Università degli Studi di Pavia
Dipartimento di Matematica “Felice Casorati”
Dottorato in Matematica e Statistica, ciclo XXVII

Friedrich-Alexander Universität Erlangen-Nürnberg
Department Mathematik
Naturwissenschaftliche Fakultät

Existence and regularity results for solutions of spectral problems

Dario Mazzoleni

Advisor:
Prof. Aldo Pratelli
Prof. Giuseppe Savaré

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Ai miei genitori,
a Valentina e a Stella

Abstract:

This Thesis is devoted to the study of some shape optimization problems for eigenvalues of the Dirichlet Laplacian. More precisely we consider the minimum problem

$$\min\left\{F(\lambda_1(\Omega),\dots,\lambda_k(\Omega))\ :\ \Omega\subset\mathbb{R}^N,\ \text{quasi-open},\ |\Omega|=1\right\},$$

with $F\colon\mathbb{R}^k\to\mathbb{R}$ increasing in each variable and lower semicontinuous.

The first result of the Thesis is a proof of the existence of an optimal set for such a problem, thus extending a well-known result due to Buttazzo and Dal Maso to the "unbounded" setting. Moreover, under a slightly stronger assumption on F, it is possible to prove that all the minimizers have a diameter uniformly bounded by a constant depending only on k, N (but *not* on the functional). The main interest of this result is the very "elementary" techniques that are used. In fact the key point consists in showing that it is always possible to choose a minimizing sequence made of sets with uniformly bounded diameter, since getting rid of "long tails" decreases the first k eigenvalues.

Then we focus on the study of the regularity of optimal sets, in particular a natural conjecture is that they should be open sets, at least. This kind of issue reveals to be quite hard to solve. With a "direct" approach we can prove, in the two dimensional setting, that minimizers for functionals like $\lambda_1(\cdot)+\dots+\lambda_k(\cdot)$ are open sets. Moreover we perform a finer analysis of the eigenfunctions of optimal sets (in generic dimension), employing techniques from the regularity of free boundary problems. In particular we prove that an optimal set Ω for the functional $\lambda_k(\cdot)$ has an eigenfunction, corresponding to the eigenvalue $\lambda_k(\Omega)$, which is Lipschitz continuous in $\mathbb{R}^N$.

At last we study the connectedness of optimal sets for convex combinations of the first three eigenvalues, and in particular we are able to prove that every minimizer for the problem

$$\min\left\{\alpha\lambda_1(\Omega)+(1-\alpha)\lambda_2(\Omega)\ :\ \Omega\subset\mathbb{R}^N,\ \text{(quasi-)open},\ |\Omega|=1\right\},$$

is connected for all $\alpha\in(0,1]$.

Sunto:

Questa Tesi tratta alcuni problemi di ottimizzazione di forma per autovalori del Laplaciano con condizioni al bordo di Dirichlet omogenee. Più precisamente consideriamo il problema di minimo

$$\min\left\{F(\lambda_1(\Omega),\dots,\lambda_k(\Omega))\ :\ \Omega\subset\mathbb{R}^N,\ \text{quasi-aperto},\ |\Omega|=1\right\},$$

ove $F\colon\mathbb{R}^k\to\mathbb{R}$ è un funzionale crescente in ciascuna variabile e semicontinuo inferiormente.

Il primo risultato della Tesi è una dimostrazione dell'esistenza di un insieme ottimale per tale problema, che estende un ben noto risultato di Buttazzo e Dal Maso al caso "non limitato". Inoltre, sotto ipotesi leggermente più forti per F, è possibile mostrare che tutti i minimi

hanno diametro uniformemente limitato da una costante che dipende solo da k, N (ma *non* dal funzionale). Il principale interesse di questo risultato è il metodo di dimostrazione utilizzato, che è molto "elementare". Infatti il punto chiave consiste nel mostrare che è sempre possibile prendere successioni minimizzanti composte da insiemi con diametro uniformemente limitato, poichè eliminare delle "lunghe code" fa decrescere i primi k autovalori.

In seguito studiamo la regolarità degli insiemi ottimali, in particolare una naturale congettura è che siano almeno aperti. Questo tipo di problema si rivela essere piuttosto difficile da risolvere. Con un approccio "diretto" possiamo dimostrare, in due dimensioni, che i minimi per funzionali come $\lambda_1(\cdot) + \cdots + \lambda_k(\cdot)$ sono aperti. Inoltre analizziamo le autofunzioni degli insiemi ottimali (in dimensione generica), utilizzando tecniche provenienti dalla teoria della regolarità per problemi con frontiera libera. In particolare, mostriamo che un insieme ottimo Ω per il funzionale $\lambda_k(\cdot)$ ammette una autofunzione, corrispondente all'autovalore $\lambda_k(\Omega)$, che è Lipschitziana in tutto $\mathbb{R}^N$.

Infine studiamo quando gli insiemi ottimali sono connessi per combinazioni convesse dei primi tre autovalori e in particolare possiamo dimostrare che ogni minimo per il problema

$$\min\left\{\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) \ : \ \Omega \subset \mathbb{R}^N, \ \text{(quasi-)aperto}, \ |\Omega| = 1\right\},$$

è connesso per ogni $\alpha \in (0, 1]$.

Zusammenfassung:

Diese Arbeit widmet sich der Untersuchung von Gestaltoptimierungsproblemen für Eigenwerte des Dirichlet-Laplace-Operators. Genauer gesagt betrachten wir das Minimierungsproblem

$$\min\left\{F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \ : \ \Omega \subset \mathbb{R}^N, \ \ \text{quasi-offen}, \ |\Omega| = 1\right\}$$

mit $F : \mathbb{R}^k \to \mathbb{R}$ wachsend in allen Variablen und unterhalbstetig.

Das erste Resultat ist der Existenzbeweis einer optimalen Menge für ein solches Problem, was ein bekanntes Resultat von Buttazzo und Dal Maso auf die "unbeschränkte" Situation erweitert. Weiterhin ist es unter einer etwas stärkeren Voraussetzung an F möglich zu zeigen, dass alle Minimierer einen gleichmäßig durch eine Konstante beschränkten Durchmesser haben, wobei die Konstante nur von k und N (jedoch *nicht* vom Funktional F) abhängt. Das Bemerkenswerteste an diesem Resultat sind die sehr "elementaren" Techniken die verwendet wurden. In der Tat ist der entscheidende Punkt zu zeigen, dass es immer möglich ist, eine Minimalfolge von Mengen mit gleichmäßig beschränktem Durchmesser auszuwählen, da das Entfernen "langer Ausläufer" die ersten k Eigenwerte verkleinert.

Danach konzentrieren wir uns auf die Untersuchung der Regularität der optimalen Mengen. Eine natürliche Vermutung dabei ist, dass diese zumindest offen sein sollten. Dies erweist sich jedoch als recht schwierig zu beweisen. Mit einer "direkten" Vorgehensweise können wir

im zweidimensionalen Fall zeigen, dass Minimierer von Funktionalen wie $\lambda_1(\cdot) + \ldots + \lambda_k(\cdot)$ offene Mengen sind. Darüber hinaus führen wir eine detailliertere Analyse der Eigenfunktionen optimaler Mengen (allgemeiner Dimension) aus, wobei Techniken aus der Regularitätstheorie freier Randwertprobleme zum Einsatz kommen. Insbesondere beweisen wir, dass eine optimale Menge Ω für das Funktional $\lambda_k(\cdot)$ eine Eigenfunktion zum Eigenwert $\lambda_k(\Omega)$ besitzt, welche Lipschitz-stetig auf $\mathbb{R}^N$ ist.

Abschließend untersuchen wir die Zusammenhangseigenschaften optimaler Mengen für Konvexkombinationen der ersten drei Eigenwerte. Genauer gelingt es uns dabei zu zeigen, dass jeder Minimierer des Problems

$$\min\left\{\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) \ : \ \Omega \subset \mathbb{R}^N, \ \ \text{quasi-offen}, \ |\Omega| = 1\right\}$$

zusammenhängend ist für alle $\alpha \in (0, 1]$.

Contents

Chapter 1

Introduction

Looking for optimal shapes is for sure one of the most fascinating topic of mathematics. This is probably due to the many different fields of mathematics which are involved: spectral theory, partial differential equations, calculus of variations, geometric measure theory, etc. Moreover, as Buttazzo highlights in [24], a shape is *something closer to human spirit than a function*, and maybe the big interest of mathematicians in shape optimization problems is motivated also by this "philosophical" reason. The study of shapes from a mathematical point of view is a very difficult subject and shape optimization problems are often, as Henrot writes in [38], *very simple to state but very hard to solve.*

A very general shape optimization problem can be written in the following way:

$$\min\{\mathcal{F}(\Omega)\ :\ \Omega\in\mathcal{A}\}, \tag{1.1}$$

where $\mathcal{A}\subset\mathcal{P}(\mathbb{R}^N)$ denotes the family of admissible shapes and $\mathcal{F}\colon\mathcal{A}\to\mathbb{R}$ is a "cost" functional. In this wide setting, many well known topics fit: from isoperimetric problems to the Newton problem of aerodynamical shapes, till spectral optimization for Schrödinger operators. Shapes are very general geometric objects (manifolds, metric spaces, etc), but in this Thesis we focus only on domains of the Euclidean space $\mathbb{R}^N$. The main issues regarding problem (1.1) are:

1. to prove the existence of a solution;

2. to describe the properties of optimal sets (e.g. openness, closedness, connectedness, convexity);

3. to provide numerical approximations of solutions.

We address only the first two points in our work and we focus on the class of shape optimization problems involving eigenvalues of the Dirichlet Laplacian: in particular, we are interested in the problem of existence of a solution, of its regularity and at last we study the property of connectedness of minimizers in some peculiar case. More precisely, we choose

$\mathcal{F}(\Omega) = F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$ and the general minimization problem that we consider is the following:

$$\min\big\{F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \ : \ \Omega \subset \mathbb{R}^N, \text{ open}, \ |\Omega| \leq 1\big\}, \tag{1.2}$$

where $k, N \in \mathbb{N}$, $|\cdot|$ denotes the Lebesgue N-dimensional measure and we call λ_i the eigenvalues of the Dirichlet Laplacian (counted with multiplicity). The bound on the measure is taken less than or equal to 1 only for simplicity: with every other positive constant everything remains unchanged. Moreover, since eigenvalues are decreasing with respect to set inclusion, it is equivalent to consider the problem with the equality constraint.

This kind of optimization problems naturally arise in the study of many physical phenomena, e.g. heat diffusion or wave propagation inside a domain $\Omega \subset \mathbb{R}^N$, and the literature is very wide (see [20, 38, 39, 24] for an overview), with many works in the last few years. Problem (1.2) was studied first by Lord Rayleigh in his treatise *The theory of sound* of 1877 (see [48]), where he focused on the case $F = \lambda_1$ and he conjectured the unit ball to be the optimal set. This was proved by Faber and Krahn (see [35, 43, 44]) in the 1920s, using techniques based on spherical decreasing rearrangements. From that result, the case $F = \lambda_2$ follows easily considering the nodal domains: Krahn and Szegö (see [43, 44, 49]) proved two disjoint equal balls of half measure each to be optimal. The situation for $k \geq 3$ becomes more complex and it is not known what are the optimal shapes, up to now. As an example of the difficulties in finding explicit minimizers, recent numerical results (see [4], [46]) shows that the minimizers for λ_3 change with the dimension. The only other functionals of eigenvalues for which the optimal shape is known are λ_1/λ_2 and λ_2/λ_3: Ashbaugh and Benguria (see [7]) proved that the minimizers are the unit ball and two equal disjoint balls of half measure each respectively.

Existence of optimal shapes. Since the search for explicit solutions did not give other results, it is natural to study at least whether a minimizer for (1.2) exists, and this subject turns out to be a difficult one, too. The main reason of difficulty is the lack of compactness for generic sequences of open sets. Moreover it is not clear, given a converging sequence of open sets with unit measure, whether the limit is a set at least in some suitable sense. The search for a "right" notion of convergence in this setting was a main problem for many years. In the 1980s Dal Maso and Mosco (see [31, 32]) proposed the notion of γ-convergence, which was the main tool used by Buttazzo and Dal Maso in 1993 (see [26]) for proving a fundamental existence result for very general functionals of eigenvalues, in the class of *quasi-open*[1] sets inside a fixed bounded box. More precisely, they fix, a priori, a bounded open set $D \subset \mathbb{R}^N$ and consider a functional $F\colon \mathbb{R}^k \to \mathbb{R}$ increasing in each variable and lower semicontinuous (l.s.c.). Then there exists a minimizer for the problem:

$$\min\{F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \ : \ \Omega \subset D, \text{ quasi-open}, \ |\Omega| \leq 1\}. \tag{1.3}$$

[1] A quasi-open set is a superlevel of an H^1 function.

The above result gives a definitive answer to the existence problem for very general class of spectral functionals in a bounded ambient space (actually, it is sufficient to suppose D to have finite measure in order to have the compact injection of $H^1_0(D)$ in $L^2(D)$, as shown in [19]). The extension of the result by Buttazzo and Dal Maso to generic domains in $\mathbb{R}^N$ is a non trivial topic because minimizing sequences, in principle, could have a significant portion of volume moving to infinity.

A first result in the direction of existence in $\mathbb{R}^N$ was obtained by Bucur and Henrot in 2000 (see [23]); they proved existence for λ_3, using a concentration-compactness argument (see [18]). Moreover, they showed that given $k \geq 1$, if there exists a bounded minimizer for λ_j for all $j = 1, \dots, k-1$, then there exists a minimizer for λ_k and also for Lipschitz functionals of the first k eigenvalues. Unfortunately, this boundedness hypothesis was not known even for λ_3. In a very recent result, Bucur (see [16]) was able to study the class of *energy shape subsolutions* with techniques coming from the theory of free boundary and to prove, for them, boundedness and finiteness of the perimeter. Since optimal sets for (1.2) can be proved to be energy shape subsolutions, it is then possible to obtain existence for λ_k for all k.

At the same time, we gave in collaboration with Aldo Pratelli, an independent proof of existence of a solution for problem (1.3) in $\mathbb{R}^N$ (see [M4]), with the very same hypotheses of Buttazzo and Dal Maso on F (increasing in each variable and l.s.c.). This different proof is more "elementary" and involves neither a concentration-compactness argument nor the regularity of energy shape subsolutions. The idea consists in showing that, given a minimizing sequence for the problem,

$$\min\big\{F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \ : \ \Omega \subset \mathbb{R}^N, \text{ quasi-open}, \ |\Omega| = 1\big\}, \tag{1.4}$$

it is then possible to find a new one made of sets with diameter bounded by a constant depending only on k, N and with all the first k eigenvalues not increased. This argument, roughly speaking, works because sets with long "tails" must have some tiny section and hence they can not have the first k eigenvalues very small. Moreover, with minor changes in the proof, it is also possible to deduce that all minimizers for (1.4) are bounded, provided that F is *weakly strictly increasing* (see [M3]).

In recent years, the existence of optimal sets was studied also for other kinds of shape optimization problem involving eigenvalues of Dirichlet Laplacian. A first example is the case of an internal constraint, that is,

$$\min\big\{\lambda_k(\Omega) \ : \ D \subset \Omega \subset \mathbb{R}^N, \text{ quasi-open}, \ |\Omega| \leq 1\big\},$$

where D is a fixed quasi open box with $|D| \leq 1$. Bucur, Buttazzo and Velichkov in [22], using a concentration compactness argument similar to the one in [18], proved existence of a solution for $k = 1$, gave a characterization of the cases when $k \geq 2$ and moreover proved some regularity of the solutions.

It is also possible to consider spectral shape optimization problems with perimeter constraint instead of volume constraint. This kind of problem was studied in the recent paper by De Philippis and Velichkov [34], where they prove that there exists a minimizer for

$$\min\left\{\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^N,\ \text{measurable},\ P(\Omega)\le 1\right\}.$$

They use techniques to some extent analogous to those used by Bucur in [16], combining a concentration compactness argument and the study of the regularity for *perimeter* shape subsolutions. The perimeter constraint turns out to have a better *regularizing effect* than the volume constraint. In fact De Philippis and Velichkov are able to give many informations about regularity of optimal shapes: first of all the optimal shapes are open, so the above problem has a solution also among open sets.

Regularity of optimal shapes and of their eigenfunctions. The results exposed above give a quite complete understanding for the problem of existence of minimizers for spectral functionals involving eigenvalues of the Dirichlet Laplacian with a measure constraint. A further question, arising from Buttazzo and Dal Maso Theorem, is the study of the regularity of optimal sets or of the corresponding eigenfunctions. For example, a natural conjecture is that minimizers for (1.3) and (1.4) are open sets and not only quasi-open. This is a quite difficult question, due to the min-max nature of the eigenvalues and to the necessity of dealing with external perturbation of sets. The only complete result in this field deals with the regularity of the free boundary of the optimal set for λ_1 inside a bounded box and was obtained by Briançon and Lamboley in [13]: in that case the free boundary turns out to be smooth. On the other hand, at least in the bounded setting, it is not always true the smoothness of a minimizer: in fact the optimal set for λ_2 has boundary not even Lipschitz, if the box is a sufficiently small rectangle. When working with higher eigenvalues, many difficulties arise since the techniques developed by Alt and Caffarelli for the study of the free boundary (see [2]) do not work for functionals defined through a min-max procedure on H^1_0.
In Chapter 5 of the Thesis we present a result obtained in collaboration with Bucur, Pratelli and Velichkov (see [M1]), in which we study the regularity of eigenfunctions on an optimal set for

$$\min\left\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^N,\ \text{(quasi-)open},\ |\Omega|\le 1\right\},\tag{1.5}$$

and we prove that the first k eigenfunctions are Lipschitz continuous in $\mathbb{R}^N$. Actually it is also possible to provide regularity of eigenfunctions of optimal sets also for more general functionals, but unfortunately, in the most interesting case of λ_k alone we are only able to prove that there exists a Lipschitz eigenfunction corresponding to the k^{th} eigenvalue. This does not imply that an optimal set is open and this question remains one major conjecture in spectral shape optimization.

We present in Chapter 4 of this Thesis also an "elementary" method that does not involve free boundary techniques in order to prove the openness of optimal sets for problem (1.5) in two dimensions. We do not obtain better results with this method, but we believe it is worth of notice.

Connectedness of optimal shapes. In [M2], we deal with a different kind of problem about connectedness of optimal sets for a spectral optimization problem. More precisely we study, for a convex combination of the first three eigenvalues of the Dirichlet Laplacian, the minimum problem

$$\inf\{\alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) \ : \ \Omega \subset \mathbb{R}^N, \text{ open}, |\Omega| \leq 1\}, \tag{1.6}$$

with $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. Our aim is to investigate for which values of α, β all the minimizers are connected. From Faber-Krahn inequality it follows that the unit ball is the minimizer when $\alpha = 1$ and $\beta = 0$. On the other hand, by the Krahn-Szegö inequality, the disjoint union of two equal balls of half measure each is the optimal set when $\alpha = 0$, $\beta = 1$, hence in this case the minimizer is disconnected.
The idea of studying such a convex combination arises from the inspiring paper by Wolf and Keller [53], in which they proved that every minimizer for λ_3 (that corresponds to $\alpha = \beta = 0$) must be connected in dimension $N = 2$ or 3. Their idea is quite simple and consists in studying the best disconnected domain and to compare it with the ball, for which the values of eigenvalues are known and depend only on the zeros of Bessel function.
The most important result that we obtain is to prove that every minimizer for the convex combination

$$\min\big\{\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) \ : \ \Omega \subset \mathbb{R}^N, \text{ (quasi)-open}, |\Omega| \leq 1\big\},$$

is connected in every dimension for $\alpha \in (0,1]$. Moreover we give some information for the other cases, mostly in $\mathbb{R}^2$. A natural conjecture is that, in two dimensions, every minimizer for (1.6) is connected unless $\alpha = 0$ and $\beta = 1$. A recent numerical work by Kao and Osting (see [41]) supports the above conjecture and moreover suggests that the ball is an optimal set in all the region $\{\alpha + 2\beta \leq 1\}$.
It is worth noticing that in the last few years many other numerical computations of the optimal shapes for single lower eigenvalues have been done. In particular Oudet [46], Antunes and Freitas [4] computed the optimal shapes for λ_k till $k = 10$ and $k = 15$, respectively. Their computations suggest that only the optimal sets for λ_2 and λ_4 should be disconnected in the two dimensional case. Moreover Berger and Oudet [11] proved that in $\mathbb{R}^2$ union of balls are never optimal for λ_k if $k \geq 5$.

Plan of the Thesis

In this thesis we will present the complete proofs only of original results obtained with our contribution. For all the other results, we will give only the statements, sometimes a sketch of the proof and the references where a detailed treatment can be found.

Chapter 2 is devoted to recall the basic concepts that we will need in the Thesis. First of all we deal with the notions of capacity, quasi-open sets and Sobolev-like spaces. Then we recall the definitions of eigenvalues for generic operators and we focus on the case of the Dirichlet Laplacian. In Section 2.3 we treat some classical results about minimization of eigenvalues with measure constraint and we present an useful bound. After that, we deal with the notion of γ-convergence and the existence result by Buttazzo and Dal Maso, which is now classical. At last, in Section 2.5, we deal with shape subsolutions and sketch the main idea of the existence result [16] by Bucur.

In Chapter 3 we deal with the existence result presented in [M4], in particular we show that it is always possible to apply the result of Buttazzo and Dal Maso also in the unbounded setting. The proof is divided in two main steps: first we consider the tails of a regular set and then the interior. Moreover, following [M3], we prove that all the optimal sets are uniformly bounded.

In Chapter 4 we start to study the regularity issue and we present a simple idea, strictly related with the existence results just exposed, that gives some informations about the regularity of optimal sets in two dimensions. In particular, we study what can be done if an optimal set has "holes too small". This approach allows also to understand what are the main difficulties in proving a regularity result.

Chapter 5 is devoted to the presentation of the results of Lipschitz regularity for eigenfunctions of optimal sets obtained in [M1]. First of all, we recall the techniques used by Briançon, Hayouni and Pierre [14] for proving Lipschitz regularity of the energy function. Then we study how to apply these techniques to shape quasi-minimizers for Dirichlet eigenvalues and at last we deal with shape supersolutions and prove the Lipschitz regularity for eigenfunctions on optimal domains. We also show for which functionals the Lipschitz regularity of eigenfunctions gives informations about the openness of an optimal set.

At last, in Chapter 6 we deal with the question of connectedness of optimal sets for convex combinations of the first three eigenvalues, following [M2]. More precisely, after recalling the values of eigenvalues for balls, we first deal with the N-dimensional case and then give more informations in two dimensions.

Table of notations

$\mathbb{N}, \mathbb{R}$	space of natural and real numbers respectively
$\mathcal{P}(E)$	the power set of E
$\mathcal{H}^s$	s-dimensional Hausdorff measure
$\dim_H(E)$	Hausdorff dimension of a set E
$\mathcal{L}^N(E) = \|E\|$	N-dimensional Lebesgue measure of a set E
$\mathrm{cap}\,(E)$	(H^1-)capacity of a set E
$P(E;\Omega)$	distributional perimeter of a set E inside Ω
$\langle\cdot,\cdot\rangle$	duality in H^1_0
$(\cdot,\cdot)$	scalar product in a generic Hilbert space H
ω_N	N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$
C_N	constant depending only on the dimension N
$E\Delta F$	symmetric difference between the sets $E, F \subseteq \mathbb{R}^N$
$\mathcal{R}(u, D)$	Rayleigh quotient of the function u in the domain D
Δ	Laplace operator
div	divergence operator
$\partial\Omega$	topological boundary of the set $\Omega \subseteq \mathbb{R}^N$
$t\Omega$	homothety of ratio $t > 0$ of a set Ω
X'	topological dual of a Banach space X
λ_i	i-th eigenvalue for $-\Delta$ with zero Dirichlet boundary conditions
u_i	i-th eigenfunction corresponding to the eigenvalue λ_i
$B_r(x)$	ball in $\mathbb{R}^N$ of radius $r \geq 0$ and centered in x
B	ball in $\mathbb{R}^N$ with measure 1
Θ	disjoint union of two balls in $\mathbb{R}^N$ of measure $\frac{1}{2}$ each
$\mathcal{M}_0(D)$	the class of capacitary measures on D
A_μ	regular set of a measure μ
R_μ, R_Ω	resolvent operator associated to a measure μ or a set Ω
$\mathcal{S}$	class of measurable sets with finite Lebesgue measure
$\mathcal{A}(D)$	class of quasi-open sets $\Omega \subset D$ with $\|\Omega\| = 1$
$\mathcal{L}(X)$	space of linear and continuous functional defined in a Banach space X
J_m	m-th Bessel function
$j_{m,k}$	k-th zero of the Bessel function J_m
$⨍_A u$	average integral of the function u over the set A
l.s.c.	lower semicontinuous
a.e.	almost everywhere
q.e.	quasi everywhere

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Chapter 2

Preliminaries and some existence results

This Chapter is devoted to briefly introduce the reader to the main tools of spectral shape optimization, which will be used throughout the Thesis. First, we deal with capacity, quasi-open sets, generalized Sobolev spaces and classical extremum problems for eigenvalues. Then, we enter more into details and we treat some recent fundamental existence results in shape optimization. More precisely in Section 2.4 we introduce the γ-convergence and the existence result by Buttazzo and Dal Maso [26]. Then in Section 2.5, we sketch the approach used by Bucur [16] for proving existence in unbounded regions.

2.1 Capacity, quasi-open sets and Sobolev spaces

We need to recall the definition of *capacity*, which is very important in the study of problems involving the space H_0^1, hence also for the study of eigenvalues of the Dirichlet Laplacian. For more details we refer to [39].

Definition 2.1. *Given a compact set $K \subset \mathbb{R}^N$, we define*

$$\mathrm{cap}\,(K) := \inf\Big\{\|v\|^2_{H^1(\mathbb{R}^N)}\ :\ v \in C_c^\infty(\mathbb{R}^N),\ v \geq 1 \textit{ in a neighborhood of } K\Big\}.$$

Then, for an open set $\Omega \subset \mathbb{R}^N$,

$$\mathrm{cap}\,(\Omega) := \sup\{\mathrm{cap}\,(K)\ :\ K \textit{ compact, } K \subset \Omega\}.$$

At last, for a generic measurable set $E \subset \mathbb{R}^N$,

$$\mathrm{cap}\,(E) := \inf\{\mathrm{cap}\,(\Omega)\ :\ \Omega \textit{ open, } \Omega \supset E\}.$$

The last definition is well-posed, since it is easy to prove that for every compact set $K \subset \mathbb{R}^N$, it is $\mathrm{cap}\,(K) = \inf\{\mathrm{cap}\,(\Omega)\ :\ \Omega \text{ open, } \Omega \supset K\}$. Moreover, it is possible to give the following characterization of the capacity: for all measurable set $E \subset \mathbb{R}^N$,

$$\mathrm{cap}\,(E) = \inf\Big\{\|v\|^2_{H^1(\mathbb{R}^N)}\ :\ v \in H^1(\mathbb{R}^N),\ v \geq 1 \text{ in a neighborhood of } E\Big\}.$$

One can also consider the *relative* capacity inside a box, with analogous definitions, which can be characterized as follows. Given an open, bounded set $D \subset \mathbb{R}^N$ and a measurable set $E \subset \mathbb{R}^N$, the *relative* capacity is:

$$\mathrm{cap}\,_D(E) = \inf\Big\{\|v\|^2_{H^1(D)}\ :\ v \in H^1(D),\ v \geq 1 \text{ in a neighborhood of } E\Big\}.$$

Mostly we are interested in sets with zero capacity, and it is immediate to check that $\mathrm{cap}\,(E) = 0$ if and only if $\mathrm{cap}\,_D(E) = 0$, for any suitable $D \supset E$.

Remark 2.2. *It is clear from the definition that if* $\mathrm{cap}\,(E) = 0$*, then* $|E| = 0$*. The opposite implication is false, for example a segment in* $\mathbb{R}^2$ *has zero Lebesgue measure, but positive capacity. In general, given* $E \subset \mathbb{R}^N$*, if* $\dim_H(E) \in [N-1, N)$*, then* $\mathrm{cap}\,(E) > 0$ *and* $|E| = 0$*, while if* $\dim_H(E) \leq N-2$*, then also* $\mathrm{cap}\,(E) = 0$*.*

We summarize some easy and useful properties of capacity.

(1) (Monotonicity) If $E \subset F$, then $\mathrm{cap}\,(E) \leq \mathrm{cap}\,(F)$;

(2) (Subadditivity) For all E, F we have $\mathrm{cap}\,(E \cap F) + \mathrm{cap}\,(E \cup F) \leq \mathrm{cap}\,(E) + \mathrm{cap}\,(F)$.

(3) Given a family of disjoints sets $(E_n)_{n\in\mathbb{N}}$, then $\mathrm{cap}\,(\cup E_n) \leq \sum \mathrm{cap}\,(E_n)$.

(4) Given an increasing sequence of sets $(E_n)_{n\in\mathbb{N}}$, then $\mathrm{cap}\,(\cup E_n) = \lim_{n\to\infty} \mathrm{cap}\,(E_n)$.

(5) Given a decreasing sequence of compact sets $(K_n)_{n\in\mathbb{N}}$, then $\mathrm{cap}\,(\cap K_n) = \lim_{n\to\infty} \mathrm{cap}\,(K_n)$.

We say that a property P holds *quasi everywhere* (q.e.) if the set for which the property does not hold has zero capacity, while we keep the usual terminology of almost everywhere (a.e.) in the case of Lebesgue measure.

Remark 2.3. *In the whole Thesis we consider sets defined up to zero capacity, hence also notions such as connectedness should be intended in this acception.*

Two very important notions related to the concept of capacity are the ones of quasi-open set and quasi-continuous function, which will be fundamental throughout the Thesis.

Definition 2.4. *We say that a set* $\Omega \subset \mathbb{R}^N$ *is* quasi-open *if for all* $\varepsilon > 0$ *there exists an open set* Ω_ε *such that* $\mathrm{cap}\,(\Omega\Delta\Omega_\varepsilon) < \varepsilon$*.*

We call $u\colon \mathbb{R}^N \to \mathbb{R}$ quasi-continuous *if for all* $\varepsilon > 0$ *there exists an open set* ω_ε *such that* $\mathrm{cap}\,(\omega_\varepsilon) < \varepsilon$ *and the restriction of* u *to* $\mathbb{R}^N \setminus \omega_\varepsilon$ *is continuous.*

Every function $u \in H^1(\mathbb{R}^N)$ has a quasi-continuous representative, which is unique up to equality q.e. and can be defined in the following way:

$$\forall x \in \mathbb{R}^N, \qquad \tilde{u}(x) := \lim_{r\to 0} \fint_{B_r(x)} u(y)\, dy.$$

In general, we will consider always the quasi-continuous representative of H^1 functions and write u instead of $\tilde{u}$. A key relation between these concepts is that, given $u\colon \mathbb{R}^N \to \mathbb{R}$ quasi-continuous and $\alpha \in \mathbb{R}$, then the set $\{u > \alpha\}$ is quasi-open. In fact, by definition, there exists, for all $\varepsilon > 0$, an open set ω_ε, with $\operatorname{cap}(\omega_\varepsilon) \leq \varepsilon$, such that $u_{|\omega_\varepsilon^c}$ is continuous. In particular, there are open sets (Ω_ε) such that $\{u > \alpha\} \cap \omega_\varepsilon^c = \Omega_\varepsilon$, that is equivalent to say that $\{u > \alpha\} \cup \omega_\varepsilon = \Omega_\varepsilon \cup \omega_\varepsilon$, which is an open set for all $\varepsilon > 0$.
We can then say that superlevels of H^1 functions are quasi-open sets and this fact will be crucial in the existence Theorem by Buttazzo and Dal Maso. Moreover for each quasi-open set Ω there is a quasi-continuous function $u \in H^1(\mathbb{R}^N)$ such that $\Omega = \{u > 0\}$. It is clear that every open set is quasi-open, and obviously one can add to an open set some pieces with zero capacity and obtain a quasi-open set. But since quasi-open sets are defined up to sets with zero capacity this is not really a new set. For an example of a quasi-open set which is not equivalent to an open set see [39, Exercice 3.6].

In view of the above concepts, we can give a new definition of the space H^1_0, which is meaningful also for a measurable set $E \subset \mathbb{R}^N$,

$$H^1_0(E) := \left\{u \in H^1(\mathbb{R}^N) \;:\; u = 0 \text{ q.e. in } \mathbb{R}^N \setminus E\right\}. \tag{2.1}$$

The extension of the space H^1 to measurable sets is crucial, because, in order to obtain existence results, it is very often necessary to work not only with open sets. We summarize some important properties in the following lemma (a proof can be found in [39, Chapter 3]).

Lemma 2.5. *(1) For a generic open set Ω, $H^1_0(\Omega)$ coincide with the usual definition as closure of the smooth functions with compact support in Ω, that is $C^\infty_c(\Omega)$, with respect to the H^1 norm.*

(2) For every measurable set $E \subset \mathbb{R}^N$ there exists a quasi-open set Ω_E such that $H^1_0(E) = H^1_0(\Omega_E)$.

(3) From the properties above we can deduce that if Ω is a quasi-open set with positive capacity, then $H^1_0(\Omega) \neq \{0\}$, and hence $|\Omega| > 0$.

Another possible extension of Sobolev spaces to measurable sets is given by the notion of *Sobolev-like* space, which is employed mostly in Chapter 5. For any measurable set $E \subset \mathbb{R}^N$ we define

$$\tilde{H}^1_0(E) := \left\{u \in H^1(\mathbb{R}^N) \;:\; u = 0 \text{ a.e. in } \mathbb{R}^N \setminus E\right\}. \tag{2.2}$$

It is clear that the definition does not coincide in general with the one given in (2.1), not even for open set: for example one can consider a ball minus a hyperplane passing through its center.

It is always true the obvious inclusion $H^1_0(E) \subseteq \tilde{H}^1_0(E)$ and equality can be proved for open sets with Lipschitz boundary (see [34]). Moreover, for all measurable E, there exists always a quasi-open set $\Omega_E \subset E$ such that

$$H^1_0(\Omega_E) = \tilde{H}^1_0(E).$$

Since $\tilde{H}^1_0(E)$ is separable, it is sufficient to consider $\Omega_E := \bigcup_{n\in\mathbb{N}} \{u_n \neq 0\}$, where $\{u_n\}_{n\in\mathbb{N}}$ is a dense sequence in $\tilde{H}^1_0(E)$.

2.2 PDEs and eigenvalues of elliptic operators

First of all we deal with the eigenvalues of general operators defined on Hilbert spaces. We remind that, given a separable Hilbert space H with scalar product $(\cdot,\cdot)$ and a linear operator $R\colon H \to H$, we say that

- R is positive if $(Rx, x) \geq 0$ for all $x \in H$,
- R is self-adjoint if $(Rx, y) = (x, Ry)$ for all $x, y \in H$,
- R is compact if the image of a bounded set has compact closure in H.

We can summarize the main informations about eigenvalues in the following theorem (see, for example, [38, Chapter 1]).

Theorem 2.6. *Let H be a separable Hilbert space and $R\colon H \to H$ be a positive, self-adjoint and positive operator. Then there exists a nonincreasing sequence of positive eigenvalues converging to zero*

$$0 \leq \cdots \leq \Lambda_{k+1}(R) \leq \Lambda_k(R) \leq \cdots \leq \Lambda_1(R),$$

and a sequence of normalized eigenvectors $(x_k)_k$, which are a basis for H and satisfy:

$$Rx_k = \Lambda_k(R)x_k, \qquad \forall k \in \mathbb{N}.$$

Moreover the eigenvalues satisfy the so called Courant-Fisher and max-min formulas:

$$\Lambda_k(R) = \min_{\phi_1,\dots,\phi_{k-1}\in H} \left\{ \max_{\phi\in\langle\phi_1,\dots,\phi_{k-1}\rangle^\perp} \left\{ \frac{(R\phi,\phi)}{(\phi,\phi)} \right\} \right\}$$
$$\Lambda_k(R) = \max_{H_k} \left\{ \min_{\phi\in H_k,\,(\phi,\phi)=1} \{(R\phi, R\phi)\} \right\},$$

where the last maximum is over subspaces $H_k \subset H$ of dimension k.

We want to focus on a special class of operators, related to *capacitary measures.* A positive Borel measure is called capacitary if, for all measurable set E, $\operatorname{cap}(E) = 0$ implies $\mu(E) = 0$. First of all, given $\mu \in \mathcal{M}_0(\mathbb{R}^N)$, we define its *regular set* A_μ (see [20, Chapter 4]) as the union of all open sets $A \subset \mathbb{R}^N$ such that $\mu(A) < \infty$. If A_μ has finite measure, we define R_μ the *resolvent operator* associated to μ as:

$$R_\mu \colon L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \qquad R_\mu(f) = u,$$

where u is the solution of

$$\min_{v \in H^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |Dv|^2 + \int_{\mathbb{R}^N} v^2 \, d\mu - \int_{\mathbb{R}^N} vf \right\},$$

where we call $L^2_\mu(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \ : \ \int u^2 \, d\mu < \infty \right\}$. Since $|A_\mu| < \infty$, then $H^1_0(A_\mu)$ is compactly embedded in $L^2(A_\mu)$, hence R_μ is well defined, compact, positive and self-adjoint. We are then able to define the eigenvalues associated to the measure μ, that is, eigenvalues of the elliptic operator $-\Delta + \mu I$, as

$$\lambda_k(\mu) = \frac{1}{\Lambda_k(R_\mu)},$$

so they form a positive nondecreasing sequence diverging to infinity as $k \to \infty$. The Rayleigh formula can be now read as

$$\lambda_k(\mu) = \min_{E_k} \left\{ \max_{v \in E_k} \left\{ \frac{\int |Dv|^2 + \int v^2 \, d\mu}{\int v^2} \right\} \right\},$$

where the minimum is over the k-dimensional subspaces of $H^1(\mathbb{R}^N) \cap L^2_\mu(\mathbb{R}^N)$.

We are interested, in this Thesis, mostly in eigenvalues of Dirichlet Laplacian on a open (or quasi-open) subset of $\mathbb{R}^N$. In order to reduce the above machinery to this easier case, for every (quasi-)open set Ω of finite volume $|\Omega|$, we consider the measure

$$\mu_\Omega(E) = \begin{cases} 0, & \text{if } \operatorname{cap}(E \setminus \Omega) = 0, \\ +\infty, & \text{if } \operatorname{cap}(E \setminus \Omega) > 0, \end{cases} \tag{2.3}$$

and we define $\lambda_k(\Omega) := \lambda_k(\mu_\Omega)$, observing that $H^1(\mathbb{R}^N) \cap L^2_{\mu_\Omega}(\mathbb{R}^N) = H^1_0(\Omega)$. We remark that this coincide with the usual definition of the k^{th} eigenvalue of the Dirichlet Laplacian (counted with multiplicity) as the k^{th} element of the spectrum of the Dirichlet Laplacian, which is discrete since $|\Omega| < \infty$ (see [36, 38]). In order to stress this equivalence, first of all we recall few definitions about elliptic PDEs, which will also be used in the whole Thesis. Given $\Omega \subset \mathbb{R}^N$ a set of finite measure and a function $f \in L^2(\Omega)$, we say that $u \in H^1(\mathbb{R}^N)$ satisfies the equation

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ \quad u &\in H^1_0(\Omega), \end{cases}$$

if for every $v \in H^1_0(\Omega)$ we have $\langle \Delta u + f, v \rangle = 0$, where we set

$$\langle \Delta u + f, v \rangle := -\int_{\mathbb{R}^N} Du \cdot Dv + \int_{\mathbb{R}^N} fv.$$

With the definition above in mind, we can say that $\lambda_k(\Omega)$ is the k^{th} smaller number such that there exists a function $u_k \in H^1_0(\Omega)$ which satisfies

$$-\Delta u_k = \lambda_k(\Omega) u_k \qquad \text{in } \Omega,$$

and u_k is called *eigenfunction* corresponding to $\lambda_k(\Omega)$. For sake of simplicity we always consider the eigenfunctions with unit L^2 norm, which is clearly possible up to rescaling. In this case the min-max formula takes the form:

$$\lambda_k(\Omega) = \min_{\substack{E_k \subset H^1_0(\Omega),\\ \text{subspace of dimension } k}} \max_{v \in E_k \setminus \{0\}} \frac{||Dv||^2_{L(\Omega)}}{||v||^2_{L^2(\Omega)}}. \tag{2.4}$$

In particular, the minimum is achieved choosing E_k the space spanned by the first k eigenfunctions $\{u_1, \dots, u_k\}$ and the above ratio is called the *Rayleigh quotient*; we denote it by

$$\mathcal{R}(u, \Omega) := \frac{||Du||^2_{L(\Omega)}}{||u||^2_{L^2(\Omega)}}.$$

In the case of measures corresponding to a set we call the resolvent operator $R_\Omega := R_{\mu_\Omega}$ and we note that for all $f \in L^2(\mathbb{R}^N)$,

$$R_\Omega(f) = \arg\min\left\{\frac{1}{2}\int_{\mathbb{R}^N} |Du|^2 - \int_{\mathbb{R}^N} uf \ : \ u \in H^1_0(\Omega)\right\}.$$

We now list some important properties of eigenvalues of Dirichlet Laplacian, for the proofs one can refer to [20]. We remind that given $t > 0$ and a set Ω, we use the notation $t\Omega := \{tx \ : \ x \in \Omega\}$.

Lemma 2.7. *The following properties hold.*

(1) (Monotonicity) Given $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ (quasi-)open set with finite measure, if $\Omega_1 \subseteq \Omega_2$, then for all $k \in \mathbb{N}$, $\lambda_k(\Omega_2) \le \lambda_k(\Omega_1)$.

(2) (Scaling) Given $\Omega \subset \mathbb{R}^N$ a (quasi-)open set and $t > 0$, then for all $k \in \mathbb{N}$, $\lambda_k(t\Omega) = t^{-2}\lambda_k(\Omega)$.

(3) The first eigenfunction is strictly positive on the connected component on which is supported, and it is zero on all the other components, if any.

Remark 2.8. *Thanks to the scaling properties of eigenvalues the following minimum problems are equivalent (for every $k \in \mathbb{N}$):*

$$\min\{\lambda_k(\Omega),\ |\Omega| = 1\}, \qquad \min\{\lambda_k(\Omega),\ |\Omega| \le 1\}, \qquad \min\left\{|\Omega|^{2/N}\lambda_k(\Omega)\right\},$$

and we will use all the different formulations. It is worth noticing that in the last formulation we have no more bound on the measure, and the quantity $\Omega \mapsto |\Omega|^{2/N}\lambda_k(\Omega)$ is invariant under homothety.

Remark 2.9. *When Ω is disconnected, for example if it has two connected components Ω_1 and Ω_2, we obtain the eigenvalues of Ω by collecting and reordering the eigenvalues of each connected component*

$$\lambda_1(\Omega) = \min\{\lambda_1(\Omega_1); \lambda_1(\Omega_2)\}$$
$$\lambda_2(\Omega) = \min\Big\{\max\{\lambda_1(\Omega_1); \lambda_1(\Omega_2)\}; \lambda_2(\Omega_1); \lambda_2(\Omega_2)\Big\},$$

and so on. More generally we can always choose every eigenfunction of a disconnected open set to vanish on all but one connected component of Ω. In fact, given λ_k, $k \geq 1$, there exists a connected component $\omega \subset \Omega$ and an index $i \leq k$, such that $\lambda_k(\Omega) = \lambda_i(\omega)$. Hence we can choose u_k to be the eigenfunction linked to $\lambda_i(\omega)$, and we can extend it to zero on $\Omega \setminus \omega$. In particular, when Ω is made by two equal connected components, we will have $\lambda_1(\Omega) = \lambda_2(\Omega)$: the first eigenvalue is double.

Another important property is about the eigenfunctions of the Dirichlet Laplacian, which have the following bound in L^∞ (for a proof we refer to [33]):

$$\|u_k\|_{L^\infty} \leq e^{1/8\pi}\lambda_k(\Omega)^{N/4}, \tag{2.5}$$

for every $k \in \mathbb{N}$. This fact will be fundamental in Chapter 5, in order to use classical results about PDEs with bounded data.

We conclude the Section with few words about eigenvalues on Sobolev-like spaces. One can define the eigenvalues of the Dirichlet Laplacian on the linear subspace $\tilde{H}^1_0(\Omega) \subseteq H^1(\mathbb{R}^N)$. In general, given a closed linear subspace H of $H^1(\mathbb{R}^N)$, which is compactly embedded in $L^2(\mathbb{R}^N)$, one defines the spectrum of the Dirichlet Laplacian on H as $(\lambda_1(H), \dots, \lambda_k(H), \dots)$, where the k^{th} eigenvalue is

$$\lambda_k(H) := \min_{E_k} \max_{u \in E_k \setminus \{0\}} \frac{\int |Du|^2\,dx}{\int u^2\,dx}, \tag{2.6}$$

and the minimum ranges over all k-dimensional subspaces E_k of H.

Given a measurable set E with finite measure and $k \in \mathbb{N}$, then $\tilde{\lambda}_k(\Omega) = \lambda_k\left(\tilde{H}^1_0(E)\right)$ and there is a sequence of eigenfunctions $u_k \in \tilde{H}^1_0(E)$ orthonormal in L^2 and satisfying the equation

$$-\Delta u_k = \tilde{\lambda}_k(\Omega) u_k \qquad \text{in } \Omega.$$

It is then clear that $\lambda_k(\tilde{H}^1_0(E)) = \lambda_k(\Omega_E) \geq \lambda_k(E)$ for some quasi-open set $\Omega_E \subset E$. Hence, thanks to the monotonicity of eigenvalues w.r.t set inclusion, it is equivalent to study the minimization problem

$$\min\big\{F(\lambda_1(E), \dots, \lambda_k(E)) \ : \ E \subset \mathbb{R}^N,\ |E| \leq 1\big\},$$

in the class of quasi-open sets or in the family of measurable sets associated to $\tilde{H}^1_0$, up to suppose the functional F to be increasing in each variable. We will use this new definition of

λ_k in Chapter 5 for the following reason. If a set of finite measure E^* is minimal with respect to exterior perturbations (and this is what we need to study the regularity issue), that is,

$$F\Big(\lambda_1(\tilde{H}^1_0(E^*)),\dots,\lambda_k(\tilde{H}^1_0(E^*))\Big) \le F\Big(\lambda_1(\tilde{H}^1_0(E)),\dots,\lambda_k(\tilde{H}^1_0(E))\Big) + |E\setminus E^*|, \qquad \forall E\supset E^*,$$

then, for every $\varepsilon>0$, E^* is the *unique* solution of

$$\min\Big\{F\Big(\lambda_1(\tilde{H}^1_0(E)),\dots,\lambda_k(\tilde{H}^1_0(E))\Big) + (1+\varepsilon)|E| \;:\; E^*\subset E\subset\mathbb{R}^N\Big\}. \tag{2.7}$$

In fact, if E^{**} is another solution of (2.7), then $|E^*\Delta E^{**}|=0$, and so $\tilde{H}^1_0(E^*)=\tilde{H}^1_0(E^{**})$.

Remark 2.10. *In the Thesis we focus only on Dirichlet boundary condition for the Laplacian, but there are other common choices of boundary conditions that lead to completely different problems (for more references one can look at the books [39, 20, 38]). One key property of the Dirichlet boundary conditions is the monotonicity of eigenvalues with respect to inclusion, which is false for the other conditions. In particular, a real number μ and a function $u\in H^1(\mathbb{R}^N)$ are an eigenvalue and the corresponding eigenfunction of the Neumann Laplacian if they solve (in the weak formulation)*

$$\begin{cases} -\Delta u &= \mu u, \qquad \text{in } \Omega,\\ \frac{\partial u}{\partial\nu} &= 0, \qquad \text{on } \partial\Omega.\end{cases}$$

More in general, given $a,b\in[0,1]$, the so called Robin boundary conditions reads as:

$$\begin{cases} -\Delta u &= \rho u, \qquad \text{in } \Omega,\\ au + b\frac{\partial u}{\partial\nu} &= 0, \qquad \text{on } \partial\Omega.\end{cases}$$

We observe that Robin conditions includes the others: for $a=1,\ b=0$ we find again the Dirichlet boundary conditions, while for $a=0,\ b=1$ we obtain the Neumann one.

2.3 Extremum problems and bounds for eigenvalues of the Dirichlet Laplacian

In this Section we consider classical minimum problems for eigenvalues of the Dirichlet Laplacian. More precisely, one looks for a set (a "shape") that minimize a single eigenvalue or a function of eigenvalues. Due to the monotonicity, a constraint on the admissible sets is needed, otherwise the minimization is not interesting. The most studied case is the one of volume constraint, but in the last years also the perimeter constraint was investigated (see [34]). This kind of shape optimization problem, in a general situation, can be written as

$$\min\big\{F(\lambda_1(\Omega),\dots,\lambda_k(\Omega)) \;:\; \Omega\subset\mathbb{R}^N,\ \text{(quasi-)open},\ |\Omega|=1\big\}, \tag{2.8}$$

and here we treat the few cases in which it is possible to find an explicit solution, considering first the minimization of single eigenvalues, that is, $F(\lambda_1,\dots,\lambda_k)=\lambda_k$, $k\in\mathbb{N}$.

The minimization of λ_1 was the first problem to be studied: Lord Rayleigh conjectured the ball to be a minimizer in [48], and in the 1920s Faber and Krahn eventually proved his conjecture to be true (see [35, 43, 44]), using spherical decreasing rearrangements. The so called Faber-Krahn inequality states, in a scale invariant form,

$$\lambda_1(\Omega) \geq \lambda_1(B) \left(\frac{|B|}{|\Omega|}\right)^{2/N}, \qquad \text{for all open sets of finite measure } \Omega \subset \mathbb{R}^N, \tag{2.9}$$

where B is the ball of unit measure in $\mathbb{R}^N$, and with equality if and only if Ω is any ball (up to sets of capacity zero). Analogously, as a minimum problem:

$$\lambda_1(B) = \min\left\{\lambda_1(\Omega), \,:\, \Omega \subset \mathbb{R}^N, \text{ open}, |\Omega| \leq 1\right\}.$$

The minimization of λ_2 was implicitly solved in Krahn's works (see [43, 44]) and then rediscovered independently by Hong [40] and Szegö [49] in the 1950s. In this case, studying the nodal sets of the first two eigenfunctions, one can prove two disjoint equal balls to be optimal. More precisely, in a scale invariant form, the (Hong-)Krahn-Szegö inequality asserts that

$$\lambda_2(\Omega) \geq 2^{2/N}\lambda_1(B) \left(\frac{|B|}{|\Omega|}\right)^{2/N}, \qquad \text{for all open sets of finite measure } \Omega \subset \mathbb{R}^N, \tag{2.10}$$

with equality if and only if Ω is any disjoint union of two balls of equal measure. Equivalently, it is

$$\lambda_2(\Theta) = \min\left\{\lambda_2(\Omega), \,:\, \Omega \subset \mathbb{R}^N, \text{ open}, |\Omega| \leq 1\right\},$$

where we denote the union of two disjoint balls each of half measure by Θ.

Unfortunately explicit minimizers for λ_k, $k \geq 3$, are not known and there is numerical evidence, at least for λ_3, that the optimal set should not be the same set in all dimensions (see [4, 11]). For the interested reader we recall here some major conjectures about optimal sets for single eigenvalues.

(1) The ball is optimal for λ_3 in two dimensions,

(2) Three disjoint balls of equal volume are optimal for λ_3 in dimension $N \geq 4$,

(3) The optimal set for λ_4 in two dimensions is made by two disjoint balls B_1, B_2 such that $\lambda_3(B_1) = \lambda_1(B_2)$,

(4) The ball of $\mathbb{R}^N$ is optimal for λ_{N+1} for all $N \geq 3$.

It is worth noticing that minimizers for single eigenvalues are not always balls or union of balls: Wolf and Keller proved in [53] that, for the minimization of λ_{13}, the optimal union of rectangles is a better candidate than the optimal union of balls in $\mathbb{R}^2$. Moreover Berger and Oudet [11], in a very recent paper, proved that for $k \geq 5$ the optimal set for λ_k in two dimensions is never a union of balls.

The only other functionals for which the optimal sets are known are λ_1/λ_2 and λ_2/λ_3, thanks to the works by Ashbaugh and Benguria [7, 8], who solved a conjecture by Payne, Pólya and Weinberger [47]. We present them in their original form as maximization results.

