An L^1-type estimate for Riesz potentials

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Abstract

In this paper, we establish new L^1-type estimates for the classical Riesz potentials of order $1/2 < \alpha < 1$. One can alternatively view this as a sharpening of a result of Stein and Weiss on the mapping properties of Riesz potentials on the real Hardy space $\mathcal{H}^1(\mathbb{R}^N)$ or a new family of L^1-Sobolev inequalities for the Riesz fractional gradient.

1 Introduction and Main Results

Let $N \geq 2$ and define the Riesz potential of order $0 < \alpha < N$ by its action on a measurable function u by convolution, i.e.

$$I_\alpha u(x) \equiv (I_\alpha * u)(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-\alpha}}\, dy,$$

whenever it is well-defined. Here, $\gamma(\alpha)$ is a normalization constant [14, p. 117] that ensures the Riesz potentials satisfy the semigroup property

$$I_{\alpha+\beta} u = I_\alpha I_\beta u, \text{ for } \alpha, \beta > 0,\ \alpha + \beta < N,$$

for u in a suitable class of functions.

The study of the mapping properties of I_α on $L^p(\mathbb{R}^N)$ was initiated by Sobolev, who proved the following *fundamental theorem about integrals of the potential type* in 1938 [11, p. 50].

Theorem 1.1 (Sobolev) *Let $0 < \alpha < N$ and $1 < p < N/\alpha$. There exists a constant $C = C(p, \alpha, N) > 0$ such that*

$$\|I_\alpha u\|_{L^{Np/(N-\alpha p)}(\mathbb{R}^N)} \leq C \|u\|_{L^p(\mathbb{R}^N)} \tag{1.1}$$

for all $u \in L^p(\mathbb{R}^N)$.

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In particular, we see that Sobolev's result concerns L^p estimates for Riesz potentials when $1 < p < \frac{N}{\alpha}$, and strictly excludes the case $p = 1$. Indeed, it is well-known that no such inequality as (1.1) can hold in this regime. One may consider, for example, $u = \chi_{B(0,1)}$. Then for $|x|$ large we have that

$$\begin{aligned} I_\alpha u(x) &= \frac{1}{\gamma(\alpha)} \int_{B(0,1)} \frac{1}{|x-y|^{N-\alpha}} \, dy \\ &\geq \frac{c}{|x|^{N-\alpha}}, \end{aligned}$$

which shows that the decay at infinity is insufficient for $I_\alpha u$ to belong to $L^{\frac{N}{N-\alpha}}(\mathbb{R}^N)$.

It is natural then to ask if there is a substitute for the inequality (1.1). One such substitute was given by Stein and Weiss [15, p. 31], where they demonstrated that if one replaces $L^p(\mathbb{R}^N)$ with the real Hardy space

$$\mathcal{H}^p(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : Ru \in L^p(\mathbb{R}^N; \mathbb{R}^N)\}$$

(where $Ru := DI_1 u$ is the vector-valued Riesz transform), one can extend the validity of Theorem 1.1 to the regime $p = 1$. For $p \in (1, \infty)$, $\mathcal{H}^p(\mathbb{R}^N) = L^p(\mathbb{R}^N)$, but for $p = 1$ the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$ is strictly smaller than $L^1(\mathbb{R}^N)$. Their result implies the following theorem of interest to our considerations.

Theorem 1.2 (Stein-Weiss) *Let $0 < \alpha < N$ and $1 \leq p < \frac{N}{\alpha}$. There exists a constant $C = C(p, \alpha, N) > 0$ such that*

$$\|I_\alpha u\|_{L^{Np/(N-\alpha p)}(\mathbb{R}^N)} \leq C \left(\|u\|_{L^p(\mathbb{R}^N)} + \|Ru\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \right)$$

for all $u \in \mathcal{H}^p(\mathbb{R}^N)$.

Actually, the approach to Sobolev inequalities due to Gagliardo [5, p. 120] and Nirenberg [8, p. 128] gives another replacement to Theorem 1.1 for $1 \leq \alpha < N$. Indeed, written in the language of potentials, one sees that the results [5, 8] assert the existence of a constant $C > 0$ such that

$$\|I_1 u\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leq C \|Ru\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)},$$

for all $u \in C_c^\infty(\mathbb{R}^N)$ such that $Ru \in L^1(\mathbb{R}^N; \mathbb{R}^N)$. Therefore, if $1 \leq \alpha < N$, the preceding inequality and Theorem 1.1 applied to $I_\alpha u = I_{\alpha-1} I_1 u$ allows us to deduce that

$$\begin{aligned} \|I_\alpha u\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} &\leq C \|I_1 u\|_{L^{N/(N-1)}(\mathbb{R}^N)} \\ &\leq C \|Ru\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)}, \end{aligned}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$ such that $Ru \in L^1(\mathbb{R}^N; \mathbb{R}^N)$.

