Hölder continuity of local minimizers

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1 Introduction

In recent years many results have appeared concerning the regularity of minimizers of integral functionals of the type

$$(1.1)\qquad \mathcal{F}(v;\Omega) := \int_\Omega F(x, v(x), Dv(x))\, dx\,,$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is an integrand satisfying the growth assumption

$$(1.2)\qquad (\mu^2 + |z|^2)^{p/2} \le F(x,u,z) \le L(\mu^2 + |z|^2)^{p/2}$$

with $L > 1$, $\mu \ge 0$, $p > 1$.
Roughly speaking two kinds of results are available.
If no other assumption is made on the integrand F, it is known (see [7]) that condition (1.2) ensures that a $W^{1,p}$ minimizer u is Hölder continuous for some exponent α depending on L, p and N. On the other hand if F is assumed to be smooth enough, for instance C^2 with respect to z, and satisfies a standard ellipticity assumption of the form

$$(1.3)\qquad \sum_{i,j=1}^N D_{ij}F(x,u,z)\xi_i\xi_j \ge \nu(\mu^2 + |z|^2)^{(p-2)/2}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^N\,,$$

one gets that Du is Hölder continuous (see *e.g.* [8], [12], [1], [3]).
If one is interested only into Lipschitz continuity properties of minimizers, the situation is somewhat different. In fact a classical result due to Hartman and Stampacchia (see [11]) says that at least when the integral depends only on Du, the convexity of F, together with the so called "bounded slope" condition, yields the global boundedness of the gradient of a minimizer u. In the same spirit in [5] it has been proved that if $F = F(z)$ satisfies (1.2) and the following strict uniform convexity assumption

$$(1.4)\quad \int_\Omega F(z+D\varphi(x))\, dx \ge \int_\Omega \Big[F(z) + \nu(\mu^2 + |z|^2 + |D\varphi(x)|^2)^{(p-2)/2}|D\varphi(x)|^2\Big]\, dx\,,$$

for all $z \in \mathbb{R}^N$ and $\varphi \in C^1_0(\Omega)$, then every local minimizer is locally Lipschitz.

At this point it is natural to investigate whether such result holds also in the general case (1.1). It is clear that now a continuity assumption with respect to x and u should be required. In fact it is well known that even in two dimensions, taking $F(x,z) = a(x)|z|^2$, with $\lambda \le a(x) \le \Lambda$, if $a(x)$ is not continuous then local minimizers are only α-Hölder continuous with $\alpha = \sqrt{\lambda/\Lambda}$ (see [13]).

In this paper we study functionals of the type (1.1), where F is uniformly continuous in (x, u) with respect to z (see condition (F_3) in Section 3). We do not make any differentiability assumption on F and in particular we do not require an ellipticity condition of the type (1.3). Instead, as in [5], we shall assume that condition (1.4) holds (uniformly with respect to (x, u)). Under these assumptions we cannot expect minimizers to be Lipschitz continuous (see Example 3.2). However we prove, see Theorem 3.1, that every minimizer of functional (1.1) is locally Hölder continuous for any $\alpha < 1$.
The proof of our result goes as follows. We consider first the case when F only depends on x and z. In this case we prove that $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for all $\alpha < 1$ and we show that the Hölder estimates on u only depend on the constants L and ν appearing in (1.2) and (1.4) (see Theorem 2.5). We notice that when $F = F(x, u, z)$ we cannot reduce to the previous case by the standard device of "freezing" the functional with respect to the variable u, since we lack the ellipticity assumption on F needed in order to make this argument work. This difficulty is instead overcome by an approximation argument based on a variational principle due to Ekeland.

2 Preliminary results

In the sequel Ω will denote a bounded open set in $\mathbb{R}^N$, Q the unit cube $(0, 1)^N$, $B_R(x_0)$ the ball $\{x \in \mathbb{R}^N : |x - x_0| < R\}$; we shall write B_R in place of $B_R(x_0)$ if no confusion may arise. If f is an integrable function we set

$$f_{x_0,R} := \fint_{B_R(x_0)} f(x)\, dx = \frac{1}{|B_R|} \int_{B_R(x_0)} f(x)\, dx$$

where $|B_R| = \omega_N R^N$ is the Lebesgue measure of the ball. The letter c will stand for a generic constant that may vary from line to line.
If u is a Hölder continuous function on $A \subset \Omega$ with exponent $0 < \alpha < 1$ we shall denote by $[u]_{\alpha,A}$ the Hölder constant of u in A, *i.e.*

$$[u]_{\alpha,A} := \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} : x, y \in A,\ x \neq y \right\} .$$

We recall the following definition.

Definition 2.1 *Let us consider the functional (1.1). A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a* Q-minimizer *of $\mathcal{F}$ if there exists $Q \geq 1$ such that*

$$\mathcal{F}(u; K) \leq Q\mathcal{F}(v; K)$$

for any $v \in W^{1,p}_{\text{loc}}(\Omega)$, with $K = \text{spt}(u - v) \subset\subset \Omega$. If the above inequality is satisfied with $Q = 1$, then u is said a local minimizer *of $\mathcal{F}$.*

In this section we shall assume that the integrand in (1.1) depends only on x and z. Under this assumption we shall prove that local minimizers are α-Hölder continuous for all $\alpha < 1$ and establish a local estimate of the Hölder constant of u which will be useful in the next section where the general case will be considered.
Let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that for any $x, y, x_0 \in \Omega$, $z \in \mathbb{R}^N$ and $\varphi \in C^1_0(Q)$ the following properties hold:

(G_1) $\quad (\mu^2 + |z|^2)^{p/2} \leq G(x, z) \leq L(\mu^2 + |z|^2)^{p/2}\,,$

(G_2)
$$\int_Q G(x_0, z + D\varphi(x))\,dx \geq \int_Q \Big[G(x_0, z) + \nu(\mu^2 + |z|^2 + |D\varphi(x)|^2)^{(p-2)/2}|D\varphi(x)|^2\Big]\,dx\,,$$

(G_3) $\quad |G(x, z) - G(y, z)| \leq \omega(|x - y|)(\mu^2 + |z|^2)^{p/2}\,,$

where $\omega : [0, +\infty) \to [0, +\infty)$ is a continuous, not decreasing, bounded function with $\omega(0) = 0$.

