Entire solutions of completely coercive quasilinear elliptic equations, II

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Abstract

In an earlier paper [3] the authors treated a broad class of quasilinear elliptic equations which have the property that any entire solution must necessarily be constant, a property of course not holding for the simple Laplace equation itself. Here we generalize the earlier class of equations to include cases where the "inhomogeneous terms" depend strongly on the gradient of the solution; see for example the model p-Laplace–type equation (2) below, as well as other more general examples discussed later.

Theorems 8 and 9 are particularly interesting in that, in contrast to the earlier conclusions, they require only the most minimal coercive behavior of the inhomogeneous terms when the solution variable lies in some arbitrarily large but bounded set; see especially the model example (15) at the end of the introduction.

1. Introduction.

We shall study entire solutions of quasilinear elliptic equations of the form

$$\operatorname{div} \mathcal{A}(x, u, Du) = \mathcal{B}(x, u, Du) \tag{1}$$

and also of the corresponding inequality

$$\operatorname{div} \mathcal{A}(x, u, Du) \geq \mathcal{B}(x, u, Du) \tag{1'}$$

under various coercive conditions on the vector-valued function $\mathcal{A}$ and the scalar function $\mathcal{B}$. These conditions, in particular, constitute significant extensions of those in our earlier paper [3].

The simplest typical example of the type of equations which we shall consider, though far from the more general ones which are treated later, is the equation

$$\Delta_p u = |u|^{q-1}\, u\, |Du|^\ell \tag{2}$$

with $p > 1$, $q > 0$, $\ell \ge 0$. When $q > p - \ell - 1$ and $\ell < p - 1$ it was shown by Filippucci, Pucci and Rigoli [5], see also Filippucci [4], that the only *non-negative* entire solutions of (2) are the identically constant functions. Here, as a consequence of our main conclusions, we strengthen this result to apply to solutions which are unrestricted in sign, moreover with the condition $\ell < p - 1$ weakened to the form

$$\ell < p - 1 + \frac{p-1}{n-1}, \tag{2'}$$

see the end of Section 3.

For the main results of the paper, we shall assume that the following general coercive (weak ellipticity) conditions hold,

$$\begin{aligned} &\mathcal{A}(x,z,\rho)\cdot\rho \ge 0, \qquad &&\mathcal{B}(x,z,\rho)z \ge 0,\\ &\mathcal{A}(x,z,0) = 0, &&\mathcal{B}(x,z,0) = 0, \end{aligned} \tag{3}$$

for all $x \in \mathbf{R}^n$, $z \in \mathbf{R}$ and $\rho \in \mathbf{R}^n$, together with the property that

$$\begin{cases} \text{If} \\ \qquad\qquad \mathcal{A}(x,z,\rho)\cdot\rho + \mathcal{B}(x,z,\rho)z = 0 \\ \text{at some point } (x,z,\rho) \in \mathbf{R}^n \times (\mathbf{R}\backslash\{0\}) \times \mathbf{R}^n, \text{ then } \rho \text{ must be } 0. \end{cases} \tag{4}$$

Further conditions on the quantities $\mathcal{A}$ and $\mathcal{B}$ will be needed only for large values of x. Specifically, we shall require the following "large radii conditions", that there exists an exponent $p > 1$ such that for all $|x| \ge R_0$, $z \in \mathbf{R}\backslash\{0\}$, $\rho \in \mathbf{R}^n$ one has the relations

$$|\mathcal{A}(x,z,\rho)|^p \le C_{\mathcal{A}}|x|^s|z|^r[\mathcal{A}(x,z,\rho)\cdot\rho]^{p-1} \tag{5}$$

and

$$\mathcal{B}(x,z,\rho)\ \operatorname{sign} z \ge C_{\mathcal{B}}|x|^{-t}|z|^q\,|\rho|^\ell, \tag{6}$$

where $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ are positive constants, and $q > 0$, $\ell \ge 0$, $r \ge 0$, s, $t \in \mathbf{R}$.

Note. If $\ell = 0$ the condition $\mathcal{B}(x,z,0) = 0$ in (3) must be replaced by $\mathcal{B}(x,0,0) = 0$.

Remark. The structural condition (5) is very general and the authors have encountered it before only in the earlier paper [3]. Indeed, to the best of our knowledge previously existing results related to the problem under consideration (see [4],[5] and the literature cited therein) deal almost exclusively with equations whose principal part is either the p-Laplacian or the mean curvature operator or variants thereof.

We recall as well that condition (6) was first introduced by Martio and Porru [6] and by Filippucci [4]. The case $\ell = 0$ of (6) was treated in [3].

Equation (2) arises as the special case $\mathcal{A} = |\rho|^{p-2}\rho$, $\mathcal{B} = |z|^{q-1} z|\rho|^{\ell}$, with the particular parameters $r = s = t = 0$. A more general model of interest is

$$\operatorname{div}\left[A(x,u,Du)\,|Du|^{p-2}Du\right] = b(x,u,Du)|u|^{q-1}\,u|Du|^{\ell} \tag{7}$$

where $p > 1$, $q > 0$, $\ell \geq 0$, and $A(x,z,\rho)$, $b(x,z,\rho)$ are non-negative measurable functions such that

$$A(x,z,\rho) \leq \text{Const.}\,|x|^{s}|z|^{r}, \quad b(x,z,\rho) \geq \text{Pos. Const.}\,|x|^{-t} \tag{7$'$}$$

for $|x| \geq R_0$, $z \neq 0$, $\rho \in \mathbf{R}^n$, with $r \geq 0$, s, $t \in \mathbf{R}$. In writing (6) and (7), and in later work, we define $|u|^{q-1}u$ to vanish at all points where $u = 0$.

A further model of importance is the equation

$$\operatorname{div}\left\{A(x)\frac{Du}{\sqrt{1+|Du|^2}}\right\} = b(x)|u|^{q-1}u\,|Du|^{\ell}, \tag{8}$$

with A, b again satisfying (7$'$), a case covered by Theorem 4 in Section 4.

In the sequel, unless otherwise explicitly mentioned, let (3), (4), (5), (6) hold, where $p > 1$ and where, $q > 0$, $\ell \geq 0$, $r \geq 0$, s, $t \in \mathbf{R}$, and define

$$\theta = p + r - 1.$$

Our first set of main results can now be stated. To begin with, note that the results cannot be entirely simple, in view of the large number of parameters present, e.g., ℓ, n, p, q, r, s, t. Nevertheless, it is exactly the four combinations $\theta = p + r - 1\,(>0)$ and

$$q + \ell, \quad s + n - p, \quad n - \ell - t$$

which play the main role in the conclusions, as might be expected in view of the form of the structural conditions (5),(6).

Theorem 1. *Assume that $p > 1$ and $q + \ell > \theta$ and that either*

$$s + t < p - \ell$$

or

$$(q+\ell)(s+n-p) - \theta(n-\ell-t) < 0.$$

If also

$$\ell \leq p - 1,$$

then any C^1 entire solution of (1) *is identically constant.*

Theorem 1′. *Under the hypotheses of Theorem* 1*, any* C^1 *entire solution of inequality* (1′) *is either identically constant or is non-positive.*

When $\ell = 0$ Theorems 1 and 1′ reduce to Theorems 1 and 2 of [3]. The first case of Theorem 1′ was obtained by Filippucci [4] for positive solutions of(1′), under a stronger version of condition (5) and assuming also $p < n$, $\ell < p-1$. It should also be observed that the hypotheses of the first case of Theorems 1 and 1′ are independent of the dimension n, *this case thus holding with no restrictions on the dimension.*

The next two results are more precise (strengthened) versions of Theorems 1 and 1′.

Theorem 2. *Let* $p > 1$*. Assume that*

$$(q+\ell)(s+n-p) - \theta(n-\ell-t) \geq 0 \tag{9}$$

and

$$\delta \equiv \frac{n-\ell-t}{s+n-p} > 1 \qquad (s+n-p>0). \tag{9'}$$

If

$$\ell < (p-1)\delta, \tag{10}$$

then any C^1 *entire solution of* (1) *is identically constant and any* C^1 *entire solution of the inequality* (1′) *must be either constant or non-positive.*

Theorem 3. *Let* $p > 1$*. Assume that*

$$(q+\ell)(s+n-p) - \theta(n-\ell-t) < 0 \tag{11}$$

and

$$\Delta \equiv \frac{q+\ell}{\theta} > 1. \tag{11'}$$

If

$$\ell < (p-1)\Delta, \tag{12}$$

then any C^1 *entire solution of* (1) *is identically constant and any* C^1 *entire solution of the inequality* (1′) *must be either constant or non-positive.*

Note that when $(q+\ell)(s+n-p) - \theta(n-\ell-t) = 0$ we have $\delta = \Delta$.

Corollary to Theorem 3. *Assume that* $q > 0$*,* $r = 0$*,* $q + \ell > p - 1$ *and*

$$(q+\ell)(s+n-p) - (p-1)(n-\ell-t) < 0. \tag{13}$$

Then any C^1 entire solution of (1) *is identically constant and any C^1 entire solution of the inequality* (1′) *must be either constant or non-positive.*

The corollary follows from the observation that (12) is automatic when $r = 0$ (recall that $q > 0$).

The final condition $u \equiv$ Constant in the above results can be improved to $u \equiv 0$ if we add to (6) the further condition

$$\mathcal{B}(x, z, 0) > 0 \qquad \text{when } z \neq 0.$$

The parameter values s, t for which Theorems 2 and 3 hold are shown in Figure 1.