Theorem 2.11. *The ball maximizes the ratio λ_2/λ_1, that is:*

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B)}{\lambda_1(B)} \qquad \text{for all open sets of finite measure } \Omega \subset \mathbb{R}^N. \tag{2.11}$$

Moreover two equal disjoint balls maximize the ratio λ_3/λ_2, hence

$$\frac{\lambda_3(\Omega)}{\lambda_2(\Omega)} \leq \frac{\lambda_3(\Theta)}{\lambda_2(\Theta)} = \frac{\lambda_2(B)}{\lambda_1(B)} \qquad \text{for all open sets of finite measure } \Omega \subset \mathbb{R}^N. \tag{2.12}$$

In other words, the ball minimize λ_1/λ_2 and two equal disjoint balls minimize λ_2/λ_3.

At last we present an easy and well-known inequality that can be seen as an extension (even if rougher) of the results by Ashbaugh and Benguria: in particular we have that the functional λ_k/λ_1 is bounded for all $k \in \mathbb{N}$. This will be very useful in Chapter 3. We present here a simple new proof given in [M4, Appendix], while another proof can be found in [6].

Theorem 2.12. *There exists a constant $M = M(k, N)$ such that for every (quasi-)open set one has*

$$\frac{\lambda_k(\Omega)}{\lambda_1(\Omega)} \leq M.$$

In order to perform our proof, we need to fix some notations, which we will use also throughout Chapter 3. First of all a generic point of $\mathbb{R}^N$ will be denoted by $z \equiv (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, or sometimes as $z \equiv (z_1, z_2, \dots, z_N)$, while a generic open set will be $\Omega \subseteq \mathbb{R}^N$.

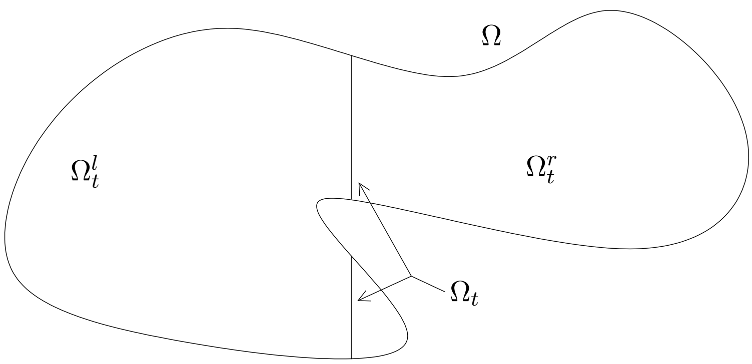

Figure 2.1: A set Ω and the corresponding sets Ω_t^l, Ω_t^r and Ω_t.

For any $t \in \mathbb{R}$, we will define

$$\Omega_t^l := \Big\{(x, y) \in \Omega : x < t\Big\}, \quad \Omega_t := \Big\{y \in \mathbb{R}^{N-1} : (t, y) \in \Omega\Big\}, \quad \Omega_t^r := \Big\{(x, y) \in \Omega : x > t\Big\};$$

notice that Ω_t^l and Ω_t^r are subsets of $\mathbb{R}^N$, while Ω_t is a subset of $\mathbb{R}^{N-1}$. Figure 2.1 shows an example of a generic set Ω with Ω_t^l, Ω_t^r and Ω_t. On the other hand, given $0 \le m \le |\Omega|$ and $0 \le m_1 \le m_2 \le |\Omega|$, we define the level $\tau(\Omega, m) \in \overline{\mathbb{R}}$ and the width $W(\Omega, m_1, m_2)$ as

$$\tau(\Omega, m) := \inf \left\{ t \in \mathbb{R} : \left|\Omega_t^l\right| \ge m \right\}, \qquad W(\Omega, m_1, m_2) := \tau(\Omega, m_2) - \tau(\Omega, m_1)\,.$$

Observe that one surely has $-\infty < \tau(\Omega, m) < +\infty$ whenever $0 < m < |\Omega|$, as well as $W(\Omega, m_1, m_2) < +\infty$ if $0 < m_1 \le m_2 < |\Omega|$.
Finally, given any set $\Omega \subseteq \mathbb{R}^N$, we define its 1-dimensional projections for $1 \le p \le N$ as

$$\pi_p(\Omega) := \left\{ t \in \mathbb{R} : \exists\,(z_1,\, z_2,\, \dots\,,\, z_N) \in \Omega,\; z_p = t \right\}.$$

For the ease of presentation, we will begin with a couple of technical lemmas, then pass to the proof of the Theorem. The first simple step of our construction states that functions with bounded Rayleigh quotients cannot concentrate too much on small regions.

Lemma 2.13. *For every $m \in (0, 1]$ and $K > 0$ there exists $\rho = \rho(m, K, N) > 0$ such that the following holds. Let $u \in H^1(\mathbb{R}^N)$ with*

$$\int_{\mathbb{R}^N} u^2 = 1\,, \qquad \int_{\mathbb{R}^N} |Du|^2 \le K\,.$$

Then for every cube $Q \subseteq \mathbb{R}^N$ with half-side ρ one has

$$\int_Q u^2 \le m\,.$$

Proof. Suppose that the claim is not true. Then there exists a sequence $\{u_n\} \subseteq H^1(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} u_n^2 = 1\,, \qquad \int_{\mathbb{R}^N} |Du_n|^2 \le K\,, \qquad \int_{Q_{1/n}} u_n^2 \ge m\,, \tag{2.13}$$

being $Q_r = [-r, r]^N$ the cube of half-side r centered at the origin. By definition, this sequence is bounded in $H^1(\mathbb{R}^N)$, hence up to a subsequence we have that u_n weakly converges to some function $u \in H^1(\mathbb{R}^N)$. In particular, for any $\varepsilon > 0$, u_n strongly converges to u in $L^2(Q_\varepsilon)$, so that, thanks to (2.13), one has $\int_{Q_\varepsilon} u^2 \ge m$. Since this is absurd, the claim follows. □

The second lemma, which is the core of our proof of Theorem 2.12, ensures that every set with bounded first eigenvalue can be split into two subregions, each of them having first eigenvalue not too large.

Lemma 2.14. *For every $K > 0$ there exists $K' = K'(K, N)$ such that, if Ω is an open subset of $\mathbb{R}^N$ with $\lambda_1(\Omega) \le K$, then there are two disjoint open subsets Ω_1, Ω_2 of Ω with $\lambda_1(\Omega_i) \le K'$ for $i = 1, 2$.*

Proof. We start applying Lemma 2.13 with K and with $m = 1/2$, thus getting a positive number ρ. Let then $\Omega \subseteq \mathbb{R}^N$ be an open set with $\lambda_1(\Omega) \leq K$, and let $u \in H^1_0(\Omega)$ be a first eigenfunction of Ω with unit L^2 norm. Extending u by 0 outside Ω, we have then by definition

$$\int_{\mathbb{R}^N} u^2 = \int_\Omega u^2 = 1\,, \qquad\qquad \int_{\mathbb{R}^N} |Du|^2 = \int_\Omega |Du|^2 \leq K\,. \tag{2.14}$$

Let now $t^- < t^+$ be identified by

$$\int_{\Omega^l_{t^-}} u^2 = \int_{\Omega^r_{t^+}} u^2 = \frac{1}{4N}\,. \tag{2.15}$$

We claim that it is possible to assume

$$t^+ - t^- \geq 2\rho\,. \tag{2.16}$$

In fact, if it is not so, this means that there is a vertical stripe of width 2ρ out of which the squared L^2 norm of u is less than $1/(2N)$ (by "vertical" we mean orthogonal to e_1). If this happens for every direction e_1, e_2, … , e_N, the intersection of the corresponding stripes is a square of half-side ρ out of which the squared L^2 norm of u is less than $1/2$. Since this is in contradiction with Lemma 2.13, we obtain the validity of (2.16), up to a rotation.

Let us now call $t = (t^+ + t^-)/2$, define $\Omega_1 = \Omega^l_t$ and $\Omega_2 = \Omega^r_t$, and let $\widetilde{u} \in H^1(\Omega_1)$ be defined as

$$\widetilde{u}(x,y) := \begin{cases} u(x,y) & \text{for } x \leq t-\rho\,,\\ \dfrac{t-x}{\rho}\,u(x,y) & \text{for } t-\rho \leq x \leq t\,. \end{cases}$$

Since $u \in H^1_0(\Omega)$, it is clear that $\widetilde{u} \in H^1_0(\Omega_1)$. Moreover, writing $Du = (D_1 u,\, D_y u)$, one has

$$D\widetilde{u}(x,y) = \left(\frac{t-x}{\rho}\,D_1u(x,y) - \frac{1}{\rho}\,u(x,y), \frac{t-x}{\rho}\,D_yu(x,y)\right)$$

for every $(x,y) \in \Omega_1$ with $x \geq t-\rho$. As a consequence, minding (2.14) one gets

$$\int_{\Omega_1} |D\widetilde{u}|^2 \leq 2\int_{\Omega_1} |Du|^2 + \frac{2}{\rho^2}\int_{\Omega_1} u^2 \leq 2K + \frac{2}{\rho^2}\,. \tag{2.17}$$

On the other hand, recalling (2.16) and (2.15) it is

$$\int_{\Omega_1} \widetilde{u}^2 \geq \int_{\Omega^l_{t^-}} u^2 = \frac{1}{4N}\,. \tag{2.18}$$

Putting together (2.17) and (2.18) one immediately obtains

$$\lambda_1(\Omega_1) \leq \mathcal{R}(\widetilde{u},\Omega_1) = \frac{\displaystyle\int_{\Omega_1} |D\widetilde{u}|^2}{\displaystyle\int_{\Omega_1} \widetilde{u}^2} \leq 8N\left(K + \frac{1}{\rho^2}\right).$$

Finally, we can set $K' = 8N\bigl(K + 1/\rho^2\bigr)$: since we have shown that $\lambda_1(\Omega_1) \leq K'$, and since by symmetry it is also $\lambda_1(\Omega_2) \leq K'$, the thesis follows. □

We are now in position to prove a first boundedness result of λ_k in terms of λ_1, from which Theorem 2.12 will then readily follow.

Lemma 2.15. *For every $K>0$ there exists $M'=M'(k,K,N)>0$ such that, for all open sets $\Omega\subseteq\mathbb{R}^N$, if $\lambda_1(\Omega)\le K$ then $\lambda_k(\Omega)\le M'$.*

Proof. Let us start by setting $K_1=K$, and then, applying Lemma 2.14, we let recursively $K_{l+1}=K'(K_l,N)$ for every $l\ge 1$. Finally, we define $M'=K_{j+1}$, where j is the smallest natural number such that $2^j\ge k$. We will show the claim of the Theorem with such constant M'.

To do so, we pick any open set Ω with $\lambda_1(\Omega)\le K=K_1$. Applying Lemma 2.14 to Ω with constant K_1, we find two disjoint open sets Ω_1, $\Omega_2\subseteq\Omega$ with $\lambda_1(\Omega_i)\le K'(K_1,N)=K_2$ for $i=1,\,2$. Then, we can apply Lemma 2.14 to Ω_1 and Ω_2 with constant K_2, finding four disjoint subsets Ω_{11}, Ω_{12}, Ω_{21}, Ω_{22} of Ω, each of them with first eigenvalue smaller than K_3. Continuing with the obvious induction, we end up with 2^j disjoint open subsets of Ω, say Ω^i for $1\le i\le 2^j$, having $\lambda_1(\Omega^i)\le K_{j+1}=M'$ for each i.

To conclude the thesis, it is thus enough to show that

$$\lambda_k(\Omega)\le\lambda_{2^j}(\Omega)\le\max\Big\{\lambda_1(\Omega^i)\,:\,1\le i\le 2^j\Big\}\le M'\,,\tag{2.19}$$

and in fact only the second inequality is to be shown, being the first and the last true by construction.

To get (2.19), for every $1\le i\le 2^j$ let u_i be a first eigenfunction of Ω^i, again extended by 0 on $\Omega\setminus\Omega^i$, so that

$$\int_\Omega u_i^2=\int_{\Omega^i}u_i^2=1\,,\qquad\qquad \int_\Omega|Du_i|^2=\int_{\Omega^i}|Du_i|^2=\lambda_1(\Omega^i)\le M'\,,$$

and then $\mathcal{R}(u_i,\Omega)\le M'$. Observe that the functions u_i are mutually orthogonal (both in the L^2 and in the H^1 sense) by construction, since they are supported on disjoint sets. Hence, the linear subspace E_{2^j} of $H^1_0(\Omega)$ spanned by the functions u_i for $1\le i\le 2^j$ is 2^j-dimensional. Thanks to the min-max formula (2.4), to prove (2.19) it is enough to show that $\mathcal{R}(w)\le\max\big\{\lambda_1(\Omega^i)\,:\,1\le i\le 2^j\big\}$ for every $w\in E_{2^j}$. And in fact, writing the generic function $w\in E_{2^j}$ as $w=\sum\beta_i u_i$, by the orthogonality of the different u_i one has clearly

$$\begin{aligned}\mathcal{R}(w,\Omega)&=\frac{\displaystyle\int_\Omega\Big|\sum\beta_i Du_i\Big|^2}{\displaystyle\int_\Omega\Big(\sum\beta_i u_i\Big)^2}=\frac{\displaystyle\sum\beta_i^2\int_\Omega|Du_i|^2}{\displaystyle\sum\beta_i^2\int_\Omega u_i^2}=\frac{\displaystyle\sum\beta_i^2\,\mathcal{R}(u_i,\Omega)\int_\Omega u_i^2}{\displaystyle\sum\beta_i^2\int_\Omega u_i^2}\\&=\frac{\displaystyle\sum\beta_i^2\,\lambda_1(\Omega^i)\int_\Omega u_i^2}{\displaystyle\sum\beta_i^2\int_\Omega u_i^2}\le\max\Big\{\lambda_1(\Omega^i)\,:\,1\le i\le 2^j\Big\}\,.\end{aligned}$$

As noticed before, this gives the validity of (2.19), hence the proof is concluded. □

To obtain Theorem 2.12, we now only need a trivial rescaling argument.

Proof of Theorem 2.12. First of all notice that, by density, it is admissible to consider only the case of the open sets. We apply Lemma 2.15 with $K = 1$, so defining $M := M'(k, 1, N)$. We will prove Theorem 2.12 with such M. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and apply the rescaling formula (property (2) of Lemma 2.7) choosing $\alpha = \lambda_1(\Omega)^{\frac{1}{2}}$, thus getting $\lambda_1(\alpha\,\Omega) = 1$. By Lemma 2.15, we derive $\lambda_k(\alpha\Omega) \leq M$, and then by scaling again we find $\lambda_k(\Omega) = \alpha^2\lambda_k(\alpha\,\Omega) \leq M\lambda_1(\Omega)$, thus the proof is concluded. □

2.4 γ-convergence and existence in a bounded box

With the present Section we begin to treat the existence theory for spectral shape optimization problems (i.e. having in mind (2.8)), which is a natural topic of interest since only in very special case it is known an explicit solution. The first fundamental concept is the one of γ-convergence, proposed by Dal Maso and Mosco [31, 32], which turns out to be a suitable notion of convergence for applying the direct method of the Calculus of Variations to this kind of problems.

We first briefly define the γ-convergence for the capacitary measures $\mathcal{M}_0(D)$, where D is an open bounded box fixed a priori, in the following way:

$$\mu_n \xrightarrow{\gamma} \mu, \qquad \text{if } R_{\mu_n}(1) \to R_\mu(1), \quad \text{in } H^1_0(D).$$

It is possible to prove (see [20, Chapter 3 and 4]) that $\mathcal{M}_0(D)$ with the topology of the γ-convergence is a compact metric space and the class of measures corresponding to sets (of the form μ_Ω) is dense in $\mathcal{M}_0(D)$.

We focus now on the case of domains of $\mathbb{R}^N$, which is our main point of interest. Given a bounded open box $D \subset \mathbb{R}^N$, we can consider the resolvent operator $R_\Omega \colon L^2(D) \to L^2(D)$ for some $\Omega \subset D$, and choose $f = 1 \in L^2(D)$. $R_\Omega(1) =: w_\Omega$ is called *torsion function* and in particular it is the (weak) solution of

$$\begin{cases} -\Delta w &= 1 \quad \text{in } \Omega, \\ \quad w &\in H^1_0(\Omega), \end{cases}$$

and hence the unique minimizer for the so called *torsion energy functional*

$$E(\Omega) := \min_{u \in H^1_0(\Omega)} \left\{ \frac{1}{2} \int_D |Du|^2 - \int_D u \right\} = \frac{1}{2} \int_D |Dw_\Omega|^2 - \int_D w_\Omega = -\frac{1}{2} \int_D w_\Omega.$$

We are now in position to give the following.

Definition 2.16. *Given a sequence of quasi-open sets contained in D, $(\Omega_n)_{n\in\mathbb{N}}$, we say that Ω_n* γ-converge *to a quasi-open set $\Omega \subset D$, as $n \to \infty$, when $w_{\Omega_n} \rightharpoonup w_\Omega$ in $H^1_0(D)$.*

Moreover Dal Maso and Mosco proved (see [31, 32]) that the convergence above implies, for all $f \in L^2(D)$, $R_{\Omega_n}(f) \to R_\Omega(f)$ in $L^2(D)$, hence also $R_{\Omega_n} \to R_\Omega$ in $\mathcal{L}(L^2(D))$ and the full spectrum converges. Thus eigenvalues of the Dirichlet Laplacian are continuous with respect to γ-convergence.

Unfortunately the γ-convergence is not compact in the class of quasi-open sets. Cioranescu and Murat in [28] built a well regarded example of a sequence of open sets γ-converging to an element of $\mathcal{M}_0(D)$, which is *not* a quasi-open set. It is then necessary, in order to prove an existence result for problems like (1.3), to introduce the so called *weak γ-convergence.*

Definition 2.17. *A sequence of quasi-open sets contained in D, $(\Omega_n)_{n\in\mathbb{N}}$ is said to* weak γ-converge *to a quasi-open domain $\Omega \subset D$ if $w_{\Omega_n} \rightharpoonup w$ in $H_0^1(D)$ as $n \to \infty$, with $\Omega := \{w > 0\}$.*

Note that w coincide with $w_\Omega = R_\Omega(1)$ if and only if the convergence is γ and not only weak γ. More precisely, for some capacitary measure μ, $w = R_\mu(1)$, since the γ-convergence is compact in the class of capacitary measures. From this characterization it is not difficult to see that $w \le w_\Omega$; hence if Ω_n weak γ-converge to Ω and $\Omega_n \subset \Omega$ for all n, then the convergence is actually γ.
Since superlevels of H^1 functions are in general only quasi-open sets, it should be now clear the importance of this class of sets in order to obtain an existence result for minimum problems involving eigenvalues of Dirichlet Laplacian.

A first good property of the (weak) γ-convergence is that it behaves well with respect to the Lebesgue measure.

Remark 2.18. *If Ω_n converges in measure to Ω, that is $|\Omega_n \Delta \Omega| \to 0$, then there exists a subsequence that γ-converges to Ω. On the other hand if Ω_n weak γ-converges to Ω, then we have the following l.s.c. with respect to the Lebesgue measure:*

$$|\Omega| \le \liminf_{n\to\infty} |\Omega_n|.$$

It is immediate from the definition and Remark 2.18 that the weak γ-convergence is compact in the class

$$\mathcal{A}(D) := \{\Omega \subset D, \text{ quasi-open}, \ |\Omega| = 1\},$$

so it seems a good candidate for applying the direct method of the Calculus of Variations in order to prove the following fundamental existence result by Buttazzo and Dal Maso. We recall here a general version of the Direct Method for completeness.

Theorem 2.19 (Direct Method)**.** *Let (X, τ) be a topological space, $J\colon X \to \overline{\mathbb{R}}$ a functional τ-l.s.c. and such that its sublevels are τ-sequentially relatively compact. Then J admits a minimum point.*

Theorem 2.20 (Buttazzo–Dal Maso). *Let $D \subset \mathbb{R}^N$ be a bounded and open set and $F\colon \mathbb{R}^k \to \mathbb{R}$ be a functional increasing in each variable and lower semicontinuous (l.s.c.). Then there exists a minimizer for the problem*

$$\min\{F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \ : \ \Omega \subset D, \ \textit{quasi-open}, \ |\Omega| \le 1\}.$$

Since the compactness of weak γ-convergence was already discussed, the other fundamental point in the proof is to show that, at least for the class of functionals non decreasing with respect to set inclusion, the weak γ-convergence has also good (semi)continuity properties.

Proposition 2.21. *A functional $J\colon \mathcal{A}(D) \to \mathbb{R}$ non decreasing with respect to set inclusion is γ l.s.c if and only if it is weak γ l.s.c..*

We remind that eigenvalues of Dirichlet Laplacian are non decreasing with respect to set inclusion and so are increasing functionals depending on them, as in the hypothesis of Theorem 2.20.
The proof of Proposition 2.21 is based on the following nontrivial claims, whose proofs make also use of the maximum principle for the Dirichlet Laplacian.

a) If w_{Ω_n} converge weakly in $H^1_0(D)$ to w and $v_N \in H^1_0(\Omega_n)$ converge weakly in $H^1_0(D)$ to v, then $v \in H^1_0(\{w > 0\})$.

b) Let $\Omega_n \subset D$ be quasi-open sets such that w_{Ω_n} converge weakly in $H^1_0(D)$ to $w \in H^1_0(\Omega)$ for some quasi-open set $\Omega \subset D$. Then there exist a subsequence (not relabeled) and a sequence of quasi-open sets $\widetilde{\Omega}_n$ that γ-converge to Ω with $\Omega_n \subset \widetilde{\Omega}_n \subset D$.

Then the Buttazzo and Dal Maso Theorem follows easily from Proposition 2.21 using the direct method of the Calculus of Variations. Given a minimizing sequence (Ω_n) of quasi-open sets for problem (1.3), by the compactness of the weak γ-convergence we can extract a subsequence (not relabeled) that weak γ-converges to a quasi-open set $\Omega \in \mathcal{A}(D)$. Using Remark 2.18 and the continuity of eigenvalues with respect to γ-convergence together with Proposition 2.21, we have that

$$|\Omega| \le \liminf_{n\to\infty} |\Omega_n| \le 1, \qquad F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) \le \liminf_{n\to\infty} F(\lambda_1(\Omega_n), \dots, \lambda_k(\Omega_n)),$$

thus Ω is an optimal set for (1.3).
This kind of approach for proving an existence result for monotone functionals is reformulated in an abstract setting in [20, Section 5.2] and in [15].

Remark 2.22. *The need of a bounded open box D in the statement of Buttazzo-Dal Maso Theorem is only to ensure that the embedding $H^1_0(D) \hookrightarrow L^2(D)$ is compact. Actually it is sufficient to ask that D is an open set of finite measure in order to get the above embedding (see [19]), so the result holds also with this weaker hypothesis.*

2.5 Concentration compactness and subsolutions

With the result by Buttazzo and Dal Maso (Theorem 2.20), looking for minimizers among sets inside a bounded box is a well understood topic for a large class of functionals. A first possible step in order to study the minimization for generic (quasi-)open sets in $\mathbb{R}^N$ is to study the concentration-compactness principle by P.L. Lions (see [45]), which tries to focus on "how" the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ can be non compact. In the case of subsets of $\mathbb{R}^N$ Bucur (see [18]) rearranged the principle in the following way, ruling out the *vanishing* case.

Theorem 2.23. *Let $\Omega_n \subset \mathbb{R}^N$ be quasi-open sets with $|\Omega_n| \leq 1$ for all $n \geq 1$. Then there exists a subsequence (not relabeled) such that one of the following situations occur:*

1) **Compactness.** *There exist $y_n \in \mathbb{R}^N$ and a capacitary measure μ such that $R_{y_n+\Omega_n} \to R_\mu$ in $\mathcal{L}(L^2(\mathbb{R}^N))$.*

2) **Dichotomy.** *There exist Ω_n^i, $i = 1, 2$ such that $|\Omega_n^i| > 0$, $d(\Omega_n^1, \Omega_n^2) \to \infty$ and $R_{\Omega_n} \to R_{\Omega_n^1 \cup \Omega_n^2}$ in $\mathcal{L}(L^2(\mathbb{R}^N))$ as $n \to \infty$.*

Thanks to the concentration compactness argument above, Bucur and Henrot in 2000 proved the following existence result for unbounded domains (see [23]), requiring a very strong hypothesis on the optimal sets for the lower order eigenvalues.

Theorem 2.24. *For $k \geq 2$ if there exists a bounded minimizer for $\lambda_1, \ldots, \lambda_k$ in the class $\mathcal{A}(\mathbb{R}^N)$, then there exists at least a minimizer for λ_{k+1} in $\mathcal{A}(\mathbb{R}^N)$.*

In particular this provides existence of a solution for the problem:

$$\min\left\{\lambda_3(\Omega) \;:\; \Omega \subset \mathbb{R}^N,\ \text{open},\ |\Omega| = 1\right\},$$

since the minimizers for λ_1 and λ_2 are respectively a ball and two balls, which are bounded. The idea of the proof is quite simple. Given a minimizing sequence made of bounded sets Ω_n for λ_{k+1} in $\mathcal{A}(\mathbb{R}^N)$, if compactness occur, existence follows from Theorem 2.20 by Buttazzo and Dal Maso. On the other hand, if dichotomy happens, then $\Omega_n^1 \cup \Omega_n^2$ is also a minimizing sequence. But it is thus possible to see that the sequence Ω_n^i must be minimizing for some lower eigenvalue in the class $\mathcal{A}(\mathbb{R}^N)$, with different measure constraints: $c_1, c_2 > 0$ such that $c_1 + c_2 \leq 1$. Hence, up to translations, a minimizer for λ_{k+1} will be the union of the two minimizers corresponding to some lower eigenvalues. Note that if we do not know that there exists a bounded minimizer for every lower eigenvalue, it is not possible to consider the union of two of them, since in principle one can be dense in $\mathbb{R}^N$.

Since not even the boundedness of the minimizers for λ_3 was known, Dorin Bucur studied the link between this kind of shape optimization problems and free boundary problems. In literature (see [2, 14]), the regularity of free boundaries is well understood only for energy-like minimizers. Bucur develops the notion of *shape subsolution* for the energy functional, in order to relate the minimization of λ_k with the regularity of free boundaries.

We need to endow the family of measurable sets of finite measure $\mathcal{S}$ with a distance induced by γ-convergence:

$$d_\gamma(A, B) := \int_{\mathbb{R}^N} |w_A - w_B|, \qquad A, B \in \mathcal{S}$$

where we considered the torsion functions $w_A, w_B \in H^1(\mathbb{R}^N)$ with the obvious zero extension.

Definition 2.25. *We say that a set $A \in \mathcal{S}$ is a* local shape subsolution *for a functional $\mathcal{F} \colon \mathcal{S} \to \mathbb{R}$ if there exist $\delta > 0$ and $\Lambda > 0$ such that*

$$\mathcal{F}(A) + \Lambda|A| \leq \mathcal{F}(\tilde{A}) + \Lambda|\tilde{A}|, \qquad \forall\, \tilde{A} \subset A,\ d_\gamma(A, \tilde{A}) < \delta.$$

Roughly speaking, working with shape subsolutions means that only inner perturbations are allowed. Bucur (see [16]) proved a very powerful regularity result for shape subsolution of the torsion energy functional E.

Lemma 2.26. *Let A be a local shape subsolution (with constants δ, Λ) for the torsion energy E. Then it is bounded, with $diam(A) \leq C(|A|, \delta, \Lambda)$, has finite perimeter and its fine interior has the same measure of A.*

The proof of the lemma for the finite perimeter part is based on controlling the term $\int_{\{0 \leq w_A \leq \varepsilon\}} |Dw_A|^2$, while the boundedness and the inner density estimate come from the following Alt-Caffarelli type estimate: there exist $r_0, C_0 > 0$ such that for all $r \leq r_0$

$$\sup_{B_{2r}(x)} w_A \leq C_0 r \qquad \text{implies} \qquad u = 0 \text{ in } B_r(x).$$

The next key point in Bucur's argument consists in linking the minimizers of eigenvalues of Dirichlet Laplacian with shape subsolution of the energy. We consider the minimization problem, equivalent to (1.4) for some $\Lambda > 0$ sufficiently small (see [39] for more details),

$$\min\left\{F(\lambda_1(A), \ldots, \lambda_k(A)) + \Lambda|A| \ :\ A \subset \mathbb{R}^N,\ \text{quasi-open}\right\}, \tag{2.20}$$

for a functional $F \colon \mathbb{R}^k \to \mathbb{R}$ which satisfies the following Lipschitz-like condition for some positive α_i, $i = 1, \ldots, k$:

$$F(x_1, \ldots, x_k) - F(y_1, \ldots, y_k) \leq \sum_{i=1}^{k} \alpha_i (x_i - y_i), \qquad \forall x_i \geq y_i,\ i = 1, \ldots, k. \tag{2.21}$$

Theorem 2.27. *Assume that A is a solution of* (2.20)*, then it is a local shape subsolution for the energy problem.*

The proof is based on [17, Theorem 3.4], which assures, for all $k \in \mathbb{N}$, the existence of a constant $c_k(A)$ such that:

$$\left| \frac{1}{\lambda_k(\tilde{A})} - \frac{1}{\lambda_k(A)} \right| \leq c_k(A) d_\gamma(A, \tilde{A}).$$

Then, up to choose δ small enough and $\tilde{A} \subseteq A$ with $d_\gamma(\tilde{A}, A) < \delta$, it follows

$$\begin{aligned}\Lambda(|A| - |\tilde{A}|) &\leq F(\lambda_1(\tilde{A}), \ldots, \lambda_k(\tilde{A})) - F(\lambda_1(A), \ldots, \lambda_k(A)) \leq \sum_i \alpha_i(\lambda_i(\tilde{A}) - \lambda_i(A)) \\ &\leq \sum_i \alpha_i c_i'(E(\tilde{A}) - E(A)) \leq \widetilde{\Lambda}^{-1}(E(\tilde{A}) - E(A)),\end{aligned}$$

with a constant $\widetilde{\Lambda}$ depending on $c_i' = c_i'(A, \delta, i)$ and α_i, for $i = 1, \ldots, k$.

Now a straightforward application of Lemma 2.24 gives the main existence result.

Theorem 2.28. *If the functional F satisfies the Lipschitz-like condition* (2.21)*, then problem* (2.20) *has at least a solution for every $k \in \mathbb{N}$. Moreover every solution is bounded and has finite perimeter.*

It is possible to give an alternative proof of the above Theorem that does not use the concentration-compactness principle, but only the regularity of energy shape subsolutions. This rearrangement of the proof is due to Bozhidar Velichkov.

Remark 2.29 (Velichkov)**.** *Let $(\Omega_n)_{n\geq 1}$ be a minimizing sequence for problem* (2.20)*, with $|\Omega_n| < \infty$ for all $n \in \mathbb{N}$, and then we consider, for all $n \in \mathbb{N}$, the minimum problem*

$$\min\{F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) + |\Omega| \;:\; \Omega \subset \Omega_n\}.$$

Theorem 2.20 by Buttazzo and Dal Maso assures that there exists a solution Ω_n^*, but this is also a subsolution and hence by Lemma 2.26 by Bucur it has diameter uniformly bounded, depending only on k, N. Hence we have a new minimizing sequence Ω_n^* uniformly bounded to which it is possible to apply again Theorem 2.20, thus obtaining existence for problem (2.20)*.*

Chapter 3

Existence of minimizers in $\mathbb{R}^N$

In this Chapter we aim to extend Theorem 2.20 by Buttazzo and Dal Maso in an unbounded setting. This is an interesting problem, because choosing a priori a bounded box seems somehow not natural. Moreover it could also happen, in principle, that minimizers inside a box D_1 are different from minimizers inside another box D_2, even if the boxes are very large. We present the existence result obtained in [M4] and then we prove that all the minimizers for problem (3.1) have diameter uniformly bounded, following [M3].

Theorem 3.1 (Existence of bounded minimizers)**.** *Let $k \in \mathbb{N}$, and let $F\colon \mathbb{R}^k \to \mathbb{R}$ be a l.s.c. functional, increasing in each variable. Then there exists a bounded minimizer for the problem*

$$\inf\Big\{F\big(\lambda_1(\Omega),\, \lambda_2(\Omega),\, \dots\, ,\, \lambda_k(\Omega)\big) : A \subseteq \mathbb{R}^N,\, |A| = 1\Big\} \tag{3.1}$$

among the quasi-open sets. More precisely, a minimizer is contained in a cube Q_R, where the size of the edges R depends on k and on N, but not *on the particular functional F.*

In the rest of this Chapter, the letter C will be always used to denote a big geometric constant, possibly increasing from line to line; the constant C will always depend *only* on N and on k (sometimes, possibly also on some constant K, which in turn will eventually be chosen only depending on N and k), thus *not* on the particular choice of F, and *not* on the set Ω. Sometimes, we will label the constants in our results as C_1, C_2, $C_3 \dots$ for successive reference.

3.1 Proof of Theorem 3.1

We immediately pass to the proof of Theorem 3.1. As already anticipated, our strategy basically consists in showing that to minimize F it is enough to concentrate on uniformly bounded sets. Roughly speaking, the basic idea why this works is that, if a set of unit volume has huge diameter, then there must be some very thin sections. This works against the smallness of the Rayleigh quotients of the eigenfunctions, since by definition they vanish on the boundary. More precisely, we will show the following result.

Proposition 3.2. *For every $K > 0$ there exists a constant $R = R(k, K, N)$, such that the following holds. If $\Omega \subseteq \mathbb{R}^N$ is an open set of unit volume and with $\lambda_k(\Omega) \leq K$, there exists another open set $\widehat{\Omega}$, still of unit volume but contained in a cube of side R, and with $\lambda_i(\widehat{\Omega}) \leq \lambda_i(\Omega)$ for every $1 \leq i \leq k$.*

Remark 3.3. *All our construction is made working on open sets, because we need to be able to "cut" and "add" pieces to the sets. This is sufficient, because we can always consider a minimizing sequence for problem* (3.1) *made of open sets, since the infimum over quasi-open sets and open (also smooth, if one wants) sets is the same.*

Let us immediately see how Theorem 3.1 follows from this proposition; then the rest of the section will be devoted to showing the proposition.

Proof of Theorem 3.1. Let us take a minimizing sequence of open sets $\{\Omega_n\}$ for problem (3.1). Fix a generic $n \in \mathbb{N}$, and assume for a moment that $\lambda_k(\Omega_n) \geq M\lambda_k(B_N)$, being B_N the ball of unit volume in $\mathbb{R}^N$ and M the constant of Theorem 2.12. If it is so, then by Theorem 2.12 one has $\lambda_1(\Omega_n) \geq \lambda_k(B_N)$, thus for every $1 \leq i \leq k$ it is $\lambda_i(\Omega_n) \geq \lambda_i(B_N)$, hence $F(\lambda_1(\Omega_n), \dots, \lambda_k(\Omega_n)) \geq F(\lambda_1(B_N), \dots, \lambda_k(B_N))$, being F increasing in each variable. Thanks to this observation, it is admissible to assume that $\lambda_k(\Omega_n) \leq K := M\lambda_k(B_N)$ for every n. By Proposition 3.2, then, there exists another sequence $\{\widehat{\Omega}_n\}$, made by open sets of unit volume contained in a cube of side R, with $\lambda_i(\widehat{\Omega}_n) \leq \lambda_i(\Omega_n)$ for every $1 \leq i \leq k$ and every $n \in \mathbb{N}$. Again by the assumption that F is increasing in each variable, we derive that also $\{\widehat{\Omega}_n\}$ is a minimizing sequence for (3.1).

We can then apply Theorem 2.20 to find a quasi-open set A, still contained in the cube of side R, and such that, up to extract a subsequence of $\{\widehat{\Omega}_n\}$, one has $\lambda_i(A) \leq \liminf_n \lambda_i(\widehat{\Omega}_n)$ for every $1 \leq i \leq k$. By the lower semi-continuity of F, we derive that A is a minimizer for F, thus the proof is concluded. □

The rest of this section is devoted to show Proposition 3.2. For the ease of presentation, we divide the construction in three sections. In the first one we obtain the boundedness of the "tails" (Lemma 3.4), while in the second one we consider the internal part (Lemma 3.10). Then, in the last section we put everything together to give the proof of Proposition 3.2.

3.1.1 Boundedness of the tails

This subsection is devoted to show that, under the assumptions of Proposition 3.2, we can reduce to the case when the "tails" of Ω are bounded. More precisely, we fix once for all a small positive number $\widehat{m} = \widehat{m}(K, N) \in (0, 1/4)$ in such a way that

$$\frac{(4\widehat{m})^{\frac{2}{N}}}{\lambda_1(B_N)} K \leq \frac{1}{2}, \tag{3.2}$$

being B_N the ball of unit volume in $\mathbb{R}^N$. We aim to show the following result.

Lemma 3.4. *For every $K > 0$ there exist $R_1 = R_1(k, K, N)$ and $\Gamma_1 = \Gamma_1(k, K, N)$ such that, for any open set $\Omega \subseteq \mathbb{R}^N$ of unit volume and with $\lambda_k(\Omega) \leq K$, there exists another open set $\widehat{\Omega} \subseteq \mathbb{R}^N$, still of unit volume, such that $\lambda_i(\widehat{\Omega}) \leq \lambda_i(\Omega)$ for every $1 \leq i \leq k$, and that for every $2 \leq p \leq N$*

$$W\big(\widehat{\Omega}, 0, \widehat{m}\big) \leq R_1 \,, \tag{3.3}$$

$$W\big(\widehat{\Omega}, \widehat{m}, 1\big) \leq \Gamma_1\Big(W\big(\Omega, \widehat{m}, 1\big)\Big) \,, \qquad \operatorname{diam}\big(\pi_p(\widehat{\Omega})\big) \leq \Gamma_1 \operatorname{diam}\big(\pi_p(\Omega)\big) \,. \tag{3.4}$$

The claim of the lemma, roughly speaking, says that it is always possible to assume that the "tail" of Ω, i.e., the set $\Omega^l_{\tau(\Omega,\widehat{m})}$ of volume $\widehat{m}$, has horizontal projection of length at most R_1. More precisely, condition (3.3) says that one can modify Ω in such a way that the tail is uniformly horizontally bounded, while condition (3.4) says that this modification does not excessively worsen the remaining part of the set Ω, nor its extension in the $N-1$ non-horizontal directions.

To prove the lemma, we start setting for brevity $\bar{t} = \tau(\Omega, 2\widehat{m})$, and for every $t \leq \bar{t}$ we define

$$\Omega^+(t) := \Omega^r_t \,, \qquad \Omega^-(t) := \Omega^l_t \,, \qquad \varepsilon(t) := \mathcal{H}^{N-1}(\Omega_t) \,. \tag{3.5}$$

Observe that

$$m(t) := \big|\Omega^-(t)\big| = \int_{-\infty}^{t} \varepsilon(s)\, ds \leq 2\widehat{m} \,. \tag{3.6}$$

Moreover, we let as usual $\big\{u_1,\, u_2,\, \dots\,,\, u_k\big\}$ be an orthonormal set of eigenfunctions with unit L^2 norm and corresponding to the first k eigenvalues of Ω. We define then also, for every $1 \leq i \leq k$ and every $t \leq \bar{t}$,

$$\delta_i(t) := \int_{\Omega_t} |Du_i(t,y)|^2\, d\mathcal{H}^{N-1}(y) \,, \qquad \mu_i(t) := \int_{\Omega_t} u_i(t,y)^2\, d\mathcal{H}^{N-1}(y) \,, \tag{3.7}$$

which makes sense since every u_i is smooth. It is convenient to give the further notation

$$\delta(t) := \sum_{i=1}^{k} \delta_i(t) = \sum_{i=1}^{k} \int_{\Omega_t} \big|Du_i(t,y)\big|^2\, d\mathcal{H}^{N-1}(y) \,,$$

and in analogy with (3.6) we also set

$$\phi(t) := \sum_{i=1}^{k} \int_{\Omega^-(t)} |Du_i|^2 = \int_{-\infty}^{t} \delta(s)\, ds \,. \tag{3.8}$$

Applying the Faber–Krahn inequality in $\mathbb{R}^{N-1}$ to the set Ω_t, and using property (2) of Lemma 2.7 on $\mathbb{R}^{N-1}$, we know that

$$\varepsilon(t)^{\frac{2}{N-1}} \lambda_1(\Omega_t) = \mathcal{H}^{N-1}(\Omega_t)^{\frac{2}{N-1}} \lambda_1(\Omega_t) \geq \lambda_1(B_{N-1}) \,,$$

calling B_{N-1} the unit ball in $\mathbb{R}^{N-1}$. As a trivial consequence, we can estimate μ_i in terms of ε and δ_i: in fact, noticing that $u_i(t,\cdot) \in H^1_0(\Omega_t)$ and writing $Du_i = (D_1 u_i, D_y u_i)$, we have

$$\mu_i(t) = \int_{\Omega_t} u_i(t,\cdot)^2\, d\mathcal{H}^{N-1} \leq \frac{1}{\lambda_1(\Omega_t)} \int_{\Omega_t} |D_y u_i(t,\cdot)|^2\, d\mathcal{H}^{N-1} \leq C\varepsilon(t)^{\frac{2}{N-1}} \delta_i(t). \tag{3.9}$$

We can now present two estimates which assure that u_i and Du_i can not be too big in $\Omega^-(t)$.

Lemma 3.5. *Under the assumptions of Lemma 3.4, for every $1 \le i \le k$ and $t \le \bar t$ the following inequalities hold:*

$$\int_{\Omega^-(t)} u_i^2 \le C_1 \varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,, \qquad\qquad \int_{\Omega^-(t)} |Du_i|^2 \le C_1 \varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,, \tag{3.10}$$

for some $C_1 = C_1(k, K, N)$.

Proof. Let us fix $t \le \bar t$. Consider the set Ω_S^- obtained by the union of $\Omega^-(t)$ and its reflection with respect to the plane $\{x = t\}$, and call $u_S \in H_0^1(\Omega_S)$ the function obtained by reflecting u_i. Calling B_N the unit ball in $\mathbb{R}^N$, we find then

$$\frac{\lambda_1(B_N)}{\big(2m(t)\big)^{\frac{2}{N}}} = \frac{\lambda_1(B_N)}{|\Omega_S^-|^{\frac{2}{N}}} \le \lambda_1(\Omega_S^-) \le \mathcal{R}(u_S, \Omega_S^-) = \mathcal{R}\big(u_i, \Omega^-(t)\big) = \frac{\displaystyle\int_{\Omega^-(t)} |Du_i|^2}{\displaystyle\int_{\Omega^-(t)} u_i^2}\,,$$

by the symmetry of Ω_S^-, and using again property (2) of Lemma 2.7. This estimate gives

$$\int_{\Omega^-(t)} u_i^2 \le \frac{\big(2m(t)\big)^{\frac{2}{N}}}{\lambda_1(B_N)} \int_{\Omega^-(t)} |Du_i|^2 \tag{3.11}$$

which in particular, being $m(t) \le 2\widehat m$ and recalling (3.2), implies

$$\int_{\Omega^-(t)} u_i^2 \le \frac{1}{2}\,. \tag{3.12}$$

On the other hand, recalling that $-\Delta u_i = \lambda_i u_i$, by Schwarz inequality and using (3.9) we have

$$\begin{aligned}\int_{\Omega^-(t)} |Du_i|^2 &= \int_{\Omega^-(t)} \lambda_i u_i^2 + \int_{\Omega_t} u_i \frac{\partial u_i}{\partial \nu} \le K \int_{\Omega^-(t)} u_i^2 + \sqrt{\int_{\Omega_t} u_i^2 \int_{\Omega_t} |Du_i|^2} \\ &\le K \int_{\Omega^-(t)} u_i^2 + C\varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,.\end{aligned} \tag{3.13}$$

It is now easy to obtain (3.10) combining (3.11) and (3.13). In fact, by inserting the latter into the first, we find

$$\int_{\Omega^-(t)} u_i^2 \le \frac{\big(2m(t)\big)^{\frac{2}{N}}}{\lambda_1(B_N)} \bigg(K \int_{\Omega^-(t)} u_i^2 + C\varepsilon(t)^{\frac{1}{N-1}} \delta_i(t) \bigg)\,,$$

which by (3.2) again yields

$$\frac{1}{2} \int_{\Omega^-(t)} u_i^2 \le \frac{\big(2m(t)\big)^{\frac{2}{N}}}{\lambda_1(B_N)} C\varepsilon(t)^{\frac{1}{N-1}} \delta_i(t) \le C\varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,. \tag{3.14}$$

The left estimate in (3.10) is then obtained. To get the right one, one has then just to insert (3.14) into (3.13). □

Let us go further into our construction, giving some definitions. For any $t \leq \bar{t}$ and $\sigma(t) > 0$, we define the cylinder $Q(t)$, shown in Figure 3.1, as

$$Q(t) := \Big\{(x,y) \in \mathbb{R}^N : t-\sigma < x < t,\ (t,y) \in \Omega\Big\} = \big(t-\sigma, t\big) \times \Omega_t\,, \tag{3.15}$$

where for any $t \leq \bar{t}$ we set

$$\sigma(t) = \varepsilon(t)^{\frac{1}{N-1}}\,. \tag{3.16}$$

We let also $\widetilde{\Omega}(t) = \Omega^+(t) \cup Q(t)$, and we introduce $\widetilde{u}_i \in H^1_0\big(\widetilde{\Omega}(t)\big)$ as

$$\widetilde{u}_i(x,y) := \left\{\begin{array}{ll} u_i(x,y) & \text{if } (x,y) \in \Omega^+(t)\,,\\ \dfrac{x-t+\sigma}{\sigma}\, u_i(t,y) & \text{if } (x,y) \in Q(t)\,. \end{array}\right. \tag{3.17}$$

The fact that $\widetilde{u}_i$ vanishes on $\partial\widetilde{\Omega}(t)$ is obvious; moreover, $Du_i = D\widetilde{u}_i$ on $\Omega^+(t)$, while on $Q(t)$ one has

$$D\widetilde{u}_i(x,y) = \left(\frac{u_i(t,y)}{\sigma}\,,\ \frac{x-t+\sigma}{\sigma}\, D_y u_i(t,y)\right)\,. \tag{3.18}$$

A simple calculation allows us to estimate the integrals of $\widetilde{u}_i$ and $D\widetilde{u}_i$ on $Q(t)$.

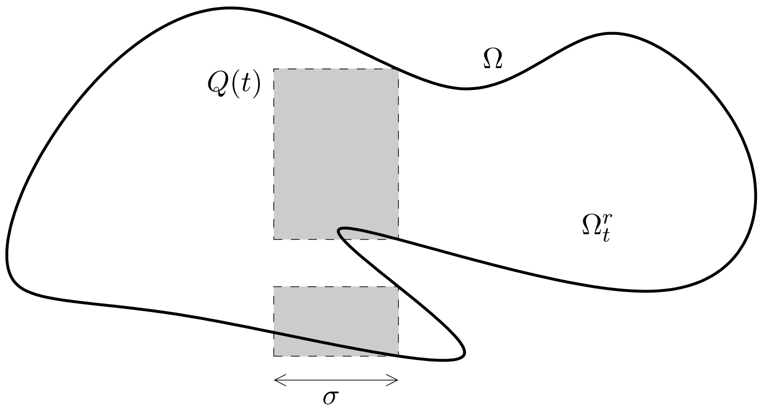

Figure 3.1: A set Ω with the cylinder $Q(t)$ (shaded).

Lemma 3.6. *For every $t \leq \bar{t}$ and $1 \leq i \leq k$, one has*

$$\int_{Q(t)} |D\widetilde{u}_i|^2 \leq C_2 \varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,, \qquad \int_{Q(t)} \widetilde{u}_i^2 \leq C_2 \varepsilon(t)^{\frac{3}{N-1}} \delta_i(t)\,, \tag{3.19}$$

for a suitable constant $C_2 = C_2(k,K,N)$.

Proof. Thanks to (3.18), and using also (3.9) and (3.16), one obtains the first estimate in (3.19) since

$$\begin{aligned}\int_{Q(t)} |D\widetilde{u}_i(x,y)|^2\,dx\,dy &= \int_{t-\sigma}^t \int_{\Omega_t} \frac{u_i^2(t,y)}{\sigma^2} + \frac{(x-t+\sigma)^2}{\sigma^2}\,|D_y u_i(t,y)|^2\,dy\,dx = \frac{\mu_i}{\sigma} + \frac{\delta_i \sigma}{3}\\ &\leq C\,\frac{\varepsilon(t)^{\frac{2}{N-1}}\delta_i(t)}{\sigma} + \frac{\delta_i\sigma}{3} = C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t)\,.\end{aligned}$$

On the other hand, the second estimate in (3.19) follows, also again using (3.9) and (3.16), by

$$\int_{Q(t)} \widetilde{u}_i(x,y)^2\,dx\,dy = \int_{t-\sigma}^{t}\int_{\Omega_t} \widetilde{u}_i^2(t,y)\,dy\,dx = \frac{\sigma\mu_i}{3} \leq C\sigma\varepsilon(t)^{\frac{2}{N-1}}\delta_i(t) = C\varepsilon(t)^{\frac{3}{N-1}}\delta_i(t)\,.$$

□

Another simple but useful estimate concerns the Rayleigh quotients of the functions $\widetilde{u}_i$ on the sets $\widetilde{\Omega}(t)$ and the integral of the products $\widetilde{u}_i\widetilde{u}_j$.

Lemma 3.7. *For every $t \leq \bar{t}$ and $1 \leq i \leq k$, one has*

$$\mathcal{R}\big(\widetilde{u}_i, \widetilde{\Omega}(t)\big) \leq \lambda_i(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t)\,. \tag{3.20}$$

Moreover, for every $i \neq j \in \{1, 2, \ldots, k\}$, one has

$$\left|\int_{\widetilde{\Omega}(t)} \widetilde{u}_i\widetilde{u}_j + D\widetilde{u}_i\cdot D\widetilde{u}_j\right| \leq C\Big(\varepsilon(t)^{\frac{3}{N-1}} + \varepsilon(t)^{\frac{1}{N-1}}\Big)\sqrt{\delta_i(t)\delta_j(t)}\,. \tag{3.21}$$

Proof. Recalling that $-\Delta u_i = \lambda_i(\Omega)u_i$, making use of (3.12) and (3.19) and arguing as in (3.13), we obtain

$$\begin{aligned}
\mathcal{R}\big(\widetilde{u}_i, \widetilde{\Omega}(t)\big) &= \frac{\displaystyle\int_{\Omega^+(t)} |D\widetilde{u}_i|^2 + \int_{Q(t)} |D\widetilde{u}_i|^2}{\displaystyle\int_{\Omega^+(t)} \widetilde{u}_i^2 + \int_{Q(t)} \widetilde{u}_i^2} \leq \frac{\displaystyle\int_{\Omega^+(t)} |Du_i|^2 + \int_{Q(t)} |D\widetilde{u}_i|^2}{\displaystyle\int_{\Omega^+(t)} u_i^2}\\
&= \frac{\displaystyle\lambda_i(\Omega)\int_{\Omega^+(t)} u_i^2 + \int_{\Omega_t} u_i\frac{\partial u_i}{\partial\nu} + \int_{Q(t)} |D\widetilde{u}_i|^2}{\displaystyle\int_{\Omega^+(t)} u_i^2} \leq \lambda_i(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t)\,,
\end{aligned}$$

hence (3.20) is proved.