Thus, although it is not possible to obtain the L^1 estimate

$$\|I_\alpha u\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} \leq C \|u\|_{L^1(\mathbb{R}^N)},$$

for all $0 < \alpha < N$, Stein and Weiss showed that one does have

$$\|I_\alpha u\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} \leq C(\|u\|_{L^1(\mathbb{R}^N)} + \|Ru\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)}).$$

However, when $1 \leq \alpha < N$, it is a consequence of Gagliardo and Nirenberg's work that one has the stronger L^1-type estimate

$$\|I_\alpha u\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} \leq C\|Ru\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)}.$$

The main result of this paper is the following theorem establishing new L^1-type estimates for the Riesz potentials for $1/2 < \alpha < 1$.

Theorem A *Let $N \geq 2$ and $1/2 < \alpha < N$. There exists a constant $C = C(\alpha, N) > 0$ such that*

$$\|I_\alpha u\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} \leq C\|Ru\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$ such that $Ru \in L^1(\mathbb{R}^N;\mathbb{R}^N)$.

Remark 1.3 *Theorem A is false when $N = 1$, which can be seen by taking Ru to approximate a Dirac mass (and in this setting $Ru = Hu$, the Hilbert transform). Then $I_\alpha u \approx c\frac{x}{|x|^{2-\alpha}}$, and therefore one sees that the integral on the left hand side diverges.*

Our motivation for such an inequality can be found in the study of certain fractional partial differential equations introduced in [10], where existence results are demonstrated for a continuous spectrum of such equations parameterized by the fractional gradient

$$D^\alpha u := DI_{1-\alpha}u,$$

when $0 < \alpha < 1$. With this notation, an alternative formulation of Theorem A is the following.

Theorem A′ *Let $N \geq 2$ and $1/2 < \alpha < 1$. There exists a constant $C = C(\alpha, N) > 0$ such that*

$$\|u\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} \leq C\|D^\alpha u\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \tag{1.2}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$.

Although in general the weak-type mapping properties of I_α in the Lorentz space $L^{N/(N-\alpha),\infty}(\mathbb{R}^N)$ are sufficient to guarantee existence of solutions to the fractional partial differential equations of interest, the sharp inequality in the integer order Sobolev setting and the regime $p > 1$ suggested the validity of Theorem A′. In fact, one might have guessed such a theorem from several additional factors. For instance, the asymptotics of the constant in Theorem 1.1 are $O(1/(p-1))$ as $p \to 1$, which agrees with the asymptotics of the operator norm of the vector-valued Riesz transform $R : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N;\mathbb{R}^N)$. More recently, we have the result of the second author and R. Garg [3, 4] which shows the logarithmic potential $I_N u$ defined for $u \in C_c^\infty(\mathbb{R}^N)$ by

$$I_N u(x) = \frac{1}{|S^{N-1}|}\int_{\mathbb{R}^N} \log\frac{1}{|x-y|}\; u(y)\, dy,$$

has for any u with sufficient decay at infinity and $\int u = 0$ the representation

$$I_N u(x) = \frac{1}{|S^{N-1}|}\int_{\mathbb{R}^N} \frac{x-y}{|x-y|}\cdot Ru(y)\, dy.$$

Therefore, when $\alpha = N$ one has the corresponding estimate

$$\|I_N u\|_{L^\infty(\mathbb{R}^N)} \leq C\|Ru\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)}.$$

In fact, we suspect that such a result holds for $0 < \alpha < N$, for which we have following conjecture.

Conjecture 1.4 *Let $N \geq 2$ and suppose $0 < \alpha < N$. There exists a constant $C = C(\alpha, N) > 0$ such that*

$$\|I_\alpha u\|_{L^{N/(N-\alpha)}(\mathbb{R}^N)} \leq C\|Ru\|_{L^1(\mathbb{R}^N;\mathbb{R}^N)}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$ such that $Ru \in L^1(\mathbb{R}^N;\mathbb{R}^N)$.

