The role of a vanishing interfacial layer in perfect elasto-plasticity

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Abstract

A two-phase elasto-plastic material is investigated. It is shown that, if the interface is modeled as the limit of a vanishing layer of a third material, then the resulting two-phase material will exhibit a smaller interfacial dissipation than that of a pure two-phase model.

Dedicated to Luc Tartar, notwithstanding his dislike for elasto-plasticity.

1 Introduction

1.1 Introductory remarks

The present contribution focusses on the behavior of a two-phase elasto-plastic material in a small strain setting.

The topic was first tackled mathematically in [10],[11], then, after a twenty five year long interlude, revisited in [3] within the framework of the rapidly expanding variational theory of rate independent evolutions (see e.g. [8]). The ensuing functional setting results in strain fields that can concentrate on sets of co-dimension at most 1 and displacement fields that can in particular jump along rectifiable hypersurfaces. So, as Luc Tartar has repeatedly pointed out, the mathematical models of small strain elasto-plasticity are *prima facie* inconsistent with the small strain assumption they were born out of. We gladly acknowledge this inconsistency which cannot be reconciled at present through the consideration of models of finite plasticity for lack of any kind of consensus of what such models could be....

In a previous contribution [5], we derived what we believe to be the first evolution model for a multi-phase heterogeneous elasto-plastic material, although earlier work [9] had previously analyzed a subclass of possible multi-phase heterogeneities. In any case, our results, specialized to a two-phase setting, demonstrate that the correct stress constraint on the interface only involves the resolved shear stress and that the corresponding set of admissible resolved shear stresses is that which corresponds to the intersection of the set of admissible stresses for each phase. This leads to a well defined interfacial flow rule which, to the best of our knowledge, cannot be found in the abundant literature on elasto-plasticity, be it on the mathematical, or on the mechanical side.

In the present contribution, we propose to investigate the impact of a vanishingly thin interface between the two phases and to demonstrate that such an interface is felt in the resulting two-phase model through an interfacial dissipation lower than that predicted by the pure two-phase problem. This question was suggested to us by Mark Peletier whom we gratefully acknowledge. Of course we cannot consider a *bona fide* thin layer of a third material because the question of the modeling of the interface between that layer and the two phases would immediately render the investigation moot. Rather, we will consider a continuously varying set of admissible stresses near and on the interface and then propose to pass to the limit in the thickness of the transition.

The result is given in Theorem 2.6 and further interpreted in Section 4. In a nutshell, we establish that any modeling of the interface as the limit of a vanishing third phase whose set of admissible stresses is smaller than the intersection of those in both phases will result in a lower interfacial dissipation, hence that the pure two-phase material is the maximally dissipating model for the interface.

So, in conclusion, it is indeed possible to model an elasto-plastic interface between two elasto-plastic phases. However, the interfacial dissipation cannot be chosen arbitrarily. It must be so that it is below that generated by the intersection of the sets of admissible stresses of both phases.

1.2 Notation and preliminaries

General notation. For $A \subseteq \mathbb{R}^N$, χ_A denotes the characteristic function of A, *i.e.*, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$. The indicator function of A, denoted by $\mathbb{I}_A$, is defined as $\mathbb{I}_A(x) = 0$ for $x \in A$, and $\mathbb{I}_A(x) = +\infty$ for $x \notin A$. The symbol $\lfloor A$ stands for "restricted to A".

We will denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure and by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure, which coincides with the usual area measure on sufficiently regular sets (see e.g. [4, Section 2.1] or [2, Section 2.8]).

Matrices. We denote by $\mathrm{M}^N_{\mathrm{sym}}$ the set of $N \times N$-symmetric matrices and by M^N_D the set of trace-free elements of $\mathrm{M}^N_{\mathrm{sym}}$. If M is an element of $\mathrm{M}^N_{\mathrm{sym}}$, then M_D is its deviatoric part, *i.e.*, its projection onto the subspace M^N_D of $\mathrm{M}^N_{\mathrm{sym}}$ orthogonal to the identity matrix for the Frobenius inner product. The symbol $\cdot$ stands for that inner product and the symbol $|\cdot|$ for the Frobenius norm. The set of symmetric endomorphisms on M^N_D is denoted by $\mathcal{L}_s(\mathrm{M}^N_D)$. For $a, b \in \mathbb{R}^N$, $a \odot b$ stands for the symmetric matrix such that $(a \odot b)_{ij} := (a_i b_j + a_j b_i)/2$.

Functional spaces. Given $E \subseteq \mathbb{R}^N$ measurable, $1 \le p < +\infty$, and M a finite dimensional normed space, $L^p(E; M)$ stands for the space of p-summable functions on E with values in M, with associated norm denoted by $\|\cdot\|_p$. Given $A \subseteq \mathbb{R}^N$ open, $H^1(A; M)$ is the Sobolev space of functions in $L^2(A; M)$ with distributional derivatives in L^2.