On the other hand, recall that u_i and u_j are orthogonal on Ω both in L^2 and in H^1_0 sense by definition, hence by using (3.10) and (3.19) we find

$$\begin{aligned}
\left|\int_{\widetilde{\Omega}(t)} \widetilde{u}_i\widetilde{u}_j\right| &\leq \left|\int_{\Omega^+(t)} u_iu_j\right| + \left|\int_{Q(t)} \widetilde{u}_i\widetilde{u}_j\right| = \left|\int_{\Omega^-(t)} u_iu_j\right| + \left|\int_{Q(t)} \widetilde{u}_i\widetilde{u}_j\right|\\
&\leq C\bigg(\varepsilon(t)^{\frac{1}{N-1}}\sqrt{\delta_i(t)\delta_j(t)} + \varepsilon(t)^{\frac{3}{N-1}}\sqrt{\delta_i(t)\delta_j(t)}\bigg)\,.
\end{aligned}$$

In the very same way, concerning $D\widetilde{u}_i$ and $D\widetilde{u}_j$, we have

$$\begin{aligned}
\left|\int_{\widetilde{\Omega}(t)} D\widetilde{u}_i\cdot D\widetilde{u}_j\right| &\leq \left|\int_{\Omega^+(t)} Du_i\cdot Du_j\right| + \left|\int_{Q(t)} D\widetilde{u}_i\cdot D\widetilde{u}_j\right|\\
&= \left|\int_{\Omega^-(t)} Du_i\cdot Du_j\right| + \left|\int_{Q(t)} D\widetilde{u}_i\cdot D\widetilde{u}_j\right| \leq C\varepsilon(t)^{\frac{1}{N-1}}\sqrt{\delta_i(t)\delta_j(t)}\,.
\end{aligned}$$

Adding up the last two estimates yields (3.21). □

In order to prove Lemma 3.4, we need to compare the eigenvalues of Ω and those of $\widetilde{\Omega}(t)$; this can be done by means of the min-max principle (2.4), which relates the eigenvalues with the Rayleigh quotients of H^1_0 functions.

Lemma 3.8. *There exist a small constant $\nu = \nu(k, K, N) < 1$ and a constant $C_3 = C_3(k, K, N)$ such that, if $\varepsilon(t)$, $\delta_i(t) \leq \nu$ for every $1 \leq i \leq k$, then*

$$\lambda_j\big(\widetilde{\Omega}(t)\big) \leq \lambda_j(\Omega) + C_3\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \qquad \forall\, 1 \leq j \leq k\,. \tag{3.22}$$

Proof. We aim to use the characterization given by (2.4). To do so, for every $1 \leq j \leq k$ we define K_j as the linear subspace of $H^1_0\big(\widetilde{\Omega}(t)\big)$ spanned by the functions $\widetilde{u}_i$ with $1 \leq i \leq j$. First of all, we state and prove the following claim.

Claim 3.A.
For every $1 \leq j \leq k$, the space K_j has dimension j.

Proof of Claim 3.A. Suppose that the claim is not true. Then, there should exist some $1 \leq \ell \leq k$ and some coefficients β_i for $i \neq \ell$ with all $|\beta_i| \leq 1$ and

$$\widetilde{u}_\ell = \sum_{1\leq i\leq k,\, i\neq\ell} \beta_i \widetilde{u}_i\,.$$

Notice now that by (3.12) we know that $\int_{\Omega^+(t)} u_\ell^2 \geq 1/2$, hence also by (3.21) we deduce

$$\frac{1}{2} \leq \int_{\widetilde{\Omega}(t)} \widetilde{u}_\ell^2 = \int_{\widetilde{\Omega}(t)} \sum_{1\leq i\leq k,\, i\neq\ell} \beta_i\widetilde{u}_i\widetilde{u}_\ell \leq \sum_{1\leq i\leq k,\, i\neq\ell} \bigg|\int_{\widetilde{\Omega}(t)} \widetilde{u}_i\widetilde{u}_\ell\bigg| \leq k\,C\Big(\nu^{\frac{3}{N-1}} + \nu^{\frac{1}{N-1}}\Big)\nu < \frac{1}{2}\,,$$

where the last inequality is true provided that $\nu = \nu(k, K, N)$ is chosen small enough. The absurd shows the validity of Claim 3.A. □

We can now show (3.22): to do so, pick a generic function $w \in K_j$, which can be written (up to a rescaling) as $w = \sum_{i=1}^j \beta_i\widetilde{u}_i$, where $\max\big\{|\beta_i|,\, 1 \leq i \leq j\big\} = 1$. We need to evaluate $\mathcal{R}\big(w, \widetilde{\Omega}(t)\big)$: we start by noticing that

$$\begin{aligned}\mathcal{R}\big(w, \widetilde{\Omega}(t)\big) &= \frac{\int_{\widetilde{\Omega}(t)} |Dw|^2}{\int_{\widetilde{\Omega}(t)} w^2} = \frac{\sum_{i=1}^j \beta_i^2 \int_{\widetilde{\Omega}(t)} |D\widetilde{u}_i|^2 + \sum_{i\neq j} \beta_i\beta_j \int_{\widetilde{\Omega}(t)} D\widetilde{u}_i \cdot D\widetilde{u}_j}{\sum_{i=1}^j \beta_i^2 \int_{\widetilde{\Omega}(t)} \widetilde{u}_i^2 + \sum_{i\neq j} \beta_i\beta_j \int_{\widetilde{\Omega}(t)} \widetilde{u}_i\widetilde{u}_j} \\ &\leq \frac{\sum_{i=1}^j \beta_i^2 \int_{\widetilde{\Omega}(t)} |D\widetilde{u}_i|^2 + Ck^2\varepsilon(t)^{\frac{1}{N-1}}\delta(t)}{\sum_{i=1}^j \beta_i^2 \int_{\widetilde{\Omega}(t)} \widetilde{u}_i^2 - Ck^2\varepsilon(t)^{\frac{1}{N-1}}\delta(t)}\,,\end{aligned} \tag{3.23}$$

where the last inequality comes by (3.21). If $\nu(k, K, N)$ is small enough, then

$$Ck^2\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \leq Ck^2\nu^{\frac{N}{N-1}} \leq \frac{1}{4}\,,$$

hence by the choice of β_i and by (3.12) the denominator in the last fraction of (3.23) is bigger than 1/4 (in particular, it is strictly positive). As a consequence, recalling also that by (3.20) one has for every $1 \le i \le j$

$$\frac{\displaystyle\int_{\widetilde\Omega(t)} |D\widetilde u_i|^2}{\displaystyle\int_{\widetilde\Omega(t)} \widetilde u_i^2} \le \lambda_i(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t) \le \lambda_j(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\,,$$

from (3.23) we deduce

$$\begin{aligned}\mathcal{R}\big(w,\widetilde\Omega(t)\big) &\le \frac{\Big(\lambda_j(\Omega)+C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\Big)\Big(\sum_{i=1}^j \beta_i^2\int_{\widetilde\Omega(t)}\widetilde u_i^2\Big)+C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)}{\sum_{i=1}^j \beta_i^2\int_{\widetilde\Omega(t)}\widetilde u_i^2 - C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)}\\ &\le \frac{\Big(\lambda_j(\Omega)+C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\Big)+2C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)}{1-2C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)} \le \lambda_j(\Omega)+C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\end{aligned}$$

(keep in mind that the constant $C = C(k, K, N)$ may increase from line to line). The validity of (3.22) is then now an immediate consequence of (2.4) and of Claim 3.A. □

We can now enter in the central part of our construction. Basically, we aim to show that either Ω already satisfies the requirements of Lemma 3.4, or some $\widetilde\Omega(t)$ does it, up to a rescaling. To do so, we need another definition, namely, for every $t \le \bar t$ we define the rescaled set

$$\widehat\Omega(t) := \big|\widetilde\Omega(t)\big|^{-\frac{1}{N}}\widetilde\Omega(t)\,,$$

so that $\big|\widehat\Omega(t)\big| = 1$. We can now show the following result.

Lemma 3.9. *Let Ω be as in the assumptions of Lemma 3.4, and let $t \le \bar t$. There exists $C_4 = C_4(k, K, N)$ such that exactly one of the three following conditions hold:*

(1) $\max\big\{\varepsilon(t),\, \delta(t)\big\} > \nu$*;*

(2) (1) does not hold and $m(t) \le C_4\big(\varepsilon(t)+\delta(t)\big)\varepsilon(t)^{\frac{1}{N-1}}$*;*

(3) (1) and (2) do not hold and for every $1 \le i \le k$, one has $\lambda_i\big(\widehat\Omega(t)\big) < \lambda_i(\Omega)$.

In particular, if condition (3) holds for t and $m(t) \ge \widehat m$, then for every $1 \le i \le k$ one has $\lambda_i\big(\widehat\Omega(t)\big) < \lambda_i(\Omega) - \eta$, being $\eta = \eta(k, K, N) > 0$.

Proof. If (1) holds true, there is of course nothing to prove. Otherwise, it is possible to apply Lemma 3.8, hence we have

$$\lambda_i\big(\widetilde\Omega(t)\big) \le \lambda_i(\Omega) + C_3\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \tag{3.24}$$

for every $1 \le i \le k$. By property (2) of Lemma 2.7 and the fact that $\big|\widehat\Omega(t)\big| = 1$, we know that

$$\lambda_i\big(\widehat\Omega(t)\big) = \big|\widetilde\Omega(t)\big|^{\frac{2}{N}}\lambda_i\big(\widetilde\Omega(t)\big)\,.$$

By construction,

$$\big|\widetilde{\Omega}(t)\big| = \big|\Omega^+(t)\big| + \big|Q(t)\big| = 1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}\,,$$

hence the above estimates and (3.24) lead to

$$\begin{aligned}
\lambda_i(\widehat{\Omega}(t)) &= \Big(1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}\Big)^{\frac{2}{N}} \lambda_i\big(\widetilde{\Omega}(t)\big) \\
&\leq \Big(1 - \frac{2}{N}\, m(t) + \frac{2}{N}\, \varepsilon(t)^{\frac{N}{N-1}}\Big)\Big(\lambda_i(\Omega) + C_3 \varepsilon(t)^{\frac{1}{N-1}} \delta(t)\Big) \\
&\leq \lambda_i(\Omega) - \frac{2\lambda_1(B_N)}{N}\, m(t) + \frac{2K}{N}\, \varepsilon(t)^{\frac{N}{N-1}} + \bigg(C_3 + \frac{2}{N}\bigg)\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\,.
\end{aligned} \tag{3.25}$$

This allows us to conclude. In fact, defining $C_4 := \frac{2(K+1)}{N} + C_3$, if $m(t) \leq C_4\big(\varepsilon(t) + \delta(t)\big)\varepsilon(t)^{\frac{1}{N-1}}$, then condition (2) holds true. Otherwise, (3.25) directly implies that $\lambda_i\big(\widehat{\Omega}(t)\big) < \lambda_i(\Omega)$.

Finally, assume that condition (3) holds and $m(t) \geq \widehat{m}$: in this case, (3.25) directly implies

$$\lambda_i\big(\widehat{\Omega}(t)\big) - \lambda_i(\Omega) \leq -\frac{2\lambda_1(B_N)}{N}\, \widehat{m} + C_4 \nu^{\frac{N}{N-1}} \leq -\eta\,,$$

where $\eta = \lambda_1(B_N)\widehat{m}/N$ and the last inequality is true up to decrease ν (notice that *decreasing* the value of the constant ν of Lemma 3.8 does not change the value of C_3). □

We are finally in position to give the proof of Lemma 3.4.

Proof of Lemma 3.4. Let us start defining

$$\hat{t} := \sup\Big\{t \leq \bar{t} :\, \text{condition (3) of Lemma 3.9 holds for } t\Big\}\,, \tag{3.26}$$

with the usual convention that, if condition (3) is false for every $t \leq \bar{t}$, then $\hat{t} = -\infty$. We introduce now the following subsets of $(\hat{t}, \bar{t})$,

$$\begin{aligned}
A :&= \Big\{t \in (\hat{t}, \bar{t}) :\, \text{condition (1) of Lemma 3.9 holds for } t\Big\}\,, \\
B :&= \Big\{t \in (\hat{t}, \bar{t}) :\, \text{condition (2) of Lemma 3.9 holds for } t \text{ and } m(t) > 0\Big\}\,,
\end{aligned}$$

and we further subdivide them as

$$\begin{aligned}
A_1 &:= \Big\{t \in A :\, \varepsilon(t) \geq \delta(t)\Big\}\,, \qquad & A_2 &:= \Big\{t \in A :\, \varepsilon(t) < \delta(t)\Big\}\,, \\
B_1 &:= \Big\{t \in B :\, \varepsilon(t) \geq \delta(t)\Big\}\,, \qquad & B_2 &:= \Big\{t \in B :\, \varepsilon(t) < \delta(t)\Big\}\,.
\end{aligned}$$

We aim to show that both A and B are uniformly bounded. Concerning A_1, observe that

$$\nu\big|A_1\big| \leq \int_{A_1} \varepsilon(t)\, dt = \Big|\Big\{(x,y) \in \Omega :\, x \in A_1\Big\}\Big| \leq \big|\Omega\big| = 1\,,$$

so that $|A_1| \leq 1/\nu$. Concerning A_2, in the same way and also recalling that $\lambda_i(\Omega) \leq K$ for every $i \leq k$, we have

$$\nu\big|A_2\big| \leq \int_{A_2} \delta(t)\, dt = \sum_{i=1}^k \int_{A_2}\int_{\Omega_t} \big|Du_i(t,y)\big|^2\, d\mathcal{H}^{N-1}(y)\, dt \leq \sum_{i=1}^k \int_\Omega \big|Du_i\big|^2 \leq kK\,,$$

so that $|A_2| \le kK/\nu$. Summarizing, we have proved that

$$\big|A\big| \le \frac{1+kK}{\nu}\,. \tag{3.27}$$

Let us then pass to the set B_1. To deal with it, we need a further subdivision, namely, we write $B_1 = \cup_{n\in\mathbb{N}} B_1^n$, where

$$B_1^n := \left\{ t \in B_1 : \, \frac{\widehat{m}}{2^n} < m(t) \le \frac{\widehat{m}}{2^{n-1}} \right\}. \tag{3.28}$$

Keeping in mind (3.6), we know that $t \mapsto m(t)$ is an increasing function, and that for a.e. $t \in \mathbb{R}$ one has $m'(t) = \varepsilon(t)$. Moreover, for every $t \in B_1$ one has by construction that

$$m(t) \le C_4\big(\varepsilon(t)+\delta(t)\big)\varepsilon(t)^{\frac{1}{N-1}} \le 2C_4\,\varepsilon(t)^{\frac{N}{N-1}}\,.$$

As a consequence, for every $t \in B_1^n$ one has

$$m'(t) = \varepsilon(t) \ge \frac{1}{C}\,m(t)^{\frac{N-1}{N}} \ge \frac{1}{C}\,\widehat{m}^{\frac{N-1}{N}}\,\frac{1}{\big(2^{\frac{N-1}{N}}\big)^n}\,.$$

This readily implies

$$\frac{1}{C}\,\widehat{m}^{\frac{N-1}{N}}\,\frac{1}{\big(2^{\frac{N-1}{N}}\big)^n}\,\big|B_1^n\big| \le \int_{B_1^n} m'(t) \le \frac{\widehat{m}}{2^n}\,,$$

which in turn gives

$$\big|B_1^n\big| \le C\widehat{m}^{\frac{1}{N}}\big(2^{-\frac{1}{N}}\big)^n\,.$$

Finally, we deduce

$$\big|B_1\big| = \sum_{n\in\mathbb{N}}\big|B_1^n\big| \le C\widehat{m}^{\frac{1}{N}}\sum_{n\in\mathbb{N}}\big(2^{-\frac{1}{N}}\big)^n = C\widehat{m}^{\frac{1}{N}}\,\frac{2^{\frac{1}{N}}}{2^{\frac{1}{N}}-1}\,. \tag{3.29}$$

Notice that basically our argument consisted in using the fact that in B_1 one has

$$m(t) \le C\varepsilon(t)^{\frac{N}{N-1}}\,, \qquad\qquad \text{with } \varepsilon(t) = m'(t)\,. \tag{3.30}$$

Concerning B_2, we can almost repeat the same argument: in fact, thanks to (3.10), for every $t \in B_2$ we have

$$\phi(t) = \sum_{i=1}^k \int_{\Omega^-(t)} |Du_i|^2 \le C_1\,\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \le C_1\,\delta(t)^{\frac{N}{N-1}}\,, \qquad\qquad \text{with } \delta(t) = \phi'(t)\,.$$

which is the perfect analogous of (3.30) with δ and ϕ in place of ε and m respectively. Since as already observed $\phi(\bar t) \le \sum\int_\Omega |Du_i|^2 \le kK$, in analogy with (3.28) we can define

$$B_2^n := \left\{ t \in B_2 : \, \frac{kK}{2^{n+1}} < \phi(t) \le \frac{kK}{2^n} \right\},$$

thus the very same argument which lead to (3.29) now gives

$$\big|B_2\big| = \sum_{n\in\mathbb{N}}\big|B_2^n\big| \le C\big(kK\big)^{\frac{1}{N}}\sum_{n\in\mathbb{N}}\big(2^{-\frac{1}{N}}\big)^n = C\big(kK\big)^{\frac{1}{N}}\,\frac{2^{\frac{1}{N}}}{2^{\frac{1}{N}}-1}\,. \tag{3.31}$$

Putting (3.27), (3.29) and (3.31) together, we find

$$|A| + |B| \leq C_5 = C_5(k, K, N). \tag{3.32}$$

We will prove the validity of the lemma with the following choice of R_1 and Γ_1,

$$R_1 = 2C_5 + 4, \qquad\qquad \Gamma_1 = 2^{[K/\eta]+1},$$

where $C_5 = C_5(k, K, N)$ and $\eta = \eta(k, K, N)$ have been introduced in (3.32) and in Lemma 3.9 respectively. To obtain our proof, we will distinguish the possible cases for Ω.

<u>Case I.</u> *One has* $\hat{t} = -\infty$.

If this case happens, then condition (3) of Lemma 3.9 never holds true, i.e., for every $t \leq \bar{t}$ either condition (1) or (2) holds. Recalling the definition of A and B and (3.32), we deduce that $W(\Omega, 0, \widehat{m}) \leq C_5$. Therefore, the claim of Lemma 3.4 is immediately obtained simply taking $\widehat{\Omega} = \Omega$, since $R_1 \geq C_5$ and $\Gamma_1 \geq 1$.

<u>Case II.</u> *One has* $\hat{t} > -\infty$.

In this case, let us notice that it must be $m(\hat{t}) > 0$, hence $(\hat{t}, \bar{t}) \subseteq A \cup B$ and thus by (3.32) $\hat{t} \geq \bar{t} - C_5$. Let us now pick some $t^\star \in [\hat{t} - 1, \hat{t}]$ for which condition (3) holds, and define $U_1 := \widehat{\Omega}(t^\star)$. By definition, U_1 has unit volume, and $\lambda_i(U_1) < \lambda_i(\Omega)$ for every $1 \leq i \leq k$, being condition (3) true for $t^\star$.

Observe now that by definition for every $2 \leq p \leq N$ one has $\pi_p\big(\widetilde{\Omega}(t^\star)\big) = \pi_p\big(\Omega^+(t^\star)\big)$, hence

$$\begin{aligned}\operatorname{diam}\big(\pi_p(U_1)\big) &= \operatorname{diam}\Big(\pi_p\big(\widehat{\Omega}(t^\star)\big)\Big) = \operatorname{diam}\Big(\pi_p\Big(\big|\widetilde{\Omega}(t^\star)\big|^{-\frac{1}{N}}\widetilde{\Omega}(t^\star)\Big)\Big) \leq 2\operatorname{diam}\Big(\pi_p\big(\widetilde{\Omega}(t^\star)\big)\Big)\\ &= 2\operatorname{diam}\Big(\pi_p\big(\Omega^+(t^\star)\big)\Big) \leq 2\operatorname{diam}\big(\pi_p(\Omega)\big),\end{aligned}$$

where we have used that $\big|\widetilde{\Omega}(t^\star)\big| \geq 1/2$. Concerning the widths of U_1 and Ω, we can start observing that

$$W\big(U_1, \widehat{m}, 1\big) = \big|\widetilde{\Omega}(t^\star)\big|^{-\frac{1}{N}}\Big(W\Big(\widetilde{\Omega}(t^\star), \widehat{m}\big|\widetilde{\Omega}(t^\star)\big|, \big|\widetilde{\Omega}(t^\star)\big|\Big)\Big).$$

Moreover, since it is admissible to assume $\nu^{\frac{N}{N-1}} < \frac{\widehat{m}}{2}$ and then

$$\big|\widetilde{\Omega}(t^\star)^l_{t^\star}\big| = \big|\varepsilon(t^\star)^{\frac{N}{N-1}}\big| < \big|\widetilde{\Omega}(t^\star)\big|\widehat{m},$$

we have

$$\tau\Big(\widetilde{\Omega}(t^\star), \widehat{m}\big|\widetilde{\Omega}(t^\star)\big|\Big) = \tau\Big(\Omega, \widehat{m}\big|\widetilde{\Omega}(t^\star)\big| + 1 - \big|\widetilde{\Omega}(t^\star)\big|\Big);$$

as a consequence, we evaluate

$$\begin{aligned}W\Big(\widetilde{\Omega}(t^\star), \widehat{m}\big|\widetilde{\Omega}(t^\star)\big|, \big|\widetilde{\Omega}(t^\star)\big|\Big) &= \tau\Big(\widetilde{\Omega}(t^\star), \big|\widetilde{\Omega}(t^\star)\big|\Big) - \tau\Big(\widetilde{\Omega}(t^\star), \widehat{m}\big|\widetilde{\Omega}(t^\star)\big|\Big)\\ &= \tau\big(\Omega, 1\big) - \tau\Big(\Omega, \widehat{m}\big|\widetilde{\Omega}(t^\star)\big| + 1 - \big|\widetilde{\Omega}(t^\star)\big|\Big) \leq \tau\big(\Omega, 1\big) - \tau\big(\Omega, \widehat{m}\big)\\ &= W\big(\Omega, \widehat{m}, 1\big),\end{aligned}$$

thus we deduce, again recalling $\big|\widetilde\Omega(t^\star)\big| \geq 1/2$, that

$$W\big(U_1,\widehat m,1\big) \leq 2W\big(\Omega,\widehat m,1\big)\,.$$

Summarizing, we have found that

$$\lambda_i(U_1)<\lambda_i(\Omega)\,,\quad \operatorname{diam}\big(\pi_p(U_1)\big)\leq 2\operatorname{diam}\big(\pi_p(\Omega)\big)\,,\quad W\big(U_1,\widehat m,1\big) \leq 2W\big(\Omega,\widehat m,1\big)\,. \tag{3.33}$$

As a consequence, the choice $\widehat\Omega=U_1$ satisfies all the requirements of Lemma 3.4, except possibly condition (3.3). To deal with this last condition, we need to further subdivide this case.

<u>Case IIa.</u> *One has $\hat t>-\infty$ and $m(t^\star)<\widehat m$.*

In this case, we can show that the choice $\widehat\Omega=U_1$ actually works. As noticed above, we have only to prove the validity of (3.3). To do so, we assume for simplicity that $\bar t=0$, which is clearly admissible by translation. Hence, $t^\star\geq \hat t-1\geq \bar t-C_5-1=-C_5-1$, and thus

$$\widetilde\Omega(t^\star)=\Omega^+(t^\star)\cup Q(t^\star)\subseteq\big\{(x,y):\, x>t^\star-1\big\}\subseteq \big\{(x,y):\, x>-C_5-2\big\}\,.$$

Recalling that $\big|\widetilde\Omega(t^\star)\big|\geq\big|\Omega^+(t^\star)\big|\geq |\Omega^+(\bar t)|=1-2\widehat m\geq 1/2$, we deduce that

$$\widehat\Omega=\widehat\Omega(t^\star)=\big|\widetilde\Omega(t^\star)\big|^{-\frac1N}\widetilde\Omega(t^\star)\subseteq 2\,\widetilde\Omega(t^\star)\subseteq\big\{(x,y):\, x>-2C_5-4\big\}\,. \tag{3.34}$$

Moreover, since $m(\hat t)<\widehat m$,

$$\begin{aligned}\big|\widehat\Omega^l_0\big|&=\big|\widehat\Omega(t^\star)^l_0\big|\geq\big|\widetilde\Omega(t^\star)^l_0\big|=\Big|\Big(\Omega^+(t^\star)\cup Q(t^\star)\Big)^l_0\Big|\geq\Big|\Big(\Omega^+(t^\star)\Big)^l_0\Big|\\&=\Big|\big\{(x,y)\in\Omega:\, t^\star<x<0\big\}\Big|=m(0)-m(t^\star)\geq m(0)-m(\hat t)\geq\widehat m\,,\end{aligned}$$

and this implies that $\tau(\widehat\Omega,\widehat m)\leq 0$. The inclusion (3.34) ensures then that $\tau(\widehat\Omega,0)\geq -2C_5-4$, and then (3.3) holds true since $R_1=2C_5+4$.

<u>Case IIb.</u> *One has $\hat t>-\infty$ and $m(t^\star)\geq\widehat m$.*

We have now to face the last possible case, namely, when $\hat t$ is finite but $m(t^\star)\geq\widehat m$. In this case, thanks to Lemma 3.9 the estimates (3.33) can be strengthened as

$$\lambda_i(U_1)<\lambda_i(\Omega)-\eta\,,\ \ \operatorname{diam}\big(\pi_p(U_1)\big)\leq 2\operatorname{diam}\big(\pi_p(\Omega)\big)\,,\ \ W\big(U_1,\widehat m,1\big) \leq 2W\big(\Omega,\widehat m,1\big)\,. \tag{3.35}$$

Concerning the validity of (3.3), it does not follow by (3.34) because the assumption $m(\hat t)\geq\widehat m$ does not imply that $\tau(\widehat\Omega,\widehat m)\leq 0$. However, we can argue as follow: if (3.3) holds true for U_1, then of course we are done by setting $\widehat\Omega=U_1$. Otherwise, we apply the above construction to the set U_1 in place of Ω: since U_1 does not satisfy (3.3), then Case I is impossible, thus we are in Case II and then by (3.33) we find an open set U_2 of unit measure such that

$$\begin{cases}\lambda_i(U_2)<\lambda_i(U_1)<\lambda_i(\Omega)-\eta\quad\forall\,1\leq i\leq k\,,\\ \operatorname{diam}\big(\pi_p(U_2)\big)\leq 2\operatorname{diam}\big(\pi_p(U_1)\big)\leq 4\operatorname{diam}\big(\pi_p(\Omega)\big)\,,\\ W\big(U_2,\widehat m,1\big)\leq 2W\big(U_1,\widehat m,1\big)\leq 4W\big(\Omega,\widehat m,1\big)\,.\end{cases}$$

If U_1 is in Case IIa, then as before we are done with the choice of $\widehat{\Omega} = U_2$. Otherwise, if we are in Case IIb, then (3.35) becomes

$$\lambda_i(U_2) < \lambda_i(\Omega) - 2\eta\,, \quad \operatorname{diam}\big(\pi_p(U_2)\big) \le 4\operatorname{diam}\big(\pi_p(\Omega)\big)\,, \quad W\big(U_2, \widehat{m}, 1\big) \le 4W\big(\Omega, \widehat{m}, 1\big)\,.$$

Going on with the obvious iteration, if the proof has not been concluded after $\ell \in \mathbb{N}$ steps then we have found an open set U_ℓ satisfying

$$\lambda_i(U_\ell) < \lambda_i(\Omega) - \ell\eta\,, \quad \operatorname{diam}\big(\pi_p(U_\ell)\big) \le 2^\ell \operatorname{diam}\big(\pi_p(\Omega)\big)\,, \quad W\big(U_\ell, \widehat{m}, 1\big) \le 2^\ell W\big(\Omega, \widehat{m}, 1\big)\,.$$

This is of course impossible if $\ell\eta \ge K$, being $\lambda_k(\Omega) \le K$: as a consequence, our iteration must stop after less than K/η steps, thus our thesis is concluded with our choice of Γ_1. □

3.1.2 Boundedness of the interior

The goal of this subsection is to obtain a uniform bound also for the interior part of a set Ω, in the sense of Lemma 3.4. Most of the arguments of this case will be identical to those that we made for the tails in Section 3.1.1, but some modifications are essential. In particular we give new definitions for ε, δ_i, μ_i, $\widetilde{\Omega}(t)$ and $\widehat{\Omega}(t)$ in order to maintain the analogy with what was done in Section 3.1.1. The result that we are going to prove is the following.

Lemma 3.10. *For every $K > 0$ there exist $R_2 = R_2(k, K, N)$ and $\Gamma_2 = \Gamma_2(k, K, N)$ such that, for any open set $\Omega \subseteq \mathbb{R}^N$ of unit volume and with $\lambda_k(\Omega) \le K$, and for any choice of $\overline{m} \in (\widehat{m}, 1 - \frac{\widehat{m}}{2})$, there exists another open set $\widehat{\Omega} \subseteq \mathbb{R}^N$, still of unit volume, such that $\lambda_i(\widehat{\Omega}) \le \lambda_i(\Omega)$ for every $1 \le i \le k$, and such that for every $2 \le p \le N$*

$$W\big(\widehat{\Omega}, 0, \overline{m}\big) \le R_2 + \Gamma_2 W\big(\Omega, 0, \overline{m} - \widehat{m}\big)\,, \qquad \operatorname{diam}\big(\pi_p(\widehat{\Omega})\big) \le \Gamma_2 \operatorname{diam}\big(\pi_p(\Omega)\big)\,. \tag{3.36}$$

To start with, we give the analogous of the definitions (3.5), (3.6) and (3.7) of Section 3.1.1 that we need now; Figure 3.2 helps to visualize the new situation. More precisely, we set for brevity

$$t_0 := \frac{\tau(\Omega, \overline{m} + \frac{\widehat{m}}{2}) + \tau(\Omega, \overline{m} - \widehat{m})}{2}\,, \qquad \bar{t} := \frac{\tau(\Omega, \overline{m} + \frac{\widehat{m}}{2}) - \tau(\Omega, \overline{m} - \widehat{m})}{2}\,;$$

keep in mind that, since $\overline{m} \in (\widehat{m}, 1 - \frac{\widehat{m}}{2})$, then $-\infty < \tau(\Omega, \overline{m} - \widehat{m}) < \tau(\Omega, \overline{m} + \frac{\widehat{m}}{2}) < +\infty$. For any $0 \le t \le \bar{t}$, we define

$$\Omega^+(t) := \Omega^l_{t_0 - t} \cup \Omega^r_{t_0 + t}\,, \qquad \Omega^-(t) := \Omega^r_{t_0 - t} \cap \Omega^l_{t_0 + t} = \Omega \setminus \Omega^+(t)\,,$$

$$\varepsilon(t) := \mathcal{H}^{N-1}(\Omega_{t_0 - t}) + \mathcal{H}^{N-1}(\Omega_{t_0 + t})\,, \qquad m(t) := \big|\Omega^-(t)\big| = \int_0^t \varepsilon(s)\, ds \le \frac{3}{2}\,\widehat{m}\,.$$

Moreover, having fixed an orthonormal set $\{u_1, u_2, \ldots, u_k\}$ of eigenfunctions with unit L^2 norm corresponding to the first k eigenvalues of Ω, for every $1 \le i \le k$ and $0 \le t \le \bar{t}$ we define

$$\delta_i(t) := \int_{\Omega_{t_0 - t}} |Du_i|^2 + \int_{\Omega_{t_0 + t}} |Du_i|^2\,, \qquad \mu_i(t) := \int_{\Omega_{t_0 - t}} u_i^2 + \int_{\Omega_{t_0 + t}} u_i^2\,.$$

In analogy with (3.8), we define again $\delta(t) = \sum_{i=1}^k \delta_i(t)$, and we set again

$$\phi(t) := \sum_{i=1}^k \int_{\Omega^-(t)} |Du_i|^2 = \int_0^t \delta(s)\, ds\,.$$

Our strategy to prove Lemma 3.10 is very similar to what we did to show Lemma 3.4; in fact,

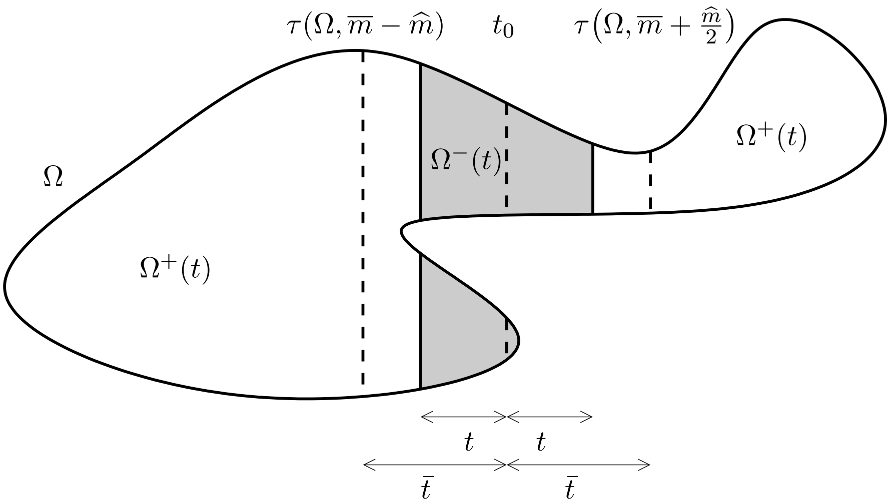

Figure 3.2: A set Ω and the corresponding quantities t_0, $\bar{t}$ and sets $\Omega^+(t)$ (white) and $\Omega^-(t)$ (shaded).

basically the only difference is that to show the analogous of Lemma 3.5 we cannot rely on the symmetrization of $\Omega^-(t)$. Let us see how to overcome this difficulty.

Lemma 3.11. *There exists a small constant $\nu = \nu(k, K, N) < 1$ such that, if Ω and $\overline{m}$ are as in the assumptions of Lemma 3.10, and $0 \le t \le \bar{t}$ is such that $\varepsilon(t), \delta(t) \le \nu$, then for every $1 \le i \le k$ one has*

$$\int_{\Omega^-(t)} u_i^2 \le C\varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,, \qquad \int_{\Omega^-(t)} |Du_i|^2 \le C\varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,. \tag{3.37}$$

Proof. Consider the "external cylinders"

$$Q_1 := \big(t_0 - t - \sigma_1, t_0 - t\big) \times \Omega_{t_0 - t}\,, \qquad Q_2 := \big(t_0 + t, t_0 + t + \sigma_2\big) \times \Omega_{t_0 + t}\,,$$

where

$$\sigma_1 = \mathcal{H}^{N-1}(\Omega_{t_0 - t})^{\frac{1}{N-1}}\,, \qquad \sigma_2 = \mathcal{H}^{N-1}(\Omega_{t_0 + t})^{\frac{1}{N-1}}\,,$$

in perfect analogy with (3.15) and (3.16). Calling $U = \Omega^-(t) \cup Q_1 \cup Q_2$, we can extend (3.17) to obtain the following definition of $\widetilde{u}_i \in H^1_0(U)$,

$$\widetilde{u}_i(x,y) := \begin{cases} u_i(x,y) & \text{if } (x,y) \in \Omega^-(t)\,, \\ \dfrac{x-(t_0-t-\sigma_1)}{\sigma_1}\, u_i(t_0-t,y) & \text{if } (x,y) \in Q_1\,, \\ \dfrac{(t_0+t+\sigma_2)-x}{\sigma_2}\, u_i(t_0+t,y) & \text{if } (x,y) \in Q_2\,. \end{cases}$$

Applying Lemma 3.6 to the two cylinders Q_1 and Q_2, and comparing the present definitions of ε and δ_i with those that we used in Section 3.1.1, (3.19) gives us

$$\int_{Q_1\cup Q_2} |D\widetilde{u}_i|^2 \leq C_2 \varepsilon(t)^{\frac{1}{N-1}} \delta_i(t)\,.$$

We can then obtain an estimate between $\int_{\Omega^-(t)} u_i^2$ and $\int_{\Omega^-(t)} |Du_i|^2$ similar to (3.11), first noticing that

$$\frac{\lambda_1(B_N)}{\big(m(t)+\varepsilon(t)^{\frac{N}{N-1}}\big)^{\frac{2}{N}}} \leq \frac{\lambda_1(B_N)}{|U|^{\frac{2}{N}}} \leq \lambda_1(U) \leq \mathcal{R}(\widetilde{u}_i, U) \leq \frac{\displaystyle\int_{\Omega^-(t)} |Du_i|^2 + \int_{Q_1\cup Q_2} |D\widetilde{u}_i|^2}{\displaystyle\int_{\Omega^-(t)} u_i^2}\,,$$

and then deducing, recalling (3.2) and choosing ν small enough,

$$\begin{aligned} \int_{\Omega^-(t)} u_i^2 &\leq \frac{\big(m(t)+\varepsilon(t)^{\frac{N}{N-1}}\big)^{\frac{2}{N}}}{\lambda_1(B_N)} \bigg(\int_{\Omega^-(t)} |Du_i|^2 + C_2\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t) \bigg) \\ &\leq \frac{1}{2^{1+\frac{1}{N}}K} \int_{\Omega^-(t)} |Du_i|^2 + \frac{C_2}{2^{1+\frac{1}{N}}K}\,\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t)\,. \end{aligned} \tag{3.38}$$

Observe that $\nu = \nu(k,K,N)$ can be chosen so small that the last estimate implies

$$\int_{\Omega^-(t)} u_i^2 \leq \frac{1}{2}\,.$$

We can also generalize (3.13); in fact, since the very same argument used in (3.9) again ensures that $\mu_i(t) \leq C\varepsilon(t)^{\frac{2}{N-1}}\delta_i(t)$, we can obtain

$$\begin{aligned} \int_{\Omega^-(t)} |Du_i|^2 &= \int_{\Omega^-(t)} \lambda_i u_i^2 + \int_{\Omega_{t_0-t}\cup\Omega_{t_0+t}} u_i\,\frac{\partial u_i}{\partial\nu} \leq K\int_{\Omega^-(t)} u_i^2 + \sqrt{\mu_i(t)\delta_i(t)} \\ &\leq K\int_{\Omega^-(t)} u_i^2 + C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t)\,. \end{aligned} \tag{3.39}$$

Putting together (3.38) and (3.39) gives (3.37). □

We need now to extend the result of Lemma 3.8 to our new setting. To do so, going on in analogy with Section 3.1.1, we give the following definition.

Definition 3.12. *Let Ω, $\overline{m}$ and $1 \leq t \leq \bar{t}$ be as in the assumptions of Lemma 3.11. Consider the "internal cylinders"*

$$Q_1 := \big(t_0 - t, t_0 - t + \sigma_1\big) \times \Omega_{t_0 - t}\,, \qquad\qquad Q_2 := \big(t_0 + t - \sigma_2, t_0 + t\big) \times \Omega_{t_0 + t}\,,$$

where

$$\sigma_1 = \mathcal{H}^{N-1}(\Omega_{t_0-t})^{\frac{1}{N-1}}\,, \qquad\qquad \sigma_2 = \mathcal{H}^{N-1}(\Omega_{t_0+t})^{\frac{1}{N-1}}\,,$$

and notice that by the assumption on $\varepsilon(t)$ and the fact that $t \geq 1$ one has $Q_1 \cap Q_2 = \emptyset$. The set $\widetilde{\Omega}(t)$ is defined as

$$\begin{aligned}\widetilde{\Omega}(t) := \Big\{(x,y) \in \mathbb{R}^N : \textit{either } x \leq t_0,\ \big(x - t + \sigma_1, y\big) \in \Omega^+(t) \cup Q_1\,,\\ \textit{or } x \geq t_0,\ \big(x + t - \sigma_2, y\big) \in \Omega^+(t) \cup Q_2\Big\}\,,\end{aligned}$$

see Figure 3.3. Notice that

$$\begin{aligned}\big|\widetilde{\Omega}(t)\big| &= \big|\Omega^+(t)\big| + \big|Q_1\big| + \big|Q_2\big| = 1 - m(t) + \mathcal{H}^{N-1}\big(\Omega_{t_0-t}\big)^{\frac{N}{N-1}} + \mathcal{H}^{N-1}\big(\Omega_{t_0+t}\big)^{\frac{N}{N-1}}\\ &\leq 1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}\,.\end{aligned}$$

Moreover, define again the rescaled set

$$\widehat{\Omega}(t) := \big|\widetilde{\Omega}(t)\big|^{-\frac{1}{N}}\widetilde{\Omega}(t)\,.$$

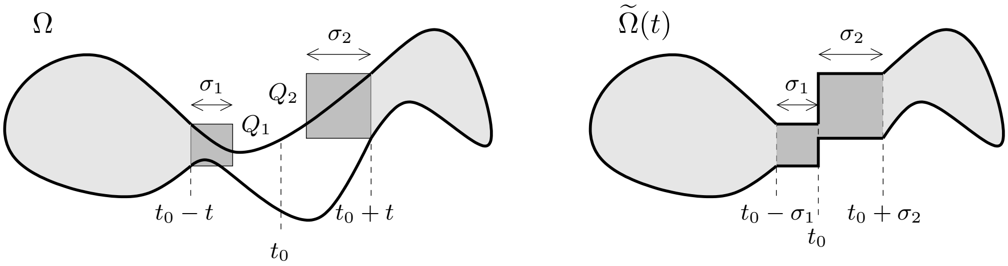

Figure 3.3: A set Ω and the corresponding set $\widetilde{\Omega}(t)$.

With this definition of the sets $\widetilde{\Omega}(t)$ and with the obvious extension of (3.17) in order to define $\widetilde{u}_i \in H^1_0(\widetilde{\Omega}(t))$, we can now literally repeat the proofs of Lemmas 3.6, 3.7, 3.8 and 3.9, the unique difference being the substitution of $Q(t)$ with $Q_1 \cup Q_2$, and of Ω_t with $\Omega_{t_0+t} \cup \Omega_{t_0-t}$. We obtain then the following result, which holds up to possibly decrease the constant $\nu = \nu(k, K, N)$ of Lemma 3.11.

Lemma 3.13. *Let Ω be as in the assumptions of Lemma 3.10, and let $1 \leq t \leq \bar{t}$. There exists $C_6 = C_6(k, K, N)$ such that exactly one of the three following conditions hold:*

(1) $\max\big\{\varepsilon(t),\, \delta(t)\big\} > \nu$;

(2) *(1) does not hold and* $m(t) \leq C_6\big(\varepsilon(t)+\delta(t)\big)\varepsilon(t)^{\frac{1}{N-1}}$*;*

(3) *(1) and (2) do not hold and for every* $1 \leq i \leq k$*, one has* $\lambda_i\big(\widehat{\Omega}(t)\big) < \lambda_i(\Omega)$*.*

In particular, if condition (3) holds for t *and* $m(t) \geq \widehat{m}/2$*, then for every* $1 \leq i \leq k$ *one has* $\lambda_i\big(\widehat{\Omega}(t)\big) < \lambda_i(\Omega) - \eta$*, being* $\eta = \eta(k, K, N) > 0$*.*

We can now conclude this section by presenting the proof of Lemma 3.10, which will be a minor modification of the proof of Lemma 3.4.

Proof of Lemma 3.10. First of all, we want to show that it is admissible to assume

$$m(t) > 0 \qquad\qquad \forall\, t > 0\,. \tag{3.40}$$

In fact, suppose that it is not so, and let $\tau = \max\{0 \leq t \leq \bar t : m(t) = 0\} > 0$. Then, Ω is the disjoint union of $\Omega \cap \{x > t_0 + \tau\}$ and $\Omega \cap \{x < t_0 - \tau\}$, and it does not intersect the whole strip $\{t_0 - \tau < x < t_0 + \tau\}$. Therefore, replacing Ω with $\big\{(x+\tau, y) : x < t_0, (x,y) \in \Omega\big\} \cup \big\{(x-\tau, y) : x > t_0, (x,y) \in \Omega\big\}$, that is, moving closer the two disjoint parts of Ω, does not change any of the eigenvalues of Ω and is clearly admissible for the proof of the lemma; moreover, the property (3.40) of course holds true for this new set. Hence, from now on we directly assume that (3.40) holds true for Ω.

Define now $\hat t$ analogously to (3.26) by setting

$$\hat t := \sup\Big\{1 \leq t \leq \bar t :\ \text{condition (3) of Lemma 3.13 holds for } t\Big\}\,,$$

with the convention that, if condition (3) is false for every $1 \leq t \leq \bar t$, then $\hat t = 1$. Again we define A and B as

$$\begin{aligned}
A :&= \Big\{t \in (\hat t, \bar t) :\ \text{condition (1) of Lemma 3.13 holds for } t\Big\}\,,\\
B :&= \Big\{t \in (\hat t, \bar t) :\ \text{condition (2) of Lemma 3.13 holds for } t \text{ and } m(t) > 0\Big\}\,.
\end{aligned}$$

The same argument of the proof of Lemma 3.4 gives then

$$\big|A\big| + \big|B\big| \leq C_7 = C_7(k, K, N)\,, \tag{3.41}$$

and we are going to show the thesis with the choice

$$R_2 = 4C_7 + 8\,, \qquad\qquad \Gamma_2 = 2^{[K/\eta]+1}\,,$$

being η the constant of Lemma 3.13. We can again subdivide the possible cases for Ω.
<u>Case I.</u> *One has* $\hat t = 1$*.*
In this case, by (3.40) one has that every $1 < t \leq \bar t$ belongs either to A or to B, thus by (3.41)

$\bar t \leq C_7+1$; as a consequence, the choice $\widehat\Omega=\Omega$ satisfies the requirements of the lemma. Indeed, while the right condition of (3.36) is obviously true, the left one follows just noticing that

$$\begin{aligned}W\big(\widehat\Omega,0,\overline{m}\big) &= W\big(\Omega,0,\overline{m}\big) = \tau(\Omega,\overline{m})-\tau(\Omega,\overline{m}-\widehat m)+W\Big(\Omega,0,\overline{m}-\widehat m\Big)\\ &\leq \tau\Big(\Omega,\overline{m}+\frac{\widehat m}{2}\Big)-\tau(\Omega,\overline{m}-\widehat m)+W\Big(\Omega,0,\overline{m}-\widehat m\Big) = 2\bar t+W\Big(\Omega,0,\overline{m}-\widehat m\Big)\\ &\leq 2\big(C_7+1\big)+W\Big(\Omega,0,\overline{m}-\widehat m\Big)\,.\end{aligned}$$

Case II. One has $\hat t>1$.
In this case, again by (3.40) we know that $A\cup B$ contains the whole segment $(\hat t,\bar t)$, thus $\bar t\leq \hat t+C_7$ by (3.41). If we choose $t^\star\in(\hat t-1,\hat t)$ for which condition (3) holds and define $U_1:=\widehat\Omega(t^\star)$, we know by construction that $|U_1|=1$ and $\lambda_i(U_1)<\lambda_i(\Omega)$ for every $1\leq i\leq k$. As in the proof of Lemma 3.4, the fact that $|\widetilde\Omega(t^\star)|\geq 1-\frac{3}{2}\widehat m\geq 1/2$ ensures that $\operatorname{diam}\big(\pi_p(U_1)\big)\leq 2\operatorname{diam}\big(\pi_p(\Omega)\big)$ for each $2\leq p\leq N$. On the other hand, concerning the width of U_1, recalling the definition of $\widetilde\Omega(t^\star)$ and observing that $(\overline{m}-\widehat m)\big|\widetilde\Omega(t^\star)\big|<\overline{m}-\widehat m$ one finds

$$\begin{aligned}W\Big(U_1,0,\overline{m}-\widehat m\Big) &= W\Big(\widehat\Omega(t^\star),0,\overline{m}-\widehat m\Big) = \big|\widetilde\Omega(t^\star)\big|^{-\frac1N}W\Big(\widetilde\Omega(t^\star),0,(\overline{m}-\widehat m)\big|\widetilde\Omega(t^\star)\big|\Big)\\ &\leq 2\bigg(\tau\Big(\widetilde\Omega(t^\star),(\overline{m}-\widehat m)\big|\widetilde\Omega(t^\star)\big|\Big)-\tau\Big(\widetilde\Omega(t^\star),0\Big)\bigg)\\ &= 2\bigg(\tau\Big(\Omega,(\overline{m}-\widehat m)\big|\widetilde\Omega(t^\star)\big|\Big)-\tau\Big(\Omega,0\Big)\bigg) = 2W\Big(\Omega,0,(\overline{m}-\widehat m)\big|\widetilde\Omega(t^\star)\big|\Big)\\ &\leq 2W\Big(\Omega,0,\overline{m}-\widehat m\Big)\,.\end{aligned}$$

Summarizing, we have found that

$$\lambda_i(U_1)<\lambda_i(\Omega)\,,\qquad \frac{\operatorname{diam}\big(\pi_p(U_1)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)}\leq 2\,,\qquad \frac{W\big(U_1,0,\overline{m}-\widehat m\big)}{W\big(\Omega,0,\overline{m}-\widehat m\big)}\leq 2\,. \tag{3.42}$$

To conclude the inspection of the validity of (3.36), we will again need to consider separately two subcases.

Case IIa. One has $\hat t>1$ and $m(t^\star)\leq\widehat m/2$.
In this case, we can quickly observe that

$$\begin{aligned}W\big(U_1,0,\overline{m}\big) &= \big|\widetilde\Omega(t^\star)\big|^{-\frac1N}W\Big(\widetilde\Omega(t^\star),0,\overline{m}\big|\widetilde\Omega(t^\star)\big|\Big)\leq 2W\Big(\widetilde\Omega(t^\star),0,\overline{m}+\frac{\widehat m}{2}-\big|\Omega^-(t^\star)\big|\Big)\\ &\leq 2W\big(\Omega,0,\overline{m}-\widehat m\big)+4\big(\bar t-t^\star+1\big)\leq 2W\big(\Omega,0,\overline{m}-\widehat m\big)+4\big(C_7+2\big)\,,\end{aligned} \tag{3.43}$$

and, together with (3.42), this concludes the proof of (3.36) and of the lemma with the choice $\widehat\Omega=U_1$.

Case IIb. One has $\hat t>1$ and $m(t^\star)>\widehat m/2$.
Let us conclude with this last case. By Lemma 3.13, in place of (3.42) we have then

$$\lambda_i(U_1)<\lambda_i(\Omega)-\eta\,,\qquad \frac{\operatorname{diam}\big(\pi_p(U_1)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)}\leq 2\,,\qquad \frac{W\big(U_1,0,\overline{m}-\widehat m\big)}{W\big(\Omega,0,\overline{m}-\widehat m\big)}\leq 2\,,$$

which still does not guarantee the validity of (3.36). However, we can apply the construction above to U_1: if U_1 is in Case I, then the choice $\widehat{\Omega} = U_1$ concludes the proof; otherwise, there exists an open set U_2 of unit measure satisfying

$$\lambda_i(U_2) < \lambda_i(\Omega) - \eta\,, \qquad \frac{\operatorname{diam}\big(\pi_p(U_2)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq 4\,, \qquad \frac{W\big(U_2, 0, \overline{m} - \widehat{m}\big)}{W\big(\Omega, 0, \overline{m} - \widehat{m}\big)} \leq 4\,. \tag{3.44}$$

If U_1 is in Case IIa then (3.43) gives

$$W\big(U_2, 0, \overline{m}\big) \leq 2W\big(U_1, 0, \overline{m} - \widehat{m}\big) + 4\big(C_7 + 2\big) \leq 4W\big(\Omega, 0, \overline{m} - \widehat{m}\big) + 4\big(C_7 + 2\big)\,,$$

so the choice $\widehat{\Omega} = U_2$ concludes the proof. Instead, if U_1 is in Case IIb then the first inequality of (3.44) becomes $\lambda_i(U_2) < \lambda_i(\Omega) - 2\eta$ for every $1 \leq i \leq k$. The obvious iteration ensures us that, if the proof has not been obtained after ℓ steps, then there must be some open set U_ℓ satisfying

$$\lambda_i(U_\ell) < \lambda_i(\Omega) - \ell\eta\,, \qquad \frac{\operatorname{diam}\big(\pi_p(U_\ell)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq 2^\ell\,, \qquad \frac{W\big(U_\ell, 0, \overline{m} - \widehat{m}\big)}{W\big(\Omega, 0, \overline{m} - \widehat{m}\big)} \leq 2^\ell\,.$$

Since this is not possible for $\ell > K/\eta$, the iteration must stop at some $\ell \leq [K/\eta]$, and thus we conclude the proof thanks to the choice of Γ_2. □

3.1.3 Proof of Proposition 3.2

We are finally in position to give the proof of Proposition 3.2, which is now a simple consequence of Lemma 3.4 and Lemma 3.10.

