Embedded area-constrained Willmore tori of small area in Riemannian three-manifolds II: Morse Theory

Norihisa Ikoma
Faculty of Mathematics and Physics
Institute of Science and Engineering
Kanazawa University
Kakuma, Kanazawa
Ishikawa 9201192, JAPAN
ikoma@se.kanazawa-u.ac.jp

Andrea Malchiodi
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa, ITALY
andrea.malchiodi@sns.it

Andrea Mondino
Mathematics Institute
Zeeman Building
University of Warwick
Coventry CV4 7AL, UK
A.Mondino@warwick.ac.uk

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Abstract

This is the second part of a series of two papers where we construct embedded Willmore tori with small area constraint in Riemannian three-manifolds. In both papers the construction relies on a Lyapunov-Schmidt reduction, the difficulty being the Möbius degeneration of the tori. In the first paper the construction was performed via minimization, here by Morse Theory. To this aim we establish new geometric expansions of the derivative of the Willmore functional on small Clifford tori (in geodesic normal coordinates) which degenerate to small geodesic spheres with a small handle under the action of the Möbius group. By using these sharp asymptotics we give sufficient conditions, in terms of the ambient curvature tensors and Morse inequalities, for having existence/multiplicity of embedded tori which are stationary for the Willmore functional under the constraint of prescribed (sufficiently small) area.

Key Words: Willmore functional, Willmore tori, nonlinear fourth order partial differential equations, Lyapunov-Schmidt reduction, Morse theory.

AMS subject classification:
49Q10, 53C21, 53C42, 35J60, 83C99.

1 Introduction

This is the second part of a series of two papers where embedded area-constrained Willmore tori in Riemannian 3-manifolds are constructed. Here the construction is performed via Morse theory, whereas in the previous paper [12] it was achieved via minimization/maximization.

Let us start by recalling the basic definitions and properties of the Willmore functional. Given an immersion $i : \Sigma \hookrightarrow (M, g)$ of a closed (compact without boundary) 2-dimensional surface Σ into a Riemannian 3-manifold (M, g), the *Willmore functional* is defined by

$$W(i) := \int_\Sigma H^2 \, d\sigma$$

where $d\sigma$ is the area form induced by the immersion and H is the mean curvature (we adopt the convention that H is the sum of the principal curvatures or, in other words, H is the trace of the second fundamental form A_{ij} with respect to the induced metric $\bar{g}_{ij}$, i.e. $H := \bar{g}^{ij} A_{ij}$).

An immersion i is called *Willmore surface* (or Willmore immersion) if it is a critical point of the Willmore functional with respect to normal perturbations or, equivalently, if it satisfies the associated Euler-Lagrange equation

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(n,n) = 0. \tag{1}$$

Here $\Delta_{\bar{g}}$ is the Laplace-Beltrami operator corresponding to the induced metric $\bar{g}$, $(\mathring{A})_{ij} := A_{ij} - \frac{1}{2} H \bar{g}_{ij}$ is the trace-free second fundamental form, n is a normal unit vector to i, and Ric is the Ricci tensor of the ambient manifold (M, g). Since of course a minimal immersion (i.e. an immersion with vanishing mean curvature) satisfies the Willmore equation, Willmore surfaces are a natural higher order generalization of minimal surfaces. Analogously, area-constrained Willmore surfaces satisfy the equation

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(n,n) = \lambda H,$$

for some $\lambda \in \mathbb{R}$ playing the role of Lagrange multiplier. These immersions are naturally linked to the *Hawking mass*

$$m_H(i) := \frac{\sqrt{Area(i)}}{64\pi^{3/2}} \left(16\pi - W(i)\right),$$

a quantity introduced in general relativity to measure the mass of a portion of space by means of the bending effect on light rays. Clearly, by the latter formula, the critical points of the Hawking mass under area constraint are exactly the area-constrained Willmore immersions (see [18] and the references therein for more material about the Hawking mass).

In case the ambient manifold is the *Euclidean three-dimensional space*, the Willmore functional is invariant under the action of the Möbius group (i.e. under composition of the immersion with isometries, homotheties and inversions with respect to spheres), so the theory of Willmore surfaces can be seen as a natural merging between *conformal invariance* and *minimal surface theory*. This was indeed the motivation of Blaschke and Thomsen in the 1920-'30 to introduce such an energy, rediscovered by Willmore [41] in the 60's and thoroughly studied in the last twenty years by a number of authors [5, 6, 15, 21, 22, 33, 35, 36, 37] (for more details see the introduction of our first paper [12]). Here let us just recall that the minimum of W among all immersed surfaces in $\mathbb{R}^3$ is achieved by the round sphere [41], the minimum among immersed surfaces of strictly positive genus is achieved by the Clifford torus and its Möbius deformations (the existence of a smooth minimum among genus one surfaces was proved by Simon [36], the characterization of the minimum was the long standing Willmore conjecture recently proved by Marques-Neves [22]), and for every positive genus the infimum is achieved by a smooth immersion (the proof of Bauer-Kuwert [5] is built on top of Simon's work [36] and some geometric ideas of Kusner [13]; see also the different approach by Rivière [33, 34]) but it is a challenging open problem to characterize such immersion.

While all the aforementioned results about Willmore surfaces concern immersions into the Euclidean space (or, equivalently by conformal invariance, for immersions into a round sphere); the results concerning Willmore immersions into curved Riemannian manifolds are much more limited and recent. In a first stage [9, 16, 17, 19, 18, 24, 25] the existence of Willmore spheres has been investigated in a perturbative setting. The global variational problem, i.e. the existence of smooth immersed spheres minimizing quadratic curvature functionals in compact Riemannian 3-manifolds, was then studied in [14] by extending the Simon's ambient approach to Riemannian manifolds (see also [29] for the non compact case). In [27]-[28], a parametric approach for weak immersions into Riemannian manifolds was developed and the existence of branched area-constrained Willmore spheres in homotopy classes established (as well as the existence of Willmore spheres under various assumptions and constraints).

Since all the above existence results in Riemannian manifolds concern surfaces *of genus* 0, a natural question is about the *existence of higher genus Willmore surfaces in general curved spaces*; in particular we will focus on the *genus one* case.

Let us mention that if the ambient space has some special symmetry then the Willmore equation (1) simplifies and it is possible to construct explicit examples (see for instance [39] for product manifolds and [4] for warped product metrics). See also [10] for the existence of stratified weak branched immersions of arbitrary genus minimizing quadratic curvature functionals under various constraints.

The goal of the present (and the previous [12]) work is to construct smooth embedded Willmore tori with small area constraint in Riemannian 3-manifolds, under some curvature/topological condition but without any symmetry assumption. More precisely we obtain the following main result.

Theorem 1.1 (Existence)**.** *Let (M,g) be a smooth closed orientable three-dimensional Riemannian manifold. Assume that the scalar curvature is a Morse function and that at every critical point P of the scalar curvature, the Ricci tensor has three distinct eigenvalues. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there exists a smooth embedded Willmore torus in (M,g) with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$. More precisely, the above surfaces are obtained as normal graphs over exponentiated (Möbius transformations of) Clifford tori and the corresponding graph functions (dilated by a factor $1/\varepsilon$) converge to 0 in $C^{4,\alpha}$-norm as $\varepsilon \to 0$ with decay rate $O(\varepsilon^2)$.*

Remark 1.1. *(i) The assumptions in Theorem 1.1 are generic in the metric g.*

(ii) If the Ricci tensor is not a multiple of the identity at all points of global maximum and minimum of the scalar curvature then we have at least two critical tori, see Remark 5.4.

We also obtain a generic multiplicity result. To state it, we need to introduce some more notation. As above assume that (M,g) is a closed connected and orientable three-manifold, that the scalar curvature $P \mapsto \mathrm{Sc}_P$ is a Morse function and that at every critical point P of the scalar curvature the Ricci tensor Ric_P has three distinct eigenvalues. For $q = 0, \dots, 3$, we set

$$C_q := \sharp\{P_i \in M \ : \ \nabla \mathrm{Sc}(P_i) = 0, \quad \mathrm{index}\,(-\nabla^2 \mathrm{Sc}(P_i)) = q\};$$

we then define

$$\tilde{C}_0 = \tilde{C}_1 := 0, \quad \tilde{C}_2 := 4\,C_0, \quad \tilde{C}_q := 4\,C_{q-2} + 2\,C_{q-3},\ q = 3,4,5, \quad \tilde{C}_6 := 2\,C_3. \tag{2}$$

Finally, considering the Betti numbers of M with $\mathbb{Z}_2$ coefficients

$$\beta_q := \mathrm{rank}_{\mathbb{Z}_2}(H_q(M;\mathbb{Z}_2)); \qquad q \geq 0,$$

we define

$$\tilde{\beta}_0 = 1; \quad \tilde{\beta}_1 = \beta_1 + 1; \quad \tilde{\beta}_2 = \tilde{\beta}_3 = \beta_1 + \beta_2 + 1; \quad \tilde{\beta}_4 = \beta_2 + 1; \quad \tilde{\beta}_5 = 1; \quad \tilde{\beta}_k = 0 \quad \text{for } k \geq 6. \tag{3}$$

Remark 1.2. (i) *The numbers $\tilde{\beta}_q$ are the Betti numbers (with $\mathbb{Z}_2$ coefficients) of the projective tangent bundle over M. By a classical result of differential topology due to Stiefel (see for instance [23, page 148]), three-dimensional oriented manifolds are parallelizable, i.e. the tangent bundle is trivial: $TM \simeq M \times \mathbb{R}^3$. As a consequence, the projective tangent bundle is homeomorphic to $M \times \mathbb{RP}^2$. Since $H_k(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2$ for $0 \leq k \leq 2$ and zero otherwise, the $\tilde{\beta}$'s can be computed as a direct application of Künneth's formula.*

(ii) *Using the homology of M with $\mathbb{Z}_2$ coefficients is more convenient than using standard $\mathbb{Z}$ coefficients for a number of reasons. First of all Künneth's formula turns out to be easier. Secondly, the Betti numbers with $\mathbb{Z}_2$ coefficients of a compact manifold X are always bounded below by the Betti numbers with $\mathbb{Z}$ coefficients, this because they also keep track of the $\mathbb{Z}_2$-torsion part. The precise relation between the two is given by the Universal Coefficients Theorem (see for instance [11, Chapter 3.A]), which implies that $H_k(X, \mathbb{Z}_2)$ consists of*

- *a $\mathbb{Z}_2$ summand for each $\mathbb{Z}$ summand of $H_k(X, \mathbb{Z})$,*
- *a $\mathbb{Z}_2$ summand for each $\mathbb{Z}_{2^n}$ summand in $H_k(X, \mathbb{Z})$, $n \geq 1$,*
- *a $\mathbb{Z}_2$ summand for each $\mathbb{Z}_{2^n}$ summand in $H_{k-1}(X, \mathbb{Z})$, $n \geq 1$.*

In particular, in our case of $X = M \times \mathbb{RP}^2$, the $\mathbb{Z}$-Betti numbers vanish in dimension larger than three while the $\mathbb{Z}_2$-Betti numbers do not vanish in dimension 4 and 5. Clearly this permits stronger conclusions in terms of existence and multiplicity of critical points via Morse-theoretic arguments.

Now we are ready to state our second main theorem.

Theorem 1.2 (Generic multiplicity)**.** *Let (M, g) be a smooth closed orientable three-dimensional Riemannian manifold. Then for generic metrics g, if $\tilde{\beta}_q - \tilde{C}_q > 0$ for some $q \in \{0, \dots, 4\}$, then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there are at least $\tilde{\beta}_q - \tilde{C}_q$ smooth embedded Willmore tori in (M, g) with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$ and with index q. In particular there are at least $\sum_{q=0}^{4}(\tilde{\beta}_q - \tilde{C}_q)^+$ area-constrained Willmore tori.*

Remark 1.3. *Notice that we always have $\tilde{\beta}_q - \tilde{C}_q > 0$, for $q = 0, 1$, so the above result implies in particular that for generic metrics there exist at least two area-constrained Willmore tori, one with index zero and the other with index one, the index being intended for critical points of the Willmore functional under area constraint. Also, as the Morse inequalities on M imply $C_q \geq \beta_q$ for generic metrics, the condition $\tilde{\beta}_q - \tilde{C}_q > 0$ is not satisfied for $q = 5$ or $q = 6$.*

Examples. If M is homeomorphic to S^3, $S^2 \times S^1$ or $S^1 \times S^1 \times S^1$, we get the following values for $\tilde{\beta}_k$.

$$\begin{aligned} M = S^3 \quad &: \quad \tilde{\beta}_k = 1 \text{ for } k = 0, \dots, 5,\ \tilde{\beta}_k = 0 \text{ for } k \geq 6. \\ M = S^2 \times S^1 \quad &: \quad \tilde{\beta}_0 = \tilde{\beta}_5 = 1,\ \tilde{\beta}_1 = \tilde{\beta}_4 = 2,\ \tilde{\beta}_2 = \tilde{\beta}_3 = 3,\ \tilde{\beta}_k = 0 \text{ for } k \geq 6. \\ M = (S^1)^3 \quad &: \quad \tilde{\beta}_0 = \tilde{\beta}_5 = 1,\ \tilde{\beta}_1 = \tilde{\beta}_4 = 4,\ \tilde{\beta}_2 = \tilde{\beta}_3 = 7,\ \tilde{\beta}_k = 0 \text{ for } k \geq 6. \end{aligned}$$

Outline of the strategy

As in our first paper [12] the proof relies on a Lyapunov-Schmidt reduction (encoding the variational structure of the problem, see [1, 2] and the book [3]). Using such techniques, together with the stability property of Clifford tori proved by Weiner [40] (see also the related gap-energy result [26]), we reduce the problem of finding area-constrained Willmore tori to a *finite dimensional* variational problem. More precisely we consider the finite dimensional space of the images, via the exponential map in (M, g), of Möbius-inverted Clifford tori with small area. Notice that since the action of the Möbius group is non-compact, such a finite dimensional space is non-compact too, with degeneracy due to the presence of a shrinking handle.

In the present work, the rough idea used to infer existence of critical points is to exploit the topology of the finite dimensional space $\mathcal{T}_\varepsilon$ of exponentiated and rotated Möbius images of Clifford tori having area $4\sqrt{2}\pi^2\varepsilon^2$ and argue via a Morse-theoretical approach. To this aim, recalling that under our assumptions M is parallelizable, we first observe that the space $\mathcal{T}_\varepsilon$ is diffeomorphic to $M \times \mathbb{B}\mathbb{RP}^2$, $\mathbb{B}\mathbb{RP}^2$ being the bundle of tangent vectors to $\mathbb{RP}^2$ with length less than 1. The geometric situation of tori degenerating to geodesic spheres with shrinking handles corresponds to approaching the boundary of $M \times \mathbb{B}\mathbb{RP}^2$, consisting in the vectors of length one in $T\mathbb{RP}^2$. In order to apply Morse theory to a manifold with boundary (see for instance the classical work of Morse-Van Schaack [31]) it is crucial to understand the normal derivative at the boundary of the manifold; this corresponds in our framework to computing the derivative with respect to the Möbius parameter. Such a computation is quite delicate since we need sharp estimates and since the torus is degenerating (as it is natural to expect, the computation involves singular integrals); this will take a large part of the present paper (for the final result see Proposition 4.2 and Remark 4.10).

A crucial role in such an expansion of the normal derivative is played by the function $\mathcal{F}$ defined below. Given $P \in M$, and an orthonormal frame $\{\mathbf{e}_{P,1}, \mathbf{e}_{P,2}, \mathbf{e}_{P,3}\}_{P\in M}$ at P, we define $\mathcal{F}(P,\cdot) : SO(3) \to \mathbb{R}$ by

$$\mathcal{F}(P,R) := \mathrm{Ric}_P(R\mathbf{e}_{P,2}, R\mathbf{e}_{P,2}) - \mathrm{Ric}_P(R\mathbf{e}_{P,3}, R\mathbf{e}_{P,3}).$$

The assumptions of Theorem 1.1 imply indeed the following non-degeneracy condition for Sc_P and $\mathcal{F}$: for the proof of the second one see Proposition 6.5.

$(ND1)$ The function $P \mapsto \mathrm{Sc}_P : M \to \mathbb{R}$ is a Morse function. In particular, Sc has finitely many critical points $P_1, \dots, P_k$.

$(ND2)$ For each $i = 1, \dots, k$, $\mathcal{F}_i(R) := \mathcal{F}(P_i, R) : SO(3) \to \mathbb{R}$ is a Morse function for every $1 \le i \le k$, and $\mathcal{F}_i(R) \neq 0$ if $\nabla \mathcal{F}_i(R) = 0$.

By $(ND2)$, every $\mathcal{F}_i$ has finitely many critical points and we call them $R_{i,1}, \dots, R_{i,\ell_i}$: recalling (2), by Proposition 6.5 it turns out that

$$\tilde{C}_q := \frac{1}{2}\sharp\{(P_i, R_{i,\ell}) \in M \times SO(3) \ : \ \mathrm{index}\,(-\nabla^2 \mathrm{Sc}(P_i)) + \mathrm{index}\,(-\nabla^2 \mathcal{F}_i(R_{i,\ell})) = q \text{ and } \mathcal{F}_i(R_{i,\ell}) < 0\}. \tag{4}$$

By our energy expansions, see Section 5, the $\tilde{C}_q$'s represent the numbers of critical points of index q for the restriction of the Willmore to the boundary of $\mathcal{T}_\varepsilon$ (defined above) such that the gradient of the energy points inwards $\mathcal{T}_\varepsilon$. Notice that the factor $\frac{1}{2}$ in the definition of $\tilde{C}_q$ is a consequence of the symmetry of the degenerate Clifford torus: indeed for every degenerate Clifford torus there exists a non trivial rotation $R \in SO(3), R \neq Id$ leaving the surface invariant (for more details see Remark 5.1). The conclusion of Theorems 1.1 and 1.2 will then follow from the general results in [31].

Besides Theorems 1.1 and 1.2, the main contribution of the present paper is the aforementioned expansion for the derivative of the Willmore energy on degenerating tori (see Proposition 4.2). We believe that it might play a role in further developments of the topic, especially in ruling-out possible degeneracy phenomena under global (non-perturbative) variational approaches to the problem, as it has already happened for the case of Willmore spheres (see for instance [14, 28, 29]).

The outline of the paper is as follows: in Section 2 we recall some preliminary results, as well as the finite-dimensional reduction of the constrained Willmore problem from [12]. In Section 3 we analyse in detail the Möbius degeneration of Clifford tori to spheres, describing their asymptotics (away from the shrinking handle) as normal graphs. In Section 4 we derive one core estimate, namely the variation of the Willmore energy on (degenerated) Clifford tori with respect to the Möbius parameter. In Section 5 we prove our main theorem via Morse theory, and finally in the Appendix we collect some explicit computations.

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2 Preliminaries

Denoting by g_0 the flat Euclidean metric, let us first state a basic property of the Willmore functional W_{g_0} for immersions $i : \Sigma \to \mathbb{R}^3$

$$W(i(\Sigma)) = \int_\Sigma H^2 d\sigma.$$

Proposition 2.1. *Let Σ be a closed surface of class C^2 and let $i:\Sigma\to\mathbb{R}^3$ be an immersion. Then, if $\lambda>0$ and if $\Phi_{x_0,\eta}$ is a Möbius inversion (see* (5)*), one has the invariance properties*

$$a)\quad W_{g_0}(\lambda i(\Sigma))=W_{g_0}(i(\Sigma))\qquad \textit{and}\qquad b)\quad W_{g_0}((\Phi_{x_0,\eta}\circ i)(\Sigma))=W_{g_0}(i(\Sigma))\quad \textit{provided } x_0\notin i(\Sigma).$$

We will next introduce some notation and recall the finite-dimensional reduction procedure from [12].

2.1 Notation and small tori in manifolds

We consider the standard Clifford torus $\mathbb{T}$ obtained via the following parametrization

$$\mathbb{T}:=\left\{X(\tilde\varphi,\tilde\theta)\ :\ \tilde\varphi,\tilde\theta\in[-\pi,\pi]\right\}$$

where

$$X(\tilde\varphi,\tilde\theta):=\left((\sqrt{2}+\cos\tilde\varphi)\cos\tilde\theta,\ (\sqrt{2}+\cos\tilde\varphi)\sin\tilde\theta,\ \sin\tilde\varphi\right).$$

For $x_0\in\mathbb{R}^3$ and $\eta>0$, the spherical inversion with respect to $\partial B_\eta(x_0)$ is defined by

$$\Phi_{x_0,\eta}(x):=\frac{\eta^2}{|x-x_0|^2}(x-x_0)+x_0. \tag{5}$$

For any smooth compact surface $\Sigma\subset\mathbb{R}^3\backslash\{x_0\}$, we set $\overline{\Sigma}:=\Phi_{x_0,\eta}(\Sigma)$ and we denote the volume elements of Σ and $\overline{\Sigma}$ by $d\sigma_\Sigma$ and $d\sigma_{\overline{\Sigma}}$ respectively. Then it is well known that

$$d\sigma_{\overline{\Sigma}}=\frac{\eta^4}{|x-x_0|^4}d\sigma_\Sigma.$$

We are interested in Möbius maps which preserve the area of $\mathbb{T}$: we first translate the torus by the vector $-(\sqrt{2}+1+\xi)\mathbf{e}_x$, $\xi>0$ where $\mathbf{e}_x:=(1,0,0)$ (so that it will be contained in $\{x_1<0\}$), and then choose $\xi=\xi_\eta>0$ depending on η so to preserve the area (see Lemma 2.1 in [12]). We set

$$\mathbb{T}_{\xi_\eta}:=\mathbb{T}-(\sqrt{2}+1+\xi_\eta)\mathbf{e}_x,\quad Y(\tilde\varphi,\tilde\theta,\eta):=X(\tilde\varphi,\tilde\theta)-(\sqrt{2}+1+\xi_\eta)\mathbf{e}_x$$

and observe that

$$4\sqrt{2}\pi^2=\mathrm{Area}(\Phi_{0,\eta}(\mathbb{T}_{\xi_\eta}))=\eta^4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sqrt{2}+\cos\tilde\varphi}{|Y(\tilde\varphi,\tilde\theta,\eta)|^4}d\tilde\varphi d\tilde\theta. \tag{6}$$

Our aim is to describe degenerating tori, namely to understand quantitatively the behaviours of ξ_η and $\Phi_{0,\eta}(\mathbb{T}_{\xi_\eta})$ as $\eta\to0$. To do so, we define the following map:

$$Z(\bar\varphi,\bar\theta,\eta):=\Phi_{0,\eta}\left(Y(\eta^2\bar\varphi,\eta^2\bar\theta,\eta)\right)=\Phi_{0,1}\left(\eta^{-2}Y(\eta^2\bar\varphi,\eta^2\bar\theta,\eta)\right)$$

for $(\bar\varphi,\bar\theta)\in\mathbb{R}^2$. In Section 2 of [12] the following result was proved.

Lemma 2.2. *([12]) For each $\eta>0$, there exists a unique $\xi_\eta>0$ such that*

$$\mathrm{Area}\left(\Phi_{0,\eta}(\mathbb{T}_{\xi_\eta})\right)=4\sqrt{2}\pi^2.$$

Moreover, the map $\eta\mapsto\xi_\eta:(0,\infty)\to(0,\infty)$ is smooth and strictly increasing in $(0,\infty)$. In addition, $\xi_\eta\to0$ as $\eta\to0$ and $\xi_\eta\to\infty$ as $\eta\to\infty$. Furthermore we have the properties

(i) $\eta^4/\xi_\eta^2=4\sqrt{2}\pi+O(\eta^2)$ *as* $\eta\to0$.

(ii) $\Phi_{0,\eta}(\mathbb{T}_{\xi_\eta})$ *converges to the sphere with radius* $\sqrt[4]{2\pi^2}$ *centred at* $-\sqrt[4]{2\pi^2}\mathbf{e}_x$ *in the following sense: for any* $R > 0$ *and* $k \in \mathbb{N}$*, if* $\eta \leq 1/R^4$*, then*

$$\|Z(\cdot,\cdot,\eta) - Z_0\|_{C^k([-R,R]^2)} \leq C_k \eta^{3/2}$$

as $\eta \to 0$*, where* C_k *depends only on* k *and* Z_0 *is defined by*

$$Z_0(\bar{\varphi},\bar{\theta}) := \Phi_{0,1}\left(-\frac{1}{2\sqrt[4]{2\pi^2}}\mathbf{e}_x + (\sqrt{2}+1)\bar{\theta}\mathbf{e}_y + \bar{\varphi}\mathbf{e}_z\right)$$

where $\mathbf{e}_y := (0,1,0)$ *and* $\mathbf{e}_z := (0,0,1)$.

A more detailed analysis of ξ_η will be carried out in Section 3. Incorporating also rotations around the z axis, we obtain a smooth two-dimensional family of tori with the same area which includes $\mathbb{T}$. Its properties can be summarized in the following result.

Proposition 2.3. *([12], Section 2) There exists a smooth family of conformal immersions* T_ω *of* $\mathbb{T}$ *into* $\mathbb{R}^3$*, parametrized by* $\omega \in \mathbb{D}$*,* $\mathbb{D}$ *being the unit disk in* $\mathbb{R}^2$*, which preserves the area of* $\mathbb{T}$ *and for which the following hold*

a) $T_0 = Id$*;*

b) for $\omega \neq 0$*,* T_ω *is an inversion with respect to a sphere centred at a point in* $\mathbb{R}^3$ *aligned to* ω *(viewed as an element of* $\mathbb{R}^3$ *with null* z*-component);*

c) as $|\omega|$ *approaches* 1*,* $T_\omega(\mathbb{T})$ *degenerates to a sphere of radius* $\sqrt[4]{2\pi^2}$ *centred at* $\sqrt[4]{2\pi^2}\frac{\omega}{|\omega|}$*.*

In what follows, we will use the symbol $\mathbb{T}_\omega$ for $T_\omega(\mathbb{T})$. We will describe next the global structure of exponential maps of scaled and rotated tori in the manifold M.
For each $P \in M$ we construct a family of surfaces from $\mathbb{T}$, $R \in SO(3)$ and T_ω:

$$\{\exp_P(\varepsilon R \mathbb{T}_\omega) \;:\; R \in SO(3),\; \omega \in \mathbb{D}\},$$

where $\varepsilon > 0$ is chosen small. Notice that, due to the rotation invariance of the Clifford torus $\mathbb{T}$, the above family is 4-dimensional and not 5-dimensional; indeed it is not difficult to see that it can be parametrized by $\mathbb{B}\mathbb{RP}^2$, the bundle of tangent vectors to $\mathbb{RP}^2$ with length less than 1. Letting then P vary, we obtain a seven-dimensional bundle over M with fiber $\mathbb{B}\mathbb{RP}^2$. We will see in the next subsection that the above tori form a family of approximate solutions to our problem, and that they may be slightly modified to become true solutions.

Remark 2.4. *In order to further simplify the notation we will sometimes parametrize the space of exponentiated tori* $\exp_P(\varepsilon R \mathbb{T}_\omega)$ *by* $(P, R, \omega) \in M \times SO(3) \times \mathbb{D}$*. Notice that in this way we are using an extra parameter; this has the advantage of simplifying our notation.*

2.2 Finite-dimensional reduction

We also recall the finite-dimensional procedure in [12, Section 3] to attack the constrained Willmore problem. This procedure consists in finding first a family of approximate solutions, which will be then adjusted to constrained Willmore surfaces up to some Lagrange multiplier.

We fix a compact set K (typically, a closed ball centred at the origin) of the unit disk $\mathbb{D}$ and we consider then the family

$$\hat{\mathcal{T}}_{\varepsilon,K} = \{\varepsilon\, R\, \mathbb{T}_\omega \;:\; R \in SO(3), \omega \in K\}\,.$$

We notice that, by construction, elements in $\hat{\mathcal{T}}_{\varepsilon,K}$ consists of Willmore surfaces in $\mathbb{R}^3$ all with area identically equal to $4\varepsilon^2\sqrt{2}\pi^2$. We then construct a family of surfaces in M defined by exponential maps of elements in $\hat{\mathcal{T}}_{\varepsilon,K}$ from arbitrary points P of M. Here we remark that since M is parallelizable (see Remark 1.2), there exist a global orthonormal frame $\{F_{P,1},F_{P,2},F_{P,3}\}_{P\in M}$ and we may identify TM with $M\times\mathbb{R}^3$. Using this identification, we may also regard the exponential map $\exp_P^g$ as a map from $\mathbb{R}^3$ into M for each $P\in M$. Then we set

$$\mathcal{T}_{\varepsilon,K}=\left\{\exp_P(\Sigma)\ :\ P\in M,\Sigma\in\hat{\mathcal{T}}_{\varepsilon,K}\right\}. \tag{7}$$

It will be convenient for us to scale coordinates in order to work with surfaces whose area is of order 1, exploiting the scaling invariance of the Willmore functional. Precisely, introduce a new metric g_ε by

$$g_\varepsilon(P):=\frac{1}{\varepsilon^2}g(P).$$

Then we have the following facts: (see Section 3 in [12])

(i) Write W_g and W_{g_ε} for the Willmore functional on (M,g) and (M,g_ε). Then Σ is a Willmore surface with the area constraint in (M,g) if and only if it is so in (M,g_ε).

(ii) The exponential maps $\exp_P^g$ on (M,g) are diffeomorphic on the Euclidean ball B_{ρ_0} for each $P\in M$ and satisfies

$$\exp_P^g(\varepsilon z)=\exp_P^{g_\varepsilon}(z)$$

for all $|z|\le\varepsilon^{-1}\rho_0$ where $\exp_P^{g_\varepsilon}$ is the exponential map on (M,g_ε).

