weakly in $\mathcal{D}_{3,2}(B^3 \times \mathcal{Y})$, then*

$$\mathbf{D}(u, B^3) + C < \liminf_{k\to+\infty} \mathbf{D}(u_k, B^3)$$

for some absolute constant $C > 0$ and hence

$$\mathbf{D}(u, B^3) < \overline{\mathbf{D}}(u, B^3) \,. \tag{4.2}$$

Example 4.3 An example of non oriented manifold for which the Hurewicz map $\rho : \pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Q})$ is not injective is the real projective plane.

Example 4.4 An example suited for our purposes is the 3-dimensional manifold given by the connected sum of two copies of the product $S^1 \times S^2$ of the 1-sphere times the 2-sphere. The second homotopy group has infinite rank, whereas the second homology group has rank 2; therefore, the Hurewicz homomorphism $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y}, \mathbb{Z})$ has an infinite dimensional kernel.

To our purposes it suffices to prove the following

Proposition 4.5 *Let $u \in \mathrm{cart}^{2,1}(B^3, \mathcal{Y})$ be given by* (4.1) *for some Lipschitz function $\varphi : S^2 \to \mathcal{Y}$ such that $\varphi_\# [\![S^2]\!] = 0$. Then the following facts are equivalent:*

(i) *φ is homotopic to a constant map in $\mathcal{Y}$;*

(ii) $\overline{\mathbf{D}}(u, B^3) = \mathbf{D}(u, B^3)$.

We first give the

Proof of Theorem 4.2: The hypotheses on $\mathcal{Y}$ yield the existence of a Lipschitz continuous function $f : S^2 \to \mathcal{Y}$ such that f is homotopically non trivial whereas the 2-cycle $f_{\#}[\![S^2]\!] \in \mathcal{Z}_2(\mathcal{Y})$ is homologically trivial. Hence

$$f_{\#}[\![S^2]\!] = \partial R \qquad \text{for some } R \in \mathcal{R}_3(\mathcal{Y})\,.$$

For $\varepsilon > 0$ small, let $h_\varepsilon : [0,1] \times S^2 \to \mathcal{Y}_\varepsilon$ be a Lipschitz homotopy such that $h_\varepsilon(0,\cdot) = f$ and if $\widetilde{f}_\varepsilon := h_\varepsilon(1,\cdot)$, then $\widetilde{f}_{\varepsilon\#}[\![S^2]\!]$ is a polyherdal cycle, where

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \operatorname{dist}(y,\mathcal{Y}) \le \varepsilon\}\,.$$

Setting then

$$\widetilde{R}_\varepsilon := R + h_{\varepsilon\#}([\![0,1]\!] \times [\![S^2]\!])\,,$$

$\widetilde{R}_\varepsilon$ is an i.m. rectifiable current in $\mathcal{R}_3(\mathcal{Y}_\varepsilon)$ with polyhedral boundary $\partial \widetilde{R}_\varepsilon = \widetilde{f}_{\varepsilon\#}[\![S^2]\!]$. As a consequence of [3, 4.2.19], we find an integral current $S_\varepsilon \in \mathcal{R}_4(\mathbb{R}^N)$, with compact support, and a C^1-diffeomorphism $g_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$R_\varepsilon := g_{\varepsilon\#}\widetilde{R}_\varepsilon - \partial S_\varepsilon$$

is polyhedral, $\mathbf{M}(S_\varepsilon) + \mathbf{M}(\partial S_\varepsilon) \le \varepsilon$, $\operatorname{spt} S_\varepsilon \subset \mathcal{Y}_{2\varepsilon}$, $\operatorname{Lip} g_\varepsilon, \operatorname{Lip} {g_\varepsilon}^{-1} \le 1+\varepsilon$ and $g_\varepsilon(x) = x$ if $x \in \operatorname{spt} \partial \widetilde{R}_\varepsilon$. In particular

$$\partial R_\varepsilon = \partial \widetilde{R}_\varepsilon = \widetilde{f}_{\varepsilon\#}[\![S^2]\!]\,.$$

Now, covering (with multiplicity and orientation) the 3-simpleces of a triangulation of R_ε, it is possible to define a Lipschitz homotopy $\widetilde{h}_\varepsilon : [0,1] \times S^2 \to \mathcal{Y}_{2\varepsilon}$ such that $\widetilde{h}_\varepsilon(0,\cdot) = \widetilde{f}_\varepsilon$ and if $f_\varepsilon := \widetilde{h}_\varepsilon(1,\cdot)$, then $f_{\varepsilon|S^2}$ is homologically trivial, i.e., $f_{\varepsilon\#}[\![S^2]\!] = 0$. Finally, since for ε small the nearest point projection $\Pi_\varepsilon : \mathcal{Y}_\varepsilon \to \mathcal{Y}$ is well defined and Lipschitz continuous, the function $\varphi := \Pi_{2\varepsilon} \circ f_\varepsilon : S^2 \to \mathcal{Y}$ is Lipschitz continuous and $\varphi_{\#}[\![S^2]\!] = 0$. Finally, since by construction φ is homotopic to f in $\mathcal{Y}$, hence it is not homotopically trivial, by Proposition 4.5 we obtain the assertion. $\square$

Proof of Proposition 4.5: To show that (i) $\Rightarrow$ (ii), we argue as in the proof of Proposition 1.12, taking into account that in this case (1.16) follows from (i). To show the converse, we first recall that if d is a positive integer, two continuous maps $u, v : B^n \to \mathcal{Y}$ are *d-homotopic*, or have the same d-homotopy type, if their restrictions to the d-dimensional skeleton of some triangulation of B^n in *generic position* are homotopic in $\mathcal{Y}$. We make use of the following result from [11], applied with $d = p = 2$.