Condition (6) is in fact more general than necessary, in that the term $|Du|^\ell$ can be replaced by the function $\sigma(Du)$ where

$$\sigma(\rho) = \begin{cases} |\rho|^{\ell_1} & \text{when } 0 \le |\rho| \le 1, \\ |\rho|^{\ell_2} & \text{when } |\rho| \ge 1, \end{cases} \tag{14}$$

with ℓ_1, ℓ_2 real parameters such that $\ell_1 \ge 0$. A condition similar to (14) was first introduced by Filippucci, Pucci and Rigoli [5], and is particularly important in that the case $\ell_2 < 0$ allows $\sigma(Du)$ to approach zero as $|Du|$ goes to ∞, which of course cannot happen with the function $|Du|^\ell$ when $\ell \ge 0$. See Section 5 for the relevant generalizations of Theorems 1 – 3 when $|Du|^\ell$ is replaced by $\sigma(Du)$.

Theorems 8 and 9 in Section 6 are particularly interesting in that, in contrast to the results given above, they require only the most minimal coercive conditions on the function $\mathcal{B}$ when z lies in a bounded set. For example, the case $r = s = t = 0$ of these theorems leads to the model equation

$$\Delta_p\, u = f(x, u)\ |Du|^\ell \operatorname{sign} u, \tag{15}$$

where f satisfies the following condition

$$\begin{aligned} &f(x,z) \text{ is continuous and positive when } |z| > 0 \textbf{ or } \text{when } |x| \le R_0, \\ &f(x,z) \ge |z|^q \qquad \text{when } |z| > d,\ |x| > R_0 \end{aligned}$$

in case $p \le n$, and the condition

$$\begin{aligned} &f(x,z) \ge 0 \qquad \text{when } |z| > 0 \textbf{ or } |x| \le R_0, \\ &f(x,z) \ge |z|^q \qquad \text{when } |z| > d,\ |x| > R_0 \end{aligned}$$

in case $p > n$.

Then, if

$$\begin{cases} q + \ell > p - 1, \quad \ell \le p - 1, & \text{when } p \le n, \\ q + \ell > p - 1, \quad \ell \le q\,\frac{p-n}{n-1} + (p-1)\,\frac{n}{n-1} & \text{when } p > n, \end{cases}$$

it follows that any C^1 entire distribution solution of (15) *must be identically constant* (*or must vanish if $p \le n$ and $\ell = 0$*).

This gives a significant generalization of a well-known theorem of Brezis [1], who considered the case $p = 2$, $d = \ell = R_0 = 0$ (see also the related Proposition 4.5 and Theorem 4.7 of [2], the main results and section 13 of [3] and Theorem 4.3 of [7]). An open question is whether in the case $p \le n$ the condition $\ell \le p - 1$ can be improved to the relation (2'), as is possible for equation (2).

In the final section of the paper, we obtain a new Liouville theorem,

Theorem 10. *Let $u = u(x)$ be a C^1 entire solution of the equation*

$$\operatorname{div} \mathcal{A}(x, u, Du) = 0,$$

where $\mathcal{A}$ satisfies conditions (3) *and* (4) *with $\mathcal{B} = 0$, and* (5) *with $n + s < p$. Assume that*

$$u(x) = O(|x|^k) \quad \text{as } |x| \to \infty \tag{16}$$

for some $k \in (0, \kappa)$, $\kappa = (p - n - s)/(p + r - 1)$.

Then u must be identically constant.

With the help of Theorem 10, one can also significantly improve Theorem B of [3], see Theorem 11 in Section 7.

Remarks. The results of the paper remain correct even when the large radii conditions (5), (6) do not hold for *all* large values of x. Indeed, as in [3] it is enough if the conditions are valid simply for a sequence of disjoint shells $R_i \le |x| \le \kappa R_i$, $\kappa = \text{const.} > 1$, where $\{R_i\}$ is an arbitrary sequence of radii tending to infinity as $i \to \infty$. For further details, see Section 6 of [3]. The delicate question of entire solutions in some Sobolev class, rather than of class C^1, is treated in Section 7 of [3].

2. Preliminaries.

We begin with several preliminary lemmas which will be of importance throughout the paper. First we make precise the meaning of a C^1 distribution solution $u = u(x)$ of (1), namely that

$$\int \{\mathcal{A}(x, u, Du) \cdot D\eta + \mathcal{B}(x, u, Du)\eta\} = 0 \tag{2.1}$$

for all functions $\eta \in C^1(\mathbf{R}^n)$ having compact support in $\mathbf{R}^n$. Naturally one must require further that the functions $\mathcal{A}(\cdot, u, Du)$, $\mathcal{B}(\cdot, u, Du)$ in (2.1) are locally integrable in $\mathbf{R}^n$. It is worth adding that, under these integrability conditions, if $u \in C^2$ is an almost everywhere ($\mathbf{R}^n$) classical solution of (1), then u is a distribution solution as well.

For the inequality ($1'$) the meaning of solution is the same, with the exception that equality in (2.1) is now replaced by $\le$ and the test function η must also be non-negative.

We suppose throughout the rest of the paper that conditions (3)-(6) are in force. Everything stands or falls, depending on the following lemma.

Lemma 2.1. *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality* $(1')$. *Then for every $\alpha > 0$, $\beta \geq 1$, $R \geq R_0 > 0$, and for every* compactly supported non-negative locally Lipschitz continuous test function φ, such that $\varphi \equiv 1$ for $|x| \leq R$, *we have*

(2.2)

$$\int_{B_R\cap\{u>0\}} [\alpha\mathcal{A}(x,u,Du)\cdot Du\ u^{\alpha-1} + \mathcal{B}(x,u,Du)u^{\alpha}]$$
$$\leq - \int_{(\mathbf{R}^n\setminus B_R)\cap\{u>0\}} \big[\alpha\,\mathcal{A}(x,u,Du)\cdot Du\ u^{\alpha-1}\varphi^{\beta}$$
$$+\beta\,\mathcal{A}(x,u,Du)\cdot D\varphi\, u^{\alpha}\varphi^{\beta-1} + \mathcal{B}(x,u,Du)u^{\alpha}\varphi^{\beta}\big]\,.$$

Proof. For $(1')$ we use the non-negative test function

$$\eta_\varepsilon = [u^+ + \varepsilon]^{\alpha}\varphi^{\beta}.$$

where $0 < \varepsilon < 1$. This is Lipschitz continuous in $\mathbf{R}^n$ so that, as is clear (trivial mollification), it can be used in the corresponding inequality version of (2.1). This gives

$$\int \mathcal{B}(x,u,Du)\eta_\varepsilon$$
$$\leq -\alpha\int \mathcal{A}(x,u,Du)\cdot Du^+\,[u^+ + \varepsilon]^{\alpha-1}\varphi^{\beta} - \beta\int \mathcal{A}(x,u,Du)\cdot D\varphi\,[u^+ + \varepsilon]^{\alpha}\varphi^{\beta-1}.$$

Since $Du^+ = 0$ a.e. in the set $\{u \leq 0\}$ we can rewrite this as

$$0 \leq \alpha\int_{\{u>0\}} \mathcal{A}(x,u,Du)\cdot Du\ [u^+ + \varepsilon]^{\alpha-1}\varphi^{\beta}$$
$$\leq -\int \mathcal{B}(x,u,Du)\eta_\varepsilon - \beta\int \mathcal{A}(x,u,Du)\cdot D\varphi\ [u^+ + \varepsilon]^{\alpha}\varphi^{\beta-1}.$$

By letting $\varepsilon \to 0$ we obtain (using Fatou's Lemma and (3) for the first integral, and Lebesgue's dominated theorem for the others)

(2.3)

$$\int \mathcal{B}(x,u,Du)[u^+]^{\alpha}\varphi^{\beta}$$
$$\leq -\alpha\int_{\{u>0\}} \mathcal{A}(x,u,Du)\cdot Du\, u^{\alpha-1}\varphi^{\beta} - \beta\int \mathcal{A}(x,u,Du)\cdot D\varphi\,[u^+]^{\alpha}\varphi^{\beta-1},$$

all the integrals being finite. Recalling that $\varphi \equiv 1$ for $|x| \leq R$, the last inequality is clearly equivalent to the stated result of the lemma.

The crucial Lemma 2.3 below is obtained by absorbing the two terms on the third line of (2.2) into the term on the second line. To this end we first obtain

Lemma 2.2. *Let* $\alpha > 0$*,* $\beta > 0$*,* $p > 1$*,* $q + \ell > \theta$*,* $s, t \in \mathbf{R}$ *and*

$$\frac{q+\ell+\alpha}{\alpha+\theta} > \frac{\ell+1}{p},\ \frac{\ell}{p-1}. \tag{2.4}$$

Then at all points x *with* $|x| \geq R_0$*,* $u = u(x) > 0$*,* $\varphi = \varphi(x) > 0$ *we have*

$$\begin{aligned} -\beta\mathcal{A}(x,u,Du)\cdot D\varphi\, u^{\alpha}\varphi^{\beta-1} \leq\ & \alpha\mathcal{A}(x,u,Du)\cdot Du\ u^{\alpha-1}\varphi^{\beta} \\ & + C_{\mathcal{B}}\,|x|^{-t}\,u^{\alpha+q}\,|Du|^{\ell}\,\varphi^{\beta} \\ & + C\,|x|^{\upsilon}\,|D\varphi/\varphi|^{\bar\gamma}\varphi^{\beta} \end{aligned} \tag{2.5}$$

where $\bar\gamma$ *depends on* α*,* ℓ*,* p*,* q*,* r*; while* υ *depends also on* s*,* t*; and* C *also on* $C_{\mathcal{A}}$*,* $C_{\mathcal{B}}$ *and* β*; see* (2.9) - (2.13)*.*

Remarks. It is worth noting that the principal condition (2.4) is satisfied whenever $\ell \leq p-1$.