Proof of Proposition 3.2. Let us pick a generic open set Ω with $\lambda_k(\Omega) \leq K$. Applying Lemma 3.4 to Ω, we find a set E_1 with

$$\lambda_i(E_1) \leq \lambda_i(\Omega)\,, \qquad W(E_1, 0, \widehat{m}) \leq R_1\,, \qquad \frac{\operatorname{diam}\big(\pi_p(E_1)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq \Gamma_1\,,$$

for every $2 \leq p \leq N$. Then, we apply Lemma 3.10 to E_1 with $\overline{m} = 2\widehat{m}$ finding E_2 which satisfies

$$\lambda_i(E_2) \leq \lambda_i(\Omega)\,, \qquad W(E_2, 0, 2\widehat{m}) \leq R_2 + \Gamma_2 R_1\,, \qquad \frac{\operatorname{diam}\big(\pi_p(E_2)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq \Gamma_2\Gamma_1\,.$$

Iterating, for any $\ell \geq 3$ such that $\ell\widehat{m} \leq 1 - \frac{\widehat{m}}{2}$ we apply Lemma 3.10 to $E_{\ell-1}$ with $\overline{m} = \ell\widehat{m}$ finding E_ℓ such that

$$\lambda_i(E_\ell) \leq \lambda_i(\Omega)\,, \quad W(E_\ell, 0, \ell\widehat{m}) \leq R_2\frac{\Gamma_2^{\ell-1} - 1}{\Gamma_2 - 1} + \Gamma_2^{\ell-1}R_1\,, \quad \frac{\operatorname{diam}\big(\pi_p(E_\ell)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq \Gamma_2^{\ell-1}\Gamma_1\,.$$

Possibly applying a last time Lemma 3.10 with $\overline{m} = 1 - \widehat{m}$, we have then found an open set E satisfying

$$\lambda_i(E) \leq \lambda_i(\Omega)\,, \quad W(E, 0, 1 - \widehat{m}) \leq R_2\frac{\Gamma_2^{[1/\widehat{m}]-1} - 1}{\Gamma_2 - 1} + \Gamma_2^{[1/\widehat{m}]-1}R_1, \quad \frac{\operatorname{diam}\big(\pi_p(E)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq \Gamma_2^{[1/\widehat{m}]-1}\Gamma_1.$$

Calling E' the set obtained by reflecting E with respect to the plane $\{x = 0\}$, the above estimates become

$$\lambda_i(E') \leq \lambda_i(\Omega)\,, \quad W(E', \widehat{m}, 1) \leq R_2 \frac{\Gamma_2^{[1/\widehat{m}]-1} - 1}{\Gamma_2 - 1} + \Gamma_2^{[1/\widehat{m}]-1} R_1\,, \quad \frac{\operatorname{diam}\big(\pi_p(E')\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq \Gamma_2^{[1/\widehat{m}]-1}\Gamma_1\,,$$

so that applying once again Lemma 3.4 to E' we find a set F_1 satisfying

$$\lambda_i(F_1) \leq \lambda_i(\Omega)\,, \qquad \operatorname{diam}\big(\pi_1(F_1)\big) = W(F_1, 0, 1) \leq R_3\,, \qquad \frac{\operatorname{diam}\big(\pi_p(F_1)\big)}{\operatorname{diam}\big(\pi_p(\Omega)\big)} \leq \Gamma_3\,,$$

having set

$$R_3 := R_1 + \Gamma_1 R_2 \frac{\Gamma_2^{[1/\widehat{m}]-1} - 1}{\Gamma_2 - 1} + \Gamma_1 \Gamma_2^{[1/\widehat{m}]-1} R_1\,, \qquad \Gamma_3 := \Gamma_2^{[1/\widehat{m}]-1}\Gamma_1^2\,.$$

We can now repeat the whole construction, using as starting set F_1 in place of Ω, and using the second coordinate in place of the first one. We will end up with a set F_2 with

$$\lambda_i(F_2) \leq \lambda_i(F_1) \leq \lambda_i(\Omega) \quad \forall\, 1 \leq i \leq k\,, \qquad \operatorname{diam}\big(\pi_2(F_2)\big) \leq R_3\,,$$

and such that for every $p \neq 2$ it is $\operatorname{diam}\big(\pi_p(F_2)\big) \leq \Gamma_3 \operatorname{diam}\big(\pi_p(F_1)\big)$. In particular, choosing $p = 1$ we discover that $\operatorname{diam}\big(\pi_1(F_2)\big) \leq \Gamma_3 R_3$. We have now to iterate also this argument: for any $3 \leq j \leq N$ we repeat the above construction starting from F_{j-1} and using the j-th coordinate in the whole procedure, obtaining a set F_j which satisfies

$$\lambda_i\big(F_j\big) \leq \lambda_i(\Omega) \quad \forall 1 \leq i \leq k\,, \qquad \operatorname{diam}\big(\pi_p(F_j)\big) \leq \Gamma_3^{j-p} R_3 \quad \forall\, 1 \leq p \leq j\,.$$

The thesis is then finally obtained by defining $\widehat{\Omega} := F_N$, being $R := \Gamma_3^{N-1} R_3$. □

3.2 Boundedness of all minimizers

With a small improvement in the proof of Theorem 3.1, it is possible to prove that all the minimizers for problem (3.1) have diameter uniformly bounded by a constant depending only on k, N, if the functional F is *weakly strictly increasing*. Without this further assumption also a constant functional would be admissible and in that case the uniform boundedness of *all* minimizers is clearly false. The results of this section come from [M3].

Definition 3.14. *A functional $F\colon \mathbb{R}^k \to \mathbb{R}$ is said to be* weakly strictly increasing *if for every $(x_1, \dots, x_N), (y_1, \dots, y_k) \in \mathbb{R}^k$ with $x_i > y_i$ for all $i = 1, \dots, k$ we have*

$$F(x_1, \dots, x_k) > F(y_1, \dots, y_k).$$

We now state the Theorem and then we pass to its prove. As in the previous section we first study the "tails", then the interior and at the end we put all the informations together.

Theorem 3.15. *Let $k, N \in \mathbb{N}$ and $F\colon \mathbb{R}^k \to \mathbb{R}$ be weakly strictly increasing and l.s.c.. Then* every *minimizer for problem* (3.1) *is contained (up to translations) in an N-cube Q_R, where $R = R(k, N)$.*

The idea of the proof is to show that, given a minimizer Ω^* for (3.1) and a sequence of open sets $(\Omega_n)_{n\in\mathbb{N}}$ which γ-converges to Ω^*, then either Ω_n are uniformly bounded (and so it is Ω^*) or it is possible to find another minimizing sequence that contradicts the minimality of Ω^*. During all the proof we will use the same notation of Section 3.1.

3.2.1 "Tails"

Throughout all this subsection and the next one, $\Omega \subset \mathbb{R}^N$ is an open set of unit volume such that $\lambda_k(\Omega) \le K$. We will choose $K := M\lambda_k(B_N)$, where M is the constant from Theorem 2.12, so this big constant actually depends only on k, N. We also remind that the constant $\widehat{m}(k, N)$ is taken as in (3.2). We begin by proving a slightly stronger version of Lemma 3.8.

Lemma 3.16. *Let Ω be an open set of unit volume, with $\lambda_k(\Omega) \le K$. There exist a constant $\widetilde{C}_3 = \widetilde{C}_3(k, N)$ such that, if $t \le \bar{t}$, then*

$$\lambda_j(\widetilde{\Omega}(t)) \le \lambda_j(\Omega) + \widetilde{C}_3 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) \qquad \forall 1 \le j \le k.$$

Proof. It is clear that, thanks to Lemma 3.8, whenever $\varepsilon(t), \delta_i(t) \le \nu$ for all $i = 1, \dots, k$, then the thesis is true with $\widetilde{C}_3 = C_3$, since $\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \le \varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}$.

We can now focus on the case when either $\varepsilon(t) > \nu$ or $\delta_i(t) > \nu$ for some i. Then, we remind that, since the first eigenfunction has not orthogonality constraints, Lemma 3.7 assures:

$$\lambda_1(\widetilde{\Omega}(t)) \le \lambda_1(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}}\delta_i(t).$$

Thanks to Theorem 2.12 there exists a constant $M > 0$ such that $\frac{\lambda_k(\Omega)}{\lambda_1(\Omega)} \le M$ for all $\Omega \subset \mathbb{R}^N$. Hence we can write, for all $1 \le j \le k$:

$$\lambda_j(\widetilde{\Omega}(t)) \le M\lambda_1(\widetilde{\Omega}(t)) \le M\left(\lambda_1(\Omega) + C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\right).$$

Moreover it is possible to find a big constant $A = A(k, N)$, such that $MK \le A\nu^{\frac{N}{N-1}}$ and then, defining $\widetilde{C}_3 = A + MC$, we can conclude the computations above:

$$\begin{aligned}\lambda_j(\widetilde{\Omega}(t)) &\le M\left(K + C\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\right) \le A\nu^{\frac{N}{N-1}} + MC\varepsilon(t)^{\frac{1}{N-1}}\delta(t)\\ &\le \lambda_j(\Omega) + A\nu^{\frac{N}{N-1}} + MC\varepsilon(t)^{\frac{1}{N-1}}\delta(t) \le \lambda_j(\Omega) + \widetilde{C}_3\left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right).\end{aligned}$$

□

We are now in position to state and prove the main Lemma of this section.

Lemma 3.17. *Let Ω be an open set of unit volume, with $\lambda_k(\Omega) \le K$ and $t \le \bar{t}$. Then there exists a constant $\widetilde{C}_4 = \widetilde{C}_4(k, N)$ such that exactly one of the following situations happens.*

(1) $m(t) \leq \widetilde{C}_4 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)$.

(2) (1) does not hold and for all $1 \leq i \leq k$, $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega)$. *Moreover for every* $\widetilde{m} > 0$ *such that* $m(t) \geq \widetilde{m}$, *there exists an* $\eta = \eta(N, \widetilde{m})$ *such that for all* $1 \leq i \leq k$,

$$\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega) - \eta.$$

Proof. From Lemma 3.16 we have

$$\lambda_i(\widetilde{\Omega}(t)) \leq \lambda_i(\Omega) + \widetilde{C}_3 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) \qquad \forall 1 \leq i \leq k,$$

moreover, putting in account that $|\widetilde{\Omega}(t)| = |\Omega^+(t)| + |Q(t)| = 1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}$ and the scaling of the eigenvalues, then for all $1 \leq i \leq k$

$$\begin{aligned} \lambda_i(\widehat{\Omega}(t)) &\leq \left(1 - m(t) + \varepsilon(t)^{\frac{N}{N-1}}\right)^{\frac{2}{N}} \left(\lambda_i(\Omega) + \widetilde{C}_3 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)\right) \\ &\leq \lambda_i(\Omega) - \frac{2}{N}\lambda_1(B_N)m(t) + \frac{2K}{N}\varepsilon(t)^{\frac{N}{N-1}} + \widetilde{C}_3 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) \\ &- \frac{2}{N}m(t)\widetilde{C}_3 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) + \frac{2}{N}\widetilde{C}_3\varepsilon(t)^{\frac{N}{N-1}} \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right). \end{aligned} \tag{3.45}$$

Then if $m(t) \leq \widetilde{C}_4 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)$, condition (1) holds true; otherwise

$$m(t) > \widetilde{C}_4 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right),$$

and we can choose $\widetilde{C}_4 \geq 1$ so that $m(t) \geq \varepsilon(t)^{\frac{N}{N-1}}$. Thus from the two last terms of (3.45), we have

$$-\frac{2}{N}m(t)\widetilde{C}_3 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) + \frac{2}{N}\widetilde{C}_3\varepsilon(t)^{\frac{N}{N-1}} \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right) \leq 0.$$

This allows us to conclude, choosing $\widetilde{C}_4 \geq \frac{2K + N\widetilde{C}_3}{\lambda_1(B_N)}$ and obtaining:

$$\lambda_i(\widehat{\Omega}(t)) - \lambda_i(\Omega) \leq -\frac{\lambda_1(B_N)}{N}m(t) < 0,$$

that is condition (2). Moreover if $m(t) \geq \widetilde{m}$, then we can improve the above estimate:

$$\lambda_i(\widehat{\Omega}(t)) - \lambda_i(\Omega) \leq -\frac{\lambda_1(B_N)}{N}\widetilde{m} = -\eta(N, \widetilde{m}) < 0,$$

and the proof is concluded. □

We introduce the following notations. Given an open set Ω as in the hypotheses of Lemma 3.17, we set

$$\hat{t} = \sup\left\{t \in (-\infty, \bar{t}) \ : \ \text{condition (2) of Lemma 3.17 holds for } t\right\}, \tag{3.46}$$

with the usual convention that $\hat{t} = -\infty$ if condition (2) is false for every $t \leq \bar{t}$. If $\hat{t} > -\infty$, then $m(\hat{t}) > 0$ and we choose some $t^\star \in [\hat{t} - 1, \hat{t}]$ for which condition (2) holds. The following Lemma concludes this Section.

Lemma 3.18. *Let $(\Omega_n)_n$ be as in the hypotheses of Lemma 3.17 and $\Omega_n \xrightarrow{\gamma} \Omega$.*

(a) If there exists a subsequence (not relabeled) such that, for all n, $m(t^\star(n)) \geq \widetilde{m} > 0$ for some $\widetilde{m} > 0$, then Ω is not optimal for problem (3.1).

(b) If there exists a subsequence such that $\hat{t}(n) = -\infty$ for all n, then there exists $R_1 = R_1(k, N) > 0$ such that $W(\Omega, 0, \widehat{m}) \leq R_1$.

(c) If there exists a subsequence such that $m(t^\star(n)) \to 0$ as $n \to \infty$, then we have again $W(\Omega, 0, \widehat{m}) \leq R_1$.

Proof. We introduce the following subsets of $(\hat{t}(n), \bar{t}(n))$ for all $n \in \mathbb{N}$:

$$A_1^n = \{t \in (\hat{t}(n), \bar{t}(n)) \ : \ \varepsilon(t) \geq \delta(t)\}, \qquad A_2^n = \{t \in (\hat{t}(n), \bar{t}(n)) \ : \ \varepsilon(t) < \delta(t)\}.$$

Then, using Lemma 3.17, it is clear that for all $t \in A_1^n$, $m(t) \leq 2C_4 \varepsilon(t)^{\frac{N}{N-1}}$, while for all $t \in A_2^n$, thanks to Lemma 3.6 and reminding (3.8), $\phi(t) \leq 2C_1 \delta(t)^{\frac{N}{N-1}}$. Hence, since $\varepsilon(t) = m'(t)$ and $\delta(t) = \phi'(t)$, we can work as in the proof of Lemma 3.4 and deduce that $|A_1^n \cup A_2^n| \leq \widetilde{C}_5 = \widetilde{C}_5(k, N)$.

If we are in case (b), since $\hat{t}(n) = -\infty$ for all n, then $W(\Omega_n, 0, \widehat{m}) \leq |A_1^n \cup A_2^n| \leq \widetilde{C}_5$ and the same is true for the γ-limit Ω.

On the other hand, if case (c) happens, in principle there could be some pieces of the limit Ω outside the bounded strip, but it is clear from property (3) of Lemma 2.5 that Ω must have zero capacity and not only zero Lebesgue measure outside the bounded strip. More precisely, we can choose (up to translations) the origin such that $m(0) = \widehat{m}$. Since Ω corresponds to a capacitary measure μ_Ω (see (2.3)), from point (c) we have that

$$\mu_\Omega \equiv \infty \qquad \text{in } \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \ : \ x < -\widetilde{C}_5\right\}.$$

Hence $W(\Omega, 0, \widehat{m}) \leq \widetilde{C}_5$.

At last we consider case (a). Thanks to Lemma 3.17, we have that for all n and for all $1 \leq i \leq k$,

$$\lambda_i(\widehat{\Omega}(t^\star(n))) < \lambda_i(\Omega) - \eta.$$

Hence, since we are supposing F to be weakly strictly increasing, we have a sequence $(\widehat{\Omega}(t^\star(n)))_n$ such that

$$\inf_n F\big(\lambda_1(\widehat{\Omega}(t^\star(n))), \ldots, \lambda_k(\widehat{\Omega}(t^\star(n)))\big) < F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)),$$

thus Ω can not be optimal for (3.1).

□

Remark 3.19. *Applying Lemma 3.18 to a sequence of open sets $(\Omega_n)_{n \in \mathbb{N}}$ satisfying the hypotheses of Lemma 3.17 and which γ-converges to Ω^*, since* (a), (b) *and* (c) *cover all the possible situations, we deduce*

$$W(\Omega^*, 0, \widehat{m}) \leq R_1(k, N).$$

3.2.2 Interior

In analogy with Lemma 3.17 we can state the following. In this case we have to keep in account also the case in which $\varepsilon(t)$ or $\delta(t)$ are greater than ν, which was necessary in the proof of Lemma 3.11 in the interior case, but for the remaining part the proof is completely equal to Lemma 3.17.

Lemma 3.20. *Let Ω be a set as in Lemma 3.17 and let $1 \leq t \leq \bar{t}$. There exists a constant $\widetilde{C}_6 = \widetilde{C}_6(k, N)$ such that exactly one of the three following conditions hold:*

(1) $\max\{\varepsilon(t), \delta(t)\} > \nu$;

(2) (1) does not hold and $m(t) \leq \widetilde{C}_6 \left(\varepsilon(t)^{\frac{N}{N-1}} + \delta(t)^{\frac{N}{N-1}}\right)$;

(3) (1) and (2) do not hold and for every $1 \leq i \leq k$, one has $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega)$. Moreover if $m(t) \geq \widetilde{m}$ for some $\widetilde{m} > 0$, then there exists $\eta = \eta(N, \widetilde{m}) > 0$ such that, for every $1 \leq i \leq k$, one has $\lambda_i(\widehat{\Omega}(t)) < \lambda_i(\Omega) - \eta$.

In order to prove the last Lemma, analogous to Lemma 3.10, we define $\hat{t}$ as in (3.46) by setting

$$\hat{t} := \sup\Big\{1 \leq t \leq \bar{t} : \text{condition (3) of Lemma 3.20 holds for } t\Big\},$$

with the convention that, if condition (3) is false for every $1 \leq t \leq \bar{t}$, then $\hat{t} = 1$. Moreover if $\hat{t} > 1$, then we choose some $t^\star \in (\hat{t} - 1, \hat{t}]$ for which condition (3) holds.

Lemma 3.21. *Let $(\Omega_n)_n$ be as in the hypotheses of Lemma 3.18, $\Omega_n \xrightarrow{\gamma} \Omega$ and $\overline{m} \in (\widehat{m}, 1 - \frac{\widehat{m}}{2})$.*

(a) If there exists a subsequence (not relabeled) such that, for all n, $m(t^\star(n)) \geq \widetilde{m} > 0$ for some $\widetilde{m}$, then Ω can not be optimal for problem (3.1).

(b) If there exists a subsequence such that $\hat{t}(n) = 1$ for all n, then there exists $R_2 = R_2(k, N) > 0$ such that $W(\Omega, \overline{m} - \widehat{m}, \overline{m}) \leq R_2$.

(c) If there exists a subsequence such that $m(t^\star(n)) \to 0$ as $n \to \infty$, then we have again $W(\Omega, \overline{m} - \widehat{m}, \overline{m}) \leq R_2$.

Proof. First of all in the proof of Lemma 3.13 we saw that it is admissible to assume

$$m(t) > 0 \qquad \forall\, t > 0\,.$$

We again define A^n and B^n as

$$\begin{aligned} A^n &:= \Big\{t \in (\hat{t}(n), \bar{t}(n)) : \text{condition (1) of Lemma 3.20 holds for } t\Big\},\\ B^n &:= \Big\{t \in (\hat{t}(n), \bar{t}(n)) : \text{condition (2) of Lemma 3.20 holds for } t \text{ and } m(t) > 0\Big\}. \end{aligned}$$

The same argument of the proof of Lemma 3.13 gives then

$$\big|A^n\big| + \big|B^n\big| \le \widetilde{C}_7 = \widetilde{C}_7(k, K, N), \qquad \forall n\,. \tag{3.47}$$

Then it is possible to conclude as in Lemma 3.18. If we are in case (b), since $\hat{t}(n) = 1$ for all n, then $W(\Omega_n, \overline{m} - \widehat{m}, \overline{m}) \le |A^n \cup B^n| \le \widetilde{C}_7 + 2$ and the same is true for the γ-limit Ω.

On the other hand, if case (c) happens, in principle there could be some pieces of the limit Ω outside the bounded strip, but property (3) of Lemma 2.5 assures that Ω must have zero capacity and not only zero Lebesgue measure outside the bounded strip. More precisely, we know that Ω corresponds to a capacitary measure μ_Ω and we call

$$\widetilde{\mu} := \mu_\Omega \llcorner (\tau(\Omega, \overline{m} - \widehat{m}), \tau(\Omega, \overline{m})) \times \mathbb{R}^{N-1},$$

in order to restrict ourselves to the strip we are interested in. In the hypothesis of case (c) we have that, up to translate together all the possible disconnected pieces,

$$\widetilde{\mu} \equiv \infty \qquad \text{in } \Big\{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} \;:\; x > \tau(\Omega, \overline{m} - \widehat{m}) + \widetilde{C}_7 + 2\Big\}.$$

Hence $W(\Omega, \overline{m} - \widehat{m}, \overline{m}) \le \widetilde{C}_7 + 2$.

At last we consider case (a). Analogously to Lemma 3.18, we have that for all n and for all $1 \le i \le k$,

$$\lambda_i(\widehat{\Omega}(t^\star(n))) < \lambda_i(\Omega) - \eta.$$

Hence, since we are supposing F to be weakly strictly increasing, we have a sequence $(\widehat{\Omega}(t^\star(n)))_n$ such that

$$\inf_n F(\lambda_1(\widehat{\Omega}(t^\star(n))), \ldots, \lambda_k(\widehat{\Omega}(t^\star(n)))) < F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)),$$

so Ω can not be optimal for (3.1). $\square$

Remark 3.22. *Applying Lemma 3.21 to a sequence of open sets $(\Omega_n)_{n\in\mathbb{N}}$ satisfying the hypotheses of Lemma 3.17 and which γ-converges to Ω^*, since* (a), (b) *and* (c) *cover all the possible situations, we deduce for all $\overline{m} \in (\widehat{m}, 1 - \frac{\widehat{m}}{2})$,*

$$W(\Omega^*, \overline{m} - \widehat{m}, \overline{m}) \le R_2(k, N).$$

3.2.3 Proof of Theorem 3.15

We are now in position to prove the main Theorem.

Proof of Theorem 3.15. Let Ω^* be a minimizer for problem (3.1); we aim to show that it is contained in an N-cube Q_R with edge of length $R = R(k, N)$. We define $K := M\lambda_k(B_N)$ and it must happen that $\lambda_k(\Omega^*) \le K$, otherwise $\lambda_i(B_N) < \lambda_i(\Omega^*)$ for all $i = 1, \ldots, k$, thus contradicting the optimality. We can then consider a sequence $(\Omega_n)_n$ of open sets with unit measure and such that $\lambda_k(\Omega_n) \le K$ for all n, which γ-converges to the set Ω^*.

First of all we apply Remark 3.19 and we have that $W(\Omega^*, 0, \widehat{m}) \leq R_1$, otherwise we contradict the optimality of Ω^*.
Then we apply Remark 3.22 with $\overline{m} = 2\widehat{m}$ and we have that $W(\Omega^*, \widehat{m}, 2\widehat{m}) \leq R_2$. We can iterate the application of Remark 3.22 with $\overline{m} = l\widehat{m}$ $(l \geq 3)$ till $l\widehat{m} \leq 1 - \frac{\widehat{m}}{2}$, thus obtaining, with a possible last application when $\overline{m} = 1 - \widehat{m}$:

$$W(\Omega^*, 0, 1 - \widehat{m}) \leq R_1 + lR_2.$$

Now we can apply the above estimate to the symmetric of the set Ω^* with respect to the plane $\{x = 0\}$, thus obtaining:

$$W(\Omega^*, \widehat{m}, 1) \leq R_1 + lR_2.$$

In conclusion we proved that $W(\Omega^*, 0, 1) \leq 2R_1 + 2lR_2$. Now we repeat the whole construction for all the other coordinates $(e_2, \ldots, e_N)$ instead of the first one. At the end, we have proved that the set Ω^* must be contained in an N-cube Q_R with edge of length $R = 2R_1 + 2lR_2$, thus the Theorem is proved. □

Remark 3.23. *The existence results by Buttazzo and Dal Maso [26], Bucur [16] and the one presented in this Chapter provide a definitive answer to the existence topic for a large class of spectral shape optimization problems. But there are still many other functionals for which it is not known the existence of an optimal set even in the class of quasi-open domains. An example of such a functional is* $F(\lambda_1, \lambda_2, \lambda_3) = \lambda_1/\lambda_3$*, which is not increasing in the third component, but for which we would conjecture existence of a minimizer, having in mind the results by Ashbaugh and Benguria.*

On the other hand, there are also functionals for which the lack of monotonicity leads to non-existence of minimizers. Here is a nice example, proposed by Bozhidar Velichkov (see [51]): we consider a spectral functional depending on an infinite *number of eigenvalues:*

$$\mathcal{F}(\Omega) := \sum_{k=1}^{\infty} a_k |\lambda_{k+1}(\Omega) - \lambda_k(\Omega)|,$$

where a_k *is an infinitesimal sequence (as* $k \to \infty$*), such that* $\sum_{k\in\mathbb{N}} a_k |\lambda_{k+1}(B) - \lambda_k(B)| = 1$*. Then the shape optimization problem*

$$\min \left\{ \mathcal{F}(\Omega) \ : \ \Omega \subset \mathbb{R}^N, \ \Omega \text{ quasi-open}, \ |\Omega| = 1 \right\},$$

has no solution. In fact, one can consider, for $n \in \mathbb{N}$, Ω_n *as the disjoint union of* n *balls of equal measure, which is a minimizing sequence such that* $\mathcal{F}(\Omega_n) \to 0$*, as* $n \to \infty$*. But it is also clear that there is no set of positive measure which has only one element in the spectrum.*

A more trivial example of functional of eigenvalues for which there exists no minimizer is $1/\lambda_1$*, which is clearly maximized by the ball, but has no minimum, unless we perform the minimization in the class* $\left\{ \Omega \subset \mathbb{R}^N, \text{ quasi-open}, |\Omega| \leq 1 \right\}$*, and we think also the empty set to be admissible, with* $\lambda_k(\emptyset) = \infty$ *for all* k*.*

Chapter 4

A two dimensional partial regularity result

The existence of minimizers for very general functionals depending on Dirichlet eigenvalues is now well understood, both in the bounded case (Theorem 2.20 by Buttazzo and Dal Maso) and in the unbounded one, thanks to the result presented in Chapter 3 and to Theorem 2.28 by Bucur. All these works provide an optimal set that, in general, is only *quasi-open*. A major problem is to understand whether optimality induces some better properties on the set, for example if minimizers are open. In this Chapter and in the next one we give some partial results in this direction and show what are the main difficulties in this analysis.

4.1 Introduction and statement of the main Theorem

We present here an 'elementary' method to prove a partial regularity result for an optimal set Ω. This method can be viewed somehow as the natural extension of the techniques used in Chapter 3 for proving a uniform boundedness estimate on minimizing sequences.

Unfortunately, in order to prove regularity of an optimal set, which means in this context its openness, we need to perform *outer* perturbations instead of the *inner* perturbations ("cuts") used for the boundedness topic. This turns out to be a lot tougher, because in order to contradict optimality, it is necessary to prove that the eigenvalues of the perturbed set decrease more than what they increase due to the rescaling to unit measure.

The results that we are able to obtain in this Chapter are valid only in a two dimensional setting and work only for a specific class of functionals. Even if in the next Chapter we will present a more general regularity result, we want to show also this different method, which seems to us worth of notice.

The basic idea, roughly speaking, is that if a set Ω, optimal for λ_k, is such that $\lambda_{k-1}(\Omega) < \lambda_k(\Omega)$, then level sets of the function u_k^2 can not have "holes" too small, otherwise it is possible to find a competitor (by "filling the holes") with lower λ_k and the other eigenvalues almost unchanged.

For this aim we need to introduce the definition of *shape supersolution*, proposed by Bucur in analogy with the one of subsolution, which is a main tool for the study of regularity by external perturbations.

Definition 4.1. *We say that a set $\Omega \subset \mathbb{R}^N$ is a* shape supersolution *for the functional $\mathcal{F}\colon \mathcal{S} \to \mathbb{R}$, defined on the class of Lebesgue measurable sets with finite measure $\mathcal{S}$, if it satisfies:*

$$\mathcal{F}(\Omega) \le \mathcal{F}(A) \qquad \forall\, A \supset \Omega.$$

Moreover, for sake of simplicity, we introduce the following notions, which will be used also in the next Chapter:

- given two points $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_p)$ in $\mathbb{R}^p$, we say that $x \ge y$ if $x_i \ge y_i$ for all $i = 1, \dots, p$;
- a function $F : \mathbb{R}^p \to \mathbb{R}$ is said to be *increasing* if $F(x) \ge F(y)$ whenever $x \ge y$;
- we say that $F : \mathbb{R}^p \to \mathbb{R}$ is *increasingly bi-Lipschitz* if F is increasing, Lipschitz, and there is a constant $L > 0$ such that
$$F(x) - F(y) \ge \frac{1}{L}\sum_{i=1}^{p}(x_i - y_i) \qquad \forall\, x \ge y\,.$$

For all the remainder of this Chapter we will work only in a two dimensional setting, for a reason that will be clear in the next pages. The key result is the following.

Theorem 4.2. *Let $k, p \in \mathbb{N}$, $F\colon \mathbb{R}^{p+1} \to \mathbb{R}$ be an increasingly bi-Lipschitz functional, $\Lambda > 0$ and $\Omega \subset \mathbb{R}^2$ be a shape supersolution for $F(\lambda_k(\cdot), \dots, \lambda_{k+p}(\cdot)) + \Lambda|\cdot|$, that is,*

$$F(\lambda_k(\Omega), \dots, \lambda_{k+p}(\Omega)) + \Lambda|\Omega| \le F(\lambda_k(A), \dots, \lambda_{k+p}(A)) + \Lambda|A| \qquad \forall\, \mathbb{R}^2 \supset A \supset \Omega, \tag{4.1}$$

and such that

1) $\lambda_k(\Omega) = \dots = \lambda_{k+p}(\Omega)$,

2) $k = 1$ *or* $\lambda_{k-1}(\Omega) < \lambda_k(\Omega)$ *and* $\lambda_{k+p}(\Omega) < \lambda_{k+p+1}(\Omega)$.

Then the set $\Omega^\star := \Big\{x \in \mathbb{R}^2 \,:\, u_k^2(x) + \dots + u_{k+p}^2(x) > 0\Big\}$ is open. Moreover $\Omega^\star$ has $\lambda_k(\Omega)$ as an eigenvalue and satisfies $|\Omega^\star| = |\Omega|$.

Theorem 4.2, in particular, assures that an optimal set Ω for (4.1) is open if $k = 1$, since in this case $\Omega^\star \subset \Omega$ is a supersolution for the functional in (6.1) with the same eigenvalues of Ω up to level $p + 1$.

From the Theorem above, we are able to treat a lot of other situations, using the fact that the notion of supersolution is very “robust” for this kind of problems. The following two lemmas will enlighten how this is possible.

Lemma 4.3. *Given $k, p \in \mathbb{N}$ and Ω a shape supersolution for the functional $\lambda_k(\cdot)+\cdots+\lambda_{k+p}(\cdot)+|\cdot|$, then Ω is also shape supersolution for the functional $\lambda_{k+i}(\cdot)+|\cdot|$, for all $i=0,\dots,p$.*

Proof. It follows from the monotonicity of eigenvalues with respect to inclusion and the definition of shape supersolution. □

Lemma 4.4. *Let $p \in \mathbb{N}$, $0 < k_0 < k_1 < \cdots < k_p$ and $F\colon \mathbb{R}^{p+1} \to \mathbb{R}$ be a functional increasingly bi-Lipschitz. If Ω is a shape supersolution for $F(\lambda_{k_0}(\cdot),\dots,\lambda_{k_p}(\cdot))+|\cdot|$, then it is also a shape supersolution for the functional $c\,(\lambda_k(\cdot)+\cdots+\lambda_{k+p}(\cdot))+|\cdot|$, for some positive constant c.*

Proof. We call L the bi-Lipschitz constant of F and it follows from the hypothesis that if $x \geq y$, then

$$F(x_0,\dots,x_p)-F(y_0,\dots,y_p) \geq \frac{1}{L}\sum_{i=0}^{p}(x_i-y_i).$$

Hence, using the definition of shape supersolution and the monotonicity of Dirichlet eigenvalues, we have

$$|A|-|\Omega| \geq F\big(\lambda_{k_0}(\Omega),\dots,\lambda_{k_p}(\Omega)\big)-F\big(\lambda_{k_0}(A),\dots,\lambda_{k_p}(A)\big) \geq \sum_{i=0}^{p}\Big(\lambda_{k_i}(\Omega)-\lambda_{k_i}(A)\Big) \qquad \forall\, A \supset \Omega,$$

from which the thesis follows easily. □

Remark 4.5. *Lemma 4.3 and Lemma 4.4 assure that in Theorem 4.2 the real fundamental hypothesis on Ω is that either $k=1$ or $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$. Unfortunately this last assumption is conjectured to be false*[1] *(see the numerical computations of [4]).*

Remark 4.6. *If $k=1$, then it is clear that $\Omega^\star$ is an open subset of Ω with the same eigenvalues up to order $p+1$ and moreover it is supersolution for the same functional. Hence the optimal set Ω is actually open, up to delete a non necessary subset. In particular, for example, optimal sets for the functional $\lambda_1(\cdot)+\cdots+\lambda_p(\cdot)+|\cdot|$ are open.*

We summarize the above extensions of Theorem 4.2 in a more general result of regularity for minimizers of an increasingly bi-Lipschitz functional of eigenvalues. We remind that, if Ω is an optimal set for λ_k with a measure constraint, then it is a supersolution for $\lambda_k+\Lambda|\cdot|$, for some $\Lambda > 0$ sufficiently small (see [39]).

Theorem 4.7. *Let $p \in \mathbb{N}$, $F\colon \mathbb{R}^{p+1} \to \mathbb{R}$ be a functional increasingly bi-Lipschitz, $0 < k_0 < \cdots < k_p$ be natural numbers and $\Omega \subset \mathbb{R}^2$ be an optimal set for*

$$\min\big\{F(\lambda_{k_0}(A),\dots,\lambda_{k_p}(A)) \;:\; A \subset \mathbb{R}^2 \text{ quasi-open, } |A|=1\big\},$$

such that $k_0=1$ or $\lambda_{k_i}(\Omega) > \lambda_{k_i-1}(\Omega)$ for some $i=0,1,\dots,p$. If we call M_i the natural number such that $\lambda_{M_i+1}(\Omega) > \lambda_{M_i}(\Omega)=\lambda_{k_i}(\Omega)$, then the set $\Omega_i^\star := \big\{x \in \mathbb{R}^2 \;:\; u_{k_i}^2(x)+\cdots+u_{M_i}^2(x) > 0\big\}$ is open.

[1]There is a strong argument supporting this fact due to Dorin Bucur, consisting in a careful *deleting* of nodal lines, which has not been published yet.

4.2 Proof of Theorem 4.2

The main instrument in order to prove Theorem 4.2 is the following fundamental proposition. We will use the following notation for squares:

$$Q_{2l}(x) := (x_1 - l, x_1 + l) \times (x_2 - l, x_2 + l), \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2, \ \forall l > 0.$$

Proposition 4.8. *Let $\Omega \subset \mathbb{R}^2$ be a connected optimal set for problem* (4.1) *satisfying the hypothesis of Theorem 4.2, $\xi \in (0,1)$ and A be a positive number sufficiently small. Then there exists $l(\xi, A) > 0$ such that for every Lebesgue point $x \in \mathbb{R}^2$ for $u_k^2 + \cdots + u_{k+p}^2$ with*

$$\lim_{l \to 0} \fint_{Q_{2l}(x)} \sum_{i=k}^{k+p} u_i^2 = 4\xi^2,$$

there is a rectangle $Q \subset \mathbb{R}^2$ containing x and with edges of length at least $l/2$ such that $u_k > \xi$ a.e. on ∂Q and either $Q \subset \Omega^\star$ or there exists a set $\widetilde{\Omega} \supset \Omega$ such that

- $\lambda_k(\widetilde{\Omega}) \leq \lambda_k(\Omega) - Xg,$
- *For all $i = k+1, \dots, k+p$, $\lambda_i(\widetilde{\Omega}) \leq \lambda_i(\Omega) + Yg,$*

where $X = \frac{1}{6k}$, $Y = 8p(1 + \lambda_k(\Omega))A$ and g is defined in (4.13).

We will call $\vartheta := \lambda_k(\Omega) - \lambda_{k-1}(\Omega) > 0$ in the following and $M(\lambda_k)$ will denote the L^∞ bound on $\|u_k\|_{L^\infty}$ (see inequality (2.5)). We observe also that $\|u_i\|_{L^\infty} \leq e^{1/(8\pi)} \lambda_i(\Omega)^{N/4} \leq e^{1/(8\pi)} \lambda_k(\Omega)^{N/4} =: M(\lambda_k)$ for all $i = 1, \dots, k-1$. Very often in the following we will call "hole" the set $Q \setminus \Omega^\star$ and it will be denoted H.
The proof of Proposition 4.8 will be carried on in many lemmas. Keep in mind that we first take $\xi \in (0,1)$, then choose $A > 0$, then $\eta = \eta(\xi, A) > 0$, and at last $l = l(\xi, \eta, A) > 0$. All these constants will be asked to be small enough during the proof; we will make clear step by step on which parameters they depend.

Let us take $\xi \in (0,1)$, $\eta = \eta(\xi) > 0$ and we consider a Lebesgue point $x = (x_1, x_2) \in \mathbb{R}^2$ for $\sum_{i=k}^{k+p} u_i^2$ such that

$$\lim_{l \to 0} \fint_{Q_{2l}(x)} \sum_{i=k}^{k+p} u_i^2 = 4\xi^2. \tag{4.2}$$

It is clear that, up to a translation, we can choose $x \in \Omega$ to be the origin and then $Q_{2l} := (-l, l)^2$, in order to simplify the notation. Moreover, since $\sum_{i=k}^{k+p} |Du_i|^2 \in L^1(\mathbb{R}^2)$, by absolute continuity of the integral we have that

$$\int_{Q_{2l}} \sum_{i=k}^{k+p} |Du_i|^2 \to 0 \qquad \text{as } l \to 0. \tag{4.3}$$

We call $S_\sigma = \{x \in Q_{2l} \,:\, x_1 = \sigma\}$ the 1−dimensional sections orthogonal to e_1, while $S^\perp_\tau = \{x \in Q_{2l} \,:\, x_2 = \tau\}$ and we consider the set

$$E_1 := \left\{ \sigma \in (0,l) \,:\, \exists\, i \in \{k,\dots,k+p\} \text{ such that } \int_{S_\sigma} |Du_i| \geq \eta \right\}.$$

We can prove the following.

Lemma 4.9. *Up to choose $l = l(\xi,\eta)$ small enough, we have*

$$|E_1| \leq \frac{l}{100}.$$

Proof. We suppose by contradiction that $|E_1| > \frac{l}{100}$ and we remind that applying the Hölder inequality we obtain:

$$\int_{S_\sigma} |Du_j| \leq \sqrt{\int_{S_\sigma} |Du_j|^2} \sqrt{\int_{S_\sigma} 1}.$$

Then for all $\sigma \in E_1$ we can compute, using $\int_{S_\sigma} |Du_j| \geq \eta$ for a $j \in \{k,\dots,k+p\}$ and $|S_\sigma| = 2l$,

$$\sum_{i=k}^{k+p} \int_{S_\sigma} |Du_i|^2 \geq \sum_{i=k}^{k+p} \frac{\left(\int_{S_\sigma} |Du_i|\right)^2}{|S_\sigma|} \geq \frac{\eta^2}{2l}.$$

So we can apply Fubini Theorem to find out:

$$\int_{Q_{2l}} \sum_i |Du_i|^2 = \int_{\sigma=-l}^{l} \int_{S_\sigma} \sum_i |Du_i|^2 \geq \int_{\sigma \in E_1} \int_{S_\sigma} \sum_i |Du_i|^2 \geq |E_1| \frac{\eta^2}{2l} \geq \frac{\eta^2}{200},$$

that is absurd, thanks to (4.3), up to choose $l(\eta)$ small enough. □

We can define, in analogy to E_1, the other three edges E_i of a "rectangle" inside Q_{2l},

$$E_2 := \left\{ \sigma \in (0,l) \,:\, \exists\, i \in \{k,\dots,k+p\} \text{ such that } \int_{S^\perp_\sigma} |Du_i| \geq \eta \right\},$$

$$E_3 := \left\{ \sigma \in (-l,0) \,:\, \exists\, i \in \{k,\dots,k+p\} \text{ such that } \int_{S_\sigma} |Du_i| \geq \eta \right\},$$

$$E_4 := \left\{ \sigma \in (-l,0) \,:\, \exists\, i \in \{k,\dots,k+p\} \text{ such that } \int_{S^\perp_\sigma} |Du_i| \geq \eta \right\},$$

and prove, in the very same way as in Lemma 4.9, that $|E_i| \leq \frac{l}{100}$ for all $i = 1,\dots,4$.
Since we are in a two dimensional setting, it is clear that $\int_{S_\sigma} |Du_i| < \eta$ implies $\operatorname{osc}_{S_\sigma}(u_i) < \eta$, where the oscillation is defined in the "essential" meaning. Hence if $\sigma \in [0,l] \setminus E_1$, then $\operatorname{osc}_{S_\sigma}\left(\sum_{i=k}^{k+p} u_k^2 + \dots + u_{k+p}^2\right) < (p+1)\eta^2$, and the same holds for the other edges.

Now we call $F_1 := \left\{ \sigma \in (0,l) \,:\, ⨍_{S_\sigma} \sum_{i=k}^{k+p} u_i^2 < 2\xi^2 \right\}$ and we can prove the following.

Lemma 4.10. *Up to choose $\eta(\xi)$ and $l(\xi,\eta)$ small enough, we have*

$$|F_1| \leq \frac{l}{50}.$$

Proof. We argue by contradiction that $|F_1| > \frac{l}{50}$. Let $\tau \in (0,l) \setminus E_2$ and $\sigma \in F_1 \setminus E_1$ such that (σ, τ) is a Lebesgue point for $\sum_{i=k}^{k+p} u_i^2$. By the previous observations, we have that $\operatorname{osc}_{S_\tau^\perp}\left(\sum_i u_i^2\right) < (p+1)\eta^2$ and $\operatorname{osc}_{S_\sigma}\left(\sum_i u_i^2\right) < (p+1)\eta^2$.
Hence, since $⨍_{S_\sigma} \sum_i u_i^2 < 2\xi^2$, we have for a.e. $\widetilde{\tau} \in [-l,l]$, $\sum_i u_i^2(\sigma, \widetilde{\tau}) < \frac{9}{4}\xi^2$, up to choose $\eta(\xi)$ small enough.
Moreover for those $\widetilde{\sigma} \in [-l,l]$ such that $(\widetilde{\sigma}, \tau)$ is a Lebesgue point for $\sum_i u_i^2$, we have $\sum_i u_i^2(\widetilde{\sigma}, \tau) < \frac{5}{2}\xi^2$, up to choose $\eta(\xi)$ small enough. Then for a.e. $\widetilde{\sigma} \in [-l,l] \setminus (E_1 \cup E_3)$,

$$⨍_{S_{\widetilde{\sigma}}} \sum_{i=k}^{k+p} u_i^2 < \frac{5}{2}\xi^2 + \operatorname{osc}_{S_{\widetilde{\sigma}}}\left(\sum_{i=k}^{k+p} u_i^2\right) < 3\xi^2, \tag{4.4}$$

up to choose $\eta(\xi)$ small enough. From Fubini Theorem we deduce

$$⨍_{Q_{2l}} \sum_{i=k}^{k+p} u_i^2 = \frac{1}{2l}\int_{\sigma=-l}^{l} ⨍_{S_\sigma} \sum_{i=k}^{k+p} u_i^2 > \frac{7}{2}\xi^2, \tag{4.5}$$

thanks to (4.2) and up to choose $l(\xi, \eta)$ small enough. Putting together (4.4) and (4.5), it must be that

$$|T| := \left|\left\{\overline{\sigma} \in [-l,l] \ : \ ⨍_{S_{\overline{\sigma}}} \sum_{i=k}^{k+p} u_i^2 > 6\xi^2\right\}\right| > 0.$$

We can then pick $\widetilde{\tau} \in [-l,l] \setminus (E_2 \cup E_4)$ and $\overline{\sigma} \in T$ such that $(\overline{\sigma}, \widetilde{\tau})$ is a Lebesgue point for $\sum_i u_i^2$ and $\sum_i u_i^2(\overline{\sigma}, \widetilde{\tau}) > 6\xi^2$. Hence for those $\sigma \in [-l,l]$ such that $(\sigma, \widetilde{\tau})$ is a Lebesgue point for $\sum_i u_i^2$, we have $\sum_i u_i^2(\sigma, \widetilde{\tau}) > 5\xi^2$, up to choose $\eta(\xi)$ small enough. This contradicts the fact that $|[-l,l] \setminus (E_1 \cup E_3)| \geq \frac{99}{50}l$ and for a.e. $\sigma \in [-l,l] \setminus (E_1 \cup E_3)$ it holds

$$\sum_i u_i^2(\sigma, \tau) < 3\xi^2 \qquad \text{for a.e. } \tau \in [-l,l].$$

So we have a contradiction and thus the proof is concluded. □

We can apply the same procedure for the other three edges, defined in analogy with E_i, and thus prove that $|F_i| \leq \frac{l}{50}$ for all $i = 1, \dots, 4$. Thanks to the bounds on the measure of the E_1, F_1 it is possible to choose a "good" $\sigma \in (\frac{l}{2}, \frac{3}{4}l)$ such that

$$⨍_{S_\sigma} \sum_{i=k}^{k+p} u_i^2 \geq 2\xi^2 \qquad \text{and} \qquad \int_{S_\sigma} |Du_i| < \eta \quad \forall\, i = k, \dots, k+p. \tag{4.6}$$

Then the same can be done for the other three edges. Moreover, in order to simplify the notation, we can suppose that the "good" edges are such that we obtain a square $Q = (-l/2, l/2)^2$ with each edge satisfying (4.6). Otherwise in general we will obtain a rectangle with edges of length at least l.

Using the previous observations about the oscillation, it is clear that

$$\operatorname{osc}_{\partial Q}\left(u_k^2 + \cdots + u_{k+p}^2\right) < 2\,(p+1)\eta^2 < \eta, \tag{4.7}$$

up to choose $\eta(\xi)$ small enough.

Remark 4.11. *We are working only in the two dimensional case because, if $N \geq 3$, then it is no more true that $\int_{S_\sigma} |Du_i| < \eta$ implies $\mathrm{osc}_{S_\sigma}(u_i) < \eta$. But if it is possible to obtain the first condition of* (4.6) *and* (4.7) *also for $N \geq 3$, then everything of the following proof will work in the very same way as in the two dimensional situation.*

Moreover, given a square Q whose edges satisfy (4.6) and (4.7), so that $u_k^2 + \cdots + u_{k+p}^2 > \xi^2$ on ∂Q (up to choose $\eta(\xi)$ small enough), then it is possible to rotate the eigenfunctions, preserving the orthogonality, in a way such that

$$u_k^2 \approx u_k^2 + \cdots + u_{k+p}^2 > \xi^2, \qquad \text{while } |u_i| < \eta \quad \forall i = k+1, \ldots, k+p \text{ on } \partial Q. \tag{4.8}$$

We call $u = u_k$ and without loss of generality we can suppose $u > \xi > 0$ on ∂Q: the case of $u < -\xi$ on ∂Q can be treated in the very same way. Then the following lemmas hold.

Lemma 4.12. *Let Ω, F be as in Theorem 4.2, Q be such that* (4.6) *and* (4.7) *hold and $\{u_k, \ldots, u_{k+p}\}$ be as in* (4.8)*. Then $u = u_k$ can not change sign inside Q.*

Proof. We have $u > 0$ on ∂Q, hence by absurd let $\widetilde{Q} \subset Q$ be a nodal domain where $u < 0$; clearly $|\widetilde{Q}| \leq |Q| \leq l^2$. Then we have

$$\lambda_{k+p}(\Omega) = \cdots = \lambda_k(\Omega) \geq \lambda_1(\widetilde{Q}) \geq \frac{\lambda_1(B)}{|\widetilde{Q}|} \geq \frac{\lambda_1(B)}{l^2} > \lambda_{k+p}(B) \geq \cdots \geq \lambda_k(B),$$

up to choose $l(\xi, \eta)$ sufficiently small. Since F is bilipschitz and increasing in each variable, the above equation contradicts the optimality of Ω, because the unit ball would be a better candidate in the minimization problem (4.1). □

Lemma 4.13. *Let Ω be as in Theorem 4.2, Q be such that* (4.6) *and* (4.7) *hold, u be as in* (4.8)*. We call $\widehat{Q} = \{x \in Q \,:\, u(x) > \xi\} \subseteq Q$. If there is a "hole" $H := Q \setminus \Omega$, then we have*

$$|H| \leq \frac{\eta^2}{8\,\lambda_1(B)\,\xi^2} |\widehat{Q}| \leq \frac{\eta^2}{8\,\lambda_1(B)\,\xi^2} |Q|. \tag{4.9}$$

Proof. By contradiction let us suppose $|H| > \frac{\eta^2}{8\,\lambda_1(B)\,\xi^2}|\widehat{Q}|$. We consider the function $v = u - \xi \in H_0^1(\widehat{Q})$. By Poincaré inequality (using also the Faber-Krahn inequality and the above hypothesis) we have

$$\begin{aligned}\int_{Q_{2l}} \sum_{i=k}^{k+p} |Du_i|^2 &\geq \int_{\widehat{Q}} |Du|^2 = \int_{\widehat{Q}} |Dv|^2 \geq \lambda_1(\widehat{Q}) \int_{\widehat{Q}} v^2 \geq \frac{\lambda_1(B)}{|\widehat{Q}|} \int_H \xi^2 \\ &\geq \frac{\lambda_1(B)}{|\widehat{Q}|} \int_{\widehat{Q}} v^2 = \frac{\lambda_1(B)}{|\widehat{Q}|} |H| \xi^2 > \frac{\lambda_1(B)}{|\widehat{Q}|} \frac{\eta^2}{8\lambda_1(B)\xi^2} |\widehat{Q}| \xi^2 = \frac{\eta^2}{8}.\end{aligned}$$

The above formula, combined with (4.3), up to choose $l(\xi, \eta)$ small enough, gives a contradiction and thus the proof is concluded. □

A last preliminary lemma assures that if an eigenfunction is “small” on the boundary of the square Q, then it can not become “too big” in the interior.