The proofs of Theorems A and A′ are a simple consequence of a duality argument and following highly non-trivial result due to Bousquet, Mironescu, and Russ [2] concerning the existence of solutions to divergence equations in the Triebel-Lizorkin spaces.

Theorem 1.5 *Suppose $-1/2 < s \leq \frac{N}{2} - 1$ and $(1+s)p = N$. There exists a constant $C > 0$ such that for any $f \in \dot{F}^{s,p}_q(B(0,1))$ with $\int_{B(0,1)} f = 0$, there is a function $\mathbf{Y} \in L^\infty(B(0,1);\mathbb{R}^N) \cap \dot{F}^{s+1,p}_q(B(0,1);\mathbb{R}^N)$ with $\operatorname{tr}(\mathbf{Y}) = 0$ that satisfies*

$$-\operatorname{div} \mathbf{Y} = f \text{ in } \mathcal{D}'(B(0,1)),$$

and

$$\|\mathbf{Y}\|_{L^\infty(B(0,1))} + \|\mathbf{Y}\|_{\dot{F}^{s+1,p}_q(B(0,1))} \leq C\|f\|_{\dot{F}^{s,p}_q(B(0,1))}.$$

Here, $\dot{F}^{s,p}_q(\mathbb{R}^N)$ denotes the homogeneous Triebel-Lizorkin space, and $\dot{F}^{s,p}_q(B(0,1))$ its restriction to the set $B(0,1)$. For the theory on Triebel-Lizorkin spaces we refer to [12, 9] and [6, Chapter 6]. We will not need to use Theorem 1.5 in this generality, as we will see in Section 2 below we only require the case $q = 2$, where $\dot{F}^{s,p}_2(\mathbb{R}^N)$ can be identified with the homogeneous fractional Sobolev space $\dot{L}^{s,p}(\mathbb{R}^N)$, [6, Remark 6.5.2, Definition 6.2.5]. This fact, combined with an appropriate scaling argument, establishes our result. We will shortly provide further details on these spaces, though let us first make several remarks concerning Theorem 1.5.

At the heart of the construction in Theorem 1.5 is a sort of one dimensional integration in each direction in Fourier space, to which one must make appropriate modification by subdividing the Littlewood-Paley representation into finer pieces and then localizing again via the Fejér kernel. The first step is somewhat reminiscent of the real space proof of the Sobolev inequality for $p = 1$ due to Gagliardo and Nirenberg, though the subsequent steps rely on harmonic analysis results that are quite deep. The construction is originally due to Bourgain and Brezis [1], who had proven the result where f is in the Lebesgue space L^N. The extension to the scale of spaces in [2] which include the homogeneous fractional Sobolev spaces is crucial to our result, and conversely, our result is informative toward the discussion in [2] over the values of s, p for which the latter result holds. There, the authors require the conditions

$$-1/2 < s,$$
$$s \leq \frac{N}{2} - 1 \quad (p \geq 2),$$

which in one dimension shows one can only apply their arguments if there exists

$$s \in (-\frac{1}{2}, -\frac{1}{2}].$$

The fact that this interval is empty is related to the failure of Theorem A when $N = 1$ mentioned in Remark 1.3. Any improvement on this interval would imply by our argument the existence of an L^1-type inequality for the Riesz potentials in one dimension, an absurdity. Thus, there is no possibility to improve their result in one dimension by lowering the values of s or p, while if one wants to improve their result in $N \geq 2$, one should use techniques not applicable in the one dimensional case.

2 Proofs of the Main Results

As Theorems A and A′ concern the Riesz potentials, let us first recall their relationship with the fractional Laplacian and homogeneous fractional Sobolev spaces. We have that the fractional laplacian of order $s > -N$ is defined by

$$(-\Delta)^{\frac{s}{2}} f := \Big((2\pi|\xi|)^s \hat{f}(\xi)\Big)^{\vee}.$$

Here, we use the convention

$$\hat{f}(\xi) := \int_{\mathbb{R}^N} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

for the Fourier transform, the notation $(\cdot)^{\vee}$ to denote its inverse. With this definition, one has the equivalence $(-\Delta)^{s/2} f \equiv I_{-s} f$, cf. [14, p. 117].