Here $L > 1$, $\nu > 0$, $\mu \geq 0$, $p > 1$. It is not restrictive, as we shall do in the sequel, to assume also $\mu \leq 1$.
If $u \in W^{1,p}_{\rm loc}(\Omega)$, $A \subset \Omega$ we set

$$\mathcal{G}(u; A) := \int_A G(x, Du(x))\,dx\,. \tag{2.1}$$

Let us start with a simple algebraic lemma.

Lemma 2.2 *If $p > 1$ there exists a constant c such that for any $\mu \geq 0$, $\xi, \eta \in \mathbb{R}^N$*

$$(\mu^2 + |\xi|^2)^{p/2} \leq c(\mu^2 + |\eta|^2)^{p/2} + c(\mu^2 + |\xi|^2 + |\eta|^2)^{(p-2)/2}|\xi - \eta|^2\,.$$

Proof. For all $p > 1$ we have the elementary inequality

$$|\xi|^p \leq 2^{p-1}\left(|\eta|^p + |\xi - \eta|^{p-2}|\xi - \eta|^2\right) \tag{2.2}$$

from which the thesis immediately follows when $p \geq 2$.
Let us consider the case $1 < p < 2$. If $|\xi| < 2|\eta| + \mu$ the claim is obvious, otherwise we have

$$|\xi - \eta| \geq |\xi| - |\eta| \geq \frac{1}{2}(|\xi| + \mu) \geq \frac{1}{4}(|\xi| + |\eta| + \mu)\,;$$

Using this inequality to estimate $|\xi - \eta|^{p-2}$ in (2.2) we get

$$|\xi|^p \leq 2^{p-1}\left[|\eta|^p + 4^{2-p}(|\xi| + |\eta| + \mu)^{p-2}|\xi - \eta|^2\right]$$

and the thesis follows. □

Proposition 2.3 *Let $G : B_R(x_0) \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying (G_1), (G_2), (G_3), of class C^2 with respect to z. Let $u \in W^{1,p}(B_R(x_0))$ be a minimizer of the functional*

$$(2.3) \quad \mathcal{H}(w; B_R(x_0)) := \int_{B_R(x_0)} G(x, Dw)\, dx + \vartheta_0 \int_{B_R(x_0)} |Dw - Du_0|\, dx$$

in its Dirichlet class $u + W_0^{1,p}(B_R(x_0))$, for some $\vartheta_0 \geq 0$ and $u_0 \in W^{1,p}(B_R(x_0))$. Then for any $\varrho < R$

$$\int_{B_\varrho(x_0)} \left(\mu^2 + |Du|^2\right)^{p/2} dx \leq c\left[\left(\frac{\varrho}{R}\right)^N + \omega(R)\right] \int_{B_R(x_0)} \left(\mu^2 + |Du|^2\right)^{p/2} dx + \frac{\vartheta_0^{p/(p-1)}}{[\omega(R)]^{1/(p-1)}} R^N .$$

Proof. We start observing that since for all $x \in B_R$ the function $z \mapsto G(x,z)$ is $C^2(\mathbb{R}^N)$ then the condition (G_2) is equivalent to require that

$$\sum_{i,j=1}^N D_{ij}G(x,z)\xi_i\xi_j \geq \nu(\mu^2 + |z|^2)^{(p-2)/2}|\xi|^2$$

for any $x \in \Omega$, $z, \xi \in \mathbb{R}^N$. Let v be the minimizer in $u + W_0^{1,p}(B_R)$ of

$$\mathcal{G}_0(w; B_R) := \int_{B_R} G(x_0, Dw)\, dx\, .$$

The function $z \mapsto G(x_0, z)$ satisfies the assumptions of Theorem 2.2 in [5], hence from this result it follows that v is locally Lipschitz in B_R and that the following estimate holds

$$\int_{B_\varrho} \left(\mu^2 + |Dv|^2\right)^{p/2} dx \leq c\left(\frac{\varrho}{R}\right)^N \int_{B_R} \left(\mu^2 + |Dv|^2\right)^{p/2} dx$$

for all $\varrho < R$. This inequality, together with the minimality of v and (G_1), implies

$$(2.4) \qquad \int_{B_\varrho} \left(\mu^2 + |Dv|^2\right)^{p/2} dx \leq c\left(\frac{\varrho}{R}\right)^N \int_{B_R} \left(\mu^2 + |Du|^2\right)^{p/2} dx\, .$$

Using Lemma 2.2 and (2.4) we have

$$(2.5) \quad \int_{B_\varrho} \left(\mu^2 + |Du|^2\right)^{p/2} dx \leq c\int_{B_\varrho} \left(\mu^2 + |Dv|^2\right)^{p/2} dx + c\int_{B_\varrho} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(p-2)/2} |Du - Dv|^2\, dx$$

$$\leq c\left(\frac{\varrho}{R}\right)^N \int_{B_R} \left(\mu^2 + |Du|^2\right)^{p/2} dx$$
$$+c\int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(p-2)/2} |Du - Dv|^2 \, dx$$

and the last integral can be controlled, using the minimality of v, as follows:

$$\begin{aligned}
\mathcal{G}_0(u; B_R) - \mathcal{G}_0(v; B_R) &= \int_{B_R} [G(x_0, Du) - G(x_0, Dv)] \, dx \\
&= \int_{B_R} D_i G(x_0, Dv)(D_i u - D_i v) \, dx \\
&+ \int_{B_R} \int_0^1 (1-t) D_{ij} G(x_0, (1-t)Dv + tDu)(D_i u - D_i v)(D_j u - D_j v) \, dt dx \\
&\geq \nu \int_{B_R} \int_0^1 (1-t) \left(\mu^2 + |(1-t)Dv + tDu|^2\right)^{(p-2)/2} |Du - Dv|^2 \, dt dx \\
&\geq c\nu \int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(p-2)/2} |Du - Dv|^2 \, dx \, .
\end{aligned}$$

