RENORMALIZATION FOR AUTONOMOUS NEARLY INCOMPRESSIBLE BV VECTOR FIELDS IN 2D

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ABSTRACT. Given a bounded autonomous vector field $b\colon \mathbb{R}^2 \to \mathbb{R}^2$, we study the uniqueness of bounded solutions to the initial value problem for the related transport equation

$$\partial_t u + b \cdot \nabla u = 0.$$

Assuming that b is of class BV and it is nearly incompressible, we prove uniqueness of weak solutions to the transport equation. The starting point is the result which has been obtained in [7] (where the *steady nearly incompressible* case is treated). Our proof is based on splitting the equation onto a suitable partition of the plane: this technique was introduced in [2], using the results on the structure of level sets of Lipschitz maps obtained in [1]. Furthermore, in order to construct the partition, we use Ambrosio's superposition principle [3].

KEYWORDS: transport equation, continuity equation, renormalization, disintegration of measures, Lipschitz functions, Superposition Principle.

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1. Introduction and notation

In this paper we consider the *continuity equation*

$$\partial_t u + \operatorname{div}(ub) = 0 \tag{1.1}$$

and the *transport equation*

$$\partial_t u + b \cdot \nabla u = 0, \tag{1.2}$$

for a scalar field $u\colon I \times \mathbb{R}^2 \to \mathbb{R}$ (where $I = (0,T)$, $T > 0$) with a vector field $b\colon I \times \mathbb{R}^2 \to \mathbb{R}^2$. We study the initial value problems for these equations with the same initial condition

$$u(0,\cdot) = \overline{u}(\cdot), \tag{1.3}$$

where $\bar{u}\colon \mathbb{R}^2 \to \mathbb{R}$ is a given scalar field.

Our aim is to investigate uniqueness of weak solutions to (1.1), (1.3) (and to (1.2), (1.3)) under weak regularity assumptions on the vector field b.

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Even if we are interested only to the two dimensional case, we present here the main definitions in $\mathbb{R}^d$, with $d \in \mathbb{N}$. When $b \in L^\infty(I \times \mathbb{R}^d)$ then (1.1) is understood in the standard sense of distributions: $u \in L^\infty(I \times \mathbb{R}^d)$ is called a *weak solution* of the continuity equation if (1.1) holds in $\mathscr{D}'(I \times \mathbb{R}^d)$. One can prove (see e.g. [12]) that, if u is a weak solution of (1.1), then there exists a map $\tilde{u} \in L^\infty([0,T] \times \mathbb{R}^d)$ such that $u(t,\cdot) = \tilde{u}(t,\cdot)$ for a.e. $t \in I$ and $t \mapsto \tilde{u}(t,\cdot)$ is weakly* continuous from $[0,T]$ into $L^\infty(\mathbb{R}^d)$. This allows us to prescribe an initial condition (1.3) for a weak solution u of the continuity equation in the following sense: we say that $u(0,\cdot) = \bar{u}(\cdot)$ holds if $\tilde{u}(0,\cdot) = \bar{u}(\cdot)$.

Definition of weak solutions of the transport equation (1.2) is slightly more delicate. If the divergence of b is absolutely continuous with respect to the Lebesgue measure then (1.2) can be written as

$$\partial_t u + \operatorname{div}(ub) - u \operatorname{div} b = 0,$$

and the latter equation can be understood in the sense of distributions (see e.g. [13] for the details). We are interested in the case when $\operatorname{div} b$ is not absolutely continuous. In this case the notion of weak solution of (1.2) can be defined for the class of *nearly incompressible vector fields*.

Definition 1.1. A bounded, locally integrable vector field $b \colon I \times \mathbb{R}^d \to \mathbb{R}^d$ is called *nearly incompressible* if there exists a function $\rho \colon I \times \Omega \to \mathbb{R}$ (called *density* of b) and a constant $C > 0$ such that $C^{-1} \le \rho(t,x) \le C$ for $\mathscr{L}^1 \times \mathscr{L}^d$-a.e. $(t,x) \in I \times \Omega$ and

$$\partial_t \rho + \operatorname{div}(\rho b) = 0 \qquad \text{in } \mathscr{D}'(I \times \Omega). \tag{1.4}$$

Nearly incompressible vector fields were introduced in connection with the hyperbolic conservation laws, namely, the Keyfitz-Kranzer system [16]. See e.g. [12] for the details. Using mollification one can prove that if $\operatorname{div} b \in L^\infty(I \times \mathbb{R}^d)$ then b is nearly incompressible. The converse implication does not hold, so near incompressibility can be considered as a weaker version of the assumption $\operatorname{div} b \in L^\infty(I \times \mathbb{R}^d)$.

Definition 1.2. Let b be a nearly incompressible vector field with density ρ. We say that a function $u \in L^\infty(I \times \mathbb{R}^2)$ is a *(ρ–)weak solution* of (1.2) if

$$(\rho u)_t + \operatorname{div}(\rho u b) = 0 \quad \text{in } \mathscr{D}'(I \times \mathbb{R}^2).$$

Thanks to Definition 1.2 one can prescribe the initial condition for a ρ–weak solution of the transport equation similarly to the case of the continuity equation, which we mentioned above (see [12] for the details).

Existence of weak solutions to initial value problem for transport equation with a nearly incompressible vector field can be proved by a standard regularization argument [12]. The problem of *uniqueness* of weak solutions is much more delicate. The theory of uniqueness in the non-smooth framework has started with the seminal paper of R.J. DiPerna and P.-L. Lions [13] where uniqueness was obtained as a corollary of so-called *renormalization property* for the vector fields with Sobolev regularity. Thanks to Definition 1.2 the renormalization property can be defined also for nearly incompressible vector fields:

Definition 1.3. We say that a nearly incompressible vector field b with density ρ has the *renormalization property* if for every ρ–weak solution $u \in L^\infty(I \times \mathbb{R}^d)$ of (1.2) and any function $\beta \in C^1(\mathbb{R})$ the function $\beta(u)$ also is a ρ-weak solution of (1.2), i.e. it satisfies

$$\partial_t (\rho\beta(u)) + \operatorname{div}(\rho\beta(u)b) = 0 \quad \text{in } \mathscr{D}'(I \times \mathbb{R}^d).$$

Nearly incompressible vector fields are related to a conjecture, made by A. Bressan in [9]:

Conjecture 1.4 (Bressan's compactness conjecture)**.** *Let $b_n : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $n \in \mathbb{N}$, be a sequence of smooth vector fields. Denote by Φ_n the solutions of the ODEs*

$$\frac{d}{dt}\Phi_n(t,x) = b_n(t, \Phi_n(t,x)),$$
$$\Phi_n(0,x) = x.$$

Assume that $\|b_n\|_\infty + \|\nabla_{t,x} b_n\|_{L^1}$ is uniformly bounded and there exists a constant $C > 0$ such that

$$C^{-1} \le \det(\nabla_x \Phi_n(t,x)) \le C$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ and all $n \in \mathbb{N}$. Then the sequence Φ_n is strongly precompact in L^1_{loc}.

It has been proved in [4] that Bressan's conjecture would follow from the next one:

Conjecture 1.5 (Renormalization conjecture)**.** *Any bounded, nearly incompressible vector field $b \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$ has the renormalization property (in the sense of Definition 1.3).*

The renormalization property can also be generalized for the systems of transport equations. Moreover, if η is another density of the nearly incompressible vector field b and b has the renormalization property with the density ρ, then any ρ–weak solution of (1.2) is also an η–weak solution and vice versa. In other words, the property of being a ρ–weak solution does not depend on the choice of the density ρ provided that renormalization holds. We refer to [12] for the details.

If the functions ρ, u and b were smooth, renormalization property would be an easy corollary of the chain rule. Out of the smooth setting, the validity of this property is a key step to get uniqueness of weak solutions. Indeed, if we for simplicity consider $\mathbb{T}^d$ instead of $\mathbb{R}^d$, then integrating the equation above over the torus we get

$$\partial_t \int_{\mathbb{T}^d} \rho\beta(u)\, dx = 0.$$

So if $\bar{u} = 0$ then for $\beta(y) = y^2$ we get

$$\int_{\mathbb{T}^d} \rho(t,x) u^2(t,x)\, dx = 0$$

for a.e. t which implies $u(t,\cdot) = 0$ for a.e. t.

The problem of uniqueness of solutions is thus shifted to prove the renormalization property for b: in [13] the authors proved that renormalization property holds under Sobolev regularity assumptions; some years later, L.

Ambrosio [3] improved this result, showing that renormalization holds for vector fields which are of class BV (locally in space) and have absolutely continuous divergence.
Another approach giving explicit compactness estimates has been introduced in [11], and further developed in [8, 15]: see also the references therein.

In the two dimensional autonomous case the problem of uniqueness is addressed in the papers [2], [1] and [7]. Indeed, in two dimensions and for divergence-free autonomous vector fields, renormalization theorems are available even under mild assumptions, because of the underlying Hamiltonian structure. In [2], the authors characterize the autonomous, divergence-free vector fields b on the plane such that the Cauchy problem for the continuity equation (1.1) admits a unique bounded weak solution for every bounded initial datum (1.3). The characterization they present relies on the so called *Weak Sard Property*, which is a (weaker) measure theoretic version of Sard's Lemma. Since the problem admits a Hamiltonian potential, uniqueness is proved following a strategy based on splitting the equation on the level sets of this function, reducing thus to a one-dimensional problem. This approach requires a preliminary study on the structure of level sets of Lipschitz maps defined on $\mathbb{R}^2$, which is carried out in the paper [1].

In [7] the *steady nearly incompressible* autonomous vector fields on $\Omega = \mathbb{R}^2$ were considered. Namely, an autonomous vector field $b\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called steady nearly incompressible if it admits a steady density $\widetilde{\rho}$, i.e. there exists a function $\widetilde{\rho}$, uniformly bounded from below and above by some strictly positive constants, such that $\operatorname{div}(\widetilde{\rho} b) = 0$. It was proved in [7] that any steady nearly incompressible BV vector field on $\mathbb{R}^2$ has the renormalization property. In the present paper we extend this result to the non-steady case. Any steady nearly incompressible vector field is nearly incompressible, but the inverse implication does not hold in general. For instance, consider a vector field $b\colon (0,2) \to \mathbb{R}$ given by $b(x) = |x-1| - 1$. If it was steady nearly incompressible, the function $\widetilde{\rho} \cdot b$ would be constant on $(0,2)$ and thus $\widetilde{\rho}$ could not be uniformly bounded from above by a positive constant. On the other hand this vector field b is nearly incompressible: the solution to the continuity equation $\partial_t \rho + \partial_x(\rho b) = 0$ with the initial condition $\rho|_{t=0} = 1$ satisfies $e^{-t} \le \rho(t,x) \le e^t$, as one can easily demostrate using the classical method of characteristics, since b is Lipschitz. This simple example can be generalized to higher dimensions.

The main result of this paper is a partial answer to the Conjecture 1.5:

Main Theorem. *Every bounded, autonomous, compactly supported, nearly incompressible* BV *vector field on* $\mathbb{R}^2$ *has the renormalization property.*

In particular, we obtain the following

Corollary 1.6. *Suppose that* $b\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is a compactly supported, nearly incompressible* BV *vector field (with density* ρ*). Then*

(1) $\forall u_0 \in L^\infty(\mathbb{R}^2)$ *there exists a unique (*ρ*-)weak solution* $u \in L^\infty(I \times \mathbb{R}^2)$ *to the transport equation* (1.2) *with the initial condition* $u|_{t=0} = u_0$.

(2) $\forall u_0 \in L^\infty(\mathbb{R}^2)$ *there exists a unique weak solution* $u \in L^\infty(I \times \mathbb{R}^2)$ *to the continuity equation* (1.1) *with the initial condition* $u|_{t=0} = u_0$.

1.1. **Structure of the paper.** The paper is organised as follows.

In Section 2 we present Ambrosio's Superposition Principle. By this Principle, the measure $\rho(t,\cdot)\mathscr{L}^2$ (where ρ is a nonnegative bounded solution of the continuity equation (1.4)) can be represented as an *image* of some probability measure η on the space of curves $C([0,T];\mathbb{R}^2)$ (concentrated on the solutions of the ODE $\gamma' = b(\gamma)$) under the evaluation map $e_t \colon \gamma \mapsto \gamma(t)$:

$$\rho(t,\cdot)\mathscr{L}^2 = (e_t)_\# \eta.$$

Using this Theorem, we construct a suitable partition of the plane and we reduce our problem *locally* to the case when the density ρ is steady, which has been studied in [7]. In this case, since $\operatorname{div}(\rho b) = 0$, there exists a Lipschitz *Hamiltonian* $H \colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\rho b = \nabla^\perp H,$$

where $\nabla^\perp = (-\partial_2, \partial_1)$.

In the general nearly incompressible case it is not possible to construct the Hamiltonian H directly as in the case of steady density. However, we reduce the problem to the steady case using the following argument. Suppose that a nonnegative bounded function ϱ solves the continuity equation

$$\varrho_t + \operatorname{div}(\varrho b) = 0,$$

$t \mapsto \varrho(t,\cdot)$ is weak* continuous and for some open set Ω and $t_{1,2} \in [0,T]$ we have $\varrho(t_1,\cdot) = \varrho(t_2,\cdot) = 0$ a.e. on Ω. Integrating the continuity equation with respect to time on $[t_1,t_2]$ it is easy to see that

$$r(x) := \int_{t_1}^{t_2} \varrho(t,x)\,dt$$

solves

$$\operatorname{div}(rb) = 0$$

in $\mathscr{D}'(\Omega)$. Therefore in Ω one can construct a *local Hamiltonian* H_Ω such that

$$rb = \nabla^\perp H_\Omega$$

in Ω.