We now connect the Triebel-Lizorkin spaces $\dot{F}^{s,p}_q$ utilized in Theorem 1.5 and the homogeneous Sobolev spaces as follows. We have that $\dot{F}^{s,p}_2(\mathbb{R}^N) = \dot{L}^{s,p}(\mathbb{R}^N)$ [6, Remark 6.5.2, Definition 6.2.5], where we recall that

$$\dot{L}^{s,p}(\mathbb{R}^N) := \{f \in \mathcal{D}'(\mathbb{R}^N) : (-\Delta)^{s/2} f \in L^p(\mathbb{R}^N)\}.$$

By definition, the (semi-)norm is given by

$$\|f\|_{\dot{L}^{s,p}(\mathbb{R}^N)} := \|(-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^N)},$$

which is a norm for $-1 < s \leq 0$, while for $0 < s < 1$ becomes a norm modulo constants. The restriction space is defined in [13, p. 59, Definition 1.95] as

$$\dot{L}^{s,p}(B(0,1)) := \{f \in \mathcal{D}'(B(0,1)) : \exists g \in \dot{L}^{s,p}(\mathbb{R}^N) \text{ s.t. } g = f \text{ on } B(0,1)\}.$$

This produces a norm for $-1 < s < 0$ on the restriction given by

$$\|f\|_{\dot{L}^{s,p}(B(0,1))} := \inf\{\|g\|_{\dot{L}^{s,p}(\mathbb{R}^N)} : g \in \dot{L}^{s,p}(\mathbb{R}^N), g = f \text{ on } B(0,1)\}.$$

Finally, $\dot{F}^{s,p}_2(B(0,1)) = \dot{L}^{s,p}(B(0,1))$, since it is similarly defined as a restriction space.

We now state and prove a consequence of Theorem 1.5 adapted to our purposes.

Corollary 2.1 *Suppose* $1/2 < \alpha < 1$, $\alpha p = N$ *and* $f \in C_c^\infty(\mathbb{R}^N)$. *Define*

$$F_n(x) := \left((-\Delta)^{\frac{1-\alpha}{2}} f\right)(nx) - \fint_{B(0,1)} (-\Delta)^{\frac{1-\alpha}{2}} f(nz)\, dz.$$

Then there exists a constant $C > 0$ *independent of* f *and* $\mathbf{Y}_n \in L^\infty(B(0,n);\mathbb{R}^N) \cap \dot{L}^{\alpha,p}(B(0,n);\mathbb{R}^N)$ *with* $\operatorname{tr}(\mathbf{Y}_n) = 0$ *which satisfies*

$$-\operatorname{div} \mathbf{Y}_n(x) = F_n(\frac{x}{n}) \text{ in } \mathcal{D}'(B(0,n))$$

and

$$\lim_{n\to\infty} \|\mathbf{Y}_n\|_{L^\infty(B(0,n))} \leq C\|f\|_{L^{N/\alpha}(\mathbb{R}^N)}.$$

Proof of Corollary 2.1. Let $f \in C_c^\infty(\mathbb{R}^N)$ and observe that from the definition of F_n, one has $F_n : B(0,1) \to \mathbb{R}$ and $\fint_{B(0,1)} F_n(x)\, dx = 0$. Moreover, the following calculation shows that $F_n(x) \in \dot{L}^{-1+\alpha,p}(B(0,1))$. By the triangle inequality we estimate

$$\|F_n\|_{\dot{L}^{-1+\alpha,p}(B(0,1))} \leq \|(-\Delta)^{\frac{1-\alpha}{2}} f(nx)\|_{\dot{L}^{-1+s,p}(B(0,1))} + \left\|\fint_{B(0,1)} (-\Delta)^{\frac{1-\alpha}{2}} f(nz)\, dz\right\|_{\dot{L}^{-1+\alpha,p}(B(0,1))}. \tag{2.1}$$

Now, $\left((-\Delta)^{(1-\alpha)/2} f\right)(nx)$ itself is an admissible extension from $B(0,1)$ to $\mathbb{R}^N$ for the computation of the norm of the first function on the right hand side, so that

$$\begin{aligned}\|(-\Delta)^{\frac{1-\alpha}{2}} f(nx)\|_{\dot{L}^{-1+\alpha,p}(B(0,1))} &\leq n^{-1+\alpha} \left(\int_{\mathbb{R}^N} |f(nx)|^p\, dx\right)^{\frac{1}{p}} \\ &\leq \frac{1}{n}\|f\|_{L^{N/\alpha}(\mathbb{R}^N)}.\end{aligned}$$