This inequality, together with (G_1), (G_3), the minimality of u and v, and (2.4), yields

$$\begin{aligned}
& c\nu \int_{B_R} (\mu^2 + |Du|^2 + |Dv|^2)^{(p-2)/2} |Du - Dv|^2 \, dx \\
&\leq \int_{B_R} [G(x_0, Du) - G(x, Du)] \, dx + \int_{B_R} [G(x, Dv) - G(x_0, Dv)] \, dx \\
&\quad + \int_{B_R} [G(x, Du) + \vartheta_0 |Du - Du_0| - G(x, Dv) - \vartheta_0 |Dv - Du_0|] \, dx \\
&\quad + \vartheta_0 \int_{B_R} (|Dv - Du_0| - |Du - Du_0|) \, dx \\
&\leq c\omega(R) \int_{B_R} \left(\mu^2 + |Du|^2\right)^{p/2} dx + \vartheta_0 \int_{B_R} |Dv - Du| \, dx \, .
\end{aligned}$$

Finally the thesis follows from this inequality and (2.5) if we observe that

$$\begin{aligned}
\tilde{c}\vartheta_0 \int_{B_R} |Dv - Du| \, dx &\leq \frac{\vartheta_0^{p/(p-1)}}{[\omega(R)]^{1/(p-1)}} R^N + \tilde{c}^p \omega_N^{p-1} \omega(R) \int_{B_R} |Dv - Du|^p \, dx \\
&\leq \frac{\vartheta_0^{p/(p-1)}}{[\omega(R)]^{1/(p-1)}} R^N + c\omega(R) \int_{B_R} (\mu^2 + |Du|^2)^{p/2} \, dx \, .
\end{aligned}$$

□

In the next proposition we prove an analogous result using an approximation argument that allows us to remove the differentiability assumption on G.

Proposition 2.4 *Let $G : B_R(x_0) \times \mathbb{R}^N \to \mathbb{R}$ satisfy (G_1), (G_2) and (G_3). If ϑ_0, u_0 and u are as in Proposition 2.3, then for all $\varrho < R$*

$$\begin{aligned}(2.6)\quad &\int_{B_\varrho(x_0)} \left(\mu^2 + |Du|^2\right)^{p/2} dx \\ &\le c\left[\left(\frac{\varrho}{R}\right)^N + \omega(R)\right]\int_{B_R(x_0)} \left(\mu^2 + |Du|^2 + |Du_0|^2\right)^{p/2} dx + \frac{2\vartheta_0^{p/(p-1)}}{[\omega(R)]^{1/(p-1)}} R^N .\end{aligned}$$

Proof. As in the proof of Proposition 2.3 when the center of a ball is not indicated it is understood that the ball is centered in x_0. Let (G_h) be the sequence of continuous functions in $B_R \times \mathbb{R}^N$ defined by

$$G_h(x, z) := \int_{B_1(0)} \rho(w) G\left(x, z + \frac{1}{h} w\right) dw$$

where ρ is a positive radially symmetric mollifier. Using the same arguments as in [5], Lemma 2.4, it is easy to check that the functions G_h satisfy the assumptions of Proposition 2.3. More precisely $G_h(x, \cdot) \in C^2(\mathbb{R}^N)$ for all $h \in \mathbb{N}$ and there exists a constant $c > 1$ not depending on h such that for any $x, y \in \Omega$, $z, \xi \in \mathbb{R}^N$

$$\begin{aligned}&\frac{1}{c}\left(\mu^2 + \frac{1}{h^2} + |z|^2\right)^{p/2} \le G_h(x, z) \le cL\left(\mu^2 + \frac{1}{h^2} + |z|^2\right)^{p/2} ,\\ &\sum_{i,j=1}^N D_{ij} G_h(x, z)\xi_i \xi_j \ge \frac{\nu}{c}\left(\mu^2 + \frac{1}{h^2} + |z|^2\right)^{(p-2)/2} |\xi|^2 ,\\ &|G_h(x, z) - G_h(y, z)| \le c\omega(|x - y|)\left(\mu^2 + \frac{1}{h^2} + |z|^2\right)^{p/2} .\end{aligned}$$

Notice that (G_h) converges uniformly to G on compact subsets of $B_R \times \mathbb{R}^N$. For every $h \in \mathbb{N}$ let u_h be the minimizer in $u + W_0^{1,p}(B_R)$ of the functional

$$w \mapsto \int_{B_R} G_h(x, Dw)\, dx + \vartheta_0 \int_{B_R} |Dw - Du_0|\, dx .$$

Then there exists C depending on L, ν, p, N and ϑ_0, but not on h, such that

$$\int_{B_R} |Du_h|^p\, dx \le C \int_{B_R} \left(|Dw|^p + |Du_0|^p + 1\right) dx$$

for any $w \in u + W_0^{1,p}(B_R)$. In particular we have that

$$(2.7)\qquad \int_{B_R} |Du_h|^p\, dx \le C \int_{B_R} (|Du|^p + |Du_0|^p + 1)\, dx ,$$

hence the sequence (u_h) is bounded in $W^{1,p}(B_R)$. Thus, passing eventually to a subsequence, we may assume that a function $u_\infty \in u + W_0^{1,p}$ exists such that $u_h \rightharpoonup u_\infty$ weakly in $W^{1,p}(B_R)$. Moreover the minimality of u_h easily implies that u_h is a Q-minimizer of

$$w \mapsto \int_{B_R} (|Dw|^p + |Du_0| + 1)\, dx\,,$$

so there exist $\tau > 1$ and $c > 0$ such that $u_h \in W^{1,p\tau}_{\rm loc}(B_R)$ for any $h \in \mathbb{N}$. More precisely for any ball $B_\varrho(x_1) \subset B_R(x_0)$ the following inequality holds (see [7], Theorem 3.1)

$$\left(\fint_{B_{\varrho/2}(x_1)} |Du_h|^{p\tau}\, dx \right)^{1/\tau} \leq c \fint_{B_\varrho(x_1)} (|Du_h|^p + |Du_0|^p + 1)\, dx$$

which, together with (2.7), implies that for any $\varrho < R$

$$\left(\int_{B_\varrho} |Du_h|^{p\tau}\, dx \right)^{1/\tau} \leq c(\varrho, R) \int_{B_R} (|Du|^p + |Du_0|^p + 1)\, dx\,. \tag{2.8}$$