Once we have constructed the local Hamiltonians, we show how we can split an equation of the form

$$\operatorname{div}(ub) = \mu, \qquad u \colon \mathbb{R}^2 \to \mathbb{R} \tag{1.5}$$

where μ is a measure on $\mathbb{R}^2$, into an equivalent family of equations along the level sets of H. This is done in Section 3, where we also recall the main results of [1, 2, 7] and adapt them to our setting. In Section 4 we establish the so-called Weak Sard Property for the Hamiltonian H.

Then we turn to study in detail the relationship between level sets of the local Hamiltonian H and the trajectories of the vector field b: in Section 5, we present some lemmas which show that (up to a η negligible set) all non constant integral curves of b are contained in "good" level sets of H.

In Section 6 we prove that the divergence operator is *local*, in the sense that the measure μ in (1.5) vanishes on the set $M := \{b = 0\}$ (Proposition

6.1). We stress that this result is true for every space dimension and it is crucial to obtain a better description of the link between the level sets and the trajectories. This is achieved in Section 7, where in particular, we prove that "good" level sets of H cover almost all the set $M^c = \{b \neq 0\}$.

Finally, in Section 8 we first show how the time-dependent problem

$$\begin{cases} u_t + b \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2). \tag{1.6}$$

can be reduced to a family of one-dimensional problems on level sets of the Hamiltonians, which can be solved explicitly. This allows to construct a η-negligible set R of trajectories with the following property, which is reminiscent of the standard Method of Characteristics (within the smooth setting): if u is a solution of 1.6, then for all $\gamma \notin R$ the function $t \mapsto u(t, \gamma(t))$ is constant. This crucial result (Lemma 8.8) combined with an elementary observation (Lemma 8.9) concludes in Section 9 the proof of the **Main Theorem** (Theorem 9.1).

1.2. **Notation.** Throughout the paper, we use the following notation:

- (X, d) is a metric space;
- $\mathbb{1}_E$ is the characteristic function of the set $E \subset X$, defined as $\mathbb{1}_E(x) = 1$ if $x \in E$ and $\mathbb{1}_E(x) = 0$ otherwise;
- Ω denotes in general a simply connected open set in $\mathbb{R}^2$;
- $\operatorname{dist}(x, E)$ is the distance of x from the set E, defined as the infimum of $d(x, y)$ as y varies in E;
- $B(x, r)$ or, equivalently, $B_r(x)$ is the open ball in $\mathbb{R}^d$ with radius r and centre x; $B(r)$ is the open ball in $\mathbb{R}^d$ with radius r and centre 0;
- $⨍_E f \, d\mu$ denotes the *average* of the function f over the set E with respect to the positive measure μ, that is
$$⨍_E f \, d\mu := \frac{1}{\mu(E)} \int_E f \, d\mu,$$
- $\mu \llcorner A$ denotes the restriction of a measure μ on a set A.
- $|\mu|$ is the total variation of a measure μ;
- μ^{sing} the singular component of μ with respect to the Lebesgue measure;
- $\mathscr{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\mathscr{H}^k$ is the k-dimensional Hausdorff measure;
- $\operatorname{Lip}(X)$ is the space of real-valued Lipschitz functions; $\operatorname{Lip}_c(X)$ is the space of real-valued compactly supported Lipschitz functions;
- $C_c^\infty(\Omega)$ is the space of smooth compactly supported functions, also called *test functions*;
- $\mathrm{BV}(\Omega)$ set of functions with bounded variation;
- $\mathscr{D}'(\Omega)$ is the space of distributions on the open set Ω;
- $\Gamma := C([0,T]; \mathbb{R}^2)$ will denote the set of continuous curves in $\mathbb{R}^2$;
- $\dot{\Gamma} := \{\gamma \in \Gamma : \gamma(t) = \gamma(0), \, \forall t \in [0,T]\}$ denotes the set of constant curves (whose graphs are fixed points);
- $\widetilde{\Gamma} := \Gamma \setminus \dot{\Gamma}$ denotes the set of non-constant curves (whose graphs have positive length);
- $e_t \colon \Gamma \to \mathbb{R}^2$ is the *evaluation map* at time t, i.e. $e_t(\gamma) = \gamma(t)$.

Moreover, if $A \subset \mathbb{R}^2$ is a measurable set,

- $\Gamma_A := \{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\}$ denotes the set of curves which stay in A for a positive amount of time;
- $\widetilde{\Gamma}_A := \widetilde{\Gamma} \cap \Gamma_A$ denotes the set of non-constant curves which stay in A for a positive amount of time;
- $\dot{\Gamma}_A := \dot{\Gamma} \cap \Gamma_A$ denotes the set of constant curves which stay in A for a positive amount of time.
- for every $s \in [0,T]$, we denote by

$$\begin{aligned} \Gamma_A^s &:= \{\gamma \in \Gamma : \gamma(s) \in A\}, \\ \widetilde{\Gamma}_A^s &:= \left\{\gamma \in \widetilde{\Gamma} : \gamma(s) \in A\right\}, \\ \dot{\Gamma}_A^s &:= \left\{\gamma \in \dot{\Gamma} :\ \gamma(s) \in A\right\} \end{aligned}$$

  accordingly the sets of all curves, non-constant curves and constant curves, which at time s belong to A;
- $\mathsf{T}_A := \{\gamma \in \Gamma_A :\ \gamma(0) \notin A, \gamma(T) \notin A\}$ denotes the set of curves which stay in A for a positive amount of time and have the endpoints outside A.

If $A \subseteq \mathbb{R}^2$, we denote by

$$\begin{aligned} \mathrm{Conn}(A) &:= \Big\{C \subset A : C \text{ is a connected component of } A\Big\}, \\ \mathrm{Conn}^\star(A) &:= \Big\{C \in \mathrm{Conn}(A) : \mathscr{H}^1(C) > 0\Big\}, \end{aligned}$$

and

$$A^\star := \bigcup_{C \in \mathrm{Conn}^\star(A)} C.$$

When the measure is not specified, it is assumed to be the Lebesgue measure, and we often write

$$\int f(x)\, dx$$

for the integral of f with respect to $\mathscr{L}^d$.

1.3. **Disintegration of a measure.** Let μ be a Radon measure on a metric space X. Let Y be a metric space and let $f \colon X \to Y$ be a Borel function. We denote by $f_\# \mu$ the *image measure* of μ under the map f. In particular, for any $\varphi \in C_c(Y)$ we have

$$\int_X \varphi(f(x))\, d\mu(x) = \int_Y \varphi(y)\, d(f_\#\mu)(y).$$

Let ν be a Radon measure on Y such that $f_\# |\mu| \ll \nu$. According to the Disintegration Theorem (Theorem 2.28 of [6] or for the most general statement Section 452 of [14]) there exists a unique measurable family of Radon measures $\{\mu_y\}_{y \in Y}$ such that for ν-a.e. $y \in Y$ the measure μ_y is concentrated on the level set $f^{-1}(y)$ and

$$\mu = \int_y \mu_y \, d\nu(y),$$

that is, for any $\varphi \in C_c(X)$
$$\int_X \varphi(x)\, d\mu(x) = \int_Y \left(\int_X \varphi(x)\, d\mu_y(x) \right) d\nu(y).$$
The family $\{\mu_y\}_{y\in Y}$ is called the *disintegration of μ with respect to f (and ν)*.

1.4. **Coarea formula.** Suppose that $H\colon \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function. Coarea formula (see e.g. [6] for the general statement) provides further information about the structure of the disintegration of $|\nabla H| \mathscr{L}^2$ with respect to H:

Lemma 1.7. *Let $\{\varpi_h\}_{h\in\mathbb{R}}$ denote the disintegration of the measure $|\nabla H|\mathscr{L}^2$ with respect to H and let $E_h := H^{-1}(h)$. Then for a.e. $h \in \mathbb{R}$ we have $\mathscr{H}^1(E_h) < \infty$ and $\varpi_h = \mathscr{H}^1 \llcorner E_h$. In other words, the disintegration of $|\nabla H|\mathscr{L}^2$ with respect to H is given by*
$$|\nabla H|\mathscr{L}^2 = \int_{\mathbb{R}} \mathscr{H}^1 \llcorner E_h \, dh.$$

2. Setting of the problem

2.1. **Ambrosio's Superposition Principle.** In [3], L. Ambrosio proved the *Superposition Principle*. Since we will use it later on in this section, we present here the statement. Let us consider the continuity equation in the form
$$\begin{cases} \partial_t \mu_t + \operatorname{div}(b\mu_t) = 0, \\ \mu_0 = \overline{\mu}, \end{cases} \tag{2.1}$$
where $[0,T] \ni t \mapsto \mu_t$ is a measure valued function and $b\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded, Borel vector field. A solution to (2.1) has to be understood in distributional sense.

We have the following

Theorem 2.1 (Superposition Principle)**.** *Let $b\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded, Borel vector field and let $[0,T] \ni t \mapsto \mu_t$ be a positive, locally finite, measure-valued solution of the continuity equation (2.1). Then there exists a family of probability measures $\{\eta_x\}_{x\in\mathbb{R}^d}$ on Γ such that*
$$\mu_t = \int e_{t\#}\eta_x d\bar{\mu}(x),$$
for any $t \in (0,T)$ and $(e_0)_\# \eta_x = \delta_x$. Moreover, η_x is concentrated on absolutely continuous integral solutions of the ODE starting from x, for $\overline{\mu}$-a.e. $x \in \mathbb{R}^d$.

In other words, any nonnegative measure-valued solution μ_t of the continuity equation (2.1) can be represented as
$$\mu_t = e_{t\#}\eta, \tag{2.2}$$
where η is some nonnegative measure on the space of continuous curves Γ, which is concentrated on the integral curves of the vector field b. In terms of Theorem 2.1 this measure η can be defined by
$$\eta = \int_{\mathbb{R}^d} \eta_x \, d\bar{\mu}(x).$$

(I.e. the family $\{\eta_x\}_{x\in\mathbb{R}^d}$ is the disintegration of η under the map e_0.)

2.2. **Partition and curves.** Let $b\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an *autonomous*, nearly incompressible vector field, with $b \in \mathrm{BV}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$; we assume b is compactly supported (with support in the unit ball of $\mathbb{R}^2$, $\mathbb{B} := B(0,1)$), defined everywhere and Borel. Let us consider the countable covering $\mathscr{B}$ of $\mathbb{R}^2$ given by

$$\mathscr{B} := \left\{ B(x,r) : \, x \in \mathbb{Q}^2, r \in \mathbb{Q}^+ \right\}.$$

For each ball $B \in \mathscr{B}$, we are interested to the trajectories of b which cross B, staying inside B for a positive amount of time. We therefore define, for every ball $B \in \mathscr{B}$ and for every rational numbers $s, t \in \mathbb{Q} \cap (0,T)$ such that $s < t$, the sets

$$\mathsf{T}_{B,s,t} := \{\gamma \in \Gamma_B : \, \gamma(s) \notin B, \gamma(t) \notin B\}.$$

We recall that (see Notations)

$$\Gamma_B := \left\{ \gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in B\}) > 0 \right\}.$$

In this first section we will work for simplicity with the sets $T_B := T_{B,0,T}$, where $B \in \mathscr{B}$ (and without any loss of generality we assume $T \in \mathbb{Q}$).

Remark 2.2. It is fairly easy to see that

$$\bigcup_{B\in\mathscr{B}} \mathsf{T}_B = \widetilde{\Gamma}.$$

Indeed, for every curve which is moving there exists a point $\gamma(t) \neq \gamma(0), \gamma(T)$, so that one has just to choose a ball in $\mathscr{B}$ containing $\gamma(t)$ but not $\gamma(0)$, $\gamma(T)$.

By Definition 1.1, there exists a function $\rho\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ which satisfies continuity equation (1.4) in $\mathscr{D}'((0,T) \times \mathbb{R}^2)$. Therefore, by Ambrosio's Superposition Principle 2.1, there exists a measure η on Γ, concentrated on the set of trajectories of b, such that

$$\rho(t,\cdot)\mathscr{L}^2 = (e_t)_\# \eta, \tag{2.3}$$

where we recall that $e_t\colon \Gamma \to \mathbb{R}^2$ is the evaluation map $\gamma \mapsto \gamma(t)$. For a fixed ball $B \in \mathscr{B}$, we consider the measure $\eta_B := \eta \llcorner \mathsf{T}_B$ and we define ρ_B by $\rho_B(t,\cdot)\mathscr{L}^2 = (e_t)_\# \eta_B$. Then we set

$$r_B(x) := \int_0^T \rho_B(t,x)dt, \qquad x \in B. \tag{2.4}$$

Lemma 2.3. *It holds* $\operatorname{div}(r_B b) = 0$ *in* $\mathscr{D}'(B)$.

Proof. For any $\phi \in C_c^\infty(B)$ we have

$$\begin{aligned}\int_B r_B b(x)\cdot\nabla\phi(x)dx &= \int_B\int_0^T \rho_B(t,x)b(x)\cdot\nabla\phi(x)\,dt\,dx\\ &= \int_0^T\int_{\mathsf{T}_B} b(\gamma(t))\cdot(\nabla\phi)(\gamma(t))\,d\eta_B\,dt\\ &= \int_0^T\int_{\mathsf{T}_B} \dot\gamma(t)\cdot(\nabla\phi)(\gamma(t))\,d\eta_B\,dt\\ &= \int_0^T\int_{\mathsf{T}_B} \frac{d}{dt}\phi(\gamma(t))\,d\eta_B\,dt\\ &= \int_{\mathsf{T}_B}\Big[\phi(\gamma(T))-\phi(\gamma(0))\Big]\,d\eta_B = 0.\end{aligned}$$

because for η_B-a.e. $\gamma\in\mathsf{T}_B$, $\gamma(0)\notin B$, $\gamma(T)\notin B$. □

3. Recent results for uniqueness in the two dimensional case

We recall here some facts about uniqueness of bounded solutions for the continuity equation in the two dimensional case, following in particular [1, 2].