Lemma 4.14. *Let Q be such that (4.6), (4.7) hold and $v \in H_0^1(\Omega)$ be an eigenfunction (of eigenvalue λ) for Ω such that $0 \le |v| \le \eta$ on ∂Q. Then $|v| \le \eta + l^2/\alpha$ in Q, where $\alpha := \frac{\pi}{\lambda M(\lambda)}$.*

Proof. We consider the set $Q' := \{v > 0\} \cap Q$ and show that $v \le \eta + l^2/\alpha$ in Q'; clearly everything can be done in the very same way in $\{v < 0\} \cap Q$. Moreover we call $Q'_\eta := \{v > \eta\} \cap Q$ in order to simplify the notations.
For all $t > \eta$ we call $Q_t = \{x \in Q \,:\, v(x) > t\}$ and $\varphi(t) = |Q_t|$. Given $t > \eta$ a Lebesgue point for the functions

$$t \mapsto \fint_{\partial Q_t} |Dv| \qquad \text{and} \qquad t \mapsto P(Q_t),$$

by Coarea formula we have, as $\varepsilon \to 0$,

$$\begin{aligned} |Q_t \setminus Q_{t+\varepsilon}| \fint_{\partial Q_t} |Dv| + o(\varepsilon) &= |Q_t \setminus Q_{t+\varepsilon}| \fint_{Q_t \setminus Q_{t+\varepsilon}} |Dv| \\ &= \varepsilon \fint_{\sigma=t}^{t+\varepsilon} \mathcal{H}^{d-1}\left(\{x \,:\, v(x) = \sigma\}\right) = \varepsilon P(Q_t) + o(\varepsilon), \end{aligned}$$

hence we deduce

$$|\varphi'(t)| = \frac{P(Q_t)}{\fint_{\partial Q_t} |Dv|}. \tag{4.10}$$

On the other hand we consider, for $t > \eta$ and $0 < \varepsilon \ll 1$, we consider

$$\widetilde{v}(x) = \begin{cases} v(x) & \text{if } v(x) < t, \\ t & \text{if } x \in Q_t \setminus Q_{t+\varepsilon}, \\ v(x) - \varepsilon & \text{if } x \in Q_{t+\varepsilon}. \end{cases}$$

Since v is the first eigenfunction in Q' (it is strictly positive there), it must hold $\mathcal{R}(v, Q') \le \mathcal{R}(\widetilde{v}, Q')$. We can compute, for $\varepsilon \to 0$

$$\int_{Q'} |Dv|^2 - \int_{Q'} |D\widetilde{v}|^2 = \int_{Q_t \setminus Q_{t+\varepsilon}} |Dv|^2 = \varepsilon \fint_{\sigma=t}^{t+\varepsilon} P(Q_\sigma) \fint_{\partial Q_\sigma} |Dv| = \varepsilon P(Q_t) \fint_{\partial Q_t} |Dv| + o(\varepsilon).$$

On the other hand we have

$$\int_{Q'} v^2 - \int_{Q'} \widetilde{v}^2 \le \int_{Q_t} v^2 - (v - \varepsilon)^2 \le \varepsilon \|v\|_{L^\infty(Q)} |Q_t| + o(\varepsilon).$$

From $\mathcal{R}(v, Q') \le \mathcal{R}(\widetilde{v}, Q')$ we then obtain

$$2\lambda\varepsilon |Q_t| \, \|v\|_{L^\infty(Q)} \ge \varepsilon P(Q_t) \fint_{\partial Q_t} |Dv|,$$

that means

$$\fint_{\partial Q_t} |Dv| \le \frac{2\lambda |Q_t| \, \|v\|_{L^\infty(Q)}}{P(Q_t)}.$$

Putting the above inequality inside (4.10), using the isoperimetric inequality and reminding $|Q_t| = \varphi(t)$, $\varphi'(t) < 0$, we obtain

$$\left|\varphi'(t)\right| \geq \frac{P(Q_t)^2}{2\lambda|Q_t|\,\|v\|_{L^\infty(\Omega)}} \geq 2\pi\frac{\varphi(t)}{2\lambda\varphi(t)\|v\|_{L^\infty(\Omega)}} = \frac{\pi}{\lambda\|v\|_{L^\infty(\Omega)}} \geq \frac{\pi}{\lambda M(\lambda)} =: \alpha, \tag{4.11}$$

defining $\alpha = \frac{\pi}{\lambda M(\lambda)}$, which depends only on λ, since $M(\lambda)$ is the L^∞ bound on the norm of an eigenfunction (see inequality (2.5)). Now we can consider the differential equation in (4.11), together with the initial condition $\varphi(\eta) = |Q'_\eta| \leq l^2$. In the very same way as in the proof of Lemma 3.4, we deduce that $\varphi(t) \leq -\alpha t + (\alpha\eta + |Q'_\eta|)$ which implies

$$v \leq \eta + \frac{l^2}{\alpha},$$

and the thesis is proved. □

Remark 4.15. *For sake of simplicity we will call in the following $\alpha_l := l^2/\alpha$.*

We are now in position to prove Proposition 4.8.

Proof of Proposition 4.8. Let Q be such that (4.6) and (4.7) hold. If $Q \subset \Omega^\star$ there is nothing to prove; otherwise let $H := Q \setminus \Omega^\star$ be a "hole" inside Ω. We remind that we are in the following situation:

$$\operatorname{osc}_{\partial Q}\left(\sum_{i=k}^{k+p} u_i^2\right) < \eta, \qquad u = u_k > \xi \text{ on } \partial Q, \qquad |u_i| < \eta \text{ on } \partial Q \quad \text{for } i = k+1, \dots, k+p. \tag{4.12}$$

Thanks to Lemma 4.12, $u \geq 0$ in Q and $u > 0$ in $Q \setminus H$; moreover by Lemma 4.13, $|H| \leq \frac{\eta^2}{8\,\lambda_1(B)\,\xi^2}|Q|$. We consider the function $\widetilde{u}\colon \Omega \cup H \to \mathbb{R}$:

$$\widetilde{u}(x) = \begin{cases} \xi & \text{if } x \in \{0 \leq u(x) \leq \xi\} \cap Q, \\ u(x) & \text{otherwise.} \end{cases}$$

We aim to compare the functions u and $\widetilde{u}$ and their Rayleigh quotients. We call $\widetilde{\Omega} = \Omega \cup H \supset \Omega$ and we apply Poincaré inequality to $\xi - u \in H_0^1(\{0 \leq u < \xi\} \cap Q)$:

$$\begin{aligned} g = \int_\Omega |Du|^2 - \int_{\widetilde{\Omega}} |D\widetilde{u}|^2 = \int_{\{0\leq u<\xi\}\cap Q} |Du|^2 &= \int_{\{0\leq u<\xi\}\cap Q} |D(\xi - u)|^2 \\ &\geq \lambda_1(\{0 \leq u < \xi\} \cap Q) \int_{\{0\leq u<\xi\}\cap Q} |(\xi - u)|^2 \\ &\geq \frac{\lambda_1(B)}{|\{0 \leq u < \xi\} \cap Q|}\xi^2|H|, \end{aligned} \tag{4.13}$$

where in the last step also the Faber-Krahn inequality was used. When we consider the L^2 norm of the functions we have:

$$\int_{\widetilde{\Omega}} \widetilde{u}^2 - \int_\Omega u^2 \geq \int_H \xi^2 = \xi^2|H|.$$

Hence we can obtain a relation between $\lambda_k(\Omega)$ and the Rayleigh quotient $\mathcal{R}(\widetilde{u},\widetilde{\Omega})$ of $\widetilde{u}$ in $\widetilde{\Omega}$:

$$\begin{aligned}\mathcal{R}(\widetilde{u},\widetilde{\Omega}) &= \frac{\int_{\widetilde{\Omega}}|D\widetilde{u}|^2}{\int_{\widetilde{\Omega}}\widetilde{u}^2} = \frac{\int_\Omega |Du|^2 - \left(\int_\Omega |Du|^2 - \int_{\widetilde{\Omega}}|D\widetilde{u}|^2\right)}{\int_\Omega u^2 + \left(\int_{\widetilde{\Omega}}\widetilde{u}^2 - \int_\Omega u^2\right)} \\ &= \frac{\int_\Omega |Du|^2 - g}{\int_\Omega u^2 + \xi^2|H|} \le \lambda_k(\Omega) - g.\end{aligned}$$

We aim now to compare g with the L^2 and H^1 scalar product of u and u_i, for $i \neq k$. First of all we note that, using the divergence formula and the definition of eigenfunction,

$$g = \int_{\{0\le u\le\xi\}\cap Q} |Du|^2 = \lambda_k(\Omega)\int_{\{0\le u\le\xi\}\cap Q} u^2 + \int_{\{u=\xi\}\cap Q} u\frac{\partial u}{\partial\nu}, \tag{4.14}$$

where we call ν the outer unit normal. Moreover we observe that both the terms in the right hand side of (4.14) are positive, hence they are both smaller than the left hand side, and that $g \le \lambda_k(\Omega)$.

Now we need to prove the following Claim, that gives a bound from below on g.

Claim 4.A.
We have that

$$g \ge \frac{\lambda_1(B)\xi^2}{200}\left|\{0\le u\le\xi\}\cap Q\right|$$

Proof of Claim 4.A. First of all we suppose that

$$\begin{aligned}&\left|\left\{0\le u\le \frac{9}{10}\xi\right\}\cap Q\right| \ge \left|\left\{\frac{9}{10}\xi\le u\le\xi\right\}\cap Q\right|, \quad \text{hence}\\ &\left|\left\{0\le u\le \frac{9}{10}\xi\right\}\cap Q\right| \ge \frac{1}{2}\left|\{0\le u\le\xi\}\cap Q\right|.\end{aligned}$$

In this case, from (4.13) we can compute

$$\begin{aligned}g &\ge \lambda_1(\{0\le u<\xi\}\cap Q)\int_{\{0\le u<\xi\}\cap Q}|(\xi-u)|^2 \\ &\ge \frac{\lambda_1(B)}{|\{0\le u\le\xi\}\cap Q|}\int_{\{0\le u\le\frac{9}{10}\xi\}\cap Q}|(\xi-u)|^2 \ge \frac{\lambda_1(B)\xi^2|\{0\le u\le\xi\}\cap Q|}{200|\{0\le u\le\xi\}\cap Q|}.\end{aligned}$$

Since $|\{0\le u\le\xi\}\cap Q| < 1$, it is clear that we have

$$g \ge \frac{\lambda_1(B)\xi^2}{200}|\{0\le u\le\xi\}\cap Q|.$$

On the other hand, it can happen that

$$\begin{aligned}&\left|\left\{0\le u\le \frac{9}{10}\xi\right\}\cap Q\right| \le \left|\left\{\frac{9}{10}\xi\le u\le\xi\right\}\cap Q\right|, \quad \text{hence}\\ &\left|\left\{\frac{9}{10}\xi\le u\le\xi\right\}\cap Q\right| \ge \frac{1}{2}\left|\{0\le u\le\xi\}\cap Q\right|.\end{aligned}$$

Then, using (4.14), we obtain

$$
\begin{aligned}
g &\geq \lambda_k(\Omega) \int_{\{0\leq u\leq \xi\}\cap Q} u^2 \geq \lambda_k(\Omega) \int_{\{\frac{9}{10}\xi\leq u\leq \xi\}\cap Q} u^2 \geq \lambda_k(\Omega) \left(\frac{9}{10}\xi\right)^2 \left|\left\{0 \leq u \leq \frac{9}{10}\xi\right\}\cap Q\right| \\
&\geq \frac{\lambda_k(\Omega) 81 \xi^2}{200} \left|\{0\leq u\leq \xi\}\cap Q\right| \geq \frac{\lambda_1(B)\xi^2}{200} \left|\{0\leq u\leq \xi\}\cap Q\right|,
\end{aligned}
$$

and the proof of the Claim is concluded. □

We can estimate, using Hölder inequality and Lemma 4.14 on u_j, for $j = k+1, \dots, k+p$:

$$
\begin{aligned}
\left|\int_{\widetilde{\Omega}} u_j (u - \widetilde{u})\right| &= \left|\int_{\{0\leq u\leq \xi\}\cap Q} u_j(\xi - u)\right| \leq \|u_j\|_{L^\infty(Q)} \|\xi - u\|_{L^1(\{0\leq u\leq \xi\}\cap Q)} \\
&\leq (\eta + \alpha_l)\xi |\{0\leq u\leq \xi\}\cap Q|.
\end{aligned}
$$

Then, in order to have

$$
\left|\int_{\widetilde{\Omega}} u_j(u - \widetilde{u})\right| \leq A\, g \qquad \forall j = k+2, \dots, k+p,
$$

for some $A(k, p, \lambda_k(\Omega), L)$ to be choosen later, it is sufficient that

$$
(\eta + \alpha_l)\xi \leq A \frac{\lambda_1(B)\xi^2}{200},
$$

that is true up to choose $\eta(\xi, A)$ and $l(\xi, \eta, A)$ small enough.
The study of the L^2 scalar product of the gradients is more complicate: first of all we note that, for all $j = k, \dots, k+p$ we have

$$
\int_{\{u=\xi\}\cap Q} u_j \frac{\partial u}{\partial \nu} = \int_{\{u=\xi\}\cap Q} u \frac{\partial u_j}{\partial \nu}, \tag{4.15}
$$

which follows from a straightforward application of the divergence Theorem, using also that u, u_j are eigenfunctions with the same eigenvalue.

Now we use again the divergence Theorem, for $j \in \{k+1, \dots, k+p\}$:

$$
\begin{aligned}
\left|\int_{\widetilde{\Omega}} Du_j \cdot (D\widetilde{u} - Du)\right| &= \left|\int_{\{0\leq u\leq \xi\}\cap Q} Du_j \cdot Du\right| \\
&\leq \lambda_j(\Omega) \left|\int_{\{0\leq u\leq \xi\}\cap Q} u_j u\right| + \left|\int_{\{u=\xi\}\cap Q} u \frac{\partial u_j}{\partial \nu}\right|.
\end{aligned} \tag{4.16}
$$

From (4.15), we observe that

$$
\left|\int_{\{u=\xi\}\cap Q} u \frac{\partial u_j}{\partial \nu}\right| = \left|\int_{\{u=\xi\}\cap Q} u_j \frac{\partial u}{\partial \nu}\right| \leq \frac{\eta + \alpha_l}{\xi} \int_{\{u=\xi\}\cap Q} u \frac{\partial u}{\partial \nu} \leq Ag,
$$

since $\frac{(\eta+\alpha_l)}{\xi} \leq A$, which holds true up to choose again $\eta(\xi, A)$, $l(\xi, \eta, A)$ small enough, and where we used also (4.14) in the last inequality.

On the other hand, when we consider the other term of (4.16), it comes out:

$$\lambda_j(\Omega)\left|\int_{\{0\le u\le\xi\}\cap Q} u_j u\right| \le \lambda_j(\Omega)(\eta+\alpha_l)\xi|\{0\le u\le\xi\}|.$$

Up to choose $\eta(\xi,A)$ and $l(\xi,\eta,A)$ small enough, we have that

$$\lambda_j(\Omega)(\eta+\alpha_l)\xi \le A\,\frac{\lambda_1(B)\xi^2}{200},$$

thus we deduce

$$\lambda_j(\Omega)\left|\int_{\{0\le u\le\xi\}\cap Q} u_j u\right| \le A\,g, \tag{4.17}$$

and

$$\left|\int_{\widetilde{\Omega}} Du_j\cdot D\widetilde{u}\right| = \left|\int_{\widetilde{\Omega}} Du_j\cdot (D\widetilde{u}-Du)\right| \le 2A\,g \qquad \forall j=k+1,\dots,k+p. \tag{4.18}$$

We need now to study the eigenfunctions related to the lower eigenvalues.

Claim 4.B.
Let $i=1,\dots,k-1$ and Q the square satisfying (4.12), with a hole $H=Q\setminus\Omega^\star$ inside. Then we have

$$\left|\int_{\{0\le u\le\xi\}\cap Q} u_i u\right| \le 2\frac{M(\lambda_k)}{\xi^2}g,$$
$$\left|\int_{\{0\le u\le\xi\}\cap Q} Du_i\cdot Du\right| \le 3\lambda_k(\Omega)\frac{M(\lambda_k)}{\xi^2}g.$$

Proof of Claim 4.B. First of all, using the same technique as in Claim 4.A, we can compute

$$g \ge \frac{\lambda_1(B)}{|\{0\le u\le\xi\}\cap Q|}\int_{\{0\le u\le\frac{9}{10}\xi\}\cap Q}(\xi-u)^2 \ge \frac{\xi^2\lambda_1(B)\left|\{0\le u\le\frac{9}{10}\xi\}\cap Q\right|}{100|\{0\le u\le\xi\}\cap Q|}. \tag{4.19}$$

We define the set, for all $i=1,\dots,k-1$

$$R_i=\left\{x\in\left\{0\le u\le\frac{9}{10}\xi\right\}\cap Q\ :\ 0\le u(x)\le\frac{u_i(x)\xi^2}{\|u_i\|_{L^\infty}}\right\},$$

and we can compute

$$\left|\int_{R_i} u_i u\right| \le \frac{\xi^2}{\|u_i\|_{L^\infty}}\int_{\{0\le u\le\frac{9}{10}\xi\}\cap Q} u_i^2 \le \xi^2 M(\lambda_k)\left|\left\{0\le u\le\frac{9}{10}\xi\right\}\cap Q\right| \le \frac{M(\lambda_k)}{\xi^2}g,$$

where the last inequality follows from (4.19) and up to choose $l(\xi,\eta)$ small enough. Having in mind (4.14), we can conclude, for all $i=1,\dots,k-1$

$$\left|\int_{\{0\le u\le\xi\}\cap Q} u_i u\right| \le \left|\int_{R_i} u_i u\right| + \left|\int_{Q\setminus R_i} u_i u\right| \le \frac{M(\lambda_k)}{\xi^2}g + \frac{\|u_i\|_{L^\infty}}{\xi^2}\int_{\{0\le u\le\xi\}\cap Q} u^2 \le 2\frac{M(\lambda_k)}{\xi^2}g.$$

We can now study the gradients, obtaining, using the divergence Theorem and again (4.14), for all $i = 1, \dots, k-1$

$$\begin{aligned}\left|\int_{\{0\le u\le\xi\}\cap Q} Du_i\cdot Du\right| &\le \left|\int_{\{u=\xi\}\cap Q} u_i\frac{\partial u}{\partial\nu}\right| + \lambda_i(\Omega)\left|\int_{\{0\le u\le\xi\}\cap Q} u_i u\right| \\ &\le \left|\frac{\|u_i\|_{L^\infty}}{\xi}\int_{\{u=\xi\}\cap Q} u\frac{\partial u}{\partial\nu}\right| + \lambda_k(\Omega)\left|\int_{\{0\le u\le\xi\}\cap Q} u_i u\right| \le \frac{3\lambda_k(\Omega)M(\lambda_k)}{\xi^2}g.\end{aligned}$$

□

We are now in position to prove the following

Claim 4.C.

Given $\xi\in(0,1)$, we define

$$X = \frac{1}{6k}, \qquad Y = 8p\,(1+\lambda_k(\Omega))\,A,$$

for some constant $A(k,p,\lambda_k(\Omega))$ sufficiently small; then we have

$$\begin{aligned}&\lambda_k(\widetilde{\Omega}) \le \lambda_k(\Omega) - X\,g,\\ &\lambda_i(\widetilde{\Omega}) \le \lambda_i(\Omega) + Y\,g \qquad \forall i = k+1,\dots,k+p,\end{aligned} \tag{4.20}$$

up to choose $\eta(\xi,A)$ and $l(\xi,\eta,A)$ small enough.

Proof of Claim 4.C. We aim to use the min-max formula for eigenvalues, so first of all let Z_m (for $k < m \le k+p$) be the linear subspace of $H^1_0(\widetilde{\Omega})$ spanned by the functions $\{u_1,\dots,u_{k-1},\widetilde{u},\dots,u_m\}$. It is easy to see that Z_m has dimension m. In fact by contradiction if it has dimension strictly less than m there should exist coefficients α_i for $i\ne k$ with all $|\alpha_i|\le 1$ and

$$\widetilde{u} = \sum_{1\le i\le m,\, i\ne k} \alpha_i u_i.$$

We can then deduce, since u_i is orthogonal to u in $L^2(\Omega)$ for all i by hypothesis,

$$\begin{aligned}1+\xi^2|H| = \int_{\widetilde{\Omega}}\widetilde{u}^2 &= \int_\Omega \sum_{1\le i\le m,\, i\ne k}\alpha_i u_i\widetilde{u} \le \sum\alpha_i\left|\int_\Omega u_i\widetilde{u}\right| \le \sum\alpha_i\left|\int_\Omega u_i u + \int_\Omega u_i(\widetilde{u}-u)\right|\\ &\le (k+p)(\eta+\alpha_l)\xi|\{0<u<\xi\}\cap Q| \le (k+p)(\eta+\alpha_l)\xi l^2,\end{aligned}$$

which is absurd up to choose $\eta(\xi,A)$ and $l(\xi,\eta,A)$ small enough. We can now show the first part of (4.20): we choose $Z_k = span(u_1,\dots,u_{k-1},\widetilde{u})$ as k-dimensional subspace of $H^1_0(\widetilde{\Omega})$, and we consider $w\in Z_k$ such that $w = \sum_{i<k}\alpha_i u_i + \widetilde{\alpha}\widetilde{u}$, with (up to rescale) $\widetilde{\alpha}=1$, $\max_i|\alpha_i|\le 1$. If there is a $i<k$ such that

$$\alpha_i \ge \frac{\xi^2}{24k^2\lambda_k(\Omega)M(\lambda_k)} = G,$$

first of all we note that, up to choose $l(\xi,\eta)$ small enough, we have

$$\begin{aligned} 2k\left|\int_{\{0\le u\le\xi\}\cap Q} Du_i\cdot Du\right| &\le 2k\|Du_i\|_{L^2(Q)}\|Du\|_{L^2(Q)} \le \frac{\vartheta}{4k}G^2,\\ 4k\lambda_k(\Omega)\left|\int_{\{0\le u\le\xi\}\cap Q} u_i u\right| &\le 4k\lambda_k(\Omega)\sqrt{\xi}\|u_i\|_{L^2(Q)}l^{d/2} \le \frac{\vartheta}{4k}G^2. \end{aligned} \tag{4.21}$$

then we can study the Rayleigh quotient,

$$\begin{aligned} \mathcal{R}(w,\widetilde{\Omega}) &\le \frac{\sum_{i<k}\alpha_i^2\int_\Omega |Du_i|^2 + \widetilde{\alpha}^2\int_{\widetilde{\Omega}}|D\widetilde{u}|^2 + 2\sum_{i<k}\alpha_i\widetilde{\alpha}\int_{\widetilde{\Omega}} Du_i D\widetilde{u}}{\sum_{i<k}\alpha_i^2\int_\Omega u_i^2 + \widetilde{\alpha}^2\int_{\widetilde{\Omega}}\widetilde{u}^2 + 2\sum_{i<k}\alpha_i\widetilde{\alpha}\int_{\widetilde{\Omega}} u_i\widetilde{u}}\\ &\le \frac{\lambda_k(\Omega)\Big(\sum_{i<k}\alpha_i^2\int_\Omega u_i^2 + \int_\Omega u^2\Big) - G^2\vartheta - \widetilde{\alpha}^2 g + \frac{\vartheta}{4k}G^2}{\Big(\sum_{i<k}\alpha_i^2\int_\Omega u_i^2 + \int_\Omega u^2\Big) - \frac{\vartheta}{8k\lambda_k(\Omega)}G^2}\\ &\le \left(\lambda_k(\Omega) - \frac{G^2\vartheta}{k} + \frac{\vartheta}{4k}G^2 - \frac{g}{k}\right)\left(1 + \frac{\vartheta}{4k\lambda_k(\Omega)}G^2\right)\\ &\le \lambda_k(\Omega) - \frac{G^2\vartheta}{4k} - \frac{g}{k} \le \lambda_k(\Omega) - Xg, \end{aligned} \tag{4.22}$$

where we used (4.21) and the definition of eigenvalues:

$$\int_\Omega |Du_i|^2 = \lambda_i(\Omega)\int_\Omega u_i^2 < \lambda_k(\Omega)\int_\Omega u_i^2 \qquad \forall i<k.$$

Moreover we have divided both the numerator and the denominator for $\big(\sum_{i<k}\alpha_i^2\int_\Omega u_i^2 + \int_\Omega u^2\big)$, which is bounded from below by 1 and from above by k, and used the fact that $G \le 1/2$, since $\xi < 1$.

On the other hand it can happen that $\alpha_i < G$ for all $i = 1,\dots,k-1$, hence

$$\sum_{i<k}\alpha_i \le kG \le \frac{\xi^2}{24k\lambda_k(\Omega)M(\lambda_k)}.$$

Thus, thanks to Claim 4.B, we have

$$\begin{aligned} 2\sum_{i<k}\alpha_i\int Du_i\cdot Du &\le \frac{1}{6k}g,\\ 4\lambda_k(\Omega)\sum_{i<k}\alpha_i\int u_i u &\le \frac{1}{2k}g,\\ \frac{1}{12\lambda_k(\Omega)k^2}g^2 &\le \frac{1}{6k}g. \end{aligned}$$

We can use the inequalities above in order to manage the Rayleigh quotient (using the same techniques as in the previous case):

$$\begin{aligned} \mathcal{R}(w,\widetilde{\Omega}) &\le \frac{\sum_{i<k}\alpha_i^2\int_\Omega |Du_i|^2 + \widetilde{\alpha}^2\int_{\widetilde{\Omega}}|D\widetilde{u}|^2 + 2\sum_{i<k}\alpha_i\widetilde{\alpha}\int_{\widetilde{\Omega}} Du_i D\widetilde{u}}{\sum_{i<k}\alpha_i^2\int_\Omega u_i^2 + \widetilde{\alpha}^2\int_{\widetilde{\Omega}}\widetilde{u}^2 + 2\sum_{i<k}\alpha_i\widetilde{\alpha}\int_{\widetilde{\Omega}} u_i\widetilde{u}}\\ &\le \left(\lambda_k(\Omega) - \frac{1}{k}g + \frac{1}{6k}g\right)\left(1 + \frac{1}{2k}g\right) \le \lambda_k(\Omega) - \frac{1}{6k}g. \end{aligned}$$

Hence, thanks to the min-max principle (2.4), we have:

$$\lambda_k(\widetilde{\Omega}) \leq \lambda_k(\Omega) - X\,g.$$

For the other eigenvalues, we choose $Z_m = span(u_1, \dots, u_{k-1}, \widetilde{u}, u_{k+1}, \dots, u_m)$ and we pick $w \in Z_m$, which can be written as

$$\sum_{1 \leq i \leq m,\ i \neq k} \alpha_i u_i + \alpha_k \widetilde{u},$$

with $\max_i \{|\alpha_i|\} = 1$, up to rescaling. Note that, this time, α_k can also be zero. In the very same way of the previous case, we first study the situation when there exists $i < k$ such that

$$\alpha_i > \frac{\xi^2}{24k^2 \lambda_k(\Omega) M(\lambda_k)} A =: \widetilde{G}.$$

Up to choose $l(\xi, \eta, A)$ small enough we have:

$$2k \left| \int_{\{0 \leq u \leq \xi\} \cap Q} Du_i \cdot Du \right| \leq \frac{\vartheta}{4k} \widetilde{G}^2, \qquad 4k\lambda_k(\Omega) \left| \int_{\{0 \leq u \leq \xi\} \cap Q} u_i u \right| \leq \frac{\vartheta}{4k} \widetilde{G}^2.$$

We evaluate now $\mathcal{R}(w, \widetilde{\Omega})$ as in (4.22), reminding that $\lambda_j(\Omega) = \lambda_k(\Omega)$ for all $j = k + 1, \dots, k+p$ and using (4.17), (4.18):

$$\begin{aligned}
\mathcal{R}(w, \widetilde{\Omega}) &= \frac{\sum_{i \neq k} \alpha_i^2 \int_\Omega |Du_i|^2 + \alpha_k^2 \int_{\widetilde{\Omega}} |D\widetilde{u}|^2 + 2 \sum_{i \neq k} \alpha_i \alpha_k \int_{\widetilde{\Omega}} Du_i D\widetilde{u}}{\sum_{i \neq k} \alpha_i^2 \int_\Omega u_i^2 + \alpha_k^2 \int_{\widetilde{\Omega}} \widetilde{u}^2 + 2 \sum_{i \neq k} \alpha_i \alpha_k \int_{\widetilde{\Omega}} u_i \widetilde{u}} \\
&\leq \frac{\sum_{i \neq k} \alpha_i^2 \int_\Omega |Du_i|^2 + \alpha_k^2 \int_\Omega |Du|^2 - \widetilde{G}^2 \vartheta + \frac{\vartheta}{4k} \widetilde{G}^2 + 4pA\,g}{\sum_{i \neq k} \alpha_i^2 \int_\Omega u_i^2 + \widetilde{\alpha}^2 \int_\Omega u^2 - \frac{\vartheta}{8k\lambda_k(\Omega)} \widetilde{G}^2 - 2pA\,g} \\
&\leq \frac{\lambda_k(\Omega) \Big(\sum_{i \neq k} \alpha_i^2 \int_\Omega u_i^2 + \int_\Omega u^2 \Big) - \widetilde{G}^2 \vartheta + \frac{\vartheta}{4k} \widetilde{G}^2 + 4pA\,g}{\Big(\sum_{i \neq k} \alpha_i^2 \int_\Omega u_i^2 + \int_\Omega u^2 \Big) - \frac{\vartheta}{8k\lambda_k(\Omega)} \widetilde{G}^2 - 2pA\,g},
\end{aligned}$$

and we note that the denominator is greater than $1/2$, up to choose $A(p, k)$ small enough. Then, we can conclude, using again the min-max principle 2.4,

$$\begin{aligned}
\lambda_j(\widetilde{\Omega}) &\leq \left(\lambda_k(\Omega) - \frac{\vartheta}{2k} \widetilde{G}^2 + 4pAg \right) \left(1 + \frac{\vartheta}{4k\lambda_k(\Omega)} \widetilde{G}^2 + 4pAg \right) \\
&\leq \lambda_k(\Omega) + 4p\,(1 + \lambda_k(\Omega))\,A\,g + 16p^2 A^2\,g^2 \leq \lambda_k(\Omega) + Y\,g,
\end{aligned}$$

up to choose $A(p, k, \lambda_k(\Omega))$ small enough.

At last we have to treat the case when, for all $i < k$, $\alpha_i < \widetilde{G}$, that implies

$$\begin{aligned}
2 \sum_{i<k} \alpha_i \int Du_i \cdot Du &\leq \frac{1}{6k} Ag, \\
4\lambda_k(\Omega) \sum_{i<k} \alpha_i \int u_i u &\leq \frac{1}{2k} Ag.
\end{aligned}$$

Having in mind also (4.17) and (4.18) we are now in position to treat the Rayleigh quotient:

$$\begin{aligned}
\mathcal{R}(w,\widetilde{\Omega}) &= \frac{\sum_{i\neq k}\alpha_i^2\int_\Omega |Du_i|^2 + \alpha_k^2\int_{\widetilde{\Omega}}|D\widetilde{u}|^2 + 2\sum_{i\neq k}\alpha_i\alpha_k\int_{\widetilde{\Omega}} Du_i D\widetilde{u}}{\sum_{i\neq k}\alpha_i^2\int_\Omega u_i^2 + \alpha_k^2\int_{\widetilde{\Omega}}\widetilde{u}^2 + 2\sum_{i\neq k}\alpha_i\alpha_k\int_{\widetilde{\Omega}} u_i\widetilde{u}} \\
&\le \frac{\sum_{i\neq k}\alpha_i^2\int_\Omega |Du_i|^2 + \alpha_k^2\int_\Omega |Du|^2 + \frac{1}{6k}Ag + 4pA\,g}{\sum_{i\neq k}\alpha_i^2\int_\Omega u_i^2 + \widetilde{\alpha}^2\int_\Omega u^2 - \frac{1}{4k\lambda_k(\Omega)}Ag - 2pA\,g} \\
&\le \frac{\lambda_k(\Omega)\left(\sum_{i\neq k}\alpha_i^2\int_\Omega u_i^2 + \widetilde{\alpha}^2\int_\Omega u^2\right) + \frac{1}{6k}Ag + 4pA\,g}{\left(\sum_{i\neq k}\alpha_i^2\int_\Omega u_i^2 + \widetilde{\alpha}^2\int_\Omega u^2\right) - \frac{1}{4k\lambda_k(\Omega)}Ag - 2pA\,g},
\end{aligned}$$

and we note that the denominator is greater than $1/2$, up to choose $A(p,k)$ small enough. Then, we can conclude, using the min-max principle

$$\begin{aligned}
\lambda_j(\widetilde{\Omega}) &\le \left(\lambda_k(\Omega) + \frac{1}{6k}Ag + 4pAg\right)\left(1 + \frac{1}{2k\lambda_k(\Omega)}Ag + 4pAg\right) \\
&\le \lambda_k(\Omega) + Y\,g,
\end{aligned}$$

up to choose again $A(p,k,\lambda_k(\Omega))$ small enough and we have thus proved the Claim. □

The previous Claim concludes the proof of Proposition 4.8. At last, we choose (the reason will be clear in the following) that

$$\Lambda|H| \le \frac{1}{10}Xg \qquad \text{and} \qquad \Lambda|H| \le \frac{\vartheta}{4L}, \tag{4.23}$$

which hold using (4.13) and up to choose $l(\xi,\eta,A)$ small enough. □

We can now prove the main Theorem of this Chapter.

Proof of Theorem 4.2. We call $L>0$ the bi-Lipschitz constant of the functional F, and Ω is a minimizer for (4.1). It is possible to assume Ω to be connected, otherwise we can work on each connected component. We consider the following sets, for $\xi\in(0,1)$:

$$O_\xi := \left\{x\in\mathbb{R}^2 \;:\; u_k^2(x)+\dots+u_{k+p}^2(x) > 2\xi^2\right\}.$$

We will prove that for every $\xi\in(0,1)$ there exists an open set $\widetilde{O}_\xi$ such that $O_\xi \subseteq \widetilde{O}_\xi \subseteq \Omega^\star$. Then the same must hold also for $\xi>1$, since there can not be more "holes" in $\Omega^\star$. Then it is clear that

$$\Omega^\star \subseteq \cup_\xi O_\xi \subseteq \cup_\xi \widetilde{O}_\xi \subseteq \Omega^\star,$$

hence $\Omega^\star = \cup_\xi\widetilde{O}_\xi$ which is an open set.

It remains only to prove the existence of such a set $\widetilde{O}_\xi$. We fix $\xi\in(0,1)$ and we apply Proposition 4.8, with A small enough depending only on $k,p,\lambda_k(\Omega),L$ to be choosen later. If there exists a "square" Q such that (4.6) and (4.7) hold, with $H=Q\setminus\Omega^\star\neq\emptyset$, we are going to show that this contradicts the minimality of Ω, since $\widetilde{\Omega}=\Omega\cup H$ is a better candidate in the minimization problem (4.1).

We remind that, by definition of ϑ,

$$\lambda_j(\widehat{\Omega}) \le \lambda_j(\Omega) - \vartheta \qquad \forall j = 1, \dots, k-1.$$

We have now two cases to treat: if $X\,g \le \vartheta$, then it is sufficient to use the hypothesis that F is increasingly bi-Lipschitz and we find that

$$\begin{aligned} F(\lambda_k(\widetilde{\Omega}), \dots, \lambda_{k+p}(\widetilde{\Omega})) + \Lambda|\widetilde{\Omega}| &\le F\left(\lambda_k(\Omega) - X\,g, \lambda_{k+1}(\Omega) + Y\,g, \dots, \lambda_{k+p}(\Omega) + Y\,g\right) + \Lambda|\widetilde{\Omega}| \\ &\le F\left(\lambda_k(\Omega), \dots, \lambda_{k+p}(\Omega)\right) + \Lambda|\Omega| + \Lambda|H| + pLY\,g - \frac{1}{L}X\,g \\ &< F(\lambda_k(\Omega), \dots, \lambda_{k+p}(\Omega)) + \Lambda|\Omega| - \frac{9}{20L}X\,g, \end{aligned}$$

up to take $A(p, \lambda_k(\Omega))$ small enough, where in the last inequality we used also (4.23).
On the other hand, if $X\,g > \vartheta$, again since F is increasingly bi-Lipschitz,

$$\begin{aligned} F(\lambda_k(\widetilde{\Omega}), \dots, \lambda_{k+p}(\widetilde{\Omega})) + \Lambda|\widetilde{\Omega}| &\le F\left(\lambda_k(\Omega) - \vartheta, \lambda_{k+1}(\Omega) + Y\,g, \dots, \lambda_{k+p}(\Omega) + Y\,g\right) + \Lambda|\widetilde{\Omega}| \\ &\le F\left(\lambda_k(\Omega), \dots, \lambda_{k+p}(\Omega)\right) + \Lambda|\Omega| + \Lambda|H| + pLY\,g - \frac{\vartheta}{L} \\ &< F(\lambda_k(\Omega), \dots, \lambda_{k+p}(\Omega)) + \Lambda|\Omega| - \frac{\vartheta}{4L}, \end{aligned}$$

up to take again $A(k, p, L, \lambda_k(\Omega))$ small enough, where we used also (4.23) in the last inequality. In both cases we contradict the minimality of Ω for problem (4.1) and thus the proof is concluded; in fact then $Q \subset \Omega^\star$ and we can choose

$$\widetilde{O}_\xi := O_\xi \cup \left\{ x \in \mathbb{R}^2 \ : \ \exists\, Q \ni x \text{ as in Proposition 4.8} \right\}.$$

$\square$

Remark 4.16. *From the proof of Theorem 4.2, it is clear that a slightly weaker assumption on F can be taken as hypothesis instead of the bi-Lipschitz condition. It is actually sufficient that there exists a constant $L > 0$ such that for all $x = (x_1, \dots, x_{p+1}) \in \mathbb{R}^{p+1}$, $a, b > 0$,*

$$F(x_1 - a, x_2 + b, \dots, x_{p+1} + b) \le F(x_1, \dots, x_{p+1}) + pLb - \frac{a}{L}.$$

Chapter 5

Lipschitz regularity of eigenfunctions on optimal domains

The main topic presented in this Chapter is a regularity result for eigenfunctions on optimal domains. In the whole Chapter we will deal with Sobolev-like spaces and with a slight abuse of notation we will call, for a measurable set Ω of finite measure and $k \in \mathbb{N}$, $\lambda_k(\Omega) = \tilde{\lambda}_k(\Omega) = \lambda_k(\tilde{H}^1_0(\Omega))$. The main result of the Chapter consists in proving that an optimal set Ω for λ_k has an eigenfunction, corresponding to the eigenvalue $\lambda_k(\Omega)$, which is Lipschitz continuous in $\mathbb{R}^N$. In the first Section we recall some definitions and results about PDEs and gradient estimates, then we treat the Lipschitz regularity of shape quasi minimizers for the Dirichlet energy, following the ideas of Briançon, Hayouni and Pierre [14]. Then, in Section 5.4, we use the "robustness" of shape supersolutions to get the main result. At last, we discuss some cases in which we can give more informations on optimal sets, in particular when we can say that they are open.

5.1 Preliminaries

First of all we summarize some preliminary result about regularity for solutions of some elliptic PDE.
We start by reformulating some basic regularity facts in the case of PDEs and eigenfunctions on measurable sets. We remark that in the present Chapter we use the quasi-Sobolev spaces $\tilde{H}^1_0$, defined in (2.2), because they appear more suitable for the study of the regularity of minimizers of spectral functionals the the usual "capacitary" H^1_0. The key reason of this unusual choice will be clear later.
Since $\tilde{H}^1_0(\Omega)$ is a close subspace of $H^1(\mathbb{R}^N)$, one can define the Dirichlet Laplacian on Ω through weak solutions of elliptic problems on Ω. More precisely, let $\Omega \subset \mathbb{R}^N$ be a set of finite Lebesgue measure and $f \in L^2(\Omega)$. We say that the function u satisfies the equation

$$-\Delta u = f, \qquad \text{in } \Omega \tag{5.1}$$

if $u \in \tilde{H}^1_0(\Omega)$ and for every $v \in \tilde{H}^1_0(\Omega)$ it is $\langle \Delta u + f, v\rangle = 0$, where for all $v \in H^1(\mathbb{R}^N)$ we set

$$\langle \Delta u + f, v\rangle := -\int_{\mathbb{R}^N} Du \cdot Dv\,dx + \int_{\mathbb{R}^N} fv\,dx. \tag{5.2}$$

Equivalently, u is a solution of (5.1) if it is a minimizer in $\tilde{H}^1_0(\Omega)$ of the functional $J_f\colon H^1(\mathbb{R}^N) \to \mathbb{R}$, defined as

$$J_f(v) := \frac{1}{2}\int_\Omega |Dv|^2\,dx - \int_\Omega vf\,dx, \qquad v \in H^1(\mathbb{R}^N).$$

It is easy to check that, if u is a solution of (5.1) in Ω, then u also belongs to $\tilde{H}^1_0(\{u \neq 0\})$, and it is a solution of

$$-\Delta u = f \quad \text{in } \{u \neq 0\}.$$

If Ω is an open set with smooth boundary and u is a solution of (5.1), then the operator $\Delta u + f\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ defined in (5.2) can be simply written as

$$\langle \Delta u + f, v\rangle = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,d\mathcal{H}^{N-1}, \qquad \forall v \in H^1(\mathbb{R}^N).$$

More generally, we can prove that, for a measurable set Ω, the operator $\Delta u + f$ is a measure concentrated on the boundary of Ω.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a set of finite measure, $f \in L^2(\Omega)$ and $u \in \tilde{H}^1_0(\Omega)$ be a solution of (5.1). Then there exists a capacitary measure μ such that, for every $v \in H^1(\mathbb{R}^N)$ one has*

$$\langle \Delta u + f, v\rangle = \int_{\mathbb{R}^N} v\,d\mu. \tag{5.3}$$

Moreover, μ satisfies the following properties:

(i) If $u \geq 0$, then the measure is positive.

(ii) The support of μ is contained in the topological boundary of Ω.

Proof. Suppose first that $u \geq 0$, and define the functions $p_n : \mathbb{R}^+ \to [0,1]$ as

$$p_n(t) = nt \quad \text{if } t \in [0, 1/n]\,, \qquad\qquad p_n(t) = 1 \quad \text{if } t \geq 1/n\,.$$

Then, for every non-negative $\varphi \in C^\infty_c(\mathbb{R}^N)$, since $p_n(u)\varphi \in \widetilde{H}^1_0(\Omega)$ we can evaluate

$$\begin{aligned} 0 = \langle \Delta u + f, p_n(u)\varphi\rangle &= \int_{\mathbb{R}^N} -Du \cdot D(p_n(u)\varphi) + fp_n(u)\varphi\,dx \\ &= \int_{\mathbb{R}^d} -p_n(u)Du \cdot D\varphi - p_n'(u)|Du|^2\varphi + fp_n(u)\varphi\,dx \\ &\leq \int_{\mathbb{R}^d} p_n(u)\left(-Du \cdot D\varphi + f\varphi\right)\,dx \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} -Du \cdot D\varphi + f\varphi\,dx = \langle \Delta u + f, \varphi\rangle\,. \end{aligned}$$

The functional $\Delta u + f$ on $C^\infty_c(\mathbb{R}^N)$ is then a positive distribution; keeping in mind that a positive distribution is always a measure, we get a positive Radon measure μ such that the equality (5.3) is true for every smooth function v. Thanks to the definition (5.2), an immediate approximation argument shows that μ is a capacitary measure, then we get at once that the right term in (5.3) makes sense also for any $v \in H^1(\mathbb{R}^N)$, and that the equation (5.3) is true in $H^1(\mathbb{R}^N)$.

Consider now the case of a generic function $u \in \tilde{H}^1_0(\Omega)$, and call $\Omega^\pm = \{u \gtrless 0\}$. It is immediate to observe that u^+ solves the equation $-\Delta u = f_{|\Omega^+}$ in Ω^+, thus the argument above implies that $\Delta u^+ + f_{|\Omega^+}$ corresponds to a positive capacitary measure μ^+, and the very same argument shows also that $\Delta u^- + f_{|\Omega^-}$ is a negative capacitary measure μ^-. Since it is straightforward to check that $\big(\Delta u^+ + f_{|\Omega^+}\big) + \big(\Delta u^- + f_{|\Omega^-}\big) = \Delta u + f$, the claim is then proved with the (signed) measure $\mu = \mu^+ + \mu^-$.

The fact that μ is concentrated on the topological boundary of Ω comes trivially by approximation, since for every smooth φ concentrated either in the interior of Ω or in the interior of $\mathbb{R}^d \setminus \Omega$ one has $\langle \Delta u + f, \varphi\rangle = 0$ by definition. □

We now quickly recall some properties of the solutions $u \in \tilde{H}^1_0(\Omega)$ of (5.1) when Ω is a measurable set of finite measure and the datum $f \in L^\infty(\Omega)$. First of all, an L^∞ estimate for u holds, namely

$$\|u\|_{L^\infty} \leq \frac{|\Omega|^{2/N}\|f\|_{L^\infty}}{2N\omega_N^{2/N}}\,,$$

and the equality achieved when Ω is a ball and $f \equiv$ const on Ω (see for instance [50]). Moreover, since the function

$$v(x) = u(x) + \frac{\|f\|_{L^\infty}}{2N}|x|^2\,,$$

is clearly subharmonic on $\mathbb{R}^N$, because $\Delta v = \Delta u + \|f\|_{L^\infty} = -f + \|f\|_{L^\infty} \geq 0$, then it is simple to notice that every point of $\mathbb{R}^N$ is a Lebesgue point for u. More in detail, whenever $v \in H^1(\mathbb{R}^N)$ is a function such that Δv is a measure on $\mathbb{R}^N$, then the following estimate holds for any $x \in \mathbb{R}^N$ and $r > 0$ (for a proof, see for instance [14, Lemma 3.6]):

$$⨍_{\partial B_r(x)} v\, d\mathcal{H}^{N-1} - v(x) = \frac{1}{N\omega_N}\int_{\rho=0}^{r} \rho^{1-N}\Delta v(B_\rho(x))\, d\rho\,, \tag{5.4}$$

where $\Delta v(B_r(x))$ is the measure of $B_r(x)$ with respect to the measure Δv.

Most of the perturbation techniques that we will use to get the Lipschitz continuity of the state functions u on the optimal sets Ω will provide us information about the mean values $⨍_{B_r} u\, dx$ or $⨍_{\partial B_r} u\, d\mathcal{H}^{N-1}$. In order to transfer this information to the gradient $|Du|$, we will make use of the following classical result (see for example [36] for a proof).

Remark 5.2 (Gradient estimate)**.** *If $u \in H^1(B_r)$ is such that $-\Delta u = f$ in B_r and $f \in L^\infty(B_r)$, then the following estimates hold*

$$\begin{aligned} \|Du\|_{L^\infty(B_{r/2})} &\leq C_N\|f\|_{L^\infty(B_r)} + \frac{2N}{r}\|u\|_{L^\infty(B_r)}\,, \\ \|u\|_{L^\infty(B_{2r/3})} &\leq \frac{r^2}{2N}\,\|f\|_{L^\infty(B_r)} + C_N ⨍_{\partial B_r} |u|\, d\mathcal{H}^{N-1}\,. \end{aligned} \tag{5.5}$$

Actually, while for the first estimate it is really important that the equation $-\Delta u = f$ is valid in the whole B_r, the second estimate holds true also for balls B_r centered in a point $x \in \partial\Omega$, where Ω is an open set such that $u \in H^1_0(\Omega)$ and $-\Delta u = f$ is valid in Ω. Even if this fact is known, we will add a simple proof of it while proving Theorem 5.5.

Thanks to the L^∞ bound on eigenfunctions of the Dirichlet Laplacian (2.5), the arguments above imply that every point of $\mathbb{R}^N$ is a Lebesgue point for u_k and that the function

$$x \mapsto |u_k(x)| + \frac{e^{1/8\pi}\lambda_k(\Omega)^{\frac{N+4}{4}}}{2N}|x|^2$$

is subharmonic in $\mathbb{R}^N$. Applying then (5.5), we get that for any ball B_r essentially contained in Ω (this means, $\Omega \setminus B_r$ is negligible) one has

$$\|Du_k\|_{L^\infty(B_{r/3})} \leq C_N \left(\lambda_k(\Omega)^{\frac{N+4}{4}} + \frac{1}{r}\fint_{\partial B_r} |u_k|\, d\mathcal{H}^{N-1}\right).$$

5.2 Lipschitz continuity of energy quasi-minimizers

In this Section we study the properties of the local quasi-minimizers for the Dirichlet integral. More precisely, let $f \in L^2(\mathbb{R}^N)$ and let $u \in H^1(\mathbb{R}^N)$ satisfies $-\Delta u = f$ in $\{u \neq 0\}$.

Definition 5.3. *We say that u is a* local quasi-minimizer *for the functional*

$$J_f(u) = \frac{1}{2}\int_{\mathbb{R}^N} |Du|^2\, dx - \int_{\mathbb{R}^N} uf\, dx, \tag{5.6}$$

if there is a positive constant C such that for every $r > 0$ we have

$$J_f(u) \leq J_f(v) + Cr^N, \qquad \forall v \in \mathcal{A}_r(u), \tag{5.7}$$

where the admissible set $\mathcal{A}_r(u)$ is defined as

$$\mathcal{A}_r(u) := \Big\{v \in H^1(\mathbb{R}^N) \,:\, \exists x_0 \in \mathbb{R}^N \text{ such that } v - u \in H^1_0(B_r(x_0))\Big\}.$$

It is equivalent to check the validity of (5.7) only for $0 \leq r \leq r_0$, for some $r_0 > 0$, since J_f is clearly bounded from below.

Remark 5.4. *We want to highlight three equivalent characterizations of local quasi-minimality. First of all it is equivalent (and this is straightforward from the definition) to the existence of a constant $C > 0$, such that for every ball $B_r(x_0)$ of radius smaller than r_0 and every $v \in H^1_0(B_r(x_0))$, we have*

$$|\langle \Delta u + f, v\rangle| \leq \frac{1}{2}\int_{B_r(x_0)} |Dv|^2\, dx + Cr^N. \tag{5.8}$$

By the nonlinearity of the right term, it is enough that for some constant $C_1, C_2 > 0$, it holds

$$|\langle \Delta u + f, v\rangle| \leq C_1\int_{B_r(x_0)} |Dv|^2\, dx + C_2 r^N. \tag{5.9}$$

Indeed it is clear that (5.8) *implies* (5.9) *and on the other hand if* (5.9) *holds true, then for every* $v \in H^1_0(B_r(x_0))$ *it is enough to apply* (5.9) *to the function* $(2C_1)^{-1}\varphi$ *to get the validity of* (5.8) *with constant* $C = 2C_1C_2$*. The third equivalent formulation is the following: there exists a constant* $C_b > 0$ *such that for any ball* $B_r(x_0)$ *of radius* $r \le r_0$ *and any* $v \in H^1_0(B_r(x_0))$*, it is*

$$|\langle \Delta u + f, v\rangle| \le C_b r^{N/2} \left(\int_{B_r(x_0)} |Dv|^2\, dx \right)^{1/2}. \tag{5.10}$$

It is clear using the mean geometric-mean quadratic inequality that (5.10) *implies* (5.9)*. On the other hand, testing* (5.9) *with* $\widetilde{\varphi} := r^{N/2}\|D\varphi\|^{-1}_{L^2}\varphi$ *gives* (5.10) *with* $C_b = C_1 + C_2$*. A last observation, coming again from the nonlinearity of the right term in* (5.9)*, is the following: if we obtain* (5.9) *only for functions* $v \in H^1_0(B_r(x_0))$ *with* $\int |Dv|^2 \le 1$*, then it is sufficient to obtain* (5.10) *for every* $v \in H^1_0(B_r(x_0))$*, just testing* (5.9) *with* $\widetilde{v} := r^{N/2}\|Dv\|^{-1}_{L^2}v$ *(this requires* $r_0 \le 1$*, which is admissible as already observed).*

We present now a Theorem concerning the Lipschitz continuity of the local quasi-minimizers, which is a consequence of the techniques introduced by Briançon, Hayouni and Pierre [14]. This will be the starting point for the regularity results for eigenfunctions on optimal domains developed in the next Sections.

Theorem 5.5. *Let* $\Omega \subset \mathbb{R}^N$ *be a measurable set of finite measure,* $f \in L^\infty(\Omega)$ *and the function* $u \in H^1(\mathbb{R}^N)$ *be a solution of the equation* $-\Delta u = f$ *in* $\tilde{H}^1_0(\Omega)$ *and a local quasi-minimizer for the functional* J_f*. Then:*

(1) u *is Lipschitz continuous on* $\mathbb{R}^N$ *and its Lipschitz constant depends on* N, $\|f\|_\infty$, $|\Omega|$, r_0 *and the constant* C_b *from* (5.10)*.*

(2) the distribution $\Delta|u|$ *is a Borel measure on* $\mathbb{R}^N$ *satisfying*

$$|\Delta|u||(B_r(x)) \le C\, r^{N-1}, \tag{5.11}$$

for every $x \in \mathbb{R}^N$ *such that* $u(x) = 0$ *and every* $0 < r < r_0/4$*, where the constant* C *depends on* N, $\|f\|_\infty$, $|\Omega|$ *and* C_b *(but not on* r_0*).*

We observe that the local quasi-minimality is also necessary for the Lipschitz continuity of u, because it expresses in a weak form the boundedness of the gradient $|Du|$ on the boundary $\partial\Omega$.

The proof of this Theorem is implicitly contained in [14, Theorem 3.1], but we reproduce it here with our notations and small changes. We will divide the proof in some Lemmas and recall the monotonicity formula of Alt, Caffarelli and Friedmann.