Let us now prove that $u_\infty = u$. Fix $\varrho < R$ and observe that the functional $\mathcal{H}$ defined in (2.3) is lower semicontinuous with respect to the weak topology of $W^{1,p}$. Remembering that (G_h) converges to G uniformly on compact subsets of $B_R \times \mathbb{R}^N$ we have that for any $k > 0$

$$\begin{aligned}
\mathcal{H}(u_\infty; B_\varrho) &\leq \liminf_{h\to\infty} \int_{B_\varrho} [G(x, Du_h) + \vartheta_0 |Du_h - Du_0|]\, dx \\
&\leq \limsup_{h\to\infty} \int_{B_\varrho \cap \{|Du_h| > k\}} G(x, Du_h)\, dx \\
&\quad + \limsup_{h\to\infty} \left[\int_{B_\varrho \cap \{|Du_h| \leq k\}} G_h(x, Du_h)\, dx + \int_{B_\varrho} \vartheta_0 |Du_h - Du_0|\, dx \right].
\end{aligned}$$

So, from the minimality of (u_h) and the uniform convergence on compact subsets again, it follows that

$$\begin{aligned}
\mathcal{H}(u_\infty; B_\varrho) \quad &\leq \quad c \limsup_{h\to\infty} \left[\int_{B_\varrho} (1 + |Du_h|^{\tau p})\, dx \right]^{1/\tau} |B_\varrho \cap \{|Du_h| > k\}|^{(\tau-1)/\tau} \\
&\qquad + \limsup_{h\to\infty} \int_{B_R} [G_h(x, Du) + \vartheta_0 |Du - Du_0|]\, dx \\
&\leq \quad c k^{p(1-\tau)} \limsup_{h\to\infty} \int_{B_\varrho} (1 + |Du_h|^{\tau p})\, dx + \int_{B_R} G(x, Du)\, dx \\
&\qquad + \int_{B_R} \vartheta_0 |Du - Du_0|\, dx\,,
\end{aligned}$$

that together with (2.8) implies

$$\mathcal{H}(u_\infty; B_\varrho) \le ck^{p(1-\tau)} + \mathcal{H}(u; B_R)\,.$$

Finally as $k \to \infty$ and then $\varrho \to R$ we obtain

$$\mathcal{H}(u_\infty; B_R) \le \mathcal{H}(u; B_R)$$

which implies $u_\infty = u$ in B_R, since by (G_2) the functional $\mathcal{H}$ is strictly convex. Now we can apply Proposition 2.3 on any u_h; moreover using the minimality of u_h and letting $h \to \infty$ we have

$$\begin{aligned}
&\int_{B_\varrho} (\mu^2 + |Du|^2)^{p/2} dx \le \liminf_{h\to\infty} \int_{B_\varrho} \left(\mu^2 + \frac{1}{h^2} + |Du_h|^2\right)^{p/2} dx \\
&\le \liminf_{h\to\infty} c\left[\left(\frac{\varrho}{R}\right)^N + \omega(R)\right]\int_{B_R} \left(\mu^2 + \frac{1}{h^2} + |Du_h|^2\right)^{p/2} dx + \frac{\vartheta_0^{p/(p-1)}}{[\omega(R)]^{1/(p-1)}} R^N \\
&\le c\left[\left(\frac{\varrho}{R}\right)^N + \omega(R)\right]\int_{B_R} \left[(\mu^2 + |Du|^2)^{p/2} + \vartheta_0 |Du - Du_0|\right] dx \\
&\qquad + \frac{\vartheta_0^{p/(p-1)}}{[\omega(R)]^{1/(p-1)}} R^N\,.
\end{aligned}$$

Estimating $\tilde{c}\vartheta_0 \int_{B_R} |Du - Du_0|$ as in the proof of Proposition 2.3 we obtain the thesis. □

As a corollary of this proposition we state the following regularity result.

Theorem 2.5 *Let G be as in Proposition 2.4 and $u \in W^{1,p}_{\rm loc}(\Omega)$ be a local minimizer of functional $\mathcal{G}$ defined as in (2.1). For any $0 < \delta < N$ there exists a constant c_δ, depending on L, ν, p, N, δ and on the diameter of Ω, such that if $B_R(x_0) \subset \Omega$ then for any $0 < \varrho < R$*

$$\int_{B_\varrho(x_0)} (\mu^2 + |Du|^2)^{p/2}\, dx \le c_\delta \left(\frac{\varrho}{R}\right)^{N-\delta} \int_{B_R(x_0)} (\mu^2 + |Du|^2)^{p/2}\, dx\,.$$

In particular $u \in C^{0,\alpha}_{\rm loc}(\Omega)$ for any $\alpha < 1$.

Proof. Proposition 2.4, applied with $\vartheta_0 = 0$, implies that for any $\varrho < R$

$$\int_{B_\varrho} (\mu^2 + |Du|^2)^{p/2}\, dx \le c\left[\left(\frac{\varrho}{R}\right)^N + \omega(R)\right] \int_{B_R} \left(\mu^2 + |Du|^2\right)^{p/2} dx\,.$$

Fixed $\delta > 0$, a standard iteration argument (see [6], page 170) leads to the existence of two positive constants R_δ, c_δ for which the assertion holds if $\varrho < R \le R_\delta$. From this the result easily follows. □

The following result, due in this form to I. Ekeland (see [2]), will be used in the proof of Theorem 3.1.