(iii) Set $g_P:=(\exp_P^g)^*g$ and $g_{\varepsilon,P}:=(\exp_P^{g_\varepsilon})^*g_\varepsilon$. Then $g_{\varepsilon,P,\alpha\beta}$ has the following expansion:

$$g_{\varepsilon,P,\alpha\beta}(y)=\delta_{\alpha\beta}+\varepsilon^2h^\varepsilon_{P,\alpha\beta}(y)\qquad\text{for each } |y|_{g_0}\le\varepsilon^{-1}\rho_0 \tag{8}$$

where $h^\varepsilon_{P,\alpha\beta}(y)$ satisfies

$$h^\varepsilon_{P,\alpha\beta}(y)=\frac{1}{3}R_{\alpha\mu\nu\beta}y^\mu y^\nu+\tilde{R}_{\alpha\beta}(\varepsilon,y),\quad \sum_{i=0}^{\ell}\left|\nabla^i\tilde{R}(\varepsilon,\cdot)\right|\le C_\ell\varepsilon^3, \tag{9}$$

$$|y|^{-2}|h^\varepsilon_{P,\alpha\beta}(y)|+|y|^{-1}|\nabla_y h^\varepsilon_{P,\alpha\beta}(y)|+\sum_{i=2}^{\ell}|\nabla^i h^\varepsilon_{P,\alpha\beta}(y)|\le h_{0,\ell}, \tag{10}$$

$$|y|^{-2}|D_P^{k+1}g_{\varepsilon,P,\alpha\beta}(y)|+|y|^{-1}|D_P^{k+1}\nabla_y g_{\varepsilon,P,\alpha\beta}(y)|+\sum_{i=2}^{\ell}|D_P^{k+1}\nabla_y^i g_{\varepsilon,P,\alpha\beta}(y)|\le C_{k,\ell}\varepsilon^2 \tag{11}$$

for all $|y|_{g_0}\le\varepsilon^{-1}\rho_0$, $k,\ell\in\mathbb{N}$. Here D_P denotes the differential by P in the original metric of M.

(iv) The family $\mathcal{T}_{\varepsilon,K}$ is expressed as

$$\mathcal{T}_{\varepsilon,K}=\{\exp_P^{g_\varepsilon}(R\mathbb{T}_\omega)\ :\ P\in M,\ R\in SO(3),\ \omega\in K\}.$$

We recall next the following well-known result concerning variations of W_{g_ε} (see for example Section 3 in [18]).

Proposition 2.5. *For an immersion $i:\Sigma\to(M,g_\varepsilon)$ one has*

$$dW_{g_\varepsilon}(i(\Sigma))[\varphi]=\int_\Sigma\left(LH+\frac{1}{2}H^3\right)\varphi\,d\sigma=-\int_\Sigma\left\{\Delta H+\left(|\mathring{A}|^2+\mathrm{Ric}(n,n)\right)H\right\}\varphi d\sigma, \tag{12}$$

where L is the elliptic, self-adjoint operator

$$L\varphi:=-\Delta\varphi-\varphi\left(|A|^2+\mathrm{Ric}(n,n)\right).$$

We also write $W'_{g_\varepsilon}(i(\Sigma)):=LH+H^3/2$.

$\mathcal{T}_{\varepsilon,K}$ form a family of approximate solutions to our problem. In fact, let us recall the following result.

Lemma 2.6. *([12], Section 3) Consider the rescaled framework described above. Fix K as before, $\ell \in \mathbb{N}$ and $\gamma \in (0,1)$. There exists a constant $C_{K,\ell}$ such that for ε small*

$$\|W'_{g_\varepsilon}(\Sigma)\|_{C^{\ell,\gamma}(\Sigma)} \le C_{K,\ell}\varepsilon^2 \qquad \text{for every } \Sigma \in \mathcal{T}_{K,\varepsilon}.$$

Next we consider small perturbations of the surfaces in $\mathcal{T}_{\varepsilon,K}$ in the following way. As in Section 3 of [12], for $(P,R,\omega) \in M \times SO(3) \times \mathbb{D}$, we denote by $g_{\varepsilon,P,R,\omega}$ the pull back of $g_{\varepsilon,P}$ via the map $R \circ T_\omega$: $g_{\varepsilon,P,R,\omega} := (R \circ T_\omega)^* g_{\varepsilon,P} = T_\omega^* \circ R^* \circ (\exp_P^{g_\varepsilon})^* g_\varepsilon$. Observe that $(\mathbb{T}, g_{\varepsilon,P,R,\omega})$ is isometric to $(\exp_P^{g_\varepsilon}(R\mathbb{T}_\omega), g_\varepsilon)$ and $(R\mathbb{T}_\omega, g_{\varepsilon,P})$. We write $n_{\varepsilon,P,R,\omega}$ for the unit outer normal to $(\mathbb{T}, g_{\varepsilon,P,R,\omega})$. Then for regular functions $\varphi : \mathbb{T} \to \mathbb{R}$, we consider perturbations of $(\mathbb{T}, g_{\varepsilon,P,R,\omega})$ as follows:

$$\begin{aligned} &(\mathbb{T}[\varphi])_{\varepsilon,P,R,\omega} := \{p + \varphi(p) n_{\varepsilon,P,R,\omega}(p) \; : \; p \in \mathbb{T}\}, \\ &(R\mathbb{T}_\omega[\varphi])_{\varepsilon,P} := \{R T_\omega(p + \varphi(p) n_{\varepsilon,P,R,\omega}(p)) \; : \; p \in \mathbb{T}\}, \quad \Sigma_{\varepsilon,P,R,\omega}[\varphi] := \exp_P^{g_\varepsilon}\Big((R\mathbb{T}_\omega[\varphi])_{\varepsilon,P}\Big). \end{aligned} \tag{13}$$

Noting that $(R\mathbb{T}_\omega[0])_{\varepsilon,P} = R\mathbb{T}_\omega$, let us also set

$$\Sigma_{\varepsilon,P,R,\omega} = \Sigma_{\varepsilon,P,R,\omega}[0] = \exp_P^{g_\varepsilon}(R\mathbb{T}_\omega). \tag{14}$$

Given a positive constant $\overline{C}$, we define next the family of functions

$$\mathcal{M}_{\varepsilon,P,R,\omega} = \left\{\varphi \in C^{4,\gamma}(\mathbb{T},\mathbb{R}) \; : \; \|\varphi\|_{C^{4,\gamma}(\mathbb{T})} \le \overline{C}\varepsilon^2 \text{ and such that } |\Sigma_{\varepsilon,P,R,\omega}[\varphi]|_{g_\varepsilon} = 4\sqrt{2}\pi^2\right\}.$$

Here we remark that since we only consider small perturbations, $\Sigma_{\varepsilon,P,R,\omega}[\varphi]$ can be expressed as a normal graph of $\mathbb{T}$. Hence, we pull back all geometric quantities of $\Sigma_{\varepsilon,P,R,\omega}[\varphi]$ onto $\mathbb{T}$. Finally, on $\mathbb{T}$, we consider Jacobi fields $Z_{i,R,\omega}$, $i = 1,\dots,7$ for $R\mathbb{T}_\omega$ which generate conformal maps preserving the area of the torus (see also the notation in [12]). Exploiting the non-degeneracy property from [40], one can prove the following result.

Proposition 2.7. *([12], Section 3) Fix a compact subset K of $\mathbb{D}$ as above. Then there exist positive constants $\bar{C}_K$ and $\bar{\varepsilon}_K$ such that for any $\varepsilon \in (0,\bar{\varepsilon}_K]$ and every $(P,R,\omega) \in M \times SO(3) \times K$, there exists a function $\varphi_\varepsilon = \varphi_\varepsilon(P,R,\omega) \in C^{5,\gamma}(\mathbb{T})$ such that*

$$a) \quad W'_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega}[\varphi_\varepsilon(P,R,\omega)]) = \beta_0 H_{\varepsilon,P,R,\omega}[\varphi_\varepsilon] + \sum_{i=1}^{7} \beta_i Z_{i,R,\omega}; \qquad b) \quad |\Sigma_{\varepsilon,P,R,\omega}[\varphi_\varepsilon]|_{g_\varepsilon} = 4\sqrt{2}\pi^2,$$

for some numbers $\beta_0,\dots,\beta_7$. Here $\Sigma_{\varepsilon,P,R,\omega}[\varphi]$ is as in (13), while $H_{\varepsilon,P,R,\omega}[\varphi_\varepsilon]$ stands for the mean curvature of $\Sigma_{\varepsilon,P,R,\omega}[\varphi_\varepsilon]$. Moreover, the map $M \times SO(3) \times K \to C^{5,\gamma}(\mathbb{T})$ defined by $(P,R,\omega) \mapsto \varphi_\varepsilon(P,R,\omega)$ is smooth and satisfies

$$\sum_{k=0}^{2} \left\|D^k_{P,R,\omega}\varphi_\varepsilon(P,R,\omega)\right\|_{C^{5,\gamma}(\mathbb{T})} \le C_K\varepsilon^2.$$

In particular, $\varphi_\varepsilon(P,R,\omega) \in \mathcal{M}_{\varepsilon,P,R,\omega}$.

We can finally encode the variational structure of the problem by means for the following result.

Proposition 2.8. *([12], Section 3) Let $K \subset\subset \mathbb{D}$, $\bar{\varepsilon}_K$ and φ_ε be as in Proposition 2.7. For $\varepsilon \in [0,\bar{\varepsilon}_K]$ define the function $\Phi_\varepsilon : \mathcal{T}_{\varepsilon,K} \to \mathbb{R}$ by*

$$\Phi_\varepsilon(P,R,\omega) := W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega}[\varphi_\varepsilon(P,R,\omega)]).$$

Then there exists $\bar{\varepsilon}'_K \in (0,\bar{\varepsilon}_K]$ such that, if $\varepsilon \in (0,\bar{\varepsilon}'_K]$ we have

$$|\Phi_\varepsilon(P,R,\omega) - W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega})| \le C_K\varepsilon^4. \tag{15}$$

For such ε's, if $(P_\varepsilon,R_\varepsilon,\omega_\varepsilon) \in M \times SO(3) \times K$ is critical for Φ_ε, then the surface $\Sigma_{\varepsilon,P_\varepsilon,R_\varepsilon,\omega_\varepsilon}[\varphi_\varepsilon(P_\varepsilon,R_\varepsilon,\omega_\varepsilon)]$ satisfies the area-constrained Willmore equation.

3 On degenerating Clifford tori

In this section we analyse Möbius-degenerating tori. In particular we improve the accuracy of the estimate (i) in Lemma 2.2 and derive the asymptotics of degenerate tori viewed as normal graphs on the limit sphere (except for the small handle), see (ii) in Lemma 2.2.

3.1 Precise asymptotics of ξ_η

The following estimate on ξ_η will be needed below.

Lemma 3.1. *In the notation of Lemma 2.2, as $\eta \to 0$, we have*

$$2\xi_\eta - \xi_\eta' \eta = O(\eta^4).$$

Proof. Recall that $\Phi_{0,\eta}(\mathbb{T}_{\xi_\eta})$ has fixed area $4\sqrt{2}\pi^2$ (see (6) and Lemma 2.2):

$$4\sqrt{2}\pi^2 = \eta^4 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sqrt{2}+\cos\tilde{\varphi}}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^4} d\tilde{\varphi} d\tilde{\theta}. \tag{16}$$

Next, we claim that

$$\frac{\xi_\eta'}{\eta} = \frac{1}{\sqrt[4]{2\pi^2}} + O(\eta^2). \tag{17}$$

Differentiating (16) with respect to η, we have

$$0 = 4\eta^3 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\sqrt{2}+\cos\tilde{\varphi}}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^4} d\tilde{\varphi} d\tilde{\theta} - 2\xi_\eta'\eta^4 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(\sqrt{2}+\cos\tilde{\varphi})f(\tilde{\varphi},\tilde{\theta},\eta)}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^6} d\tilde{\varphi} d\tilde{\theta}$$

where

$$f(\tilde{\varphi},\tilde{\theta},\eta) := 2\left\{(\sqrt{2}+1) - (\sqrt{2}+\cos\tilde{\varphi})\cos\tilde{\theta} + \xi_\eta\right\}.$$

Multiplying η by the above equality, it follows from (16) that

$$\frac{\xi_\eta'}{\eta}\eta^6 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(\sqrt{2}+\cos\tilde{\varphi})f(\tilde{\varphi},\tilde{\theta},\eta)}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^6} d\tilde{\varphi} d\tilde{\theta} = 8\sqrt{2}\pi^2.$$

Therefore, to prove (17), it suffices to show

$$\eta^6 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(\sqrt{2}+\cos\tilde{\varphi})f(\tilde{\varphi},\tilde{\theta},\eta)}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^6} d\tilde{\varphi} d\tilde{\theta} = 2^{15/4}\pi^{5/2} + O(\eta^2). \tag{18}$$

To this end, we use the following decomposition:

$$I_\eta := \left\{(\tilde{\varphi},\tilde{\theta}) \in [-\pi,\pi]^2 \ : \ \tilde{\varphi}^2 + \left(\sqrt{2}+1\right)^2\tilde{\theta}^2 \le \eta^2\right\}, \qquad J_\eta := [-\pi,\pi]^2 \setminus I_\eta.$$

First, we show

$$\eta^6 \int_{J_\eta} \frac{(\sqrt{2}+\cos\tilde{\varphi})f(\tilde{\varphi},\tilde{\theta},\eta)}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^6} d\tilde{\varphi} d\tilde{\theta} = O(\eta^4). \tag{19}$$

By a Taylor expansion at the origin, we notice that

$$\begin{aligned} |Y(\tilde{\varphi},\tilde{\theta},\eta)|^2 &= (\sqrt{2}+1)^2\tilde{\theta}^2 + \tilde{\varphi}^2 + \xi_\eta^2 + \left(\tilde{\varphi}^2 + (\sqrt{2}+1)\tilde{\theta}^2\right)\xi_\eta + O(\tilde{\varphi}^4+\tilde{\theta}^4), \\ f(\tilde{\varphi},\tilde{\theta},\eta) &= \tilde{\varphi}^2 + (\sqrt{2}+1)\tilde{\theta}^2 + 2\xi_\eta + O(\tilde{\varphi}^4+\tilde{\theta}^4). \end{aligned} \tag{20}$$

Since by Lemma 2.2 (i) it holds $\xi_\eta = A\eta^2 + O(\eta^4)$, where $A > 0$, there exist $C_0, C_1 > 0$, which are independent of $\eta \in (0,1]$, such that

$$|Y(\tilde{\varphi}, \tilde{\theta}, \xi_\eta)|^2 \geq C_0 \left(\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2\right),$$
$$|f(\tilde{\varphi}, \tilde{\theta}, \xi_\eta)| \leq C_1(\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2)$$

for every $(\tilde{\varphi}, \tilde{\theta}) \in J_\eta$. Thus, using the change of variables $(\tilde{\varphi}, \tilde{\theta}) = \left(r\cos\Theta, (\sqrt{2}+1)^{-1} r \sin\Theta\right)$ and noting that $J_\eta \subset \{(r,\Theta) \mid \eta \leq r \leq \pi,\ 0 \leq \Theta \leq 2\pi\}$, we have (by definition of J_η)

$$\int_{J_\eta} \frac{(\sqrt{2}+\cos\tilde{\varphi}) f(\tilde{\varphi}, \tilde{\theta}, \eta)}{|Y(\tilde{\varphi}, \tilde{\theta}, \eta)|^6} d\tilde{\varphi} d\tilde{\theta} \leq C_2 \int_{J_\eta} \frac{d\tilde{\varphi} d\tilde{\theta}}{\left\{\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2\right\}^2} \leq C_2 \int_\eta^\pi \int_0^{2\pi} \frac{1}{r^3} dr d\Theta \leq C_3 \eta^{-2}.$$

Multiplying by η^6, we get (19).

For the integral on I_η, we consider the following two quantities:

$$\widehat{I}_1 := \eta^6 \int_{I_\eta} \frac{(\sqrt{2}+1)(\tilde{\varphi}^2 + (\sqrt{2}+1)\tilde{\theta}^2)}{|Y(\tilde{\varphi}, \tilde{\theta}, \eta)|^6} d\tilde{\varphi} d\tilde{\theta}, \qquad \widehat{I}_2 := \eta^6 \int_{I_\eta} \frac{(\sqrt{2}+1)2\xi_\eta}{|Y(\tilde{\varphi}, \tilde{\theta}, \eta)|^6} d\tilde{\varphi} d\tilde{\theta}.$$

We first claim $\widehat{I}_1 = O(\eta^2)$. In fact, noting (20) and $\xi_\eta = A\eta^2 + O(\eta^4)$, and recalling the above computations together with $I_\eta = \{(r,\Theta) : 0 \leq r \leq \eta,\ 0 \leq \Theta \leq 2\pi\}$, one has

$$\widehat{I}_1 \leq C_4 \eta^6 \int_{I_\eta} \frac{d\tilde{\varphi} d\tilde{\theta}}{\left\{\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2 + \xi_\eta^2\right\}^2} \leq C_5 \eta^6 \int_0^\eta \frac{r}{(r^2 + \xi_\eta^2)^2} dr \leq C_6 \eta^6 \frac{1}{\xi_\eta^2} \leq C_7 \eta^2.$$

Next, we compute $\widehat{I}_2$. First, it follows from (20) and $\xi_\eta = O(\eta^2)$ that

$$\left\{1 - O(\eta^2)\right\}\left\{\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2\right\} + \xi_\eta^2 \leq |Y(\tilde{\varphi}, \tilde{\theta}, \xi_\eta)|^2 \leq \left\{1 + O(\eta^2)\right\}\left\{\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2\right\} + \xi_\eta^2$$

for all $(\tilde{\varphi}, \tilde{\theta}) \in I_\eta$. Therefore, instead of $\widehat{I}_2$, it suffices to compute

$$\widehat{I}_3 := \eta^6 \int_{I_\eta} \frac{(\sqrt{2}+1)2\xi_\eta}{\left[\{1 + O(\eta^2)\}\left\{\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2\right\} + \xi_\eta^2\right]^3} d\tilde{\varphi} d\tilde{\theta}.$$

Using the same change of variables as above, we get

$$\begin{aligned}
\widehat{I}_3 &= \eta^6 \int_0^\eta \int_0^{2\pi} \frac{2(\sqrt{2}+1)\xi_\eta r}{\left[\{1+O(\eta^2)\} r^2 + \xi_\eta^2\right]^3} \frac{dr d\Theta}{\sqrt{2}+1} = \eta^6 4\pi \xi_\eta \int_0^\eta \frac{r}{\left[\{1+O(\eta^2)\} r^2 + \xi_\eta^2\right]^3} dr \\
&= \frac{\pi}{1 + O(\eta^2)} \eta^6 \xi_\eta \left\{\frac{1}{\xi_\eta^4} - \frac{1}{\left(\eta^2 + \xi_\eta^2 + O(\eta^4)\right)^2}\right\} = \pi\left(1 + O(\eta^2)\right)\left\{\frac{\eta^6}{\xi_\eta^3} - \frac{\eta^6 \xi_\eta}{\left(\eta^2 + \xi_\eta^2 + O(\eta^4)\right)^2}\right\}.
\end{aligned}$$

Recalling Lemma 2.2 (i), there holds

$$\frac{\eta^6}{\xi_\eta^3} = 2^{15/4}\pi^{3/2} + O(\eta^2), \quad \frac{\eta^6 \xi_\eta}{\left\{\eta^2 + \xi_\eta^2 + O(\eta^4)\right\}^2} = O(\eta^4).$$

Hence, one observes that

$$\widehat{I}_2 = \widehat{I}_3 + O(\eta^2) = 2^{15/4}\pi^{5/2} + O(\eta^2).$$

Since we have

$$(\sqrt{2}+\cos\tilde{\varphi}) f(\tilde{\varphi}, \tilde{\theta}, \eta) = (\sqrt{2}+1)(\tilde{\varphi}^2 + (\sqrt{2}+1)\tilde{\theta}^2 + 2\xi_\eta) + O(\eta^4) \qquad \text{on } I_\eta,$$

noting $\xi_\eta = O(\eta^2)$ and the estimates of $\widehat{I}_1$ and $\widehat{I}_2$, (18) follows. Since (18) implies (17), Lemma 3.1 holds. □

3.2 Jacobi field generated by Möbius inversions

Here we analyse the variation of Möbius inversions on degenerating tori. In particular we derive the asymptotics of the normal vector field induced by this variation. Define

$$\Phi_\eta(x) := \frac{\eta^2}{|x|^2}x; \qquad \Psi_\eta(x) := (\mathrm{Ref}_{\mathbf{e}_x} \circ \Phi_\eta)(x),$$

where $\mathrm{Ref}_{\mathbf{e}_x}$ stands for the reflection $\mathrm{Ref}_{\mathbf{e}_x}(y) := y - 2\langle y, \mathbf{e}_x\rangle \mathbf{e}_x$. Recall that, for $\xi > 0$, we have set

$$\mathbb{T}_\xi := \mathbb{T} - (\sqrt{2} + 1 + \xi)\mathbf{e}_x.$$

Recall also that we used the following parametrizations of $\mathbb{T}$ and $\mathbb{T}_{\xi_\eta}$: for $(\tilde\varphi, \tilde\theta) \in [0, 2\pi]^2$,

$$\begin{aligned} X(\tilde\varphi, \tilde\theta) &= \Big((\sqrt{2} + \cos\tilde\varphi)\cos\tilde\theta,\ (\sqrt{2} + \cos\tilde\varphi)\sin\tilde\theta,\ \sin\tilde\varphi\Big), \\ Y(\tilde\varphi, \tilde\theta, \eta) &= X(\tilde\varphi, \tilde\theta) - \Big(\sqrt{2} + 1 + \xi_\eta\Big)\mathbf{e}_x \\ &= \Big((\sqrt{2} + \cos\tilde\varphi)\cos\tilde\theta - (\sqrt{2} + 1 + \xi_\eta),\ (\sqrt{2} + \cos\tilde\varphi)\sin\tilde\theta,\ \sin\tilde\varphi\Big). \end{aligned} \tag{21}$$

As unit normal to $\mathbb{T}_{\xi_\eta}$, we choose the outward one

$$n(\tilde\varphi, \tilde\theta) = (\cos\tilde\varphi\cos\tilde\theta,\ \cos\tilde\varphi\sin\tilde\theta,\ \sin\tilde\varphi).$$

We put also

$$\mathcal{Z}(\tilde\varphi, \tilde\theta, \eta) := \Psi_\eta(Y(\tilde\varphi, \tilde\theta, \eta)), \quad n_{0,\eta}(\tilde\varphi, \tilde\theta) := \frac{(D_x\Psi_\eta)(Y(\tilde\varphi, \tilde\theta, \eta))[n]}{|(D_x\Psi_\eta)(Y(\tilde\varphi, \tilde\theta, \eta))[n]|} = \frac{\Big(\mathrm{Ref}_{\mathbf{e}_x} \circ (D_x\Phi_\eta)(Y(\tilde\varphi, \tilde\theta, \eta))\Big)[n]}{|(D_x\Phi_\eta)(Y(\tilde\varphi, \tilde\theta, \eta))[n]|}.$$

Recalling Lemma 2.2, we easily see that $\mathcal{Z}(\eta^2\bar\varphi, \eta^2\bar\theta, \eta) \to \mathrm{Ref}_{\mathbf{e}_x} \circ Z_0(\bar\varphi, \bar\theta)$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ as $\eta \to 0$ and $n_{0,\eta}$ is an outward unit normal to $\Psi_\eta(\mathbb{T}_{\xi_\eta})$ since Ψ_η is a conformal map. Finally, we define the normal component of the variation of Ψ_η by

$$\varphi_\eta(\tilde\varphi, \tilde\theta) := \left\langle \frac{\partial \mathcal{Z}}{\partial \eta}(\tilde\varphi, \tilde\theta, \eta),\ n_{0,\eta}(\tilde\varphi, \tilde\theta) \right\rangle. \tag{22}$$

Our aim here is to prove the next proposition.

Proposition 3.2. *Set* $2\tilde A := \lim_{\eta\to 0} \eta^2/\xi_\eta > 0$ *and*

$$\psi_\eta(\bar\varphi, \bar\theta) := \frac{\varphi_\eta(\eta^2\bar\varphi, \eta^2\bar\theta)}{\eta} \quad \text{for } (\bar\varphi, \bar\theta) \in \mathbb{R}^2.$$

Then there holds

$$\psi_\eta(\bar\varphi, \bar\theta) \to \psi_0(\bar\varphi, \bar\theta) = -\frac{1}{\bar\varphi^2 + (\sqrt{2}+1)^2\bar\theta^2 + 1/(4\tilde A^2)} \left\{ \bar\varphi^2 + (\sqrt{2}+1)\bar\theta^2 - \frac{\sqrt{2}}{8\tilde A^2} \right\}$$
$$\text{in } C^\infty_{\mathrm{loc}}(\mathbb{R}^2).$$

Noting that as $\eta \to 0$, $\mathcal{Z}(\mathbb{T}_{\xi_\eta})$ *converges to the sphere of radius* $\tilde A$ *and centred at* $\tilde A\mathbf{e}_x$ *(denoted by* $S^2_{\tilde A}$*), we observe that* $\psi_0 \in C^\infty(S^2_{\tilde A} \setminus \{0\}) \cap L^\infty(S^2_{\tilde A})$*. Moreover, using the following polar coordinates*

$$x = \tilde A(1 + \cos\theta), \quad y = \tilde A\sin\theta\cos\varphi, \quad z = \tilde A\sin\theta\sin\varphi, \quad (\theta, \varphi) \in [0, \pi] \times [0, 2\pi],$$

ψ_0 *is expressed as follows:*

$$\begin{aligned}\psi_0 &= -\frac{1}{2\tilde{A}x}\left(z^2+y^2-(2-\sqrt{2})y^2\right)+\frac{\sqrt{2}}{4\tilde{A}}x \\ &= \frac{\sqrt{2}}{2}\cos\theta+\frac{2-\sqrt{2}}{4}(1-\cos\theta)\cos 2\varphi \\ &= \frac{1}{2}(\cos\theta-1)+\frac{2-\sqrt{2}}{2}(1-\cos\theta)\cos^2\varphi+\frac{\sqrt{2}}{4}(1+\cos\theta).\end{aligned} \tag{23}$$

To prove the above proposition we need two preliminary lemmas.

Lemma 3.3. *For each $k\in\mathbb{N}$ and $R>0$, (ψ_η) is bounded in $C^k([-R,R]^2)$ as $\eta\to 0$.*

Proof. We first show that φ_η is expressed as follows:

$$\begin{aligned}\varphi_\eta &= -\frac{\eta}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^2}\left\{2\left(\sqrt{2}\cos\tilde{\varphi}+1-(\sqrt{2}+1)\cos\tilde{\varphi}\cos\tilde{\theta}\right)+(\xi'_\eta\eta-2\xi_\eta)\cos\tilde{\varphi}\cos\tilde{\theta}\right\} \\ &=: -\frac{\eta}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^2}\left(h(\tilde{\varphi},\tilde{\theta})+(\xi'_\eta\eta-2\xi_\eta)\cos\tilde{\varphi}\cos\tilde{\theta}\right).\end{aligned} \tag{24}$$

To this end, from the definition of $\mathcal{Z}$, we have

$$\begin{aligned}\frac{\partial\mathcal{Z}}{\partial\eta} &= 2\eta\frac{\mathrm{Ref}_{\mathbf{e}_x}(Y(\tilde{\varphi},\tilde{\theta},\eta))}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^2}+\left(\mathrm{Ref}_{\mathbf{e}_x}\circ D_x\Phi_\eta(Y(\tilde{\varphi},\tilde{\theta},\eta))\right)\left[\frac{\partial Y}{\partial\eta}\right] \\ &= 2\eta\frac{\mathrm{Ref}_{\mathbf{e}_x}(Y(\tilde{\varphi},\tilde{\theta},\eta))}{|Y(\tilde{\varphi},\tilde{\theta},\eta)|^2}+\left(\mathrm{Ref}_{\mathbf{e}_x}\circ D_x\Phi_\eta(Y(\tilde{\varphi},\tilde{\theta},\eta))\right)\left[-\xi'_\eta\mathbf{e}_x\right].\end{aligned}$$

From the fact that $\mathrm{Ref}_{\mathbf{e}_x}\in O(3)$ and the formula

$$D_x\Phi_\eta(x)=\frac{\eta^2}{|x|^2}\left(\mathrm{Id}_{\mathbb{R}^3}-2\frac{x}{|x|}\otimes\frac{x}{|x|}\right), \tag{25}$$

it follows that

$$|D_x\Psi_\eta(Y)[n]|=|D_x\Phi_\eta(Y)[n]|=\frac{\eta^2}{|Y|^2}.$$

Since $D_x\Phi_\eta$ is conformal (cf. (25)), we obtain

$$\begin{aligned}\varphi_\eta &= \left\langle\frac{\partial\mathcal{Z}}{\partial\eta},\ n_{0,\eta}\right\rangle=\left\langle 2\eta\frac{\mathrm{Ref}_{\mathbf{e}_x}(Y)}{|Y|^2}+(\mathrm{Ref}_{\mathbf{e}_x}\circ D\Phi_\eta(Y))\left[-\xi'_\eta\mathbf{e}_x\right],\ n_{0,\eta}\right\rangle \\ &= \left\langle 2\eta\frac{\mathrm{Ref}_{\mathbf{e}_x}(Y)}{|Y|^2},\ n_{0,\eta}\right\rangle-\xi'_\eta\left\langle D\Phi_\eta(Y)[\mathbf{e}_x],\ \frac{D\Phi_\eta(Y)[n]}{|D\Phi_\eta(Y)[n]|}\right\rangle \\ &= 2\eta\left\langle\frac{\mathrm{Ref}_{\mathbf{e}_x}(Y)}{|Y|^2},\ n_{0,\eta}\right\rangle-\xi'_\eta\frac{\eta^4}{|Y|^4}\frac{1}{|D\Phi_\eta(Y)[n]|}\langle\mathbf{e}_x,n\rangle \\ &= 2\eta\left\langle\frac{\mathrm{Ref}_{\mathbf{e}_x}(Y)}{|Y|^2},\ n_{0,\eta}\right\rangle-\frac{\eta^2}{|Y|^2}\xi'_\eta\langle\mathbf{e}_x,n\rangle.\end{aligned}$$

On the other hand, using (25), one sees that

$$n_{0,\eta}=\mathrm{Ref}_{\mathbf{e}_x}\left(\frac{D\Phi_\eta(Y)[n]}{|D\Phi_\eta(Y)[n]|}\right)=\mathrm{Ref}_{\mathbf{e}_x}\left(n-2\left\langle\frac{Y}{|Y|},\ n\right\rangle\frac{Y}{|Y|}\right).$$

Thus it follows that

$$\left\langle\frac{\mathrm{Ref}_{\mathbf{e}_x}(Y)}{|Y|^2},\ n_{0,\eta}\right\rangle=\frac{1}{|Y|^2}\left\langle Y,\ n-2\left\langle\frac{Y}{|Y|},n\right\rangle\frac{Y}{|Y|}\right\rangle=-\frac{1}{|Y|^2}\langle Y,n\rangle,$$

which implies

$$\varphi_\eta = \frac{1}{|Y|^2}\left(-2\eta\langle Y, n\rangle - \eta^2\xi_\eta'\langle \mathbf{e}_x, n\rangle\right) = -\frac{\eta}{|Y|^2}\left(2\langle Y, n\rangle + \eta\xi_\eta'\langle \mathbf{e}_x, n\rangle\right).$$

Noting that

$$\langle \mathbf{e}_x, n\rangle = \cos\tilde\varphi\cos\tilde\theta, \quad \langle Y, n\rangle = \left\langle \begin{pmatrix} (\sqrt{2}+\cos\tilde\varphi)\cos\tilde\theta - (\sqrt{2}+1+\xi_\eta) \\ (\sqrt{2}+\cos\tilde\varphi)\sin\tilde\theta \\ \sin\tilde\varphi \end{pmatrix}, \begin{pmatrix} \cos\tilde\varphi\cos\tilde\theta \\ \cos\tilde\varphi\sin\tilde\theta \\ \sin\tilde\varphi \end{pmatrix} \right\rangle$$
$$= \sqrt{2}\cos\tilde\varphi + 1 - (\sqrt{2}+1)\cos\tilde\varphi\cos\tilde\theta - \xi_\eta\cos\tilde\varphi\cos\tilde\theta,$$

we get (24).