As is apparent from the proof below, Lemma 2.2 remains true even when the parameters q and ℓ are allowed to take negative values (of course subject to the conditions of the lemma).

Proof. By (3) we have $\mathcal{A}(x,u,0) = 0$, so (2.5) trivially holds when $Du = 0$. We can thus assume that $Du \neq 0$ for the remainder of the proof.

Let

$$I = -\mathcal{A}(x,u,Du)\cdot D\varphi\, u^{\alpha}\varphi^{\beta-1}.$$

By the triple Young inequality, if

$$I \leq d^{1/\bar\alpha}\, e^{1/\bar\beta}\, f^{1/\bar\gamma} \tag{2.6}$$

where d, e, $f > 0$ and $\bar\alpha$, $\bar\beta$, $\bar\gamma > 1$ are such that

$$\frac{1}{\bar\alpha} + \frac{1}{\bar\beta} + \frac{1}{\bar\gamma} = 1, \tag{2.7}$$

then

$$I \leq d + e + f.$$

By choosing d, e, f in the obvious way, we can then obtain (2.5).

More precisely, it is necessary to verify (2.6), (2.7) with

$$\begin{aligned} d &= \frac{\alpha}{\beta}\mathcal{A}(x,u,Du)\cdot Du\ u^{\alpha-1}\varphi^{\beta} \\ e &= \frac{C_{\mathcal{B}}}{\beta}|x|^{-t}\,u^{\alpha+q}\,|Du|^{\ell}\,\varphi^{\beta} \\ f &= \frac{C}{\beta}\,|x|^{\upsilon}\,|D\varphi|^{\bar\gamma}\varphi^{\beta-\bar\gamma}. \end{aligned} \tag{2.8}$$

This in fact requires careful calculation.

To begin, we write $\mathcal{A} = \mathcal{A}(x, u, Du)$ and

$$\begin{aligned}|\mathcal{A}| &= |\mathcal{A}|^{1-p/(p-1)\bar{\alpha}} \cdot |\mathcal{A}|^{p/(p-1)\bar{\alpha}} \\ &\le (C_{\mathcal{A}}|x|^s\, u^r\, |Du|^{p-1})^{1-p/(p-1)\bar{\alpha}} \cdot (C_{\mathcal{A}}|x|^s\, u^r\, [\mathcal{A}\cdot Du]^{p-1})^{1/(p-1)\bar{\alpha}} \\ &= (C_{\mathcal{A}}|x|^s\, u^r)^{1-1/\bar{\alpha}}\, |Du|^{p-1-p/\bar{\alpha}}\, [\mathcal{A}\cdot Du]^{1/\bar{\alpha}},\end{aligned}$$

using (5) at the second step. Therefore from the definition of I,

$$\begin{aligned}I &\le |\mathcal{A}|\, |D\varphi|\, u^{\alpha}\, \varphi^{\beta-1} \\ &\le (C_{\mathcal{A}}|x|^s)^{1-1/\bar{\alpha}}\, u^{\alpha+r-r/\bar{\alpha}}\, |Du|^{p-1-p/\bar{\alpha}}\, [\mathcal{A}\cdot Du]^{1/\bar{\alpha}}\, |D\varphi|\, \varphi^{\beta-1}.\end{aligned}$$

The validity of (2.6) now entails the following exponent balances:

$$\text{Powers of } u: \quad \alpha + r - r/\bar{\alpha} = \frac{\alpha-1}{\bar{\alpha}} + \frac{\alpha+q}{\bar{\beta}},$$

$$\text{Powers of } |Du|: \quad p - 1 - \frac{p}{\bar{\alpha}} = \frac{\ell}{\bar{\beta}},$$

$$\text{Powers of } |x|: \quad s(1-1/\bar{\alpha}) = -\frac{t}{\bar{\beta}} + \frac{\upsilon}{\bar{\gamma}},$$

$$\text{Powers of } \varphi: \quad \beta - 1 = \frac{\beta}{\bar{\alpha}} + \frac{\beta}{\bar{\beta}} + \frac{\beta-\bar{\gamma}}{\bar{\gamma}}.$$

The terms $|D\varphi|$ and $\mathcal{A}\cdot Du$ already balance, while finally we have the coefficient balance

$$C_{\mathcal{A}}^{1-1/\bar{\alpha}} = \alpha^{1/\bar{\alpha}}\, C_{\mathcal{B}}^{1/\bar{\beta}}\, C^{1/\bar{\gamma}}/\beta,$$

which determines C in terms of $C_{\mathcal{A}}$, $C_{\mathcal{B}}$, α, β and $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, that is

$$C = C_{\mathcal{A}} \left(\frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}\right)^{\bar{\gamma}/\bar{\beta}} \left(\frac{\beta}{\alpha^{1/\bar{\alpha}}}\right)^{\bar{\gamma}}. \tag{2.9}$$

At the same time, the above balances place heavy restrictions on the exponents $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ themselves. To begin with, eliminating $\bar{\alpha}$ from the first two power balances yields

$$\bar{\beta} = p\frac{q+\ell+\alpha}{\alpha+\theta} - \ell. \tag{2.10}$$

In turn, from the second power balance,

$$\frac{1}{\bar{\alpha}} = \frac{p-1-\ell/\bar{\beta}}{p} \tag{2.11}$$

and

$$\frac{1}{\bar\gamma} = 1 - \frac{1}{\bar\alpha} - \frac{1}{\bar\beta} = \frac{1}{p} + \frac{\ell}{p\bar\beta} - \frac{1}{\bar\beta} = \frac{q+\ell-\theta}{\alpha+\theta}\cdot\frac{1}{\bar\beta} \tag{2.12}$$

by (2.10). Also from the third power balance

$$v = \bar\gamma\left\{s\left(\frac{1}{\bar\beta}+\frac{1}{\bar\gamma}\right)+\frac{t}{\bar\beta}\right\} = s+(s+t)\frac{\bar\gamma}{\bar\beta}, \tag{2.13}$$

while the fourth balance is automatic by (2.7).

With $\bar\alpha$, $\bar\beta$, $\bar\gamma$ thus determined, it is still requisite to verify that $\bar\alpha$, $\bar\beta$, $\bar\gamma > 1$. The condition $\bar\beta > 1$ is guaranteed by the first hypothesis of (2.4). For $\bar\alpha$ one sees from (2.11) that necessarily

$$-1 < \ell/\bar\beta < p-1, \tag{2.14}$$

this being verified (after a short calculation) in view of the second hypothesis of (2.4). That $\bar\gamma > 1$ is now clear from (2.12), since $1/\bar\alpha + 1/\bar\beta > 0$ and $q+\ell-\theta > 0$, giving $0 < 1/\bar\gamma < 1$.

Remark. In case $\ell = 0$ and $p > 1$ we can take $\bar\alpha = p/(p-1)$, this in fact being the case already treated in [3].

We can now prove the key

Lemma 2.3. *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality* $(1')$ *and assume $p > 1$, $q+\ell > \theta$ and that* (2.4) *is valid, where $\alpha > 0$. Then for $R \ge R_0$ we have*

$$\begin{aligned}\min\{\alpha,1\}\int_{B_{R_1}\cap\{u>0\}} &[\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1}\\ &\le C_1 R^{(q+\ell+\alpha)\nu+n-\ell-t},\end{aligned} \tag{2.15}$$

where

$$\nu = (s+t-p+\ell)/(q+\ell-\theta) \tag{2.16}$$

and C_1 depends only on α, ℓ, n, p, q, r, s, t and the structural parameters $C_{\mathcal{A}}$, $C_{\mathcal{B}}$ in (5) *and* (6) *(see (2.19).*

Proof. Choose $\beta = \bar\gamma\,(>1)$. Then from (2.2), by using (2.5) and (6) to eliminate obvious terms, we obtain at once

$$\begin{aligned}\min\{\alpha,1\}\int_{B_R\cap\{u>0\}} &[\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1}\\ &\le C\int_{\mathbf{R}^n\setminus B_R} |x|^{v}\, |D\varphi|^{\bar\gamma}.\end{aligned} \tag{2.17}$$

We now take for φ the explicit function

$$\varphi(x) = \varphi_R(x) = \psi\left(\frac{|x|}{R}\right), \tag{2.18)}$$

where

$$\psi(\tau) = \begin{cases} 1, & 0 \le \tau \le 1, \\ 2 - \tau, & 1 < \tau < 2, \\ 0, & \tau \ge 2. \end{cases}$$

Then $|D\varphi| = 1/R$ when $R \le |x| \le 2R$ and $D\varphi = 0$ otherwise (of course $\varphi \equiv 1$ in B_R). Thus the right hand integral in (2.17) satisfies

$$\int_{\mathbf{R}^n \setminus B_R} |x|^{\upsilon}\, |D\varphi|^{\bar\gamma} \le 2^{n+|\upsilon|}\omega_n\, R^{\upsilon+n-\bar\gamma}.$$

From (2.13) we find that

$$\upsilon + n - \bar\gamma = s + (s+t)\frac{\bar\gamma}{\bar\beta} + n - \bar\gamma = (s+t+\ell)\left(1 + \frac{\bar\gamma}{\bar\beta}\right) - (\ell + \bar\beta)\,\frac{\bar\gamma}{\bar\beta} + n - \ell - t,$$

and then from (2.10), (2.12) and a little calculation,

$$\upsilon + n - \bar\gamma = (q + \ell + \alpha)\,\nu + n - \ell - t.$$

This completes the proof of (2.15), with

$$C_1 = 2^{n+|\upsilon|}\omega_n\, C, \tag{2.19}$$

where C is given by (2.9) with $\beta = \bar\gamma$.