Theorem 4.6 (White) *Let $d := [p]$ be the greatest integer less than or equal to p. Then each map in the strong $W^{1,p}$-closure of smooth maps $u_k : B^n \to \mathcal{Y}$ has a well-defined d-homotopy type, and these d-homotopy types are preserved by strong $W^{1,p}$-convergence.*

If (ii) holds, by Remark 4.1 there exists a sequence $u_k : B^3 \to \mathcal{Y}$ of smooth maps such that $u_k \to u$ strongly in $W^{1,2}$. By the coarea formula, for a.e. $r \in (0,1)$ we have that $u_{k|\partial B^3_r} \to u_{|\partial B^3_r}$ strongly in $W^{1,2}(\partial B^3_r, \mathcal{Y})$. By applying White's Theorem, for a.e. such choice of r we infer that $u_{|\partial B^3_r}$ has the same homotopy type of $u_{k|\partial B^3_r}$. Since $u_{k|\partial B^3_r}$ is homotopically trivial, a homotopy to zero being given by $h_k(t,x) := u(tx)$, $0 \le t \le 1$, $x \in \partial B^3_r$, we conclude that φ is homotopic to a constant map in $\mathcal{Y}$. $\square$

5 An example for the area functional

In this section we aim to extend to the area functional the results of the previous section. We recall that if $u : B^2 \to \mathcal{Y}$ is a sufficiently smooth map, the area of the graph of u is given by

$$\mathrm{A}(u, B^2) := \int_{B^2} \sqrt{1 + |Du|^2 + |M_{(2)}(Du)|^2}\, dx\,, \tag{5.1}$$

where $|M_{(2)}(Du)|$ denotes the square root of the sum of the square of the determinants of all the (2×2) minors of the Jacobian matrix of Du. We will show the existence of maps $u : B^2 \to \mathcal{Y}$, smooth outside the origin, with finite area $\mathrm{A}(u,B^2)<+\infty$, which belong to the weak sequential closure of smooth graphs, but for which one cannot find a sequence of smooth maps $u_k : B^2 \to \mathcal{Y}$ with graphs G_{u_k} converging to G_u, weakly in the sense of currents in $\mathcal{D}_2(B^2\times\mathcal{Y})$, and $\mathrm{A}(u_k,B^2)$ converging to the area of u. Since the area functional coincides with the mass of the current G_u carried by the graph, i.e.,

$$\mathrm{A}(u,B^2)=\mathbf{M}(G_u)\,,$$

by lower semicontinuity we equivalently have

$$\mathbf{M}(G_u)+C<\liminf_{k\to+\infty}\mathbf{M}(G_{u_k})$$

for some absolute constant $C>0$.

The homological condition $\partial G_u \llcorner B^2\times\mathbb{R}^N=0$ being necessary to have approximability in the weak sense of currents by graphs of smooth maps, we will consider maps in the class

$$\mathrm{cart}^1(B^2,\mathcal{Y}):=\{u\in W^{1,1}(B^2,\mathcal{Y})\mid |M_{(2)}(Du)|\in L^1(B^2)\ \text{ and }\ \partial G_u\llcorner B^2\times\mathbb{R}^N=0\}\,,$$

compare [6]. We will consider the homogeneous extension $u : B^2\to\mathcal{Y}$ of Lipschitz continuous maps $\varphi : S^1\to\mathcal{Y}$, where $S^1:=\partial B^2$ is the unit circle, given by (4.1). Of course $u\in W^{1,p}(B^2,\mathcal{Y})$ for every $p<2$. Moreover, since the image of u is 1-dimensional, by the area formula we infer that $|M_{(2)}(Du)|=0$ a.e. in B^2. As a consequence, (5.1) is well defined and finite for any such u. Finally

$$\partial G_u\llcorner B^2\times\mathbb{R}^N=-\delta_0\times\varphi_\#[\![\,S^1\,]\!]\,,$$

so that $u\in\mathrm{cart}^1(B^2,\mathcal{Y})$ if and only if $\varphi_\#[\![\,S^1\,]\!]=0$.