Next, recalling the definition of h in (24) and using a Taylor expansion, one observes that

$$Y(\tilde\varphi, \tilde\theta, \eta) = \begin{pmatrix} -\xi_\eta \\ (\sqrt{2}+1)\tilde\theta \\ \tilde\varphi \end{pmatrix} + R_Y(\tilde\varphi, \tilde\theta), \qquad h(\tilde\varphi, \tilde\theta) = \tilde\varphi^2 + (\sqrt{2}+1)\tilde\theta^2 + R_h(\tilde\varphi, \tilde\theta) \tag{26}$$

where R_Y, R_h are smooth functions satisfying

$$\left|D^\alpha R_Y(\tilde\varphi, \tilde\theta)\right| \le C_k\left(\tilde\varphi^{(2-|\alpha|)_+} + \tilde\theta^{(2-|\alpha|)_+}\right), \qquad \left|D^\beta R_h(\tilde\varphi, \tilde\theta)\right| \le C_k\left(\tilde\varphi^{(4-|\beta|)_+} + \tilde\theta^{(4-|\beta|)_+}\right) \tag{27}$$

for all $\alpha, \beta \in \mathbb{Z}_+^2$ with $|\alpha|, |\beta| \le k$ in a neighbourhood of the origin. Therefore, for $(\bar\varphi, \bar\theta) \in \mathbb{R}^2$, we have

$$Y(\eta^2\bar\varphi, \eta^2\bar\theta, \eta) = \eta^2\left(-\frac{1}{2\tilde A},\ (\sqrt{2}+1)\bar\theta,\ \bar\varphi\right) + R_Y(\eta^2\bar\varphi, \eta^2\bar\theta) + \left(\frac{\eta^2}{2\tilde A} - \xi_\eta\right)\mathbf{e}_x,$$
$$h(\eta^2\bar\varphi, \eta^2\bar\theta) = \eta^4(\bar\varphi^2 + (\sqrt{2}+1)\bar\theta^2) + R_h(\eta^2\bar\varphi, \eta^2\bar\theta).$$

Notice that from (27) it follows that if $\eta \le 1/R^4$, then

$$|D^\alpha_{(\bar\varphi,\bar\theta)}\left(R_Y(\eta^2\bar\varphi, \eta^2\bar\theta)\right)| \le C_{|\alpha|}\eta^{7/2}, \qquad |D^\alpha_{(\bar\varphi,\bar\theta)}\left(R_h(\eta^2\bar\varphi, \eta^2\bar\theta)\right)| \le C_{|\alpha|}\eta^7$$

for all $\alpha \in \mathbb{Z}_+^2$ and $(\bar\varphi, \bar\theta) \in [-R, R]^2$ where $C_{|\alpha|}$ depends only on $|\alpha|$. By $2\tilde A = \lim_{\eta\to 0}(\eta^2/\xi_\eta)$ and Lemma 2.2 (i), we have $\eta^2/(2\tilde A) - \xi_\eta = O(\eta^4)$. Hence,

$$\eta^{-2}Y(\eta^2\bar\varphi, \eta^2\bar\theta, \eta) = \left(-\frac{1}{2\tilde A},\ (\sqrt{2}+1)\bar\theta,\ \bar\varphi\right) + \bar R_Y(\bar\varphi, \bar\theta), \quad \eta^{-4}h(\eta^2\bar\varphi, \eta^2\bar\theta) = \bar\varphi^2 + (\sqrt{2}+1)\bar\theta^2 + \bar R_h(\bar\varphi, \bar\theta)$$

where $\bar R_Y = O_k(\eta^{3/2})$ and $\bar R_h = O_k(\eta^3)$ in $C^k([-R, R]^2)$ sense provided $\eta \le 1/R^4$. Here $O_k(\eta^i)$ means $\|O_k(\eta^i)\|_{C^k([-R,R]^2)} \le C_k\eta^i$ and C_k does not depend on R. Hence, one observes that

$$\psi_\eta(\bar\theta, \bar\varphi) = \frac{\varphi_\eta(\eta^2\bar\theta, \eta^2\bar\varphi)}{\eta} = -\frac{\bar\varphi^2 + (\sqrt{2}+1)\bar\theta^2 + \eta^{-4}(\xi_\eta'\eta - 2\xi_\eta)\cos(\eta^2\bar\varphi)\cos(\eta^2\bar\theta)}{(\sqrt{2}+1)^2\bar\theta^2 + \bar\varphi^2 + \frac{1}{4\tilde A^2}} + R_\psi(\bar\varphi, \bar\theta)$$

where $R_\psi(\bar\varphi, \bar\theta) = O_k(\eta^{5/4})$. By Lemma 3.1, there holds $\xi_\eta'\eta - 2\xi_\eta = O(\eta^4)$, hence,

$$\psi_\eta(\bar\varphi, \bar\theta) = -\frac{\bar\varphi^2 + (\sqrt{2}+1)\bar\theta^2 + \eta^{-4}(\xi_\eta'\eta - 2\xi_\eta)}{(\sqrt{2}+1)^2\bar\theta^2 + \bar\varphi^2 + \frac{1}{4\tilde A^2}} + O_k(\eta^{5/4}), \tag{28}$$

which implies that (ψ_η) is bounded in $C^k([-R, R]^2)$. □

From Lemmas 3.1 and 3.3, and (28), taking a subsequence (η_k), we may assume

$$\psi_{\eta_k} \to \psi_0 = -\frac{\bar{\varphi}^2 + (\sqrt{2}+1)\bar{\theta}^2 + c_0}{(\sqrt{2}+1)\bar{\theta}^2 + \bar{\varphi}^2 + \frac{1}{4\tilde{A}^2}} \qquad \text{in } C^{\ell}_{\text{loc}}(\mathbb{R}^2) \tag{29}$$

for every $\ell \in \mathbb{N}$, where

$$c_0 = \lim_{k\to\infty} \eta_k^{-4}(\xi'_{\eta_k}\eta_k - 2\xi_{\eta_k}).$$

Note that ψ_0 is a bounded function. Furthermore, using the map

$$(\bar{\varphi}, \bar{\theta}) \mapsto \Psi_1\left(-\frac{1}{2\tilde{A}}\mathbf{e}_x + (\sqrt{2}+1)\bar{\theta}\mathbf{e}_y + \bar{\varphi}\mathbf{e}_z\right) : \mathbb{R}^2 \to S^2_{\tilde{A}}$$

as the parametrization of $S^2_{\tilde{A}}$, by the conformality of Ψ_1 and (25), we have

$$d\sigma = \frac{\sqrt{2}+1}{\left\{(\sqrt{2}+1)^2\bar{\theta}^2 + \bar{\varphi}^2 + \frac{1}{4\tilde{A}^2}\right\}^2} d\bar{\varphi}d\bar{\theta}.$$

Hence, ψ_0 is integrable on $S^2_{\tilde{A}}$.

Next we prove that the function ψ_0 has null mean value on the limit sphere $S^2_{\tilde{A}}$.

Lemma 3.4. *Viewed as a real function on the limit sphere (through the above parameterization), the function ψ_0 in (29) satisfies*

$$\int_{S^2_{\tilde{A}}} \psi_0 \, d\sigma = 0. \tag{30}$$

Remark 3.5. *From (30) we shall prove $c_0 = -\sqrt{2}/(8\tilde{A}^2)$ in the proof of Proposition 3.2. Here we remark that since c_0 is independent of the choice of subsequence (η_k), as $\eta \to 0$, we obtain*

$$\eta^{-4}(\xi'_\eta \eta - 2\xi_\eta) \to -\frac{\sqrt{2}}{8\tilde{A}^2}, \qquad \psi_\eta \to \psi_0 \quad \text{in } C^{\ell}_{\text{loc}}(\mathbb{R}^2).$$

Proof of Lemma 3.4. We argue by contradiction and suppose that

$$\int_{S^2_{\tilde{A}}} \psi_0 \, d\sigma = A \neq 0.$$

Set $\Sigma_\eta := \Psi_\eta(\mathbb{T}_{\xi_\eta})$. Since Ψ_η preserves the area of $\mathbb{T}_{\xi_\eta}$, it follows from the definition of φ_η that

$$0 = \frac{d}{d\eta}\text{Area}(\Sigma_\eta) = \int_{\Sigma_\eta} H_{\Sigma_\eta}\varphi_\eta d\sigma,$$

where H_{Σ_η} is the mean curvature of Σ_η.

Let $\delta > 0$ and decompose Σ_η into two parts:

$$\Sigma_\eta = (\Sigma_\eta \cap B_\delta(0)) \cup (\Sigma_\eta \cap (B_\delta(0))^c) =: \Sigma_{\eta,1} + \Sigma_{\eta,2}.$$

First we prove that there exist $C > 0$, independent of η and δ, and $\eta_0, \delta_0 > 0$ such that

$$\left|\int_{\Sigma_{\eta,1}} H_{\Sigma_\eta}\varphi_\eta d\sigma\right| \leq C\delta\eta \tag{31}$$

for each $\eta \le \eta_0$ and $\delta \le \delta_0$. In fact, from Hölder's inequality, one observes that

$$\left| \int_{\Sigma_{\eta,1}} H_{\Sigma_\eta} \varphi_\eta d\sigma \right| \le \left(\int_{\Sigma_{\eta,1}} H^2_{\Sigma_\eta} d\sigma \right)^{1/2} \left(\int_{\Sigma_{\eta,1}} \varphi^2_\eta d\sigma \right)^{1/2} \le \left(\int_{\Sigma_\eta} H^2_{\Sigma_\eta} d\sigma \right)^{1/2} \left(\int_{\Sigma_{\eta,1}} \varphi^2_\eta d\sigma \right)^{1/2}.$$

By the conformal invariance of the Willmore functional (see Proposition 2.1), we have

$$\int_{\Sigma_\eta} H^2_{\Sigma_\eta} d\sigma = W_{g_0}(\Sigma_\eta) = W_{g_0}(\mathbb{T}).$$

On the other hand, we remark that $|\mathcal{Z}(\tilde\varphi, \tilde\theta, \eta)| \le \delta$ is equivalent to $|Y(\tilde\varphi, \tilde\theta, \eta)| \ge \eta^2 \delta^{-1}$. Since Σ_η is parametrized by $\mathcal{Z}(\tilde\varphi, \tilde\theta, \eta)$ ($(\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2$) and Φ_η is conformal, we observe that the area element of $\mathcal{Z}$ is given by

$$d\sigma = \left(\sqrt{2} + \cos\tilde\varphi \right) \frac{\eta^4}{|Y(\tilde\varphi, \tilde\theta, \eta)|^4} d\tilde\varphi d\tilde\theta.$$

Hence, we have

$$\int_{\Sigma_{\eta,1}} \varphi^2_\eta d\sigma = \int_{\mathbb{T}_{\xi_\eta} \cap \Psi_\eta^{-1}(B^c_{\eta^2\delta^{-1}})} \varphi^2_\eta(\tilde\varphi, \tilde\theta)(\sqrt{2} + \cos\tilde\varphi) \frac{\eta^4}{|Y(\tilde\varphi, \tilde\theta, \eta)|^4} d\tilde\theta d\tilde\varphi.$$

Next, recalling (20) (or (26)), we may find $C_0, C_1 > 0$ and $\eta_0 > 0$ such that

$$C_1(\tilde\varphi^2 + (\sqrt{2}+1)^2\tilde\theta^2 + \xi^2_\eta) \ge |Y(\tilde\varphi, \tilde\theta, \eta)|^2 \ge C_0(\tilde\varphi^2 + (\sqrt{2}+1)^2\tilde\theta^2) \tag{32}$$

for all $\tilde\varphi, \tilde\theta \in [-\pi, \pi]$ and $\eta \le \eta_0$. Noting that $\xi_\eta = \eta^2/(2\tilde A) + O(\eta^4)$ by (17), we may assume that there exists $C_2 > 0$ (independent of δ) satisfying

$$\eta \le \eta_0, \quad |Y(\tilde\varphi, \tilde\theta, \eta)|^2 \ge \frac{\eta^4}{\delta^2} \quad \Rightarrow \quad \tilde\varphi^2 + (\sqrt{2}+1)^2\tilde\theta^2 \ge C_2 \frac{\eta^4}{\delta^2}$$

for $0 < \delta \le \bar\delta(C_1)$. Moreover, we claim that

$$|\varphi_\eta(\tilde\varphi, \tilde\theta)| \le C_3 \eta \qquad \text{for every } (\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2. \tag{33}$$

In fact, recall (24):

$$\varphi_\eta(\tilde\varphi, \tilde\theta) = -\frac{\eta}{|Y(\tilde\varphi, \tilde\theta, \eta)|^2} \left[h(\tilde\varphi, \tilde\theta) + (\eta\xi'_\eta - 2\xi_\eta) \cos\tilde\varphi \cos\tilde\theta \right].$$

From Lemmas 2.2 and 3.1, and $|Y(\tilde\varphi, \tilde\theta, \eta)| \ge |\langle Y, \mathbf{e}_x \rangle| \ge \xi_\eta \ge C_4 \eta^2$ for each $(\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2$, there holds

$$\frac{|(\eta\xi'_\eta - 2\xi_\eta)\cos\tilde\varphi\cos\tilde\theta|}{|Y(\tilde\varphi, \tilde\theta, \eta)|^2} \le C_5 \qquad \text{for each } (\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2 \text{ and } \eta \le \eta_0.$$

On the other hand, by (26), we have

$$\left| h(\tilde\varphi, \tilde\theta) \right| \le C_6 \left(\tilde\varphi^2 + (\sqrt{2}+1)^2\tilde\theta^2 \right) \qquad \text{for all } (\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2.$$

Thus by (32), we obtain

$$\frac{\left| h(\tilde\varphi, \tilde\theta) \right|}{|Y(\tilde\varphi, \tilde\theta, \eta)|^2} \le C_7$$

for all $(\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2$ and $\eta \le \eta_0$. Combining the two estimates above, (33) holds.

Now setting

$$A_{\eta,\delta}=\{(\tilde{\varphi},\tilde{\theta})\ :\ C_2\eta^4\delta^{-2}\leq\tilde{\varphi}^2+(\sqrt{2}+1)^2\tilde{\theta}^2\leq 8\pi^2\},$$

and using (33) and polar coordinates as in the proof of Lemma 3.1, we get

$$\int_{\Sigma_{\eta,1}}\varphi_\eta^2 d\sigma\leq C_8\eta^6\int_{A_{\eta,\delta}}\frac{1}{|Y|^4}d\tilde{\varphi}d\tilde{\theta}\leq C_9\eta^6\int_{A_{\eta,\delta}}\left\{\tilde{\varphi}^2+(\sqrt{2}+1)^2\tilde{\theta}^2\right\}^{-2}d\tilde{\theta}d\tilde{\varphi}\leq C_{10}\eta^6\left(\frac{\delta}{\eta^2}\right)^2=C_{10}\eta^2\delta^2$$

for all $\eta\leq\eta_0$. Thus (31) holds for some $\eta_0>0$ and $\delta_0=\bar{\delta}(C_1)>0$.

Next, we consider the integral on $\Sigma_{\eta,2}$. We first remark that $|\mathcal{Z}(\tilde{\varphi},\tilde{\theta},\eta)|\geq\delta$ is equivalent to $|Y(\tilde{\varphi},\tilde{\theta},\eta)|\leq\eta^2\delta^{-1}$ and the following holds: (see (32))

$$|Y(\tilde{\varphi},\tilde{\theta},\eta)|\leq\eta^2\delta^{-1}\quad\Rightarrow\quad\tilde{\varphi}^2+(\sqrt{2}+1)^2\tilde{\theta}^2\leq C_{11}^2\eta^4\delta^{-2}$$

for every $\eta\leq\eta_0$. Since

$$\mathcal{Z}(\eta_k^2\bar{\varphi},\eta_k^2\bar{\theta},\eta_k)\to\Psi_1\left(-\frac{1}{2\tilde{A}},(\sqrt{2}+1)\bar{\theta},\bar{\varphi}\right),\quad\eta_k^{-1}\varphi_{\eta_k}(\eta_k^2\bar{\varphi},\eta_k^2\bar{\theta})\to\psi_0(\bar{\varphi},\bar{\theta})\text{ in }C^\ell([-C_{11}\delta^{-1},C_{11}\delta^{-1}]^2)$$

for any $\ell\in\mathbb{N}$ and noticing that the maps

$$\mathcal{Z}(\eta_k^2\bar{\varphi},\eta_k^2\bar{\theta},\eta_k)=\Psi_{\eta_k}(Y(\eta_k^2\bar{\varphi},\eta_k^2\bar{\theta},\eta_k))=\Psi_1\left(\eta_k^{-2}Y(\eta_k^2\bar{\varphi},\eta_k^2\bar{\theta},\eta_k)\right),\quad(\bar{\varphi},\bar{\theta})\mapsto\Psi_1\left(-\frac{1}{2\tilde{A}},(\sqrt{2}+1)\bar{\theta},\bar{\varphi}\right)$$

are parametrizations of $\Sigma_{\eta_k,2}$ and $S^2_{\tilde{A}}$, we obtain

$$H_{\Sigma_{\eta_k}}(\mathcal{Z}(\eta_k^2\bar{\varphi},\eta_k^2\bar{\theta},\eta_k))\to\frac{2}{\tilde{A}}\qquad\text{in }C^0([-C_{11}\delta^{-1},C_{11}\delta^{-1}]^2)$$

and

$$\lim_{k\to\infty}\eta_k^{-1}\left|\int_{\Sigma_{\eta_k,2}}H_{\Sigma_{\eta_k}}\varphi_{\eta_k}d\sigma\right|=\frac{2}{\tilde{A}}\left|\int_{S^2_{\tilde{A}}\cap B^c_\delta(0)}\psi_0 d\sigma\right|.$$

Since ψ_0 is integrable, we may find $0<\delta_2\leq\delta_0=\bar{\delta}_0(C_1)$ so that if $\delta\leq\delta_2$, then

$$\frac{2}{\tilde{A}}\left|\int_{S^2_{\tilde{A}}\cap B^c_\delta(0)}\psi_0 d\sigma\right|\geq\frac{|A|}{\tilde{A}}>0.$$

Therefore, by (31), for all $\eta_k\leq\eta_0$ and $\delta\leq\delta_2$, it follows that

$$0=\eta_k^{-1}\int_{\Sigma_{\eta_k}}H_{\Sigma_{\eta_k}}\varphi_{\eta_k}d\sigma=\eta_k^{-1}\int_{\Sigma_{\eta_k,2}}H_{\Sigma_{\eta_k}}\varphi_{\eta_k}d\sigma+\eta_k^{-1}\int_{\Sigma_{\eta_k,1}}H_{\Sigma_{\eta_k}}\varphi_{\eta_k}d\sigma\begin{cases}\geq\frac{A}{\tilde{A}}-C\delta & \text{if }A>0,\\ \leq-\frac{A}{\tilde{A}}+C\delta & \text{if }A<0.\end{cases}$$

Noting that C does not depend on δ, choosing $\delta>0$ sufficiently small and k sufficiently large, we get a contradiction and the Lemma holds. □

We are now ready to prove Proposition 3.2

Proof of Proposition 3.2. Notice first that

$$S^2_{\tilde{A}}\backslash\{0\}=\tilde{A}(S^2+\mathbf{e}_x)\backslash\{0\}=\bigcup\left\{\Psi_1\left(-\frac{1}{2\tilde{A}},(\sqrt{2}+1)y,z\right)\ :\ y,z\in\mathbb{R}\right\}$$

and set

$$B := \bar{\varphi}^2 + (\sqrt{2}+1)^2\bar{\theta}^2 + \frac{1}{4\tilde{A}^2}, \quad (x,y,z) := \Psi_1\left(-\frac{1}{2\tilde{A}}, (\sqrt{2}+1)\bar{\theta}, \bar{\varphi}\right) = \left(\frac{1}{B}\frac{1}{2\tilde{A}}, \frac{(\sqrt{2}+1)\bar{\theta}}{B}, \frac{\bar{\varphi}}{B}\right).$$

Since

$$(x-\tilde{A})^2 + y^2 + z^2 = \tilde{A}^2 \quad \Leftrightarrow \quad x^2 - 2\tilde{A}x + y^2 + z^2 = 0,$$

by the definition of B and (x,y,z), we have

$$B = (Bz)^2 + (By)^2 + (Bx)^2, \qquad \text{therefore} \qquad B = \frac{1}{x^2+y^2+z^2} = \frac{1}{2\tilde{A}x}.$$

Recalling $c_0 = \lim_{\eta_k\to 0}\eta_k^{-4}(\xi'_{\eta_k}\eta_k - 2\xi_{\eta_k})$, it follows from the above formulas and (29) that

$$(34)\qquad \begin{aligned}\psi_0 &= -\frac{1}{B}\left(B^2z^2 + \frac{B^2}{\sqrt{2}+1}y^2 + c_0\right) = -B\left(z^2 + \frac{y^2}{\sqrt{2}+1}\right) - 2\tilde{A}c_0x\\ &= -\frac{1}{2\tilde{A}x}\left(z^2 + y^2 - (2-\sqrt{2})y^2\right) - 2\tilde{A}c_0x\\ &= -\frac{1}{2\tilde{A}x}\left(-x^2 + 2\tilde{A}x - (2-\sqrt{2})y^2 + 4\tilde{A}^2c_0x^2\right)\\ &= -\frac{1}{2\tilde{A}x}\left((4\tilde{A}^2c_0 - 1)x^2 + 2\tilde{A}x - (2-\sqrt{2})y^2\right).\end{aligned}$$

Now substituting $x = \tilde{A}(\cos\theta+1)$, $y = \tilde{A}\sin\theta\cos\varphi$ and $z = \tilde{A}\sin\theta\sin\varphi$, and using that $\sin^2\theta = 1-\cos^2\theta$, $\cos^2\varphi = (1+\cos 2\varphi)/2$, we have

$$(35)\qquad \begin{aligned}\psi_0 &= -\frac{1}{2(1+\cos\theta)}\left\{(4\tilde{A}^2c_0-1)(1+\cos\theta)^2 + 2(1+\cos\theta) - (2-\sqrt{2})\sin^2\theta\cos^2\varphi\right\}\\ &= -\frac{1}{2}\left\{(4\tilde{A}^2c_0-1)(1+\cos\theta) + 2 - (2-\sqrt{2})(1-\cos\theta)\frac{1+\cos 2\varphi}{2}\right\}\\ &= -\frac{1}{2}\left\{4\tilde{A}^2c_0 + \frac{\sqrt{2}}{2} + \left(4\tilde{A}^2c_0 - \frac{\sqrt{2}}{2}\right)\cos\theta - \frac{2-\sqrt{2}}{2}(1-\cos\theta)\cos 2\varphi\right\}.\end{aligned}$$

Integrating this equality over $S^2_{\tilde{A}}$ and noting the area element in the above coordinate is given by $d\sigma = \tilde{A}^2\sin\theta$, Lemma 3.4 yields

$$(36)\qquad c_0 = -\frac{\sqrt{2}}{8\tilde{A}^2}.$$

In particular, we may observe that c_0 is independent of choices of subsequence (η_k). Hence, as $\eta\to 0$, we obtain

$$\eta^{-4}(\xi'_\eta\eta - 2\xi_\eta) \to -\frac{\sqrt{2}}{8\tilde{A}^2}, \quad \psi_\eta \to \psi_0 \quad \text{in } C^\infty_{\mathrm{loc}}(\mathbb{R}^2).$$

Now substituting (36) into (34) and (35), we get

$$\begin{aligned}\psi_0 &= -\frac{1}{\bar{\varphi}^2 + (\sqrt{2}+1)^2\bar{\theta}^2 + 1/(4\tilde{A}^2)}\left\{\bar{\varphi}^2 + (\sqrt{2}+1)\bar{\theta}^2 - \frac{\sqrt{2}}{8\tilde{A}^2}\right\}\\ &= -\frac{1}{2\tilde{A}x}\left\{z^2 + y^2 - (2-\sqrt{2})y^2\right\} + \frac{\sqrt{2}}{4\tilde{A}}x\\ &= \frac{\sqrt{2}}{2}\cos\theta + \frac{2-\sqrt{2}}{4}(1-\cos\theta)\cos 2\varphi\\ &= \frac{1}{2}(\cos\theta - 1) + \frac{2-\sqrt{2}}{2}(1-\cos\theta)\cos^2\varphi + \frac{\sqrt{2}}{4}(1+\cos\theta).\end{aligned}$$

This completes the proof. □

4 Asymptotics of Willmore energy on degenerating tori

In this section we consider inverted tori embedded in manifolds, which degenerate to a sphere joint to a small handle. We estimate then the derivative of the Willmore energy with respect to the variation of the Möbius parameter. We first recall some basic facts, and separate the handle contribution to the derivative from the spherical one. We then compute the leading order term arising from the curvature of the ambient metric, postponing some explicit computations to an appendix.

4.1 Basic material and handle decomposition

The goal of this section is to estimate the derivative of the Willmore energy on degenerating tori with respect to the Möbius parameter ω for $|\omega|$ close to 1, namely to prove Proposition 4.2 below.
Let us recall the following result from [12], which regards the asymptotics of Willmore energy for degenerating tori of small area. In the degenerate limit, apart from the handle contribution, one recovers up to high order the energy of a small geodesic sphere (see [24]).

Proposition 4.1. *([12], Proposition 4.6) There exists $C_0 > 0$, which is independent of ε, such that*

$$\limsup_{r\uparrow 1} \sup_{P\in M, R\in SO(3), |\omega|=r} \left| \frac{1}{\varepsilon^2} \left(W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega}) - 8\pi^2 + \frac{8\sqrt{2}}{3}\pi^2\varepsilon^2 \mathrm{Sc}_P \right) \right| \leq C_0 \varepsilon$$

for all sufficiently small $\varepsilon > 0$.

In the next proposition we state one of the main technical results of the paper; to this aim recall the notation introduced in (14) for the surfaces $\Sigma_{\varepsilon,P,Id,\omega}$.

Proposition 4.2. *Let $\delta \in (0, 1/2)$, $R = Id$ and $\omega = |\omega|\mathbf{e}_x$ with $1 - |\omega| = \eta$. Then there exist $0 < \eta_\delta$, C_0 and C_δ such that for every $\tilde{\eta} \in (0, \eta_\delta)$, one may find $C_{\tilde{\eta}} > 0$ satisfying*

$$\left| \frac{\partial}{\partial\omega} W_{g_\varepsilon}(\Sigma_{\varepsilon,P,Id,\omega}) - \eta\varepsilon^2 \frac{16}{3}\pi B\tilde{A}(R_{22} - R_{33}) \right| \leq \left[C_0\delta + C_\delta \left\{ o_\eta(1) + \varepsilon \right\} \right] \eta\varepsilon^2 + C_{\tilde{\eta}}\varepsilon^4$$

for all $\varepsilon \in (0, 1/2]$ and $\eta \in [\tilde{\eta}, \eta_\delta]$ where C_0 is independent of $\delta, \eta, \varepsilon$, C_δ depends only on $\delta > 0$, $o_\eta(1) := |\eta^{-4}(\xi'_\eta \eta - 2\xi_\eta) - c_0)| + \eta^{3/2}$, $c_0 := -\sqrt{2}/(8\tilde{A}^2)$ (See Remark 3.5 and (36)), $o_\eta(1) \to 0$ as $\eta \to 0$, $\tilde{A} = \sqrt[4]{2\pi^2}$, $B = (2 - \sqrt{2})/4$ and R_{ij} are the components of the Ricci tensor Ric_P.