3. Main Results, I.

Here we prove Theorems 1, 2 and 3. It is convenient to treat Theorems 2 and 3 first.

Proof of Theorem 2. We observe from (9′) that $s + t < p - \ell$, and also from (9) and (9′) that $q + \ell > \theta$. Thus $\nu < 0$, where ν is defined by (2.16).

Let $\varepsilon > 0$ and let α be chosen so that the exponent in (2.15) equals $-\varepsilon$, that is

$$(q + \ell + \alpha)\,\nu + n - \ell - t = -\varepsilon.$$

With the help of (2.16) and (9) this gives, specifically,

$$\alpha = \frac{(q+\ell)(s+n-p) - \theta(n-\ell-t)}{p - \ell - s - t} + \frac{\varepsilon}{|\nu|} > 0.$$

With this choice of α, a short calculation then shows that

$$\frac{q+\ell+\alpha}{\alpha+\theta} = \frac{n-\ell-t+\varepsilon}{s+n-p+\varepsilon} = \delta - \varepsilon\frac{\delta-1}{s+n-p+\varepsilon}$$

Condition (10) shows that $\delta > \ell/(p-1)$. Then $p > \ell/\delta + 1 > (\ell+1)/\delta$ since $\delta > 1$, so also $\delta > (\ell+1)/p$. If now ε is taken to be sufficiently small, we conclude that

$$\frac{q+\ell+\alpha}{\alpha+\theta} > \frac{\ell+1}{p}, \frac{\ell}{p-1}.$$

Consequently the main hypothesis (2.4) of Lemma 2.3 is valid, and (2.15) holds (with negative exponent).

In turn, letting $R \to \infty$ in (2.15) one obtains

$$\int_{u>0} [\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1} = 0. \tag{3.1}$$

Because of (3) we get

$$\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u = 0 \quad \text{a.e. in the set } \{u>0\}. \tag{3.2}$$

From (4) it follows that $Du = 0$ almost everywhere in the set $\{u > 0\}$. Therefore $Du^+ = 0$ almost everywhere in $\mathbf{R}^n$, and thus $u^+ \equiv$ Const. in $\mathbf{R}^n$. Since u^+ is continuous, this implies that any C^1 entire solution of the inequality $(1')$ must be either constant or non-positive. A similar argument based on the function $-u$ shows also that $u^- \equiv$ Const. in $\mathbf{R}^n$. Since $u = u^+ - u^-$, this completes the proof.

Proof of Theorem 3. Let $\alpha = \varepsilon$, where ε is a positive constant to be chosen later. In particular, if ε is suitably small then from (11)

$$(q+\ell+\alpha)\,\nu + n - \ell - t < 0,$$

so the exponent in (2.15) is negative.

Moreover with $\alpha = \varepsilon$ we get

$$\frac{q+\ell+\alpha}{\alpha+\theta} = \frac{q+\ell+\varepsilon}{\varepsilon+\theta} = \Delta - \varepsilon\,\frac{q+\ell-\theta}{\theta\,(\theta+\varepsilon)}.$$

Condition (12) implies $\Delta > \ell/(p-1)$; then as in the proof of Theorem 2 also $\Delta > (\ell+1)/p$. If ε is even smaller, if necessary, we conclude that

$$\frac{q+\ell+\alpha}{\alpha+\theta} > \frac{\ell+1}{p}, \frac{\ell}{p-1},$$

that is (2.4) holds. We can thus apply Lemma 2.3 with a negative exponent in (2.15). The rest of the proof is then the same as the final part of the proof of Theorem 2.

Proof of Theorems 1 and 1′. Clearly, it is enough to prove Theorem 1′. For the second case of Theorem 1′, it is easy to see that (11), (11′) and (12) are satisfied. Theorem 3 then implies that any $C1$ entire solution of the inequality (1′) is constant. Thus the second case of Theorem 1′ is a consequence of Theorem 3.

Conversely, when the second case of Theorem 1′ is unavailable, that is, if (11) fails, then (9) holds. In turn the conditions $q+\ell>\theta$, $s+t<p-\ell$ in the first case of Theorem 1′ then imply without difficulty that $s+n-p$ is positive.[1] and moreover that $\delta>1$. The condition $\ell\le p-1$ now gives $\ell<(p-1)\delta$, so Theorem 2 implies that any C^1 entire solution of (1) is constant. Thus (when the second case of Theorem 1′ is unavailable) the first case of Theorem 1′ follows from Theorem 2. In consequence, Theorem 1′ is a corollary of Theorems 2 and 3.

3.1 The special case $r=0$. The $p-$Laplacian.

We consider in more detail the special case $r=0$ of (5). Here for convenience let

$$\sigma=s+n, \qquad \tau=s+t.$$

We observe that if $s=0$ this includes the p-Laplacian, with $\sigma=n$ and $\tau=t$.

Case 1. $p\ge\sigma>1$. Condition (13) can be rewritten

$$\ell<(p-1)\,\frac{\sigma-\tau}{\sigma-1}+q\,\frac{p-\sigma}{\sigma-1}, \qquad \ell>p-1-q. \tag{3.3}$$

Then by the corollary to Theorem 3, $u\equiv$ Constant for any entire solution of (1).

Case 2. $p<\sigma$, $\tau\le 1$. Here by direct calculation[2] we have

$$(p-1)\frac{\sigma-\tau}{\sigma-1}\le p-\tau. \tag{3.4}$$

Now if

$$\ell<(p-1)\,\frac{\sigma-\tau}{\sigma-1}, \qquad \ell>p-1-q, \tag{3.5}$$

then (10) is satisfied, and by (3.4) also $\ell<p-\tau$. But then $\delta>1$. Hence by Theorem 2 (when (9) holds) or by the corollary of Theorem 3 (when (13) holds), we find $u\equiv$ Constant.

[1] We have

$$(q+\ell-\theta)(s+n-p)\ge\theta(n-\ell-t)-\theta(s+n-p)=\theta\,(p-\ell-s-t)>0.$$

[2] Or simply note that (3.4) holds when $\tau\to-\infty$ and when $\tau=1$, and hence for all $\tau\le 1$.

Case 2. $p < \sigma$, $1 < \tau < p$. Then in analogy with Case 2 we have

$$(p-1)\frac{\sigma-\tau}{\sigma-1} > p-\tau. \tag{3.6}$$

Now if

$$\ell < p-\tau, \qquad \ell > p-1-q, \tag{3.7}$$

then (3.5) is satisfied (see (3.6)), that is, (10) is valid, and also, as above, $\delta 1$. Therefore as in Case 2 we find $u \equiv$ Constant.

Conditions (3.4) and (3.6) are shown in Figure 2 for the case $s = t = 0$.

Equation (2) is an example when $r = s = t = 0$. Thus for this equation with the restriction $q > p - \ell - 1$, it follows that $u \equiv$ Constant when $(2')$ is verified.

4. Operators allowing multiple values of p.

In this section we study the case where the function $\mathcal{A}$ satisfies the large radii condition (5) for multiple values of the exponent p. Thus, when condition (6) is in force, we can use this information to improve the previous results.

For simplicity, in fact, we shall consider only the case of functions $\mathcal{A}$ which satisfy the large radii condition (5) for values p such that

$$1 \le p \le 2.$$

Note that the classical case (8) satisfies (5) in exactly this case, see also [3], Section 6. The first main result of the present section is:

Theorem 4. *Assume $q > 0$, $r \ge 0$, $s, t \in \mathbf{R}$. Let condition (5) hold for all $p \in [1, 2]$. Also suppose that (6) is in force with $0 \le \ell \le 1$.*

(a) Assume $q + \ell > 1 + r$ and either

$$(i) \qquad s + t \le 2 - \ell$$

or

$$(ii) \qquad s < \max\left\{\frac{1+r}{q+\ell}(n-\ell-t)+2-n,\ \frac{r+\ell}{q+\ell}(n-\ell-t)+1+\ell-n\right\}$$

Then any entire C^1 distribution solution of equation (1) must vanish everywhere.

(b) Assume $q > r$, $q + \ell \le 1 + r$ and either

$$(i) \qquad s + t < 1 + q - r$$

or

$$(ii) \qquad s < \left\{\frac{r+\ell}{q+\ell}(n-\ell-t)+\ell+1-n\right\}. \tag{4.2}$$

Then any entire C^1 distribution solution of equation (1) must be constant.

Theorem 5. *Under the assumptions of Theorem 4 any entire C^1 distribution solution of the inequality $(1')$ must be either constant or non-positive.*

Note that in the conditions of Theorems 4 and 5 there is no appearance of the parameter p, since the main exponents are now 1 and 2.

Conditions (i), (ii) of Theorem 4 (a) can be combined and written alternatively as

$$s < \max\left\{2-\ell-t,\ \frac{1+r}{q+\ell}(n-\ell-t)+2-n,\ \frac{r+\ell}{q+\ell}(n-\ell-t)+1+\ell-n\right\}.$$

The graph of the borderline condition has two corners, at $s=2-n$, $t=n-\ell$ and at $s=1-n-r$, $t=n+q$, see Figure 3. Similarly, conditions (i), (ii) of Theorem 4(b) can be written

$$s < \max\left\{1+q-r-t,\ \frac{r+\ell}{q+\ell}(n-\ell-t)+1+\ell-n\right\}.$$

Here there is a (single) corner at $s=1-n-r$, $t=n+q$.