First of all we note that if the state function u, quasi-minimizer for the functional J_f, is positive, then the classical approach of Alt and Caffarelli (see [2]) can be applied to obtain the Lipschitz continuity of u. This approach is based on an external perturbation and on the following inequality (see [2, Lemma 3.2])

$$\frac{\big|B_r(x_0)\cap\{u=0\}\big|}{r^2}\Big(\fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{N-1}\Big)^2 \le C_N\int_{B_r(x_0)}|D(u-v)|^2\,dx, \tag{5.12}$$

which holds for every $x_0\in\mathbb{R}^N$, $r>0$, $u\in H^1(\mathbb{R}^N)$, $u\ge 0$ and $v\in H^1(B_r)$ that solves

$$\min\left\{\int_{B_r(x_0)}|Dv|^2\,dx:\ v-u\in H^1_0(B_r(x_0)),\ v\ge u\right\}. \tag{5.13}$$

Since for sign-changing state functions u, the inequality (5.12) is not known, one needs a more careful analysis on the common boundary of $\{u>0\}$ and $\{u<0\}$, which is based on the monotonicity formula proved by Alt, Caffarelli and Friedmann in [3].

Theorem 5.6. *Let $U^+,U^-\in H^1(B_1)$ be continuous non-negative functions such that $\Delta U^\pm\ge -1$ on B_1 and $U^+U^-=0$. Then there is a dimensional constant C_N such that for each $r\in(0,\frac{1}{2})$*

$$\left(\frac{1}{r^2}\int_{B_r}\frac{|DU^+(x)|^2}{|x|^{N-2}}\,dx\right)\left(\frac{1}{r^2}\int_{B_r}\frac{|DU^-(x)|^2}{|x|^{N-2}}\,dx\right)\le C_N\left(1+\int_{B_1}|U^++U^-|^2\,dx\right). \tag{5.14}$$

For our purposes we will need the following rescaled version of this formula.

Corollary 5.7. *Let $\Omega\subset\mathbb{R}^N$ be a quasi-open set of finite measure, $f\in L^\infty(\Omega)$ and $u:\mathbb{R}^N\to\mathbb{R}$ be a continuous function such that*

$$-\Delta u=f\quad in\quad \Omega,\qquad u\in H^1_0(\Omega). \tag{5.15}$$

Setting $u^+=\sup\{u,0\}$ and $u^-=\sup\{-u,0\}$, there is a dimensional constant C_N such that for each $0<r\le 1/2$

$$\left(\frac{1}{r^2}\int_{B_r}\frac{|Du^+(x)|^2}{|x|^{N-2}}\,dx\right)\left(\frac{1}{r^2}\int_{B_r}\frac{|Du^-(x)|^2}{|x|^{N-2}}\,dx\right)\le C_N\left(\|f\|^2_\infty+\int_\Omega u^2\,dx\right)\le C_m, \tag{5.16}$$

where $C_m=C_N\|f\|^2_\infty\left(1+|\Omega|^{\frac{N+4}{N}}\right)$.

Proof of Corollary 5.7. We apply Theorem 5.6 to $U^\pm=\|f\|^{-1}_\infty u^\pm$ and substituting in (5.14) we obtain the first inequality in (5.16). The second one follows, using the equation (5.15):

$$\|u\|^2_{L^2}\le C_N|\Omega|^{2/N}\|Du\|^2_{L^2}=C_N|\Omega|^{2/N}\int_\Omega fu\,dx\le C_N|\Omega|^{2/N+1/2}\|f\|_\infty\|u\|_{L^2},$$

and this concludes the proof. □

The proof of the Lipschitz continuity of the quasi-minimizers for J_f needs two preliminary results, precisely in Lemma 5.8 we prove the continuity of u and in Lemma 5.10, we give an estimate on the Laplacian of u as a measure on the boundary $\partial\{u\ne 0\}$.

Lemma 5.8. *Suppose that u satisfies the assumptions of Theorem 5.5, then it is continuous.*

Proof. Let $x_n \to x_\infty \in \mathbb{R}^N$ and set $\delta_n := |x_n - x_\infty|$. If for some n, $|B_{\delta_n}(x_\infty) \cap \{u = 0\}| = 0$, then $-\Delta u = f$ in $B_{\delta_n}(x_\infty)$ and so u is continuous in x_∞.
Assume now that for all n, $|B_{\delta_n}(x_\infty) \cap \{u = 0\}| \neq 0$ and consider the function $u_n : \mathbb{R}^N \to \mathbb{R}$ defined by $u_n(\xi) = u(x_\infty + \delta_n \xi)$. Since $\|u_n\|_\infty = \|u\|_\infty$, for any n, and $\|u\|_{L^\infty} < \infty$, we can assume, up to a subsequence, that u_n converges weakly-$*$ in L^∞ to some function $u_\infty \in L^\infty(\mathbb{R}^N)$. We will prove that $u_\infty = 0$ and that $u_n \to u_\infty$ uniformly on B_1, which implies the continuity of u and $u(x_\infty) = 0$.
Step 1. u_∞ is a constant.
For all $R \geq 1$ and $n \in \mathbb{N}$, we introduce the function $v_{R,n}$ such thay:

$$\begin{cases} -\Delta v_{R,n} = f, & \text{in } B_{R\delta_n}(x_\infty), \\ v_{R,n} = u, & \text{on } \partial B_{R\delta_n}(x_\infty). \end{cases}$$

Setting $v_n(\xi) := v_{R,n}(x_\infty + \delta_n \xi)$, we have that

$$\begin{aligned} \int_{B_R} |D(u_n - v_n)|^2 d\xi &= \delta_n^{2-N} \int_{B_{R\delta_n}(x_\infty)} |D(u - v_{R,n})|^2 dx \\ &= \delta_n^{2-N} \int_{B_{R\delta_n}(x_\infty)} Du \cdot D(u - v_{R,n}) dx - \delta_n^{2-N} \int_{B_{R\delta_n}(x_\infty)} f(u - v_{R,n}) dx \\ &\leq C_b \delta_n^{2-N} \left(\int_{B_{R\delta_n}(x_\infty)} |D(u - v_{R,n})|^2 dx \right)^{1/2} R^{N/2} \delta_n^{N/2} \\ &\leq C_b R^{N/2} \delta_n \left(\int_{B_R} |D(u_n - v_n)|^2 d\xi \right)^{1/2}, \end{aligned}$$

where C_b is the constant from (5.10), and then for $\delta_n \leq r_0$, we have

$$\fint_{B_R} |D(u_n - v_n)|^2 \, d\xi \leq C_b^2 \delta_n^2,$$

In particular, $u_n - v_n \to 0$ in $H^1(B_R)$ for any $R \geq 1$. On the other hand, calling $f_n(\xi) := f(x_\infty + \delta_n \xi)$, we have that

$$\begin{cases} -\Delta v_n = \delta_n^2 f, & \text{in } B_R, \\ v_n \leq \|u\|_\infty, & \text{on } \partial B_R. \end{cases}$$

Since $\|f_n\|_{L^\infty} = \|f\|_{L^\infty}$, the maximum principle implies that the v_n are equi-bounded in B_R; then the estimate (5.5) implies that they are equi-Lipschitz and thus equi-continuous in $B_{R/2}$. So, up to a subsequence, v_n uniformly converges to some function which is harmonic on $B_{R/2}$. Since $u_n - v_n \to 0$ in $H^1(B_R)$ and u_n converges weakly-$*$ in $L^\infty(\mathbb{R}^N)$ to u_∞, we deduce that v_n converges uniformly to u_∞ on B_R and that u_∞ is harmonic on $B_{R/2}$ for every $R \geq 1$. Therefore u_infty is a bounded harmonic function in $\mathbb{R}^N$, so it is constant.

Step 2. $u_n \to u_\infty$ *in* $H^1_{loc}(\mathbb{R}^N)$.
In fact, for the functions $\widetilde{v}_n = v_n - u_\infty$, we have that

$$\begin{cases} -\Delta \widetilde{v}_n = \delta_n^2 f, & \text{in } B_R, \\ \widetilde{v}_n \leq 2\|u\|_\infty, & \text{on } \partial B_R, \end{cases}$$

and $\widetilde{v}_n \to 0$ uniformly on $B_{R/2}$. Again by (5.5), we have that $\|D\widetilde{v}_n\|_{L^\infty(B_{R/4})} \to 0$ and so, $v_n \to u_\infty$ in $H^1(B_{R/4})$. Since $u_n - v_n \to 0$ in $H^1(B_R)$ we have completed also this Step.
Step 3. If $u_\infty \geq 0$, *then* $u_n^- \to 0$ *uniformly on balls.*
Since on $\{u_n < 0\}$, the equality $-\Delta u_n^- = -\delta_n^2 f$ holds, we have that $-\Delta u_n^- \leq -\delta_n^2 f I_{\{u_n<0\}} \leq \delta_n^2 |f|$ on $\mathbb{R}^N$. Thus, it is enough to prove that for each $R \geq 1$, $\widetilde{u}_n \to 0$ uniformly on $B_{2R/3}$, where

$$\begin{cases} -\Delta \widetilde{u}_n = \delta_n^2 |f|, & \text{in } B_R, \\ \widetilde{u}_n = u_n^-, & \text{on } \partial B_R. \end{cases}$$

Since $u_n^- \to 0$ in $H^1(B_R)$, we have that $\int_{\partial B_R} u_n^- \to 0$ and the claim follows again from estimate (5.5)
Step 4. $u_\infty = 0$
Suppose, without loss of generality, that $u_\infty \geq 0$. Let $y_n = x_\infty + \delta_n \xi_n$, where $\xi_n \in B_1$, be a Lebesgue point of u such that $u(y_n) = 0$. For each $s > 0$ consider the function $\phi_s \in C_c^\infty(B_{2s}(y_n))$ such that $0 \leq \phi_s \leq 1$, $\phi_s = 1$ on $B_s(y_n)$ and $\|D\phi_s\|_{L^\infty} \leq \frac{2}{s}$. Thus, we have that

$$|\langle \Delta u + f, \phi_s \rangle| \leq C_N C_b s^{N-1},$$

where C_b is the constant from (5.10). Denote with μ_1 and μ_2 the positive Borel measures $\Delta u^+ + f I_{\{u>0\}}$ and $\Delta u^- - f I_{\{u<0\}}$. Then, we have

$$\mu_1(B_s(y_n)) \leq \langle \mu_1, \phi_s \rangle = \langle \mu_1 - \mu_2, \phi_s \rangle + \langle \mu_2, \phi_s \rangle \leq C_N C_b s^{N-1} + \mu_2(B_{2s}(y_n)).$$

Moreover, since $f \in L^\infty$, we have that for each $s \leq 1$,

$$\begin{aligned} \Delta u^+(B_s(y_n)) &\leq C_N C_b s^{N-1} + \Delta u^-(B_{2s}(y_n)) + C_N \|f\|_{L^\infty} s^N \\ &\leq C_N \big(C_b + \|f\|_\infty\big) s^{N-1} + \Delta u^-(B_{2s}(y_n)). \end{aligned} \tag{5.17}$$

We recall now the standard estimate

$$\frac{\partial}{\partial s} \fint_{\partial B_s(y_n)} u^+ = \fint_{\partial B_s(y_n)} \frac{\partial u^+}{\partial \nu} = \frac{1}{N\omega_N s^{N-1}} \Delta u^+(B_s),$$

and observe that since y_n is a Lebesgue point for u with $u(y_n) = 0$, then

$$\lim_{s \to 0} \fint_{\partial B_s(y_n)} u^+ = 0.$$

Thanks to the above observation, multiplying the estimate (5.17) by s^{1-N} and integrating, we obtain

$$\fint_{\partial B_{\delta_n}(y_n)} u^+ \, d\mathcal{H}^{N-1} \leq C_N \big(C_b + \|f\|_\infty\big) \delta_n + \frac{1}{2} \fint_{\partial B_{2\delta_n}(y_n)} u^- \, d\mathcal{H}^{N-1},$$

or, equivalently,

$$\fint_{\partial B_1} u_n^+(\xi_n+\cdot)\,d\mathcal{H}^{N-1} \le C_N\big(C_b+\|f\|_\infty\big)\delta_n + \frac{1}{2}\fint_{\partial B_2} u_n^-(\xi_n+\cdot)\,d\mathcal{H}^{N-1}.$$

Since by Step 3 the right-hand side goes to zero as $n\to\infty$, so does the left-hand side. Up to a subsequence, we may assume that $\xi_n\to\xi_\infty$ and so, $u_n(\xi_n+\cdot)\to u_\infty(\xi_\infty+\cdot)=u_\infty$ in $H^1_{loc}(\mathbb{R}^N)$. Thus $u_\infty=0$.

Step 5. The convergence $u_n\to 0$ is uniform on the ball B_1.

Since $u_\infty=0$, this follows just applying twice Step 3, once to u and the other to $-u$. □

Remark 5.9. *In $\mathbb{R}^2$ the continuity of the state function u, instead of using Theorem 5.5, can be deduced by the classical Alt-Caffarelli argument, which we apply after reducing the problem to the case when u is positive. This is the analogous argument to the one exposed in Chapter 4, rewritten with the notation of free boundary problems. For example, if $u\in H^1(\mathbb{R}^2)$ is a function satisfying*

$$J_{\lambda u}(u)+c|\{u\neq 0\}|\le J_{\lambda u}(v)+c|\{v\neq 0\}|,\qquad \forall v\in H^1(\mathbb{R}^2),$$

for some $\lambda>0$, then u is continuous. Indeed, let $x_0\in\mathbb{R}^2$ be such that $u(x_0)>0$ and let $r_0>0$ and $\varepsilon>0$ be small enough such that, for every $x\in\mathbb{R}^2$ and every $r\le r_0$, we have $\int_{B_r(x)}|Du|^2\,dx\le\varepsilon$. As a consequence, for every $x\in\mathbb{R}^2$ there is some $r_x\in[r_0/2,r_0]$ such that $\int_{\partial B_{r_x}(x)}|Du|^2\,dx\le 2\varepsilon/r_0$ and

$$osc_{\partial B_{r_x}(x)}u\le\int_{\partial B_{r_x}(x)}|Du|\,d\mathcal{H}^1\le\sqrt{2\pi r_0}\sqrt{2\varepsilon/r_0}\le\sqrt{4\pi\varepsilon}. \tag{5.18}$$

On the other hand, the positive part $u^+=\sup\{u,0\}$ of u satisfies $\Delta u^++\lambda\|u\|_\infty\ge 0$ on $\mathbb{R}^2$ and so, there is a constant $C>0$ such that

$$u(x_0)\le\fint_{\partial B_{r_{x_0}}(x_0)}u\,d\mathcal{H}^1+Cr_{x_0}^2,$$

which together with (5.18) gives that, choosing $r_0>0$ small enough, we can construct a ball $B_r(x_0)$ of radius $r\le r_0$ such that $u\ge u(x_0)/2>0$ on $\partial B_r(x_0)$.

We then notice that the set $\{u<0\}\cap B_r(x_0)$ has measure 0. Indeed, if this is not the case, then the function $\widetilde{u}=\sup\{-u,0\}I_{B_r(x_0)}\in H^1_0(B_r(x_0))$ is such that $J_{\lambda u}(u)=J_{\lambda u}(-\widetilde{u})+J_{\lambda u}(u+\widetilde{u})$. By the maximum principle $\|\widetilde{u}\|_\infty\le Cr_0^2$ and so, for some constant $C>0$, we have

$$\big|J_{\lambda u}(-\widetilde{u})\big|\le Cr_0^2\big|\{u<0\}\cap B_r(x_0)\big|<c\big|\{u<0\}\cap B_r(x_0)\big|,$$

for r_0 small enough. Hence we have $J_{\lambda u}(-\widetilde{u})+c|\{u<0\}\cap B_r(x_0)|>0$, that contradicts the quasi-minimality of u.

We conclude the proof by showing that the set $\{u=0\}\cap B_r(x_0)$ *has measure* 0. *We compare* u *with the function* $w = I_{B_r^c(x_0)}u + I_{B_r(x_0)}v$, *where* v *is the function from* (5.13).

$$\begin{aligned}
c\big|\{u=0\}\cap B_r(x_0)\big| &\geq J_{\lambda u}(u) - J_{\lambda u}(w)\\
&= \frac{1}{2}\int_{B_r(x_0)} \left(|Du|^2 - |Dv|^2\right) dx - \int_{B_r(x_0)} \lambda u(u-v)\, dx\\
&\geq \frac{1}{2}\int_{B_r(x_0)} |D(u-v)|^2\, dx\\
&\geq \frac{C_2}{r^2}\big|\{u=0\}\cap B_r(x_0)\big|\Big(\fint_{\partial B_r(x_0)} u\, d\mathcal{H}^1\Big)^2,
\end{aligned}$$

where the last inequality is due to (5.12). *If we suppose that* $|\{u=0\}\cap B_r(x_0)| > 0$, *then for some constant* $C>0$, *we would have* $u(x_0) \leq C r_0^2$, *which is absurd choosing* $r_0 > 0$ *small enough.*

Lemma 5.10. *Let* $u \in H^1(\mathbb{R}^N)$ *satisfies the assumptions of Theorem 5.5 and in particular let* r_0 *and* C_b *be as in* (5.10). *Then, for each* $x_0 \in \mathbb{R}^N$, *such that* $u(x_0) = 0$ *and for every* $0 < r \leq r_0/4$, *it is*

$$|\Delta|u||(B_r(x_0)) \leq C\, r^{N-1},$$

where the constant C *depends only on* N, $|\Omega|$, $\|f\|_{L^\infty}$ *and* C_b.

Proof. Without loss of generality we can suppose $x_0 = 0$. For each $r>0$, consider the functions

$$v^r := v_+^r - v_-^r, \qquad w^r := w_+^r - w_-^r,$$

where $v_\pm^r$ and $w_\pm^r$ are solutions of the following equations on B_r

$$\begin{cases} -\Delta v_\pm^r = f^\pm, \text{ in } B_r,\\ \quad v_\pm^r = u^\pm, \text{ on } \partial B_r, \end{cases} \qquad \begin{cases} -\Delta w_\pm^r = f^\pm, \text{ in } B_r,\\ \quad v_\pm^r = 0, \text{ on } \partial B_r. \end{cases}$$

Thus we have that $v_\pm^r - w_\pm^r$ is harmonic in B_r and so, the estimate

$$\int_{B_r} |D(v_\pm^r - w_\pm^r)|^2\, dx \leq \int_{B_r} |Du^\pm|^2\, dx. \tag{5.19}$$

Since $u^\pm - v_\pm^r + w_\pm^r \in H_0^1(B_r)$ and $v_\pm^r - w_\pm^r$ is harmonic, we have

$$\begin{aligned}
\int_{B_r} |D(u^\pm - v_\pm^r + w_\pm^r)|^2\, dx &= \int_{B_r} Du^\pm \cdot D(u^\pm - v_\pm^r + w_\pm^r)\, dx\\
&= \int_{B_r} |Du^\pm|^2\, dx + \int_{B_r} Du^\pm \cdot D(w_\pm^r - v_\pm^r)\, dx \leq 2\int_{B_r} |Du^\pm|^2\, dx,
\end{aligned}$$

where the last inequality is due to (5.19). Thus, for $r \leq 1/2$ we obtain from the monotonicity formula (5.16) the estimate

$$\begin{aligned}
&\left(\fint_{B_r} |D(u^+ - v_+^r + w_+^r)|^2\, dx\right)\left(\fint_{B_r} |D(u^- - v_-^r + w_-^r)|^2\, dx\right)\\
&\qquad \leq 4\left(\fint_{B_r} |Du^+|^2\, dx\right)\left(\fint_{B_r} |Du^-|^2\, dx\right) \leq 4C_m\,.
\end{aligned} \tag{5.20}$$

On the other hand, for $0 < r \le r_0 \le 1$ and using also (5.10), we have

$$\begin{aligned} \int_{B_r} |D(u - v^r + w^r)|^2\,dx &\le 2\int_{B_r} |D(u - v^r)|^2\,dx + 2\int_{B_r} |Dw^r|^2\,dx \\ &= 2\int_{B_r} [Du\cdot D(u - v^r) + f(u - v^r)]\,dx + 2\int_{B_r} |Dw^r|^2\,dx \\ &\le C_b^2 r^N + C_b\int_{B_r} |D(u - v^r)|^2 + 2\int_{B_r} |Dw^r|^2\,dx \le Cr^N, \end{aligned} \tag{5.21}$$

where the constant C depends on N, $|\Omega|$, $\|f\|_{L^\infty}$ and C_b. Using (5.20) and (5.21), we have

$$\begin{aligned} &\int_{B_r} |D(u^+ - v_+^r + w_+^r)|^2\,dx + \int_{B_r} |D(u^- - v_-^r + w_-^r)|^2\,dx \\ &\le 2\left(\int_{B_r} |D(u^+ - v_+^r + w_+^r)|^2\,dx\right)^{1/2}\left(\int_{B_r} |D(u^- - v_-^r + w_-^r)|^2\,dx\right)^{1/2} \\ &+ \int_{B_r} |D(u - v^r + w^r)|^2\,dx \le Cr^N, \end{aligned}$$

where the constant C might have incresed, but have the same dependences as before, since C_m depends on N, $|\Omega|$, $\|f\|_{L^\infty}$. Putting together the last estimate with (5.21), we eventually get:

$$\int_{B_r} |D(u^\pm - v_\pm^r)|^2\,dx \le Cr^N. \tag{5.22}$$

Let us now define

$$U := u^+ - v_+^r, \qquad \mu_1 := \Delta u^+ + fI_{u>0}, \qquad \mu_2 := \Delta u^- - fI_{u<0}.$$

By definition $U \in H_0^1(B_r)$ and we can see that it is also subharmonic:

$$\Delta U = \Delta(u^+ - v_+^r) = \Delta u^+ + f^+ \ge \Delta u^+ + fI_{\{u>0\}} = \mu_1 \ge 0, \tag{5.23}$$

and so $U \le 0$. Then, using also (5.22), we have

$$\int_{B_r} v_+^r\,d\mu_1 = \int_{B_r} (v_+^r - u^+)\,d\mu_1 \le \int_{B_r} |DU|^2\,dx \le Cr^N. \tag{5.24}$$

Recalling that $U \le 0$ on B^r, we have that for each $z \in B_{r/4}$

$$⨍_{\partial B_{3r/4}(z)} U\,d\mathcal{H}^{N-1} \le 0 \le u^+(z) = U(z) + v_+^r(z).$$

Applying then (5.4) to $U \in H^1(B_r)$ (which is admissible because every signed distribution is a measure) and using (5.23), we obtain

$$\begin{aligned} v_+^r(z) &\ge -U(z) = ⨍_{\partial B_{3r/4}(z)} U\,d\mathcal{H}^{N-1} + \frac{1}{N\omega_N}\int_0^{3r/4} s^{1-N}\Delta U(B_s(z))\,ds \\ &\ge \frac{1}{N\omega_N}\int_0^{3r/4} s^{1-N}\mu_1(B_s(z))\,ds. \end{aligned}$$

By (5.24), we obtain eventually

$$
\begin{aligned}
C(r/4)^N &\geq \int_{B_{r/4}} v_+^r(z)\,d\mu_1(z) \geq \frac{1}{N\omega_N}\int_{B_{r/4}} d\mu_1(z)\int_0^{3r/4} s^{1-N}\mu_1(B_s(z))\,ds \\
&\geq \frac{1}{N\omega_N}\int_{B_{r/4}} d\mu_1(z)\int_{r/2}^{3r/4} s^{1-N}\mu_1(B_s(z))\,ds \\
&\geq \frac{1}{N\omega_N}\int_{B_{r/4}} d\mu_1(z)\int_{r/2}^{3r/4} s^{1-N}\mu_1(B_{r/4})\,ds \geq C_N r^{2-N}\left(\mu_1(B_{r/4})\right)^2,
\end{aligned}
$$

i.e. $\mu_1(B_r) \leq Cr^{N-1}$ for $0 < r < r_0/4$. Since the same clearly holds for μ_2 and $|\Delta|u|| \leq \mu_1 + \mu_2 + f$, and $f \in L^\infty$, we get the thesis. □

We are finally in position to give the proof of Theorem 5.5.

Proof of Theorem 5.5. By Lemma 5.8 we know that u is continuous, so we can assume that Ω coincides with the open set $\{u \neq 0\}$. Thanks to Lemma 5.10, we already know the validity of (5.11) for x such that $u(x) = 0$ and $0 < r < r_0/4$, hence to prove (2) of Theorem 5.5 we only need to check that $\Delta|u|$ is a Borel measure on $\mathbb{R}^N$. Since $\Delta|u| \equiv 0$ outside of $\overline{\Omega}$, we have only to take care of $\overline{\Omega}$. But $\Delta|u|$ coincides with $\pm f \in L^\infty$ inside Ω, thus just covering the compact set $\partial\Omega$ with finitely many balls of radius $r_0/5$ centered at points of $\partial\Omega$ we immediately obtain that $\Delta|u|$ is a Borel measure on the whole $\mathbb{R}^N$.

Let us now prove (1). For any $r > 0$, denote with $\Omega_r \subseteq \Omega$ the set $\{x \in \Omega : d(x, \Omega^c) < r\}$. Choose $x \in \Omega_{r_0/12}$ and let $y \in \partial\Omega$ be such that $R_x := d(x, \Omega^c) = |x - y|$. We claim now that

$$
\|u\|_{L^\infty(B_{2R_x}(y))} \leq \frac{9R_x^2}{2N}\,\|f\|_{L^\infty(B_{3R_x(y)})} + C_N \fint_{\partial B_{3R_x}(y)} |u|\,d\mathcal{H}^{N-1}. \tag{5.25}
$$

Notice that this is exactly the second gradient estimate (5.5) applied to u in the ball $B_{3R_x}(y)$, but actually we cannot apply this estimate because on that ball the equation $-\Delta u = f$ is not satisfied. To prove the validity of (5.25), assume then without loss of generality that $\|u\|_{L^\infty(B_{2R_x}(y))} = \|u^+\|_{L^\infty(B_{2R_x}(y))}$, and define v^+, as in Lemma 5.10, the solution of

$$
\begin{cases}
-\Delta v^+ = f^+ & \text{in } B_{3R_x}(y)\,, \\
v^+ = u^+ & \text{on } \partial B_{3R_x}(y)\,.
\end{cases}
$$

As already observed during the proof of Lemma 5.10, in (5.23), the function $u^+ - v^+$ is subharmonic hence, since it belongs to $H_0^1\big(B_{3R_x}(y)\big)$, it is negative in $B_{3R_x}(y)$. By this observation, and applying (5.5) in $B_{3R_x}(y)$ to the function v^+, which is admissible, we get

$$\begin{aligned}\|u\|_{L^\infty(B_{2R_x}(y))} = \|u^+\|_{L^\infty(B_{2R_x}(y))} &\leq \|v^+\|_{L^\infty(B_{2R_x}(y))}\\ &\leq \frac{9R_x^2}{2N}\,\|f^+\|_{L^\infty(B_{3R_x(y)})} + C_N \fint_{\partial B_{3R_x}(y)} |v^+|\,d\mathcal{H}^{N-1}\\ &\leq \frac{9R_x^2}{2N}\,\|f\|_{L^\infty(B_{3R_x(y)})} + C_N \fint_{\partial B_{3R_x}(y)} |u^+|\,d\mathcal{H}^{N-1}\\ &\leq \frac{9R_x^2}{2N}\,\|f\|_{L^\infty(B_{3R_x(y)})} + C_N \fint_{\partial B_{3R_x}(y)} |u|\,d\mathcal{H}^{N-1}\,,\end{aligned}$$

thus the validity of (5.25) is established. Hence, applying the first gradient estimate (5.5) to u in the ball $B_{R_x}(x)$, using (5.25), and then applying the estimate (5.4), which is possible because $\Delta|u|$ is a measure (mind also that $u(y) = 0$), we get

$$\begin{aligned}|\nabla u(x)| &\leq C_N\|f\|_{L^\infty} + \frac{2N}{R_x}\|u\|_{L^\infty(B_{R_x}(x))} \leq C_N\|f\|_{L^\infty} + \frac{2N}{R_x}\|u\|_{L^\infty(B_{2R_x}(y))}\\ &\leq (C_N + r_0)\|f\|_{L^\infty} + \frac{C_N}{R_x}\fint_{\partial B_{3R_x}(y)} |u|\,d\mathcal{H}^{N-1}\\ &\leq (C_N + r_0)\|f\|_{L^\infty} + \frac{C_N}{R_x}\int_0^{3R_x} s^{1-d}|\Delta|u||(B_s(y))\,ds \leq (C_N + r_0)\|f\|_{L^\infty} + 3C_N C\,,\end{aligned}$$

where C is the constant from Lemma 5.10. Since for $x \in \Omega \setminus \Omega_{r_0/12}$ we have, still by (5.5), that

$$|\nabla u(x)| \leq C_N\|f\|_{L^\infty} + \frac{24N}{r_0}\|u\|_{L^\infty}\,,$$

we obtain that u is Lipschitz and its Lipschitz constant can be estimated as desired. □

We prove now that, in the case of eigenfunctions, something better can be obtained, in particular the Lipschitz constant does *not* depend on r_0.

Theorem 5.11. *Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set of finite measure, and let u be a normalized eigenfunction on Ω with eigenvalue λ, as well as a local quasi-minimizer of J_f, being $f = \lambda u$; in particular, u satisfies* (5.10) *with some constant C_u and for r smaller than some $r_0 = r_0(u) \leq 1$. Then u is Lipschitz continuous in $\mathbb{R}^N$ and the Lipschitz constant depends* only *on N, $|\Omega|$, λ, and C_u, but* not *on r_0.*

Proof. By Theorem 5.5, applied to u and $f := \lambda u$, we already know that u is Lipschitz continuous, then we must only show that its Lipschitz constant is independent on r_0.

Let us then set $\widetilde{\Omega} := \{u \neq 0\}$ (note that $\widetilde{\Omega}$ is open); let also $\bar{x}$ be any point with $R := d(\bar{x}, \widetilde{\Omega}^c) < r_0/8$ and let $y \in \partial\widetilde{\Omega}$ be such that $|y - \bar{x}| = R$. Using the first estimate (5.5) on the ball $B_{R/2}(\bar{x}) \subseteq \widetilde{\Omega}$ we know that

$$|\nabla u(\bar{x})| \leq C_N\lambda\|u\|_{L^\infty} + \frac{4N}{R}\|u\|_{L^\infty(B_{R/2}(\bar{x}))}\,. \tag{5.26}$$

Let now $z \in B_{R/2}(\bar{x})$ be a point such that $\|u\|_{L^\infty(B_{R/2}(\bar{x}))} \leq |u(z)|$. For any $0 < r < R/2$, the ball $B_r(z)$ is contained in $\widetilde{\Omega}$, so we can apply the second estimate (5.5) on it, to get

$$\|u\|_{L^\infty(B_{R/2}(\bar{x}))} \leq |u(z)| \leq \|u\|_{L^\infty(B_{2r/3}(z))} \leq \frac{R^2}{8N}\,\lambda\|u\|_{L^\infty} + C_N \fint_{\partial B_r(z)} |u|\,d\mathcal{H}^{N-1}\,.$$

Being this estimate valid for every $0 < r < R/2$, then of course it is also

$$\|u\|_{L^\infty(B_{R/2}(\bar{x}))} \leq \frac{R^2}{8N}\,\lambda\|u\|_{L^\infty} + C_N \fint_{B_{R/2}(z)} |u|\,d\mathcal{H}^N\,. \tag{5.27}$$

Since $B_{R/2}(z) \subseteq B_{2R}(y)$ by construction, we deduce

$$\fint_{B_{R/2}(z)} |u|\,d\mathcal{H}^N = \frac{2^N}{\omega_N R^N}\int_{B_{R/2}(z)} |u| \leq \frac{2^N}{\omega_N R^N}\int_{B_{2R}(y)} |u| = 4^N \fint_{B_{2R}(y)} |u|\,d\mathcal{H}^N\,. \tag{5.28}$$

Finally, there exists some $r \in (0, 2R)$ such that

$$\fint_{B_{2R}(y)} |u|\,d\mathcal{H}^N \leq \fint_{\partial B_r(y)} |u|\,d\mathcal{H}^{N-1}\,. \tag{5.29}$$

Putting together (5.26), (5.27), (5.28) and (5.29), we then get

$$|\nabla u(\bar{x})| \leq C_N\lambda\|u\|_{L^\infty} + \frac{C_N}{R}\fint_{\partial B_r(y)} |u|\,d\mathcal{H}^{N-1}\,. \tag{5.30}$$

Observe now that the ball $B_r(y)$ is not contained in $\widetilde{\Omega}$, hence in this ball we could not apply the gradient estimates (5.5). Nevertheless, $|u|$ belongs to $H^1(\mathbb{R}^N)$, because u does, and Theorem 5.5 ensures that $\Delta|u|$ is a measure on $\mathbb{R}^N$; thus, we are in position to apply the estimate (5.4) with $v = |u|$ and $x = y$: keeping in mind that $u(y) = 0$ because $y \in \partial\widetilde{\Omega}$ and u is Lipschitz continuous, and using also (5.11) from Theorem 5.5, we get

$$\fint_{\partial B_r(y)} |u|\,d\mathcal{H}^{N-1} = \frac{1}{N\omega_N}\int_{\rho=0}^{r} \rho^{1-N}\Delta|u|(B_\rho(y))\,d\rho \leq \frac{Cr}{N\omega_N} \leq \frac{2CR}{N\omega_N}\,,$$

which inserted in (5.30), and recalling again Theorem 5.5, finally gives

$$|\nabla u(\bar{x})| \leq C'\,, \tag{5.31}$$

for some constant C' depending on N, λ, $\|u\|_{L^\infty}$, $|\Omega|$ and C_u, but not on r_0. In turn, since from (2.5) we have that $\|u\|_{L^\infty}$ can be bounded only in terms of λ and N, we have that C' actually depends only on N, λ, $|\Omega|$ and C_u, and of course still not on r_0. Summarizing, up to now we have shown that the Lipschitz constant of u is independent of r_0 in a $r_0/8$-neighborhood of the boundary of $\widetilde{\Omega}$; we will conclude the proof by showing that an estimate near the boundary implies a (worse) estimate in the whole set $\widetilde{\Omega}$.

To do so, consider the auxiliary function $P \in C^\infty(\widetilde{\Omega})$ defined as

$$P := |\nabla u|^2 + \lambda u^2 - 2\lambda^2\|u\|^2_{L^\infty} w_{\widetilde{\Omega}}\,,$$

where $w_{\widetilde{\Omega}} \in H^1_0(\widetilde{\Omega})$ is the solution of the equation $-\Delta w_{\widetilde{\Omega}} = 1$ in $\widetilde{\Omega}$. A direct computation gives that P is sub-harmonic on the open set $\widetilde{\Omega}$, indeed

$$\Delta P = \big(2[Hess(u)]^2 - 2\lambda|\nabla u|^2\big) + \big(2\lambda|\nabla u|^2 - 2\lambda^2 u^2\big) + 2\lambda^2\|u\|^2_{L^\infty} \geq 0\,.$$

Thus, by the maximum principle we get

$$\sup\big\{P(x)\,:\,x\in\widetilde{\Omega}\big\} \leq \sup\big\{P(x)\,:\,x\in\widetilde{\Omega},\ \mathrm{dist}(x,\partial\widetilde{\Omega}) < r_0/8\big\}\,,$$

and so, recalling the estimate (5.31) near the boundary, we immediately obtain

$$\|\nabla u\|^2_{L^\infty} \leq \|P\|_{L^\infty} + 2\lambda^2\|u\|^2_{L^\infty}\|w_{\widetilde{\Omega}}\|_{L^\infty} \leq C'^2 + \lambda\|u\|^2_{L^\infty} + 2\lambda^2\|u\|^2_{L^\infty}\|w_{\widetilde{\Omega}}\|_{L^\infty}\,.$$

We finally conclude the proof, just recalling again that $\|u\|_{L^\infty}$ can be bounded only in terms of λ and N as shown in (2.5), and also by the classical bound $\|w_{\widetilde{\Omega}}\|_{L^\infty} \leq C_N|\widetilde{\Omega}|^{2/N}$ (see, for example, [50, Theorem 1]). □

5.3 Shape quasi-minimizers for Dirichlet eigenvalues

In this section we discuss the regularity of the eigenfunctions on sets which are minimal with respect to a given (spectral) shape functional; in particular, we will show in Lemma 5.17 that the k-th eigenfunction of a set which is a shape quasi-minimizer for λ_k is Lipschitz as soon as λ_k is a simple eigenvalue for Ω. In what follows we denote by $\mathcal{S}$ the family of subset of $\mathbb{R}^N$ with finite Lebesgue measure endowed with the equivalence relation $\Omega \sim \widetilde{\Omega}$, whenever $|\Omega\Delta\widetilde{\Omega}| = 0$.

Definition 5.12. *We say that the measurable set $\Omega \in \mathcal{S}$ is a* shape quasi-minimizer *for the functional $\mathcal{F} : \mathcal{S} \to \mathbb{R}$ if there exists a constant $C > 0$ such that for every ball $B_r(x)$ and every set $\widetilde{\Omega} \in \mathcal{S}$ with $\Omega\Delta\widetilde{\Omega} \subseteq B_r(x)$ it is*

$$\mathcal{F}(\Omega) \leq \mathcal{F}(\widetilde{\Omega}) + C|B_r|\,.$$

Of course, whenever $\mathcal{F}$ is positive (as is almost always the case in the applications) then we can restrict ourselves in considering balls with radius r smaller than some given $r_0 > 0$.

Remark 5.13. *If the functional $\mathcal{F}$ is non-increasing with respect to inclusions, then Ω is a shape quasi-minimizer if and only if for every ball $B_r(x)$ it is*

$$\mathcal{F}(\Omega) \leq \mathcal{F}\big(\Omega \cup B_r(x)\big) + C|B_r|\,.$$

One may expect that the property of shape quasi-minimality contains some information on the regularity of Ω, or of the eigenfunctions on Ω. Let us quickly see what happens with a very simple example, namely, let us consider the Dirichlet Energy

$$E(\Omega) := \min\Big\{J_1(u) :\ u \in \tilde{H}^1_0(\Omega)\Big\}\,,$$

where the functional J_1 is intended in the sense of (5.6) with $f \equiv 1$, and let Ω be a shape quasi-minimizer for E. Then, calling w_Ω the energy function, it is clear that for any ball $B_r(x)$ and any $\widetilde{\Omega} \in \mathcal{S}$ such that $\widetilde{\Omega}\Delta\Omega \subseteq B_r(x)$ it is

$$J_1(w_\Omega) = E(\Omega) \leq E(\widetilde{\Omega}) + C|B_r| \leq J_1(w_\Omega + \varphi) + C|B_r| \qquad \forall\, \varphi \in H^1_0\big(B_r(x)\big)\,.$$

This means that w_Ω is a local quasi-minimizer for the functional J_1, according to Definition 5.7, and then Theorem 5.5 ensures that w_Ω is Lipschitz continuous on $\mathbb{R}^N$.

The case $\mathcal{F} = \lambda_k$ is more involved, since the k-th eigenvalue is not defined through a single state function, but is variationally characterized by a min-max procedure involving an entire linear subspace of $\tilde{H}^1_0(\Omega)$; therefore, we will need to transfer information from the minimality of Ω to the variation of the eigenvalues of Ω, then from this to the variation of the eigenfunctions, and finally from this to the regularity of Ω itself.

The main tool to prove Lemma 5.17 is the technical Lemma 5.14 below. There, we consider a generic set $\Omega \in \mathcal{S}$, we take $k \geq l \geq 1$ so that

$$\lambda_k(\Omega) = \cdots = \lambda_{k-l+1}(\Omega) > \lambda_{k-l}(\Omega)\,, \tag{5.32}$$

where by consistence we mean $\lambda_0(\Omega) = 0$, and we fix l normalized orthogonal eigenfunctions corresponding to eigenvalue $\lambda_k(\Omega)$, that we call $u_{k-l+1}, \ldots, u_k$. We will use the following notation: given a vector $\alpha = (\alpha_{k-l+1}, ..., \alpha_k) \in \mathbb{R}^l$, we denote by $\boldsymbol{u}_\alpha$ the corresponding linear combination

$$\boldsymbol{u}_\alpha := \alpha_{k-l+1}u_{k-l+1} + ... + \alpha_k u_k\,.$$

Lemma 5.14. *Let $\Omega \subseteq \mathbb{R}^N$ be a set of finite measure and $k \geq l \geq 1$ be such that* (5.32) *holds. For every $\eta > 0$ there is a constant $r_0 > 0$ such that, for every $x \in \mathbb{R}^N$, every $0 < r < r_0$, and every l-uple of functions $v_{k-l+1}, \ldots, v_k \in H^1_0(B_r(x))$, there is a unit vector $\alpha \in \mathbb{R}^l$ such that*

$$\lambda_k(\Omega \cup B_r(x)) \leq \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2} + \eta \int |\nabla \boldsymbol{v}_\alpha|^2\,. \tag{5.33}$$

The constant r_0 depends on Ω and in particular, if the gap $\lambda_{k-l+1}(\Omega) - \lambda_{k-l}(\Omega)$ vanishes, then so does r_0 as well.

Proof. For the sake of shortness, let us simply write B_r in place of $B_r(x)$, as well as λ_j in place of $\lambda_j(\Omega)$. By the definition of the k-th eigenvalue, we know that

$$\lambda_k(\Omega \cup B_r) \leq \max\left\{\frac{\int |\nabla u|^2}{\int u^2} : u \in \operatorname{span}\big\langle u_1, ..., u_{k-l}, u_{k-l+1} + v_{k-l+1}, ..., u_k + v_k\big\rangle\right\}\,, \tag{5.34}$$

and the maximum is attained for some linear combination

$$\alpha_1 u_1 + ... + \alpha_{k-l}u_{k-l} + \alpha_{k-l+1}(u_{k-l+1} + v_{k-l+1}) + ... + \alpha_k(u_k + v_k)\,.$$

One can immediately notice that the vector $\alpha = (\alpha_{k-l+1}, ..., \alpha_k) \in \mathbb{R}^l$ must be non-zero if $\lambda_{k-l}(\Omega) < \lambda_k(\Omega \cup B_r)$. And in turn, an immediate argument by contradiction shows that this is always the case if r_0 is small enough; we can then assume that α is a unitary vector. On the other hand, consider the vector $(\alpha_1, \ldots, \alpha_{k-l})$: if it is the null vector, then (5.33) comes directly from (5.34), hence we have nothing to prove. Otherwise, let us call

$$u := \frac{\alpha_1 u_1 + \ldots + \alpha_{k-l} u_{k-l}}{\sqrt{\alpha_1^2 + \ldots + \alpha_{k-l}^2}},$$

so that $\int u^2 = 1$, $\int |\nabla u|^2 \leq \lambda_{k-l}$, and from (5.34) we derive

$$\lambda_k(\Omega \cup B_r) \leq \max_{t\in\mathbb{R}} \left\{ \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu)|^2}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu|^2} \right\}. \tag{5.35}$$

We can now quickly evaluate, keeping in mind that u and $\boldsymbol{u}_\alpha$ are orthogonal,

$$\begin{aligned}\frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu)|^2}{\int (\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu)^2} &\leq \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2 + 2t\int \nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)\cdot \nabla u + t^2\lambda_{k-l}}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2 + 2t\int (\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)u + t^2} \\ &= \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2 + 2t\int_{B_r} \nabla \boldsymbol{v}_\alpha \cdot \nabla u + t^2\lambda_{k-l}}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2 + 2t\int_{B_r} \boldsymbol{v}_\alpha u + t^2} = \frac{A + 2ta + t^2\lambda_{k-l}}{B + 2tb + t^2},\end{aligned}$$

where by shortness we write

$$A = \int_{\mathbb{R}^N} |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2, \qquad a = \int_{B_r} \nabla \boldsymbol{v}_\alpha \cdot \nabla u$$
$$B = \int_{\mathbb{R}^N} |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2, \qquad b = \int_{B_r} \boldsymbol{v}_\alpha u.$$

Calling now for simplicity $D = \sqrt{\int |\nabla \boldsymbol{v}_\alpha|^2}$, and picking a small number $\delta = \delta(\eta) > 0$ to be chosen later, it is clear by the Hölder inequality and the embedding of $H_0^1(B_r)$ into $L^2(B_r)$ that, if r_0 is small enough, then

$$|a| \leq \|\nabla u\|_{L^2(B_r)} D \leq \delta D, \qquad |b| \leq \|\boldsymbol{v}_\alpha\|_{L^2(B_r)} \leq \delta D, \qquad B \geq \int_{\mathbb{R}^N \setminus B_r} \boldsymbol{u}_\alpha^2 \geq 1 - \delta. \tag{5.36}$$

On the other hand, we can estimate the quotient A/B as

$$\begin{aligned}\frac{A}{B} &= \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2\,dx}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2\,dx} = \frac{\lambda_k + 2\int \nabla \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{v}_\alpha\,dx + \int |\nabla \boldsymbol{v}_\alpha|^2\,dx}{1 + 2\int \boldsymbol{u}_\alpha \boldsymbol{v}_\alpha\,dx + \int \boldsymbol{v}_\alpha^2\,dx} \\ &\geq \frac{\lambda_k - 2D\left(\int_{B_r} |\nabla \boldsymbol{u}_\alpha|^2\,dx\right)^{1/2} + D^2}{1 + 2\left(\int_{B_r} \boldsymbol{u}_\alpha^2\,dx\right)^{1/2}\left(\int_{B_r} \boldsymbol{v}_\alpha^2\,dx\right)^{1/2} + \int \boldsymbol{v}_\alpha^2\,dx} \\ &\geq \frac{\lambda_k - 2D\left(\int_{B_{r_0}} |\nabla \boldsymbol{u}_\alpha|^2\,dx\right)^{1/2} + D^2}{1 + 2\left(\int_{B_r} \boldsymbol{v}_\alpha^2\,dx\right)^{1/2} + \int_{B_r} \boldsymbol{v}_\alpha^2\,dx} \geq \frac{\lambda_k - 2D\left(\int_{B_{r_0}} |\nabla \boldsymbol{u}_\alpha|^2\,dx\right)^{1/2} + D^2}{1 + 2C_N |B_{r_0}|^{1/d} D + C_N^2 |B_{r_0}|^{2/d} D^2} \\ &> \lambda_{k-l},\end{aligned} \tag{5.37}$$

where the last inequality is again true as soon as r_0 is small enough. Moreover, we also have

$$B + 2tb + t^2 \geq (1 - \delta)B \qquad \forall\, t \in \mathbb{R}. \tag{5.38}$$

Indeed, if $\int_{B_r} \boldsymbol{v}_\alpha^2 \leq 100$, then for r_0 small enough we have

$$|b| \leq \sqrt{\int_{B_r} u^2}\sqrt{\int_{B_r} \boldsymbol{v}_\alpha^2} \leq \delta\,,$$

thus $2tb + t^2 \geq -b^2 \geq -\delta^2 \geq -\delta B$ also by (5.36) and (5.38) holds. Instead, if $\int_{B_r} \boldsymbol{v}_\alpha^2 > 100$, then $b \leq \delta\sqrt{B}$ and thus again $2tb + t^2 \geq -b^2 \geq -\delta^2 B$ and (5.38) is again true.

We are finally in position to conclude. Indeed, if

$$|t| \leq \sqrt{\delta} D \qquad \text{and} \qquad D^2 \geq \lambda_k\,,$$

then $A \leq 3D^2$ and then, recalling (5.36) and (5.38), we have

$$\frac{A + 2ta + t^2\lambda_{k-l}}{B + 2tb + t^2} \leq \frac{A + 2\delta^{3/2}D^2 + \delta D^2\lambda_{k-l}}{B(1-\delta)} \leq \frac{A}{B} + \eta D^2$$

as soon as δ is small enough with respect to η. Keeping in mind (5.35), this estimate gives (5.33). Instead, if

$$|t| \leq \sqrt{\delta} D \qquad \text{and} \qquad D^2 \leq \lambda_k\,,$$

then

$$\frac{A + 2ta + t^2\lambda_{k-l}}{B + 2tb + t^2} \leq \frac{A + 2\delta^{3/2}D^2 + \delta D^2\lambda_{k-l}}{B - 2\delta^{3/2}D^2} \leq \frac{A}{B} + \eta D^2$$

and we again deduce (5.33). Finally, if

$$|t| \geq \sqrt{\delta} D\,,$$

then by (5.36) $|at| \leq \sqrt{\delta}t^2$ and $|bt| \leq \sqrt{\delta}t^2$, which in view of (5.37) if $\delta \ll 1$ gives

$$\frac{A + 2ta + t^2\lambda_{k-l}}{B + 2tb + t^2} \leq \frac{A + t^2\big(\lambda_{k-l} + 2\sqrt{\delta}\big)}{B + t^2\big(1 - 2\sqrt{\delta}\big)} \leq \frac{A}{B} \leq \frac{A}{B} + \eta D^2\,.$$

We have then deduced (5.33) in any case, and the proof is concluded. □

Remark 5.15. *The preceding lemma enlightens one of the main difficulties in the study of the regularity of spectral minimizers. Indeed, let Ω^* be a solution of a spectral optimization problem of the form* (2.8) *involving λ_k and such that* (5.32) *holds for some $l > 1$. Then every perturbation $\widetilde{u}_k = u_k + v$ of the eigenfunction $u_k \in \tilde{H}^1_0(\Omega^*)$ gives information on some linear combination $\boldsymbol{u}_\alpha$ of eigenfunctions $u_k, \dots, u_{k-l+1}$, and not simply on the function u_k. Recovering some information on u_k from estimates on $\boldsymbol{u}_\alpha$ is a difficult task, since the combination is a priori unknown, and anyway it depends on the perturbation v.*

Remark 5.16. *In case $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$, the estimate (5.33) reads as*

$$\lambda_k(\Omega \cup B_r(x)) \leq \frac{\int |\nabla(u_k + v)|^2\,dx}{\int |u_k + v|^2\,dx} + \eta \int |\nabla v|^2\,dx \tag{5.39}$$

for every ball $B_r(x)$ with $r < r_0$ and every $v \in H^1_0(B_r(x))$.

Lemma 5.17. *Let $\Omega \subseteq \mathbb{R}^N$ be a shape quasi-minimizer (with constant C) for λ_k such that $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$. Then every eigenfunction $u_k \in \tilde{H}^1_0(\Omega)$, normalized in L^2 and corresponding to the eigenvalue $\lambda_k(\Omega)$, is Lipschitz continuous on $\mathbb{R}^N$. Moreover, the Lipschitz constant depends only on $\lambda_k(\Omega)$, $|\Omega|$, N, and on the constant C, but not on u_k nor on Ω.*

Proof. Let u_k be a normalized eigenfunction corresponding to λ_k. Applying first the shape quasi-minimality of Ω and then the estimate (5.39) for $v \in H^1_0(B_r(x))$, with $r \le r_0 \le 1$ and $\int |\nabla v|^2 \le 1$, we obtain

$$\lambda_k(\Omega) \le \lambda_k\big(\Omega \cup B_r(x)\big) + C|B_r| \le \frac{\int |\nabla(u_k+v)|^2\,dx}{\int |u_k+v|^2\,dx} + \eta \int |\nabla v|^2\,dx + C|B_r|\,. \tag{5.40}$$

We now observe that, using Poincaré inequality and the hypotheses $r \le 1$, $\int |\nabla v|^2\,dx \le 1$, we have

$$\int |u_k+v|^2\,dx \le 2\int u_k^2\,dx + 2\int v^2\,dx \le 2 + \frac{2}{\lambda_1(B_r)}\int |\nabla v|^2\,dx \le 4.$$

Then we multiply both sides of (5.40) by $\int |u_k+v|^2\,dx$, so to get

$$-2\int \nabla u_k \cdot \nabla v + 2\lambda_k(\Omega)\int u_k v\,dx + \lambda_k(\Omega)\int v^2\,dx \le (4\eta+1)\int |\nabla v|^2\,dx + 4C|B_r|\,,$$

from which we deduce

$$\big|\langle \Delta u_k + \lambda_k(\Omega)u_k, v\rangle\big| = \left|-\int \nabla u_k \cdot \nabla v\,dx + \lambda_k(\Omega)\int u_k v\,dx\right| \le \frac{4\eta+1}{2}\int |\nabla v|^2\,dx + 2C|B_r|\,.$$

Hence the function u_k is a quasi-minimizer for the functional J_f, with $f = \lambda_k(\Omega)u_k$, thanks to (5.9) with $C_1 = \frac{4\eta+1}{2}$ and $C_2 = 2C$. Since u_k is bounded by (2.5), the claim follows directly by Theorem 5.11. □

It is important to underline something: if Ω is a minimizer of λ_k, then one expects the eigenvalue not to be simple; nevertheless, in the next sections we will be able to extract some information on optimal sets by using the above result.