The proof of this proposition is quite involved and will be worked out in the present and the next subsection and in the Appendix. After scaling the metric as in (8), we will apply formula (12) to the case $\Sigma \in \mathcal{T}_{\varepsilon,K}$, for a surface corresponding to a value ω_0 of the parameter ω which is very close to 1 in modulus. We shall write

$$W'_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega}) := LH + \frac{1}{2}H^3, \qquad dW_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega})[\varphi] := \int_{\Sigma_{\varepsilon,P,R,\omega}} W'_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega})\varphi d\sigma.$$

Using the notation

$$\omega = (1-\eta)\mathbf{e}_x; \qquad \eta \simeq 0,$$

we will take the function φ_η in (22) as normal variation φ.

It is again convenient to exploit the conformal invariance of the Euclidean Willmore functional W_{g_0} in order to write that

$$dW_{g_\varepsilon}[\varphi] = dW_{g_0}[\varphi] + (dW_{g_\varepsilon}[\varphi] - dW_{g_0}[\varphi]) = (dW_{g_\varepsilon}[\varphi] - dW_{g_0}[\varphi]).$$

The right-hand side is easier to deal with because some cancellations will occur, but on the other hand we will pick up terms of order ε^2 from the curvature of the ambient metric g_ε, see (9).

As already seen in Lemma 2.2, degenerating tori geometrically look like spheres with small handles attached near the origin of geodesic normal coordinates. In order to evaluate the above derivative it is convenient to localize the normal variation near the handle and away from it. For a small but fixed $\delta > 0$ we then choose a radial cut-off function χ_δ on the degenerate torus such that

$$\chi_\delta(x) = \begin{cases} 1 & \text{for } |x| \le \delta; \\ 0 & \text{for } |x| \ge 2\delta, \end{cases}$$

and write

$$\varphi_\eta = \varphi_{1,\delta,\eta} + \varphi_{2,\delta,\eta} := \chi_\delta \varphi_\eta + (1-\chi_\delta)\varphi_\eta. \tag{37}$$

We then have

$$(dW_{g_\varepsilon}[\varphi_\eta] - dW_{g_0}[\varphi_\eta]) = (dW_{g_\varepsilon}[\varphi_{1,\delta,\eta}] - dW_{g_0}[\varphi_{1,\delta,\eta}]) + (dW_{g_\varepsilon}[\varphi_{2,\delta,\eta}] - dW_{g_0}[\varphi_{2,\delta,\eta}]) . \tag{38}$$

Next we compute the contribution of the handle region to the derivative.

Proposition 4.3. *There exists $C_0 > 0$ such that for any $\delta, \varepsilon, \eta \in (0, 1/2)$ one has*

$$|dW_{g_\varepsilon}[\varphi_{1,\delta,\eta}] - dW_{g_0}[\varphi_{1,\delta,\eta}]| \le C_0\, \varepsilon^2\, \eta\, \delta.$$

To prove the above proposition, we first prepare the notation, recalling Section 3. Let us denote by $(g_{\varepsilon,P,\eta})_{ij}$ the induced metric on $\Psi_\eta(\mathbb{T}_{\xi_\eta})$ from $g_{\varepsilon,P}$ in the coordinate $\mathcal{Z}(\tilde\varphi, \tilde\theta, \eta)$ where $\partial_1 = \partial_{\tilde\varphi}$ and $\partial_2 = \partial_{\tilde\theta}$. Furthermore, we write $d\sigma_{\varepsilon,P,\eta}$, $(\Gamma_{\varepsilon,P,\eta})^k_{ij}$, $(A_{\varepsilon,P,\eta})^j_i$, $\Delta_{\varepsilon,P,\eta}$ and $n_{\varepsilon,P,\eta}$ for the area element of $\Psi_\eta(\mathbb{T}_{\xi_\eta})$, the Christoffel symbols, the second fundamental form, the Laplace-Beltrami operator and unit outer normal, respectively. Finally, let us denote by $\mathrm{Ric}_{g_{\varepsilon,P}}$ the Ricci tensor for the ambient space $(B_{10}, g_{\varepsilon,P})$. For these quantities, we have

Lemma 4.4. *Recalling* (21)*, there exists $C_0 > 0$ such that for all $P \in M$ and $\varepsilon, \eta \in (0, 1/2)$,*

(i) *The area elements satisfy*

$$d\sigma_{0,\eta} = (\sqrt{2}\cos\tilde\varphi + 1)\frac{\eta^4}{|Y|^4} d\tilde\varphi d\tilde\theta, \qquad |d\sigma_{\varepsilon,P,\eta} - d\sigma_{0,\eta}| \le C_0 \varepsilon^2 \frac{\eta^8}{|Y|^6} d\tilde\varphi d\tilde\theta.$$

(ii) $|(g_{\varepsilon,P,\eta})^{ij}| \le C_0|Y|^4/\eta^4$ *and* $|(g_{\varepsilon,P,\eta})^{ij} - (g_{0,\eta})^{ij}| \le C_0\varepsilon^2|Y|^2$.

(iii) $|(\Gamma_{\varepsilon,P,\eta})^k_{ij}| \le C_0/|Y|$ *and* $|(\Gamma_{\varepsilon,P,\eta})^k_{ij} - (\Gamma_{0,\eta})^k_{ij}| \le C_0\varepsilon^2\eta^4/|Y|^3$.

(iv) $|(A_{\varepsilon,P,\eta})^i_j| \le C_0|Y|/\eta^2$ *and* $|(A_{\varepsilon,P,\eta})^i_j - (A_{0,\eta})^i_j| \le C_0\varepsilon^2\eta^2/|Y|$.

(v) $|\mathrm{Ric}_{g_{\varepsilon,P}}(x)|_{g_0} \le C_0\varepsilon^2$.

(vi) $|n_{\varepsilon,P,\eta} - n_{0,\eta}| \le C_0\varepsilon^2\eta^4/|Y|^2$, $|\partial_i(n_{\varepsilon,P,\eta} - n_{0,\eta})| \le C_0\varepsilon^2\eta^4/|Y|^3$ *and* $|\partial_i\partial_j(n_{\varepsilon,P,\eta} - n_{0,\eta})| \le C_0\varepsilon^2\eta^4/|Y|^4$.

Proof. Since $\mathcal{Z}(\tilde\varphi, \tilde\theta, \eta) = \Psi_\eta(Y(\tilde\varphi, \tilde\theta, \eta))$ and $|D^\alpha_{(\tilde\varphi,\tilde\theta)} Y|$ are uniformly bounded with respect to ε and η for any $\alpha \in \mathbb{Z}^2_+$, we have

$$|\mathcal{Z}| = \frac{\eta^2}{|Y|}, \quad |\partial_i \mathcal{Z}| = \frac{\eta^2}{|Y|^2}|\partial_i Y| \sim C_0\frac{\eta^2}{|Y|^2}, \quad |\partial_i\partial_j \mathcal{Z}| \le C_0\left(\frac{\eta^2}{|Y|^3} + \frac{\eta^2}{|Y|^2}\right) \le C_0\frac{\eta^2}{|Y|^3}. \tag{39}$$

Furthermore, by the conformality of Ψ_η and $g_0(\partial_i Y, \partial_j Y) = |\partial_i Y||\partial_j Y|\delta_{ij}$, one also sees that

$$g_0[\partial_i \mathcal{Z}, \partial_j \mathcal{Z}] = |\partial_i \mathcal{Z}|_{g_0} |\partial_j \mathcal{Z}|_{g_0} \delta_{ij}. \tag{40}$$

Notice also that $f_{0,i} = \partial_i \mathcal{Z}/|\partial_i \mathcal{Z}|_{g_0}$ $(i = 1, 2)$ form an orthonormal basis of $T_{\mathcal{Z}}\Psi_\eta(\mathbb{T}_{\xi_\eta})$ and $n_{0,\eta}$ is given by (see Subsection 3.2 for the definition of n)

$$n_{0,\eta}(\tilde\varphi, \tilde\theta) = \frac{(D_x\Psi_\eta)(Y(\tilde\varphi, \tilde\theta, \eta))[n(\tilde\varphi, \tilde\theta)]}{|(D_x\Psi_\eta)(Y(\tilde\varphi, \tilde\theta, \eta))[n(\tilde\varphi, \tilde\theta)]|_{g_0}}.$$

Since $g_{\varepsilon,P,\alpha\beta}(x) = \delta_{\alpha\beta} + \varepsilon^2 h^\varepsilon_{P,\alpha\beta}(x)$ and $h^\varepsilon_{P,\alpha\beta}$ satisfies (10) uniformly with respect to ε and P, using (39) and (40) the above claims follow from direct computations. □

We now prove Proposition 4.3.

Proof of Proposition 4.3. We first recall (24):

$$\begin{aligned}\varphi_\eta &= -\frac{\eta}{|Y(\tilde\varphi, \tilde\theta, \eta)|^2}\left\{2\left(\sqrt{2}\cos\tilde\varphi + 1 - (\sqrt{2}+1)\cos\tilde\varphi\cos\tilde\theta\right) + (\xi_\eta'\eta - 2\xi_\eta)\cos\tilde\varphi\cos\tilde\theta\right\}\\ &= -\frac{\eta}{|Y(\tilde\varphi, \tilde\theta, \eta)|^2}\left(h(\tilde\varphi, \tilde\theta) + (\xi_\eta'\eta - 2\xi_\eta)\cos\tilde\varphi\cos\tilde\theta\right).\end{aligned}$$

From Lemma 3.1 and a Taylor expansion of Y and h (see (26), (27) and (32)), one may find $C_0 > 0$ such that

$$|\varphi_\eta| \le C_0\eta, \qquad |\partial_i\varphi_\eta| \le C_0\frac{\eta}{|Y|}, \qquad |\partial_i\partial_j\varphi_\eta| \le C_0\frac{\eta}{|Y|^2} \tag{41}$$

for all $(\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2$ and $\eta \in (0, 1/2)$.

Next, denote by $H_{\varepsilon,P,\eta}$ the mean curvature of $\Psi_\eta(\mathbb{T}_{\xi_\eta})$ with the ambient metric $g_{\varepsilon,P}$. Recalling (12) and noting

$$\begin{aligned}&dW_{g_\varepsilon}[\varphi_{1,\delta,\eta}] - dW_{g_0}[\varphi_{1,\delta,\eta}]\\ &= \int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left\{-H_{\varepsilon,P,\eta}\Delta_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta} - H_{\varepsilon,P,\eta}\left(|A_{\varepsilon,P,\eta}|^2 + \mathrm{Ric}_{g_{\varepsilon,P}}(n_{\varepsilon,P,\eta}, n_{\varepsilon,P,\eta}) - \frac{1}{2}H^2_{\varepsilon,P,\eta}\right)\varphi_{1,\delta,\eta}\right\}\\ &\qquad \times (d\sigma_{\varepsilon,P,\eta} - d\sigma_{0,\eta})\\ &\quad - \int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}(H_{\varepsilon,P,\eta}\Delta_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta} - H_{0,\eta}\Delta_{0,\eta}\varphi_{1,\delta,\eta})\,d\sigma_{0,\eta}\\ &\quad - \int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})} H_{\varepsilon,P,\eta}\mathrm{Ric}_{g_{\varepsilon,P}}(n_{\varepsilon,P,\eta}, n_{\varepsilon,P,\eta})\varphi_{1,\delta,\eta}d\sigma_{0,\eta}\\ &\quad - \int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left\{H_{\varepsilon,P,\eta}|A_{\varepsilon,P,\eta}|^2 - H_{0,\eta}|A_{0,\eta}|^2\right\}\varphi_{1,\delta,\eta}d\sigma_{0,\eta} + \frac{1}{2}\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left(H^3_{\varepsilon,P,\eta} - H^3_{0,\eta}\right)\varphi_{1,\delta,\eta}d\sigma_{0,\eta},\end{aligned} \tag{42}$$

we estimate each term in the above using Lemma 4.4. For this purpose, we first remark that

$$|\mathcal{Z}| \le 2\delta \qquad \Leftrightarrow \qquad |Y| \ge \frac{\eta^2}{2\delta}.$$

Moreover, as in the proof of Lemma 3.4 (see (32)), by $\xi_\eta = O(\eta^2)$, we may find a $C_1 > 0$, which is independent of δ and η, such that

$$I_\delta := \left\{(\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2 \ :\ \left(\tilde\varphi^2 + (\sqrt{2}+1)^2\tilde\theta^2\right) \ge C_1\frac{\eta^4}{\delta^2}\right\} \supset \left\{(\tilde\varphi, \tilde\theta) \in [-\pi, \pi]^2 \ :\ |Y(\tilde\varphi, \tilde\theta, \eta)| \ge \frac{\eta^2}{2\delta}\right\}$$

for all $\delta, \eta \in (0, 1/2)$.

First, we estimate the last two terms in (42). Since $H_{\varepsilon,P,\eta} = (A_{\varepsilon,P,\eta})^i_i$ and $|A_{\varepsilon,P,\eta}|^2 = (A_{\varepsilon,P,\eta})^i_j(A_{\varepsilon,P,\eta})^j_i$, by Lemma 4.4, it is easily seen that

$$\left|H_{\varepsilon,P,\eta}|A_{\varepsilon,P,\eta}|^2 - H_{0,\eta}|A_{0,\eta}|^2\right| + \left|H^3_{\varepsilon,P,\eta} - H^3_{0,\eta}\right| \le C_0\varepsilon^2\frac{|Y|}{\eta^2}.$$

Hence, from (41), (32), $\operatorname{supp}\varphi_{1,\delta,\eta} \subset \overline{B_{2\delta}(0)}$ and a change of variables, it follows that

$$\begin{aligned}(43)\quad &\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left|H_{\varepsilon,P,\eta}\,|A_{\varepsilon,P,\eta}|^2 - H_{0,\eta}\,|A_{0,\eta}|^2\right||\varphi_{1,\delta,\eta}|d\sigma_{0,\eta} + \frac{1}{2}\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left|H^3_{\varepsilon,P,\eta} - H^3_{0,\eta}\right||\varphi_{1,\delta,\eta}|d\sigma_{0,\eta}\\ &\le C_0\varepsilon^2\int_{I_\delta}\frac{\eta^3}{|Y|^3}d\tilde{\varphi}d\tilde{\theta} \le C_0\varepsilon^2\int_{I_\delta}\frac{\eta^3}{(\tilde{\varphi}^2 + (\sqrt{2}+1)^2\tilde{\theta}^2)^{3/2}}d\tilde{\varphi}d\tilde{\theta} \le C_0\varepsilon^2\eta^3\int_{C_1\eta^2/\delta}^{10} r^{-2}dr \le C_2\varepsilon^2\eta\delta,\end{aligned}$$

where C_2 is independent of ε, η and δ. Similarly, for the Ricci tensor, we have

$$(44)\qquad \int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}|H_{\varepsilon,P,\eta}||\mathrm{Ric}_{g_{\varepsilon,P}}(n_{\varepsilon,P,\eta}, n_{\varepsilon,P,\eta})||\varphi_{1,\delta,\eta}|d\sigma_{0,\eta} \le C_0\varepsilon^2\int_{I_\delta}\frac{\eta^3}{|Y|^3}d\tilde{\varphi}d\tilde{\theta} \le C_2\varepsilon^2\eta\delta.$$

In order to deal with the first two terms in (42), we estimate

$$\Delta_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta} \qquad \text{and} \qquad (\Delta_{\varepsilon,P,\eta} - \Delta_{0,\eta})\,\varphi_{1,\delta,\eta}.$$

First, by (39) and the definition of χ_δ, there holds

$$\left|\partial_i\left(\chi_\delta\left(\mathcal{Z}(\tilde{\varphi},\tilde{\theta},\eta)\right)\right)\right| \le C_0\frac{1}{\delta}\frac{\eta^2}{|Y|^2}, \qquad \left|\partial_i\partial_j\left(\chi_\delta\left(\mathcal{Z}(\tilde{\varphi},\tilde{\theta},\eta)\right)\right)\right| \le C_0\left(\frac{1}{\delta^2}\frac{\eta^4}{|Y|^4} + \frac{1}{\delta}\frac{\eta^2}{|Y|^3}\right).$$

Write $\mathrm{Hess}_{\varepsilon,P,\eta}$ for the Hessian of $(\Psi_\eta(\mathbb{T}_{\xi_\eta}), g_{\varepsilon,P,\eta})$. From (41) and Lemma 4.4, it follows that

$$\begin{aligned}\left|(\mathrm{Hess}_{\varepsilon,P,\eta}(\varphi_{1,\delta,\eta}))_{ij}\right| &= \left|\partial_i\partial_j(\varphi_{1,\delta,\eta}) - (\Gamma_{\varepsilon,P,\eta})^k_{ij}\partial_k\varphi_{1,\delta,\eta}\right|\\ &\le C_0\left\{\left(\frac{1}{\delta^2}\frac{\eta^4}{|Y|^4} + \frac{1}{\delta}\frac{\eta^2}{|Y|^3}\right)\eta + \frac{1}{\delta}\frac{\eta^2}{|Y|^2}\frac{\eta}{|Y|} + \frac{\eta}{|Y|^2}\right\} + \frac{C_0}{|Y|}\left(\frac{1}{\delta}\frac{\eta^2}{|Y|^2}\eta + \frac{\eta}{|Y|}\right)\\ &\le C_0\eta\left(\frac{1}{\delta^2}\frac{\eta^4}{|Y|^4} + \frac{1}{\delta}\frac{\eta^2}{|Y|^3} + \frac{1}{|Y|^2}\right)\end{aligned}$$

and

$$\begin{aligned}|(\mathrm{Hess}_{\varepsilon,P,\eta} - \mathrm{Hess}_{0,\eta})\,\varphi_{1,\delta,\eta}| &= \left|(\Gamma_{\varepsilon,P,\eta})^k_{ij} - (\Gamma_{0,\eta})^k_{ij}\right||\partial_k\varphi_{1,\delta,\eta}| \le C_0\varepsilon^2\frac{\eta^4}{|Y|^3}\left(\frac{\eta^3}{\delta|Y|^2} + \frac{\eta}{|Y|}\right)\\ &= C_0\varepsilon^2\eta\left(\frac{\eta^6}{\delta|Y|^5} + \frac{\eta^4}{|Y|^4}\right).\end{aligned}$$

Recalling $\Delta_{\varepsilon,P,\eta}f = (g_{\varepsilon,P,\eta})^{ij}(\mathrm{Hess}_{\varepsilon,P,\eta}f)_{ij}$, we get

$$|\Delta_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta}| \le C_0\eta\left(\frac{1}{\delta^2} + \frac{|Y|}{\delta\eta^2} + \frac{|Y|^2}{\eta^4}\right)$$

and

$$\begin{aligned}|(\Delta_{\varepsilon,P,\eta} - \Delta_{0,\eta})\,\varphi_{1,\delta,\eta}| &\le \left|(g_{\varepsilon,P,\eta})^{ij} - (g_{0,\eta})^{ij}\right||\mathrm{Hess}_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta}| + \left|(g_{\varepsilon,P,\eta})^{ij}\right||(\mathrm{Hess}_{\varepsilon,P,\eta} - \mathrm{Hess}_{0,\eta})\,\varphi_{1,\delta,\eta}|\\ &\le C_0\varepsilon^2\eta\left(\frac{1}{\delta^2}\frac{\eta^4}{|Y|^2} + \frac{1}{\delta}\frac{\eta^2}{|Y|} + 1\right) + C_0\varepsilon^2\eta\left(\frac{\eta^2}{\delta|Y|} + 1\right)\\ &\le C_0\varepsilon^2\eta\left(\frac{1}{\delta^2}\frac{\eta^4}{|Y|^2} + \frac{1}{\delta}\frac{\eta^2}{|Y|} + 1\right).\end{aligned}$$

From these estimates and Lemma 4.4, one may observe that

$$\begin{aligned}
&\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})} \left| H_{\varepsilon,P,\eta}\Delta_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta} + H_{\varepsilon,P,\eta}\left(|A_{\varepsilon,P,\eta}|^2 + \mathrm{Ric}_{g_{\varepsilon,P}}(n_{\varepsilon,P,\eta}, n_{\varepsilon,P,\eta}) - \frac{1}{2}H^2_{\varepsilon,P,\eta}\right)\varphi_{1,\delta,\eta}\right| \\
&\qquad \times |d\sigma_{\varepsilon,P,\eta} - d\sigma_{0,\eta}| \\
&\le C_0\varepsilon^2\eta\int_{I_\delta}\left\{\frac{|Y|}{\eta^2}\left(\frac{1}{\delta^2}+\frac{|Y|}{\delta\eta^2}+\frac{|Y|^2}{\eta^4}\right)+\frac{|Y|^3}{\eta^6}+\varepsilon^2\frac{|Y|}{\eta^2}\right\}\frac{\eta^8}{|Y|^6}d\tilde{\varphi}d\tilde{\theta} \qquad (45)\\
&\le C_0\varepsilon^2\eta\int_{C_1\eta^2/\delta}^{10}\left\{\frac{\eta^6}{\delta^2 r^4}+\frac{\eta^4}{\delta r^3}+\frac{\eta^2}{r^2}+\varepsilon^2\frac{\eta^6}{r^4}\right\}dr \le C_2\varepsilon^2\eta\delta
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})} |H_{\varepsilon,P,\eta}\Delta_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta} - H_{0,\eta}\Delta_{0,\eta}\varphi_{1,\delta,\eta}|\, d\sigma_{0,\eta} \\
&\le \int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})} \{|H_{\varepsilon,P,\eta} - H_{0,\eta}|\,|\Delta_{\varepsilon,P,\eta}\varphi_{1,\delta,\eta}| + |H_{0,\eta}|\,|(\Delta_{\varepsilon,P,\eta} - \Delta_{0,\eta})\varphi_{1,\delta,\eta}|\}\, d\sigma_{0,\eta} \qquad (46)\\
&\le C_0\varepsilon^2\eta\int_{I_\delta}\left\{\frac{\eta^2}{\delta^2|Y|}+\frac{1}{\delta}+\frac{|Y|}{\eta^2}\right\}\frac{\eta^4}{|Y|^4}d\tilde{\varphi}d\tilde{\theta} \le C_2\varepsilon^2\eta\delta.
\end{aligned}$$

The conclusion of Proposition easily follows from (42), (43), (44), (45) and (46). □

4.2 Metric dependence

The goal of this subsection is to estimate the contribution from $\varphi_{2,\delta,\eta}$ (see (37)) to the derivative of the Willmore energy. $\varphi_{2,\delta,\eta}$ is supported in a region of the degenerating torus where the curvature stays bounded. The main contribution of $\varphi_{2,\delta,\eta}$ will be due to the curvature of M and to the deviation of the tori from a purely spherical shape. Our aim is to prove the following result, which quantifies both effects.

Proposition 4.5. *Let the limit sphere $S^2_{\tilde{A}}$ and ψ_0 be as in Proposition 3.2. For $\delta \in (0, 1/2]$ and $\varphi_{\eta,\delta,2}$ as in (37), there exist $C_0 > 0$, $C_\delta > 0$ and $\eta_\delta > 0$ such that*

$$\begin{aligned}
&\left| dW_{g_\varepsilon}\left[\frac{\varphi_{\eta,\delta,2}}{\eta}\right] - dW_{g_0}\left[\frac{\varphi_{\eta,\delta,2}}{\eta}\right] + \varepsilon^2\left[\int_{S^2_{\tilde{A}}}(1-\chi_\delta)\left(F\Delta_{S^2_{\tilde{A}}}\psi_0 + \mathrm{Ric}_P(n_0,n_0)H_{S^2_{\tilde{A}}}\psi_0\right)d\sigma_0\right]\right| \\
&\le C_0\delta\varepsilon^2 + C_\delta\left(o_\eta(1)\varepsilon^2 + \varepsilon^3\right)
\end{aligned}$$

holds for any $\eta \in (0, \eta_\delta]$ and $\varepsilon \in (0, 1/2]$ where C_0 is independent of δ, C_δ depends only on δ, $o_\eta(1)$ is as in Proposition 4.2, and F is given by

$$\begin{aligned}
F &:= -\sum_{i=1}^{2} e_i(h_{ni}) + \sum_{i,j=1}^{2} h_{nj}\langle\nabla^{\mathbb{R}^3}_{e_i}e_i, e_j\rangle - \frac{1}{2}h_{nn}H_{S^2_{\tilde{A}}} + \frac{1}{2}\sum_{i=1}^{2}\frac{\partial h}{\partial n_0}(e_i,e_i), \\
H_{S^2_{\tilde{A}}} &:= \textit{the mean curvature of } S^2_{\tilde{A}} \textit{ in } (\mathbb{R}^3, g_0), \\
\mathcal{X}(\theta,\varphi) &:= \tilde{A}\,(\cos\theta + 1, \sin\theta\cos\varphi, \sin\theta\sin\varphi) \qquad (\theta,\varphi) \in (0,\pi)\times[0,2\pi], \\
e_1 &:= \tilde{A}^{-1}\partial_\theta\mathcal{X} = (-\sin\theta, \cos\theta\cos\varphi, \cos\theta\sin\varphi), \\
e_2 &:= (\tilde{A}\sin\theta)^{-1}\partial_\varphi\mathcal{X} = (0, -\sin\varphi, \cos\varphi), \\
n_0 &:= (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi), \\
(h(x))_{\alpha\beta} &:= \frac{1}{3}R_{\alpha\mu\nu\beta}x^\mu x^\nu, \quad h_{ni} := h(x)(n_0, e_i), \quad h_{nn} := h(x)(n_0,n_0).
\end{aligned}$$

Remark 4.6. *The term F above will turn out to be the metric derivative of the mean curvature of $(S^2_{\tilde{A}}, g_t)$ at $t = 0$ where $g_{t,\alpha\beta}(x) := \delta_{\alpha\beta} + th_{\alpha\beta}(x)$. Hence, F is smooth on $S^2_{\tilde{A}}$.*

Before proving Proposition 4.5 we collect some useful preliminary material and lemmas. Recalling the expansion of the metric g in the normal coordinates and setting $t = \varepsilon^2$, we observe that

$$g_{t,P,\alpha\beta}(x) := g_{\varepsilon,P,\alpha\beta}(x) = \delta_{\alpha\beta} + th_{P,\alpha\beta}(t,x)$$

and $t \mapsto g_{t,P,\alpha\beta}(x) : [0, t_0] \to C^k(B_{10})$ is of class $C^{1,1/2}$ for each $k \in \mathbb{N}$. Moreover,

$$\frac{\partial}{\partial t} g_{t,P,\alpha\beta}(x)\big|_{t=0} = h_{P,\alpha\beta}(0,x) = \frac{1}{3} R_{\alpha\mu\nu\beta} x^\mu x^\nu. \tag{47}$$

Next, we denote by $\Delta_{g_{t,P},\eta}$, $A_{g_{t,P},\eta}$, $\mathring{A}_{g_{t,P},\eta}$, $H_{g_{t,P},\eta}$ and $n_{g_{t,P},\eta}$ the Laplace-Beltrami operator, the second fundamental form, its traceless part, the mean curvature and the unit outer normal of $(\Psi_\eta(\mathbb{T}), g_{t,P})$. We also write $\mathrm{Ric}_{g_{t,P}}$ and $dW(t,P,\eta)$ for the Ricci tensor of $(B_{10}, g_{t,P})$ and the derivative of the Willmore functional at $(\Psi_\eta(\mathbb{T}), g_{t,P})$.

Lemma 4.7. *For each $\delta \in (0, 1/2)$, one may find $\eta_\delta > 0$ and C_δ so that if $0 < \eta \le \eta_\delta$ and $0 < t \le 1/2$, then*

$$\left| dW(t,P,\eta)[\psi_{2,\delta,\eta}] - dW(0,\eta)[\psi_{2,\delta,\eta}] + t\tilde{W}_P[\psi_{2,\delta,0}] \right| \le C_\delta (o_\eta(1)t + t^{3/2}), \tag{48}$$

where C_δ depends only on δ, $o_\eta(1)$ is as in Proposition 4.2 and

$$\psi_{2,\delta,\eta} := \frac{\varphi_{2,\delta,\eta}}{\eta} = \frac{(1-\chi_\delta)\varphi_\eta}{\eta}, \quad \psi_{2,\delta,0} := (1-\chi_\delta)\psi_0,$$
$$\tilde{W}_P[\psi] := \int_{S^2_{\tilde{A}}} \left\{ \left(\Delta_{S^2_{\tilde{A}}} \frac{dH_{g_{t,P},0}}{dt}\Big|_{t=0} \right) \psi + \mathrm{Ric}_P(n_{0,0}, n_{0,0}) H_{S^2_{\tilde{A}}} \psi \right\} d\sigma_{g_0}.$$

Proof. We first fix a $\delta \in (0, 1/2)$. Recall from Proposition 2.5 that

$$dW(t,P,\eta)[\psi] = -\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})} \left\{ \Delta_{g_{t,P},\eta} H_{g_{t,P},\eta} + \left(|\mathring{A}_{g_{t,P},\eta}|^2 + \mathrm{Ric}_{g_{t,P}}(n_{g_{t,P},\eta}, n_{g_{t,P},\eta}) \right) H_{g_{t,P},\eta} \right\} \psi d\sigma_{g_{t,P},\eta}. \tag{49}$$

Since $|\mathcal{Z}| \ge \delta$ is equivalent to $|Y| \le \eta^2/\delta$, from the parameterization of $\mathcal{Z}(\eta^2\bar{\varphi}, \eta^2\bar{\theta}, \eta)$ for $\Psi_\eta(\mathbb{T}_{\xi_\eta})$ and (32), it is easily seen that there exist $C_1 > 0$, which is independent of δ and η, such that if $0 < \eta \le 1/\delta$, then

$$\Psi_\eta(\mathbb{T}_{\xi_\eta}) \cap (B_\delta(0))^c \subset \left\{ \mathcal{Z}(\eta^2\bar{\varphi}, \eta^2\bar{\theta}, \eta) \mid (\bar{\varphi}, \bar{\theta}) \in I_\delta \right\} \qquad \text{where } I_\delta := \left[-\frac{C_1}{\delta}, \frac{C_1}{\delta} \right]^2.$$

We apply Lemma 2.2 (ii) for $R = C_1/\delta$. Then one may find a $\eta_\delta > 0$ such that for every $k \in \mathbb{N}$, there exists a $C_k > 0$ satisfying

$$\left\| \mathcal{Z}(\eta^2 \cdot, \eta^2 \cdot) - \mathrm{Ref}_{\mathbf{e}_x} \circ Z_0 \right\|_{C^k([-R,R]^2)} \le C_k \eta^{3/2} \tag{50}$$

provided $\eta \in (0, \eta_\delta]$.