Let us introduce for every $u\in\mathrm{cart}^1(B^2,\mathcal{Y})$ the relaxed functional

$$\overline{\mathrm{A}}(u,B^2):=\inf\{\liminf_{k\to+\infty}\mathrm{A}(u_k,B^2)\mid\{u_k\}\subset\mathrm{Lip}(B^2,\mathcal{Y})\,,\ G_{u_k}\rightharpoonup G_u\quad\text{weakly in }\mathcal{D}_2(B^2\times\mathcal{Y})\}\,.$$

By lower semicontinuity of the mass we trivially have

$$\mathrm{A}(u,B^2)\le\overline{\mathrm{A}}(u,B^2)\,,$$

with equality false, in general. Moreover, equality holds if there exists a smooth sequence $u_k : B^2\to\mathcal{Y}$ such that $G_{u_k}\rightharpoonup G_u$ weakly in $\mathcal{D}_2(B^2\times\mathcal{Y})$ and $\mathrm{A}(u_k,B^2)\to\mathrm{A}(u,B^2)$.

Similarly to the previous section, we have the following

Proposition 5.1 *Let $u\in\mathrm{cart}^1(B^2,\mathcal{Y})$ be given by* (4.1) *for some Lipschitz function $\varphi : S^1\to\mathcal{Y}$ such that $\varphi_\#[\![\,S^1\,]\!]=0$. Then the following facts are equivalent:*

(i) *φ is homotopic to a constant map in $\mathcal{Y}$;*

(ii) $\overline{\mathrm{A}}(u,B^2)=\mathrm{A}(u,B^2)$.

We postpone its proof and we first give an example of a smooth compact oriented 2-dimensional Riemannian manifold $\mathcal{Y}$, with first homotopy group $\pi_1(\mathcal{Y})\neq 0$, for which we can find a Lipschitz function $\varphi : S^1\to\mathcal{Y}$ such that $\varphi_\#[\![\,S^1\,]\!]=0$ and φ is not homotopically trivial in $\mathcal{Y}$. By Proposition 5.1, this does prove the assertion in the beginning of this section, i.e.,

$$\mathrm{A}(u,B^2)<\overline{\mathrm{A}}(u,B^2)\,.$$

Example 5.2 We consider two copies of the standard torus in $\mathbb{R}^3$, remove a small disk D from each of them and connect with a collar C attached on the boundary of such disks. Then $\mathcal{Y}$ is homeomorphic to a compact surface of genus two. If $\widetilde{\varphi}$ maps the circle S^1 onto a loop wrapped around the collar C, then $\widetilde{\varphi}_\#[\![\,S^1\,]\!]$ is homologically trivial, being the boundary of the current given by the integration of one "halph" of $\mathcal{Y}$. Moreover $\widetilde{\varphi}$ is homotopically non trivial, since it belongs to the homotopy class given by $ABA^{-1}B^{-1}$, where A and B are the generators of one of such tori. Arguing as in the proof of Theorem 4.2, we find that $\widetilde{\varphi}$ is homotopically equivalent to a Lipschitz function φ such that $\varphi_\#[\![\,S^1\,]\!]=0$, since the image of φ covers the generators A and B with the order $ABA^{-1}B^{-1}$.

Proof of Proposition 5.1: To show that (i) $\Rightarrow$ (ii), since the 1-cycle $\varphi_{|S^1}$ is homotopically trivial and homologically zero, we infer that it is contractible in $\mathcal{Y}$. Therefore, there exists a Lipschitz continuous extension $v : B^2 \to \mathcal{Y}$ of φ, i.e., $v_{|\partial B^2} = \varphi$, such that $v(B^2) \subset \varphi(S^1)$. In particular, by the area formula we have that $|M_{(2)}(Dv)| = 0$ a.e. in B^2. We then define, for $0 < r < 1$,

$$u_r(x) := \begin{cases} \varphi\left(\dfrac{x}{|x|}\right) & \text{if} \quad r \le |x| < 1 \\ v\left(\dfrac{x}{r}\right) & \text{if} \quad |x| < r\,. \end{cases}$$

Since

$$\int_{B_r^2} |Du_r|\,dx = r\int_{B^2} |Dv|\,dx \qquad \text{and} \qquad \int_{B_r^2} |M_{(2)}(Du_r)|\,dx = \int_{B^2} |M_{(2)}(Dv)|\,dx = 0\,,$$

letting $r \to 0^+$, we obtain that $G_{v_r} \rightharpoonup G_u$ with $\mathrm{A}(v_r, B^2) \to \mathrm{A}(v_r, B^2)$ and hence the assertion.

To show the converse, if (ii) holds, there exists a sequence $u_k : B^2 \to \mathcal{Y}$ of smooth maps such that $u_k \to u$ strongly in $W^{1,1}$. By the coarea formula, for a.e. $r \in (0,1)$ we have that $u_{k|\partial B_r^2} \to u_{|\partial B_r^2}$ strongly in $W^{1,1}(\partial B_r^2, \mathcal{Y})$. By applying White's result, Theorem 4.6, with $d = p = 1$ this time, for a.e. such choice of r we infer that $u_{|\partial B_r^2}$ has the same homotopy type of $u_{k|\partial B_r^2}$. Since $u_{k|\partial B_r^2}$ is homotopically trivial, we conclude that φ is homotopic to a constant map in $\mathcal{Y}$. $\square$

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