3.1. **Structure of level sets of Lipschitz functions.** Let $\Omega\subset\mathbb{R}^2$ be a bounded, open set and let $f\colon\Omega\to\mathbb{R}$ be a Lipschitz function. For any $r\in\mathbb{R}$, we denote by $E_r := f^{-1}(r)$ the corresponding level set.

Theorem 3.1 ([1, Thm. 2.5])**.** *Suppose that $f\colon\Omega\to\mathbb{R}$ is a compactly supported Lipschitz function. For any $r\in\mathbb{R}$, let $E_r := f^{-1}(r)$. Then the following statements hold for $\mathscr{L}^1$-a.e. $r\in f(\Omega)$:*

1. *$\mathscr{H}^1(E_r)<\infty$ and E_r is countable $\mathscr{H}^1$-rectifiable;*
2. *for $\mathscr{H}^1$-a.e. $x\in E_r$ the function f is differentiable at x with $\nabla f(x)\neq 0$;*
3. *$\mathrm{Conn}^\star(E_r)$ is countable and every $C\in\mathrm{Conn}^*(E_r)$ is a closed simple curve;*
4. *$\mathscr{H}^1(E_r\setminus E_r^*)=0$.*

For brevity, we will say that the level set E_r is *regular with respect to* Ω if it satisfies conditions (1)-(2)-(3)-(4) (or it is empty). In this way, the theorem above can be stated by saying that for a.e. $r\in\mathbb{R}$ the level sets E_r are regular with respect to Ω.

3.2. **Disintegration of Lebesgue measure with respect to Hamiltonians.** From Lemma 2.3 we have $\mathrm{div}(rb)=0$ in B; since B is simply connected, there exists a Lipschitz potential $H_B\colon B\to\mathbb{R}$ such that

$$\nabla^\perp H_B(x) = r_B(x)b(x),\qquad \text{for } \mathscr{L}^2\text{-a.e. } x\in B.$$

Using Theorem 3.1 on the Lipschitz function H_B, we can define the negligible set N_1 such that E_h is regular in B whenever $h\notin N_1$; moreover, let N_2 denote the negligible set on which the measure $((H_B)_\#\mathscr{L}^2)^{\mathrm{sing}}$ is concentrated, where $((H_B)_\#\mathscr{L}^2)^{\mathrm{sing}}$ is the singular part of $((H_B)_\#\mathscr{L}^2)$ with respect to $\mathscr{L}^1$. Then we set

$$N := N_1\cup N_2 \qquad\text{and}\qquad E^* := \cup_{h\notin N}E_h^* \tag{3.1}$$

Therefore we can associate to B a triple (H_B, N, E). For any $x \in E$ let C_x denote the connected component of E such that $x \in C_x$. By definition of E for any $x \in E$ the corresponding connected component C_x has strictly positive length.

Let us fix an arbitrary ball $B \in \mathscr{B}$. For brevity let H denote the corresponding Hamiltonian H_B.

Lemma 3.2 ([2, Lemma 2.8]). *There exist Borel families of measures σ_h, κ_h, $h \in \mathbb{R}$, such that*

$$\mathscr{L}^2 \llcorner B = \int \left(c_h \mathscr{H}^1 \llcorner E_h + \sigma_h \right) dh + \int \kappa_h \, d\zeta(h), \tag{3.2}$$

where

(1) $c_h \in L^1(\mathscr{H}^1 \llcorner E_h^\star)$, $c_h > 0$ *a.e.; moreover, by Coarea formula, we have* $c_h = 1/|\nabla H|$ *a.e. (w.r.t.* $\mathscr{H}^1 \llcorner E_h^\star$*);*

(2) σ_h *is concentrated on* $E_h^\star \cap \{\nabla H = 0\}$*;*

(3) κ_h *is concentrated on* $E_h^\star \cap \{\nabla H = 0\}$*;*

(4) $\zeta := H_\# \mathscr{L}^2 \llcorner (B \setminus E^\star)$ *is concentrated on N (hence* $\zeta \perp \mathscr{L}^1$*);*

(5) σ_h *is concentrated on* $E_h \cap \{b \neq 0, r_B = 0\}$*.*

Proof. Points (1)-(4) are exactly [2, Lemma 2.8]. Concerning (5), it can be proved using minor modifications of the proof of [7, Theorem 8.2]: indeed, we have that, being b of class BV and hence approximately differentiable a.e., $H_\# \mathscr{L}^2 \llcorner \{b = 0\} \perp \mathscr{L}^1$: by comparing two disintegrations of $\mathscr{L}^2 \llcorner \{b = 0\}$ we conclude that σ_h is concentrated on $\{b \neq 0\}$ for a.e. h. □

Remark 3.3. Using Coarea formula (see Lemma 1.7), we can show

$$\mathscr{H}^1(E_h \cap \{\nabla H = 0\}) = 0$$

for $\mathscr{L}^1$-a.e. $h \notin N$. Therefore $\sigma_h \perp \mathscr{H}^1$ for $\mathscr{L}^1$-a.e. $h \notin N$.

Remark 3.4. Thanks to (3.2) we always can add to N, if necessary, an $\mathscr{L}^1$-negligible set so that for any $h \notin N$ for $\mathscr{H}^1$-a.e. $x \in E_h^\star$ we have $r(x) > 0$, $b(x) \neq 0$ and $r(x)b(x) = \nabla^\perp H(x)$.

3.3. **Reduction of the equation on the level sets.** Our goal is now to study the equation $\operatorname{div}(ub) = \mu$, where u is a bounded Borel function on $\mathbb{R}^2$ and μ is a Radon measure on $\mathbb{R}^2$, inside a ball from the collection $\mathscr{B}$.

Lemma 3.5. *Suppose that μ is a Radon measure on $\mathbb{R}^2$ and $u \in L^\infty(\mathbb{R}^2)$. Then equation*

$$\operatorname{div}(ub) = \mu \tag{3.3}$$

holds in $\mathscr{D}'(B)$ if and only if:

- *the disintegration of μ with respect to H has the form*

$$\mu = \int \mu_h \, dh + \int \nu_h \, d\zeta(h), \tag{3.4}$$

  where ζ is defined in Point (4) of Lemma 3.2;
- *for $\mathscr{L}^1$-a.e. h*

$$\operatorname{div}\left(u c_h b \mathscr{H}^1 \llcorner E_h \right) + \operatorname{div}(u b \sigma_h) = \mu_h; \tag{3.5}$$

- *for ζ-a.e. h*

$$\operatorname{div}(ub\kappa_h) = \nu_h. \tag{3.6}$$

Proof. Let λ^s be a measure on $\mathbb{R}$ such that $H_\#|\mu| \ll \mathscr{L}^1 + \zeta + \lambda^s$, where ζ is defined as in Lemma 3.2 and $\lambda^s \perp \mathscr{L}^1 + \zeta$. Applying the Disintegration Theorem, we have that

$$\mu = \int \mu_h dh + \int \nu_h d\zeta(h) + \int \lambda_h d\lambda^s(h), \tag{3.7}$$

with μ_h, ν_h, λ_h concentrated on $\{H = h\}$. Writing equation (3.3) in distribution form we get

$$\int_{\mathbb{R}^2} u(b \cdot \nabla \phi)\, dx + \int \phi d\mu = 0, \qquad \forall \phi \in C_c^\infty(B).$$

By an elementary approximation argument, it is clear that we can use as test functions ϕ Lipschitz with compact support.

Using the disintegration of Lebesgue measure (3.2) and the disintegration (3.7) we thus obtain

$$\begin{aligned} \int \Big[\int_{\mathbb{R}^2} uc_h (b\cdot\nabla\phi)\, d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{R}^2} u(b\cdot\nabla\phi)\, d\sigma_h\Big]\, dh \\ + \int\int_{\mathbb{R}^2} u(b\cdot\nabla\phi)\, d\kappa_h\, d\zeta(h) + \int\int_{\mathbb{R}^2} \phi\, d\mu_h\, dh \\ + \int\int_{\mathbb{R}^2} \phi\, d\nu_h\, d\zeta(h) + \int\int_{\mathbb{R}^2} \phi\, d\lambda_h\, d\lambda^s(h) = 0, \end{aligned} \tag{3.8}$$

for every $\phi \in \mathrm{Lip}_c(B)$. In particular we can take

$$\phi = \psi(H(x))\varphi(x), \qquad \psi \in C^\infty(\mathbb{R}),\ \varphi \in C_c^\infty(B),$$

so that we can rewrite (3.8) as

$$\begin{aligned} \int \psi(h) \Big[\int_{\mathbb{R}^2} uc_h (b\cdot\nabla\varphi)\, d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{R}^2} u(b\cdot\nabla\varphi)\, d\sigma_h\Big)\, dh \\ + \int \psi(h) \int_{\mathbb{R}^2} u(b\cdot\nabla\varphi)\, d\kappa_h\, d\zeta(h) + \int \psi(h)\int_{\mathbb{R}^2} \varphi\, d\mu_h\, dh \\ + \int \psi(h) \int_{\mathbb{R}^2} \varphi\, d\nu_h\, d\zeta(h) + \int \psi(h)\int_{\mathbb{R}^2} \varphi\, d\lambda_h\, d\lambda^s(h) = 0, \end{aligned}$$

because

$$b\cdot\nabla\phi = \psi(H(x)) b\cdot\nabla\varphi(x)$$

for $\mathscr{L}^2$-a.e. $x \in \mathbb{R}^2$.

Since the equalities above hold for all $\psi \in C^\infty(\mathbb{R})$ we have

$$\int \Big[\int_{\mathbb{R}^2} uc_h (b\cdot\nabla\varphi)\, d\mathscr{H}^1 \llcorner E_h + \int_{\mathbb{R}^2} u(b\cdot\nabla\varphi)\, d\sigma_h\Big]\, dh + \int\int_{\mathbb{R}^2} \varphi\, d\mu_h\, dh = 0,$$

$$\int \Big[\int_{\mathbb{R}^2} u(b\cdot\nabla\varphi)\, d\kappa_h + \int_{\mathbb{R}^2} \varphi\, d\nu_h\Big]\, d\zeta(h) = 0,$$

$$\int\int_{\mathbb{R}^2} \varphi\, d\lambda_h\, d\lambda^s(h) = 0,$$

which give, respectively, (3.5), (3.6) and (3.4).

□

3.4. **Reduction on connected components of level sets.** If $K \subset \mathbb{R}^d$ is a compact then, in general, not any connected component C of K can be separated from $K \setminus C$ by a smooth function. However, it can be separated by a sequence of such functions:

Lemma 3.6 ([1, Section 2.8], [7, Lemma 5.3]). *If $K \subset \mathbb{R}^d$ is compact then for any connected component C of K there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that*

(1) $0 \le \phi_n \le 1$ on $\mathbb{R}^d$ and $\phi_n \in \{0,1\}$ on K for all $n \in \mathbb{N}$;
(2) for any $x \in C$, we have $\phi_n(x) = 1$ for every $n \in \mathbb{N}$;
(3) for any $x \in K \setminus C$, we have $\phi_n(x) \to 0$ as $n \to +\infty$;
(4) for any $n \in \mathbb{N}$, we have $\operatorname{supp} \nabla \phi_n \cap K = \emptyset$.

With the aid of this lemma we can now study the equation (3.5) on the nontrivial connected components of the level sets. In view of Lemma 3.5 in what follows we always assume that $h \notin N$ (see (3.1)).

Lemma 3.7. *The equation (3.5) holds iff*

- *for any nontrivial connected component C of E_h it holds*

$$\operatorname{div}\left(uc_h b \mathscr{H}^1 \llcorner C\right) + \operatorname{div}(ub\sigma_h \llcorner C) = \mu_h \llcorner C; \tag{3.9}$$

- *it holds*

$$\operatorname{div}(ub\sigma_h \llcorner (E_h \setminus E_h^\star)) = \mu_h \llcorner (E_h \setminus E_h^\star). \tag{3.10}$$

Proof. For any Borel set $A \subset \mathbb{R}^2$ we introduce the following functional

$$\Lambda_A(\psi) := \int_A uc_h (b\cdot\nabla\psi)\, d\mathscr{H}^1 \llcorner E_h + \int_A u(b\cdot\nabla\psi)\, d\sigma_h + \int_A \psi\, d\mu_h,$$

for all $\psi \in C_c^\infty(B)$.

Now fix a connected component C of E_h and take a sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ given by Lemma 3.6 (applied with $K := E_h$). By assumption, we have that $\Lambda(\psi\phi_n) = 0$ for every $\psi \in C_c^\infty(B)$ and for every n. Let us pass to the limit as $n \to \infty$.

On one hand we have

$$\int \psi\phi_n\, d\mu_h = \int_C \psi\, d\mu + \int_{E_h\setminus C} \psi\phi_n\, d\mu \to \int_C \psi\, d\mu$$

because the second term converges to 0 since $\phi_n \to 0$ pointwise on $E_h \setminus C$.