Now, due to the cut-off function χ_δ, it is sufficient to consider the quantities on I_δ. We also suppose $0 < \eta \le \eta_\delta$. Since $t \mapsto g_{t,P,\alpha\beta}(x)$ is of class $C^{1,1/2}$ and the convergence (50) holds, we observe that

$$\Delta_{g_{t,P},\eta} f = \Delta_{g_0,\eta} f + \frac{d}{dt} \Delta_{g_{t,P},\eta} f\big|_{t=0} t + O_{\delta,1}(t^{3/2} \|f\|_{C^2(I_\delta)}), \quad H_{g_{t,P},\eta} = H_{g_0,\eta} + \frac{d}{dt} H_{g_{t,P},\eta}\big|_{t=0} t + O_{\delta,2}(t^{3/2}),$$
$$|\mathring{A}_{g_{t,P},\eta}|^2 = |\mathring{A}_{g_0,\eta}|^2 + \frac{d}{dt} |\mathring{A}_{g_{t,P},\eta}|^2\big|_{t=0} t + O_{\delta,2}(t^{3/2}), \qquad \mathrm{Ric}_t = \mathrm{Ric}_{g_0} + \frac{d}{dt} \mathrm{Ric}_{g_{t,P}}\big|_{t=0} t + O_{\delta,2}(t^{3/2}),$$
$$d\sigma_{g_{t,P},\eta} = d\sigma_{g_0,\eta} + \frac{d}{dt} d\sigma_{g_{t,P},\eta}\big|_{t=0} t + O_{\delta,2}(t^{3/2}), \qquad n_{g_{t,P},\eta} = n_{g_0,\eta} + \frac{d}{dt} n_{g_{t,P},\eta}\big|_{t=0} t + O_{\delta,2}(t^{3/2})$$

where $|O_{\delta,1}(t^{3/2}\|f\|_{C^2})| \le C_{1,\delta}\|f\|_{C^2}t^{3/2}$ and $\|O_{\delta,2}(t^{3/2})\|_{C^2(I_\delta)} \le C_{2,\delta}t^{3/2}$, and $C_{\delta,i}$ depend only on δ. Substituting these formula into (49) and noting $\mathrm{Ric}_0 = 0$, we obtain

(51)

$$\begin{aligned}
&dW(t,P,\eta)[\psi_{2,\delta,\eta}] \\
=&dW(0,\eta)[\psi_{2,\delta,\eta}] - t\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left\{\Delta_{0,\eta}\frac{d}{dt}H_{g_{t,P},\eta}\big|_{t=0} + \frac{d}{dt}\Delta_{g_{t,P},\eta}H_{g_0,\eta}\big|_{t=0}\right\}\psi_{2,\delta,\eta}d\sigma_{g_0,\eta} \\
&- t\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left\{\frac{d}{dt}|\mathring{A}_{g_{t,P},\eta}|^2\big|_{t=0}H_{g_0,\eta} + |\mathring{A}_{g_0,\eta}|^2\frac{d}{dt}H_{g_{t,P},\eta}\big|_{t=0} + \frac{d}{dt}\mathrm{Ric}_{g_{t,P}}\big|_{t=0}(n_{g_0,\eta},n_{g_0,\eta})H_{g_0,\eta}\right\}\psi_{2,\delta,\eta}d\sigma_{g_0,\eta} \\
&- t\int_{\Psi_\eta(\mathbb{T}_{\xi_\eta})}\left\{\Delta_{g_0,\eta}H_{g_0,\eta} + |\mathring{A}_{g_0,\eta}|^2H_{g_0,\eta}\right\}\psi_{2,\delta,\eta}\frac{d}{dt}d\sigma_{g_{t,P},\eta}\big|_{t=0} + O_\delta(t^{3/2})
\end{aligned}$$

where $|O_\delta(t^{3/2})| \le C_{\delta,3}t^{3/2}$.

Next, we observe the behaviours of the above quantities as $\eta \to 0$. By (50) and the fact that $\mathrm{Ref}_{\mathbf{e}_x} \circ Z_0$ is a position vector of $S^2_{\tilde{A}}$, it follows from $H_{g_0,0} = 2/\tilde{A}$ and $(A_{g_0,0})^i_j = \delta^i_j/\tilde{A}$ that

$$\left\|H_{g_0,\eta} - \frac{2}{\tilde{A}}\right\|_{C^2(I_\delta)} + \left\|(A_{g_0,\eta})^i_j - \frac{\delta^i_j}{\tilde{A}}\right\|_{C^2(I_\delta)} + \left\|\mathring{A}_{g_0,\eta}\right\|_{C^0(I_\delta)} \le C_{\delta,4}\eta^{3/2}.$$

Hence,

$$\begin{aligned}
&\left\|\frac{d}{dt}\Delta_{g_{t,P},\eta}H_{g_0,\eta}\big|_{t=0}\right\|_{C^0(I_\delta)} + \left\|\frac{d}{dt}|\mathring{A}_{g_{t,P},\eta}|^2\big|_{t=0}\right\|_{C^0(I_\delta)} \\
&\qquad + \|\Delta_{g_0,\eta}H_{g_0,\eta}\|_{C^0(I_\delta)} + \left\||\mathring{A}_{g_0,\eta}|^2H_{g_0,\eta}\right\|_{C^0(I_\delta)} \le C_{\delta,4}\eta^{3/2}.
\end{aligned}$$

Recalling (28), we also observe that

$$\begin{aligned}
&\|\psi_{2,\delta,\eta} - (1-\chi_\delta)\psi_0\|_{C^0(I_\delta)} + \left\|\Delta_{g_0,\eta}\frac{d}{dt}H_{g_{t,P},\eta}\big|_{t=0} - \Delta_{S^2_{\tilde{A}}}\frac{d}{dt}H_{g_{t,P},0}\big|_{t=0}\right\|_{C^0(I_\delta)} \\
&\quad + \left\|\frac{d}{dt}\mathrm{Ric}_{g_{t,P}}|_{t=0}(n_{g_0,\eta},n_{g_0,\eta}) - \frac{d}{dt}\mathrm{Ric}_{g_{t,P}}|_{t=0}(n_{g_0,0},n_{g_0,0})\right\|_{C^0(I_\delta)} \le C_{\delta,4}o_\eta(1)
\end{aligned}$$

where $o_\eta(1) = |\eta^{-4}(\xi'_\eta\eta - 2\xi_\eta) - c_0| + \eta^{3/2} \to 0$ as $\eta \to 0$ by Lemma 3.1 and Remark 3.5. Therefore, by (51), in order to show (48), it is sufficient to prove

$$\frac{d}{dt}\mathrm{Ric}_{g_{t,P}}\big|_{t=0}(n_{g_0,0},n_{g_0,0}) = \mathrm{Ric}_P(n_{g_0,0},n_{g_0,0}). \tag{52}$$

To this end, let (x^1,x^2,x^3) denote the coordinates of $(B_{10},g_{t,P})$ with $g_0((\partial_\alpha)_x,(\partial_\beta)_x) = \delta_{\alpha\beta}$ and define $R_{t,\alpha\beta}$ as the component of the Ricci tensor in these coordinates:

$$R_{g_{t,P},\alpha\beta}(x) = \mathrm{Ric}_{g_{t,P}}(x)\left((\partial_\alpha)_x,(\partial_\beta)_x\right).$$

We also write $\Gamma^\gamma_{g_{t,P},\lambda\nu}$ for the Christoffel symbol in the above coordinates. Then arguing as in the proof of Lemma 4.2 in [12], we obtain

$$\frac{d}{dt}\Gamma^\kappa_{g_{t,P},\lambda\mu}\Big|_{t=0} = \frac{1}{2}\delta^{\kappa\xi}\left(\partial_\lambda h_{P,\xi\mu} + \partial_\mu h_{P,\xi\lambda} - \partial_\xi h_{P,\lambda\mu}\right). \tag{53}$$

We also remark that $\Gamma^\kappa_{g_0,\lambda\nu} \equiv 0$. Hence, from the formula

$$R_{g_{t,P},\alpha\beta} = \partial_\rho\Gamma^\rho_{g_{t,P},\beta\alpha} - \partial_\beta\Gamma^\rho_{g_{t,P},\rho\alpha} + \Gamma^\rho_{g_{t,P},\rho\lambda}\Gamma^\lambda_{g_{t,P},\beta\alpha} - \Gamma^\rho_{g_{t,P},\beta\lambda}\Gamma^\lambda_{g_{t,P},\rho\alpha}$$

it follows that

$$\frac{d}{dt}R_{g_{t,P},\alpha\beta}\Big|_{t=0} = \partial_\rho\frac{d}{dt}\Gamma^\rho_{g_{t,P},\beta\alpha}\Big|_{t=0} - \partial_\beta\frac{d}{dt}\Gamma^\rho_{g_{t,P},\rho\alpha}\Big|_{t=0}.$$

Now, by(47), one observes that

$$\partial_\zeta \partial_\xi h_{P,\alpha\beta} = \frac{1}{3} R_{\alpha\mu\nu\beta}(\delta_\xi^\mu \delta_\zeta^\nu + \delta_\xi^\nu \delta_\zeta^\mu) = \frac{1}{3}(R_{\alpha\xi\zeta\beta} + R_{\alpha\zeta\xi\beta}).$$

Thus from (53), we see that

$$\begin{aligned}
\partial_\rho \frac{d}{dt} \Gamma^\rho_{g_{t,P},\beta\alpha}\Big|_{t=0} &= \frac{1}{2}\delta^{\rho\xi}(\partial_\rho\partial_\beta h_{P,\xi\alpha} + \partial_\rho\partial_\alpha h_{P,\xi\beta} - \partial_\rho\partial_\xi h_{P,\alpha\beta}) \\
&= \frac{1}{2}\sum_{\rho=1}^{3}(\partial_\rho\partial_\beta h_{P,\rho\alpha} + \partial_\alpha\partial_\rho h_{P,\rho\beta} - \partial_\rho\partial_\rho h_{P,\alpha\beta}) \\
&= \frac{1}{6}\sum_{\rho=1}^{3}(R_{\rho\rho\beta\alpha} + R_{\rho\beta\rho\alpha} + R_{\rho\rho\alpha\beta} + R_{\rho\alpha\rho\beta} + 2R_{\rho\alpha\rho\beta}) = \frac{2}{3}R_{\alpha\beta}, \\
\partial_\beta \frac{d}{dt}\Gamma^\rho_{g_t,\rho\alpha}\Big|_{t=0} &= \frac{1}{2}\delta^{\rho\kappa}(\partial_\beta\partial_\rho h_{\kappa\alpha} + \partial_\beta\partial_\alpha h_{\kappa\rho} - \partial_\beta\partial_\kappa h_{\rho\alpha}) = \frac{1}{2}\partial_\beta\partial_\alpha\sum_{\rho=1}^{3} h_{\rho\rho} = -\frac{1}{3}R_{\alpha\beta}.
\end{aligned}$$

Hence, we have

$$\frac{d}{dt} R_{g_{t,P},\alpha\beta}\Big|_{t=0} = R_{\alpha\beta},$$

which yields (52), and we complete the proof. □

Proof of Proposition 4.5. By Lemma 4.7 and $t = \varepsilon^2$, for every $\delta \in (0, 1/2]$, we find $\eta_\delta > 0$ and C_δ such that

$$\left| dW_{g_\varepsilon}[\psi_{2,\delta,\eta}] - dW_{g_0}[\psi_{2,\delta,\eta}] + \varepsilon^2 \tilde{W}_P[\psi_{2,\delta,0}] \right| \le C_\delta \left(o_\eta(1)\varepsilon^2 + \varepsilon^3 \right)$$

for all $0 < \eta \le \eta_\delta$ and $0 < \varepsilon \le 1/2$. We remark that $dH_{g_{t,P},0}/dt|_{t=0}$ is smooth on $S^2_{\tilde{A}}$ and it follows from the proof of [12, Lemma 4.2] that

$$F(q) = \frac{d}{dt} H_{g_{t,P},0}\big|_{t=0}(q) \qquad (q \in S^2_{\tilde{A}}).$$

Noting that $0 \in S^2_{\tilde{A}}$ and $\psi_{2,\delta,0} = (1-\chi_\delta)\psi_0$ is also smooth on $S^2_{\tilde{A}}$, one has

$$\int_{S^2_{\tilde{A}}} (\Delta_{S^2_{\tilde{A}}} F)\psi_{2,\delta,0} d\sigma = \int_{S^2_{\tilde{A}}} \left\{ \Delta_{S^2_{\tilde{A}}}(F - F(0)) \right\} (1-\chi_\delta)\psi_0 d\sigma = \int_{S^2_{\tilde{A}}} (F - F(0)) \Delta_{S^2_{\tilde{A}}} \left\{ (1-\chi_\delta)\psi_0 \right\}. \tag{54}$$

Therefore, to prove Proposition 4.5, it is enough to show that

$$\int_{S^2_{\tilde{A}}} (F - F(0))\Delta_{S^2_{\tilde{A}}} \left\{(1-\chi_\delta)\psi_0\right\} d\sigma = \int_{S^2_{\tilde{A}}} (1-\chi_\delta) F \Delta_{S^2_{\tilde{A}}} \psi_0 d\sigma + O(\delta).$$

Since

$$\Delta_{S^2_{\tilde{A}}}((1-\chi_\delta)\psi_0) = -(\Delta_{S^2_{\tilde{A}}}\chi_\delta)\psi_0 - 2g_{S^2_{\tilde{A}}}(\nabla_{S^2_{\tilde{A}}}\chi_\delta, \nabla_{S^2_{\tilde{A}}}\psi_0) + (1-\chi_\delta)\Delta_{S^2_{\tilde{A}}}\psi_0,$$

it suffices to prove that

$$\int_{S^2_{\tilde{A}}} |F - F(0)| \left\{ \left|\Delta_{S^2_{\tilde{A}}}\chi_\delta\right| |\psi_0| + \left| g_{S^2_{\tilde{A}}}(\nabla_{S^2_{\tilde{A}}}\chi_\delta, \nabla_{S^2_{\tilde{A}}}\psi_0)\right| \right\} d\sigma = O(\delta) = \int_{S^2_{\tilde{A}}} F(0)(1-\chi_\delta)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma. \tag{55}$$

Since we may suppose that χ_δ is radially symmetric, i.e. $\chi_\delta(x) = \chi_\delta(|x|)$, we observe that $\chi_\delta(|\mathcal{X}(\theta,\varphi)|)$ depends only on θ. For the definition of $\mathcal{X}(\theta,\varphi)$, see Proposition 4.5. Furthermore, we may also assume

$$\begin{aligned}
&|\chi'_\delta(|x|)| \le C_0\delta^{-1}, \qquad |\chi''_\delta(|x|)| \le C_0\delta^{-2}, \\
&\mathrm{supp}\,(\chi_\delta) \cap S^2_{\tilde{A}} \subset \{(\theta,\varphi) \in [0,\pi]\times[0,2\pi] \;:\; |\theta - \pi| \le C_0\delta\} =: I_\delta
\end{aligned} \tag{56}$$

for all $\delta \in (0, 1/2)$ where $C_0 > 0$ is independent of δ. Using $\mathcal{X}(\theta, \varphi)$ as a coordinate of $S^2_{\tilde{A}}$ and writing $\psi_0 = A\cos\theta + B(1-\cos\theta)\cos 2\varphi$ where $(A, B) = (\sqrt{2}/2, (2-\sqrt{2})/4)$ by (23), it is easily seen that

$$\begin{aligned}\Delta_{S^2_{\tilde{A}}} \psi_0 &= \frac{1}{\tilde{A}^2}\left\{2\cos\theta(-A + B\cos 2\varphi) - \frac{4B(1-\cos\theta)\cos 2\varphi}{\sin^2\theta}\right\} \\ &= \frac{1}{\tilde{A}^2}\left[-2A\cos\theta + 2B\cos 2\varphi\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right].\end{aligned}$$

Thus, by (56) and the fact that $\chi_\delta(|\mathcal{X}(\theta, \varphi)|)$ depends only on θ, we have

$$\begin{aligned}&\int_{S^2_{\tilde{A}}} F(0)(1-\chi_\delta)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma \\ =& F(0)\int_0^\pi\int_0^{2\pi}(1-\chi_\delta)\left[-2A\cos\theta + 2B\cos 2\varphi\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right]\sin\theta d\varphi d\theta \\ =& -4\pi A F(0)\int_0^\pi (1-\chi_\delta)\sin\theta\cos\theta d\theta = 4\pi A F(0)\int_0^\pi \chi_\delta \sin\theta\cos\theta d\theta = O(\delta^2).\end{aligned}$$

On the other hand, since $\mathcal{X} = \tilde{A}(n_0 + \mathbf{e}_x)$ where n_0 is the outer unit normal to $S^2_{\tilde{A}}$, $|\mathcal{X}(\theta, \varphi)|^2 = 2\tilde{A}^2(1+\cos\theta)$ and $\sin\theta \sim \pi - \theta \sim \sqrt{1+\cos\theta}$ for $|\theta - \pi| \leq C_0\delta$, one observes that for $(\theta, \varphi) \in I_\delta$,

$$\begin{aligned}|\partial_\theta\{\chi_\delta(|\mathcal{X}(\theta,\varphi)|)\}| &= \left|\chi'_\delta(|\mathcal{X}|)\left\langle\frac{\mathcal{X}}{|\mathcal{X}|}, \partial_\theta\mathcal{X}\right\rangle\right| = \frac{\tilde{A}}{\sqrt{2}}\left|\chi'_\delta(|\mathcal{X}|)\frac{\sin\theta}{\sqrt{1+\cos\theta}}\right| \leq C_1\delta^{-1}, \\ \left|\partial^2_\theta\{\chi_\delta(|\mathcal{X}(\theta,\varphi)|)\}\right| &\leq \left|\chi''_\delta(|\mathcal{X}|)\left(\frac{\langle\mathcal{X}, \partial_\theta\mathcal{X}\rangle}{|\mathcal{X}|}\right)^2\right| + \left|\chi'_\delta(|\mathcal{X}|)\partial_\theta\left\langle\frac{\mathcal{X}}{|\mathcal{X}|}, \partial_\theta\mathcal{X}\right\rangle\right| \\ &\leq C_1\left(\frac{1}{\delta^2} + \frac{1}{\delta\sin\theta}\right).\end{aligned} \tag{57}$$

By (23), we have $\partial_\theta\psi_0 = -A\sin\theta + B\sin\theta\cos 2\varphi$. Therefore, we obtain

$$\begin{aligned}\left|\Delta_{S^2_{\tilde{A}}}\{\chi_\delta(|\mathcal{X}(\theta,\varphi)|)\}\right| &= \left|\frac{1}{\tilde{A}^2\sin\theta}\partial_\theta[\sin\theta\partial_\theta\{\chi_\delta(|\mathcal{X}|)\}]\right| \leq C_2\left(\frac{1}{\delta^2} + \frac{1}{\delta\sin\theta}\right), \\ \left|g_{S^2_{\tilde{A}}}(\nabla_{S^2_{\tilde{A}}}\chi_\delta, \nabla_{S^2_{\tilde{A}}}\psi_0)\right| &= \left|\frac{1}{\tilde{A}^2}\partial_\theta\chi_\delta\partial_\theta\psi_0\right| \leq C_2\delta^{-1}.\end{aligned}$$

Finally, by the continuity of F at the origin, one sees $|F(\mathcal{X}(\theta, \varphi)) - F(0)| \leq C_2\delta$ in I_δ. Hence, noting $\psi_0 \in L^\infty(S^2_{\tilde{A}})$, we get

$$\begin{aligned}\int_{S^2_{\tilde{A}}}|F(\mathcal{X}(\theta,\varphi)) - F(0)|\left|\Delta_{S^2_{\tilde{A}}}\{\chi_\delta(\mathcal{X}(\theta,\varphi))\}\right||\psi_0|d\sigma &\leq C_3\int_{I_\delta}\left(\frac{1}{\delta} + \frac{1}{\sin\theta}\right)\tilde{A}^2\sin\theta d\theta d\varphi \\ &\leq C\int_0^{C_0\delta}(\delta^{-1}\theta + 1)d\theta \leq C\delta.\end{aligned}$$

Similarly, we obtain

$$\int_{S^2_{\tilde{A}}}|F(\mathcal{X}(\theta,\varphi)) - F(0)|\left|g_{S^2_{\tilde{A}}}(\nabla_{S^2_{\tilde{A}}}\chi_\delta, \nabla_{S^2_{\tilde{A}}}\psi_0)\right|d\sigma \leq C\int_0^{C_0\delta}\theta d\theta \leq C\delta.$$

Thus (55) holds and we complete the proof. □

4.3 Proof of Proposition 4.2

By (38), $W'_{g_0} = 0$ and Propositions 4.3 and 4.5, for each $\delta \in (0, 1/2)$, there exists $\eta_\delta > 0$ such that

$$dW_{g_\varepsilon}[\varphi_\eta] = -\eta\varepsilon^2 \int_{S^2_{\tilde{A}}} (1-\chi_\delta)\left(F\Delta_{S^2_{\tilde{A}}}\psi_0 + \mathrm{Ric}_P(n_0, n_0)H_{S^2_{\tilde{A}}}\psi_0\right) d\sigma + O(\delta\eta\varepsilon^2) + O_\delta\left(o_\eta(1)\eta\varepsilon^2 + \eta\varepsilon^3\right)$$

holds for all $\varepsilon \in (0, 1/2)$ and $\eta \in (0, \eta_\delta)$. The next proposition evaluates the first term in the right-hand side of the above formula and will be proved in an appendix as it consists of long explicit computations.

Proposition 4.8. *One has*

$$\int_{S^2_{\tilde{A}}} (1-\chi_\delta)\left(F\Delta_{S^2_{\tilde{A}}}\psi_0 + \mathrm{Ric}_P(n_0, n_0)H_{S^2_{\tilde{A}}}\psi_0\right) d\sigma = \frac{16}{3}\pi B\tilde{A}(R_{22} - R_{33}) + O(\delta^2). \tag{58}$$

where $\tilde{A} = \sqrt[4]{2}\sqrt{\pi}$ *and* $B = \frac{2-\sqrt{2}}{4}$.

Remark 4.9. *From* (54) *and* (55) *in the proof of Proposition 4.5, Proposition 4.8 yields*

$$\int_{S^2_{\tilde{A}}} (\Delta_{S^2_{\tilde{A}}} F)\psi_0 + \mathrm{Ric}_P(n_0, n_0)H_{S^2_{\tilde{A}}}\psi_0 d\sigma = \frac{16}{3}\pi B\tilde{A}(R_{22} - R_{33}).$$

When considering the variation in η of the surface $\Sigma_{\varepsilon,P,Id,\omega}$ with $\eta = 1 - |\omega|$, the normal component of the variation vector field will be given by

$$\varphi_{\varepsilon,\eta} := g_{\varepsilon,P}\left(\mathcal{Z}\right)\left[\frac{\partial \mathcal{Z}}{\partial\eta}, n_{\varepsilon,P,\eta}\right],$$

where $\mathcal{Z}$ is defined in Subsection 3.2 and where $n_{\varepsilon,P,\eta}$ stands for the unit outer normal to $\Sigma_{\varepsilon,P,Id,\omega}$ in $(\mathbb{R}^3, g_{\varepsilon,P})$. By (8), for any compact set $K \subset \mathbb{D}$, one finds that

$$\varphi_{\varepsilon,\eta} = g_0\left[\frac{\partial \mathcal{Z}}{\partial\eta}, n_{0,\eta}\right] + \kappa_{\varepsilon,\eta} = \varphi_\eta + \kappa_{\varepsilon,\eta},$$

where $\kappa_{\varepsilon,\eta}$ is a smooth function satisfying

$$\|\kappa_{\varepsilon,\eta}\|_{C^4(\Sigma_{\varepsilon,P,Id,\omega})} \le C_K\varepsilon^2 \qquad \text{for } \omega \in K.$$

Using Lemma 2.6 and $|\omega| = 1 - \eta$, we find that

$$\begin{aligned}\frac{\partial}{\partial\omega}W_{g_\varepsilon}(\Sigma_{\varepsilon,P,Id,\omega}) &= -dW_{g_\varepsilon}[\varphi_{\varepsilon,\eta}] = -dW_{g_\varepsilon}[\varphi_\eta + \kappa_{\varepsilon,\eta}] \\ &= -dW_{g_\varepsilon}[\varphi_\eta] - dW_{g_\varepsilon}[\kappa_{\varepsilon,\eta}] = -dW_{g_\varepsilon}[\varphi_\eta] + O_K(\varepsilon^4).\end{aligned}$$

By Proposition 4.8 and the formula before it, the conclusion follows.

Remark 4.10. *Let* $R \in SO(3)$ *and* $r \in [0, 1)$. *As in Proposition 4.2, we have the following estimate: ($\eta := 1 - r$, $\tilde{\eta} \in (0, \eta_\delta)$)*

$$\left|\frac{\partial}{\partial r}W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - \eta\varepsilon^2\frac{16}{3}\pi B\tilde{A}\mathcal{F}(P, R)\right| \le \left[C_0\delta + C_\delta\left\{o_\eta(1) + \varepsilon\right\}\right]\eta\varepsilon^2 + C_{\tilde{\eta}}\varepsilon^4 \tag{59}$$

for all $\eta \in (\tilde{\eta}, \eta_\delta]$ *and* $\varepsilon \in (0, 1/2)$ *where* $\mathcal{F}(P, R) := \mathrm{Ric}_P(R\mathbf{e}_y, R\mathbf{e}_y) - \mathrm{Ric}_P(R\mathbf{e}_z, R\mathbf{e}_z)$.

To see that (59) *holds, we first remark that from the definitions of* $\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}$ *and the map* $T_{r\mathbf{e}_x}$ *in Proposition 2.3, we have*

$$\Sigma_{\varepsilon,P,R,r\mathbf{e}_x} = \exp_P^{g_\varepsilon}(R\mathbb{T}_{r\mathbf{e}_x}).$$

Notice that $(R\mathbb{T}_{r\mathbf{e}_x}, g_{\varepsilon,P})$ *is isometric to* $(\mathbb{T}_{r\mathbf{e}_x}, R^* g_{\varepsilon,P})$*. Putting* $t = \varepsilon^2$ *and* $g_{t,P,R} = R^* g_{\varepsilon,P}$*, it follows from the proof of* (52) *that*

$$\frac{d}{dt}\mathrm{Ric}_{g_{t,P,R}}(x)\Big|_{t=0}(n_{0,0}, n_{0,0}) = \frac{d}{dt}\mathrm{Ric}_{g_{t,P}}(Rx)\Big|_{t=0}(Rn_{0,0}, Rn_{0,0}) = \mathrm{Ric}_P(Rn_{0,0}, Rn_{0,0}).$$

Moreover, from the proof of Proposition 4.8 in Appendix I, we also observe that

$$\int_{S^2_{\tilde{A}}}(1-\chi_\delta)\left(F_{P,R}\Delta_{S^2_{\tilde{A}}}\psi_0 + \mathrm{Ric}_P(Rn_{0,0}, Rn_{0,0})\right)\mathrm{d}\sigma = \frac{16}{3}\pi B\tilde{A}\mathcal{F}(P,R) + O(\delta^2)$$

where $F_{P,R} = d/dt|_{t=0} H_{g_{t,P,R}, S^2_{\tilde{A}}}$ *the metric derivative of the mean curvature of* $S^2_{\tilde{A}}$*. Thus, we obtain*

$$\frac{\partial}{\partial r}W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) = \frac{\partial}{\partial r}W_{g_{\varepsilon,P,R}}(\mathbb{T}_{r\mathbf{e}_x}) = -\frac{\partial}{\partial \eta}W_{g_{\varepsilon,P,R}}(\mathbb{T}_{\xi_\eta})$$

where $\eta := 1 - |\omega|$*. Combining these facts, arguing as in the proof of Proposition 4.2, one can check that* (59) *holds.*

5 Proof of the main theorems

In this section we collect all the estimates and expansions established so far in order to prove our main results, namely Theorems 1.1 and 1.2.