Clearly Theorem 4 is a consequence of Theorem 5, see the proof of Theorem 2.

Proof of Theorem 5. We suppose first that $\ell > 0$. Case(a), (i) of Theorem 5 then follows at once from Theorem $1'$, case (i) by taking $p=2$. Case (a), (ii) is a consequence of Theorem $1'$, case (ii) by taking first $p=1+\ell$ and then $p=2$ (note in both cases $p>1$).

To obtain case (b), (i) we take $p=q+\ell-r+1-\varepsilon$, with $\varepsilon>0$ so small that $\ell+1\le p\le 2$ and $q+\ell>\theta$. In fact the condition $\ell+1\le p$ requires $\varepsilon<q-r$, which by the hypothesis $q>r$ obviously holds for suitably smallε. Both the remaining conditions $p\le 2$ and $q+\ell>\theta$ directly follow form the hypothesis $q+\ell\le 1+r$ and the given form of p. This being the case, the result again follows from Theorem $1'$, case (i), after letting $\varepsilon\to 0$. (It is worth adding that case (b), (i) can occur only if $\ell<1$.) Case (b), (ii) is a consequence of Theorem $1'$, case (ii), again by taking $p=1+\ell$.

Remarks.

i) The special case $\ell=0$ is exactly Theorem 2 of [3].

ii) The possibility of negative values of ℓ is taken up in the next section.

5. A generalization of condition (6).

In this section we consider the situation when conditions (3), (4), (5) hold , while (6) is replaced by the weaker assumption

$$\mathcal{B}(x,z,\rho)\,\operatorname{sign} z \geq C_{\mathcal{B}}|x|^{-t} f(z)\,\sigma(\rho) \tag{6'}$$

for $|x| \geq R_0$, where f, σ are non-negative functions subject to the conditions

$$f(z) \geq \begin{cases} |z|^{q_1} & \text{when } |z| < 1 \\ |z|^{q_2} & \text{when } |z| \geq 1, \end{cases} \tag{5.1}$$

$$\sigma(\rho) \geq \begin{cases} |\rho|^{\ell_1} & \text{when } |\rho| < 1 \\ |\rho|^{\ell_2} & \text{when } |\rho| \geq 1, \end{cases} \tag{5.2}$$

with q_1, $\ell_1 \geq 0$, q_2, $\ell_2 \in \mathbf{R}$. We also define

$$q = \min(q_1,\, q_2), \qquad \ell = \min(\ell_1,\, \ell_2)$$

and

$$\bar{q} = \max(q_1,\, q_2), \qquad \bar{\ell} = \max(\ell_1,\, \ell_2).$$

Conditions similar to (5.1) were previously introduced in [5] and in [7].

Theorem 6. *Assume $p > 1$ and $q + \ell > \theta$ where $\theta = p + r - 1$. If also $\bar{\ell} \leq p - 1$ and*

$$s + t < p - \bar{\ell}, \tag{5.3}$$

then any entire C^1 distribution solution of equation (1) *must be constant.*

Proof. We shall apply Lemma 2.2 in the four sets

$$\{|u| < 1,\ |\rho| < 1\}, \qquad \{|u| < 1,\ |\rho| \geq 1\}$$

$$\{|u| \geq 1,\ |\rho| < 1\}, \qquad \{|u| \geq 1,\ |\rho| \geq 1\}.$$

Since $q + \ell > \theta$, it follows that for any $\alpha > 0$

$$\frac{q_i + \ell_j + \alpha}{\alpha + \theta} > 1, \qquad i = 1,2, \quad j = 1,2.$$

Together with the condition $\bar{\ell} \leq p - 1$, we see that inequality (2.4) holds for each region above and for any $\alpha > 0$.

Now arguing as in Lemma 2.3 we obtain the principle inequality

$$\begin{aligned}\min\{\alpha,1\}\int_{B_R\cap\{u>0\}} & [\mathcal{A}(x,u,Du)\cdot Du+\mathcal{B}(x,u,Du)u]\,u^{\alpha-1}\\ &\le C_1\Big(R^{(q_1+\ell_1+\alpha)\nu_1+n-\ell_1-t}+R^{(q_1+\ell_2+\alpha)\nu_{12}+n-\ell_2-t}\\ &\quad+R^{(q_2+\ell_1+\alpha)\nu_{21}+n-\ell_1-t}+R^{(q_2+\ell_2+\alpha)\nu_2+n-\ell_2-t}\Big)\end{aligned} \tag{5.4}$$

for $R\ge R_0$, where

$$\nu_1=(s+t-p+\ell_1)/(q_1+\ell_1-\theta)\qquad \nu_{12}=(s+t-p+\ell_2)/(q_1+\ell_2-\theta)$$

$$\nu_{21}=(s+t-p+\ell_1)/(q_2+\ell_1-\theta)\qquad \nu_2=(s+t-p+\ell_2)/(q_2+\ell_2-\theta)$$

and C_1 depends only on α, ℓ_1, ℓ_2, n, p, q_1, q_2, r, s, t and the structural parameters $C_{\mathcal{A}}$, $C_{\mathcal{B}}$ in (5) and (6′).

It follows from the conditions $q+\ell>\theta$ and $s+t-p+\bar{\ell}<0$ that all the quantities ν_1, ν_{12}, ν_{21}, ν_2 are well-defined and negative. If α is now taken sufficiently large then all four exponents on the right side of (5.3) are negative. Hence letting $R\to\infty$ in (5.4) we obtain (3.1). The rest of the proof is then the same as for Theorem 2.

Remark. A result corresponding to Theorem 6 can also be obtained in the case of multiple values of p, but can be left to the interested reader.

For the case of the p-Laplacian we have the following more specific result, covering, e.g., the equation

$$\Delta_p\,u=f(u)\,\sigma(Du)\ \text{sign}\ u.$$

Theorem 7. *Let $r=s=t=0$ and $q+\ell>p-1$. Suppose that*

$$(i)\qquad p\ge n,\quad \bar{\ell}<q\,\frac{p-n}{n-1}+(p-1)\,\frac{n}{n-1},$$

or

$$(ii)\qquad 1<p<n\quad\text{and either}$$

$$\bar{\ell}<\bar{q}\,\frac{p-n}{n-1}+(p-1)\,\frac{n}{n-1}$$

or

$$\ell\ge q\,\frac{p-n}{n-1}+(p-1)\,\frac{n}{n-1},\quad \bar{\ell}<(p-1)\,\frac{n}{n-1}$$

or

$$\bar{\ell}\le p-1.$$

Then any C^1 entire solution of (1) *is identically constant.*

The case $q = 0$ is perhaps the most important example of (5.1). Here the most interesting parts of Theorem 7 are when $\ell > p-1$ and either

$$(i) \qquad p \geq n, \quad \bar{\ell} < (p-1)\,\frac{n}{n-1}$$

or

$$(ii) \qquad p < n, \quad \bar{\ell} < \bar{q}\,\frac{p-n}{n-1} + (p-1)\,\frac{n}{n-1}.$$

Proof of Theorem 7. The idea is that in each of the cases of the theorem one can choose a (single) value for α so that (5.4) holds with all the exponents being negative, in which case the conclusion $u \equiv$ Const. is immediate by letting $R \to \infty$.

The idea is obvious in the final case, this being just Theorem 6 (note $\bar{\ell} \leq p-1 < p$ so $p - \bar{\ell} > 0 = s+t$). Also in the first case of the theorem, that is (i), it is easy to see that condition (13) of the Corollary of Theorem 3 is then satisfied for each of the principal regions, together with the condition $q + \ell > p-1$. Therefore, as in the proof of Theorem 3, if α is sufficiently small then (2.4) holds for each of the regions, and, equally, (5.4) holds with all the exponents being negative.

The first case of (ii) is proved in the same way, noting that now $p < n$ rather than $p \geq n$.

Finally the second case of (ii) is a consequence of Theorem 2.Indeed, a short calculation shows that (9), for both the values of ℓ and both the values of $q$in question, is a consequence of the first condition of this case.Also from the second condition for this case we get $\bar{\ell} < p$ and so $\delta = (n-\ell)/(n-p)1$, that is condition (9') holds for both values of ℓ. Finally condition (10) is also implied by the second condition, and the required choice of α follows as in the proof of Theorem 2.

6. Main results, II : a further generalization of condition (6).

Here we replace condition (5.1) by the much weaker requirement

$$f(z) \geq \begin{cases} 0 & \text{when } |z| > 0 \\ |z|^q & \text{when } |z| > d, \end{cases} \tag{6.1}$$

where d is a given positive constant. As always, the conditions (3), (4) are assumed to be in force.

Theorem 8. *Assume the "large radii conditions"* (5) *and*

$$\mathcal{B}(x,z,\rho)\ \mathrm{sign}\, z \geq C_{\mathcal{B}}|x|^{-t} f(z)\,|\rho|^{\ell}, \qquad |x| \geq R_0,\ |z| > 0, \tag{6.2}$$

where the function f satisfies (6.1)*; $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ are positive constants; $\ell \geq 0$, $r \geq 0$, s, $t \in \mathbf{R}$ and $q + \ell > \theta$.*

Suppose also

$$s + n - p < 0, \qquad (q+\ell)(s+n-p) - \theta(n-\ell-t) < 0, \tag{6.3}$$

and

$$\ell < (p-1)\,\frac{q+\ell}{\theta}. \tag{6.4}$$

Then any entire C^1 distribution solution of equation (1) *must be constant.*

Corollary to Theorem 8. *When $q > 0$, $r = 0$ the result of Theorem* 8 *remains valid without the additional condition* (6.4).