On the other hand $\nabla(\psi\phi_n) = \psi\nabla\phi_n + \phi_n\nabla\psi$. In the terms with $\phi_n\nabla\psi$ we pass to the limit as above. The terms with the product $\psi\nabla\phi_n$ identically vanish thanks to the condition (4) on ϕ_n in Lemma 3.6. Therefore, we have that for every $\psi \in C_c^\infty(B)$

$$\Lambda_{E_h}(\psi\phi_n) \to \int_C uc_h(b\cdot\nabla\psi)\, d\mathscr{H}^1 + \int_C u(b\cdot\nabla\psi)\, d\sigma_h + \int_C \psi\, d\mu_h = \Lambda_C(\psi),$$

as $n \to +\infty$. Since $\Lambda_{E_h}(\psi\phi_n) = 0$ for every n, we deduce that $\Lambda_C(\psi) = 0$ and this gives (3.9).

In order to get (3.10), it is enough to observe that $E_h^\star$ is a countable union of connected components C, therefore (from the previous step) we deduce that

$$\int_{E_h^\star} uc_h(b\cdot\nabla\psi)\,d\mathscr{H}^1+\int_{E_h^\star} u(b\cdot\nabla\psi)\,d\sigma_h+\int_{E_h^\star}\psi\,d\mu_h=0,\qquad \forall\psi\in C_c^\infty(B).$$

Hence

$$\Lambda_{E_h\setminus E_h^\star}:=\int_{E_h^\star\setminus E_h} uc_h(b\cdot\nabla\psi)\,d\mathscr{H}^1+\int_{E_h^\star\setminus E_h} u(b\cdot\nabla\psi)\,d\sigma_h+\int_{E_h^\star\setminus E_h}\psi\,d\mu_h=0,$$

for every $\psi\in C_c^\infty(B)$. Remembering that $\mathscr{H}^1(E_h^\star\setminus E_h)=0$ by Theorem 3.1 we get (3.10) and this concludes the proof.

The converse implication can be easily obtained by summing the equations (3.9) and (3.10). □

Lemma 3.8. *Equation* (3.9) *holds iff*

$$\operatorname{div}\left(uc_h b\mathscr{H}^1 \llcorner C\right)=\mu_h\llcorner C, \tag{3.11a}$$

$$\operatorname{div}(ub\sigma_h\llcorner C)=0. \tag{3.11b}$$

The proof of Lemma 3.8 would be fairly easy in the case when γ is a straight line. Roughly saying, in this case (3.9) would read as

$$\int u(x)c_h(x)b(x)\psi'(x)\,dx+\int u(x)c_h(x)b(x)\psi'(x)\,d\sigma_h(x)+\int\psi(x)\,d\mu(x)=0,$$

$\psi\in C_0^\infty(\mathbb{R})$. Since σ_h is concentrated on a $\mathscr{L}^1$-negligible set S, any $\phi\in C_0^1$ can be approximated in C^0-norm with a sequence of C^1-functions ϕ_n having 0-derivative on S. Consequently, ϕ_n' converge to ϕ' weak* in L^∞ as $n\to\infty$. Then, substituting $\psi=\phi_n$ and passing to the limit as $n\to\infty$ we get

$$\int u(x)c_h(x)b(x)\phi'(x)\,dx+\int\phi(x)\,d\mu(x)=0.$$

Hence the only technicality here is to repeat this argument on a curve.

Before presenting the formal proof of Lemma 3.8 we would like to discuss the parametric version of the equation (3.11a).

Let $\gamma\colon I\to\mathbb{R}^2$ be an injective Lipschitz parametrization of C, where $I=\mathbb{R}/\ell\mathbb{Z}$ or $I=(0,\ell)$ for some $\ell>0$ is the domain of γ. In view of Remark 3.4) we can assume that the directions of b and $\nabla^\perp H$ agree $\mathscr{H}^1$-a.e. on C. So there exists a constant $\varpi\in\{+1,-1\}$ such that

$$\frac{b(\gamma(s))}{|b(\gamma(s))|}=\varpi\frac{\gamma'(s)}{|\gamma'(s)|} \tag{3.12}$$

for a.e. $s\in I$. We will say that γ is an *admissible parametrization* of C if $\varpi=+1$. In the rest of the text we will consider only admissible parametrizations of the connected components C.

Lemma 3.9. *Equation* (3.11a) *holds iff for any admissible parametrization γ of C*

$$\partial_s(\hat{u}\hat{c}_h|\hat{b}|)=\hat{\mu}_h \tag{3.13}$$

where $\gamma_\#\hat{\mu}_h=\mu_h\llcorner C$, $\hat{u}=u\circ\gamma$, $\hat{c}_h=c_h\circ\gamma$ and $\hat{b}=b\circ\gamma$.

In the proof of Lemma 3.9 we will use the following result:

Lemma 3.10 ([1, Section 7])**.** *Let $a \in L^1(I)$ and μ a Radon measure on I, where $I = \mathbb{R}/\ell\mathbb{Z}$ or $I = (0,\ell)$ for some $\ell > 0$. Suppose that $\gamma\colon I \to \Omega$ is an injective Lipschitz function such that $\gamma' \neq 0$ a.e. on I and $\gamma(0,\ell) \subset \Omega$. Consider the functional*

$$\Lambda(\phi) := \int_I \phi' a\, dt + \int_I \phi\, d\mu, \qquad \forall \phi \in \mathrm{Lip}_c(I).$$

If $\Lambda(\varphi \circ \gamma) = 0$ for any $\varphi \in C_c^\infty(\Omega)$ then $\Lambda(\phi) = 0$ for any $\phi \in \mathrm{Lip}_c(I)$.

Proof of Lemma 3.9. Let us recall a corollary from Area formula: if $\gamma\colon I \to \mathbb{R}^2$ is an injective Lipschitz parametrization of C then

$$\mathscr{H}^1 \llcorner C = \gamma_\# \left(|\gamma'| \mathscr{L}^1 \right).$$

Using this formula the distributional version of (3.11a),

$$\int_C u c_h b \cdot \nabla \phi d\mathscr{H}^1 \llcorner C + \int_C \phi d\mu_h = 0, \quad \forall \phi \in C_c^\infty(B),$$

can be written as

$$\int_I u(\gamma(s)) c_h(\gamma(s)) b(\gamma(s)) \cdot (\nabla\phi)(\gamma(s)) |\gamma'(s)|\, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0$$

where $\hat{\mu}_h$ is defined by $\hat{\mu}_h := \left(\gamma^{-1}\right)_\# \mu_h$.

Using (3.12) we can write the equation above as

$$\int_I u(\gamma(s)) c_h(\gamma(s)) \gamma'(s) (\nabla\phi)(\gamma(s)) |b(\gamma(s))|\, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0,$$

which reads as

$$\int_I \hat{u}(s) \hat{c_h}(s) \partial_s \phi(\gamma(s)) |\hat{b}(s)|\, ds + \int_I \phi(\gamma(s)) d\hat{\mu}_h(s) = 0.$$

Since the equation above holds for any $\phi \in C_c^\infty(B)$ it remains to apply Lemma 3.10. ☐

Proof of Lemma 3.8. Let us write $\Lambda(\phi) = M(\phi) + N(\phi)$, where

$$M(\phi) := \int_C u c_h (b \cdot \nabla \phi)\, d\mathscr{H}^1 + \int_C \phi\, d\mu_h$$

and

$$N(\phi) := \int_C u b \cdot \nabla \phi d\sigma_h$$

for every $\phi \in C_c^\infty(B)$.

Fix a test function ϕ: we are going to “perturb” ϕ in such a way that $N(\phi)$ becomes arbitrarily small and $M(\phi)$ remains almost unchanged. Since $\Lambda(\phi) = 0$ we will obtain that $|M(\phi)| < \varepsilon$ and this will imply that $M(\phi) = N(\phi) = 0$.

By Lemma 3.2, we have $\sigma_h \perp \mathscr{H}^1 \llcorner C$ therefore there exists a $\mathscr{H}^1$-negligible set $S \subset C$ such that σ_h is concentrated on S. Moreover, by inner regularity, for every $n \in \mathbb{N}$, we can find a compact $K \subset S$ such that

$$\sigma_h(S \setminus K) < \frac{1}{n}.$$

Using the fact that $\mathscr{H}^1(K) = 0$, for every $n \in \mathbb{N}$, we can find countably many open balls $\{B_{r_j}(z_j)\}_{j\in\mathbb{N}}$ which cover K and whose radii r_j satisfy

$$\sum_{j\in\mathbb{N}} r_j < \frac{1}{n}.$$

Furthermore, by compactness, we can extract from $\{B_{r_j}(z_j)\}_{l\in\mathbb{N}}$ a finite subcovering, $\{B_{r_j}(z_j)\}$ with $j = 1, \dots, \nu$ where $\nu = \nu(n) \in \mathbb{N}$ (we stress that ν depends on n).

For every $j \in \{1, \dots, \nu\}$, let

$$P_i^{j,n} := (z_{j,i} - r_j, z_{j,i} + r_j)$$

denote the projection of $B_{r_j}(z_j)$ onto the x_i-axis, with $i = 1, 2$. Since $P_i^{j,n}$ is an open interval we can find a smooth function $\psi_i^{j,n} \colon \mathbb{R} \to \mathbb{R}$ such that

$$\psi_i^{j,n}(\xi) = \begin{cases} 0 & \xi \in P_i^{j,n}, \\ 1 & \operatorname{dist}(\xi, \partial P_i^{j,n}) > 2r_i, \end{cases}$$

and $0 \le \psi_i^{j,n} \le 1$ for every $\xi \in \mathbb{R}$. Now we consider the product $\psi_i^n := \psi_i^{1,n}\psi_i^{2,n} \cdots \psi_i^{\nu,n}$ and we define the functions $\chi_i^n \colon \mathbb{R} \to \mathbb{R}$ as

$$\chi_i^n(\xi) := \int_0^\xi \psi_i^n(w)\, dw$$

for $i = 1, 2$ and $n \in \mathbb{N}$. Now we set $\chi^n(x) := (\chi_1^n(x), \chi_2^n(x))$ and $\phi_n := \phi \circ \chi^n$. Since $\|\chi^n - \mathrm{id}\|_\infty \le 4\sum_i r_i \le \frac{4}{n}$ we deduce that $\phi_n \to \phi$ uniformly in C because

$$|\phi_n(x) - \phi(x)| \le \|\nabla\phi\|_\infty \|\chi^n - \mathrm{id}\|_\infty \to 0$$

as $n \to +\infty$.

Let us now take an admissible parametrization of C, $\gamma \colon I \to \mathbb{R}$, and let us introduce the functions $\hat{\phi}_n := \phi_n \circ \gamma$. Using for instance the density of C^1 functions in $L^1(I)$, we can actually show that $\partial_s\hat{\phi}_n \rightharpoonup^\star \partial_s\hat{\phi}$ in weak* topology of L^∞. Passing to the parametrization as in the proof of Lemma 3.9 we get

$$\int_C u c_h (b \cdot \nabla\phi_n)\, d\mathscr{H}^1 = \int_I \hat{u}\hat{c}_h\hat{b}\, \partial_s\hat{\phi}_n\, ds,$$

where we denote by $\hat{\cdot}$ the composition with γ.

Using weak* convergence, we obtain that

$$\int_C u c_h (b \cdot \nabla\phi_n)\, d\mathscr{H}^1 \to \int_C u c_h (b \cdot \nabla\phi)\, d\mathscr{H}^1.$$

On the other hand, by uniform convergence, we immediately get

$$\int \phi_n\, d\mu_h \to \int \phi\, d\mu_h,$$

as $n \to +\infty$. In particular, we have that $M(\phi_n) \to M(\phi)$.

Now observe that $\nabla\phi_n = 0$ on K by construction, hence we get

$$N(\phi_n) \le \int_{S\setminus K} |ub||\nabla\phi_n| d\sigma_h \le \|ub\|_\infty \|\nabla\phi\|_\infty \frac{1}{n} \to 0$$

and this implies that $N(\phi) = 0$. Therefore, $0 = \Lambda(\phi) = M(\phi)$, which concludes the proof. □

We note, in particular, that from (3.11b), being $b \in \mathrm{BV}$ and taking $u \equiv 1$ in (3.3), we have that $\operatorname{div}(b\sigma_h \llcorner E_h) = 0$ for a.e. h.

Let

$$F := \{b \neq 0, r_B = 0\} \cap E. \tag{3.14}$$

By Point (5) of Lemma 3.2, σ_h is concentrated on $F \cap E_h$ hence we have

$$\operatorname{div}(\mathbb{1}_F b\sigma_h) = 0, \qquad \text{for } \mathscr{L}^1\text{-a.e. } h. \tag{3.15}$$

This important piece of information is very useful to prove the following

Lemma 3.11. *We have* $\operatorname{div}(\mathbb{1}_F b) = 0$ *in* $\mathscr{D}'(B)$*.*

Proof. For every test function $\phi \in C_c^\infty(B)$, we have

$$\int_F (b(x)\nabla\phi(x))\,dx = \int\int_{F\cap E_h} (b(x)\cdot\nabla\phi(x))\,d\sigma_h(x)dh.$$

Using again Point (5) of Lemma 3.2 and (3.15), we get that

$$\int_{F\cap E_h} (b(x)\nabla\phi(x))\,d\sigma_h(x) = 0$$

and then we conclude. □

Finally, let us mention a covering property of the set $E^\star$:

Lemma 3.12. *Let* $E^\star$ *be the set defined in* (3.1)*. Then*

$$E^\star \supset \{\nabla H \neq 0\} \mod \mathscr{L}^2.$$

Proof. Suppose that $P := \{\nabla H \neq 0\} \setminus E$ has positive measure. Then

$$0 < \int_P |\nabla H|\,dx = \int\int \mathbb{1}_P\,d\mathscr{H}^1 \llcorner E_h\,dh = 0$$

where the first equality is due to Coarea formula (Lemma 1.7) and the second equality holds since $\mathbb{1}_P$ is zero on E_h for a.e. h. □

Note that in general $E^\star$ can contain a subset of $\{\nabla H = 0\}$ with positive measure (see [1]). However, in the next section we show that, if H has the so-called *weak Sard property*, then in fact $E^\star = \{\nabla H \neq 0\} \mod \mathscr{L}^2$.