Lemma 2.6 *Let (V,d) be a complete metric space and $\mathcal{I} : V \to (-\infty,+\infty]$ a lower semicontinuous functional such that*

$$\inf_V \mathcal{I} < +\infty \,.$$

Given $\epsilon > 0$, let $u \in V$ be such that

$$\mathcal{I}(u) \leq \inf_V \mathcal{I} + \epsilon \,.$$

Then there exists $v \in V$ satisfying the following properties:

(i) $d(u,v) \leq 1$,

(ii) $\mathcal{I}(v) \leq \mathcal{I}(u)$,

(iii) *v is a minimizer of the functional $w \mapsto \mathcal{I}(w) + \epsilon d(v,w)$.*

We conclude this section by proving a higher integrability result up to the boundary (see also [10]).

Lemma 2.7 *Let $G : B_{2R}(x_0) \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying (G_1). Let us consider $u \in W^{1,q}(B_{2R}(x_0))$ for a certain $q > p$. If v is a minimizer of the functional $\mathcal{G}$ in the Dirichlet class $u + W_0^{1,p}(B_R(x_0))$ then there exist $r \in (p,q)$ and c depending on L, p, N, but not on u or R, such that $v \in W^{1,r}(B_R(x_0))$ and*

$$\left(\fint_{B_R(x_0)} |Dv|^r \, dx \right)^{1/r} \leq c \left(\fint_{B_{2R}(x_0)} (1 + |Du|^q) \, dx \right)^{1/q} .$$

Proof. As usual, whenever the center of a ball is not indicated it will be understood that the ball is centered in x_0. Let us set

$$w(x) := \begin{cases} v(x) & \text{if } x \in B_R \,, \\ u(x) & \text{if } x \in B_{2R} \setminus B_R \,. \end{cases}$$

If $B_{2\varrho}(x_1) \subset B_R$ the standard Caccioppoli inequality gives

$$\int_{B_\varrho(x_1)} |Dv|^p \, dx \leq c \int_{B_{2\varrho}(x_1)} \left(1 + \frac{|v - v_{x_1,2\varrho}|^p}{\varrho^p} \right) dx \,,$$

which implies

$$\fint_{B_\varrho(x_1)} |Dv|^p \, dx \leq c \left(\fint_{B_{2\varrho}(x_1)} (1 + |Dv|^{p_*}) \, dx \right)^{p/p_*} , \tag{2.9}$$

with $p_* = Np/(N+p)$, if $p \geq N/(N-1)$, $p_* = 1$ otherwise.
Let now consider $B_{2\varrho}(x_1) \subset B_{2R}$ and $x_1 \in \partial B_R$. Let us fix $\varrho \leq s < t \leq 2\varrho$ and η a cut-off function between $B_s(x_1)$ and $B_t(x_1)$, with $|D\eta| \leq 2/(t-s)$. Observing that $u = v$ on ∂B_R, we easily obtain

$$\begin{aligned}\int_{B_s(x_1)\cap B_R} |Dv|^p\, dx \quad &\leq \quad c\int_{B_{2\varrho}(x_1)\cap B_R}\left(1+|Du|^p+\frac{|v-u|^p}{(t-s)^p}\right)dx \\ &\qquad +c\int_{(B_t(x_1)\setminus B_s(x_1))\cap B_R} |Dv|^p\, dx\,.\end{aligned}$$

From this inequality, arguing in a standard way (see the proof of Theorem 3.1 in [7]), we get

$$\begin{aligned}&\fint_{B_\varrho(x_1)\cap B_R} |Dv|^p\, dx \leq c\fint_{B_{2\varrho}(x_1)\cap B_R}\left(1+\frac{|v-u|^p}{\varrho^p}\right)dx + c\fint_{B_{2\varrho}(x_1)} |Du|^p\, dx \\ &\leq c\left(\fint_{B_{2\varrho}(x_1)\cap B_R}(1+|Dv-Du|^{p_*})\, dx\right)^{p/p_*} + c\fint_{B_{2\varrho}(x_1)} |Du|^p\, dx\,,\end{aligned}$$

hence it follows that

$$\fint_{B_\varrho(x_1)} |Dw|^p\, dx \leq c\left(\fint_{B_{2\varrho}(x_1)} |Dw|^{p_*}\, dx\right)^{p/p_*} + c\fint_{B_{2\varrho}(x_1)}(1+|Du|^p)\, dx. \tag{2.10}$$

By (2.9) it then follows that (2.10) holds not only if $B_{2\varrho}(x_1) \subset B_R$ or $B_{2\varrho}(x_1) \cap B_R = \emptyset$, but also when $x_1 \in \partial B_R$ and $B_{2\varrho}(x_1) \subset B_{2R}$. Let consider now the case of a ball such that $B_{2\varrho}(x_1) \cap \partial B_R$ is not empty and $B_{8\varrho}(x_1) \subset B_{2R}$. Fixed x_2 in $B_{2\varrho}(x_1) \cap \partial B_R$ we easily have that

$$\begin{aligned}\fint_{B_\varrho(x_1)} |Dw|^p\, dx \quad &\leq \quad 3^N\fint_{B_{3\varrho}(x_2)} |Dw|^p\, dx \\ &\leq \quad c\left(\fint_{B_{6\varrho}(x_2)} |Dw|^{p_*}\, dx\right)^{p/p_*} + c\fint_{B_{6\varrho}(x_2)}(1+|Du|^p)\, dx \\ &\leq \quad c\left(\fint_{B_{8\varrho}(x_1)} |Dw|^{p_*}\, dx\right)^{p/p_*} + c\fint_{B_{8\varrho}(x_1)}(1+|Du|^p)\, dx.\end{aligned}$$