Let us observe that

$$(-\Delta)^{\frac{1-\alpha}{2}} f(x) = -\operatorname{div}\ RI_\alpha f.$$

so that if $\operatorname{supp} f \subset B(0,R_1)$, then changing variables $x = nz$ and applying the divergence theorem we can estimate

$$\begin{aligned}\left|\fint_{B(0,1)} (-\Delta)^{\frac{1-\alpha}{2}} f(nz)\, dz\right| &= \frac{1}{n^N}\left|\int_{B(0,n)} (-\Delta)^{\frac{1-\alpha}{2}} f(x)\, dx\right| \\ &= \frac{c}{n^N}\left|\int_{\partial B(0,n)} \int_{B(0,R_1)} f(y)\frac{x-y}{|x-y|^{N+1-\alpha}}\, dy \cdot \mathbf{n}\, d\mathcal{H}^{N-1}(x)\right| \\ &\leq \frac{C'}{n^{N+1-\alpha}},\end{aligned}$$

whenever $n > R_1$. Therefore, we can estimate the second term of (2.1) by

$$\begin{aligned}\left\|\fint_{B(0,1)} (-\Delta)^{\frac{1-\alpha}{2}} f(nz)\, dz\right\|_{\dot{L}^{-1+\alpha,p}(B(0,1))} &= \left|\fint_{B(0,1)} (-\Delta)^{\frac{1-\alpha}{2}} f(nz)\, dz\right| \|1\|_{\dot{L}^{-1+\alpha,p}(B(0,1))} \\ &\leq \frac{C'}{n^{N+1-\alpha}}\|1\|_{\dot{L}^{-1+\alpha,p}(B(0,1))}.\end{aligned}$$

It only remains to check that this last norm is finite. However, if we make an extension by a usual cutoff-function $g_0 \in C_c^\infty(B(0,2))$ with $g_0 \equiv 1$ in $B(0,1)$, by Theorem 1.1 we have

$$||I_{1-\alpha} g||_{L^{N/\alpha}(\mathbb{R}^N)} \leq ||g||_{L^N(\mathbb{R}^N)} \leq C,$$

from which the estimate follows. We thus conclude that

$$\|F_n\|_{\dot{L}^{-1+\alpha,p}(B(0,1))} \leq \frac{1}{n}\left(\|f\|_{L^{N/\alpha}(\mathbb{R}^N)} + \frac{C'}{n^{N-\alpha}}\right),$$

whenever $n > R_1$. Therefore, as F_n satisfies the hypothesis of Theorem 1.5 (with $s = -1+\alpha$), we may find a function $\tilde{\mathbf{Y}}_n$ such that $\operatorname{tr}(\tilde{\mathbf{Y}}_n) = 0$, (on $\partial B(0,1)$)

$$-\operatorname{div} \tilde{\mathbf{Y}}_n(x) = F_n(x) \text{ in } \mathcal{D}'(B(0,1))$$

and

$$\|\tilde{\mathbf{Y}}_n\|_{L^\infty(B(0,1))} + \|\tilde{\mathbf{Y}}_n\|_{\dot{L}^{\alpha,p}(B(0,1))} \leq C\|F_n\|_{\dot{L}^{-1+\alpha,p}(B(0,1))}.$$

We then define $\mathbf{Y}_n(x) := n\tilde{\mathbf{Y}}_n(\frac{x}{n})$, observing that $\operatorname{tr}(\mathbf{Y}_n) = 0$ (on $\partial B(0,n)$), while

$$\begin{aligned} -\operatorname{div} \mathbf{Y}_n(x) &= -\operatorname{div} \tilde{\mathbf{Y}}_n(\frac{x}{n}) \\ &= F_n(\frac{x}{n}) \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|\mathbf{Y}_n\|_{L^\infty(B(0,n))} &= n\|\tilde{\mathbf{Y}}_n\|_{L^\infty(B(0,1))} \\ &\leq Cn\|F_n\|_{\dot{L}^{-1+\alpha,p}(B(0,1))} \\ &= C(\|f\|_{L^{N/\alpha}(\mathbb{R}^N)} + \frac{C}{n^{N-\alpha}}), \end{aligned}$$

from which the desired estimate follows by taking the limit as $n \to \infty$. ■

We now proceed to prove Theorem A, while the proof of Theorem A′ is similar.