4. Weak Sard Property of Hamiltonians

4.1. **Matching properties.** As we have seen at the beginning of Section 3.2, to every Hamiltonian H we can associate a triple (H, N, E) where N is the set given by Theorem 3.1 and $E = \cup_{h\notin N} E_h^\star$.

Suppose now we have another triple $(\widetilde{H}, \widetilde{N}, \widetilde{E})$; we ask whether, given $x \in E \cap \widetilde{E}$ it is true that $C_x = \widetilde{C}_x$. This is essentially the definition of matching property; moreover, we will prove the "Matching Lemma", which states that gradients of H and $\widetilde{H}$ being parallel (in a simply connected set) is a sufficient condition for matching.

4.2. **Matching of two Hamiltonians.** Let us consider two Lipschitz Hamiltonians H_1 and H_2, defined on the same open, simply connected set A; according to Theorem 3.1, we have two negligible sets N_1 and N_2 such that the level sets E^1_h and $E^2_{h'}$ of H_1 and H_2 are regular for $h \notin N_1$ and $h' \notin N_2$. We set $E_1 := \cup_{h \notin N_1} E^1_h$ and $E_2 := \cup_{h' \notin N_2} E^2_{h'}$.

Definition 4.1. The Hamiltonians H_1 and H_2 *match* in an open subset $A' \subset A$ if $C^1_x = C^2_x$ for $\mathscr{L}^2$-a.e. $x \in A' \cap E_1 \cap E_2$, where C^i_x denotes the connected component in A' of the level sets $H_i^{-1}(H_i(x))$ which contains x.

As usual, given two vectors a and b in $\mathbb{R}^2$ we write $a \parallel b$ if $a = \alpha b$ or $b = \alpha a$ for some real number α.

We now state and prove the following

Lemma 4.2 (Matching lemma). *Let H_1, H_2 be defined as above. If $\nabla H_1 \parallel \nabla H_2$ a.e. on $A' \subset A$ open, then the Hamiltonians H_1 and H_2 match in A'.*

Proof. Let $b_1 := \nabla^\perp H_1$. Then $\operatorname{div} b_1 = 0$. Let us prove that

$$\operatorname{div}(H_2 b_1) = 0 \tag{4.1}$$

in the sense of distributions. Indeed, we have for every $\varphi \in \mathrm{Lip}_c(A')$

$$\int H_2 (b_1 \cdot \nabla\varphi)\, dx = \int \left[b_1 \cdot \nabla(H_2\varphi) - \varphi(b_1 \cdot \nabla H_2)\right] dx.$$

The first term is zero because $\operatorname{div} b_1 = 0$ (and φH_2 can be used as test function since it is Lipschitz); the second term is also zero because $\nabla H_2 \parallel \nabla H_1$ a.e. on A', hence $b_1 \perp \nabla H_2$ a.e. on A'.

From (4.1), using [7, Theorem 4.1 and 6.1], we obtain that there exists a $\mathscr{L}^1$ negligible set N such that H_2 is constant on every non trivial connected components $C \cap A'$ of the level sets of H_1 which do not correspond to values in N. By disintegration, we have that the sets of points $x \in A' \cap E_1$ such that $H_1(x) \notin N$ are a negligible set and therefore we can infer that for a.e. $x \in A' \cap E_1$, H_2 is constant along the connected components in A' of the level sets of H_1. By repeating the same argument for H_2 we get the claim. □

4.3. **The Weak Sard property.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function and let S be the critical set of f, defined as the set of all $x \in \mathbb{R}^2$ where f is not differentiable or $\nabla f(x) = 0$. We are interested in the following property: *the push-forward according to f of the restriction of $\mathscr{L}^2$ to S is singular with respect to $\mathscr{L}^1$, that is*

$$f_\# \left(\mathscr{L}^2 \llcorner S \right) \perp \mathscr{L}^1.$$

This property clearly implies the following *Weak Sard Property*, which is used in [2, Section 2.13]:

$$f_\# \left(\mathscr{L}^2 \llcorner (S \cap E^\star) \right) \perp \mathscr{L}^1,$$

where the set $E^\star$ is the union of all connected components with positive length of all level sets of f. We point out that the relevance of the Weak Sard Property in the framework of transport and continuity equation is explained is [2, Theorem 4.7].

Remark 4.3. Informally, the weak Sard property means that the "good" level sets of H do not intersect the critical set S, apart from a negligible set. In terms of the disintegration of the Lebesgue measure (3.2), we can say that H has the weak Sard property if and only if $\sigma_h = 0$ for a.e. h.

Now we give the following

Definition 4.4. We set

$$\widetilde{r}_B := r_B + \mathbb{1}_F,$$

where we recall that r_B is the function defined in (2.4) and F is the set defined in (3.14).

By linearity of divergence, by Lemma 2.3 and Lemma 3.11, we have

$$\operatorname{div}(\widetilde{r}_B b) = 0$$

in $\mathscr{D}'(B)$. Therefore, we conclude that there exists a Lipschitz potential $\widetilde{H}$ such that $\nabla \widetilde{H}^\perp = \widetilde{r}_B b$.

Moreover, we observe that $\nabla H \parallel \nabla \widetilde{H}$ a.e. in B: therefore we can apply Matching Lemma 4.2 to get that the regular level sets of H and of $\widetilde{H}$ agree. In particular, we obtain $E = \widetilde{E} \mod \mathscr{L}^2$, directly from the definition of $\widetilde{H}$. We note also that the function $\widetilde{H}$ has the Weak Sard property: indeed, directly from the construction, we have $\nabla \widetilde{H} \neq 0$ on E hence, since $E = \widetilde{E}$ mod $\mathscr{L}^2$, it follows that $\mathscr{L}^2(\widetilde{E} \cap \widetilde{S}) = 0$.

Finally, disintegrating $\mathscr{L}^2 \llcorner E$ with respect to H we get

$$\mathscr{L}^2 \llcorner E = \int_{\mathbb{R}} (c_h \mathscr{H}^1 \llcorner E_h + \sigma_h)\, dh,$$

while using the Hamiltonian $\widetilde{H}$

$$\mathscr{L}^2 \llcorner E = \int_{\mathbb{R}} \widetilde{c}_h \mathscr{H}^1 \llcorner \widetilde{E}_h\, dh.$$

In particular, it follows that $\sigma_h = 0$ for a.e. h, which means that $H = \widetilde{H}$ (up to additive constants) and H has the Weak Sard Property.

We collect this result in the following

Lemma 4.5. *The Hamiltonian H_B has the weak Sard property.*

We conclude this section with the following corollary concerning the covering properties of the set $E^\star$ defined in (3.1):

Corollary 4.6. *Suppose that H has the weak Sard property. Let $E^\star$ be the set defined in (3.1). Then*

$$E^\star = \{\nabla H \neq 0\} \mod \mathscr{L}^2.$$

Proof. The argument is similar to Lemma 3.12. Let $Q = E^\star \setminus \{\nabla H \neq 0\}$. By (3.2)

$$\mathscr{L}^2(Q) = \int \left(\int_Q d\sigma_h \right) dh = 0$$

since by Remark 4.3 $\sigma_h = 0$ for a.e. h. □

Remark 4.7. If we do not assume BV regularity of b, but $b(x) \neq 0$ for $\mathscr{L}^2$-a.e. $x \in \mathbb{R}^2$ the conclusion of Lemma 4.5 still holds. This can be proved using minor modifications of the above argument. More precisely, since b is nearly incompressible the function $m(x) := \int_0^T \rho(\tau, x)\, d\tau$, where ρ is the density of b, solves

$$\operatorname{div}(mb) = \rho(T, \cdot) - \rho(0, \cdot) \tag{4.2}$$

in $\mathscr{D}'(B)$, being $\rho(T,\cdot)$ and $\rho(0,\cdot)$ the weak-$\star$ limits in L^∞ of $\rho(t,\cdot)$ as $t \to T$ and $t \to 0$ respectively. Applying Lemmas 3.5, 3.7, 3.8 with $u = m$, from (3.11b) we obtain

$$\operatorname{div}(mb\sigma_h \llcorner C) = 0. \tag{4.3}$$

Hence Lemma 3.11 holds replacing $\mathbb{1}_F b$ with $m\mathbb{1}_F b$: in particular, setting

$$\widetilde{r}_B := r_B + m\mathbb{1}_F$$

we can repeat the argument of Section 4.

5. Level sets and trajectories I

In this section, we assume that H_B is defined on all $\mathbb{R}^2$ (using standard theorems for the extension of Lipschitz maps).

5.1. **Trajectories.** We now present some lemmas which relate the trajectories $\gamma \in \mathsf{T}_B$ to the level sets of the Hamiltonian. The first result we prove is that η-a.e. γ is contained in a level set.

Lemma 5.1. *Let $B \in \mathscr{B}$, $t_1, t_2 \in [0,T]$ and set $\mathsf{T} := \{\gamma : \gamma((t_1,t_2)) \subset B\}$. Then η-a.e. $\gamma \in \mathsf{T}$ we have $(t_1,t_2) \ni t \mapsto H(\gamma(t))$ is a constant function.*

Proof. Let $(\varrho_\varepsilon)_\varepsilon$ be the standard family of convolution kernels in $\mathbb{R}^2$. We set $H_\varepsilon(x) := H \star \varrho_\varepsilon(x)$ for any $x \in B$.

For every $t \in [t_1, t_2]$ define

$$I(t) := \int_{\mathsf{T}} |H(\gamma(t)) - H(\gamma(0))| d\eta(\gamma)$$

and we will prove $I \equiv 0$.

First note that I is positive because the integrand is non-negative and η is positive. On the other hand,

$$I(t) \leq \underbrace{\int_{\mathsf{T}} |H(\gamma(t)) - H_\varepsilon(\gamma(t))| d\eta(\gamma)}_{I_1^\varepsilon} + \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H_\varepsilon(\gamma(0))| d\eta(\gamma)}_{I_2^\varepsilon} + \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(0)) - H(\gamma(0))| d\eta(\gamma)}_{I_3^\varepsilon}.$$

Now for a.e. $x \in \mathbb{R}^2$ we have $H_\varepsilon(x) \to H(x)$: hence

$$\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H(\gamma(t))|\, d\eta(\gamma) \leq \int_B |H_\varepsilon(x) - H(x)| \rho(t,x) dx \to 0$$

as $\varepsilon \to 0$. Therefore, we can infer that

$$I_1^\varepsilon \to 0, \qquad I_3^\varepsilon \to 0$$

as $\varepsilon \downarrow 0$.

Let us study I_2^ε. We have

$$\begin{aligned}
I_2^\varepsilon(t) &\le \int_{\mathsf{T}} \int_{t_1}^t |\partial_s H_\varepsilon(\gamma(s))| \, ds \, d\eta(\gamma) \\
&= \int_{\mathsf{T}} \int_{t_1}^t |\nabla H_\varepsilon(\gamma(s)) \cdot b(\gamma(s))| \, ds \, d\eta(\gamma) \\
&= \int_{t_1}^t \int |\nabla H_\varepsilon(x) \cdot b(x)| \, d(e_t \# \eta \llcorner \mathsf{T})(x) \, ds \\
&\le \int_0^T \int |\nabla H_\varepsilon(x) \cdot b(x)| \rho_{\mathsf{T}}(t,x) \, dx \, ds \\
&= \int |\nabla H_\varepsilon(x) \cdot b(x)| r_{\mathsf{T}}(x) \, dx \to \int |\nabla H(x) \cdot b(x)| r_{\mathsf{T}}(x) \, dx = 0
\end{aligned}$$

where we have used $\nabla H_\varepsilon(x) \to \nabla H(x)$ for a.e. x. In the end, we have that $I_2^\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and this concludes the proof. □

We now show that Lemma 5.1 can be improved, showing indeed that η_B-a.e. γ is contained in a *regular* level set of H.

Lemma 5.2. *Up to a η_B negligible set, the image of every $\gamma \in \mathsf{T}_B$ is contained in a connected component of a* regular *level set of H_B.*

Proof. Using Lemma 5.1, we remove η_B-negligible set of trajectories along which H_B is not constant. Set $E^c := B \setminus E$ and consider the set

$$\mathscr{P} := \{\gamma \in \mathsf{T}_B : \gamma((0,T)) \cap B \subset E^c\}.$$

It is enough to show that $\eta(\mathscr{P}) = 0$: this means that for η-a.e. γ the image $\gamma(0,T)$ is not contained in the complement of E and thus we must have (in the ball) $\gamma(0,T) \subset E$ for η-a.e. $\gamma \in \mathsf{T}_B$ (this follows remembering that a.e. γ is contained in a level set).