Since this estimate is true for any $B_\varrho(x_1)$ such that $B_{8\varrho}(x_1) \subset B_{2R}$, it follows with an easy argument that (2.10) holds for any $B_\varrho(x_1)$ such that $B_{2\varrho}(x_1) \subset B_{2R}$, possibly with a different constant c. The Gehring lemma proved in [6] yields now that if $B_{2\varrho}(x_1) \subset B_{2R}$ then

$$\left(\fint_{B_\varrho(x_1)} |Dw|^r\, dx\right)^{1/r} \leq c\left(\fint_{B_{2\varrho}(x_1)} |Dw|^p\, dx\right)^{1/p} + c\left(\fint_{B_{2\varrho}(x_1)}(1+|Du|^q)\, dx\right)^{1/q},$$

with suitable c and $p < r < q$. In particular we have proved that

$$\begin{aligned}\left(\fint_{B_R} |Dv|^r \, dx\right)^{1/r} &\leq c\left(\fint_{B_{2R}} |Dw|^p \, dx\right)^{1/p} + c\left(\fint_{B_{2R}} (1+|Du|^q) \, dx\right)^{1/q} \\ &\leq c\left(\fint_{B_R} |Dv|^p \, dx\right)^{1/p} + c\left(\fint_{B_{2R}} (1+|Du|^q) \, dx\right)^{1/q} \\ &\leq c\left(\fint_{B_R} (1+|Du|^p) \, dx\right)^{1/p} + c\left(\fint_{B_{2R}} (1+|Du|^q) \, dx\right)^{1/q}\end{aligned}$$

and finally the thesis follows. □

3 Regularity of local minimizers

In this section we study the regularity of local minimizers of a functional of the type (1.1), where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the following assumptions: for any $x, y, x_0 \in \Omega$, $u, v, u_0 \in \mathbb{R}$, $z \in \mathbb{R}^N$ and $\varphi \in C_0^1(Q_1)$

(F_1) $\quad (\mu^2 + |z|^2)^{p/2} \leq F(x,u,z) \leq L(\mu^2 + |z|^2)^{p/2}$,

(F_2) $$\int_Q F(x_0, u_0, z + D\varphi(x)) \, dx \geq \int_Q \left[F(x_0,u_0,z) + \nu(\mu^2 + |z|^2 + |D\varphi(x)|^2)^{(p-2)/2} |D\varphi(x)|^2\right] dx\,,$$

(F_3) $\quad |F(x,u,z) - F(y,v,z)| \leq \omega(|x-y| + |u-v|)(\mu^2 + |z|^2)^{p/2}$,
where $\omega : [0, +\infty) \to [0, +\infty)$ is a continuous, not decreasing, bounded function with $\omega(0) = 0$.

As before, $L > 1$, $\nu > 0$, $0 \leq \mu \leq 1$, $p > 1$. Since it is not restrictive, we shall henceforth assume ω to be concave.

We can now state our main result.

Theorem 3.1 *Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function verifying assumptions (F_1), (F_2) and (F_3) above. If $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a local minimizer of the functional*

$$\mathcal{F}(w;\Omega) := \int_\Omega F\left(x, w(x), Dw(x)\right) \, dx$$

then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for all $\alpha < 1$.

Proof. Since we want to prove a local result, it is not restrictive to assume that (see [7]) $u \in W^{1,q}(\Omega)$ for some $q > p$ and that for any ball $B_R(x_1) \subset \Omega$

$$\left[\fint_{B_{R/2}(x_1)} \left(\mu^2 + |Du|^2\right)^{q/2} \, dx\right]^{1/q} \leq c\left[\fint_{B_R(x_1)} \left(1 + |Du|^2\right)^{p/2} \, dx\right]^{1/p}. \tag{3.1}$$

Moreover (see [7]) we can assume that $u \in C^{0,\gamma}(\Omega)$ for some $\gamma \in (0,1)$; thus let us denote simply by $[u]_\gamma$ the Hölder constant of u in Ω. Let us fix $B_R(x_0)$ such that $B_{4R}(x_0) \subset \Omega$. As before we shall not indicate the center of a ball when it is x_0.

Step 1. For any $x \in B_R$, $z \in \mathbb{R}^N$ we set

$$G(x,z) := F(x, u(x), z)\,.$$

Let $\mathcal{G}$ denote the functional defined in (2.1). Let v be the minimizer of $\mathcal{G}$ in $u + W_0^{1,1}(B_R)$. Using the minimality of u, we have

$$\begin{aligned}(3.2)\qquad \mathcal{G}(u) &\le \mathcal{F}(v; B_R)\\ &\le \mathcal{G}(v) + \int_{B_R} [F(x, v(x), Dv(x)) - F(x, u(x), Dv(x)]\,dx\\ &\le \mathcal{G}(v) + \int_{B_R} (\mu^2 + |Dv|^2)^{p/2}\omega(|v(x) - u(x)|)\,dx\,.\end{aligned}$$

Let $r \in (p,q)$ be the exponent given by Lemma 2.7. Using the boundedness and concavity of ω, together with (3.1), we can control the last integral as follows:

$$\begin{aligned}(3.3)\quad &\int_{B_R} (\mu^2 + |Dv|^2)^{p/2}\omega(|v(x) - u(x)|)\,dx\\ &\le |B_R| \left[\fint_{B_R} (\mu^2 + |Dv|^2)^{r/2} dx \right]^{p/r} \left[\fint_{B_R} \omega^{r/(r-p)}(|v(x) - u(x)|)\,dx \right]^{(r-p)/r}\\ &\le c|B_R| \left[\fint_{B_{2R}} (1 + |Du|^q)\,dx \right]^{p/q} \left[\fint_{B_R} \omega(|v(x) - u(x)|)\,dx \right]^{(r-p)/r}\\ &\le c\omega^\sigma \left(\fint_{B_R} |v(x) - u(x)|\,dx \right) \int_{B_{4R}} (1 + |Du|^p)\,dx\,,\end{aligned}$$