For $r \in (0,1)$, we consider the compact set of the unit disk $\mathbb{D}$

$$K_r = \{|\omega| \le r\}.$$

Then by the definition of (7) one sees that

$$\partial\mathcal{T}_{\varepsilon,K_r} = \{\exp_P^{g_\varepsilon}(R\mathbb{T}_\omega) \ : \ P \in M,\ R \in SO(3),\ |\omega| = r\} = \{\exp_P^{g_\varepsilon}(R\mathbb{T}_{r\mathbf{e}_x}) \ : \ P \in M,\ R \in SO(3)\}$$

and $\partial\mathcal{T}_{\varepsilon,K_r}$ is parametrised by $M \times SO(3)$ through the map $(P,R) \mapsto \exp_P^{g_\varepsilon}(R\mathbb{T}_{r\mathbf{e}_x})$.

Remark 5.1. *Notice that, from the geometric point of view, the above parametrization of* $\partial\mathcal{T}_{\varepsilon,K_r}$ *is counting twice each torus: indeed, due to planar symmetry, for every* $r < 1$ *there exists a nontrivial rotation* $R \in SO(3)$ *such that* $RT_{r\mathbf{e}_x}$ *and* $T_{r\mathbf{e}_x}$ *are just different parametrizations of the same torus. This is the reason for the appearance of the factor* $\frac{1}{2}$ *in the definition of* $\tilde{C}_q$ *in* (4)*.*

Using this map, we have the following estimate for W_{g_ε} on $\partial\mathcal{T}_{\varepsilon,K_r}$:

Proposition 5.2. *Fix* $\delta \in (0,1/2)$*. Then there exist* $C_0 > 0$*,* $r_\delta \in (0,1)$ *and* $C_\delta > 0$ *satisfying the following property: for every* $r_\delta \le r < \tilde{r} < 1$*, one may find* $C_{\tilde{r}} > 0$ *such that*

$$\begin{aligned}&\left\| W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - 8\pi^2 + \frac{8\sqrt{2}\pi^2\varepsilon^2}{3}\left(\mathrm{Sc}_P + \frac{B\tilde{A}}{\sqrt{2}\pi}(1-r)^2\mathcal{F}(P,R)\right)\right\|_{C^2(M\times SO(3))}\\ &\le \varepsilon^2\left[C_0\varepsilon + o_{\tilde{r}}(1) + \left[C_0\delta + C_\delta\left\{o_r(1)+\varepsilon\right\}\right](1-r)^2 + C_{\tilde{r}}\varepsilon^2(\tilde{r}-r)\right]\end{aligned}$$

for all $\varepsilon \in (0,1/2)$*. Here* $o_{\tilde{r}}, o_r(1) \to 0$ *as* $\tilde{r}, r \uparrow 1$*.*

Proof. Fix $\delta \in (0,1/2)$. We claim that there exist $C_0 > 0$, $r_\delta \in (0,1)$ and $C_\delta > 0$ satisfying the following properties: for every $r_\delta \le r < \tilde{r} < 1$ one may find $C_{\tilde{r}} > 0$ such that

$$\limsup_{r\uparrow 1}\frac{1}{\varepsilon^2}\left\| W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - 8\pi^2 + \frac{8\sqrt{2}}{3}\varepsilon^2\pi^2\mathrm{Sc}_P\right\|_{C^2(M\times SO(3))} \le C_0\,\varepsilon, \tag{60}$$

$$\begin{aligned}&\left\|\frac{\partial}{\partial r}W_{g_\varepsilon}\left(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}\right) - \frac{16}{3}\pi B\tilde{A}(1-r)\varepsilon^2\mathcal{F}(P,R)\right\|_{C^2(M\times SO(3))}\\ &\le \left[C_0\delta + C_\delta\left\{o_r(1)+\varepsilon\right\}\right](1-r)\varepsilon^2 + C_{\tilde{r}}\varepsilon^4\end{aligned} \tag{61}$$

for each $\varepsilon \in (0, 1/2)$.

We remark that in [12, Proposition 4.6] (see also Proposition 4.1 above) and in Proposition 4.2 (Remark 4.10) we have shown (60) and (61) in C^0-sense. To prove (60) and (61) in C^2-sense with respect to (P, R), put $g_{\varepsilon,P,R}(x) := (\exp_P^{g_\varepsilon} \circ R)^* g_\varepsilon$. Then $(\mathbb{T}_{r\mathbf{e}_x}, g_{\varepsilon,P,R})$ is isometric to $(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}, g_\varepsilon)$. Moreover, it follows from (8) and (11) that

$$\left|D^k_{P,R}\, g_{\varepsilon,P,R}(x)\right| \le C_k \varepsilon^2 |x|^2. \tag{62}$$

For (60), we argue as in the proof of Proposition 4.6 in [12], namely, using the Willmore functional in the Euclidean space and decomposing the functional into the handle part and the sphere part. By (62), we may observe that the contribution of the handle part is negligible in C^2-sense. On the other hand, the sphere part depends smoothly on P and R. Combining these facts, we see that (60) holds.

For (61), we also proceed as in the proof of Proposition 4.2 and use the cut-off function χ_δ. By (62) and the fact that $\mathbb{T}$ and its conformal deformations are critical points for the Euclidean Willmore functional, one sees that the handle part is negligible in C^2-sense. Then since the sphere part depends smoothly on P and R, it follows from the expressions in Remarks 4.9 and 4.10 that a counterpart of Proposition 4.8 in C^2-sense holds:

$$\left\| \int_{S^2_{\tilde A}} \left\{ \left(\Delta_{S^2_{\tilde A}} \frac{dH_{g_{t,P,R}}}{dt}\Big|_{t=0} \right) + \mathrm{Ric}_P(Rn_{0,0}, Rn_{0,0}) \right\} (1-\chi_\delta)\psi_0 d\sigma - \frac{16}{3}\pi B\tilde A \mathcal{F}(P,R) \right\|_{C^2} \le C_0 \delta^2.$$

Thus, arguing as in the proof of Proposition 4.2, we can show the estimate (61).

Now integrate (61) on $[r, \tilde r]$ to obtain

$$\begin{aligned}&\left\| W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\tilde r\mathbf{e}_x}) - W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) + \frac{8}{3}\pi B\tilde A \varepsilon^2 \mathcal{F}(P,R)\left\{(1-\tilde r)^2 - (1-r)^2\right\} \right\|_{C^2(M\times SO(3))} \\ &\le \left[C_0\delta + C_\delta\left\{o_r(1)+\varepsilon\right\}\right]\varepsilon^2\left\{(1-r)^2 - (1-\tilde r)^2\right\} + C_{\tilde r}\varepsilon^4(\tilde r - r).\end{aligned}$$

From (60) it follows that

$$\left\| W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\tilde r\mathbf{e}_x}) - 8\pi^2 + \frac{8\sqrt2}{3}\pi^2\varepsilon^2 \mathrm{Sc}_P \right\|_{C^2(M\times SO(3))} \le C_0\varepsilon^2\left\{\varepsilon + o_{\tilde r}(1)\right\}$$

Therefore, we have

$$\begin{aligned}&\left\| W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - 8\pi^2 + \frac{8\sqrt2\pi^2}{3}\varepsilon^2 \left(\mathrm{Sc}_P + \frac{B\tilde A}{\sqrt2\pi}\mathcal{F}(P,R)(1-r)^2 \right) \right\|_{C^2(M\times SO(3))} \\ &\le \varepsilon^2\left\{C_0\varepsilon + o_{\tilde r}(1)\right\} + \left[C_0\delta + C_\delta\left\{o_r(1)+\varepsilon\right\}\right]\varepsilon^2(1-r)^2 + C_{\tilde r}\varepsilon^4(\tilde r - r)\end{aligned}$$

and Proposition 5.2 follows. □

Recall that, in the spirit of [31], a C^2 function defined on a manifold with boundary is said to satisfy the *general boundary conditions* if its gradient never vanishes at the boundary and if its restriction to the boundary is a Morse function. We have then the following result.

Lemma 5.3. *For $\varepsilon > 0$ small, let Φ_ε be defined as in Proposition 2.8. Then, under the assumptions of Theorem 1.1, there exists $r_0 \in (0,1)$ satisfying the following property: for all $r \in [r_0, 1)$ one may find $\varepsilon_r > 0$ such that if $\varepsilon \in (0, \varepsilon_r]$, Φ_ε satisfies the general boundary conditions on $\partial\mathcal{T}_{\varepsilon,K_r}$*

Proof. For $0 < r < 1$, set

$$\mathcal{G}_r(P,R) := -\mathrm{Sc}_P - \frac{B\tilde A}{\sqrt2\pi}\mathcal{F}(P,R)(1-r)^2.$$

We divide our arguments into several steps:

Step 1: *There exist $r_1 \in (0,1)$ and $\zeta_0 > 0$ such that if $r_1 \le r < 1$ and a function $\mathcal{H}(P,R) \in C^2(M \times SO(3))$ satisfies*

$$\begin{aligned} &\frac{1}{\varepsilon^2} \left\| D_P^\ell \left(\mathcal{H}(P,R) - \frac{8\sqrt{2}\pi^2}{3} \varepsilon^2 \mathcal{G}_r(P,R) \right) \right\|_{L^\infty(M\times SO(3))} \\ &+ \frac{1}{\varepsilon^2(1-r)^2} \left\| D_R D_{P,R}^m \left(\mathcal{H}(P,R) - \frac{8\sqrt{2}\pi^2}{3} \varepsilon^2 \mathcal{G}_r(P,R) \right) \right\|_{L^\infty(M\times SO(3))} \le \zeta_0 \end{aligned} \tag{63}$$

for $1 \le \ell \le 2$ and $0 \le m \le 1$, then $\mathcal{H}$ is a Morse function on $M \times SO(3)$ and

$$\begin{aligned} &\sharp\{(P,R) \in M \times SO(3) \ : \ \nabla \mathcal{G}_r(P,R) = 0, \quad \mathrm{index}(\nabla^2 \mathcal{G}_r(P,R)) = q\} \\ =&\sharp\{(P,R) \in M \times SO(3) \ : \ \nabla \mathcal{H}(P,R) = 0, \quad \mathrm{index}(\nabla^2 \mathcal{H}(P,R)) = q\} \end{aligned} \tag{64}$$

for each $0 \le q \le 6$. Moreover, when $\zeta_0 \to 0$, any critical point (Q,R) of $\mathcal{H}(P,R)$ satisfies $\min_{1\le i\le k} |Q - P_i|_g \to 0$ where P_i are the critical points of Sc_P which by assumption satisfy $(ND1)$ of the Introduction.

In fact, since Sc_P is a Morse function by assumption and

$$\left\| D_P^\ell \left\{ \frac{\mathcal{H}(P,R)}{\varepsilon^2} + \frac{8\sqrt{2}\pi^2}{3} \left(\mathrm{Sc}_P + \frac{B\tilde{A}}{\sqrt{2}\pi} \mathcal{F}(P,R)(1-r)^2 \right) \right\} \right\|_{L^\infty(M\times SO(3))} \le \zeta_0,$$

holds, we may find some $r_1 \in (0,1)$ and $\zeta_0 > 0$ so that when $r \in [r_1, 1)$, the function $P \mapsto \mathcal{H}(P,R)$ is a Morse function on M, the number of critical points is the same as that of $P \mapsto \mathrm{Sc}_P$ and if $D_P \mathcal{H}(Q,R) = 0$ for some $Q \in M$, then Q must be close to one of P_i $(1 \le i \le k)$ by $(ND1)$. Therefore, enlarging $r_1 \in (0,1)$ and shrinking $\zeta_0 > 0$ if necessary, $(ND2)$ of the Introduction implies that the function $R \mapsto \mathcal{F}(Q,R)$ is a Morse function provided $D_P \mathcal{F}(Q,R_0) = 0$. Since it follows from (63) that

$$\left\| D_R D_{P,R}^m \left(\frac{\mathcal{H}(P,R)}{\varepsilon^2(1-r)^2} - \frac{8\pi}{3} B\tilde{A}\mathcal{F}(P,R) \right) \right\|_{L^\infty(M\times SO(3))} \le \zeta_0 \quad (0 \le m \le 1),$$

one sees that $R \mapsto \mathcal{H}(Q,R)$ is a Morse function on $SO(3)$ and the number of critical points is the same as of $R \mapsto \mathcal{F}(P_i,R)$ where $|Q - P_i|_g = \min\{|Q-P_j|_g \ : \ 1 \le j \le k\}$ provided $D_P\mathcal{H}(Q,R) = 0$ and $\zeta_0 > 0$ is sufficiently small. Therefore, if $r_1 \le r < 1$ and $\mathcal{H}$ satisfies (63), then the number of critical points of $\mathcal{H}$ are the same as that of $\mathcal{G}_r(P,R)$.

Next, let $\nabla \mathcal{H}(P,R) = 0$ and observe from (63) that

$$\begin{aligned} &\det\left(\nabla^2 \mathcal{H}(P,R)\right) \\ =&\varepsilon^{12}(1-r)^6 \left\{ \det\left(D_P^2 \frac{\mathcal{H}(P,R)}{\varepsilon^2} \right) \det\left(D_R^2 \frac{\mathcal{H}(P,R)}{\varepsilon^2(1-r)^2} \right) + O((1-r)^2) \right\} \\ =&\varepsilon^{12}(1-r)^6 \left(\frac{8\sqrt{2}\pi^2}{3} \right)^6 \left\{ \det\left(-D_P^2 \mathrm{Sc}_P\right) \det\left(D_R^2 \frac{B\tilde{A}}{\sqrt{2}\pi} \mathcal{F}(P,R) \right) + O((1-r)^2) + O(\zeta_0) \right\} \end{aligned}$$

Thus replacing r_1 and ζ_0 by larger and smaller one respectively, we observe that $\mathcal{H}$ is a Morse function on $M \times SO(3)$ and the indices of $\mathcal{H}$ and $\mathcal{G}_r(P,R)$ coincide if $r \in [r_1, 1)$ and $\mathcal{H}$ satisfies (63).

Finally, it is easily seen that $\min_{1\le i\le k} |Q - P_i|_g \to 0$ for any $Q \in M$ satisfying $D_P \mathcal{H}(Q,R) = 0$ as $\zeta_0 \to 0$.

Step 2: *One may find $r_0 \in [r_1, 1)$ and $\varepsilon_1 > 0$ such that for all $r \in [r_0, 1)$, there exists $\tilde{\varepsilon}_r > 0$ so that if $0 < \varepsilon \le \tilde{\varepsilon}_r$, then the function $W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x})$ satisfies (63) with ζ_0 replaced by $\zeta_0/2$.*

We first recall Proposition 5.2: if $r_\delta \le r < s < 1$, then

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\left\| D_P^\ell \left(W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - \frac{8\sqrt{2}\pi^2}{3}\varepsilon^2 \mathcal{G}_r(P,R)\right)\right\|_{L^\infty(M\times SO(3))} \\
&+ \frac{1}{\varepsilon^2(1-r)^2}\left\| D_R D_{P,R}^m \left(W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - \frac{8\sqrt{2}\pi^2}{3}\varepsilon^2 \mathcal{G}_r(P,R)\right)\right\|_{L^\infty(M\times SO(3))} \\
\le\,& C_0(1-r)^{-2}\varepsilon + (1-r)^{-2} o_s(1) + [C_0\delta + C_\delta\{o_r(1)+\varepsilon\}] + C_s(1-r)^{-2}(s-r)\varepsilon^2
\end{aligned} \tag{65}
$$

for $1 \le \ell \le 2$ and $0 \le m \le 1$. We first fix $\delta_0 > 0$ so that $C_0\delta_0 \le \zeta_0/8$. Next we select $r_0 \ge \max\{r_\delta, r_1\}$ so that $C_{\delta_0} o_r(1) \le \zeta_0/8$ for each $r \in [r_0,1)$. Choose s_r sufficiently close to 1 to hold $(1-r)^{-2} o_{s_r}(1) \le \zeta_0/8$. Finally, find $\tilde{\varepsilon}_r > 0$ so that $C_0(1-r)^{-2}\varepsilon + C_{\delta_0}\varepsilon + C_{s_r}(1-r)^{-2}(s_r - r)\varepsilon^2 \le \zeta_0/8$ for all $\varepsilon \le \tilde{\varepsilon}_r$. Then for $r \in [r_0,1)$, it follows from (65) that

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\left\| D_P^\ell \left(W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - \frac{8\sqrt{2}\pi^2}{3}\varepsilon^2 \mathcal{G}_r(P,R)\right)\right\|_{L^\infty(M\times SO(3))} \\
&+ \frac{1}{\varepsilon^2(1-r)^2}\left\| D_R D_{P,R}^m \left(W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x}) - \frac{8\sqrt{2}\pi^2}{3}\varepsilon^2 \mathcal{G}_r(P,R)\right)\right\|_{L^\infty(M\times SO(3))} \le \frac{\zeta_0}{2}
\end{aligned}
$$

for all $\varepsilon \in (0, \tilde{\varepsilon}_r]$. Thus Step 2 holds.

Step 3: *Let $r \in [r_0, 1)$ where r_0 is the constant appearing in Step 2. Then we have the following estimate:*

$$
\|\Phi_\varepsilon(P,R,\omega) - W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega})\|_{C^2(M\times SO(3)\times \overline{B_r(0)})} \le C_r \varepsilon^4 \tag{66}
$$

for all ε where C_r depends only on r.

Put $A_r := M \times SO(3) \times \overline{B_r}$. For $(P,R,\omega) \in A_r$, we define

$$
\begin{aligned}
&g_{0,R,\omega} := (RT_\omega)^* g_0, \quad g_{\varepsilon,P,R,\omega} := (\exp_P^{g_\varepsilon} \circ R \circ T_\omega)^* g_\varepsilon = (RT_\omega)^* g_{\varepsilon,P}, \\
&Z_{\varepsilon,P,R,\omega}(s;p) := p + s\varphi_\varepsilon(P,R,\omega;p) n_{\varepsilon,P,R,\omega}(p), \quad \mathbb{T}[s\varphi_\varepsilon(P,R,\omega)] := \{Z_{\varepsilon,P,R,\omega}(s;p) \ :\ p \in \mathbb{T}\}
\end{aligned}
$$

where $p \in \mathbb{T}$, $s \in [0,1]$ and $n_{\varepsilon,P,R,\omega}$ denotes the unit outer normal to $(\mathbb{T}, g_{\varepsilon,P,R,\omega})$. Remark that $(\Sigma_{\varepsilon,P,R,\omega}[s\varphi_\varepsilon(P,R,\omega)], g_\varepsilon)$ is isometric to $(\mathbb{T}[s\varphi_\varepsilon(P,R,\omega)], g_{\varepsilon,P,R,\omega})$. Then it follows from Proposition 2.5 that

$$
\begin{aligned}
\Phi_\varepsilon(P,R,\omega) - W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega}) &= \int_0^1 \frac{d}{ds} W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega}[s\varphi_\varepsilon(P,R,\omega)])ds \\
&= \int_0^1 \int_{\mathbb{T}} W'_{g_{\varepsilon,P,R,\omega}}(s) \psi_{\varepsilon,P,R,\omega} d\sigma ds
\end{aligned} \tag{67}
$$

where $W'_{g_{\varepsilon,P,R,\omega}}(s)$ stands for the derivative of the Willmore functional for $(\mathbb{T}[s\varphi_\varepsilon(P,R,\omega)], g_{\varepsilon,P,R,\omega})$,

$$
\begin{aligned}
\psi_{\varepsilon,P,R,\omega}(s;p) &:= g_\varepsilon(Z_{\varepsilon,P,R,\omega}(s;p))\left[\frac{d}{ds} Z_{\varepsilon,P,R,\omega}(s;p), n_{\varepsilon,P,R,\omega}(s;p)\right] \\
&= \varphi_\varepsilon(P,R,\omega;p) g_{\varepsilon,P,R,\omega}(Z_{\varepsilon,P,R,\omega}(s;p))[n_{\varepsilon,P,R,\omega}(p), n_{\varepsilon,P,R,\omega}(s;p)]
\end{aligned}
$$

and $n_{\varepsilon,P,R,\omega}(s;p)$ is the unit outer normal for $(\mathbb{T}[s\varphi_\varepsilon(P,R,\omega)], g_{\varepsilon,P,R,\omega})$. Remark that since $\varphi_\varepsilon(P,R,\omega)$ is small, $\mathbb{T}[s\varphi_\varepsilon(P,R,\omega)]$ and $\mathbb{T}$ are diffeomorphic and we pull back all geometric quantities of $\mathbb{T}[s\varphi_\varepsilon(P,R,\omega)]$ on $\mathbb{T}$.

Now by Proposition 2.7, one can easily check that

$$
\sup_{(P,R,\omega)\in A_r} \left\| D^k_{P,R,\omega} \psi_{\varepsilon,P,R,\omega}(s;\cdot)\right\|_{C^{4,\gamma}(\mathbb{T})} \le C_r \varepsilon^2 \tag{68}
$$

for $k=0,1,2$. On the other hand, $W'_{g_{0,R,\omega}}(\mathbb{T})=0$ holds for all $(R,\omega)\in SO(3)\times\overline{B_r}$ thanks to Proposition 2.1. In particular, $D^k_{R,\omega}W'_{g_{0,R,\omega}}(\mathbb{T})=0$ for $k=0,1,2$. Since (8) and (11) yield

$$\sup_{(P,R,\omega)\in A_r}\left\|D^k_{P,R,\omega}(g_{\varepsilon,P,R,\omega}-g_{0,R,\omega})\right\|_{C^\ell(\overline{B_{10}(0)})}\le C_{r,\ell}\varepsilon^2$$

for each $\ell\in\mathbb{N}$, combining the estimates of $\varphi_\varepsilon(P,R,\omega)$ in Proposition 2.7, we obtain

$$\sup_{(P,R,\omega)\in A_r,s\in[0,1]}\left\|D^k_{P,R,\omega}W'_{g_{\varepsilon,P,R,\omega}}(s)\right\|_{C^{0,\gamma}(\mathbb{T})}\le C_r\varepsilon^2 \tag{69}$$

for $k=0,1,2$. Thus by (67), (68) and (69), we have

$$\sup_{(P,R,\omega)\in A_r}\left|D^k_{P,R,\omega}\left(\Phi_\varepsilon(P,R,\omega)-W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,\omega})\right)\right|\le C_r\varepsilon^4$$

for $k=0,1,2$. Thus Step 3 holds.

Step 4: *Conclusion*

Fix $r_0\in(0,1)$ and $\tilde{\varepsilon}_r>0$ from Step 2 and let $r_0\le r<1$. By (66), we choose $\varepsilon_r\in(0,\tilde{\varepsilon}_r]$ so that

$$\frac{1}{\varepsilon^2(1-r)^2}\left\|\Phi_\varepsilon(P,R,r\mathbf{e}_x)-W_{g_\varepsilon}(\Sigma_{\varepsilon,P,R,r\mathbf{e}_x})\right\|_{C^2(M\times SO(3))}\le\frac{\zeta_0}{2}$$

for all $\varepsilon\in(0,\varepsilon_r]$. Hence, by Step 2, we observe that $\Phi_\varepsilon(P,R,r\mathbf{e}_x)$ satisfies (63) with r. Thus if $\varepsilon\in(0,\varepsilon_r]$, then $\Phi_\varepsilon(P,R,r\mathbf{e}_x)$ satisfies the general boundary condition on $\partial\mathcal{T}_{\varepsilon,K_r}$ and we complete the proof. □

Proof of Theorem 1.1. We apply the finite-dimensional reduction as described in Subsection 2.2. By Proposition 2.8 it is sufficient to find critical points of the reduced energy Φ_ε. For doing this we employ the Morse inequalities for manifolds with boundary from [31]: these relate the q-th Betti numbers (we choose here $\mathbb{Z}_2$ coefficients) of the underlying manifold to the number of critical points with index q of Morse functions satisfying the general boundary conditions. Concerning the latter critical points, one has to count those at the interior, plus the ones (still, of index q) for the function restricted to the boundary such that the gradient (which is non-zero by the general boundary conditions) is pointing inwards.

For our purpose we choose to work with the manifold $\mathcal{T}_{\varepsilon,K_r}$ (see (7) and the beginning of this section) where $r\in[r_0,1)$ and r_0 appears in Lemma 5.3, whose homology can be described as follows. By deforming r to 0 one can see that $\mathcal{T}_{\varepsilon,K_r}$ retracts to the family of (exponentiated) rotated (but not Möbius inverted) small Clifford tori centred at arbitrary points of M. By the invariances of the Clifford torus, this set is homeomorphic to the family of lines (axes of the symmetric tori) in TM passing through the base points of the tangent spaces, namely an $\mathbb{RP}^2$-bundle E over M. By Remark 1.2, since M is parallelizable, we can compute the homology of E using Künneth's formula

$$H_k(E;\mathbb{Z}_2)=\bigoplus_{i+j=k}H_i(M;\mathbb{Z}_2)\oplus H_j(\mathbb{RP}^2;\mathbb{Z}_2).$$

As $H_k(\mathbb{RP}^2,\mathbb{Z}_2)=\mathbb{Z}_2$ for $0\le k\le 2$ and zero otherwise, it follows that the Betti numbers (with $\mathbb{Z}_2$ coefficients) of $\mathcal{T}_{\varepsilon,K_r}$ are given be the $\tilde{\beta}$'s as in (3).

To prove the existence result, we set

$$\Psi_{\varepsilon,r}=\Phi_\varepsilon\big|_{\partial\mathcal{T}_{\varepsilon,K_r}},$$

and define

$$\tilde{C}_q:=\frac{1}{2}\sharp\{(P,R)\in M\times SO(3)\ :\ \nabla\Psi_{\varepsilon,r}(P,R)=0,\ \nabla\Phi_\varepsilon(P,R,r\mathbf{e}_x)\text{ is inward, index}\left(\nabla^2\Psi_{\varepsilon,r}(P,R)\right)=q\}.$$

Notice that due to Lemma 5.3, for any $r \in [r_0, 1)$ and $\varepsilon \in (0, \varepsilon_r]$, Φ_ε satisfies the general boundary condition on $\partial \mathcal{T}_{\varepsilon, K_r}$, so $\tilde{C}_q$ is well-defined. Now we claim that $\tilde{C}_q$'s in the above formula coincide with the numbers in (2) when we fix $r \in (0,1)$ sufficiently close to 1 and $\varepsilon \in (0, \varepsilon_r]$.

First we remark that when $r \in [r_0, 1)$ and $\varepsilon \in (0, \varepsilon_r]$, it follows from the proof of Lemma 5.3 and (64) that

$$\begin{aligned} & \sharp\{(P,R) \in M \times SO(3) \ : \ \nabla \Psi_{\varepsilon,r}(P,R) = 0, \ \mathrm{index}(\nabla^2 \Psi_{\varepsilon,r}(P,R)) = q\} \\ = & \sharp\{(P,R) \in M \times SO(3) \ : \ \nabla \mathcal{G}_r(P,R) = 0, \ \mathrm{index}(\nabla^2 \mathcal{G}_r(P,R)) = q\} \\ = & \sharp\{(P_i, R_{i,\ell}) \in M \times SO(3) \ : \ 1 \le i \le k, \ 1 \le \ell \le \ell_i, \ \mathrm{index}(-D_P^2 \mathrm{Sc}(P_i)) + \mathrm{index}(-D_R^2 \mathcal{F}_i(R_{i,\ell})) = q\} \end{aligned}$$

since $1 - r$ is small. Moreover, we may also observe that if $\nabla \Psi_{\varepsilon,r}(P,R) = 0$, then (P,R) must be close to $(P_i, R_{i,\ell})$, (i,ℓ) is uniquely determined and the correspondence is one-to-one. Hence, $(ND2)$ implies that $\mathcal{F}(P,R) \neq 0$ provided $\nabla \Psi_{\varepsilon,r}(P,R) = 0$. Combining this fact with (59) (or (61)) and (66), enlarging r and shrinking ε_r, if $\varepsilon \in (0, \varepsilon_r]$ and $\nabla \Psi_{\varepsilon,r}(P,R) = 0$, then $\nabla \Phi_\varepsilon(P, R, r\mathbf{e}_x)$ points inward if and only if $\mathcal{F}(P,R) < 0$. Therefore, noting Remark 5.1, our claim holds.

In order to prove the existence of critical points of Φ_ε, let us assume by contradiction that there is no critical point of Φ_ε in the interior to $\mathcal{T}_{\varepsilon, K_r}$: then the Morse inequalities in [31] $\tilde{C}_q \ge \tilde{\beta}_q$ would be violated for $q = 0, 1$, see (3). This concludes the proof.

Notice also that the index of the constructed area-constrained Willmore torus coincides with the index of the corresponding critical point of the reduced functional, since the second variation of the Willmore functional is positive definite on the orthogonal of the Kernel thanks to the work of Weiner [40]. □

Remark 5.4. *This remark contains a shorter proof of Theorem 1.1 under milder assumptions: however, in view of Theorem 1.2, it was convenient for us to use the above Morse-theoretical general framework.*

Suppose (M, g) is as in Remark 1.1 (ii). Then, by Proposition 5.2 and Lemma 5.3, for $r \in (r_0, 1)$ where r_0 from Lemma 5.3 and $\varepsilon > 0$ sufficiently small, the maximum (resp. minimum) of Φ_ε restricted to $\partial \mathcal{T}_{\varepsilon, K_r}$ is attained for some (P,R) such that P is close to a global minimum (resp. maximum) point for the scalar curvature on M. At each of these points P the Ricci tensor is not a multiple of the identity, so we may find $R_{P,+}, R_{P,-} \in SO(3)$ satisfying $\mathcal{F}(P, R_{P,-}) < 0 < \mathcal{F}(P, R_{P,+})$. By Proposition 4.2 or Remark 4.10, the inward derivative of Φ_ε is positive (resp. negative) for $R_{P,-}$ (resp. $R_{P,+}$). Therefore the global maximum (resp. minimum) of Φ_ε on the closure of $\mathcal{T}_{\varepsilon, K_r}$ is attained at the interior.