By Coarea formula (see Lemma 3.2), $|\nabla H| \mathscr{L}^2 \llcorner E^c = 0$, i.e.

$$\int \mathbb{1}_{E^c}(x) |\nabla H(x)| \, dx = 0.$$

Since $\nabla H = r_B b^\perp$ in B and $r_B \ge 0$ (since $\rho_B > 0$), we have

$$\begin{aligned}
0 &= \int \mathbb{1}_{E^c}(x) |r_B(x) b(x)| \, dx \\
&= \int \mathbb{1}_{E^c}(x) r_B(x) |b(x)| \, dx \\
&= \int \int_0^T \mathbb{1}_{E^c}(x) \rho_B(t,x) |b(x)| \, dx \, dt.
\end{aligned}$$

Using (2.3) we have

$$0 = \int_0^T \int \mathbb{1}_{E^c}(\gamma(t)) |b(\gamma(t))| \, d\eta(\gamma) \, dt = \int_0^T \int_{\mathscr{P}} |b(\gamma(t))| \, d\eta(\gamma) \, dt$$

which implies (by Fubini) that for η-a.e. $\gamma \in \mathscr{P}$ we have

$$\int_0^T |b(\gamma(t))| \, dt = 0.$$

This gives $|b(\gamma(t))| = 0$ for a.e. $t \in [0, T]$ and this contradicts the definition of T_B. Hence $\eta(\mathscr{P}) = 0$. $\square$

6. Locality of the divergence

In this section we prove that the if $\operatorname{div}(ub)$ is a measure, then it is 0 on the set

$$M := \Big\{ x \in \mathbb{R}^2 : b(x) = 0,\ x \in \mathcal{D}_b \text{ and } \nabla^{\mathrm{appr}} b(x) = 0 \Big\}, \tag{6.1}$$

where $\mathcal{D}_b$ is the set of approximate differentiability points and $\nabla^{\mathrm{appr}} b$ is the approximate differential, according to Definition [6, Def. 3.70]. For shortness, we will call this property *locality of the divergence*.

Let U be an open set in $\mathbb{R}^d$, $d \in \mathbb{N}$. The main result of this section is the following

Proposition 6.1. *$u \in L^\infty(U)$ and suppose that $\operatorname{div}(ub) = \lambda$ in the sense of distributions, where λ is a Radon measure on U. Then $|\lambda| \llcorner M = 0$.*

Note that we do not assume any weak differentiability of u or ub, so the conclusion of Proposition 6.1 does not follow immediately from the standard locality properties of the approximate derivative (see e.g. [6], Proposition 3.73). Moreover, we also mention a related counterexample (contained in [1]), where the authors construct a bounded vector field V on the plane whose (distributional) divergence belongs to L^∞, is non-trivial, and is supported in the set where V vanishes. Our proof is based on Besicovitch-Vitali covering Lemma ([6, Thm. 2.19]) and uses some basic facts about the trace properties of L^∞ vector fields whose divergence is a measure (we refer to [10, 5] or [12]). In particular, we recall the following Theorem (for the proof, see [12, Prop 7.10]):

Theorem 6.2 (Fubini's Theorem for traces)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and $B \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ be a vector field whose distributional divergence $\operatorname{div} B =: \mu$ is a Radon measure with locally finite variation in Ω. Let $F \in C^1(\Omega)$. Then for a.e. $t \in \mathbb{R}$ we have*

$$Tr(B, \partial\{F > t\}) = B \cdot \nu \qquad \mathscr{H}^{d-1}\text{-a.e. on } \Omega \cap \partial\{F > t\}, \tag{6.2}$$

where ν denotes the exterior unit normal to $\partial\{F > t\}$ and the distribution $Tr(B, \partial\Omega')$ is defined by

$$\langle Tr(B, \partial\Omega'), \phi \rangle := \int_{\Omega'} \phi \, d\mu + \int_{\Omega'} \nabla\phi \cdot B \, dx, \qquad \forall \phi \in C^\infty_c(\Omega).$$

for every open subset $\Omega' \subset \Omega$ with C^1 boundary.

Furthermore, we will use the following elementary

Lemma 6.3. *Let $G \colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, Borel function. For every $r > 0$ there exists a set of positive measure of real numbers $s = s(r) \in [r, 2r]$ such that*

$$\int_{\partial B_{s(r)}} |G(x)| \, d\mathscr{H}^{d-1}(x) \le \frac{1}{r} \int_{B_{2r}} |G(y)| \, dy.$$

Proof of Proposition 6.1. Fix an arbitrary $x \in M$. For brevity let $B_r := B_r(x)$. By (6.2) with $F(y) := |x-y|^2$, there exists an $\mathscr{L}^1$-negligible set N_x such that for any positive number $r \notin N_x$ we have

$$|\lambda(B_r)| = \left| \int_{\partial B_r} ub \cdot \nu \, d\mathscr{H}^{d-1} \right| \le C \int_{\partial B_r} |b| \, d\mathscr{H}^{d-1},$$

where ν denotes the exterior unit normal to ∂B_r. By Lemma 6.3

$$C \int_{\partial B_r} |b| \, d\mathscr{H}^{d-1} \le \frac{C}{r} \int_{B_{2r}} |b(x)| \, dx = o(r^d)$$

because, by definition of M, we have $⨍_{B_r} |b| \, dx = o(r)$. Therefore

$$|\lambda(B_r)| = o(r^d). \tag{6.3}$$

Fix $\varepsilon > 0$. By (6.3) for any $x \in M$ there exists $\delta_x > 0$ such that for any positive number $r < \delta_x$ such that $r \notin N_x$ we have

$$|\lambda(B_r(x))| \le \varepsilon r^d. \tag{6.4}$$

Let $S \subset M$ be an arbitrary bounded subset.

By regularity of λ, there exists a bounded open set $O \supset S$ such that $|\lambda|(O \setminus S) < \varepsilon$. Hence, for any $x \in S$ there exists $\rho_x > 0$ such that $B(x,r) \subset O$ for any positive number $r < \rho_x$. Consequently

$$\mathscr{F} := \{B(x,r) : x \in S, r < \min(\rho_x, \delta_x), r \notin N_x\}$$

is a fine covering of S.

Hence we can apply Besicovitch-Vitali covering Lemma ([6, Thm. 2.19]): there exists a countable disjoint subfamily $\{B_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ such that

$$|\lambda| \left(S \setminus \bigcup_i B_i \right) = 0.$$

On the other hand, since $\bigcup_i B_i \subset O$ by construction, we have

$$|\lambda| \left(\bigcup_i B_i \setminus S \right) < \varepsilon.$$

Using (6.4), since the balls B_i are disjoint, we have

$$\lambda \left(\bigcup_i B_i \right) = \sum_i \lambda(B_i) \le \varepsilon \mathscr{L}^2 \left(\bigcup_i B_i \right).$$

Hence

$$\lambda(S) = \lambda \left(\bigcup_i B_i \right) - \lambda \left(\bigcup_i B_i \setminus S \right) \to 0$$

as $\varepsilon \downarrow 0$. Hence $\lambda \llcorner S = 0$ and, by arbitrariness of $S \subset M$, $\lambda \llcorner M = 0$. $\square$

6.1. **Comparison between $\mathscr{L}^2$ and η.** We present here two general lemmas which relate the Lebesgue measure $\mathscr{L}^2$ and the measure η and are based on nearly incompressibility of the vector field b.

Lemma 6.4. *Let $A \subset \mathbb{R}^2$ be a measurable set. Then $\mathscr{L}^2(A) = 0$ if and only if $\eta(\Gamma_A) = 0$ where*

$$\Gamma_A := \Big\{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\Big\}.$$

Proof. Let us prove first that $\mathscr{L}^2(A) = 0$ implies $\eta(\Gamma_A) = 0$. We denote by ρ_A the density such that $\rho_A(t,\cdot)\mathscr{L}^2 = e_{t\#}(\eta \llcorner \Gamma_A)$ and $r_A(x) := \int_0^T \rho_A(t,x)\,dt$. We have, using Fubini,

$$\begin{aligned}
0 = \mathscr{L}^2(A) = r_A\mathscr{L}^2(A) &= \int_0^T \int_\Gamma \mathbb{1}_A(x)\rho_A(t,x)\,dx\,dt \\
&= \int_0^T \int_\Gamma \mathbb{1}_A(\gamma(t))\,d\eta(\gamma)\,dt \\
&= \int_\Gamma \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \mathscr{L}^1(\{t \in [0,T] :\, \gamma(t) \in A\})\,d\eta(\gamma),
\end{aligned}$$

hence, $\mathscr{L}^1(\{t \in [0,T] :\, \gamma(t) \in A\}) = 0$ for η-a.e. $\gamma \in \Gamma_A$.

For the opposite direction, using that ρ is uniformly bounded from below by $1/C$, we get

$$\begin{aligned}
\frac{T}{C}\mathscr{L}^2(A) = \frac{T}{C}\int \mathbb{1}_A(x)\,dx &= \frac{1}{C}\int_0^T \int \mathbb{1}_A(x)\,dx\,dt \\
&\le \int_0^T \int \mathbb{1}_A(x)\rho(t,x)\,dx\,dt \\
&= \int_0^T \int_\Gamma \mathbb{1}_A(\gamma(t))\,d\eta(\gamma)\,dt \\
&= \int_\Gamma \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \mathscr{L}^1(\{t \in [0,T] :\, \gamma(t) \in A\})\,d\eta(\gamma) = 0.
\end{aligned}$$ □

Lemma 6.5. *We have $\mathscr{L}^2(A) = 0$ if and only if $\eta(\Gamma_A^s) = 0$ for every $s \in [0,T]$.*

Proof. For direct implication

$$\begin{aligned} 0 = \mathscr{L}^2(A) &= \int \mathbb{1}_A(x)\rho(s,x)\,dx \\ &= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\ &= \int_{\Gamma^s_A} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma^s_A). \end{aligned}$$

For the opposite direction,

$$\begin{aligned} \frac{1}{C}\mathscr{L}^2(A) &= \frac{1}{C}\int \mathbb{1}_A(x)\,dx \\ &\le \int \mathbb{1}_A(x)\rho(s,x)\,dx \\ &= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\ &= \int_{\Gamma^s_A} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma^s_A) = 0. \end{aligned}$$

□

We now recall the set M, defined in (6.1) as

$$M := \Big\{x \in \mathbb{R}^2 : b(x) = 0,\ x \in \mathcal{D}_b \text{ and } \nabla^{\text{appr}} b(x) = 0\Big\},$$

and we consider the sets

$$\widetilde{\Gamma}_M := \widetilde{\Gamma} \cap \Gamma_M$$

and

$$\widetilde{\Gamma}^s_M := \Big\{\gamma \in \widetilde{\Gamma} : \gamma(s) \in M\Big\}.$$

Using Proposition 6.1, we can show the following

Lemma 6.6. *Let M be the set defined in (6.1) and for every fixed $s \in [0,T]$ let $\widetilde{\Gamma}^s_M := \{\gamma \in \widetilde{\Gamma} \colon \gamma(s) \in M\}$. Then:*

- $\eta(\widetilde{\Gamma}^s_M) = 0$ *for a.e.* $s \in [0,T]$*;*
- $\eta(\widetilde{\Gamma}_M) = 0$.

Proof. Let us denote by $\eta^s_M := \eta \llcorner \widetilde{\Gamma}^s_M$ and consider the Borel function

$$\rho^s_M(t,\cdot)\mathscr{L}^2 = e_{t\#}\eta^s_M.$$

It is easy to see that ρ^s_M solves continuity equation

$$\partial_t \rho^s_M + \operatorname{div}(\rho^s_M b) = 0. \tag{6.5}$$

Integrating in time on $[0,t]$ we get

$$\operatorname{div}\left(b\int_0^t \rho^s_M(\tau,\cdot)d\tau\right) = (\rho^s_M(t,\cdot) - \rho^s_M(0,\cdot))\mathscr{L}^2.$$

In particular, thanks to Proposition 6.1, we have that

$$(\rho^s_M(t,\cdot) - \rho^s_M(0,\cdot))\mathscr{L}^2 \llcorner M = 0, \tag{6.6}$$

hence $\rho^s_M(t,\cdot) = \rho^s_M(0,\cdot)$, for a.e. x. Furthermore, integrating in space the continuity equation (6.5) we get the conservation of mass:

$$\frac{d}{dt}\int_{\mathbb{R}^2} \rho^s_M(t,x)\,dx = 0. \tag{6.7}$$

Therefore, using (6.6) and (6.7), we have

$$\begin{aligned}\int_{\mathbb{R}^2\setminus M} \rho^s_M(t,x)dx &= \int_{\mathbb{R}^2} \rho^s_M(t,x)dx - \int_M \rho^s_M(t,x)dx = \\ &= \int_{\mathbb{R}^2} \rho^s_M(s,x)dx - \int_M \rho^s_M(s,x)dx = \int_{\mathbb{R}^2\setminus M} \rho^s_M(s,x)dx = \\ &= \int \mathbb{1}_{\mathbb{R}^2\setminus M}(\gamma(s))d\eta_M(\gamma) = 0,\end{aligned}$$

which gives us $\rho^s_M(t,\cdot) = 0$ a.e. on $\mathbb{R}^2 \setminus M$. Hence

$$0 = \int_0^T \int_{\mathbb{R}^2\setminus M} \rho^s_M(t,x)\,dx = \int_0^T \int 1_{\mathbb{R}^2\setminus M}(\gamma(t))\,d\eta^s_M(\gamma)\,dt$$

and this implies that $\eta^s_M(\widetilde{\Gamma}^s_M) = 0$ for $s \in [0,T]$, since $\gamma \in \widetilde{\Gamma}$ are not constant functions (by definition) and $b = 0$ on M.