The corollary follows from the observation that (6.4), and hence (2.4), is automatic when $q > 0$, $r = 0$, see the corollary to Theorem 3.

Proof of Theorem 8. In analogy to the proof of Theorem 6 we consider the regions

$$|u| > d, \qquad |u| \le d,$$

with the purpose to obtain an inequality corresponding to (5.4). For the first region, we argue as in the proof of Lemma 2.3, taking $\alpha = \varepsilon < 1$ so small that (2.4) holds (see the proof of Theorem 3).

For the second region we use (2.5) with $C_{\mathcal{B}}$ replaced by $(2d)^{-q_1}$ and q replaced by q_1, where q_1 is a constant sufficiently large that $q_1 + \ell > \theta$ and (2.4) holds. Then as in the proof of Theorem 6 there results

(6.5)

$$\begin{aligned}\min\{\alpha,1\} \int_{B_{R_1}\cap\{u>0\}} &[\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1} \\ &\le (2d)^{-q_1} \int_{0<u\le d,\, |x|\le 2R} |x|^{-t}\, |u|^{\alpha+q_1} |Du|^{\ell} \\ &\qquad + C_1\, R^{(q+\ell+\alpha)\nu+n-\ell-t} + C_2\, R^{(q_1+\ell+\alpha)\nu_1+n-\ell-t}\end{aligned}$$

for $R \ge R_0$, where

$$\nu = \frac{s+t-p+\ell}{q+\ell-\theta}, \qquad \nu_1 = \frac{s+t-p+\ell}{q_1+\ell-\theta};$$

C_1 is given by (2.19), (2.9) with $\beta = \bar{\gamma}$; C_2 is the same as C_1 with the two exceptions that q is replaced by q_1 and $C_{\mathcal{B}}$ by $(2d)^{-q_1}$; and where the integral on the right side arises because, in the present case where (6.1) is in force, there is no corresponding quantity from the function $\mathcal{B}$ to balance it.

Relying on the second part of (6.3), we have

$$\begin{aligned}(q+\ell+\alpha)\nu + n - \ell - t &= (q+\ell+\alpha)\frac{s+t-p+\ell}{q+\ell-\theta} + (n-\ell-t) \\ &= \frac{1}{q+\ell-\theta}\,\{(q+\ell+\alpha)(s+n-p) - \theta(n-\ell-t) + \alpha(t+\ell-n)\} < 0\end{aligned}$$

provided that $\alpha\,(=\varepsilon)$ is even smaller if necessary, say finally $\alpha = \alpha_0 < 1$. That is, the first exponent in (6.5)) is negative provided that $\alpha = \alpha_0$ is sufficiently small.

We can now directly let $q_1 \to \infty$ in (6.5). In preparation for this, the coefficients C_1 and C_2 are crucial. In fact,

$$C_1 = 2^{n+|\upsilon|}\,\omega_n\,C_{\mathcal{A}}\left(\frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}\right)^{\bar\gamma/\bar\beta}\frac{\bar\gamma^{\bar\gamma}}{\alpha^{\bar\gamma/\bar\alpha}},$$

$$C_2 = 2^{n+|\upsilon|}\,\omega_n\,C_{\mathcal{A}}\left(\frac{C_{\mathcal{A}}}{(2d)^{-q_1}}\right)^{\bar\gamma/\bar\beta}\frac{\bar\gamma^{\bar\gamma}}{\alpha^{\bar\gamma/\bar\alpha}} \qquad (\text{with } q \text{ replaced by } q_1).$$

For q fixed and $\alpha = \alpha_0$ the coefficient C_1 is a fixed finite constant.

To evaluate C_2 as $q_1 \to \infty$ we have by (2.10), (2.11), (2.12),

$$\bar\beta = p\frac{q_1+\ell+\alpha_0}{\alpha_0+\theta} - \ell \to \infty,$$

$$\bar\gamma = \frac{\alpha_0+\theta}{q_1+\ell-\theta}\left\{p\frac{q_1+\ell+\alpha_0}{\alpha_0+\theta} - \ell\right\} \to p,$$

$$\frac{\bar\gamma}{\bar\beta} \to 0, \qquad q_1\,\frac{\bar\gamma}{\bar\beta} = q_1\frac{\alpha_0+\theta}{q_1+\ell-\theta} \to \alpha_0+\theta,$$

$$\frac{1}{\bar\alpha} = \frac{p-1-\ell/\bar\beta}{p} \to \frac{p-1}{p}.$$

Also by (2.13) we get $\upsilon \to s$. Thus

$$C_2 \to 2^{n+|s|}\,\omega_n\,C_{\mathcal{A}}\,(2d)^{\alpha_0+\theta}\,p^p/\alpha_0^{p-1} = C_3.$$

Moreover, as $q \to \infty$,

$$(q_1+\ell+\alpha)\nu_1 + n - t = \frac{q_1+\ell+\alpha}{q_1+\ell-\theta}\,(s+t-p+\ell) + n - \ell - t \to s+n-p,$$

while for the integral on the right side of (6.5) we have the estimate

$$(2d)^{-q_1}\int_{0<u\le d,\,|x|\le 2R} |x|^{-t}\,u^{\alpha+q_1}|Du|^{\ell} \le 2^{-q_1}\;2^{n+|t|}\omega_n\,R^{n-t}\;d^{\alpha_0}\,\mathrm{sup}_{|x|\le 2R}\,|Du(x)|^{\ell}.$$

With the above estimates in hand, letting $q_1 \to \infty$ in the the inequality (6.5) leads to

$$\alpha_0\,J \le C_1\,R^{\,\text{negative}} + C_3\,R^{s+n-p},$$

where J denotes the integral on the left side of (6.5), and with C_1, C_3 being constants as described above; note here that the integral on the right side of (6.5) tends to 0 since the factor $2^{-q_1} \to 0$ while the other terms are finite.

Now letting $R \to \infty$ and using the first condition of (6.3) we obtain $J = 0$, that is (3.1), and the proof is now completed as usual.

A result corresponding to Theorem 8 but with $s + n \geq p$ is also possible, though with somewhat stronger hypotheses, specifically with (6.1) replaced by

$$\begin{cases} f(0) = 0, \\ f(z) \text{ is continuous and positive when } |z| > 0, \\ f(z) \geq |z|^q \text{ when } |z| > d, \end{cases} \tag{6.6}$$

where d is a non-negative constant.

Theorem 9. *Assume the "large radii conditions"* (5) *and* (6.2), *where the function* f *satisfies* (6.6); $C_{\mathcal{A}}$ *and* $C_{\mathcal{B}}$ *are positive constants;* $\ell \geq 0$, $r \geq 0$, s, $t \in \mathbf{R}$ *and* $q + \ell > \theta$. *If*

$$s + n \geq p \qquad s + t < p - \ell, \qquad \ell \leq p - 1,$$

then any entire C^1 *distribution solution of equation* (1) *must be constant in* $\mathbf{R}^n$ (*or must vanish if* $\ell = 0$).

Before giving the proof, it is convenient first to have another version of Lemma 2.2.

Lemma 2.2′. *Let* $\alpha > 1$, $\beta > 1$, *p1*, $q + \ell > \theta$. *Then at all points* x *with* $|x| \geq R_0$, $u = u(x) > 0$, $\varphi = \varphi(x) > 0$ *we have*

$$\begin{aligned} -\beta \mathcal{A}(x, u, Du) \cdot D\varphi\, u^{\alpha} \varphi^{\beta-1} \leq \alpha \mathcal{A}(x, u, Du) \cdot Du\, u^{\alpha-1} \varphi^{\beta} \\ + |x|^{-t}\, u^{q+\ell+\alpha} \varphi^{\beta} \\ + C\, |x|^{v}\, |D\varphi/\varphi|^{\bar{\gamma}} \varphi^{\beta} \end{aligned} \tag{6.8}$$

where

$$v = s + (s + t)\, \frac{\alpha + \theta}{q + \ell - \theta}, \qquad \bar{\gamma} = p\, \frac{q + \ell + \alpha}{q + \ell - \theta}, \qquad C = C_{\mathcal{A}}^{\bar{\gamma}/p}\, \beta^{\bar{\gamma}}. \tag{6.9}$$

Proof. This is essentially the same as the proof of Lemma 2.2, with the exceptions that we take $\ell = 0$, $C_{\mathcal{B}} = 1$ and replace q (without confusion) by $q + \ell$. The proof of Lemma 2.2 then supplies in place of (2.10) the new value

$$\bar{\beta} = p\, \frac{q + \ell + \alpha}{\alpha + \theta}.$$

By the assumption $q + \ell > \theta$, this is greater than $p\,(> 1)$; that is, condition (2.4) is no longer required in the proof.

Also from the proof of Lemma 2.2, see (2.12), we find

$$\bar{\gamma} = \frac{\alpha + \theta}{q + \ell - \theta} = p\, \frac{q + \ell + \alpha}{q + \ell - \theta},$$

and, unchanged from Lemma 2.2, see (2.13),

$$\upsilon = s + (s+t)\,\frac{\bar\gamma}{\bar\beta} = s + (s+t)\,\frac{\alpha+\theta}{q+\ell-\theta}.$$

Finally, for simplicity, the term $\alpha^{1/\bar\alpha}\,(>1)$ is deleted from the formula for C, see (2.9).