5.4 Shape supersolutions of spectral functionals

In what follows we use the results of the preceding sections to derive the existence of Lipschitz eigenfunctions for sets which are shape supersolutions (see Definition 4.1) of suitable spectral functionals.

Let us recall immediately some obvious but useful observations about shape supersolutions that will be fundamental in our analysis.

Remark 5.18. • *If Ω^* is a shape supersolution for $\mathcal{F} + \Lambda|\cdot|$ and $\Lambda > 0$ then, for every $\Lambda' > \Lambda$, Ω^* is the unique solution of*

$$\min\Big\{\mathcal{F}(\Omega) + \Lambda'|\Omega| :\ \Omega \text{ Lebesgue measurable},\ \ \Omega \supseteq \Omega^*\Big\}.$$

Note that this point was already highlighted when we introduced the Sobolev-like spaces in Section 2.2. This property is the main reason for this choice of spaces instead of the capacitary ones.

- *If $\mathcal{F}$ is non-increasing with respect to the inclusion, then every shape supersolution of the functional $\mathcal{F} + \Lambda|\cdot|$, where $\Lambda > 0$, is also a shape quasi-minimizer for the functional $\mathcal{F}$, with constant $C = \Lambda$ in Definition 5.12 (this immediately follows by Remark 5.13).*

- *If Ω^* is a shape supersolution for the functional $\sum_{i=1}^m c_i\lambda_i + \Lambda|\cdot|$, then it is also a shape supersolution for the functional $\sum_{i=1}^m \tilde{c}_i\lambda_i + \widetilde{\Lambda}|\cdot|$ whenever $0 \leq \tilde{c}_i \leq c_i$ for every $1 \leq i \leq m$, and $\widetilde{\Lambda} \geq \Lambda \geq 0$ (this is immediate from the definition).*

- *If Ω^* minimizes λ_k among all the sets of given volume, then it is also a shape quasi-minimizer for the functional $\mathcal{F} = \lambda_k$, as well as a shape supersolution of $\lambda_k + \Lambda|\cdot|$ for some positive Λ (this follows just by rescaling).*

In Lemma 5.17 it was shown that the k-th eigenfunctions of the the shape quasi-minimizers for λ_k are Lipschitz continuous under the assumption $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$. In the next Theorem, we show that for shape supersolutions of $\lambda_k + \Lambda|\cdot|$ the later assumption can be dropped.

Theorem 5.19. *Let $\Omega^* \subseteq \mathbb{R}^N$ be a bounded shape supersolution for the functional $\lambda_k + \Lambda|\cdot|$, being $\Lambda > 0$. Then there is an eigenfunction $u_k \in \tilde{H}^1_0(\Omega^*)$, normalized in L^2 and corresponding to the eigenvalue $\lambda_k(\Omega^*)$, which is Lipschitz continuous on $\mathbb{R}^N$.*

Proof. The core of the proof of this theorem is the following claim.
Claim 5.A. *Let Ω^* be a bounded shape supersolution for $\lambda_j + \Lambda_j|\cdot|$, with some $\Lambda_j > 0$. Then, either there exists a Lipschitz eigenfunction u_j for $\lambda_j(\Omega^*)$, or $\lambda_j(\Omega^*) = \lambda_{j-1}(\Omega^*)$ and there exists some constant Λ_{j-1} such that Ω^* is also a shape supersolution for $\lambda_{j-1} + \Lambda_{j-1}|\cdot|$.*
We show now first that the claim implies the thesis, and then the validity of the claim.
<u>Step I.</u> *The claim implies the thesis.*
By hypothesis, we can apply the claim with $j = k$. If we find a Lipschitz eigenfunction u_k for $\lambda_k(\Omega^*)$ we are done; otherwise, we can apply the claim with $j = k - 1$. If we find a Lipschitz eigenfunction u_{k-1} for $\lambda_{k-1}(\Omega^*)$ we are again done, since if we are in this situation then $\lambda_{k-1}(\Omega^*) = \lambda_k(\Omega^*)$. Otherwise, we pass to $j = k - 2$ and so on, with a finite recursive argument (which surely concludes since at least for $j = 1$ the first alternative of the claim must hold true). Summarizing, there is always some $1 \leq \bar{j} \leq k$ such that a Lipschitz eigenfunction for $\lambda_{\bar{j}}(\Omega^*)$ exists, and by construction $\lambda_{\bar{j}}(\Omega^*) = \lambda_k(\Omega^*)$. Therefore, the thesis comes from the claim.
<u>Step II.</u> *The claim holds true.*
First of all, since λ_j is non-increasing with respect to the inclusion, then by Remark 5.18 we know that Ω^* is a shape quasi-minimizer for λ_j, with constant $C = \Lambda_j$ in Definition 5.12. As a

consequence, if $\lambda_j(\Omega^*) > \lambda_{j-1}(\Omega^*)$, then Lemma 5.17 already ensures the Lipschitz property for any normalized eigenfunction u_j corresponding to $\lambda_j(\Omega^*)$, and the claim is already proved.

Let us instead assume that $\lambda_j(\Omega^*) = \lambda_{j-1}(\Omega^*)$ and, for every $\varepsilon \in (0,1)$, consider the problem

$$\min\Big\{(1-\varepsilon)\lambda_j(\Omega) + \varepsilon\lambda_{j-1}(\Omega) + 2\Lambda_j|\Omega| :\ \Omega \supseteq \Omega^*\Big\}. \tag{5.41}$$

It is well-known that a minimizer Ω_ε of this problem exists, and it is clear that any such minimizer is a shape supersolution of the functional $\lambda_j + 2(1-\varepsilon)^{-1}\Lambda_j|\cdot|$.

Suppose then that, for some sequence $\varepsilon_n \to 0$, there is a corresponding sequence of solutions Ω_{ε_n} to (5.41) which satisfy $\lambda_j(\Omega_{\varepsilon_n}) > \lambda_{j-1}(\Omega_{\varepsilon_n})$. Again by Lemma 5.17, we deduce the existence of normalized eigenfunctions u_j^n for $\lambda_j(\Omega_{\varepsilon_n})$, which are Lipschitz with a constant depending only on N, $\lambda_j(\Omega_{\varepsilon_n})$, $|\Omega_{\varepsilon_n}|$, and on Λ_j. Since the sets Ω_{ε_n} are uniformly bounded (see for instance [22, Proposition 5.12]), a suitable subsequence γ-converges to some set $\widetilde{\Omega} \supseteq \Omega^*$, which is then a minimizer of the functional $\lambda_j + 2\Lambda_j|\cdot|$ among sets containing Ω^*, and thus in turn it is $\widetilde{\Omega} = \Omega^*$ by Remark 5.18. Still up to a subsequence, the functions u_j^n uniformly and weakly-H^1_0 converge to a function $u_j \in H^1_0(\Omega^*)$; moreover, since for every $v \in H^1_0(\Omega^*)$ we have

$$\int \nabla u_j \cdot \nabla v\, dx = \lim_{n\to\infty} \int \nabla u_j^n \cdot \nabla v\, dx = \lim_{n\to\infty} \lambda_j(\Omega_{\varepsilon_n}) \int u_j^n v\, dx = \lambda_j(\Omega^*) \int u_j v\, dx\,,$$

we deduce that u_j is a normalized Lipschitz eigenfunction for $\lambda_j(\Omega^*)$, and then the claim has been proved also in this case.

We are then left to consider the case when $\lambda_j(\Omega^*) = \lambda_{j-1}(\Omega^*)$ and, for some small $\bar\varepsilon > 0$, every solution $\Omega_{\bar\varepsilon}$ of (5.41) has $\lambda_j(\Omega_{\bar\varepsilon}) = \lambda_{j-1}(\Omega_{\bar\varepsilon})$. This implies that $\Omega_{\bar\varepsilon}$ minimizes also $\lambda_j(\Omega) + 2\Lambda_j|\Omega|$ for sets $\Omega \supseteq \Omega^*$, so that actually $\Omega_{\bar\varepsilon} = \Omega^*$ again by Remark 5.18. In other words, Ω^* itself is a solution of (5.41) for $\bar\varepsilon$. As an immediate consequence, Ω^* is a shape supersolution for the functional $\lambda_{j-1} + 2\Lambda_j\bar\varepsilon^{-1}|\cdot|$: indeed, for any $\Omega \supseteq \Omega^*$ one has

$$\bar\varepsilon\lambda_{j-1}(\Omega^*) + 2\Lambda_j|\Omega^*| \le \bar\varepsilon\lambda_{j-1}(\Omega^*) + (1-\bar\varepsilon)\big(\lambda_j(\Omega^*) - \lambda_j(\Omega)\big) + 2\Lambda_j|\Omega^*| \le \bar\varepsilon\lambda_{j-1}(\Omega) + 2\Lambda_j|\Omega|$$

by (5.41), and thus the claim has been proved also in this last case. □

A particular case of the above theorem concerns the optimal sets for the k-th eigenvalue.

Corollary 5.20. *Let Ω^* be a minimizer of the k-th eigenvalue among all the quasi-open sets of a given volume. Then, there exists an eigenfunction $u_k \in \widetilde{H}^1_0(\Omega^*)$, corresponding to the eigenvalue $\lambda_k(\Omega^*)$, which is Lipschitz continuous on $\mathbb{R}^N$.*

Proof. Since it is known that such a minimizer exists and is bounded (see [16, M4, M3]), and since we have already observed in Remark 5.18 that any such minimizer is also a shape quasi-minimizer for $\lambda_k + \Lambda|\cdot|$, the claim follows just by applying Theorem 5.19. □

It is important to observe that, if Ω^* is a minimizer of the k-th eigenvalue and the k-th eigenvalue of Ω^* is not simple (which actually seems always to be the case, unless when $k = 1$), then the above corollary only states the existence of a Lipschitz eigenfunction for $\lambda_k(\Omega^*)$, but not that the whole eigenspace of λ_k in $\widetilde{H}^1_0(\Omega^*)$ is done by Lipschitz functions.

Our next aim is to improve Theorem 5.19 by considering functionals depending on more than just a single eigenvalue, hence of the form $F\big(\lambda_{k_1}(\Omega),\dots,\lambda_{k_p}(\Omega)\big)$. To do so, we need the following preliminary result.

Lemma 5.21. *Let $\Omega^*\subseteq\mathbb{R}^N$ be a bounded shape supersolution for the functional*

$$\lambda_k+\lambda_{k+1}+\cdots+\lambda_{k+p}+\Lambda|\cdot|\,,$$

for some constant $\Lambda>0$. Then there are L^2-orthonormal eigenfunctions $u_k,\dots,u_{k+p}\in\tilde{H}^1_0(\Omega^*)$, corresponding to the eigenvalues $\lambda_k(\Omega^*),\dots,\lambda_{k+p}(\Omega^*)$, which are Lipschitz continuous on $\mathbb{R}^N$.

Proof. For any $k\le j\le k+p$, the set Ω^* is a shape supersolution for $\lambda_j+\Lambda|\cdot|$, thus also a shape quasi-minimizer for λ_j with constant Λ, by Remark 5.18; hence, if $\lambda_j(\Omega^*)>\lambda_{j-1}(\Omega^*)$, by Lemma 5.17 we already know that the whole eigenspace corresponding to $\lambda_j(\Omega^*)$ is done by Lipschitz functions, and then for every $j\le l\le k+p$ such that $\lambda_j(\Omega^*)=\lambda_l(\Omega^*)$ we have orthogonal eigenfunctions u_j, u_{j+1}, … , u_l corresponding to the eigenvalues $\lambda_j(\Omega^*)=\lambda_{j+1}(\Omega^*)=\cdots=\lambda_l(\Omega^*)$.

Since eigenfunctions corresponding to different eigenvalues are always orthogonal, the above observation concludes the proof of the lemma if $\lambda_k(\Omega^*)>\lambda_{k-1}(\Omega^*)$.

Otherwise, we can use an argument very similar to that of the proof of Theorem 5.19: for every $\varepsilon\in(0,1)$ we consider a solution Ω_ε of the problem

$$\min\Big\{\sum_{j=k+1}^{k+p}\lambda_j(\Omega)+(1-\varepsilon)\lambda_k(\Omega)+\varepsilon\lambda_{k-1}(\Omega)+2\Lambda|\Omega|:\ \Omega\supseteq\Omega^*\Big\}\,, \tag{5.42}$$

which is in turn also a shape supersolution for the functional

$$\sum_{j=k}^{k+p}\lambda_j+\frac{2\Lambda}{1-\varepsilon}\,|\cdot|\,,$$

again using Remark 5.18. If there is a sequence $\varepsilon_n\to0$ such that $\lambda_k(\Omega_{\varepsilon_n})>\lambda_{k-1}(\Omega_{\varepsilon_n})$, then we can apply the above argument to every set Ω_{ε_n} finding orthogonal eigenfunctions u^n_j for $k\le j\le k+p$ which are Lipschitz continuous, with constants not depending on ε. Then, exactly as in the proof of Theorem 5.19, one immediately obtains that Ω_{ε_n} γ-converges to Ω^*, and that weak-H^1_0 limits u_j of the functions u^n_j are the desired Lipschitz eigenfunctions.

We must now only face the case that, for some small $\bar\varepsilon$, every solution $\Omega_{\bar\varepsilon}$ of (5.42) satisfies $\lambda_k(\Omega_{\bar\varepsilon})=\lambda_{k-1}(\Omega_{\bar\varepsilon})$, and thus $\Omega_{\bar\varepsilon}$ actually coincides with Ω^*. Since this implies in particular that Ω^* is a shape supersolution for the functional

$$\lambda_{k-1}+\lambda_k+\lambda_{k+1}+\cdots+\lambda_{k+p}+\frac{2\Lambda}{\bar\varepsilon}\,|\cdot|\,,$$

then we are in the same situation as at the beginning, with k replaced by $k-1$. With a finite recursion argument (which surely has an end, because we conclude when $\lambda_k > \lambda_{k-1}$, which emptily holds when $k = 1$), we obtain the thesis. □

Before stating the main result of this section, we recall the following terminology from Chapter 4:

- given two points $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ in $\mathbb{R}^p$, we say that $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, p$;
- a function $F : \mathbb{R}^p \to \mathbb{R}$ is said to be *increasing* if $F(x) \geq F(y)$ whenever $x \geq y$;
- we say that $F : \mathbb{R}^p \to \mathbb{R}$ is *increasingly bi-Lipschitz* if F is increasing, Lipschitz, and there is a constant $c > 0$ such that

$$F(x) - F(y) \geq c|x - y| \qquad \forall\, x \geq y\,.$$

- an increasing and locally Lipschitz function $F : \mathbb{R}^p \to \mathbb{R}$ is said *locally increasingly bi-Lipschitz* if for every x there is a constant $c(x)$ and a neighborhood $U \subseteq \mathbb{R}^p$ of x such that, for every $y_1 \geq y_2$ in U, one has $F(y_1) - F(y_2) \geq c(x)|y_1 - y_2|$.

Theorem 5.22. *Let $F : \mathbb{R}^p \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function, and let $0 < k_1 < k_2 < \cdots < k_p \in \mathbb{N}$ and $\Lambda > 0$. Then for every bounded shape supersolution Ω^* of the functional*

$$\Omega \mapsto F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big) + \Lambda|\Omega|\,,$$

there exists a sequence of orthonormal eigenfunctions $u_{k_1}, \ldots, u_{k_p}$, corresponding to the eigenvalues $\lambda_{k_j}(\Omega^*)$, $j = 1, \ldots, p$, which are Lipschitz continuous on $\mathbb{R}^N$. Moreover,

- *if for some k_j we have $\lambda_{k_j}(\Omega^*) > \lambda_{k_j-1}(\Omega^*)$, then the full eigenspace corresponding to $\lambda_{k_j}(\Omega^*)$ consists only on Lipschitz functions;*
- *if $\lambda_{k_j}(\Omega^*) = \lambda_{k_{j-1}}(\Omega^*)$, then there exist at least $k_j - k_{j-1} + 1$ orthonormal Lipschitz eigenfunctions corresponding to $\lambda_{k_j}(\Omega^*)$.*

Proof. Since the eigenspaces corresponding to different eigenvalues are orthogonal, we can restrict ourselves to consider the case when $\lambda_{k_1}(\Omega^*) = \lambda_{k_p}(\Omega^*)$. Moreover, the local bi-Lipschitz property ensures that Ω^* is also shape supersolution of the functional

$$\sum_{j=1}^{p} \lambda_{k_j} + \Lambda'|\cdot|\,,$$

for a suitable positive constant Λ'. As a consequence, Ω^* is shape supersolution also for the functional

$$\left(\sum_{j=1}^{p-1} \frac{1}{k_{j+1} - k_j} \sum_{i=k_j}^{k_{j+1}-1} \lambda_i\right) + \lambda_{k_p} + \Lambda'|\cdot|\,,$$

and then finally, using again Remark 5.18, also for the functional

$$\sum_{j=k_1}^{k_p} \lambda_j + \Lambda''|\cdot|\,.$$

The claim then directly follows from Lemma 5.21. □

5.5 Optimal sets for functionals depending on the first k eigenvalues

In this last Section we will be able to show that, at least for some specific functionals, a minimizer is actually an open set, instead of a quasi-open set. The following results are, essentially, consequences of Theorem 5.22.

Theorem 5.23. *Let $F:\mathbb{R}^k\to\mathbb{R}$ be a locally increasingly bi-Lipschitz function. Then every solution Ω^* of the problem*

$$\min\Big\{F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big):\ \Omega\subseteq\mathbb{R}^N \text{ measurable},\ |\Omega|=1\Big\}\,, \tag{5.43}$$

is an essentially open set. Moreover, the eigenfunctions of the Dirichlet Laplacian on Ω^*, corresponding to the eigenvalues $\lambda_1(\Omega^*),\dots,\lambda_k(\Omega^*)$, are Lipschitz continuous on $\mathbb{R}^N$.

Proof. We first note that the existence of a solution of (5.43) follows by the results from [16] and [M4]. Then, we claim that every solution Ω^* is a shape supersolution of the functional

$$\Omega\mapsto F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big)+\Lambda|\Omega|\,, \tag{5.44}$$

for some suitably chosen $\Lambda>0$. Indeed, let us take a generic set $\Omega\supseteq\Omega^*$ and let us call $t:=(|\Omega|/|\Omega^*|)^{1/N}>1$; we can assume that t is as close to 1 as we wish, since otherwise the claim is emptily true, up to increase the constant Λ. Thus, calling L the Lipschitz constant of F in a neighborhood of $\big(\lambda_1(\Omega^*),\,\dots\,,\,\lambda_k(\Omega^*)\big)$, by the optimality of Ω^* we have

$$\begin{aligned}
F\big(\lambda_1(\Omega^*),\dots,\lambda_k(\Omega^*)\big) &\leq F\big(\lambda_1(\Omega/t),\dots,\lambda_k(\Omega/t)\big)=F\big(t^2\lambda_1(\Omega),\dots,t^2\lambda_k(\Omega)\big)\\
&\leq F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big)+L(t^2-1)\sum_{i=1}^k\lambda_i(\Omega)\\
&\leq F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big)+L(t^N-1)\sum_{i=1}^k\lambda_i(\Omega^*)\\
&= F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big)+\frac{L}{|\Omega^*|}\sum_{i=1}^k\lambda_i(\Omega^*)\Big(|\Omega|-|\Omega^*|\Big)\,.
\end{aligned}$$

Then, Ω^* is a shape supersolution for the functional (5.44), as claimed, and thus the Lipschitz continuity of an orthonormal set $\{u_1,\,\dots\,,\,u_k\}$ of eigenfunctions follows by Theorem 5.22.

The openness of the set Ω^* follows by the observation that the open set

$$\Omega^{**} := \bigcup_{i=1}^{k} \{u_k \neq 0\}$$

is essentially contained in Ω^* and has the same first k eigenvalues as Ω^*: indeed, these eigenvalues are smaller than those of Ω^* by the characterization (2.6) of the eigenvalues and thanks to the functions u_i, but also greater than those of Ω^* because Ω^{**} is essentially contained in Ω^*. By the optimality of Ω^* we deduce that $|\Omega^* \Delta \Omega^{**}| = 0$, i.e., Ω^{**} is equivalent to Ω^* and the proof is completed. □

Observe that, by the definition of the open set Ω^{**} in the above proof, it follows that the first k eigenvalues defined on the space $\tilde{H}^1_0(\Omega^{**})$, and those defined on the classical Sobolev space $H^1_0(\Omega^{**})$, coincide. Thus, we have a solution of the shape optimization problem (5.43) in its classical formulation.

Corollary 5.24. *Let $F : \mathbb{R}^k \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function. Then there is a solution Ω^* of the problem*

$$\min \Big\{ F\big(\lambda_1(\Omega), \dots, \lambda_k(\Omega)\big) : \ \Omega \subseteq \mathbb{R}^N \ \textit{open}, \ |\Omega| = 1 \Big\} .$$

Moreover, the eigenfunctions of the Dirichlet Laplacian on Ω^*, corresponding to the eigenvalues $\lambda_1(\Omega^*), \dots, \lambda_k(\Omega^*)$, are Lipschitz continuous on $\mathbb{R}^N$.

The openness can be obtained not only for sets minimizing (5.43), but also for shape supersolutions.

Proposition 5.25. *Let $F : \mathbb{R}^k \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function, and let Ω^* be a shape supersolution for the functional*

$$\Omega \mapsto F\big(\lambda_1(\Omega), \dots, \lambda_k(\Omega)\big) + \Lambda |\Omega| . \tag{5.45}$$

Then there is an open set $\Omega^{**} \subseteq \Omega^*$ such that $\lambda_i(\Omega^{**}) = \lambda_i(\Omega^*)$ for $i = 1, \dots, k$, and which is still a supersolution for the functional (5.45). *Moreover, there exists a sequence of Lipschitz orthonormal eigenfunctions corresponding to the first k eigenvalues in Ω^{**}.*

Proof. Applying Theorem 5.22 to Ω^*, we find an orthonormal set of Lipschitz eigenfunctions u_1, u_2, ... , u_k for Ω^*; then, as in Theorem 5.23, we define

$$\Omega^{**} := \bigcup_{i=1}^{k} \{u_i \neq 0\} ,$$

which is open since the functions u_i are Lipschitz continuous. As before, Ω^{**} is essentially contained in Ω^*, thus it has bigger eigenvalues, and on the other hand the definition of eigenvalues –together with the fact that each u_i is by definition in $H^1_0(\Omega^{**})$– gives the opposite inequality.

As a consequence, we conclude that $\lambda_i(\Omega^*) = \lambda_i(\Omega^{**})$ for every $i = 1, \dots, k$. It is now immediate to show that Ω^{**} is also a shape supersolution for (5.45): indeed, for every $\Omega \supseteq \Omega^{**}$, we just compute

$$\begin{aligned} F\big(\lambda_1(\Omega^{**}), \dots, \lambda_k(\Omega^{**})\big) + \Lambda|\Omega^{**}| &= F\big(\lambda_1(\Omega^*), \dots, \lambda_k(\Omega^*)\big) + \Lambda|\Omega^*| - \Lambda|\Omega^* \setminus \Omega^{**}| \\ &\leq F\big(\lambda_1(\Omega \cup \Omega^*), \dots, \lambda_k(\Omega \cup \Omega^*)\big) + \Lambda|\Omega \cup \Omega^*| - \Lambda|\Omega^* \setminus \Omega^{**}| \\ &\leq F\big(\lambda_1(\Omega), \dots, \lambda_k(\Omega)\big) + \Lambda|\Omega| \,. \end{aligned}$$

Being then Ω^{**} a shape supersolution for (5.45), and being the functions u_i also eigenfunctions for Ω^{**}, the proof is concluded. □

For functionals of the form

$$\Omega \mapsto F\big(\lambda_{k_1}(\Omega), \dots, \lambda_{k_p}(\Omega)\big)\,,$$

depending on some non-consecutive eigenvalues $\lambda_{k_1}, \dots, \lambda_{k_p}$, it is still possible to obtain that an optimal sets Ω^* for the problem

$$\min\Big\{F\big(\lambda_{k_1}(\Omega), \dots, \lambda_{k_p}(\Omega)\big)\ :\ \Omega \subseteq \mathbb{R}^N \text{ measurable, } |\Omega| = 1\Big\}\,, \tag{5.46}$$

is open, provided that an additional condition on the eigenvalues of Ω^* is satisfied.

Proposition 5.26. *Let $F : \mathbb{R}^p \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function, $0 < k_1 < k_2 < \cdots < k_p$ be natural numbers, and Ω^* be a minimizer for the problem (5.46). If for all $j = 1, \dots, p$ one has $\lambda_{k_j}(\Omega^*) > \lambda_{k_j - 1}(\Omega^*)$, then Ω^* is essentially open. Moreover all the eigenfunctions corresponding to $\lambda_{k_j}(\Omega^*)$, for $j = 1, \dots, p$ are Lipschitz continuous on $\mathbb{R}^N$.*

Proof. First of all, we remind that a minimizer for the problem (5.46) exists and is bounded, and moreover it is also a shape supersolution of the functional $F\big(\lambda_{k_1}(\Omega), \dots, \lambda_{k_p}(\Omega)\big)\big) + \Lambda|\Omega|$ for a suitable Λ, exactly as in the proof of Theorem 5.23; thus, the second part of the claim simply follows by Theorem 5.22, and it only remains to show that Ω^* is essentially open.

Let us fix an orthonormal set of eigenfunctions $\{u_i,\, 1 \leq i \leq k_p\}$ for the first k_p eigenvalues in Ω^*, and consider the family of indices

$$I := \Big\{i \leq k_p : \lambda_i(\Omega^*) = \lambda_{k_j}(\Omega^*), \text{ for some } j\Big\}\,.$$

Recalling again Theorem 5.22, we know that for every $i \in I$ the eigenfunction u_i is Lipschitz continuous, thus the set

$$\Omega_A := \left\{x \in \mathbb{R}^N\ :\ \sum_{i \in I} u_i(x)^2 > 0\right\}$$

is open, and of course essentially contained in Ω^*. Our aim is then to prove that $N = \Omega^* \setminus \Omega_A$ is negligible. Suppose, by contradiction, that $|N| > 0$ and let $x \in N$ be a point of density one for N, i.e.

$$\lim_{\rho \to 0} \frac{|N \cap B_\rho(x)|}{|B_\rho(x)|} = 1\,.$$

Since, for $\rho \to 0$, the sets $\Omega^* \setminus (N \cap B_\rho(x))$ γ-converge to Ω^* we have the convergence of the spectra $\lambda_k(\Omega^* \setminus (N \cap B_\rho(x))) \to \lambda_k(\Omega^*)$, for every $k \in \mathbb{N}$. Then, being $\lambda_{k_j}(\Omega^*) > \lambda_{k_j-1}(\Omega^*)$, we can choose ρ small enough such that the set $\widetilde{\Omega} = \Omega^* \setminus (N \cap B_\rho(x))$ satisfies

$$\lambda_{k_j-1}(\widetilde{\Omega}) < \lambda_{k_j}(\Omega^*), \qquad \forall\, j = 1, \ldots, p. \tag{5.47}$$

We note now that for $i \in I$ the eigenfunction u_i belongs to $\tilde{H}^1_0(\widetilde{\Omega})$, and since $\widetilde{\Omega} \subseteq \Omega^*$ we get that u_i satisfies the equation

$$-\Delta u_i = \lambda_{k_j}(\Omega^*) u_i, \qquad u_i \in \tilde{H}^1_0(\widetilde{\Omega}).$$

Thus, for each $i \in I$ the number $\lambda_i(\Omega^*)$ is also in the spectrum of the Dirichlet Laplacian on $\widetilde{\Omega}$. Combined with (5.47) and with the fact that $\widetilde{\Omega} \subseteq \Omega^*$, this gives

$$\lambda_k(\widetilde{\Omega}) = \lambda_k(\Omega^*), \qquad \forall\, k = 1, \ldots, k_p.$$

Since for every $\rho > 0$ we have $|N \cap B_\rho(x)| > 0$, it follows that $|\widetilde{\Omega}| < |\Omega^*| = 1$; by the strict monotonicity of F, if we rescale $\widetilde{\Omega}$ till volume 1 we get a competitor strictly better than Ω^* in (5.46), which is a contradiction with the optimality of Ω^*. □

Remark 5.27. *Unfortunately, Proposition 5.26 provides the openness of optimal sets only up to zero Lebesgue measure. Hence we have that* $\tilde{H}^1_0(\Omega^*) = \tilde{H}^1_0(\Omega_A)$, *but we do not know in general if* $H^1_0(\Omega^*) = H^1_0(\Omega_A)$; *thus, it is not clear whether an open "classical" solution exists, where by "classical" we refer to the case when the eigenvalues are considered in the standard* H^1_0 *spaces, and not in the modified* $\tilde{H}^1_0$ *spaces. Keep in mind that this problem did not occur with the situation of Theorem 5.23, as noticed right after its proof.*

Chapter 6

Connectedness of minimizers for convex combinations of low eigenvalues

6.1 Introduction

In this last Chapter of the Thesis, we deal with the problem of connectedness of minimizers for a convex combination of the first three eigenvalues of Dirichlet Laplacian. In particular, for $\alpha, \beta \in [0,1]$ and $\alpha+\beta \leq 1$, we consider

$$\inf\{\alpha\lambda_1(\Omega)+\beta\lambda_2(\Omega)+(1-\alpha-\beta)\lambda_3(\Omega)\ :\ \Omega \subset \mathbb{R}^N,\ \text{open},\ |\Omega| \leq 1\}, \tag{6.1}$$

and we aim to understand for which values of α, β all the minimizers for (6.1) are connected. When $\alpha = 1$ and $\beta = 0$, problem (6.1) reduces to the minimization of the first eigenvalue, and by the Faber-Krahn inequality (2.9), it is well known that the ball is the unique minimizer. Hence, in this case, the minimizer -the ball- is connected in every dimension. On the other hand, if we consider the case of $\alpha = 0$ and $\beta = 1$, which means that we are minimizing the second eigenvalue, the Krahn-Szegö inequality (2.10) asserts that two equal disjoint balls are the unique minimizer. In this case the disconnectedness of the optimal set is somehow intrinsic in the problem: it is easy to see that if we impose a connectedness constraint and consider,

$$\inf\left\{\lambda_2(\Omega)\ :\ \Omega \subset \mathbb{R}^N,\ \text{open, connected},\ |\Omega| \leq 1\right\},$$

then there are no minimizers and the infimum equals the value of λ_2 for two equal disjoint balls of half measure each. It is in fact sufficient to consider as a minimizing sequence the two equal balls connected with a very thin strip (of volume decreasing to zero).

Very little is known about the other cases: Wolf and Keller in [53] proved that in dimension $N = 2, 3$ any minimizer for $\lambda_3(\Omega)$ (which corresponds to $\alpha = \beta = 0$ in (6.1)) is connected, by showing that the ball has lower third eigenvalue than any disconnected set of the same measure. For $N \geq 4$ it is not known whether minimizers for λ_3 are connected; Wolf and Keller conjectured that three disjoint equal balls should be optimal.

The Chapter is organized as follow: after Section 6.2 in which we recall how eigenvalues of balls can be computed, in Section 6.3 we state the main results and propose a conjecture for the cases that we are not able to treat. In Section 6.4 we give the proof of the results for generic dimension, while Section 6.5 is devoted to the two dimensional case, in which more informations can be obtained.

6.2 Explicit computations of eigenvalues for balls

We recall here some explicit computations of the eigenvalues for balls in the two dimensional setting. Let $B_R \subset \mathbb{R}^2$ be the ball of radius R centered in the origin. We look for eigenvalues (and eigenfunctions) in polar coordinates (ρ, ϑ), that means we search a real number λ and a function u such that,

$$-\Delta u(\rho, \vartheta) = \lambda u(\rho, \vartheta) \qquad \text{and} \qquad u(\rho = R, \vartheta) = 0.$$

The first equation can be rewritten using the expression of Laplace operator in polar coordinates:

$$-u_{\rho\rho}(\rho, \vartheta) - \frac{1}{\rho} u_\rho(\rho, \vartheta) - \frac{1}{\rho^2} u_{\vartheta\vartheta}(\rho, \vartheta) = \lambda u(\rho, \vartheta).$$

Using the usual technique of separation of variables, we look for a solution of the form $u(\rho, \vartheta) = v(\rho)w(\vartheta)$, that leads to the equation:

$$-w(\vartheta)v''(\rho) - \frac{1}{\rho} w(\vartheta)v'(\rho) - \frac{1}{\rho^2} w''(\vartheta)v(\rho) = \lambda v(\rho)w(\vartheta).$$

Forgetting the zero solution, we can divide both the right and the left hand side by $v(\rho)w(\vartheta)$:

$$-\rho^2 \frac{v''(\rho)}{v(\rho)} - \rho \frac{v'(\rho)}{v(\rho)} - \rho^2 \lambda = \frac{w''(\vartheta)}{w(\vartheta)}.$$

Since the two sides of the equality depends on different variables, the only chance to achieve a solution is that they are both equals to a constant k:

$$w''(\vartheta) + kw(\theta) = 0 \qquad \text{with } w(\vartheta) = w(\vartheta + 2m\pi) \ \forall m \in \mathbb{Z} \tag{6.2}$$

$$v''(\rho) + \frac{1}{\rho} v'(\rho) + (\lambda - \frac{k}{\rho^2}) v(\rho) = 0 \qquad \text{with } v(R) = 0 \text{ and } v'(0) = 0 \tag{6.3}$$

The periodicity condition in equation (6.2) implies $k = m^2$ and $w(\vartheta) = a_m \cos(m\vartheta) + b_m \sin(m\vartheta)$ for $m \in \mathbb{N}$. Hence equation (6.3) for $k = m^2$ becomes a Bessel differential equation.

If we call J_m the m-th Bessel function and $j_{m,k}$ its k-th zero, it can be proved the following relations between Bessel functions and eigenvalues for Dirichlet Laplacian in the case of the disk:

$$\lambda_{0,k} = \frac{j_{0,k}}{R^2} \qquad k \geq 1$$

$$u_{0,k}(\rho,\vartheta) = \frac{1}{\sqrt{\pi}R|J_0(j'_{0,k})|} J_0(\frac{j'_{0,k}\rho}{R}) \qquad k \geq 1$$

$$\lambda_{m,k} = \frac{j_{m,k}}{R^2} \qquad m,k \geq 1 \text{ double eigenvalue}$$

$$u_{m,k} = \begin{cases} \frac{\sqrt{2}j'_{m,k}}{\sqrt{\pi}R\sqrt{(j'_{m,k})^2 - m^2|J_m(j'_{m,k})|}} J_m(\frac{j'_{m,k}\rho}{R}) \cos(m\vartheta) & m,k \geq 1 \\ \frac{\sqrt{2}j'_{m,k}}{\sqrt{\pi}R\sqrt{(j'_{m,k})^2 - m^2|J_m(j'_{m,k})|}} J_m(\frac{j'_{m,k}\rho}{R}) \sin(m\vartheta) & m,k \geq 1 \end{cases}$$

Then we reorder the $\{\lambda_{m,k}\}$ in a non increasing sequence and we rename them with the usual notation $\lambda_1 < \lambda_2 \leq \lambda_3...$
In the following table one can find an array with the first approximate (we will always limit to three decimal digits) values for zeros of Bessel functions $j_{m,k}$ numerically computed. For more values, one can refer to [1].

$m \setminus k$	1	2	3	4
0	2.405	5.520	8.654	11.791
1	3.832	7.016	10.173	13.324
2	5.136	8.417	11.620	14.796
3	6.380	9.761	13.015	16.223

Remark 6.1. *The case in dimension $N \geq 3$ is similar: the eigenvalues for the N-ball involve the Bessel functions $J_{N/2-1}, J_{N/2}$; for example*

$$\lambda_1(B_R) = \frac{j^2_{N/2-1,1}}{R^2}, \qquad \lambda_2(B_R) = \cdots = \lambda_{N+1}(B_R) = \frac{j^2_{N/2,1}}{R^2}$$

The values for the ball with unit measure B, which we will use many times for calculations in Chapter 6, are:

$$\lambda_1(B) = \omega_N^{2/N} j^2_{N/2-1,1}, \qquad \lambda_2(B) = \cdots = \lambda_{N+1}(B) = \omega_N^{2/N} j^2_{N/2,1}.$$

In particular, in $\mathbb{R}^2$ we have $\lambda_1(B) \approx 18.168$ and $\lambda_2(B) = \lambda_3(B) \approx 46.125$.

6.3 Statement of the main Theorems

Before stating the main results, we define some constants which will be useful in the following. Throughout this paper let α_N satisfy

$$\alpha_N = \frac{\lambda_2(B) - 2^{2/N}\lambda_1(B)}{\lambda_2(B) - \lambda_1(B)}$$

for $N = 2, 3, 4$, and be the infimum of the numbers that satisfies

$$\alpha\left[\left(\left(\frac{1-\alpha}{\alpha}\right)^{N/(N+2)}+1\right)^{2/N}-1\right]+(1-\alpha)\left[\left(\left(\frac{\alpha}{1-\alpha}\right)^{N/(N+2)}+1\right)^{2/N}-\frac{\lambda_2(B)}{\lambda_1(B)}\right]>0,$$

for $N \geq 5$. Let $\beta_2 = 2 - \frac{\lambda_2(B)}{2\lambda_1(B)}$, β_3 be the supremum of the numbers in the range $\left(\frac{1}{3}, \frac{2^{2/3}}{1+2^{2/3}}\right)$, satisfying

$$\beta\left[2^{2/5}\left(\frac{1-\beta}{\beta}\right)^{3/5}+1\right]^{2/3}+2^{2/3}(1-\beta)\left[2^{-2/5}\left(\frac{\beta}{1-\beta}\right)^{3/5}+1\right]^{2/3}-\frac{\lambda_2(B)}{\lambda_1(B)}>0,$$

and let $\beta_N = 0$ for $N \geq 4$. Finally let $\gamma_2 = \gamma_3 = 0$ and let γ_N for $N \geq 4$ be the infimum of the numbers satisfying

$$\gamma\left[\left(1+\left(\frac{\lambda_1(B)}{\lambda_2(B)}\right)^{N/2}\right)^{2/N}-1\right]+(1-\gamma)\left[3^{2/N}-\frac{\lambda_2(B)}{\lambda_1(B)}\right]>0.$$

The approximate values for $N = 2, 3, 4$ are:

N	α_N	β_N	γ_N
2	0.350	0.730	0
3	0.439	0.476	0
4	0.479	0	0.311

We are now in position to state the main Theorems, the first one in general dimension, with the second giving additional information about the two dimensional case.

Theorem 6.2. *Any minimizer of* (6.1) *is connected for each of the cases*

(i) $\alpha + \beta = 1$, $\alpha > 0$,

(ii) $\alpha_N < \alpha \leq 1$,

(iii) $0 < \beta < \beta_N(1 - \alpha)$,

(iv) $\beta = 0$, $\gamma_N < \alpha \leq 1$.

Remark 6.3. *For* $N = 2$*, Theorem 6.2 states that every minimizer for* (6.1) *is connected in each of the cases:*

(i) $\alpha + \beta = 1$, $\alpha > 0$,

(ii) $0.350 \approx \alpha_2 < \alpha \leq 1$,

(iii) $0 \leq \beta < \beta_2(1 - \alpha) \approx 0.730(1 - \alpha)$,

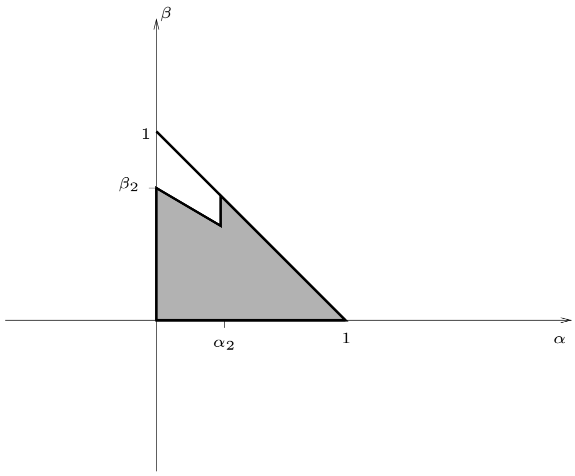

Figure 6.1: In the two dimensional case, the (α, β) for which we have all minimizers connected.

We represent in Figure 6.1 the two dimensional situation: the grey part of the $\alpha - \beta$ triangle denotes the cases in which all minimizers are connected: notice that the edge corresponding to $\beta = 1 - \alpha$ is completely covered, for $\alpha > 0$.

Theorem 6.4. *Let $N = 2$.*

(a) Any disconnected minimizer of (6.1) *satisfies $\lambda_1(\Omega) = \lambda_2(\Omega)$ and has exactly two components.*

(b) If any minimizer of (6.1) *is connected for $\alpha = 0$ and each $\beta \in [0, 1)$, then any minimizer is connected unless $\beta = 1$.*

From Remark 6.3 and Theorem 6.4 it is quite natural to make the following conjecture.

Conjecture 6.5. *Let $N = 2$; a minimizer for the problem* (6.1) *can not be disconnected unless $\beta = 1$.*

In a recent paper by Osting and Kao [41], there are numerical results that support Conjecture 6.5: the numerically computed optimal domain for problem (6.1) has one connected component unless $\alpha = 0$ and $\beta = 1$. Moreover the numerical computations also suggest that in the region $\{(\alpha, \beta) : \alpha + 2\beta \leq 1\}$ the optimal solution is a ball, while for all the other (α, β)-values (except of course $(\alpha, \beta) = (0, 1)$) the first four eigenvalues of the optimal domain are each simple. It is interesting to note that this last numerical result, together with Theorem 6.4 (a) supports Conjecture 6.5, too.

6.4 The general case

This section is completely devoted to the proof of Theorem 6.2.

Throughout all this Chapter Ω will often denote an optimal disconnected candidate for a minimizer. Since for disconnected sets the eigenvalues are obtained by collecting and reordering the eigenvalues of the components (see Remark 2.9), we give the following definition.

Definition 6.6. *We say that the k-th eigenfunction u_k is* supported on a component G of Ω *when $\lambda_k(\Omega) = \lambda_i(G)$, for some $i \leq k$. Moreover we write that G* supports l eigenvalues *if it has l eigenvalues less than or equal to the largest eigenvalue of Ω that we are minimizing.*

Note that a minimizer for (6.1) must have at least one of the first three eigenfunctions supported on each component.
An important step in order to prove Theorem 6.2 is the following lemma which rules out disjoint union of balls if $\beta < 1$, and its proof relies on a recent result by van den Berg (see [9]).

Lemma 6.7. *Let $N \geq 2$. The disjoint union of two balls can be optimal for (6.1) only if $\beta = 1$.*

Proof. The idea of the proof is that letting the two disjoint equal balls slightly overlap we obtain a better candidate for a minimizer of (6.1), because the increase in the second eigenvalue is less than the decrease in the first and the third. We divide the proof in two steps, treating first the case of two balls with equal measure, then the case of balls with different size.
Step I. Let $B(\varepsilon) = B(0,1) \cap \{x \;:\; x_1 < 1-\varepsilon\}$ and $\Omega(\varepsilon) = B(0,1) \cup B(2(1-\varepsilon)e_1, 1)$, where $x = (x_1, x_2, \ldots, x_N)$ and e_1 is the unit vector in the x_1 direction. Moreover $\tilde{\Omega}(\varepsilon) = |\Omega(\varepsilon)|^{-\frac{1}{N}}\Omega(\varepsilon)$ is the set rescaled to unit measure. It follows from Theorem 1 in [9] that

$$\lambda_1(B(\varepsilon))|B(\varepsilon)|^{2/N} = \lambda_1(B)|B|^{2/N} + o\left(\varepsilon^{(N+1)/2}\right). \tag{6.4}$$

Since eigenvalues of Dirichlet Laplacian are monotone with respect to set inclusion, we have $\lambda_1(\Omega(\varepsilon)) < \lambda_1(B)$ and $\lambda_3(\Omega(\varepsilon)) < \lambda_2(B)$. Thus taking scaling into account gives $\lambda_1(\tilde{\Omega}(\varepsilon)) < \lambda_1(\Theta) - c_1\varepsilon^{(N+1)/2}$ and $\lambda_3(\tilde{\Omega}(\varepsilon)) < \lambda_3(\Theta) - c_2\varepsilon^{(N+1)/2}$, for some positive constants c_1, c_2, reminding that $\lambda_2(B) = \lambda_3(B)$, $\lambda_1(\Theta) = 2^{2/N}\lambda_1(B)$ and $\lambda_3(\Theta) = 2^{2/N}\lambda_2(B)$. By the min-max principle (2.4) we can obtain an upper bound for $\lambda_2(\Omega(\varepsilon))$ by choosing the subspace $E_2 \subset H^1_0(\Omega)$ spanned by the first eigenfunction of $B(\varepsilon)$ and the first eigenfunction of $\Omega(\varepsilon) \cap \{x \mid x_1 > 1-\varepsilon\}$. Hence $\lambda_2(\Omega(\varepsilon)) \leq \lambda_1(B(\varepsilon))$, so we can apply (6.4) and use the scaling (see property (2) of Lemma 2.7) to obtain $\lambda_2(\tilde{\Omega}(\varepsilon)) \leq \lambda_2(\Theta) + o(\varepsilon^{(N+1)/2})$.

For $\beta < 1$ and for sufficiently small $\varepsilon > 0$, this gives

$$\alpha\lambda_1(\tilde{\Omega}(\varepsilon)) + \beta\lambda_2(\tilde{\Omega}(\varepsilon)) + (1-\alpha-\beta)\lambda_3(\tilde{\Omega}(\varepsilon)) < \alpha\lambda_1(\Theta) + \beta\lambda_2(\Theta) + (1-\alpha-\beta)\lambda_3(\Theta).$$

Step II. Let $r_1 > r_2$ and Ω be the disjoint union of two balls with radii r_1, r_2 such that the first two eigenfunctions are supported on different components. We write $\tilde{\Omega} = |\Omega|^{-\frac{1}{N}}\Omega$ for the set rescaled to unit measure. Then we define

$$B_{r_1} = B(0, r_1), \quad B_{r_2} = B\left(\left(r_1 + r_2 - \frac{\varepsilon}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right)e_1, r_2\right), \quad \Omega(\varepsilon) = B_{r_1} \cup B_{r_2},$$
$$B_1(\varepsilon) = B_{r_1} \cap \left\{x \;:\; x_1 < r_1 - \frac{\varepsilon}{2r_1}\right\}, \quad B_2(\varepsilon) = B_{r_2} \cap \left\{x \;:\; x_1 > r_2 - \frac{\varepsilon}{2r_2}\right\}.$$

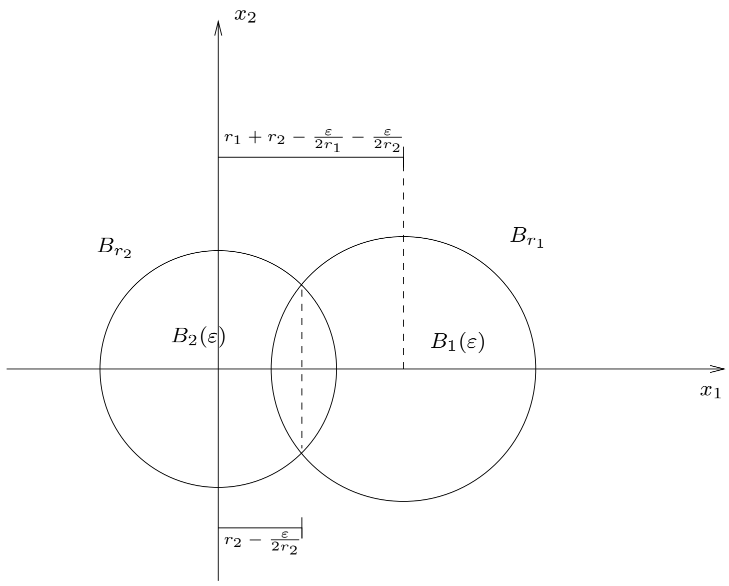

Figure 6.2: The sets B_{r_1}, B_{r_2}, $B_1(\varepsilon)$ and $B_2(\varepsilon)$.

Figure 6.2 represents a possible configuration of the sets above.

By monotonicity of Dirichlet eigenvalues with respect to set inclusion, we have $\lambda_1(\Omega(\varepsilon)) < \lambda_1(B_{r_1})$ and $\lambda_3(\Omega(\varepsilon)) < \lambda_2(B_{r_1})$, and again taking account of the scaling we have $\lambda_1(\tilde{\Omega}(\varepsilon)) < \lambda_1(\tilde{\Omega}) - c_1\varepsilon^{(N+1)/2}$ and $\lambda_3(\tilde{\Omega}(\varepsilon)) < \lambda_3(\tilde{\Omega}) - c_2\varepsilon^{(N+1)/2}$, for some positive constants c_1, c_2.

By the min-max principle (2.4), we can obtain an upper bound for $\lambda_2(\Omega(\varepsilon))$ by choosing $E_2 \subset H_0^1(\Omega)$ spanned by the first eigenfunction of B_{r_2} and the first eigenfunction of $\Omega(\varepsilon) \setminus B_{r_2}$, and for ε small enough we have $\lambda_1(B_{r_2}) \geq \lambda_1(\Omega(\varepsilon) \setminus B_{r_2})$. Hence $\lambda_2(\Omega(\varepsilon)) \leq \lambda_1(B_{r_2})$, and taking account of the scaling $\lambda_2(\tilde{\Omega}(\varepsilon)) \leq \lambda_2(\tilde{\Omega}) - c_3\varepsilon^{(N+1)/2}$, for some positive c_3. In conclusion, for $\beta < 1$ and ε small enough, $\tilde{\Omega}(\varepsilon)$ is a better candidate than $\tilde{\Omega}$ for problem (6.1). $\square$

This last remark will be useful in the following and in Section 6.5.

Remark 6.8. *Let $N = 2, 3$. A disconnected set Ω can never be optimal for (6.1) if $\lambda_2(\Omega) \geq \lambda_2(B)$. Here the ball is better, since $\lambda_1(B) < \lambda_1(\Omega)$ by the Faber-Krahn inequality and $\lambda_3(B) < \lambda_3(\Omega)$ by [38, Corollary 5.2.2].*

Before starting the proof of Theorem 6.2 we note that if a connected component of the optimal disconnected set supports only one of the first three eigenfunctions, then by the Faber-Krahn inequality it must be a ball of the same measure.

Proof of Theorem 6.2 (i). We deal with the case $\alpha + \beta = 1$, that is, we consider the functional $\alpha\lambda_1(\cdot) + (1-\alpha)\lambda_2(\cdot)$. Note that this result for $\mathbb{R}^2$ is also discussed in [52, Chapter 2], but the details of a proof are not given. A disconnected minimizer Ω must by the Faber-Krahn inequality be the union of two disjoint balls, since we are considering only the first two eigenvalues. Hence an immediate application of Lemma 6.7 rules out this configuration in any dimension when $\alpha > 0$, so we conclude. □

Remark 6.9. *It is actually possible to give a different proof of the connectedness of the minimizers for the problem*

$$\inf\left\{\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) \ : \ \Omega \subseteq \mathbb{R}^N, \ \text{open, with } |\Omega| \le 1\right\}, \tag{6.5}$$

for $\alpha > 0$, that does not involve the use of Theorem 1 *from [9]. It is easy to see that the optimal disconnected set for problem (6.5) if $\alpha \in (1/2, 1]$ is made by two disjoint ball of different measure and hence can never be optimal by Step II of Lemma 6.7. On the other hand when $\alpha \in (0, 1/2]$ the best disconnected set is the disjoint union of two equal balls with half measure, Θ. We give here a different proof of the fact that Θ can not be optimal for (6.5) unless $\alpha = 0$. It is clearly sufficient to focus on the case $\alpha \in (0, 1/2]$.*

We introduce the set

$$\mathcal{E} = \left\{(\lambda_1(\Omega), \lambda_2(\Omega)) \ : \ \Omega \subseteq \mathbb{R}^N \ \text{open, with } |\Omega| = 1\right\}.$$

For a description of many properties of this set and a numerical approximation of it we refer to [21] or to [38, Chapter 6.4]. The property which interests us deals with the lower part of the boundary of $\mathcal{E}$, the curve $\mathcal{C}$ that joins the point $A = (\lambda_1(\Theta), \lambda_2(\Theta))$ and $B = (\lambda_1(B), \lambda_2(B))$ (see Figure 6.3).