with $\sigma = (r-p)/r$. Recalling the Caccioppoli inequality for the minimizer u (see [7]), we have

$$\begin{aligned}&\omega^\sigma \left(\fint_{B_R} |v - u|\,dx \right) \le \omega^\sigma \left(cR \fint_{B_R} |Dv - Du|\,dx \right)\\ &\le \omega^\sigma \left[\left(cR^p \fint_{B_R} |Dv - Du|^p\,dx \right)^{1/p} \right] \le \omega^\sigma \left[\left(cR^p \fint_{B_R} (1 + |Du|^p)\,dx \right)^{1/p} \right]\\ &\le \omega^\sigma \left[\left(cR^p \fint_{B_{2R}} \left(1 + \frac{|u - u_{2R}|^p}{R^p} \right) dx \right)^{1/p} \right] \le \omega^\sigma \left[\left(cR^p + c[u]_\gamma^p R^{p\gamma} \right)^{1/p} \right]\\ &\le \omega^\sigma\,(c_0 R^\gamma)\,.\end{aligned}$$

Finally this relation, together with (3.2), (3.3) and the minimality of v, implies

$$\mathcal{G}(u) \le \inf_{u + W_0^{1,1}(B_R)} \mathcal{G} + c\omega^\sigma(c_0 R^\gamma) \int_{B_{4R}} (1 + |Du|^p)\,dx\,.$$

STEP 2. We argue as in [4]. Let us define

$$H(R) := c\omega^{\sigma}(c_0 R^{\gamma}) \int_{B_{4R}} (1+|Du|^p)\, dx$$

and apply Lemma 2.6 to the space $V = u + W_0^{1,1}(B_R)$ endowed with the distance

$$d(w_1, w_2) := H^{-1/p}(R) R^{-N(p-1)/p} \int_{B_R} |Dw_1 - Dw_2|\, dx\,.$$

Then there exists a function $v_0 \in u + W_0^{1,p}(B_R)$ such that

$$\int_{B_R} |Du - Dv_0|\, dx \le H^{1/p}(R) R^{N(p-1)/p}, \quad \mathcal{G}(v_0) \le \mathcal{G}(u)\,, \tag{3.4}$$

$$v_0 \quad \text{is a minimizer of} \quad \mathcal{G}(w) + \left[\frac{H(R)}{R^N}\right]^{(p-1)/p} \int_{B_R} |Dw - Dv_0|\, dx\,. \tag{3.5}$$

The minimality of v_0 implies that for any $\varphi \in W_0^{1,p}(B_R)$

$$\begin{aligned} \mathcal{G}(v_0; \mathrm{spt}\varphi) &\le \mathcal{G}(v_0+\varphi; \mathrm{spt}\varphi) \\ &\quad + \left[\frac{H(R)}{R^N}\right]^{(p-1)/p} \int_{\mathrm{spt}\varphi} |(Dv_0 + D\varphi) - Dv_0|\, dx \\ &\le \mathcal{G}(v_0+\varphi; \mathrm{spt}\varphi) + \frac{1}{2}\int_{\mathrm{spt}\varphi} |Dv_0|^p\, dx \\ &\quad + \frac{1}{2}\int_{\mathrm{spt}\varphi} |Dv_0 + D\varphi|^p\, dx + c(p)\frac{H(R)}{R^N}|\mathrm{spt}\varphi|\,. \end{aligned}$$

From this inequality it easily follows that v_0 is a Q-minimizer, with Q depending only on L and p, of the functional

$$w \mapsto \int_{B_R} \left(|Dw|^p + \frac{H(R)}{R^N} + 1\right) dx$$

and then (see [7]) there exist $s \in (p, q)$ and $c > 0$, independent on v_0, such that

$$\begin{aligned} \left(\fint_{B_{R/2}} |Dv_0|^s\, dx\right)^{p/s} &\le c\fint_{B_R} |Dv_0|^p\, dx + c\left(1 + \frac{H(R)}{R^N}\right) \\ &\le c\fint_{B_{4R}} (1+|Du|^p)\, dx\,. \end{aligned} \tag{3.6}$$

We remark that the function G satisfies (G_1), (G_2) and (G_3) with ω replaced by the function $\tilde{\omega}$ given by

$$\tilde{\omega}(t) := \max\left\{\omega\left(t + [u]_{\gamma} t^{\gamma}\right), [\omega^{\sigma}(c_0 t^{\gamma})]^{(p-1)/2}\right\}\,. \tag{3.7}$$

Applying Proposition 2.4 to the functional in (3.5), with $\vartheta_0 = \left[\dfrac{H(R)}{R^N}\right]^{(p-1)/p}$ and $u_0 = v_0$, we have that for any $\varrho \le R/2$

$$
\begin{aligned}
\int_{B_\varrho} |Du|^p\,dx &\le 2^{p-1}\int_{B_\varrho} |Dv_0|^p\,dx + 2^{p-1}\int_{B_\varrho} |Du - Dv_0|^p\,dx \\
&\le c\left[\left(\frac{\varrho}{R}\right)^N + \tilde\omega(R)\right]\int_{B_R} (1+|Dv_0|^p)\,dx + c\frac{H(R)}{[\tilde\omega(R)]^{1/(p-1)}} \\
&\qquad\qquad + c\int_{B_{R/2}} |Du - Dv_0|^p\,dx \\
&\le c\left[\left(\frac{\varrho}{R}\right)^N + \tilde\omega(R)\right]\int_{B_R} (1+|Du|^p)\,dx \\
&\quad + c\,[\tilde\omega(R)]^{1/(p-1)}\int_{B_{4R}} (1+|Du|^p)\,dx + c\int_{B_{R/2}} |Du - Dv_0|^p\,dx\,.
\end{aligned}
$$