Proof of Theorem 9. From (6.6) it is easy to see that for any constant $c \in (0,d)$ there exists $C_c > 0$ such that

$$f(z) \ge C_c|z|^q \qquad \text{when } |z| \ge c. \tag{6.10}$$

Our purpose again is to obtain an inequality corresponding to (5.4). In view of the third condition of (6.6), condition (2.4) is automatically satisfied for all $\alpha > 0$. Therefore, as in the proof of Theorem 8, for the set $u > c$ we can use (2.5) and argue as in Lemma 2.3; on the other hand, for the set $0 < u \le c$ we proceed somewhat differently. In particular we use the inequality (6.8) rather than (2.5), with again $\beta = \bar\gamma$. By this means, in place of (6.5) there results (assume $\alpha > 1$)

(6.11)

$$\begin{aligned}\int_{B_R\cap\{u>0\}} &[\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\,u^{\alpha-1} \\ &\le \int_{0<u<c,\,|x|\le 2R} |x|^{-t}\,u^{q+\ell+\alpha} \\ &\qquad\qquad + \hat C_1\,R^{(q+\ell+\alpha)\nu+n-\ell-t} + \hat C_2\,R^{(q+\ell+\alpha)\bar\nu+n-t}\end{aligned}$$

for $R \ge R_0$; here the integral on the right side appears because, as in the proof of Theorem 8, there is no corresponding quantity from the function $\mathcal{B}$ to balance it; the second exponent of R in the last line arises since, by (6.9),

$$\upsilon + n - t = (q+\ell+\alpha)\bar\nu + n - t; \qquad \bar\nu = (s+t-p)/(q+\ell-\theta);$$

finally

$$\hat C_1 = 2^{n+|\upsilon|}\,\omega_n\,C_{\mathcal{A}}\left(\frac{C_{\mathcal{A}}}{C_c\,C_{\mathcal{B}}}\right)^{\bar\gamma/\bar\beta}\bar\gamma^{\bar\gamma}, \qquad \hat C_2 = 2^{n+|\upsilon|}\,\omega_n\,C_{\mathcal{A}}^{\bar\gamma/p}\,\bar\gamma^{\bar\gamma},$$

for $\hat C_1$ we use $\bar\beta$, $\bar\gamma$ given by (2.10), (2.12), while for $\hat C_2$ we use $\bar\upsilon$, $\bar\gamma$ given by (6.9).

The coefficients $\hat C_1$ and $\hat C_2$ are again crucial. From (2.12), (2.10) for the case of $\hat C_2$ we have

$$\frac{\bar\gamma}{\bar\beta} = \frac{\alpha+\theta}{q+\ell-\theta}, \qquad \bar\gamma = \frac{\alpha+\theta}{q+\ell-\theta}\,\bar\beta \ge p\frac{q+\ell+\alpha}{q+\ell-\theta};$$

therefore, with $\alpha \ge q+\ell > \theta$,

$$\frac{\bar\gamma}{\bar\beta} \le \frac{2}{q+\ell-\theta}\,\alpha, \qquad \bar\gamma^{\bar\gamma} \le \left(\frac{2p}{q+\ell-\theta}\,\alpha\right)^{\frac{p(q+\ell)}{q+\ell-\theta}+\frac{p\alpha}{q+\ell-\theta}}.$$

In turn, from (2.13),

$$n + |v| \le n + |s| + |s+t|\,\frac{\bar{\gamma}}{\bar{\beta}} \le \text{Const.}\,(1+\alpha).$$

In combination, this gives

$$\hat{C}_1 \le \text{Const.} \cdot \text{Const.}^{\alpha} \cdot \alpha^{\text{Const.}} \cdot \alpha^{p\alpha/(q+\ell-\theta)} \tag{6.12}$$

for appropriate constants depending only on n, ℓ, r, s, t; p, q, θ; $C_{\mathcal{A}}, C_{\mathcal{B}}$ and c. Similarly

$$\hat{C}_2 \le \text{Const.} \cdot \text{Const.}^{\alpha} \cdot \alpha^{\text{Const.}} \cdot \alpha^{p\alpha/(q+\ell-\theta)}. \tag{6.13}$$

We now take $R = \alpha^{\mu}$ in (6.10), where μ is a positive constant which remains to be determined. This choice of R satisfies $R \ge R_0$ provided also $\alpha \ge R_0^{1/\mu}$. Then

$$\begin{aligned} R^{(q+\ell+\alpha)\nu+n-\ell-t} &= \alpha^{\mu((q+\ell)\nu+n-\ell-t)} \cdot \alpha^{\mu(s+t-p+\ell)\alpha/(q+\ell-\theta)} \\ R^{(q+\ell+\alpha)\bar{\nu}+n-t} &= \alpha^{\mu(q+\ell)\bar{\nu}+n-t} \cdot \alpha^{\mu(s+t-p)\alpha/(q+\ell-\theta)}. \end{aligned} \tag{6.14}$$

With the definition

$$\hat{J} = \int_{B_R \cap \{u \ge 2c\}} [\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\,,$$

we have

$$\int_{B_R \cap \{u>0\}} [\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1} \ge (2c)^{\alpha-1}\,\hat{J}. \tag{6.15}$$

Inserting the estimates (6.12) – (6.15) into (6.11) and dividing through by $(2c)^{\alpha-1}$ then gives the following principal inequality

(6.16)

$$\begin{aligned} \hat{J} \le\ & 2^{n+|t|}\omega_n\, \alpha^{\mu(n-t)}\, c^{\alpha+q+\ell}\,/(2c)^{\alpha-1} \\ & + \text{Const.} \cdot \text{Const.}^{\alpha} \cdot \alpha^{\text{Const.}} \cdot [\alpha^{(\mu(s+t-p+\ell)+p)\alpha/(q+\ell-\theta)} + \alpha^{(\mu(s+t-p)+p)\alpha/(q+\ell-\theta)}]. \end{aligned}$$

Now choose

$$\mu = \frac{p+q+\ell-\theta}{p-\ell-s-t}$$

so that

$$\frac{\mu(s+t-p+\ell)+p}{q+\ell-\theta} = -1, \quad \frac{\mu(s+t-p)+p}{q+\ell-\theta} = -1 - \frac{\ell\mu}{q+\ell-\theta}.$$

Both the final exponents in (6.16) therefore become $\le -\alpha$. Thus letting $\alpha \to \infty$ in (6.16) we get $\hat{J} = 0$ (!); note here that

$$2^{n+|t|}\omega_n\, \alpha^{\mu(n-t)}\, c^{q+\ell+\alpha}\,/(2c)^{\alpha-1} = 2^{n+1+|t|}\omega_n\, \alpha^{\mu(n-t)}\, c^{q+\ell+1} \cdot 2^{-\alpha} \to 0$$

as $\alpha \to \infty$.**

Since $\hat{J} = 0$ and c can be arbitrarily small, it follows at once from (3) that (3.2) holds, and the proof is then completed as in Theorem 2.

7. A new Liouville theorem.

Here we prove Theorem 10, stated at the end of the introduction.

Let $k < \kappa$. Then by (16) there exist positive constants M, $R_1 \geq R_0$ depending on k such that

$$u(x) \leq M|x|^k \qquad \text{for } |x| \geq R_1. \tag{7.1}$$

Then, as in the proof of Lemma 2.3 and Theorem 8 (taking $t = 0$ and $\ell = 0$) we obtain in place of (6.5) the inequality

$$\begin{aligned} \min\{\alpha, 1\} \int_{B_{R_1} \cap \{u>0\}} & [\mathcal{A}(x,u,Du) \cdot Du + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1} \\ & \leq (2d)^{-q_1} \int_{R \leq |x| \leq 2R} |u|^{\alpha+q_1} |Du|^{\ell} + C_2\, R^{(q_1+\alpha)\nu_1+n} \end{aligned} \tag{7.2}$$

for $R \geq R_1$, where $\nu_1 = (s-p)/(q_1-\theta)$ and C_2 is the same as in the proof of Theorem 8 (but with $\ell = 0$).

We can now take $d = M(2R)^k$ and use (7.1) to get

$$\begin{aligned} \min\{\alpha, 1\} \int_{B_{R_1} \cap \{u>0\}} & [\mathcal{A}(x,u,Du) \cdot Du + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1} \\ & \leq 2^{-q_1} (2R)^n \omega_n \{M(2R)^k\}^{\alpha} \\ & \quad + C_4 \{2M(2R)^k\}^{q_1(\alpha+\theta)/(q_1-\theta)} R^{(q_1+\alpha)\nu_1+n} \end{aligned} \tag{7.3}$$

** The right side of (6.16) can be extremely large for intermediate values of α, before ultimately approaching 0 as $\alpha \to \infty$. Since in (6.16) the relevant value of the Constant in the term Const.$^{\alpha}$ is

$$\frac{1}{c} \left\{ \frac{C_{\mathcal{A}}}{C_c\, C_{\mathcal{B}}} \left(\frac{2p}{q+\ell-\theta} \right)^p \right\}^{1/(q+\ell-\theta} = m,$$

we have, since the function $m^{\alpha}\, \alpha^{-\alpha}$ takes its maximum value when $\alpha = m/e$,

$$\max\{\text{Const.}^{\alpha}\, \alpha^{-\alpha}\} = e^{m/e} \approx 10^{m/6.3}.$$

When c is small, the value of m can be quite large, so the right side of (6.16) can easily reach tens of trillions (national debt) before ultimately decreasing and approaching 0.

with

$$C_4 = 2^{n+|v|}\,\omega_n\,C_{\mathcal{A}}\,C_{\mathcal{A}}^{\bar\gamma/\bar\beta}\,\frac{\bar\gamma^{\bar\gamma}}{\alpha^{\bar\gamma/\bar\alpha}} \qquad \text{(with } q \text{ replaced by } q_1 \text{ and } \ell = 0).$$

Here we can let $q_1 \to \infty$ (with α and R fixed). Following the calculations in the proof of Theorem 8 then gives

$$\min\{\alpha, 1\}\, J \le C_5\, R^{k(\alpha+\theta)+s+n-p}.$$

with

$$C_5 = 2^{n+|s|}\,\omega_n\,C_{\mathcal{A}}\,(2^{k+1}M)^{\alpha+\theta}\,p^p/\alpha^{p-1}.$$

But $k = \kappa - \varepsilon$ for some $\varepsilon \in (0, \kappa)$, so that

$$\begin{aligned} k(\alpha+\theta) + s + n - p &= k\alpha + \kappa\,\theta + s + n - p - \varepsilon\theta \\ &= k\alpha - \varepsilon\theta \end{aligned}$$

using the definition $\kappa = (p-n-s)/\theta$. Thus if α is taken so small that $0 < \alpha < -\varepsilon\theta/k$, it follows that

$$\min\{\alpha, 1\}\, J \le C_5\, R^{\,\text{negative}}.$$

Letting $R \to \infty$ gives $J = 0$, and the proof is then completed as usual.