Now the second part easily follows from the first one by a Fubini-like argument: indeed, we set

$$I := \int_0^T \eta(\widetilde{\Gamma}^s_M)\,ds = 0.$$

Since $\eta(\widetilde{\Gamma}^s_M) = \int_{\widetilde{\Gamma}} 1_M(\gamma(s))\,d\eta(\gamma)$ and using Fubini's theorem we get

$$I = \int_{\widetilde{\Gamma}} \int_0^T 1_M(\gamma(s))\,ds\,d\eta(\gamma) = 0$$

i.e. $\mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in M\}) = 0$ for η-a.e. $\gamma \in \widetilde{\Gamma}_M$ and this concludes the proof. □

7. Level sets and trajectories II

The results obtained in the Section 6 provide us with a better description of the relantionship between the trajectories $\gamma \in \Gamma_B$ and the level sets of H_B, thus improving the results of Section 5.

7.1. Trajectories and level sets coincide up to a translation in time. Let $B \in \mathscr{B}$ a fixed ball of the collection and, as usual, let H_B denote its Hamiltonian. Thanks to Lemma 5.2, there exists a η-negligible set N such that for every $\gamma \in \Gamma_B \setminus N$ the image $\gamma(0,T)$ is contained in a connected component $\mathfrak{c}$ of a regular level set of H_B.

Recalling [1, Theorem 2.5(iv)], there exists a parametrization $\gamma_\mathfrak{c}$ of $\mathfrak{c}$ with the following properties:

- $\gamma_\mathfrak{c} \colon I_\mathfrak{c} \to \mathbb{R}^2$ is a Lipschitz map, where $I_\mathfrak{c} = \mathbb{R}/\ell\mathbb{Z}$ or $I_\mathfrak{c} = [0,\ell]$ for some $\ell > 0$ is the domain of γ;
- $\gamma_\mathfrak{c}$ is injective;
- $\gamma_\mathfrak{c}'(s) = b(\gamma_\mathfrak{c}(s))$ for $\mathscr{L}^1$-a.e. $s \in I_\mathfrak{c}$.

Thus it makes sense to wonder about the relationship between the trajectory $\gamma \in \Gamma_B \setminus N$ and the parametrization $\gamma_{\mathfrak{c}}$ of the corresponding connected component. The following proposition precises this relation, showing that γ and $\gamma_{\mathfrak{c}}$ coincide up to a translation in time.

Proposition 7.1. *Let N be the set given by Lemma 5.2 and $\gamma \in \widetilde{\Gamma} \setminus N$. Then (a suitable restriction of) γ coincides with $\gamma_{\mathfrak{c}}$ up to a translation in time.*

In order to prove Proposition 7.1, we need the following auxiliary

Lemma 7.2. *Let $\gamma \colon I \to \mathbb{R}^2$ be a solution of the ordinary differential equation*

$$\gamma'(t) = b(\gamma(t)), \qquad t \in I \subset \mathbb{R},$$

where $I = [0,T]$ and $\frac{1}{|b|} \in L^1_{\rm loc}(\mathscr{H}^1 \llcorner \gamma(I))$. Assume that there exists a injective curve $\hat\gamma$ defined on I such that $\gamma(I) \subset \hat\gamma(I)$ and that $\dot{\hat\gamma} = b(\hat\gamma)$. Then

$$\int_{\gamma([0,T])} \frac{d\mathscr{H}^1(w)}{|b(w)|} = T - \mathscr{L}^1(\{t \in [0,T] : \gamma'(t) = 0\}).$$

Proof. Observe that

$$\begin{aligned}\int_{\gamma([0,T])} \frac{d\mathscr{H}^1(w)}{|b(w)|} &\overset{(1)}{=} \int_{\gamma([0,T])} \frac{\mathbb{1}_{\{b\neq 0\}}(w)\, d\mathscr{H}^1(w)}{|b(w)|} \\ &\overset{(2)}{=} \int_{\{t\in[0,T]:\, \gamma'(t)\neq 0\}} \frac{|\gamma'(\tau)|}{|b(\gamma(\tau))|} d\tau \\ &= T - \mathscr{L}^1(\{t \in [0,T] : \gamma'(t) = 0\}),\end{aligned}$$

where

(1) follows by definition;
(2) is the Area formula, i.e. $\mathscr{H}^1 \llcorner C = \gamma_\#(|\gamma'|\mathscr{L}^1)$, where $C = \gamma((0,T))$, which can be applied because there exists $\hat\gamma$ by hypothesis.

This concludes the proof. □

Now we can prove Proposition 7.1.

Proof. Let $\bar t \in [0,T]$ such that $\gamma_{\mathfrak{c}}(0) = \gamma(\bar t)$. By Lemma 7.2, we have that for any s in a suitable subinterval of $[0,T]$ it holds

$$\int_{\gamma\left([\bar t, \bar t + s]\right)} \frac{d\mathscr{H}^1(w)}{|b(w)|} = (\bar t + s) - \bar t - \mathscr{L}^1([\bar t, \bar t + s] \cap \gamma^{-1}(\{b = 0\})). \tag{7.1}$$

By Lemma 6.6 and the fact that $\mathscr{L}^2(\{b = 0\} \setminus M) = 0$, where M is defined in (6.1), we know that for η-a.e. $\gamma \in \widetilde{\Gamma}$,

$$\mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in \{b = 0\}\}) = 0,$$

hence (7.1) is actually

$$\int_{\gamma\left([\bar t, \bar t + s]\right)} \frac{d\mathscr{H}^1(w)}{|b(w)|} = s. \tag{7.2}$$

On the other hand, applying again Lemma 7.2 to $\gamma_{\mathfrak{c}}$, which is injective, we get

$$\int_{\gamma_{\mathfrak{c}}(0,s)} \frac{d\mathscr{H}^1(w)}{|b(w)|} = s. \tag{7.3}$$

Since, by definition, $\gamma_{\mathfrak{c}}(0) = \gamma(\bar{t})$, comparing (7.2) and (7.3) and using the fact that $|b| > 0$ $\mathscr{H}^1$-a.e. on γ, we deduce that

$$\gamma(\bar{t} + s) = \gamma_{\mathfrak{c}}(s)$$

which means that γ (restricted to a suitable time subinterval of $[0,T]$) and $\gamma_{\mathfrak{c}}$ coincide up to a translation in time. □

7.2. **Covering property of the regular level sets.** Let us recall that for each ball $B \in \mathscr{B}$ and for any rational numbers $s,t \in \mathbb{Q} \cap (0,T)$ with $s < t$ we have set

$$\mathsf{T}_{B,s,t} := \{\gamma \in \Gamma_B : \gamma(s) \notin B, \gamma(t) \notin B\}.$$

Remark 7.3. In the same way as in Remark 2.2, we can easily see that

$$\bigcup_{\substack{B\in\mathscr{B} \\ s,t\in\mathbb{Q}\cap[0,T]}} \mathsf{T}_{B,s,t} = \widetilde{\Gamma}. \tag{7.4}$$

For each $B \in \mathscr{B}$, $s \in \mathbb{Q} \cap (0,T)$, $t \in \mathbb{Q} \cap (s,T)$ restricting η to $\mathsf{T}_{B,s,t}$, we can construct the local Hamiltonian $H_{B,s,t}$ as in Sections 2.2-3.2.

We now set

$$\hat{E} := \bigcup_{\substack{B\in\mathscr{B} \\ s,t\in\mathbb{Q}\cap[0,T]}} E^{\star}_{B,s,t}. \tag{7.5}$$

The following covering property is a global analog of Lemma 3.12:

Lemma 7.4. *It holds that* $\hat{E} \supset \{b \neq 0\} \mod \mathscr{L}^2$.

Proof. Let $P := \{b \neq 0\} \setminus \hat{E}$. Then for any $B \in \mathscr{B}$ it holds that $P \subset \{\nabla H_B = 0\} \mod \mathscr{L}^2$. Since $b \neq 0$ on P and $\nabla H^{\perp} = r_B b$ it holds that $r_B = 0$ a.e. on P for all $B \in \mathscr{B}$. Then for any $B \in \mathscr{B}$

$$\begin{aligned} 0 &= \int_{P\cap B} r_B \, dx \\ &= \int_0^T \int \mathbb{1}_{P\cap B}(x) \rho_B(t,x) \, dx \, dt \\ &= \int_{\widetilde{\Gamma}} \int_0^T \mathbb{1}_{P\cap B}(\gamma(t)) \, d\eta(\gamma) \, dt, \end{aligned}$$

hence η-a.e. $\gamma \in \widetilde{\Gamma}$ spends zero amount of time in $P \cap B$. Since B is arbitrary and $\mathscr{B}$ is countable, we can generalize this claim to the whole set P:

$$\int_{\widetilde{\Gamma}} \int_0^T \mathbb{1}_P(\gamma(t)) \, dt \, d\eta(\gamma) = 0. \tag{7.6}$$

By nearly incompressibility

$$\begin{aligned}
\mathscr{L}^2(P) &\leq C\int_0^T\int \mathbb{1}_P(x)\rho(t,x)\,dx\,dt \\
&= C\int_0^T\int_{\dot\Gamma\cup\widetilde\Gamma} \mathbb{1}_P(\gamma(t))\,d\eta(\gamma)\,dt \\
&\overset{(*)}{=} C\int_0^T\int_{\dot\Gamma} \mathbb{1}_P(\gamma(t))\,d\eta(\gamma)\,dt \\
&\overset{(**)}{=} C\int_0^T\int_{\dot\Gamma} \mathbb{1}_P(\gamma(t))\mathbb{1}_{\{b=0\}}(\gamma(t))\,d\eta(\gamma)\,dt \\
&\leq C\int_0^T\int \mathbb{1}_P(\gamma(t))\mathbb{1}_{\{b=0\}}(\gamma(t))\,d\eta(\gamma)\,dt \\
&\leq C\int_0^T\int \mathbb{1}_P(x)\mathbb{1}_{\{b=0\}}(x)\rho(t,x)\,dx\,dt \\
&\overset{(***)}{=} 0,
\end{aligned}$$

where

- (*) holds by (7.6);
- (**) holds because $\mathbb{1}_{\{b=0\}}(\gamma(t)) = 1$ for any $t \in [0,T]$ and any $\gamma \in \dot\Gamma$: indeed, for any $\gamma \in \dot\Gamma$ which is an integral curve of b we have $0 = \gamma'(t) = b(\gamma(t))$, hence $\gamma(t) \in \{b = 0\}$;
- (***) holds because P and $\{b = 0\}$ are disjoint. □

In view of Corollary 4.6 the proof above actually leads to a stronger statement:

Lemma 7.5. $\hat E = \{b \neq 0\} \mod \mathscr{L}^2$.

8. Solution of the transport equation on integral curves

8.1. **Splitting on the level sets of the time-dependent problem.** We now present the time-dependent version of Lemmas 3.5-3.7-3.8-3.9.

Lemma 8.1. *Fix a ball $B \in \mathscr{B}$ and the corresponding Hamiltonian H_B. Let $v \in L^\infty([0,T]\times B)$ be a solution to the problem*

$$\begin{cases} v_t + \operatorname{div}(vb) = 0, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times B) \tag{8.1}$$

Then $\hat v(t,s) := v(t,\gamma(s))$ solves

$$\begin{cases} \partial_t(\hat v \hat c_h|\hat b|) + \partial_s(\hat v \hat c_h|\hat b|) = 0, \\ \hat v(0,\cdot) = \hat v_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times I).$$

for $\mathscr{L}^1$-a.e. h, where $\gamma\colon I \to \mathbb{R}^2$ is an admissible parametrization of a connected component of the level set E_h of the Hamiltonian H_B.

Proof. Multiplying by a function $\psi \in C_c^\infty([0,T))$ and formally integrating by parts we get

$$v_t\psi + \operatorname{div}(v\psi b) = \psi\nu \Rightarrow \operatorname{div}\left(\int_0^T v\psi\,dt\,b\right) = \int_0^T v\psi_t\,dt - \psi(0)v_0,$$

i.e.

$$\operatorname{div}(wb) = \mu,$$

where $w := \int_0^T v\psi \, dt$ and

$$\mu := \left(\int_0^T v\psi_t \, dt - \psi(0) v_0 \right) \mathscr{L}^2.$$

Applying Lemma 3.5, we obtain that continuity equation implies

$$\operatorname{div}\left(w c_h b \mathscr{H}^1 \llcorner E_h\right) = \mu_h \quad \text{in } \mathscr{D}'(\mathbb{R}^2) \text{ for } \mathscr{L}^1\text{-a.e. } h \in \mathbb{R}. \tag{8.2}$$

The measure μ_h can be computed explicitly, using Coarea Formula:

$$\mu_h = \left(\int_0^T v\psi_t \, dt - \psi(0) v_0 \right) \mathscr{H}^1 \llcorner E_h.$$

Thanks to Lemma 3.9, equation (8.2) is *equivalent* to

$$\partial_s(\hat{v} \hat{c}_h |\hat{b}|) = \hat{\mu}_h,$$

in $\mathscr{D}'((0,T) \times I)$. Now being γ_h Lipschitz and injective, we have

$$(\gamma_h^{-1})_\# (\mathscr{H}^1 \llcorner E_h) = |\gamma_h'| \mathscr{L}^1,$$

and this allows us to compute explicitly

$$\begin{aligned}
\hat{\mu}_h &= (\gamma_h^{-1})_\# \mu_h \\
&= (\gamma_h^{-1})_\# \left(\int_0^T v\psi_t \, dt \, c_h \mathscr{H}^1 \llcorner E_h - \int_{\mathbb{R}^2} \psi(0) v_0 c_h d\mathscr{H}^1 \llcorner F_h \right) \\
&= \int_0^T v(\tau, \gamma(s)) \psi_\tau(\tau) c_h(\gamma_h(s)) |b(\gamma_h(s))| \, d\tau - \psi(0) v_0(\gamma_h(s)) c_h(\gamma(s)),
\end{aligned}$$

which formally means

$$\hat{\mu}_h = -\int_0^T \partial_t(\hat{v} |\hat{b}| \hat{c}_h).$$

To sum up, we have obtained that Problem (8.13) implies that

$$\begin{cases} \partial_t(\hat{v} \hat{c}_h |\hat{b}|) + \partial_s(\hat{v} \hat{c}_h |\hat{b}|) = 0, \\ \hat{v}(0, \cdot) = \hat{v}_0(\cdot), \end{cases}$$

in $\mathscr{D}'((0,T) \times I)$ for $\mathscr{L}^1$-a.e. $h \in \mathbb{R}$. □

Lemma 8.2. *Fix $\sigma \in \mathbb{Q} \cap (0,T)$, $\theta \in \mathbb{Q} \cap (\sigma, T)$ and $B \in \mathscr{B}$. Let $H := H_{B,\sigma,\theta}$. Let $u \in L^\infty([0,T] \times \mathbb{R}^2)$ be a ρ-weak solution of the problem*

$$\begin{cases} u_t + b \cdot \nabla u = 0, \\ u(0, \cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2).$$

Then there exists a negligible set $Z = Z_{B,\sigma,\theta} \subset \mathbb{R}$ such that

- *for any $h \in Z$ the level set $E_h := H^{-1}(h)$ is regular;*
- *if $h \notin Z$ and E_h is regular then for any nontrivial connected component C of E_h with admissible parametrization $\gamma_C \colon I \to \mathbb{R}^2$, any $t \in (0,T)$ and any $s \in I$ there exists a constant w such that*

$$u(t + \xi, \gamma_C(s + \xi)) = w \tag{8.3}$$

for a.e. $\xi \in \mathbb{R}$ *such that* $s+\xi \in I$ *and* $t+\xi \in (0,T)$.
In particular, for any $s \in I$ *it holds that*

$$u(\xi, \gamma_C(s+\xi)) = u_0(s) \tag{8.4}$$

for a.e. $\xi \in \mathbb{R}$ *such that* $s+\xi \in I$.