To prove Theorem 1.2 we need the following transversality result, see Theorem 1.1 in [32].

Theorem 5.5. *Let X, Y, Z be Banach spaces and let $U \subset X$, $V \subset Y$ be open subsets. Let $F : V \times U \to Z$ be a C^k map with $k \ge 1$ such that*

(i) for any $y \in V$, $F(y, \cdot) : x \mapsto F(y,x)$ is a Fredholm map of index l with $l \le k$;

(ii) the operator $F'(y_0, x_0) : Y \times X \to Z$ is onto at any point (y_0, x_0) such that $F(y_0, x_0) = z_0$;

(iii) the set of $x \in U$ such that $F(y,x) = z_0$ with y in a compact set of Y is relatively compact in U. Then the set $\{y \in V : z_0$ is a regular value of $F(y, \cdot)\}$ is a dense open subset of V.

Proof of Theorem 1.2. By the Morse inequalities in [31] it will be sufficient to show that for generic metrics the reduced functional Φ_ε is a Morse function. We will apply Theorem 5.5 with $X = \mathbb{R}^7$ being a local coordinate system for $\mathcal{T}_{\varepsilon, K_r}$, $Z = \mathbb{R}^7$, Y the set of C^2-symmetric $(2,0)$ tensors h on M and

$$F(h, x) = \nabla \Phi_{\varepsilon, g_\varepsilon + h_\varepsilon}(x),$$

where we highlighted the metric dependence in Φ_ε, and where we are scaling coordinates as in Subsection 2.2. Given any torus in $\mathcal{T}_{\varepsilon, K_r}$, one can use formula (49) (where $t = \varepsilon^2$) to compute the gradient of $W_{g_\varepsilon + h_\varepsilon}|_{\mathcal{T}_{\varepsilon, K_r}}$. Localizing the metric variation h_ε near finitely-many points of the ($1/\varepsilon$-dilated) torus, one can arbitrarily vary $\nabla W_{g_\varepsilon + h_\varepsilon}|_{\mathcal{T}_{\varepsilon, K_r}}$ by vectors of order ε^2. By (66), this property will hold true also for $\nabla \Phi_{\varepsilon, g_\varepsilon + h_\varepsilon}$, so *(ii)* in Theorem 5.5 will be satisfied. *(i)* and *(iii)* are trivially satisfied as X is finite-dimensional. □

6 Appendices

6.1 Appendix I: proof of Proposition 4.8

In this appendix we compute each term in (58). We first recall our notation here:

$$
\begin{aligned}
F &= -\sum_{i=1}^{2} e_i(h_{ni}) + \sum_{i,j=1}^{2} h_{nj}\langle \nabla^{\mathbb{R}^3}_{e_i} e_i, e_j\rangle - \frac{1}{2} h_{nn} H_{S^2_{\tilde{A}}} + \frac{1}{2}\sum_{i=1}^{2} \frac{\partial h}{\partial n_0}(e_i, e_i),\\
\mathcal{X}(\theta,\varphi) &= \tilde{A}\,(\cos\theta + 1, \sin\theta\cos\varphi, \sin\theta\sin\varphi) \qquad (\theta,\varphi)\in(0,\pi)\times[0,2\pi],\\
e_1 &= \tilde{A}^{-1}\partial_\theta \mathcal{X} = (-\sin\theta, \cos\theta\cos\varphi, \cos\theta\sin\varphi),\\
e_2 &= (\tilde{A}\sin\theta)^{-1}\partial_\varphi X = (0, -\sin\varphi, \cos\varphi),\\
n_0 &= (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi),\\
(h(x))_{\alpha\beta} &= \frac{1}{3} R_{\alpha\mu\nu\beta} x^\mu x^\nu, \quad h_{ni} = h(x)(n_0, e_i), \quad h_{nn} = h(x)(n_0, n_0),\\
\psi_0 &= A\cos\theta + B(1-\cos\theta)\cos 2\varphi, \quad (A,B) = \left(\frac{\sqrt{2}}{2}, \frac{2-\sqrt{2}}{4}\right).
\end{aligned}
$$

We remark that the Riemann curvature tensor can be expressed by the Ricci curvature and the scalar curvature as follows: (see [20]):

$$
h_{\alpha\beta}(x) := \frac{\mathrm{Sc}_P}{6}(|x|^2\delta_{\alpha\beta} - x_\alpha x_\beta) - \frac{1}{3}\delta_{\alpha\beta}\mathrm{Ric}_P(x,x) - \frac{1}{3}|x|^2 R_{\alpha\beta} + \frac{1}{3}(x_\alpha R_{\beta\mu}x^\mu + x_\beta R_{\alpha\mu}x^\mu). \tag{70}
$$

We first collect some facts which will be useful in the proof of the computations below:

Lemma 6.1. *The following hold:*

(i) $\langle \nabla^{\mathbb{R}^3}_{e_i} e_i, e_j\rangle = 0$ *except for* $(i,j) = (2,1)$ *and* $\langle \nabla^{\mathbb{R}^3}_{e_2} e_2, e_1\rangle = -\cos\theta/(\tilde{A}\sin\theta)$.

(ii) *For* $\mathcal{X}(\theta,\varphi)$,

$$
\begin{aligned}
\mathcal{X} &= \tilde{A}(n_0 + \mathbf{e}_x), \quad \langle \mathcal{X}, n_0\rangle = \tilde{A}(1 + \langle n_0, \mathbf{e}_x\rangle), \quad \langle \mathcal{X}, n_0\rangle^2 = \tilde{A}^2(1 + 2\langle n_0, \mathbf{e}_x\rangle + \langle n_0, \mathbf{e}_x\rangle^2)\\
|\mathcal{X}|^2 &= \tilde{A}^2\langle n_0 + \mathbf{e}_x, n_0 + \mathbf{e}_x\rangle = \tilde{A}^2(2 + 2\langle n_0, \mathbf{e}_x\rangle),
\end{aligned}
$$

(iii)

$$
\begin{aligned}
0 &= \int_0^\pi \cos^3\theta\sin\theta d\theta = \int_0^\pi \sin\theta\cos\theta d\theta = \int_0^\pi \cos\theta\sin^3\theta d\theta,\\
\frac{\pi}{2} &= \int_0^{2\pi} \cos^2\varphi\cos 2\varphi d\varphi = -\int_0^{2\pi}\cos 2\varphi \sin^2\varphi d\varphi,\\
0 &= \int_0^{2\pi} \cos 2\varphi\sin\varphi\cos\varphi d\varphi = \int_0^{2\pi}\cos 2\varphi\cos\varphi d\varphi = \int_0^{2\pi}\cos 2\varphi\sin\varphi d\varphi,\\
\frac{4}{3} &= \int_0^\pi \sin^3\theta d\theta, \quad \frac{2}{5} = \int_0^\pi \cos^4\theta\sin\theta d\theta, \quad \frac{4}{15} = \int_0^\pi \sin^3\theta\cos^2\theta d\theta, \quad \frac{2}{3} = \int_0^\pi \sin\theta\cos^2\theta d\theta.
\end{aligned}
$$

Proof. Noting that

$$
\nabla_{e_1}e_1 = \frac{1}{\tilde{A}}\nabla_{\partial_\theta \mathcal{X}} e_1 = \frac{1}{\tilde{A}}\partial_\theta e_1 = -\frac{n_0}{\tilde{A}}, \quad \nabla_{e_2}e_2 = \frac{1}{\tilde{A}\sin\theta}\nabla_{\partial_\varphi \mathcal{X}} e_2 = \frac{1}{\tilde{A}\sin\theta}\partial_\varphi e_2 = \frac{1}{\tilde{A}\sin\theta}\begin{pmatrix} 0 \\ -\cos\varphi \\ -\sin\varphi, \end{pmatrix}
$$

we have

$$
\langle \nabla_{e_1}e_1, e_j\rangle = 0 = \langle \nabla_{e_2}e_2, e_2\rangle, \quad \langle \nabla_{e_2}e_2, e_1\rangle = -\frac{\cos\theta}{\tilde{A}\sin\theta}.
$$

Thus (i) holds. (ii) and (iii) can be proven by direct computations. □

Next we compute the second term in the left hand side of (58).

Lemma 6.2. *The following holds:*

$$\int_{S^2_{\tilde{A}}} H_{S^2_{\tilde{A}}} \mathrm{Ric}_P(n_0, n_0)(1-\chi_\delta)\psi_0 d\sigma = \frac{4}{3}\pi\tilde{A}B(R_{22}-R_{33}) + O(\delta^2).$$

Proof. Since ψ_0 is bounded, it is enough to show

$$\int_{S^2_{\tilde{A}}} H_{S^2_{\tilde{A}}} \mathrm{Ric}_P(n_0, n_0)\psi_0 d\sigma = \frac{4}{3}\pi\tilde{A}B(R_{22}-R_{33}).$$

First we expand $\mathrm{Ric}_P(n_0, n_0)$ as follows:

$$\begin{aligned}\mathrm{Ric}_P(n_0, n_0) =& (R_{11}\cos^2\theta + R_{22}\sin^2\theta\cos^2\varphi + R_{33}\sin^2\theta\sin^2\varphi) \\ &+ (2R_{12}\sin\theta\cos\theta\cos\varphi + 2R_{23}\sin^2\theta\sin\varphi\cos\varphi + 2R_{31}\sin\theta\cos\theta\sin\varphi).\end{aligned}$$

From $H_{S^2_{\tilde{A}}} \equiv 2/\tilde{A}$ and Lemma 6.1 (iii), it follows that

$$\begin{aligned}&\int_{S^2_{\tilde{A}}} H_{S^2_{\tilde{A}}} \mathrm{Ric}_P(n_0, n_0)\psi_0 d\sigma \\ =& 2\tilde{A}B\int_0^{\pi}\int_0^{2\pi}(1-\cos\theta)\cos 2\varphi\sin\theta \mathrm{Ric}_P(n_0, n_0)d\varphi d\theta \\ =& 2\tilde{A}B\int_0^{\pi}\int_0^{2\pi}(1-\cos\theta)\sin\theta\cos 2\varphi(R_{22}\sin^2\theta\cos^2\varphi + R_{33}\sin^2\theta\sin^2\varphi)d\varphi d\theta \\ =& \pi\tilde{A}B(R_{22}-R_{33})\int_0^{\pi}(1-\cos\theta)\sin^3\theta d\theta = \frac{4}{3}\pi\tilde{A}B(R_{22}-R_{33}),\end{aligned}$$

which completes the proof. □

Next, we show

Lemma 6.3. *There holds*

$$\int_{S^2_{\tilde{A}}}(1-\chi_\delta)F\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma = 4\pi\tilde{A}B(R_{22}-R_{33}) + O(\delta^2).$$

To show Lemma 6.3, we first rewrite F as follows:

Lemma 6.4. *One has*

$$\begin{aligned}F = -\sum_{i=1}^{2} e_i(h_{ni}) + \tilde{A}\Big\{ -\frac{\mathrm{Sc}_P}{6}(1+\cos\theta) - \frac{1}{3}\mathrm{Ric}_P(\mathbf{f}_1, e_1)\cos\theta - \frac{1}{3}(1+\cos\theta)\mathrm{Ric}_P(n_0, \mathbf{e}_x) \\ + \frac{1}{3}\mathrm{Ric}_P(n_0, n_0)\cos\theta + \frac{1}{3}\mathrm{Ric}_P(\mathbf{e}_x, \mathbf{e}_x)\Big\}\end{aligned} \tag{71}$$

where

$$\mathbf{f}_1 = \big(\sin\theta,\ -(1+\cos\theta)\cos\varphi,\ -(1+\cos\theta)\sin\varphi\big).$$

Proof. First, we express $h_{nn}(\mathcal{X})$ in terms of n_0 and $\mathbf{e}_x$. By (70), notice that

$$h_{nn}(\mathcal{X}) = \frac{\mathrm{Sc}_P}{6}(|\mathcal{X}|^2 - \langle\mathcal{X}, n_0\rangle^2) - \frac{1}{3}\mathrm{Ric}_P(\mathcal{X}, \mathcal{X}) - \frac{1}{3}|\mathcal{X}|^2\mathrm{Ric}_P(n_0, n_0) + \frac{2}{3}\langle\mathcal{X}, n_0\rangle\mathrm{Ric}_P(\mathcal{X}, n_0).$$

Using Lemma 6.1, one gets

$$
\begin{aligned}
h_{nn}(\mathcal{X}) = \tilde{A}^2\Big[&\frac{\mathrm{Sc}_P}{6}(1-\langle n_0,\mathbf{e}_x\rangle^2) - \frac{1}{3}\left\{\mathrm{Ric}_P(n_0,n_0)+2\mathrm{Ric}_P(n_0,\mathbf{e}_x)+\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\right\} \\
&- \frac{2}{3}(1+\langle n_0,\mathbf{e}_x\rangle)\mathrm{Ric}_P(n_0,n_0) + \frac{2}{3}(1+\langle n_0,\mathbf{e}_x\rangle)\left\{\mathrm{Ric}_P(n_0,n_0)+\mathrm{Ric}_P(\mathbf{e}_x,n_0)\right\}\Big] \\
= \tilde{A}^2&\left\{\frac{\mathrm{Sc}_P}{6}(1-\langle n_0,\mathbf{e}_x\rangle^2) - \frac{1}{3}\mathrm{Ric}_P(n_0,n_0) + \frac{2}{3}\langle n_0,\mathbf{e}_x\rangle\mathrm{Ric}_P(n_0,\mathbf{e}_x) - \frac{1}{3}\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\right\}.
\end{aligned}
\tag{72}
$$

Next, we show

$$
\sum_{i=1}^{2}\frac{\partial h}{\partial n_0}(e_i,e_i) = -\frac{2\tilde{A}}{3}\mathrm{Ric}_P(n_0+\mathbf{e}_x,n_0). \tag{73}
$$

Since

$$
\begin{aligned}
\partial_\eta h_{\alpha\beta}(x) = &\frac{\mathrm{Sc}_P}{6}(2x_\eta\delta_{\alpha\beta}-\delta_{\alpha\eta}x_\beta-\delta_{\beta\eta}x_\alpha) - \frac{2}{3}\delta_{\alpha\beta}R_{\eta\mu}x^\mu - \frac{2}{3}x_\eta R_{\alpha\beta} \\
&+ \frac{1}{3}(\delta_{\alpha\eta}R_{\beta\mu}x^\mu + x_\alpha R_{\beta\eta} + \delta_{\beta\eta}R_{\alpha\mu}x^\mu + x_\beta R_{\alpha\eta}),
\end{aligned}
$$

we observe that

$$
\begin{aligned}
\frac{\partial h_{\alpha\beta}}{\partial n_0}(X) &= \langle\nabla h_{\alpha\beta}(\mathcal{X}),n_0\rangle \\
&= \frac{\mathrm{Sc}_P}{6}(2\langle\mathcal{X},n_0\rangle\delta_{\alpha\beta} - n_{0,\alpha}\mathcal{X}_\beta - n_{0,\beta}\mathcal{X}_\alpha) - \frac{2}{3}\delta_{\alpha\beta}\mathrm{Ric}_P(n_0,\mathcal{X}) - \frac{2}{3}\langle\mathcal{X},n_0\rangle R_{\alpha\beta} \\
&\quad + \frac{1}{3}(n_{0,\alpha}R_{\beta\mu}\mathcal{X}^\mu + \mathcal{X}_\alpha R_{\beta\eta}n_0^\eta + n_{0,\beta}R_{\alpha\mu}\mathcal{X}^\mu + \mathcal{X}_\beta R_{\alpha\eta}n_0^\eta).
\end{aligned}
$$

Thus, there holds

$$
\mathrm{tr}_{\mathbb{R}^3}\left(\frac{\partial h}{\partial n_0}\right) = \frac{2}{3}\mathrm{Sc}_P\langle\mathcal{X},n_0\rangle - 2\mathrm{Ric}_P(n_0,\mathcal{X}) - \frac{2}{3}\langle\mathcal{X},n_0\rangle\mathrm{Sc}_P + \frac{4}{3}\mathrm{Ric}_P(\mathcal{X},n_0) = -\frac{2}{3}\mathrm{Ric}_P(\mathcal{X},n_0).
$$

We also note

$$
\begin{aligned}
\frac{\partial h}{\partial n_0}(n_0,n_0) &= \frac{\mathrm{Sc}_P}{6}(2\langle\mathcal{X},n_0\rangle - 2\langle\mathcal{X},n_0\rangle) - \frac{2}{3}\mathrm{Ric}_P(\mathcal{X},n_0) - \frac{2}{3}\langle\mathcal{X},n_0\rangle\mathrm{Ric}_P(n_0,n_0) \\
&\quad + \frac{2}{3}\{\mathrm{Ric}_P(n_0,\mathcal{X}) + \langle\mathcal{X},n_0\rangle\mathrm{Ric}_P(n_0,n_0)\} \\
&= 0.
\end{aligned}
$$

Since $\{e_1,e_2,n_0\}$ forms an orthonormal basis of $\mathbb{R}^3$, we conclude

$$
\sum_{i=1}^{2}\frac{\partial h}{\partial n_0}(e_i,e_i) = \mathrm{tr}_{\mathbb{R}^3}\left(\frac{\partial h}{\partial n_0}\right) - \frac{\partial h}{\partial n_0}(n_0,n_0) = -\frac{2}{3}\mathrm{Ric}_P(\mathcal{X},n_0) = -\frac{2\tilde{A}}{3}\mathrm{Ric}_P(n_0+\mathbf{e}_x,n_0)
$$

and (73) holds.

By (70) and Lemma 6.1, we also have

(74)

$$
\begin{aligned}
h_{ni}(\mathcal{X}) &= \frac{\mathrm{Sc}_P}{6}\left(-\langle\mathcal{X},n_0\rangle\langle\mathcal{X},e_i\rangle\right) - \frac{1}{3}|\mathcal{X}|^2\mathrm{Ric}_P(n_0,e_i) + \frac{1}{3}\left\{\langle\mathcal{X},n_0\rangle\mathrm{Ric}_P(\mathcal{X},e_i) + \langle\mathcal{X},e_i\rangle\mathrm{Ric}(\mathcal{X},n_0)\right\} \\
&= \tilde{A}^2\Big[-\frac{\mathrm{Sc}_P}{6}\left(1+\langle n_0,\mathbf{e}_x\rangle\right)\langle\mathbf{e}_x,e_i\rangle - \frac{2}{3}\left(1+\langle n_0,\mathbf{e}_x\rangle\right)\mathrm{Ric}_P(n_0,e_i) \\
&\qquad + \frac{1}{3}\left(1+\langle n_0,\mathbf{e}_x\rangle\right)\mathrm{Ric}_P(n_0+\mathbf{e}_x,e_i) + \frac{1}{3}\langle\mathbf{e}_x,e_i\rangle\mathrm{Ric}_P(n_0+\mathbf{e}_x,n_0)\Big] \\
&= \tilde{A}^2\left[-\frac{\mathrm{Sc}_P}{6}\left(1+\langle n_0,\mathbf{e}_x\rangle\right)\langle\mathbf{e}_x,e_i\rangle + \frac{1}{3}\left(1+\langle n_0,\mathbf{e}_x\rangle\right)\mathrm{Ric}_P(\mathbf{e}_x-n_0,e_i) + \frac{1}{3}\langle\mathbf{e}_x,e_i\rangle\mathrm{Ric}_P(n_0+\mathbf{e}_x,n_0)\right].
\end{aligned}
$$

Now using Lemma 6.1, (72) and (73), we have

$$
\begin{aligned}
F =& -\sum_{i=1}^{2} e_i(h_{ni}) + \sum_{i,j=1}^{2} h_{nj}\langle \nabla^{\mathbb{R}^3}_{e_i} e_i, e_j\rangle - \frac{1}{2}h_{nn}H_{S^2_{\tilde{A}}} + \frac{1}{2}\sum_{i=1}^{2}\frac{\partial h}{\partial n_0}(e_i,e_i) \\
=& -\sum_{i=1}^{2} e_i(h_{ni}) + h_{n1}\langle \nabla^{\mathbb{R}^3}_{e_2} e_2, e_1\rangle - \frac{1}{2}h_{nn}H_{S^2_{\tilde{A}}} - \frac{\tilde{A}}{3}\mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0) \\
=& -\sum_{i=1}^{2} e_i(h_{ni}) - \tilde{A}\frac{\cos\theta}{\sin\theta}\Big\{ -\frac{\mathrm{Sc}_P}{6}(1+\langle n_0,\mathbf{e}_x\rangle)\langle \mathbf{e}_x, e_1\rangle + \frac{1}{3}(1+\langle n_0,\mathbf{e}_x\rangle)\mathrm{Ric}_P(\mathbf{e}_x - n_0, e_1) \\
&+ \frac{1}{3}\langle \mathbf{e}_x, e_1\rangle \mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0)\Big\} \\
&- \tilde{A}\left\{\frac{\mathrm{Sc}_P}{6}(1-\langle n_0,\mathbf{e}_x\rangle^2) - \frac{1}{3}\mathrm{Ric}_P(n_0,n_0) + \frac{2}{3}\langle n_0,\mathbf{e}_x\rangle \mathrm{Ric}_P(n_0,\mathbf{e}_x) - \frac{1}{3}\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\right\} \\
&- \frac{\tilde{A}}{3}\mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0) \\
=& -\sum_{i=1}^{2} e_i(h_{ni}) + \tilde{A}\frac{\cos\theta}{\sin\theta}\Big\{ \frac{\mathrm{Sc}_P}{6}(1+\langle n_0,\mathbf{e}_x\rangle)\langle \mathbf{e}_x, e_1\rangle - \frac{1}{3}(1+\langle n_0,\mathbf{e}_x\rangle)\mathrm{Ric}_P(\mathbf{e}_x - n_0, e_1) \\
&- \frac{1}{3}\langle \mathbf{e}_x, e_1\rangle \mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0)\Big\} \\
&- \tilde{A}\left\{\frac{\mathrm{Sc}_P}{6}(1-\langle n_0,\mathbf{e}_x\rangle^2) + \frac{1}{3}(1+2\langle n_0,\mathbf{e}_x\rangle)\mathrm{Ric}_P(n_0,\mathbf{e}_x) - \frac{1}{3}\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\right\}.
\end{aligned}
\tag{75}
$$

Noting that

$$
\langle n_0, \mathbf{e}_x\rangle = \cos\theta, \quad \langle \mathbf{e}_x, e_1\rangle = -\sin\theta,
$$

one obtains

$$
\begin{aligned}
&\frac{\cos\theta}{\sin\theta}\frac{\mathrm{Sc}_P}{6}(1+\langle n_0,\mathbf{e}_x\rangle)\langle \mathbf{e}_x, e_1\rangle - \frac{\mathrm{Sc}_P}{6}(1-\langle n_0,\mathbf{e}_x\rangle^2) \\
=&\frac{\mathrm{Sc}_P}{6}(1+\langle n_0,\mathbf{e}_x\rangle)\left\{\frac{\cos\theta}{\sin\theta}\langle \mathbf{e}_x, e_1\rangle - 1 + \langle n_0,\mathbf{e}_x\rangle\right\} = -\frac{\mathrm{Sc}_P}{6}(1+\cos\theta).
\end{aligned}
$$

Similarly, since

$$
\begin{aligned}
(1+\langle n_0,\mathbf{e}_x\rangle)(\mathbf{e}_x - n_0) &= \big(1-\cos^2\theta,\ -(1+\cos\theta)\sin\theta\cos\varphi,\ -(1+\cos\theta)\sin\theta\sin\varphi\big) \\
&= \sin\theta\big(\sin\theta,\ -(1+\cos\theta)\cos\varphi,\ -(1+\cos\theta)\sin\varphi\big) \\
&=: \mathbf{f}_1\sin\theta
\end{aligned}
$$

where

$$
\mathbf{f}_1 = \big(\sin\theta,\ -(1+\cos\theta)\cos\varphi,\ -(1+\cos\theta)\sin\varphi\big),
$$

we have

$$
-\frac{1}{3}\frac{\cos\theta}{\sin\theta}(1+\langle n_0,\mathbf{e}_x\rangle)\mathrm{Ric}_P(\mathbf{e}_x - n_0, e_1) = -\frac{1}{3}\mathrm{Ric}_P(\mathbf{f}_1, e_1)\cos\theta.
$$

Finally, from

$$
-\frac{1}{3}\frac{\cos\theta}{\sin\theta}\langle \mathbf{e}_x, e_1\rangle \mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0) = \frac{1}{3}\mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0)\cos\theta,
$$

it follows that

$$
\begin{aligned}
&-\frac{1}{3}\frac{\cos\theta}{\sin\theta}\langle \mathbf{e}_x, e_1\rangle \mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0) - \frac{1}{3}(1+2\langle n_0,\mathbf{e}_x\rangle)\mathrm{Ric}_P(n_0,\mathbf{e}_x) \\
=&\frac{1}{3}\mathrm{Ric}_P(n_0,n_0)\cos\theta - \frac{1}{3}(1+\cos\theta)\mathrm{Ric}_P(n_0,\mathbf{e}_x).
\end{aligned}
$$

Substituting these formulas into (75), we have (71). □

Now we complete the proof of Lemma 6.3.