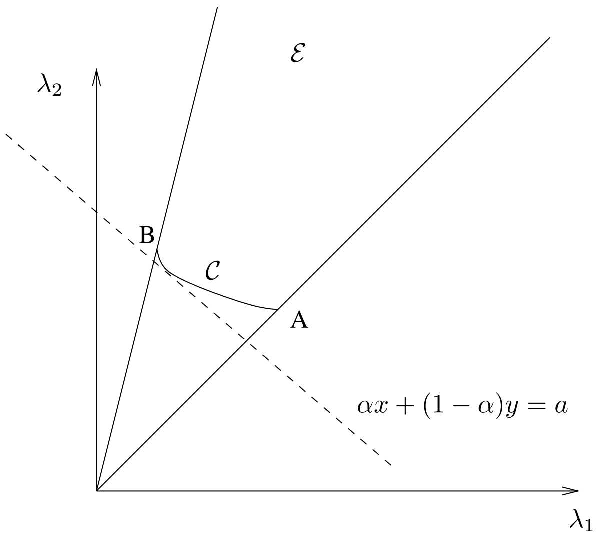

Figure 6.3: The set $\mathcal{E}$.

Wolf and Keller in [53] proved that the curve $\mathcal{C}$ must be vertical at the point B by a perturbation argument with nearly circular domains. They also suggested that $\mathcal{C}$ should be horizontal at A, and this was proved recently by Brasco, Nitsch and Pratelli in [12]. This is the crucial point of our proof, as a minimizer for the convex combination $\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega)$ is given *by the set corresponding to the first point in which the straight line $\alpha x + (1-\alpha)y = a$ touches $\mathcal{E}$, by increasing a. In particular, for $\alpha = 0$ this line is $y = \lambda_2(\Theta) = 2\lambda_1(B)$ by the Krahn-Szegö inequality. On the other hand, for all $\alpha \in (0, 1/2)$, it is possible to find a set $\widetilde{\Omega}$ that is linked to a line of the form $\alpha x + (1-\alpha)y = a_\alpha$, with $a_\alpha < \lambda_2(\Theta) = 2\lambda_1(B)$, since the curve $\mathcal{C}$ has horizontal tangent. Hence Θ can not be the minimizer for (6.5) unless $\alpha = 0$.*

Proof of Theorem 6.2 (ii). We need a different argument from the one used in the proof of part (i), but start again from the case $\alpha + \beta = 1$. The case $\alpha = 1$ was already solved by the Faber-Krahn inequality (2.9). Again a disconnected minimizer Ω must be the union of two disjoint balls. Without loss of generality, we suppose that the ball supporting u_1 has measure $m \le 1$, while the other one (supporting u_2) has measure $1 - m$. Having in mind the scaling of eigenvalues, it is clear that $m \ge 1/2$ and we have

$$\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) = \lambda_1(B)\left(\frac{\alpha}{m^{2/N}} + \frac{1-\alpha}{(1-m)^{2/N}}\right).$$

First of all we consider the case when $\alpha \in (\frac{1}{2}, 1)$. By minimizing in m in the previous expression to obtain a lower bound and comparing with the value for the unit ball rules out this configuration if

$$\alpha\lambda_1(B)\left[\left(\frac{1-\alpha}{\alpha}\right)^{N/(N+2)} + 1\right]^{2/N} + (1-\alpha)\lambda_1(B)\left[\left(\frac{\alpha}{1-\alpha}\right)^{N/(N+2)} + 1\right]^{2/N} > \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B), \tag{6.6}$$

that is, when the function $f_N \colon (0,1) \to \mathbb{R}$,

$$f_N(\alpha) = \alpha\left\{\left[\left(\frac{1-\alpha}{\alpha}\right)^{\frac{N}{N+2}} + 1\right]^{\frac{2}{N}} - 1\right\} + (1-\alpha)\left\{\left[\left(\frac{\alpha}{1-\alpha}\right)^{\frac{N}{N+2}} + 1\right]^{\frac{2}{N}} - \frac{\lambda_2(B)}{\lambda_1(B)}\right\}$$

is positive. The following property of $f_N(\alpha)$ is important for our analysis.

Claim 6.A.

For every $N \in \mathbb{N}$, there exists $\alpha_N \in (0,1)$ such that

$$f_N(\alpha) < 0 \quad \text{if } \alpha \in (0, \alpha_N) \qquad \text{and} \qquad f_N(\alpha) > 0 \quad \text{if } \alpha \in (\alpha_N, 1).$$

Proof of Claim 6.A. We introduce the increasing function $\psi \colon [0,1) \to [0,\infty)$:

$$\psi(\alpha) = \frac{\alpha}{1-\alpha},$$

and $\phi_N \colon (0,\infty) \to \mathbb{R}$

$$\phi_N(t) = t\left\{\left[\left(\frac{1}{t}\right)^{\frac{N}{N+2}} + 1\right]^{\frac{2}{N}} - 1\right\} + \left\{\left[t^{\frac{N}{N+2}} + 1\right]^{\frac{2}{N}} - \frac{\lambda_2(B)}{\lambda_1(B)}\right\},$$

so that $f_N(\alpha) = (1-\alpha)\phi_N(\psi(\alpha))$, for $\alpha \in (0,1)$. Hence the sign of f_N in the interval $(0,1)$ is the same as that of ϕ_N in the interval $(0,\infty)$. We note that ϕ_N is the sum of two functions, where the second one,

$$t \mapsto \left(t^{\frac{N}{N+2}} + 1\right)^{\frac{2}{N}} - \frac{\lambda_2(B)}{\lambda_1(B)},$$

is strictly increasing. The first one,

$$t \mapsto t\left\{\left[\left(\frac{1}{t}\right)^{\frac{N}{N+2}} + 1\right]^{\frac{2}{N}} - 1\right\} =: \phi_N^1(t),$$

is also strictly increasing, since we have:

$$\frac{d}{dt}\phi_N^1(t) = \left\{\left[\left(\frac{1}{t}\right)^{\frac{N}{N+2}} + 1\right]^{\frac{2}{N}} - 1\right\} - \frac{2}{N+2}t^{-\frac{N}{N+2}}\left[\left(\frac{1}{t}\right)^{\frac{N}{N+2}} + 1\right]^{\frac{2}{N}-1},$$

that is positive when

$$1 + \frac{N}{N+2}t^{-\frac{N}{N+2}} > \left[\left(\frac{1}{t}\right)^{\frac{N}{N+2}} + 1\right]^{1-\frac{2}{N}}.$$

The above inequality holds, since $t \mapsto t^{1-2/N}$ is concave and thus

$$\left[\left(\frac{1}{t}\right)^{\frac{N}{N+2}} + 1\right]^{1-\frac{2}{N}} < 1 + \frac{N-2}{N}\left(\frac{1}{t}\right)^{\frac{N}{N+2}} < 1 + \frac{N}{N+2}\left(\frac{1}{t}\right)^{\frac{N}{N+2}},$$

where we used also the inequality $(N-2)/N < N/(N+2)$. So ϕ_N is the sum of two strictly increasing functions and thus it has the same property. Hence ϕ_N can change sign only once in $(0,\infty)$. We note that

$$\lim_{t\to 0^+}\phi_N(t) = 1 - \frac{\lambda_2(B)}{\lambda_1(B)} < 0 \quad \text{and} \quad \lim_{t\to\infty}\phi_N(t) = +\infty,$$

so there exists an unique $t_N \in (0,\infty)$ such that

$$\phi_N(t) < 0 \quad \text{if } t \in (0, t_N) \quad \text{and} \quad \phi_N(t) > 0 \quad \text{if } t \in (t_N, \infty).$$

In conclusion the thesis follows setting $\alpha_N = \psi^{-1}(t_N)$. □

Keep in mind that, for all $N \geq 2$, $f_N(\alpha) > 0$ if and only if (6.6) holds for α. We can compute

$$f_2\left(\frac{1}{2}\right) = \frac{1}{2}\Big((1+1) - 1\Big) + \frac{1}{2}\left[(1+1) - \frac{\lambda_2(B)}{\lambda_1(B)}\right] \approx 0.230 > 0,$$
$$f_3\left(\frac{1}{2}\right) = \frac{1}{2}\left((1+1)^{2/3} - 1\right) + \frac{1}{2}\left[(1+1)^{2/3} - \frac{\lambda_2(B)}{\lambda_1(B)}\right] \approx 0.065 > 0,$$
$$f_4\left(\frac{1}{2}\right) = \frac{1}{2}\left((1+1)^{1/2} - 1\right) + \frac{1}{2}\left[(1+1)^{1/2} - \frac{\lambda_2(B)}{\lambda_1(B)}\right] \approx 0.016 > 0,$$

where we used that

$$\frac{\lambda_2(B)}{\lambda_1(B)} \approx \begin{cases} 2.539 & \text{if } N = 2, \\ 2.044 & \text{if } N = 3, \\ 1.796 & \text{if } N = 4. \end{cases}$$

Then, by Claim 6.A, for $N \leq 4$ we have that $\alpha_N < 1/2$, since $f_N(1/2) > 0$. Hence $f_N(\alpha) > 0$ for all $\alpha \in (1/2, 1)$ and thus (6.6) holds in that range.
When $N \geq 5$, we define $\widetilde{\alpha}_N = \inf\{\alpha \in (1/2, 1) \,:\, (6.6) \text{ holds}\}$ and Claim 6.A assures that (6.6) is true for all $\alpha \in (\widetilde{\alpha}_N, 1)$.
Now we deal with the case when $\alpha \in (0, \frac{1}{2}]$. The constraint $m \geq 1/2$ implies that the optimal disconnected configuration consists in two disjoint balls of equal measure, since in this case we have

$$\lambda_1(B)2^{2/n} = \alpha\lambda_1(\Theta) + (1-\alpha)\lambda_2(\Theta) < \lambda_1(B)\left(\frac{\alpha}{m^{2/n}} + \frac{1-\alpha}{(1-m)^{2/n}}\right).$$

This is ruled out by comparison with the unit ball when

$$\lambda_1(B)2^{2/N} > \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B), \tag{6.7}$$

and we call, for $N \leq 4$, $\widehat{\alpha}_N = \frac{\lambda_2(B) - 2^{\frac{2}{N}}\lambda_1(B)}{\lambda_2(B) - \lambda_1(B)}$, so that if $\alpha \in (\widehat{\alpha}_N, 1/2]$, then (6.7) holds. On the other hand, if $N \geq 5$, (6.7) is never true.
In conclusion, putting together the discussion for $\alpha \in (0, 1/2]$ and for $\alpha \in (1/2, 1)$, we have that

(a) For $N \leq 4$ the ball is a better candidate for problem (6.1) than any disconnected minimizer if $\alpha \in (\widehat{\alpha}_N, 1/2] \cup (1/2, 1)$.

(b) For $N \geq 5$ the ball is a better candidate for problem (6.1) than any disconnected minimizer if $\alpha \in (\widetilde{\alpha}_N, 1)$.

Thus we define $\alpha_N = \widetilde{\alpha}_N$ if $N \geq 5$ and $\alpha_N = \widehat{\alpha}_N$ if $N \leq 4$; hence we have that the ball is better than any disconnected set for $1 \geq \alpha > \alpha_N$.

Finally to extend beyond the situation $\alpha + \beta = 1$, just note that for $1 \geq \alpha > \alpha_N$

$$\begin{aligned} \alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) &\geq \alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) \\ &> \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B) = \alpha\lambda_1(B) + \beta\lambda_2(B) + (1-\alpha-\beta)\lambda_3(B), \end{aligned}$$

using the fact that $\lambda_2(\Omega) \leq \lambda_3(\Omega)$ while $\lambda_2(B) = \lambda_3(B)$, so we conclude. □

Proof of Theorem 6.2 (iii). The case $\alpha = 0$. We first consider the case $\alpha = 0$, that is, we deal with the functional $\beta\lambda_2(\cdot) + (1-\beta)\lambda_3(\cdot)$. First of all we find out the best disconnected configuration.

Claim 6.B.

Let $\alpha = 0$. A disconnected minimizer is made by a ball supporting u_2 and another set supporting u_1 and u_3.

Proof of Claim 6.B. At first, we consider the configuration made by a set supporting u_1, u_2 and a ball supporting u_3. Since λ_1 does not appear in the functional, it is better to have three balls by applying Krahn-Szegö inequality (2.10) to the set supporting u_1 and u_2. Note that the new set, made by three disjoint balls, is as in the statement of the claim.
On the other hand an optimal configuration made by a ball supporting u_1 and by another set supporting u_2, u_3 should satisfy $\lambda_1(\Omega) = \lambda_2(\Omega)$, since λ_1 is not involved in the minimization. Up to switch λ_1 and λ_2, we are in a configuration made by a ball supporting u_2 and another set supporting u_1, u_3 so the claim is proved. □

Thanks to Claim 6.B, it remains only to rule out a disconnected minimizer made by a ball supporting u_2, which we suppose to have measure m (hence $\lambda_2(\Omega) = \frac{\lambda_1(B)}{m^{2/n}}$) and a set supporting u_1 and u_3 (which must have measure $1-m$). Unfortunately, we are able to find out informations only when $n = 2, 3$. First of all we note that $m \leq 1/2$, otherwise, by the Faber-Krahn inequality, $\lambda_1(\Omega) > \lambda_2(\Omega)$. Moreover, using Remark 6.8, it must happen that $\lambda_2(B) > \lambda_2(\Omega)$, which implies

$$\frac{\lambda_1(B)}{m^{2/N}} < \lambda_2(B), \qquad \text{that is,} \qquad m > \left(\frac{\lambda_1(B)}{\lambda_2(B)}\right)^{N/2},$$

that assures $m \geq 1/3$ for $N = 2, 3$. For $\beta \in [0, 1)$, the Krahn-Szegö inequality gives the lower bound

$$\beta\lambda_2(\Omega) + (1-\beta)\lambda_3(\Omega) \geq \lambda_1(B)\left(\frac{\beta}{m^{2/N}} + \frac{2^{2/N}(1-\beta)}{(1-m)^{2/N}}\right). \tag{6.8}$$

We are interested in minimizing the left hand side of (6.8) with respect to m, in order to improve the lower bound. We define

$$m(\beta) = \left[2^{2/(N+2)}\left(\frac{1-\beta}{\beta}\right)^{N/(N+2)} + 1\right]^{-1},$$

and note that the right hand side of (6.8) is decreasing in $(0, m(\beta))$ and increasing in $(m(\beta), \infty)$. Moreover, we have that

$$m(\beta) \leq \frac{1}{3} \quad \text{if } \beta \in \left[0, \frac{1}{3}\right] \quad \text{and} \quad m(\beta) \geq \frac{1}{2} \quad \text{if } \beta \in \left[\frac{2^{\frac{2}{N}}}{1+2^{\frac{2}{N}}}, 1\right),$$

so that (from the constraints on m) in this two ranges the right hand side of (6.8) is minimal for $m = 1/3$ and $m = 1/2$ respectively. On the other hand, when $\beta \in \left(\frac{1}{3}, \frac{2^{\frac{2}{N}}}{1+2^{\frac{2}{N}}}\right)$, the right hand side of (6.8) is minimal for $m = m(\beta)$.
We are now in position to compare the lower bound for $\beta\lambda_2(\Omega) + (1-\beta)\lambda_3(\Omega)$ with $\beta\lambda_2(B) + (1-\beta)\lambda_3(B) = \lambda_2(B)$.
For $\beta \in [0, 1/3]$ we have

$$\lambda_2(B) < 3^{\frac{2}{N}}\lambda_1(B) \leq \beta\lambda_2(\Omega) + (1-\beta)\lambda_3(\Omega),$$

hence in this range, for $N = 2, 3$, the ball is a better candidate than any disconnected set for problem (6.1).

Now we deal with the case when $\beta \in \left(\frac{1}{3}, \frac{2^{2/N}}{1+2^{2/N}}\right)$. Substituting the optimal values for m in (6.8) and comparing with $\beta\lambda_2(B) + (1-\beta)\lambda_3(B) = \lambda_2(B)$ gives connectedness for the whole range when $N = 2$, since

$$\lambda_2(B) < \lambda_1(B)\left[\beta\left(\sqrt{\frac{2(1-\beta)}{\beta}}+1\right) + 2(1-\beta)\left(\sqrt{\frac{\beta}{2(1-\beta)}}+1\right)\right] \qquad \forall \beta \in \left(\frac{1}{3}, \frac{2}{3}\right).$$

On the other hand, for $n = 3$, we have connectedness when

$$\lambda_2(B) < \lambda_1(B)\left\{\beta\left[2^{2/5}\left(\frac{1-\beta}{\beta}\right)^{3/5}+1\right]^{2/3} + 2^{2/3}(1-\beta)\left[2^{-2/5}\left(\frac{\beta}{1-\beta}\right)^{3/5}+1\right]^{2/3}\right\}. \tag{6.9}$$

In order to study this situation, we start by considering the function $g\colon (0,1) \to \mathbb{R}$

$$g(\beta) = \beta\left[2^{2/5}\left(\frac{1-\beta}{\beta}\right)^{3/5}+1\right]^{2/3} + 2^{2/3}(1-\beta)\left[2^{-2/5}\left(\frac{\beta}{1-\beta}\right)^{3/5}+1\right]^{2/3} - \frac{\lambda_2(B)}{\lambda_1(B)},$$

and note that (6.9) holds for a $\beta \in \left(\frac{1}{3}, \frac{2^{2/3}}{1+2^{2/3}}\right)$ if and only if $g(\beta) > 0$. The function g is concave in its whole domain $(0,1)$, since it is possible to compute

$$\begin{aligned}
g''(\beta) = &-\left(\frac{2^{9/5}}{25}\beta^{-11/5}(1-\beta)^{-2/5}\right)\left[2^{2/5}\left(\frac{1-\beta}{\beta}\right)^{3/5}+1\right]^{-4/3} \\
&- \frac{2^{9/5}}{25}\left(\beta^{-3/5}(1-\beta)^{-7/5} + \beta^{-8/5}(1-\beta)^{-2/5}\right)\left[2^{2/5}\left(\frac{1-\beta}{\beta}\right)^{3/5}+1\right]^{-1/3} \\
&- \frac{2^{13/15}}{25}\beta^{-4/5}(1-\beta)^{-11/5}\left[2^{-2/5}\left(\frac{\beta}{1-\beta}\right)^{3/5}+1\right]^{-4/3} \\
&- \frac{2^{34/15}}{25}\left(\beta^{-2/5}(1-\beta)^{-8/5} + \beta^{-7/5}(1-\beta)^{-3/5}\right)\left[2^{-2/5}\left(\frac{\beta}{1-\beta}\right)^{3/5}+1\right]^{-1/3} < 0.
\end{aligned}$$

Moreover we have that

$$g(1/3) \approx 0.036 > 0, \qquad \text{while} \qquad g\left(\frac{2^{2/3}}{1+2^{2/3}}\right) \approx -0.094 < 0,$$

hence we define $\beta_3 = \sup\left\{\beta \in \left(\frac{1}{3}, \frac{2^{2/3}}{1+2^{2/3}}\right) \,:\, g(\beta) > 0\right\}$ and notice that the set

$$\left\{\beta \in \left(\frac{1}{3}, \frac{2^{2/3}}{1+2^{2/3}}\right) \,:\, (6.9) \text{ holds}\right\} = (1/3, \beta_3)$$

is a nonempty interval.

At last, we consider also the case when $\beta \in \left[\frac{2^{2/N}}{1+2^{2/N}}, 1\right)$. As we pointed out above, here we obtain the minimum in the right hand side of (6.8) for $m = 1/2$. Comparing again with the functional for the ball, in the two dimensional case, we have connectedness when $2/3 \leq \beta < \beta_2$, since we have

$$\lambda_2(B) < 2(2-\beta)\lambda_1(B).$$

On the other hand, unfortunately, we do not obtain additional informations when $N = 3$.
Putting all the above information together, we conclude connectedness for:

$N = 2: \quad \beta \in [0, \beta_2) \approx [0, 0.730)$,

$N = 3: \quad \beta \in [0, \beta_3) \approx [0, 0.476)$.

The case $0 < \beta < \beta_N(1-\alpha)$. Recall from the above (case $\alpha = 0$) that

$$\eta\lambda_2(\Omega) + (1-\eta)\lambda_3(\Omega) > \eta\lambda_2(B) + (1-\eta)\lambda_3(B),$$

for $\eta \in [0, \beta_N)$. This implies, with the choice $\eta = \frac{\beta}{1-\alpha}$,

$$\frac{\beta}{1-\alpha}\lambda_2(\Omega) + \frac{(1-\alpha-\beta)}{1-\alpha}\lambda_3(\Omega) > \frac{\beta}{1-\alpha}\lambda_2(B) + \frac{(1-\alpha-\beta)}{1-\alpha}\lambda_3(B),$$

for $\frac{\beta}{1-\alpha} \in [0, \beta_N)$, and so

$$\beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) > \beta\lambda_2(B) + (1-\alpha-\beta)\lambda_3(B),$$

for $\beta \in [0, \beta_N(1-\alpha))$. Together with $\lambda_1(\Omega) \geq \lambda_1(B)$ this concludes the proof of Theorem 6.2 (*iii*). □

Proof of Theorem 6.2 (iv): the case $\beta = 0$. We finally consider the case $\beta = 0$, that is, we deal with the functional $\alpha\lambda_1(\cdot) + (1-\alpha)\lambda_3(\cdot)$. To prove connectedness we first look for the best disconnected set.

Claim 6.C.

Let $\beta = 0$. A disconnected minimizer is made by a ball supporting u_3 and another set supporting u_1, u_2.

Proof of Claim 6.C. First of all we note that a configuration with a ball supporting the first eigenfunction and a set supporting the others would be three balls using the Krahn-Szegö inequality (2.10) on this last set. A set made by three balls is as required in the claim.
On the other hand an optimal configuration with a ball supporting the second eigenvalue would have $\lambda_2(\Omega) = \lambda_3(\Omega)$, as scaling down the ball to obtain this does not effect $\lambda_1(\Omega), \lambda_3(\Omega)$. Up to switch λ_2 and λ_3 we are in a configuration with a ball supporting u_3 and another set supporting u_1, u_2. So the claim is proved. □

We can now focus on a disconnected set made by a ball with measure m supporting u_3 and a set (with measure $1-m$) supporting u_1, u_2; we aim to rule it out. This is done by obtaining lower bounds for the first and third eigenvalues and using comparison with a ball. By the scaling (see property (2) of Lemma 2.7), we have $\lambda_3(\Omega) = \frac{\lambda_1(B)}{m^{2/N}}$, while the Faber-Krahn and the Krahn-Szegö inequalities respectively give

$$\lambda_1(\Omega) \geq \frac{\lambda_1(B)}{(1-m)^{2/N}} \quad \text{and} \quad \frac{\lambda_1(B)}{m^{2/N}} = \lambda_3(\Omega) \geq \lambda_2(\Omega) \geq 2^{2/N} \frac{\lambda_1(B)}{(1-m)^{2/N}}, \tag{6.10}$$

which implies $\frac{1}{m} \geq 3$, and so $m \leq \frac{1}{3}$. By [38, Corollary 5.2.2] we have that, since Ω is disconnected, for $N = 2, 3$

$$\lambda_3(B) = \lambda_2(B) \leq \lambda_3(\Omega).$$

By the Faber-Krahn inequality, the ball strictly lowers the first eigenvalue, so we rule out this configuration for all $\alpha \in [0, 1]$ when $N = 2, 3$.

For $N \geq 4$ we must be more precise and obtain only partial estimates. If $\lambda_3(\Omega) = \frac{\lambda_1(B)}{m^{2/N}} \geq \frac{\lambda_2(B)}{(1-m)^{2/N}}$, then as we assume $\beta = 0$, the set supporting the first two eigenvalues should be a ball B_1. This would contradict the optimality of Ω, as we would have $\lambda_3(B_1) = \lambda_2(B_1) \leq \lambda_3(\Omega)$ and $|B_1| < 1$. So we conclude $\frac{\lambda_1(B)}{m^{2/N}} < \frac{\lambda_2(B)}{(1-m)^{2/N}}$, hence $m > \frac{m_0^{N/2}}{1+m_0^{N/2}}$, which gives the bound

$$\frac{1}{1-m} > 1 + m_0^{N/2}. \tag{6.11}$$

Taking into account (6.10), (6.11) and the estimate $m \geq 1/3$, we obtain

$$\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_3(\Omega) \geq \alpha\lambda_1(B)\left(1 + m_0^{N/2}\right)^{2/N} + (1-\alpha)\lambda_1(B)3^{2/N}. \tag{6.12}$$

By comparing the lower bound (6.12) with $\alpha\lambda_1(B) + (1-\alpha)\lambda_3(B)$ we deduce that a minimizer is connected when $\gamma_N < \alpha \leq 1$ and the proof of Theorem 6.2 (*iv*) is concluded. We have connectedness, for example, in the following ranges:

$N = 2: \quad \alpha \in [0, 1]$,

$N = 3: \quad \alpha \in [0, 1]$,

$N = 4: \quad \alpha \in (\gamma_4, 1] \approx (0.311, 1]$,

$N = 5: \quad \alpha \in (\gamma_5, 1] \approx (0.467, 1]$,

$N = 6: \quad \alpha \in (\gamma_6, 1] \approx (0.547, 1]$.

□

6.5 The two dimensional case

We start this section with a lemma which rules out a minimizer for problem (6.1) with three connected components when $n = 2$. We remind that we call $\mathcal{F}(\cdot) = \alpha\lambda_1(\cdot) + \beta\lambda_2(\cdot) + (1 - \alpha - \beta)\lambda_3(\cdot)$, while $\mathcal{G}$ is the same functional for $\alpha = 0$, in order to avoid confusion.

Lemma 6.10. *Let $N = 2$. Any disconnected minimizer of* (6.1) *has at most two connected components.*

Proof. For the case $\alpha + \beta = 1$, which corresponds to the functional $\alpha\lambda_1(\cdot) + (1 - \alpha)\lambda_2(\cdot)$, it is clear that a minimizer has at most two components since only the first two eigenvalues are into play. For $\alpha + \beta < 1$ the Faber-Krahn inequality implies that a disconnected minimizer with three components would be the union of three disjoint balls. If $\alpha > 0$, it is possible to apply Lemma 6.7 to the union of the balls supporting the second and the third eigenfunctions, thus ruling out this configuration. For $\alpha = 0$ (that is, for the functional $\beta\lambda_2(\cdot) + (1 - \beta)\lambda_3(\cdot)$) this argument does not work, since we can lower only λ_1 which is not into play, while neither λ_2 nor λ_3 are lowered. Hence we rule out the configuration with three connected components only for $n = 2$, by comparing it with B and Θ.

Let $\mathcal{G}(\cdot) = \beta\lambda_2(\cdot) + (1 - \beta)\lambda_3(\cdot)$, and write Ω_i, $i = 1, 2, 3$, for the three components of Ω. Assuming $\lambda_i(\Omega) = \lambda_1(\Omega_i)$ for $i = 1, 2, 3$ gives $|\Omega_1| \geq |\Omega_2| \geq |\Omega_3|$. We write $m = |\Omega_1|$ and note that $|\Omega_2| = m$, as for $|\Omega_1| > |\Omega_2|$ we could enlarge Ω_2 and shrink Ω_1, lowering the functional. Thus $|\Omega_3| = 1 - 2m$, and the following constraints on m hold:

1) Remark 6.8 implies $\lambda_2(B) > \lambda_2(\Omega) = \lambda_1(\Omega_2) = \frac{\lambda_1(B)}{m}$, so $m > \frac{\lambda_1(B)}{\lambda_2(B)} = m_1 \approx 0.394$.

2) We must have $\frac{\lambda_2(B)}{m} = \lambda_2(\Omega_1) \geq \lambda_1(\Omega_3) = \frac{\lambda_1(B)}{1-2m}$, as otherwise we can reduce to only two components. This inequality implies
$$m \leq \frac{\lambda_2(B)}{\lambda_1(B) + 2\lambda_2(B)} = m_2 \approx 0.418.$$

Coming back to the study of $\mathcal{G}$, we can use the scaling properties of eigenvalues and the bounds above to obtain
$$\mathcal{G}(\Omega) = \beta\lambda_2(\Omega)+(1-\beta)\lambda_3(\Omega) = \left\{\frac{\beta}{m} + \frac{(1-\beta)}{(1-2m)}\right\}\lambda_1(B) \geq \left\{\frac{\beta}{m_2} + \frac{(1-\beta)}{(1-2m_1)}\right\}\lambda_1(B). \quad (6.13)$$

Now we look for those β for which the unit ball B gives a lower value of $\mathcal{G}$ than this lower bound. In particular we are looking for those β that satisfy
$$\mathcal{G}(B) - \mathcal{G}(\Omega) \leq \lambda_2(B) - \left\{\frac{\beta}{m_2} + \frac{(1-\beta)}{(1-2m_1)}\right\}\lambda_1(B) < 0,$$
i.e.
$$\beta < \frac{\frac{1}{(1-2m_1)} - \frac{1}{m_1}}{\left\{\frac{1}{(1-2m_1)} - \frac{1}{m_2}\right\}} \approx 0.936.$$

For this range of β three balls can not be optimal when minimizing $\mathcal{G}$.
The remaining β are ruled out by comparing Ω with Θ. Using (6.13), three connected components can not be optimal when

$$\mathcal{G}(\Theta)-\mathcal{G}(\Omega)\le 2\beta\lambda_1(B)+2(1-\beta)\lambda_2(B)-\left\{\frac{\beta}{m_2}+\frac{(1-\beta)}{(1-2m_1)}\right\}\lambda_1(B)<0,$$

i.e. when

$$\beta>\frac{2\lambda_2(B)-\frac{\lambda_1(B)}{1-2m_1}}{2\lambda_2(B)-\left(2+\frac{1}{1-2m_1}-\frac{1}{m_2}\right)\lambda_1(B)}\approx 0.479.$$

Since the two ranges we obtained on β cover all cases, a minimizer for (6.1) can never have three components in $\mathbb{R}^2$. □

We prove now an important lemma, which asserts that a disconnected minimizer must have multiple eigenvalues. The idea of the proof is that if every eigenvalue is simple, then small variations of the connected components (in the sense of shrinking one and enlarging the other) contradict the optimality of such a disconnected set. For simplicity we will often write $\lambda_i=\lambda_i(\Omega)$, $\gamma=1-\alpha-\beta$, and as before define $m_0=\frac{\lambda_1(B)}{\lambda_2(B)}\approx 0.394$.

Lemma 6.11. *A disconnected minimizer Ω for (6.1) in $\mathbb{R}^2$ can not have both $\lambda_1(\Omega)\neq\lambda_2(\Omega)$ and $\lambda_2(\Omega)\neq\lambda_3(\Omega)$.*

Proof. Note that we only need to consider the cases for problem (6.1) that are not covered by Remark 6.3. Additionally, the case of three components is ruled out by Lemma 6.10. The analysis of the remaining cases is divided into three steps.

Step I. We consider the case of a set $\Omega=\Omega_1\cup\Omega_2$, with Ω_1 supporting u_1, while Ω_2 supports u_2 and u_3. From the hypotheses of the Step, $\lambda_1=\lambda_1(\Omega_1)$, $\lambda_2=\lambda_1(\Omega_2)$ and $\lambda_3=\lambda_2(\Omega_2)$, and by Faber-Krahn Ω_1 is a ball. We define $m=|\Omega_1|$, so $1-m=|\Omega_2|$. The following constraints on m hold:

1) $m>\frac{\lambda_1(B)}{\lambda_2(B)}=m_1\approx 0.394$, since $\frac{\lambda_1(B)}{m}=\lambda_1(\Omega_1)\le\lambda_1(\Omega_2)=\lambda_2<\lambda_2(B)$ (see Remark 6.8).

2) $\frac{\lambda_2(B)}{m}=\lambda_2(\Omega_1)\ge\lambda_2(\Omega_2)>\frac{\lambda_2(\Theta)}{(1-m)}=\frac{2\lambda_1(B)}{(1-m)}$, so $m<\frac{\lambda_2(B)}{2\lambda_1(B)+\lambda_2(B)}=m_2\approx 0.559$.

Now we can shrink Ω_1 and enlarge Ω_2, in order to obtain two new sets of the same shape $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$, such that $|\widetilde{\Omega}_1|=m-\varepsilon$, while $|\widetilde{\Omega}_2|=1-m+\varepsilon$. Writing $\widetilde{\lambda}_i$ for the eigenvalues of $\widetilde{\Omega}_1\cup\widetilde{\Omega}_2$ we obtain the following ratios (for $\varepsilon\ll 1$):

$$\frac{\widetilde{\lambda}_1}{\lambda_1}=\frac{m}{m-\varepsilon}\approx 1+\frac{\varepsilon}{m};\qquad \frac{\widetilde{\lambda}_2}{\lambda_2}=\frac{\widetilde{\lambda}_3}{\lambda_3}=\frac{(1-m)}{1-m+\varepsilon}\approx 1-\frac{\varepsilon}{1-m}.$$

The optimality of Ω implies $\mathcal{F}(\Omega)\le\mathcal{F}(\widetilde{\Omega}_1\cup\widetilde{\Omega}_2)$, that means

$$\left(\frac{\alpha\lambda_1}{m}-\frac{\beta\lambda_2+\gamma\lambda_3}{1-m}\right)\varepsilon+o(\varepsilon)\ge 0.$$

Taking either $\varepsilon > 0$ or $\varepsilon < 0$ (this is possible since we are supposing that the eigenvalues are simple) gives that the expression in the brackets must be zero, hence $\frac{\alpha\lambda_1}{m} = \frac{\beta\lambda_2+\gamma\lambda_3}{1-m}$.
In order to conclude this first step it suffices to consider $\varepsilon > 0$. Since $\widetilde{\lambda}_3 \leq \lambda_3$, we have a contradiction if $\mathcal{F}(\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2) < \mathcal{F}(\Omega)$, i.e. when

$$\alpha\widetilde{\lambda}_1 + \beta\widetilde{\lambda}_2 < \alpha\lambda_1 + \beta\lambda_2. \tag{6.14}$$

Equation (6.14) holds if and only if

$$\beta\lambda_2\left(1-\frac{\widetilde{\lambda}_2}{\lambda_2}\right) > \alpha\lambda_1\left(\frac{\widetilde{\lambda}_1}{\lambda_1}-1\right) \qquad \Longleftrightarrow \qquad \beta > \alpha\left(\frac{\lambda_1}{\lambda_2}\right)\left(\frac{1-m}{m}\right).$$

Using $\frac{\lambda_1}{\lambda_2} \leq \frac{\lambda_1(B)}{m}\frac{1-m}{\lambda_2(\Theta)} = \frac{1-m}{2m}$ and the above constraints on m gives $\frac{1-m}{m} \leq \frac{1-m_1}{m_1}$. So if $\beta > \frac{\alpha}{2}\left(\frac{1-m_1}{m_1}\right)^2 \approx 1.18\alpha$, the set $\Omega = \Omega_1 \cup \Omega_2$ can not be optimal. The case $\beta \leq 1.18\alpha$ was treated in Remark 6.3, and so this concludes Step I.

Step II. We now consider the case of a set $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 supporting u_1 and u_3, while Ω_2 supports u_2. Clearly $\lambda_1 = \lambda_1(\Omega_1)$, $\lambda_2 = \lambda_1(\Omega_2)$ and $\lambda_3 = \lambda_2(\Omega_1)$, and again it is better to take Ω_2 to be a ball. Write $m = |\Omega_1|$ and $1-m = |\Omega_2|$. The following constraints on m hold:

1) $\frac{\lambda_1(B)}{m} < \lambda_1(\Omega_1) \leq \lambda_1(\Omega_2) = \frac{\lambda_1(B)}{(1-m)}$, so $m > 1/2 = m_1$.

2) $\frac{\lambda_1(B)}{(1-m)} = \lambda_2(\Omega) < \lambda_2(B)$ by Remark 6.8, so $m < \frac{\lambda_2(B)-\lambda_1(B)}{\lambda_2(B)} = 1-m_0 = m_2 \approx 0.606$.

As in the previous case we shrink Ω_1 to $\widetilde{\Omega}_1$ and we enlarge Ω_2 to $\widetilde{\Omega}_2$, so that $|\widetilde{\Omega}_1| = m-\varepsilon$, while $|\widetilde{\Omega}_2| = 1-m+\varepsilon$. With the same arguments of the previous Step, if Ω is optimal then $\mathcal{F}(\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2) \geq \mathcal{F}(\Omega)$ and so

$$\left(\frac{\alpha\lambda_1+\gamma\lambda_3}{m} - \frac{\beta\lambda_2}{1-m}\right)\varepsilon + o(\varepsilon) \geq 0.$$

Taking again either $\varepsilon > 0$ or $\varepsilon < 0$ gives $\frac{\beta\lambda_2}{1-m} = \frac{\alpha\lambda_1+\gamma\lambda_3}{m}$. Now, since Ω_2 is a ball and thanks to the bounds on m, we can rewrite the complete functional in a more interesting way

$$\mathcal{F}(\Omega) = \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 = \frac{m}{1-m}\beta\lambda_2 + \beta\lambda_2 = \frac{\beta\lambda_2}{1-m} = \frac{\beta\lambda_1(B)}{(1-m)^2} \geq \frac{\beta\lambda_1(B)}{(1-m_1)^2} \geq 4\beta\lambda_1(B).$$

Comparing this lower bound with the case of the ball gives a contradiction for β such that

$$\mathcal{F}(B) - \mathcal{F}(\Omega) \leq \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B) - 4\beta\lambda_1(B) < 0,$$

i.e. for

$$\beta > \frac{1}{4m_0} - \frac{\alpha}{4}\left(\frac{1}{m_0}-1\right) \approx 0.635 - 0.385\alpha. \tag{6.15}$$

In order to consider the cases that are not covered by Remark 6.3, we look at the equations $\alpha = \alpha_2$, $\beta = \frac{1}{4m_0} - \frac{\alpha}{4}\left(\frac{1}{m_0}-1\right)$, and $\beta = \beta_2(1-\alpha)$. The remaining cases can be viewed as

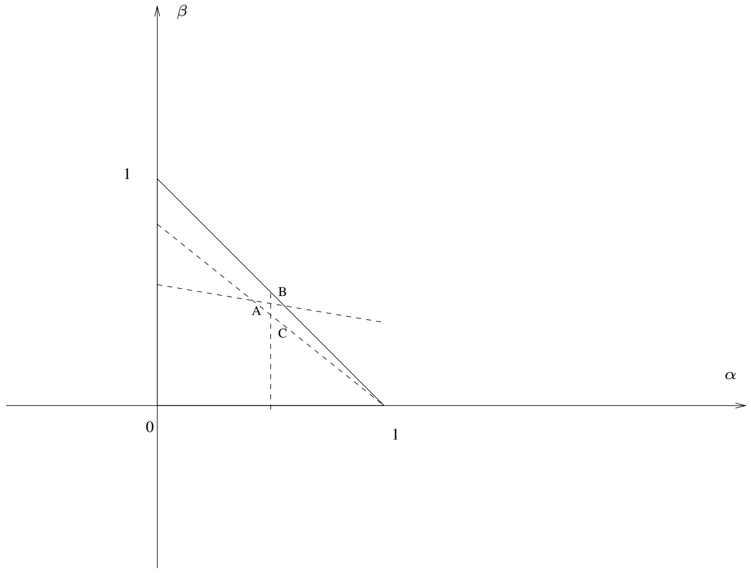

Figure 6.4: The small triangle in the α-β plane

inside a small triangle in the α-β plane, with vertices approximately given by $A = (0.275; 0.529)$, $B = (0.350; 0.500)$ and $C = (0.350; 0.474)$ (See Figure 6.4).

For these remaining points, it is possible to show that a ball is better than Ω, i.e. that $\mathcal{F}(B) - \mathcal{F}(\Omega) < 0$. In fact, the following relations hold between the eigenvalues of the ball and those of Ω (using $m \in (m_1, m_2)$, the Faber-Krahn and the Krahn-Szegö inequalities):

$$\begin{aligned} \lambda_1(B) - \lambda_1 &\le \left(1 - \frac{1}{m}\right)\lambda_1(B) \le \left(1 - \frac{1}{m_2}\right)\lambda_1(B), \\ \lambda_2(B) - \lambda_2 &= \lambda_2(B) - \frac{\lambda_1(B)}{1-m} \le \lambda_2(B) - 2\lambda_1(B), \\ \lambda_3(B) - \lambda_3 &\le \lambda_2(B) - \frac{2\lambda_1(B)}{m} \le \lambda_2(B) - \frac{2\lambda_1(B)}{m_2}. \end{aligned} \tag{6.16}$$

From (6.16) we get

$$\mathcal{F}(B) - \mathcal{F}(\Omega) \le \alpha\lambda_1(B)\left(1 - \frac{1}{m_2}\right) + \beta\left(\lambda_2(B) - 2\lambda_1(B)\right) + (1-\alpha-\beta)\left(\lambda_2(B) - \frac{2\lambda_1(B)}{m_2}\right).$$

Hence the ball is better than Ω if

$$\beta < \frac{2\lambda_1(B) - m_2\lambda_2(B)}{2\lambda_1(B) - 2m_2\lambda_1(B)} + \alpha\frac{(-\lambda_1(B) + m_2(\lambda_2(B) - \lambda_1(B)))}{2\lambda_1(B) - 2m_2\lambda_1(B)} \approx 0.58 - 0.08\alpha.$$

This inequality together with (6.15) concludes Step II.

Step III. We now consider the case of a set $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 supporting u_1 and u_2, while Ω_2 supports u_3. Clearly $\lambda_1 = \lambda_1(\Omega_1)$, $\lambda_2 = \lambda_2(\Omega_1)$ and $\lambda_3 = \lambda_1(\Omega_2)$, and it is better to take Ω_2 to be a ball. Let $m = |\Omega_1|$ and $1 - m = |\Omega_2|$. Note that if $\lambda_3(\Omega) = \lambda_1(\Omega_2) \ge m\lambda_3(\Omega_1)$, then Ω can not be optimal. In fact in this case it is better to take the connected set obtained by enlarging Ω_1 till unit measure, since this lowers both λ_1 and λ_2 (by monotonicity), while also the third eigenvalue is lower, by hypothesis. The following constraints on m hold:

1) $\lambda_2(B) > \lambda_2(\Omega) = \lambda_2(\Omega_1) \geq \frac{2\lambda_1(B)}{m}$ (see Remark 6.8), so $m > 2m_0 = m_1 \approx 0.788$.

2) In order to have Ω optimal, an upper bound on m follows from inequality $m\lambda_3(\Omega_1) > \lambda_1(\Omega_2)$ explained above and from the fact that $\lambda_2(\Omega) < \lambda_2(B)$ (see Remark 6.8). Using also (2.12) from Theorem 2.11 gives

$$\frac{\lambda_1(B)}{(1-m)} = \lambda_1(\Omega_2) < m\lambda_3(\Omega_1) \leq m\frac{\lambda_2(B)}{\lambda_1(B)}\lambda_2(\Omega_1) < m\frac{\lambda_2(B)^2}{\lambda_1(B)}.$$

This means $m^2 - m + m_0^2 < 0$, which gives the upper bound

$$m < \frac{\lambda_2(B) + \sqrt{\lambda_2(B)^2 - 4\lambda_1(B)^2}}{2\lambda_2(B)} = m_2 \approx 0.808.$$

As in the previous steps we can enlarge Ω_1 to $\widetilde{\Omega}_1$ and we can shrink Ω_2 to $\widetilde{\Omega}_2$, in order that $|\widetilde{\Omega}_1| = m + \varepsilon$, while $|\widetilde{\Omega}_2| = 1 - m - \varepsilon$. The following ratios between the eigenvalues of $\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$ and those of Ω hold (for $\varepsilon \ll 1$) :

$$\frac{\widetilde{\lambda}_1}{\lambda_1} = \frac{\widetilde{\lambda}_2}{\lambda_2} = \frac{m}{m+\varepsilon} \approx 1 - \frac{\varepsilon}{m}; \qquad \frac{\widetilde{\lambda}_3}{\lambda_3} = \frac{1-m}{1-m-\varepsilon} \approx 1 + \frac{\varepsilon}{1-m}.$$

In order to be optimal, Ω must satisfy

$$\alpha\widetilde{\lambda}_1 + \beta\widetilde{\lambda}_2 + (1-\alpha-\beta)\widetilde{\lambda}_3 \geq \alpha\lambda_1 + \beta\lambda_2 + (1-\alpha-\beta)\lambda_3.$$

An analogous argument to that in Step I and Step II gives a contradiction for $\beta \gtrsim 0.914 - 0.948\alpha$. Actually we can obtain a better result observing that Ω is worse than Ω_1 enlarged to unit measure (which we will call $\overline{\Omega}$ in the following) if β is suitably large. We denote by $\{\overline{\lambda}_i\}$ the eigenvalues of $\overline{\Omega}$ and we again write $\gamma = 1-\alpha-\beta$ for the sake of simplicity. The following relations between the eigenvalues hold: $\overline{\lambda}_1 = m\lambda_1$, $\overline{\lambda}_2 = m\lambda_2$ and $\overline{\lambda}_3 = m\lambda_3(\Omega_1) \leq \frac{m}{m_0}\lambda_2$, using (2.12) from Theorem 2.11 by Ashbaugh and Benguria. This gives

$$\mathcal{F}(\overline{\Omega}) = \alpha\overline{\lambda}_1 + \beta\overline{\lambda}_2 + \gamma\overline{\lambda}_3 \leq \mathcal{F}(\Omega) + \alpha(m-1)\lambda_1 + \beta(m-1)\lambda_2 + \gamma\left(\frac{m}{m_1} - 1\right)\lambda_2.$$

Clearly Ω can not be optimal when $\mathcal{F}(\overline{\Omega}) - \mathcal{F}(\Omega) < 0$, which holds if

$$\alpha(m-1)\lambda_1 + \left[\beta(m-1) + \gamma\left(\frac{m}{m_0} - 1\right)\right]\lambda_2 < 0.$$

The first part (2.11) of Theorem 2.11 gives that the result follows if

$$\left[\alpha(m-1)m_0 + \beta(m-1) + \gamma\left(\frac{m}{m_0} - 1\right)\right]\lambda_2 < 0.$$

Since $m \in (m_1, m_2)$ and the function in brackets is clearly increasing in m, Ω can not be optimal when

$$\frac{m_2}{m_0} - 1 + \alpha\left((m_2-1)m_0 + 1 - \frac{m_2}{m_0}\right) + \beta\left(m_2 - \frac{m_2}{m_0}\right) < 0,$$

i.e. for

$$\beta > \frac{\left(\frac{m_2}{m_0} - 1\right) + \alpha\left((m_2-1)m_0 + 1 - \frac{m_2}{m_0}\right)}{\frac{m_2}{m_0} - m_2} \approx 0.845 - 0.906\alpha. \tag{6.17}$$

In conclusion we have an estimate that tells us that when β is suitably big, then $\Omega = \Omega_1 \cup \Omega_2$ can not be optimal. Writing $\gamma = 1-\alpha-\beta$, we now finally show that Ω can not be optimal also when γ is not very small. We use a technique very similar to the case β big. For this suppose Ω is optimal for the problem (6.1) and let $|\varepsilon| \ll 1$. Then if we enlarge Ω_1 to $\widetilde{\Omega}_1$ with measure $m+\varepsilon$ and we shrink Ω_2 to $\widetilde{\Omega}_2$ with measure $1-m-\varepsilon$, calling λ_i the eigenvalues of $\Omega = \Omega_1 \cup \Omega_2$, while $\widetilde{\lambda}_i$ are the eigenvalues of $\widetilde{\Omega} = \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$, we must have $\mathcal{F}(\Omega) - \mathcal{F}(\widetilde{\Omega}) \leq 0$. On the other hand, with analogous computations to those in Step II,

$$\mathcal{F}(\Omega) - \mathcal{F}(\widetilde{\Omega}) = \left(\frac{\gamma\lambda_3}{1-m} - \frac{\alpha\lambda_1 + \beta\lambda_2}{m}\right)\varepsilon + o(\varepsilon), \tag{6.18}$$

and hence the expression in brackets must be zero, as otherwise taking $\varepsilon > 0$ or $\varepsilon < 0$ (this is possible since we are treating only the case of simple eigenvalues) contradicts the optimality of Ω. So if Ω is optimal then $\alpha\lambda_1 + \beta\lambda_2 = \gamma\lambda_3\frac{m}{1-m}$. Since $m \mapsto \frac{1}{(1-m)^2}$ is increasing we have the lower bound

$$\mathcal{F}(\Omega, \alpha, \beta) = \gamma\lambda_3\frac{m}{1-m} + \gamma\lambda_3 \geq \gamma\lambda_1(B)\frac{1}{(1-2m_1)^2}.$$

We can show that, for γ suitably big, comparing the functional for Θ with the lower bound above gives an absurd. In fact, the functional for the two balls is given by

$$\mathcal{F}(\Theta, \alpha, \beta) = (\alpha+\beta)2\lambda_1(B) + \gamma 2\lambda_2(B) = 2\lambda_1(B) + \gamma(2\lambda_2(B) - 2\lambda_1(B)).$$

Hence $\mathcal{F}(\Theta, \alpha, \beta) < \mathcal{F}(\Omega, \alpha, \beta)$ for $\gamma > \overline{\gamma} \approx 0.104$, in which case two balls are better than our set Ω. Combining the cases in which either $\gamma > \overline{\gamma}$ or (6.17) holds concludes Step III and hence the proof of the lemma. □

6.5.1 Proof of Theorem 6.4 (a).

Proof of Theorem 6.4 (a). It is proved in Lemma 6.11 that any disconnected minimizer Ω has multiple eigenvalues. By Remark 6.3 every minimizer for

$$\inf\{\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_3(\Omega) \ : \ \Omega \text{ open in } \mathbb{R}^2,\ |\Omega| \leq 1\}$$

is connected for all $\alpha \in [0,1]$, and we call $\tilde{\Omega}$ such a connected minimizer. The case $\lambda_2(\Omega) = \lambda_3(\Omega)$ is then ruled out, as it would give

$$\begin{aligned}\alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) &= \alpha\lambda_1(\Omega) + (1-\alpha)\lambda_3(\Omega) > \alpha\lambda_1(\tilde{\Omega}) + (1-\alpha)\lambda_3(\tilde{\Omega}) \\ &\geq \alpha\lambda_1(\tilde{\Omega}) + \beta\lambda_2(\tilde{\Omega}) + (1-\alpha-\beta)\lambda_3(\tilde{\Omega}).\end{aligned}$$

Therefore any disconnected minimizer must satisfy $\lambda_1(\Omega) = \lambda_2(\Omega)$ and can be viewed as the union of a disk supporting the first eigenvalue with a connected set supporting the second and third, since a minimizer with three connected components was ruled out by Lemma 6.10. □

6.5.2 Proof of Theorem 6.4 (b).

Proof of Theorem 6.4 (b). Let $\alpha + \beta < 1$, and let $\tilde{\Omega}$ be a connected minimizer for $\inf\{(\alpha + \beta)\lambda_2(\Omega) + (1 - \alpha - \beta)\lambda_3(\Omega) \,:\, \Omega \text{ open in } \mathbb{R}^2,\ |\Omega| \leq 1\}$, while there are no disconnected minimizers by hypothesis. Theorem 6.4 (a) then gives $\lambda_1(\Omega) = \lambda_2(\Omega)$ for a disconnected minimizer Ω for problem (6.1), whereby

$$\begin{aligned}\alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1 - \alpha - \beta)\lambda_3(\Omega) &= (\alpha + \beta)\lambda_2(\Omega) + (1 - \alpha - \beta)\lambda_3(\Omega)\\ &> (\alpha + \beta)\lambda_2(\tilde{\Omega}) + (1 - \alpha - \beta)\lambda_3(\tilde{\Omega}),\\ &\geq \alpha\lambda_1(\tilde{\Omega}) + \beta\lambda_2(\tilde{\Omega}) + (1 - \alpha - \beta)\lambda_3(\tilde{\Omega}),\end{aligned}$$

which contradicts the minimality of Ω and thus the proof is concluded. □

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The thesis is based on the following papers

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