Remarks. It is not hard to check that Theorem 10 also holds when $p = 1$ and $r > 0$.

A result similar to Theorem 10 was proved in [9] for a more special class of operators $\mathcal{A}$, but with a weaker growth condition than (17), namely $u(x) = o(|x|^\kappa)$ as $|x| \to \infty$, see Theorem 1.1 of [9].

Theorem 10 obviously does not apply when $1 < p \le n$. In fact, at least when $1 < p < n$, there exist solutions of the inequality $\Delta_p\, u \ge 0$ which are, somewhat surprisingly, both *positive and non-constant* and yet have at the same time an *arbitrarily small* L^∞ *norm.* Indeed, by direct calculation (see [3], Section 9) the function

$$v = v(x) = \varepsilon\, e^{-1/|x|^\beta}, \qquad \epsilon > 0,\ \beta > 0,$$

has $||v||_{L^\infty} = \epsilon$, while

$$\Delta_p\, v = \{n - p - \beta(p-1) + (p-1)\beta\,|x|^{-\beta}\}\,|x|^{-p-\beta\,(p-1)}\{\beta\, v(x)\}^{p-1}.$$

The right side is then positive if $\beta = (n-p)/(p-1)$. (Also then $||\Delta_p\, v||_{L^\infty} \le \text{Const.}\,\varepsilon^{p-1}$ so even $\Delta_p\, v$ can be arbitrarily small.)

In the following theorem we consider the case $q + \ell < \theta$, previously untreated. Here we recall that

$$\nu = \frac{p - \ell - s - t}{\theta - q - \ell}, \qquad \kappa = \frac{p - n - s}{\theta}.$$

Theorem 11. *Let u be a C^1 entire distribution solution of* (1)*, with* (3) - (6) *in force. Suppose that $q + \ell < \theta$, $\ell \le p - 1$. There are two cases:*

(*i*) $s + t < p - \ell$, *and*

$$(q + \ell)\,\kappa + n - \ell - t \geq 0,$$

with

$$u(x) = O(|x|^k) \quad \text{as } |x| \to \infty \tag{7.4}$$

for some $k \in (0, \nu)$.

(*ii*) $s + n < p$, *and*

$$(q + \ell)\,\kappa + n - \ell - t \leq 0,$$

with

$$u(x) = O(|x|^k) \quad \text{as } |x| \to \infty \tag{7.5}$$

for some $k \in (0, \kappa)$.

If either case (*i*) *or* (*ii*) *holds, then* u *must be identically constant in* $\mathbf{R}^n$.

Remarks. In the interior of region (i) we have $\nu\kappa$, while $\nu < \kappa$ in the interior of region (ii). On the boundary between regions (i) and (ii) one has $\nu = \kappa$. See Figure 4.

Theorem 11 provides a significant generalization of Theorem B of [3], except that Theorem B allows the slightly weaker condition $u(x) = o(|x|^\nu)$ as $|x| \to \infty$ in case (i). On the other hand, for the region (ii) Theorem B either gives a weaker result than Theorem 11 or no result at all. Of course, even more, Theorem B applies only for the special case $\ell = 0$.

Proof of Theorem 11. *Case* (i). Let $k < \nu$. Then by the growth condition (7.4) there exist positive constants M, $R_1 \geq R_0$ depending on k such that

$$u(x) \leq M|x|^k \qquad \text{for } |x| \geq R_1. \tag{7.6}$$

Put $\tau = \theta - q - \ell + \varepsilon'$, where $\varepsilon' > 0$ is a small constant to be determined later. As in the proof of Theorem 10, we use the calculations of Lemma 2.3, with however in (2.5) the value q being replaced by $q + \tau$, t replaced by $t + k\tau$ and $C_{\mathcal{B}}$ replaced by $C_{\mathcal{B}}/M^\tau$. In this case we have, first, $(q + \tau) + \ell = \theta + \varepsilon'\theta$, as required. Also the left side of (2.4) takes the form

$$\frac{q + \ell + \alpha + \tau}{\alpha + \theta} = \frac{\alpha + \theta + \varepsilon'}{\alpha + \theta} > 1$$

so that (2.4) now holds because of the condition $\ell \leq p - 1$. Then from (2.2), (6), and the revised inequality (2.5), and with φ as in the proof of Lemma 2.3, there results in place of (7.2)

$$\begin{aligned} \min\{\alpha, 1\} \int_{B_{R_1} \cap \{u>0\}} & [\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)u]\, u^{\alpha-1} \\ & \leq \frac{C_{\mathcal{B}}}{M^\tau} \int_{R \leq |x| \leq 2R \cap \{u>0\}} |x|^{-t-k\tau}\, u^{q+\alpha+\tau} |Du|^\ell \\ & \quad - C_{\mathcal{B}} \int_{R \leq |x| \geq R \cap \{u>0\}} |x|^{-t} u^{q+\alpha} |Du|^\ell + C_2\, R^\lambda \end{aligned} \tag{7.7}$$

for $R \geq R_1$, where

$$C_2 = 2^{n+|v|}\,\omega_n\, C_{\mathcal{A}} \left(\frac{C_{\mathcal{A}}\, M^\tau}{C_{\mathcal{B}}} \right)^{\bar\gamma/\bar\beta} \left(\frac{\bar\gamma}{\alpha^{1/\bar\alpha}} \right)^{\bar\gamma}$$

(with q replaced by $q+\tau$ and t by $t+k\tau$) and where

$$\lambda = (q+\ell+\alpha+\tau)\,\hat\nu + n - \ell - t - k\tau, \qquad \hat\nu = \frac{s+t+\ell-p+k\tau}{q+\ell+\tau-\theta}.$$

Using $\tau = \theta - \; q - \ell + \varepsilon'$ then gives, after a short calculation,

$$\lambda = \frac{\alpha+\theta}{\varepsilon'}\,(s+t+\ell-p+k\tau) + s + n - p. \tag{7.8}$$

By (7.6) the term u^τ in the first integral on the right in (7.7) is dominated by $M^\tau |x|^{k\tau}$; that is, the first integral is dominated by the second, so that (7.7) becomes

$$\min(\alpha, 1)\, J \leq C_2\, R^\lambda \tag{7.9}$$

where λ is given by (7.8).

Since $k < \nu$ we can write $k = \nu - \varepsilon$ with $\varepsilon \in (0, \nu)$. Another short calculation gives

$$s+t+\ell-p+k\tau = s+t+\ell-p+\nu\,\tau - \tau\varepsilon = \nu\varepsilon' - \tau\varepsilon.$$

Now take

$$\varepsilon' = \varepsilon\,\frac{\theta - q - \ell}{2\nu - \varepsilon}.$$

Then $\nu\varepsilon' - \tau\varepsilon = -\nu\epsilon'$, whence finally

$$\lambda = -(\alpha+\theta)\,\nu + s + n - p.$$

Thus, whether $s+n-p$ is negative or not, if α is suitably large we have $\lambda < 0$. But then letting $R \to \infty$ yields $J = 0$ in view of (7.9), and the proof is then completed as usual.

Case (ii). Since $\mathcal{B}(x, u, Du)\ \mathrm{sign}\, u \geq 0$ it follows that

$$\operatorname{div} \mathcal{A}(x, u, Du) \begin{cases} \geq 0 & \text{if } u \geq 0, \\ \leq 0 & \text{if } u \leq 0. \end{cases} \tag{7.10}$$

But then a trivial modification of the proof of Theorem 10 shows that if $u(x) = O(|x|^k)$ as $|x| \to \infty$ with $k < \kappa$, then any entire C^1 solution of (7.1) is constant.

The following theorem extends Theorem A of [3] to the case $\ell > 0$. The proof is essentially the same as for Theorem 11.

Theorem 12. *Let u be a C^1 entire distribution solution of* (1)*, with* (3) - (6) *in force. Suppose that $q+\ell = \theta$, $\ell \leq p-1$. There are two cases:*

(i) $s + t < p - \ell$, *with* $u(x)$ *having algebraic growth as* $|x| \to infty$.

(ii) $s + n < p$, *and* $s + t \geq p - \ell$,
with

$$u(x) = O(|x|^k) \quad \text{as } |x| \to \infty$$

for some $k \in (0, \kappa)$.

If either case (i) or (ii) holds, then u must be identically constant in $\mathbf{R}^n$.

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