Proof of Lemma 6.3. Due to (71), we first compute the following quantities one by one:

$$\int_{S^2_{\tilde{A}}} (1-\chi_\delta)(\Delta_{S^2_{\tilde{A}}}\psi_0)\tilde{A}\Big\{-\frac{\mathrm{Sc}_P}{6}(1+\cos\theta) - \frac{1}{3}\mathrm{Ric}_P(\mathbf{f}_1, e_1)\cos\theta - \frac{1}{3}(1+\cos\theta)\mathrm{Ric}_P(n_0, \mathbf{e}_x) + \frac{1}{3}\mathrm{Ric}_P(n_0,n_0)\cos\theta + \frac{1}{3}\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\Big\}d\sigma.$$

Here we recall that $\chi_\delta(\mathcal{X}(\theta,\varphi))$ does not depend on φ and we use this fact repeatedly in the following computations. We also recall that

$$\Delta_{S^2_{\tilde{A}}}\psi_0 = \frac{1}{\tilde{A}^2}\left[-2A\cos\theta + 2B\cos 2\varphi\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right].$$

- $\displaystyle\int_{S^2_{\tilde{A}}} \frac{1}{3}(1-\chi_\delta)\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma = O(\delta^2).$

  Since $\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x) = R_{11}$, it follows from Lemma 6.1 that

$$\begin{aligned}
&\int_{S^2_{\tilde{A}}} \frac{1}{3}(1-\chi_\delta)\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma\\
=&\frac{R_{11}}{3}\int_0^\pi (1-\chi_\delta)\int_0^{2\pi} \sin\theta\left[-2A\cos\theta + 2B\cos 2\varphi\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right]d\varphi d\theta\\
=&-\frac{4\pi R_{11}A}{3}\int_0^\pi (1-\chi_\delta)\cos\theta\sin\theta d\theta = \frac{4\pi R_{11}A}{3}\int_0^\pi \chi_\delta\cos\theta\sin\theta d\theta = O(\delta^2).
\end{aligned}$$

- $\displaystyle\int_{S^2_{\tilde{A}}} \frac{1}{3}(1-\chi_\delta)\mathrm{Ric}_P(n_0,n_0)\cos\theta\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma = -\frac{8}{15}\pi AR_{11} - \frac{8}{45}\pi A(R_{22}+R_{33}) + \frac{8}{15}\pi B(R_{22}-R_{33}) + O(\delta^2).$

  We first note that

$$\begin{aligned}
\mathrm{Ric}_P(n_0,n_0) =& (R_{11}\cos^2\theta + R_{22}\sin^2\theta\cos^2\varphi + R_{33}\sin^2\theta\sin^2\varphi)\\
&+ (2R_{12}\cos\theta\sin\theta\cos\varphi + 2R_{23}\sin^2\theta\cos\varphi\sin\varphi + 2R_{13}\cos\theta\sin\theta\sin\varphi)\\
=:& R_I + R_{II}.
\end{aligned}$$

Using Lemma 6.1, we get

$$\begin{aligned}
&\int_{S^2_{\tilde{A}}} (1-\chi_\delta)\mathrm{Ric}_P(n_0,n_0)\cos\theta\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma\\
=&\int_0^\pi (1-\chi_\delta)\int_0^{2\pi} R_I\sin\theta\cos\theta\left[-2A\cos\theta + 2B\cos 2\varphi\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right]d\varphi d\theta\\
=&-2A\int_0^\pi (1-\chi_\delta)\sin\theta\cos^2\theta\left\{2\pi R_{11}\cos^2\theta + \pi(R_{22}+R_{33})\sin^2\theta\right\}d\theta\\
&+2B\int_0^\pi (1-\chi_\delta)\sin\theta\cos\theta\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\frac{\pi}{2}(R_{22}-R_{33})\sin^2\theta d\theta\\
=&-\frac{8}{5}\pi AR_{11} - \frac{8}{15}\pi A(R_{22}+R_{33}) + \pi B(R_{22}-R_{33})\left\{\frac{4}{15}+\frac{4}{3}\right\} + O(\delta^2).
\end{aligned}$$

Hence,

$$\int_{S^2_{\tilde{A}}} \frac{1}{3}(1-\chi_\delta)\mathrm{Ric}_P(n_0,n_0)\cos\theta\Delta_{S^2_{\tilde{A}}}\psi d\sigma = -\frac{8}{15}\pi AR_{11} - \frac{8}{45}\pi A(R_{22}+R_{33}) + \frac{8}{15}\pi B(R_{22}-R_{33}) + O(\delta^2).$$

- $\displaystyle\int_{S^2_{\tilde{A}}} -\frac{1}{3}(1-\chi_\delta)(1+\cos\theta)\mathrm{Ric}_P(n_0,\mathbf{e}_x)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma = \frac{8}{9}\pi AR_{11} + O(\delta^2).$

From

$$\mathrm{Ric}_P(n_0,\mathbf{e}_x) = R_{11}\cos\theta + R_{12}\sin\theta\cos\varphi + R_{13}\sin\theta\sin\varphi,$$

it follows that

$$\begin{aligned}
&\int_{S^2_{\tilde{A}}}(1-\chi_\delta)(1+\cos\theta)\mathrm{Ric}_P(n_0,\mathbf{e}_x)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma \\
=&\int_0^\pi (1-\chi_\delta)\int_0^{2\pi}(1+\cos\theta)R_{11}\cos\theta\sin\theta(-2A\cos\theta)d\varphi d\theta \\
=&-4\pi AR_{11}\int_0^\pi (1-\chi_\delta)\cos^2\theta\sin\theta(1+\cos\theta)d\theta + O(\delta^2) = -\frac{8}{3}\pi AR_{11} + O(\delta^2).
\end{aligned}$$

- $\displaystyle\int_{S^2_{\tilde{A}}} -\frac{1}{3}(1-\chi_\delta)\mathrm{Ric}_P(\mathbf{f}_1,e_1)\cos\theta\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma = -\frac{16}{45}\pi AR_{11} - \frac{4}{15}\pi A(R_{22}+R_{33}) - \frac{14}{45}\pi B(R_{22}-R_{33}) + O(\delta^2).$

From $\mathbf{f}_1 = (\sin\theta, -(1+\cos\theta)\cos\varphi, -(1+\cos\theta)\sin\varphi)$ and

$$\begin{aligned}
\mathrm{Ric}_P(\mathbf{f}_1,e_1) =& \mathrm{Ric}_P\left(\begin{pmatrix}\sin\theta \\ -(1+\cos\theta)\cos\varphi \\ -(1+\cos\theta)\sin\varphi\end{pmatrix}, \begin{pmatrix}-\sin\theta \\ \cos\theta\cos\varphi \\ \cos\theta\sin\varphi\end{pmatrix}\right) \\
=& \left\{-R_{11}\sin^2\theta - R_{22}(1+\cos\theta)\cos\theta\cos^2\varphi - R_{33}(1+\cos\theta)\cos\theta\sin^2\varphi\right\} \\
&+ \left\{(R_{12}\cos\varphi + R_{13}\sin\varphi)(1+2\cos\theta)\sin\theta - 2R_{23}(1+\cos\theta)\cos\theta\sin\varphi\cos\varphi\right\},
\end{aligned} \tag{76}$$

one observes that

$$\begin{aligned}
&\int_{S^2_{\tilde{A}}}(1-\chi_\delta)\mathrm{Ric}_P(\mathbf{f}_1,e_1)\cos\theta\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma \\
=&\int_0^\pi(1-\chi_\delta)\int_0^{2\pi}\left\{-R_{11}\sin^2\theta - R_{22}(1+\cos\theta)\cos\theta\cos^2\varphi - R_{33}(1+\cos\theta)\cos\theta\sin^2\varphi\right\} \\
&\qquad\times\cos\theta\sin\theta\left[-2A\cos\theta + 2B\cos 2\varphi\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right]d\varphi d\theta \\
=&2\pi A\int_0^\pi(1-\chi_\delta)\left\{2R_{11}\sin^2\theta + (R_{22}+R_{33})(1+\cos\theta)\cos\theta\right\}\cos^2\theta\sin\theta d\theta \\
&+2B\int_0^\pi(1-\chi_\delta)\frac{\pi}{2}(-R_{22}+R_{33})(1+\cos\theta)\cos^2\theta\sin\theta\left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}d\theta \\
=&2\pi A\left\{\frac{8}{15}R_{11} + \frac{2}{5}(R_{22}+R_{33})\right\} \\
&+\pi B(-R_{22}+R_{33})\int_0^\pi(1-\chi_\delta)\cos^2\theta\sin\theta\left\{(1+\cos\theta)\cos\theta - 2\right\}d\theta + O(\delta^2) \\
=&\frac{16}{15}\pi AR_{11} + \frac{4}{5}\pi A(R_{22}+R_{33}) + \pi B(-R_{22}+R_{33})\left(\frac{2}{5} - \frac{4}{3}\right) + O(\delta^2) \\
=&\frac{16}{15}\pi AR_{11} + \frac{4}{5}\pi A(R_{22}+R_{33}) + \frac{14}{15}\pi B(R_{22}-R_{33}) + O(\delta^2).
\end{aligned}$$

- $\displaystyle\int_{S^2_{\tilde{A}}} -(1-\chi_\delta)\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma = \frac{4}{9}\pi A\mathrm{Sc}_P + O(\delta^2).$

By
$$\tilde{A}^2(1+\cos\theta)\Delta_{S^2_{\tilde{A}}}\psi_0 = \left[-2A(1+\cos\theta)\cos\theta + 2B\cos 2\varphi\left\{(1+\cos\theta)\cos\theta - 2\right\}\right],$$
we obtain
$$\begin{aligned}
&\int_{S^2_{\tilde{A}}}(1-\chi_\delta)(1+\cos\theta)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma\\
=&\int_0^\pi (1-\chi_\delta)\int_0^{2\pi}\left[-2A(1+\cos\theta)\cos\theta + 2B\cos 2\varphi\left\{(1+\cos\theta)\cos\theta - 2\right\}\right]\sin\theta d\varphi d\theta\\
=&-4\pi A\int_0^\pi (1-\chi_\delta)(1+\cos\theta)\cos\theta\sin\theta d\theta = -\frac{8}{3}\pi A + O(\delta^2).
\end{aligned}$$

Collecting the above results, we have
$$\begin{aligned}
&\int_{S^2_{\tilde{A}}}(1-\chi_\delta)(\Delta_{S^2_{\tilde{A}}}\psi_0)\tilde{A}\bigg\{-\frac{\mathrm{Sc}_P}{6}(1+\cos\theta) - \frac{1}{3}\mathrm{Ric}_P(\mathbf{f}_1, e_1)\cos\theta - \frac{1}{3}(1+\cos\theta)\mathrm{Ric}_P(n_0,\mathbf{e}_x)\\
&\qquad + \frac{1}{3}\mathrm{Ric}_P(n_0,n_0)\cos\theta + \frac{1}{3}\mathrm{Ric}_P(\mathbf{e}_x,\mathbf{e}_x)\bigg\}d\sigma\\
=&\tilde{A}\left\{\frac{4}{9}\pi A\mathrm{Sc}_P - \frac{4}{9}\pi A(R_{22}+R_{33}) + \frac{2}{9}\pi B(R_{22}-R_{33})\right\} + O(\delta^2).
\end{aligned}\tag{77}$$

Next, we compute $-\sum_{i=1}^2\int_{S^2_{\tilde{A}}} e_i(h_{ni})(1-\chi_\delta)\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma$. We recall the following expressions of h_{ni} from (74):
$$h_{ni}(X) = \tilde{A}^2\left\{-\frac{\mathrm{Sc}_P}{6}(1+\langle n_0,\mathbf{e}_x\rangle)\langle\mathbf{e}_x, e_i\rangle + \frac{1}{3}(1+\langle n_0,\mathbf{e}_x\rangle)\mathrm{Ric}_P(\mathbf{e}_x - n_0, e_i) + \frac{1}{3}\langle\mathbf{e}_x, e_i\rangle\mathrm{Ric}_P(n_0+\mathbf{e}_x, n_0)\right\}.$$

- $\displaystyle\int_{S^2_{\tilde{A}}} -(1-\chi_\delta)e_2(h_{n2})\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma = \frac{20}{9}\pi\tilde{A}B(R_{22}-R_{33}) + O(\delta^2).$

By $\langle\mathbf{e}_x, e_2\rangle = 0$ and $(1+\langle n_0,\mathbf{e}_x\rangle)(\mathbf{e}_x - n_0) = \mathbf{f}_1\sin\theta$, we have
$$h_{n2} = \frac{\tilde{A}^2}{3}\mathrm{Ric}_P(\mathbf{f}_1, e_2)\sin\theta.$$
By simple calculations, one may see
$$\begin{aligned}
\mathrm{Ric}_P(\mathbf{f}_1, e_2) =&\frac{1}{2}(R_{22}-R_{33})(1+\cos\theta)\sin 2\varphi - R_{23}(1+\cos\theta)\cos 2\varphi - R_{12}\sin\theta\sin\varphi + R_{13}\sin\theta\cos\varphi,\\
\frac{\partial}{\partial\varphi}\mathrm{Ric}_P(\mathbf{f}_1, e_2) =&(R_{22}-R_{33})(1+\cos\theta)\cos 2\varphi + 2R_{23}(1+\cos\theta)\sin 2\varphi - R_{12}\sin\theta\cos\varphi - R_{13}\sin\theta\sin\varphi.
\end{aligned}$$

Since $e_2(f) = (\tilde{A}\sin\theta)^{-1}\partial_\varphi f$, we get

$$\begin{aligned}
&\int_{S^2_{\tilde{A}}} (1-\chi_\delta) e_2(h_{n2}) \Delta_{S^2_{\tilde{A}}} \psi_0 d\sigma \\
=&\int_0^\pi (1-\chi_\delta) \int_0^{2\pi} \frac{\tilde{A}^2}{3} \frac{1}{\tilde{A}\sin\theta} \frac{\partial}{\partial\varphi} \left(\mathrm{Ric}_P(\mathbf{f}_1, e_2)\sin\theta\right) \\
&\qquad \times \left[-2A\cos\theta + 2B\cos 2\varphi \left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right] \sin\theta d\varphi d\theta \\
=&\frac{2}{3}\tilde{A}B \int_0^\pi (1-\chi_\delta) \int_0^{2\pi} (R_{22}-R_{33})(1+\cos\theta)\sin\theta\cos^2 2\varphi \left\{\cos\theta - \frac{2(1-\cos\theta)}{\sin^2\theta}\right\} d\varphi d\theta \\
=&\frac{2}{3}\pi\tilde{A}B(R_{22}-R_{33}) \int_0^\pi (1-\chi_\delta)\sin\theta \left\{\cos\theta(1+\cos\theta) - 2\right\} d\theta \\
=&\frac{2}{3}\pi\tilde{A}B(R_{22}-R_{33}) \left(\frac{2}{3} - 4\right) + O(\delta^2) = -\frac{20}{9}\pi\tilde{A}B(R_{22}-R_{33}) + O(\delta^2).
\end{aligned}$$

- $\displaystyle\int_{S^2_{\tilde{A}}} -(1-\chi_\delta) e_1(h_{n1}) \Delta_{S^2_{\tilde{A}}} \psi_0 d\sigma = \tilde{A}\left\{-\frac{4}{9}\pi A R_{11} + \frac{14}{9}\pi B(R_{22}-R_{33})\right\} + O(\delta^2).$

First, we notice that

$$h_{n1} = \tilde{A}^2 \left\{\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta + \frac{1}{3}\mathrm{Ric}_P(\mathbf{f}_1, e_1)\sin\theta - \frac{1}{3}\mathrm{Ric}_P(n_0 + \mathbf{e}_x, n_0)\sin\theta\right\}$$

and that by (76),

$$\begin{aligned}
\mathrm{Ric}_P(\mathbf{f}_1, e_1)\sin\theta =& -\left\{R_{11}\sin^2\theta + (R_{22}\cos^2\varphi + R_{33}\sin^2\varphi)(1+\cos\theta)\cos\theta\right\}\sin\theta \\
&+ R_1(\theta,\varphi) =: R_{\mathbf{f}_1 e_1} + R_1(\theta,\varphi), \\
\mathrm{Ric}_P(n_0 + \mathbf{e}_x, n_0)\sin\theta =& \left\{R_{11}\cos\theta(1+\cos\theta) + R_{22}\sin^2\theta\cos^2\varphi + R_{33}\sin^2\theta\sin^2\varphi\right\}\sin\theta \\
&+ R_2(\theta,\varphi) =: R_{n\mathbf{e}_x}(\theta,\varphi) + R_2(\theta,\varphi)
\end{aligned}$$

where

$$0 = \int_0^{2\pi} \partial_\theta R_1(\theta,\varphi) d\varphi = \int_0^{2\pi} \partial_\theta R_1(\theta,\varphi)\cos 2\varphi d\varphi = \int_0^{2\pi} \partial_\theta R_2(\theta,\varphi) d\varphi = \int_0^{2\pi} \partial_\theta R_2(\theta,\varphi)\cos 2\varphi d\varphi.$$

We also remark that in a neighbourhood of $\theta = -\pi$, one sees that

$$\left|\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta\right| + |R_{\mathbf{f}_1 e_1}(\theta,\varphi)| + |R_{n\mathbf{e}_x}(\theta,\varphi)| \le C_0 \sin^3\theta. \tag{78}$$

Thus since $e_1(f) = \tilde{A}^{-1}\partial_\theta f$ and $|\partial_\theta(\chi_\delta(\mathcal{X}))| \le C_0\delta^{-1}$ by (57), it follows from (78) and integration by parts in θ that

$$
\begin{aligned}
&\int_{S^2_{\tilde{A}}}(1-\chi_\delta)e_1(h_{n1})\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma\\
=&\tilde{A}\int_0^\pi\int_0^{2\pi}(1-\chi_\delta)\partial_\theta\left\{\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta+\frac{1}{3}\left(R_{\mathbf{f}_1e_1}+R_1\right)-\frac{1}{3}\left(R_{n\mathbf{e}_x}+R_2\right)\right\}\\
&\qquad\times\left[-2A\cos\theta+2B\cos 2\varphi\left\{\cos\theta-\frac{2(1-\cos\theta)}{\sin^2\theta}\right\}\right]\sin\theta d\varphi d\theta\\
=&\tilde{A}\int_0^\pi\int_0^{2\pi}(1-\chi_\delta)\partial_\theta\left\{\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta+\frac{1}{3}R_{\mathbf{f}_1e_1}-\frac{1}{3}R_{n\mathbf{e}_x}\right\}\\
&\qquad\times\left[-A\sin 2\theta+2B\cos 2\varphi\left\{\frac{\sin 2\theta}{2}-\frac{2(1-\cos\theta)}{\sin\theta}\right\}\right]d\varphi d\theta\\
=&-\tilde{A}\int_0^\pi\int_0^{2\pi}(1-\chi_\delta)\left\{\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta+\frac{1}{3}R_{\mathbf{f}_1e_1}-\frac{1}{3}R_{n\mathbf{e}_x}\right\}\\
&\qquad\times\left[-2A\cos 2\theta+2B\cos 2\varphi\left\{\cos 2\theta-2+2\frac{(1-\cos\theta)\cos\theta}{\sin^2\theta}\right\}\right]d\varphi d\theta+O(\delta^2).
\end{aligned}
$$

Next, from

$$
\begin{aligned}
\int_0^{2\pi}R_{\mathbf{f}_1e_1}d\varphi&=-\pi\left\{2R_{11}\sin^2\theta+(R_{22}+R_{33})(1+\cos\theta)\cos\theta\right\}\sin\theta,\\
\int_0^{2\pi}R_{n\mathbf{e}_x}d\varphi&=\pi\left\{2R_{11}(1+\cos\theta)\cos\theta+(R_{22}+R_{33})\sin^2\theta\right\}\sin\theta,\\
\int_0^{2\pi}R_{\mathbf{f}_1e_1}\cos 2\varphi d\varphi&=-\frac{\pi}{2}(R_{22}-R_{33})(1+\cos\theta)\cos\theta\sin\theta,\\
\int_0^{2\pi}R_{n\mathbf{e}_x}\cos 2\varphi d\varphi&=\frac{\pi}{2}(R_{22}-R_{33})\sin^3\theta,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\int_0^{2\pi}(R_{\mathbf{f}_1e_1}-R_{n\mathbf{e}_x})d\varphi&=-\pi\left\{2R_{11}(1+\cos\theta)+(R_{22}+R_{33})(1+\cos\theta)\right\}\sin\theta\\
&=-\pi\left\{\mathrm{Sc}_P+R_{11}\right\}(1+\cos\theta)\sin\theta,\\
\int_0^{2\pi}(R_{\mathbf{f}_1e_1}-R_{n\mathbf{e}_x})\cos 2\varphi d\varphi&=-\frac{\pi}{2}(R_{22}-R_{33})(1+\cos\theta)\sin\theta.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\int_0^{2\pi}\left\{\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta+\frac{1}{3}(R_{\mathbf{f}_1e_1}-R_{n\mathbf{e}_x})\right\}d\varphi&=\frac{\pi}{3}\left\{\mathrm{Sc}_P-(\mathrm{Sc}_P+R_{11})\right\}(1+\cos\theta)\sin\theta\\
&=-\frac{\pi}{3}R_{11}(1+\cos\theta)\sin\theta,\\
\int_0^{2\pi}\left\{\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta+\frac{1}{3}(R_{\mathbf{f}_1e_1}-R_{n\mathbf{e}_x})\right\}\cos 2\varphi d\varphi&=-\frac{\pi}{6}(R_{22}-R_{33})(1+\cos\theta)\sin\theta.
\end{aligned}
$$

Thus

$$\begin{aligned}
& -\tilde{A}\int_0^{\pi}\int_0^{2\pi}(1-\chi_\delta)\left\{\frac{\mathrm{Sc}_P}{6}(1+\cos\theta)\sin\theta+\frac{1}{3}(R_{\mathbf{f}_1 e_1}-R_{\boldsymbol{n}\mathbf{e}_x})\right\} \\
& \qquad\qquad \times\left[-2A\cos 2\theta+2B\cos 2\varphi\left\{\cos 2\theta-2+2\frac{(1-\cos\theta)\cos\theta}{\sin^2\theta}\right\}\right]d\varphi d\theta \\
= & -\frac{2}{3}\pi\tilde{A}AR_{11}\int_0^{\pi}(1-\chi_\delta)(1+\cos\theta)\sin\theta\cos 2\theta d\theta \\
& +\frac{\pi}{3}\tilde{A}B(R_{22}-R_{33})\int_0^{\pi}(1-\chi_\delta)(1+\cos\theta)\sin\theta\left\{\cos 2\theta-2+2\frac{(1-\cos\theta)\cos\theta}{\sin^2\theta}\right\}d\theta \\
= & -\frac{2}{3}\pi\tilde{A}AR_{11}\int_0^{\pi}(1-\chi_\delta)(1+\cos\theta)(2\cos^2\theta-1)\sin\theta d\theta \\
& +\frac{\pi}{3}\tilde{A}B(R_{22}-R_{33})\int_0^{\pi}(1-\chi_\delta)\sin\theta\left\{(2\cos^2\theta-1)(1+\cos\theta)-2(1+\cos\theta)+2\cos\theta\right\}d\theta \\
= & -\frac{2}{3}\pi\tilde{A}R_{11}A\int_0^{\pi}(2\cos^2\theta-1)\sin\theta d\theta+\frac{\pi}{3}\tilde{A}B(R_{22}-R_{33})\int_0^{\pi}\sin\theta(2\cos^2\theta-3)d\theta+O(\delta^2) \\
= & -\frac{2}{3}\pi\tilde{A}AR_{11}\left(\frac{4}{3}-2\right)+\frac{\pi}{3}\tilde{A}B(R_{22}-R_{33})\left(\frac{4}{3}-6\right)+O(\delta^2) \\
= & \frac{4}{9}\pi\tilde{A}AR_{11}-\frac{14}{9}\pi\tilde{A}B(R_{22}-R_{33})+O(\delta^2).
\end{aligned}$$

Thus we get

$$-\int_{S^2_{\tilde{A}}}(1-\chi_\delta)e_1(h_{n1})\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma=\tilde{A}\left\{-\frac{4}{9}\pi AR_{11}+\frac{14}{9}\pi B(R_{22}-R_{33})\right\}+O(\delta^2).$$

Combining (77), we have

$$\int_{S^2_{\tilde{A}}}(1-\chi_\delta)F\Delta_{S^2_{\tilde{A}}}\psi_0 d\sigma=4\pi\tilde{A}B(R_{22}-R_{33})+O(\delta^2)$$

and we complete the proof. □

Proof of Proposition 4.8. From Lemmas 6.2 and 6.3, we have

$$\begin{aligned}
\int_{S^2_{\tilde{A}}}(1-\chi_\delta)\left\{F\Delta_{S^2_{\tilde{A}}}\psi_0+H_{S^2_{\tilde{A}}}\mathrm{Ric}_P(n_0,n_0)\psi_0\right\}d\sigma &= 4\pi\tilde{A}B(R_{22}-R_{33})+\frac{4}{3}\pi\tilde{A}B(R_{22}-R_{33})+O(\delta^2) \\
&= \frac{16}{3}\pi\tilde{A}B(R_{22}-R_{33})+O(\delta^2),
\end{aligned}$$

which gives (58). Thus we completed the proof. □

6.2 Appendix II: study of $\mathcal{F}(P,R)$

In this appendix we prove the following result, which guarantees condition $(ND2)$ in the introduction.

Proposition 6.5. *Let S be a symmetric bilinear form on $\mathbb{R}^3$, and denote by $\alpha_1,\alpha_2,\alpha_3$ the eigenvalues corresponding to the eigenvectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. Consider the function $F: SO(3)\to\mathbb{R}$ defined by*

$$F(R):=S(R\mathbf{e}_2,R\mathbf{e}_2)-S(R\mathbf{e}_3,R\mathbf{e}_3). \tag{79}$$

Then F is a Morse function if and only if the eigenvalues are distinct: $\alpha_i\neq\alpha_j$ for $i\neq j$.
In this case F has exactly 24 critical points $\{R_{(ij)}\}$ satisfying $R\mathbf{e}_2=\pm\mathbf{e}_i$, $R\mathbf{e}_3=\pm\mathbf{e}_j$ for $i,j=1,2,3$ with $i\neq j$ and the eigenvalues of the Hessian $\nabla^2F(R_{(ij)})$ are $\alpha_k-\alpha_i, 2(\alpha_j-\alpha_i), \alpha_j-\alpha_k$, where $\{k\}=\{1,2,3\}\setminus\{i,j\}$, and $F(R_{(ij)})=\alpha_i-\alpha_j$. In particular

- *F has exactly 4 critical points of index 3 given by $\{R_{(ij)}\}$ with $i = \pm 3$ and $j = \pm 1$. They all satisfy $F(R_{(ij)}) > 0$.*
- *F has exactly 8 critical points of index 2 given by $\{R_{(ij)}\}$ with $i = \pm 3$ and $j = \pm 2$, or $i = \pm 2$ and $j = \pm 1$. They all satisfy $F(R_{(ij)}) > 0$.*
- *F has exactly 8 critical points of index 1 given by $\{R_{(ij)}\}$ with $i = \pm 2$ and $j = \pm 3$, or $i = \pm 1$ and $j = \pm 2$. They all satisfy $F(R_{(ij)}) < 0$.*
- *F has exactly 4 critical points of index 0 given by $\{R_{(ij)}\}$ with $i = \pm 1$ and $j = \pm 3$. They all satisfy $F(R_{(ij)}) < 0$.*

Proof. First of all let us show that if S has multiple eigenvalues then F cannot be Morse. Up to relabelling we can assume $\alpha_1 = \alpha_2$. Let $\bar{R} \in SO(3)$ be the rotation such that $\bar{R}\mathbf{e}_2 = \mathbf{e}_1$ and $\bar{R}\mathbf{e}_3 = \mathbf{e}_2$; then for every rotation R_θ, $\theta \in S^1$, with axis $\mathbf{e}_3$ (i.e. a rotation of the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$) we have

$$F(R_\theta \circ \bar{R}) = S(R_\theta \circ \bar{R}\mathbf{e}_2) - S(R_\theta \circ \bar{R}\mathbf{e}_3) = S(R_\theta \mathbf{e}_1) - S(R_\theta \mathbf{e}_2) = \alpha_1 - \alpha_1 = 0.$$

Since F is constant on a one-dimensional submanifold it cannot be Morse.

From now on we therefore assume $\alpha_i \neq \alpha_j$ for $i \neq j$. Throughout the proof all the vectors and matrices of $\mathbb{R}^3$ will be expressed in coordinates with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of eigenvectors of S. Notice that a rotation $R \in SO(3)$ is uniquely determined by the coordinates (x_1, x_2, x_3) of $R\mathbf{e}_2$ and by the coordinates (x_4, x_5, x_6) of $R\mathbf{e}_3$, note also that such coordinates satisfy the following non degenerate system of three constraints:

$$\left\{ \sum_{i=1}^{3} x_i^2 = 1,\ \sum_{j=4}^{6} x_j^2 = 1,\ \sum_{i=1}^{3} x_i x_{i+3} = 0 \right\}. \tag{80}$$

Therefore finding a critical point of $F : SO(3) \to \mathbb{R}$ is equivalent to finding critical points of the corresponding function defined on $\mathbb{R}^6$ under the constraints (80) which, by the Lagrange multipliers rule, is in turn equivalent to look for free critical points, in $x \in \mathbb{R}^6$, of the Lagrange function

$$L(x_1, \dots, x_6, \lambda, \mu, \nu) := \sum_{i=1}^{3} \alpha_i (x_i^2 - x_{i+3}^2) - \lambda \sum_{i=1}^{3} x_i^2 - \mu \sum_{i=4}^{6} x_i^2 - \nu \sum_{i=1}^{3} x_i x_{i+3}. \tag{81}$$

This corresponds to solving the following system of nine equations in $(x_1, \dots, x_6, \lambda, \mu, \nu)$. Notice that the first six equations are linear and correspond to the optimization of L in x, the last three equations are quadratic and correspond to the constraints (80):

$$\begin{cases} 2(\alpha_i - \lambda)x_i - \nu x_{i+3} = 0, \quad i = 1,2,3 \\ \nu x_i + 2(\mu + \alpha_i)x_{i+3} = 0 \quad i = 1,2,3 \\ \sum_{i=1}^{3} x_i^2 = 1,\ \sum_{j=4}^{6} x_j^2 = 1,\ \sum_{i=1}^{3} x_i x_{i+3} = 0. \end{cases} \tag{82}$$

As the first step, we show $\nu = 0$. Let $x = (x_1, \dots, x_6)$ satisfy (82). Then it follows from (82) that

$$\nu = \sum_{i=1}^{3} \nu x_{i+3}^2 = \sum_{i=1}^{3} 2(\alpha_i - \lambda) x_i x_{i+3} = 2 \sum_{i=1}^{3} \alpha_i x_i x_{i+3},$$
$$\nu = \sum_{i=1}^{3} \nu x_i^2 = \sum_{i=1} (-2)(\mu + \alpha_i) x_i x_{i+3} = -2 \sum_{i=1}^{3} \alpha_i x_i x_{i+3}.$$

Summing these two equations, we obtain $\nu = 0$.

Since we know $\nu = 0$, by the assumptions $\alpha_i \neq \alpha_j$ for $i \neq j$, it is immediate to check that the solutions of (82) are given by

$$\{(x_1, \dots, x_6) \ : \ x_i = \pm 1, x_j = \pm 1, x_k = 0, \lambda = \alpha_i, \mu = -\alpha_j, \nu = 0\}, \tag{83}$$

for exactly one $i \in \{1,2,3\}$, one $j \in \{4,5,6\}$ with $j-3 \neq i$ and for all $k \in \{1,\cdots,6\} \setminus \{i,j\}$. Notice that these 24 solutions correspond to the rotations $R_{(ij)} \in SO(3)$ described in the statement of the proposition.

In order to know the index of these 24 critical points, observe that it is enough to perform a second order analysis at $R = Id = R_{(23)} \in SO(3)$: indeed, the index of F at $R_{(ij)}$ is the same as $F \circ R_{(ij)}$ at Id, so the general case just follows by a suitable relabelling of the indices.

By using (83), at the critical point $R = R_{(23)} = Id$ the Lagrange function (81) takes the form

$$L(x, \lambda = \alpha_2, \mu = -\alpha_3, \nu = 0) = (\alpha_1 - \alpha_2)x_1^2 + (\alpha_3 - \alpha_2)x_3^2 + (\alpha_3 - \alpha_1)x_4^2 + (\alpha_3 - \alpha_2)x_5^2.$$

Since $v \in \mathbb{R}^6$ is tangent to the constraints (80) at $\bar{x} = (0,1,0,0,0,1)$ if and only if it has the form $v = (v_1, 0, v_3, v_4, -v_3, 0)$, the Hessian in x of L on the tangent space to the constraint manifold at $\bar{x}$ is

$$\nabla_x^2 L(\bar{x}, \lambda = \alpha_2, \mu = -\alpha_3, \nu = 0)[v] = (\alpha_1 - \alpha_2)v_1^2 + 2(\alpha_3 - \alpha_2)v_3^2 + (\alpha_3 - \alpha_1)v_4^2.$$

But such a constrained Hessian corresponds to the Hessian of F at $R_{(23)} = Id$: $\nabla^2 F(R_{(23)})$. It follows that $\nabla^2 F(R_{(23)})$ is non degenerate if and only if $\alpha_i \neq \alpha_j$ for $i \neq j$, that the eigenvalues of $\nabla^2 F(R_{(23)})$ are $\alpha_1 - \alpha_2, 2(\alpha_3 - \alpha_2), \alpha_3 - \alpha_1$. Moreover, assuming $\alpha_1 < \alpha_2 < \alpha_3$, the index of $\nabla^2 F(R_{(23)})$ is one and $F(R_{(23)}) = \alpha_2 - \alpha_3 < 0$. As mentioned above, the second order analysis of F at the general critical point $R_{(ij)}$ follows then by a relabelling argument. □

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