Proof. Setting $v := u\rho \in L^\infty([0,T]\times\mathbb{R}^2)$ and $v_0(\cdot) = u_0(\cdot)\rho(0,\cdot)$, by definition of ρ-weak solution we have

$$\begin{cases} v_t + \operatorname{div}(vb) = 0, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{R}^2). \tag{8.5}$$

Hence we can apply Lemma 8.1 in B to get

$$\begin{cases} \partial_t(\hat v \hat c_h |\hat b|) + \partial_s(\hat v \hat c_h |\hat b|) = 0, \\ \hat v(0,\cdot) = \hat v_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times I). \tag{8.6}$$

for all $h \in H(B)\setminus N_1$, where $\mathscr{L}^1(N_1) = 0$.

From (8.6) it immediately follows that the function

$$\xi \mapsto \Big(\hat\rho \hat u \hat c_h |\hat b|\Big)(t+\xi, s+\xi) \tag{8.7}$$

is equal a.e. to some constant w_1.

Applying the same argument to the problem

$$\begin{cases} \rho_t + \operatorname{div}(\rho b) = 0, \\ \rho(0,\cdot) = \rho_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{R}^2), \tag{8.8}$$

(which holds thanks to nearly incompressibility assumption) we obtain a negligible set N_2 such that for all $h \in H(B)\setminus N_2$, for any connected component of E_h the map

$$\xi \mapsto \Big(\hat\rho \hat c_h |\hat b|\Big)(t+\xi, s+\xi) \tag{8.9}$$

is equal a.e. to some constant w_2.

Let $N := N_1 \cup N_2$ and fix $h \notin N$.

Comparing (8.10) and (8.8), using that $\rho c_h |b| > 0$ $\mathscr{H}^1$-a.e. on E_h (for a.e. h), we obtain that

$$\xi \mapsto \Big(\hat v \hat c_h |\hat b|\Big)(t+\xi, s+\xi) \tag{8.10}$$

is equal a.e. to the constant $w = w_1/w_2$ for a.e. $h \notin N$. □

8.2. **Selection of appropriate trajectories.**

Lemma 8.3. *There exists an* η*-negligible set* $N \subset \Gamma$ *such that any integral curve* $\gamma \in \widetilde\Gamma \setminus N$ *of the vector field* b *has the following properties:*

(1) *for any* $B \in \mathscr{B}$, *if* $\gamma \in \mathsf{T}_{B,s,t}$ *then each connected component of* $\gamma([s,t]) \cap B$ *is contained in a regular level set of* H_B;
(2) *for any* $\tau \in (0,T)$ *there exist a ball* $B \in \mathscr{B}$, $s \in \mathbb{Q}\cap(0,T)$ *and* $t \in \mathbb{Q}\cap(\tau,T)$ *such that* $\gamma \in \mathsf{T}_{B,s,t}$.

Proof. First of all, using Lemma 6.6 we can remove a negligible set of integral curves of b which stay in the set $\{b = 0\}$ for a positive amount of time. Applying Lemmas 5.1 and 5.2 countably many times (for each ball $B \in \mathscr{B}$

and all rationals $s \in \mathbb{Q} \cap (0,T)$ and $t \in \mathbb{Q} \cap (s,T)$) we obtain the set $N \subset \Gamma$ such that the first property holds.

Next, for any $\tau \in (0,T)$ there exists $s \in \mathbb{Q} \cap (0,\tau)$ such that $\gamma(s) \neq \gamma(\tau)$. (Otherwise, since γ is an integral curve of b, it would have to stay in $\{b = 0\}$ for a positive amount of time). Similarly there exists $t \in (s,T)$ such that $\gamma(t) \neq \gamma(\tau)$. Then for any ball $B \in \mathscr{B}$ with sufficiently small radius, containing $\gamma(\tau)$ and not containing $\gamma(s)$ and $\gamma(t)$ it clearly holds that $\gamma \in T_{B,s,t}$. □

Lemma 8.4. *Let $Z_{B,s,t}$ denote negligible set given by Lemma 8.2. Then for η-a.e. $\gamma \in \widetilde{\Gamma}$ it holds that*

$$H_{B,s,t}(\gamma([0,T])) \cap Z_{B,s,t} = \emptyset. \tag{8.11}$$

Proof. Set $A := H_{B,s,t}^{-1}(Z_{B,s,t})$: by Coarea Formula, $\mathscr{L}^2(A) = 0$. Applying Lemma 6.4 we deduce that

$$\eta\Big(\Big\{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\Big\} \Big) = 0.$$

On the other hand $b \neq 0$ a.e. on $E_{B,s,t}$, hence

$$\begin{aligned} &\Big\{\gamma \in \widetilde{\Gamma} \setminus N :\, \gamma([0,T]) \cap E_{B,s,t} \subset E_h,\, h \in Z\Big\} \\ &= \Big\{\gamma \in \widetilde{\Gamma} \setminus N :\, \gamma([0,T]) \cap E_{B,s,t} \subset A\Big\} \\ &\subset \Big\{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\Big\}. \end{aligned}$$ □

From the Lemma 8.4 it does not follow immediately that the endpoints $\gamma(0)$ and $\gamma(T)$ are contained in regular level sets of some Hamiltonians. But now we are going to establish this property. Being $Z_{B,s,t}$ given by Lemma 8.2, let $\widetilde{E}_{B,s,t} := E_{B,s,t} \setminus H_{B,s,t}^{-1}(Z_{B,s,t})$ and

$$\widetilde{E} := \bigcup_{\substack{B\in\mathscr{B},\\ s,t\in\mathbb{Q}\cap(0,T):\, s<t}} \widetilde{E}_{B,s,t}. \tag{8.12}$$

Note that since $\widetilde{E}_{B,s,t} = E_{B,s,t} \mod \mathscr{L}^2$ (by Coarea formula), it follows that $\widetilde{E} = \hat{E} \mod \mathscr{L}^2$.

The following lemma shows that η-a.e. nontrivial trajectory of b starts from the set $\widetilde{E}$ (and also stops in $\widetilde{E}$):

Lemma 8.5. *For η-a.e. $\gamma \in \widetilde{\Gamma}$ it holds that $\gamma(0) \in \widetilde{E}$ and $\gamma(T) \in \widetilde{E}$.*

Proof. Consider the set X of $\eta \in \widetilde{\Gamma}$ such that $\gamma(0) \notin \widetilde{E}$. By Lemma 7.5 it holds that $b = 0$ a.e. on the complement of $\widetilde{E}$. Hence by Lemma 6.6 we have $\eta(X) = 0$. The argument for $\gamma(T)$ is similar. □

In the lemmas above we have been removing η-negligible sets of trajectories of b. Let us summarize some properties of the remaining ones:

Lemma 8.6. *There exists a η-negligible set $R \subset \widetilde{\Gamma}$ such that for any $\tau \in [0,T]$ and any $\gamma \in \widetilde{\Gamma} \setminus R$ there exist $s \in \mathbb{Q} \cap (0,T)$, $t \in \mathbb{Q} \cap (s,T)$ and $B \in \mathscr{B}$ such that $\gamma(\tau) \in \widetilde{E}_{B,s,t}$.*

Proof. We define R as the union of η-negligible sets given by Lemmas 8.3, 8.4 and 8.5. If $\tau \in (0,T)$ the claim follows from Lemma 8.3 since we can always find s and t such that $\tau \in (s,t)$ and the desired property holds. If $\tau = 0$ or $\tau = T$ then the result follows from Lemma 8.5. □

Corollary 8.7. *For any $\gamma \in \widetilde{\Gamma} \setminus R$ and any $\tau \in [0,T]$ there exists $\delta > 0$ and a constant w such that the function $\xi \mapsto u(\xi, \gamma(\xi))$ is equal to w for a.e. $\xi \in (\tau - \delta, \tau + \delta) \cap [0,T]$. Moreover, if $\tau = 0$ then the constant w is equal to $u_0(\gamma(0))$.*

Proof. The result follows directly from Lemma 8.6, Proposition 7.1 and Lemma 8.2. □

8.3. **Solutions are constant along η-a.e. trajectory.** Now we are in a position to recover the method of characteristics in our weak setting:

Lemma 8.8. *Suppose that b is a bounded, autonomous,* BV *compactly supported, nearly incompressible (with density ρ) vector field on $\mathbb{R}^2$ and let $u \in L^\infty([0,T] \times \mathbb{R}^2)$ be a ρ-weak solution of the problem*

$$\begin{cases} u_t + b \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2). \tag{8.13}$$

Then for η-a.e. $\gamma \in \Gamma$ for a.e. $t \in [0,T]$ it holds that

$$u(t, \gamma(t)) = u_0(\gamma(0)).$$

Proof. It is clear that the thesis holds for any $\gamma \in \dot{\Gamma}$. Indeed, by Proposition 6.1

$$\partial_t(\rho u \mathbb{1}_M) = 0$$

in $\mathscr{D}'$, where the set M is defined in (6.1).

Hence it is sufficient to consider only the moving trajectories, i.e. $\gamma \in \widetilde{\Gamma}$. Let R be the set given by Lemma 8.6. Let $\gamma \in \widetilde{\Gamma} \setminus R$. By Corollary 8.7 for any $\tau \in [0,T]$ there exists $\delta > 0$ such that the function $t \mapsto u(t, \gamma(t))$ is equal to some constant w_τ for a.e. $t \in (\tau - \delta, \tau + \delta) \cap [0,T]$. Moreover, if $\tau = 0$ then $w_\tau = u_0(\gamma(0))$. It remains to extract a finite covering of $[0,T]$. □

The following lemma is elementary, we prove it for sake of completeness.

Lemma 8.9. *Let $u \in L^\infty([0,T] \times \mathbb{R}^2)$. If for η-a.e. γ and a.e. $t \in [0,T]$ it holds that $u(t, \gamma(t)) = u_0(\gamma(t))$, then u solves the transport equation with the initial condition u_0, i.e.*

$$\begin{cases} u_t + b \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

Proof. Let $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^2)$ be a smooth test function which vanishes at T. Then

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} (\rho u \varphi_t + \rho u b \nabla \varphi)\, dx\, dt + \int_{\mathbb{R}^2} \rho(0,x) u_0(x) \varphi(0,x)\, dx \\
&= \int_0^T \int_\Gamma u(t,\gamma(t)) \partial_t \varphi(t,\gamma(t))\, d\eta(\gamma)\, dt + \int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) \\
&= \int_0^T \int_\Gamma u_0(\gamma(0)) \partial_t \varphi(t,\gamma(t))\, d\eta(\gamma)\, dt + \int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) \\
&= -\int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) + \int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0))\, d\eta(\gamma) = 0. \qquad \square
\end{aligned}$$

9. Renormalization: proof of the Main Theorem

We are finally ready to state and prove the main result of the paper, which is the following

Theorem 9.1. *Every bounded, autonomous, compactly supported and nearly incompressible* BV *vector field on* $\mathbb{R}^2$ *has the renormalization property.*

Proof. Let $u \in L^\infty([0,T] \times \mathbb{R}^2)$ be a solution of

$$\begin{cases} u_t + b \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2).$$

By Lemma 8.8 the function $t \mapsto u(t,\gamma(t))$ is constant for η-a.e. γ. Then for any $\beta \in C^1(\mathbb{R},\mathbb{R})$ the function $t \mapsto \beta(u(t,\gamma(t)))$ is constant for η-a.e. γ. Hence by Lemma 8.9 the function $\beta(u)$ is a solution of

$$\begin{cases} (\beta(u))_t + b \cdot \nabla \beta(u) = 0, \\ \beta(u)(0,\cdot) = \beta(u_0)(\cdot). \end{cases}$$

This concludes the proof. $\square$

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