Finally we have to estimate the last integral. Choosing $\theta \in (0,1)$ such that $\theta/s + 1 - \theta = 1/p$, using (3.1), (3.6), (3.4) and (3.7) we get

$$
\begin{aligned}
&\int_{B_{R/2}} |Du - Dv_0|^p\,dx \\
&\le |B_{R/2}|\left(\fint_{B_{R/2}} |Du - Dv_0|^s\,dx\right)^{\theta p/s}\left(\fint_{B_{R/2}} |Du - Dv_0|\,dx\right)^{(1-\theta)p} \\
&\le cR^N\left[\fint_{B_{4R}} (1+|Du|^p)\,dx\right]^\theta \left(\frac{H(R)}{R^N}\right)^{1-\theta} \\
&\le c\,[\tilde\omega(R)]^{2(1-\theta)/(p-1)}\int_{B_{4R}} (1+|Du|^p)\,dx\,.
\end{aligned}
$$

So we have proved that if $B_{4R}(x_0) \subset \Omega$ and if $\varrho < R/2$ then

$$
\int_{B_\varrho(x_0)} |Du|^p\,dx \le c\left\{\left(\frac{\varrho}{R}\right)^N + [\tilde\omega(R)]^\delta\right\}\int_{B_{4R}(x_0)} (1+|Du|^p)\,dx\,,
$$

for a certain $\delta > 0$ independent on R. From this inequality the thesis easily follows by a standard iteration argument (see [6], page 170). □

We observe that the result stated in Theorem 3.1 is sharp in the sense that even when F depends only on x and z we cannot expect in general that local minimizers are locally Lipschitz, as it is shown by the following example, which is a suitable modification of a well known example concerning the regularity of classical solutions of Poisson equation (see [9], chap.4).

Example 3.2 Let D be the unit disk in $\mathbb{R}^2$. We define two functions $w, f : D \to \mathbb{R}$ as follows:

$$w(x,y) := \sum_{k=0}^{\infty} \frac{1}{k+1} \eta\left(2^k x, 2^k y\right) xy\,,$$

$$f(x,y) := \sum_{k=0}^{\infty} \Bigg[\frac{2^{2k}}{k+1} \Delta\eta\left(2^k x, 2^k y\right) xy + \frac{2^{k+1}}{k+1} \left(\frac{\partial \eta}{\partial x}\left(2^k x, 2^k y\right) y + \frac{\partial \eta}{\partial y}\left(2^k x, 2^k y\right) x \right) \Bigg],$$

with $\eta \in C_0^\infty(\mathbb{R}^2)$, $\eta = 1$ on D, $\eta(x,y) \le 1$ for all (x,y) and $\mathrm{spt}\eta \subset D_2$, where D_2 is the disk of radius 2 centered at the origin. It is easy to prove that $\Delta w(x,y) = f(x,y)$ in the classical sense and that f is a continuous function. Let now $v \in W_0^{1,2}(D)$ be a weak solution of

$$\Delta v = \frac{\partial f}{\partial x}\,. \tag{3.8}$$

Since the function $u(x,y) = \frac{\partial w}{\partial x}(x,y)$ is a distributional solution of (3.8) it follows that $u - v$ is a distributional solution of Laplace equation, hence it is harmonic in the classical sense. In particular it follows that $u \in W^{1,2}(D)$. Hence u is a local minimizer of the functional

$$\mathcal{F}(v) = \int_D \left[|Dv(x,y)|^2 - \langle g(x,y), Dv(x,y)\rangle \right] dxdy\,,$$

with $g = (2f, 0)$, which satisfies the assumptions of Theorem 3.1. However u is not a Lipschitz function, since

$$\lim_{t\to 0^+} \frac{|u(0,t) - u(0,0)|}{t} = \infty\,.$$

It is clear from the proof of Theorem 3.1 that this result can be generalized in various directions. A possible extension is provided by the next result. Here p^* denotes the Sobolev exponent $Np/(N-p)$ if $p < N$ and any number greater than 1 if $p \ge N$.

Theorem 3.3 *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional*

$$\tilde{\mathcal{F}}(w;\Omega) = \int_\Omega F\left(x, w(x), Dw(x)\right) dx + \int_\Omega g(x, w(x))\, dx,$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $|g(x,u)| \le L(1 + |u|^t)$ with $t < p^*$, F satisfies the assumptions of Theorem 3.1 and moreover

$$F(x,u,0) = \min_{z\in\mathbb{R}^N} F(x,u,z) \qquad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{3.9}$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for all $\alpha < 1$.

PROOF. The proof of the result closely follows the one of Theorem 3.1. Henceforth we shall only indicate the necessary changes. Define G and v as in the proof of Theorem 3.1. Since u is bounded, from (3.9) we easily get by a truncation argument that v is bounded too and $||v||_{L^\infty(B_R)} \leq ||u||_{L^\infty(B_R)}$. Arguing as before we obtain that

$$\mathcal{G}(u) \leq \inf_{u+W_0^{1,1}(B_R)} \mathcal{G} + c\omega^\sigma(c_0 R^\gamma) \int_{B_{4R}} (1+|Du|^p)\, dx + cR^N .$$

Defining now

$$H(R) := c\omega^\sigma(c_0 R^\gamma) \int_{B_{4R}} (1+|Du|^p)\, dx + cR^N$$

and fixed $0 < \beta < N$ one can set now

$$\tilde{\omega}(t) := \max\left\{\omega\left(t+[u]_\gamma t^\gamma\right), [\omega^\sigma(c_0 t^\gamma)]^{(p-1)/2}, R^{\beta(p-1)}\right\} .$$

With this choice the final estimate becomes

$$\int_{B_\varrho} |Du|^p\, dx \leq c\left\{\left(\frac{\varrho}{R}\right)^N + [\tilde{\omega}(R)]^\delta\right\} \int_{B_{4R}} (1+|Du|^p)\, dx + cR^{N-\beta}$$

and again the result follows by the iteration argument in [6], page 170, and by the arbitrary choice of β. □

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