Proof of Theorem A. Let $N \geq 2$. From the discussion in the introduction, it suffices to consider the case $1/2 < \alpha < 1$. By duality, we have that

$$\begin{aligned} \|I_\alpha u\|_{L^{N/(N-\alpha)}} &= \sup_{\|f\|_{L^{N/\alpha}} \leq 1} \int_{\mathbb{R}^N} (I_\alpha u) f \\ &\leq 2\int_{\mathbb{R}^N} (I_\alpha u) f, \end{aligned}$$

for some $f \in C_c^\infty(\mathbb{R}^N)$ with $\|f\|_{\frac{N}{\alpha}} \leq 1$. Then the identity $f = I_{1-\alpha}(-\Delta)^{\frac{1-\alpha}{2}} f$, Fubini's theorem, and changing variables gives the estimate

$$\|I_\alpha u\|_{L^{N/(N-\alpha)}} \leq 2\int_{\mathbb{R}^N} I_1 u \, (-\Delta)^{\frac{1-\alpha}{2}} f.$$

The main point now is that we would like to use Corollary 2.1 to introduce $-\operatorname{div}\mathbf{Y}_n$ and integrate by parts. To this end, we recall that in the proof of Corollary 2.1 we observed that if $\operatorname{supp} f \subset B(0,R_1)$ then

$$\left| \int_{B(0,n)} (-\Delta)^{\frac{1-\alpha}{2}} f(z)\, dz \right| \leq \frac{C}{n^{1-\alpha}},$$

for n sufficiently large. Meanwhile, Jensen's inequality and Theorem 1.1 imply that for any $1 < r < N$, denoting $r^* = \frac{Nr}{N-r}$, we can estimate

$$\lim_{n\to\infty} \left| \fint_{B(0,n)} I_1 u\, dx \right| \leq \lim_{n\to\infty} \left(\fint_{B(0,n)} |I_1 u|^{r^*}\, dx \right)^{\frac{1}{r^*}} \leq \frac{C}{n^{1/r^*}} \|u\|_{L^r(\mathbb{R}^N)}.$$

Therefore,

$$\lim_{n\to\infty} \fint_{B(0,n)} I_1 u(z)\, dz \int_{B(0,n)} (-\Delta)^{\frac{1-\alpha}{2}} f(z)\, dz = 0,$$

and so if we define

$$F_n(x) := (-\Delta)^{\frac{1-\alpha}{2}} f(nx) - \fint_{B(0,1)} (-\Delta)^{\frac{1-\alpha}{2}} f(nz)\, dz,$$

we see that

$$\|I_\alpha u\|_{L^{N/(N-\alpha)}} \leq 2 \lim_{n\to\infty} \int_{B(0,n)} I_1 u(x)\, F_n(x/n)\, dx.$$

Now, applying Corollary 2.1, we can find $\mathbf{Y}_n \in L^\infty(B(0,n))$ with $\operatorname{tr}(\mathbf{Y}_n) = 0$ such that

$$-\operatorname{div}\mathbf{Y}_n(x) = F_n(\frac{x}{n}) \text{ in } \mathcal{D}'(B(0,n))$$

and

$$\lim_{n\to\infty} \|\mathbf{Y}_n\|_{L^\infty(B(0,n))} \leq C\|f\|_{L^{N/\alpha}}.$$

Since $I_1 u \in \dot{L}^{\alpha,p}(B(0,n))$ and $\operatorname{tr}(\mathbf{Y}_n) = 0$, we have

$$\int_{B(0,n)} I_1 u(x)\, F_n(x/n)\, dx = \int_{B(0,n)} \nabla I_1 u(x) \cdot \mathbf{Y}_n(x)\, dx,$$

and so

$$\begin{aligned}
\|I_\alpha u\|_{L^{N/(N-\alpha)}} &\leq 2 \lim_{n\to\infty} \int_{B(0,n)} Ru \cdot \mathbf{Y}_n\, dx \\
&\leq 2 \lim_{n\to\infty} \|Ru\|_{L^1(B(0,n))} \|\mathbf{Y}_n\|_{L^\infty(B(0,n))} \\
&\leq C\|Ru\|_{L^1(\mathbb{R}^N)} \|f\|_{L^{N/\alpha}(\mathbb{R}^N)},
\end{aligned}$$

which implies the thesis. ■

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