Finally, let X be a normed space. We denote by $BV(a, b; X)$ and $AC(a, b; X)$ the space of functions with bounded variation and the space of absolutely continuous functions from $[a, b]$ to X, respectively. The total variation of $f \in BV(a, b; X)$ is defined as

$$\mathcal{V}_X(f; a, b) := \sup \left\{ \sum_{j=1}^{k} \|f(t_j) - f(t_{j-1})\|_X \,:\, a = t_0 < t_1 < \cdots < t_k = b \right\}.$$

Measures. If E is a locally compact separable metric space, and X a finite dimensional normed space, $\mathcal{M}_b(E; X)$ will denote the space of finite Radon measures on E with values in X. For $\mu \in \mathcal{M}_b(E; X)$, we denote by $|\mu|$ its variation measure. The space $\mathcal{M}_b(E; X)$ is the topological dual of $C^0_0(E; X^*)$, the set of continuous functions u from E to the vector dual X^* of X which "vanish at the boundary", *i.e.*, such that for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ with $|u(x)| < \varepsilon$ for $x \notin K$.

The (kinematic) space BD. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. The displacement field u lies in the space of functions of bounded deformations

$$BD(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^N) \,:\, Eu := 1/2(\nabla u + \nabla u^T) \in \mathcal{M}_b(\Omega; \mathrm{M}^N_{\mathrm{sym}})\}$$

endowed with the norm

$$\|u\|_{BD} := \|u\|_1 + \|Eu\|_{\mathcal{M}_b}.$$

We refer the reader to e.g. [12, Chapter 2], and [1] for background material.

Besides elementary properties of $BD(\Omega)$, we will only appeal to the structure of Eu as a Radon measure: more precisely, as is the case for functions of bounded variation, the measure Eu decomposes as

$$Eu = E^a u + E^j u + E^c u.$$

Here $E^a u$ denotes the part of the measure adsolutely continuous w.r.t. $\mathcal{L}^N$, so that

$$E^a u = \mathcal{E}u \, d\mathcal{L}^N, \quad \text{with } \mathcal{E}u \in L^1(\Omega; \mathrm{M}^N_{\mathrm{sym}}).$$

The singular part is further decomposed into a *jump part* $E^j u$ and a *Cantor part* $E^c u$. Specifically,

$$E^j u = [u] \odot \nu_u \, d\mathcal{H}^{N-1} \lfloor J_u,$$

where J_u stands for the *jump set* of u (see [2, Definition 3.67]), $[u]$ being the jump of u across J_u, while $E^c u$ vanishes on Borel sets which are σ-finite with respect to the area measure $\mathcal{H}^{N-1}$ (see [1, Proposition 4.4]).

Finally, we say that

$$u_n \overset{*}{\rightharpoonup} u \qquad \text{weakly}^* \text{ in } BD(\Omega)$$

iff

$$u_n \to u, \ \text{ strongly in } L^1(\Omega; \mathbb{R}^N) \text{ and } Eu_n \overset{*}{\rightharpoonup} Eu \ \text{ weakly}^* \text{ in } \mathcal{M}_b(\Omega; \mathrm{M}^N_{\mathrm{sym}}).$$

The (static) space Σ. Let $\Omega \subset \mathbb{R}^N$ be open and bounded with a Lipschitz boundary. We set

$$\Sigma := \left\{ \sigma \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}) : \operatorname{div} \sigma \in L^2(\Omega; \mathbb{R}^N) \text{ and } \sigma_D \in L^\infty(\Omega; \mathbb{R}^N) \right\}.$$

It is classical that, if $\sigma \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ with $\operatorname{div} \sigma \in L^2(\Omega; \mathbb{R}^N)$, $\sigma\nu$ is well defined as an element of $H^{-1/2}(\partial\Omega; \mathbb{R}^N)$, ν being the outer normal to $\partial\Omega$.

More generally, consider an arbitrary Lipschitz subdomain $A \subset \overline{\Omega}$ with outer normal ν, and $\Delta \subset \partial A$ open in the relative topology. We can define the restriction of $\sigma\nu$ "on Δ" by testing against functions in $H^{1/2}(\partial A; \mathbb{R}^N)$ with compact support in Δ. This amounts to viewing $\sigma\nu$ as an element of the dual to $H^{1/2}_{00}(\Delta; \mathbb{R}^N)$.

If $\sigma \in \Sigma$, then, in the spirit of [6, Lemma 2.4], we can define a tangential component $[\sigma\nu]_\tau$ of $\sigma\nu$ on Δ such that

$$[\sigma\nu]_\tau \in L^\infty(\Delta; \mathbb{R}^N). \tag{1.1}$$

Indeed, consider any regularization $\sigma_n \in C^\infty(\bar{A}; \mathrm{M}^N_{\mathrm{sym}})$ of σ on $\bar{A}$ such that

$$\begin{cases} \sigma_n \to \sigma & \text{strongly in } L^2(A; \mathrm{M}^N_{\mathrm{sym}}) \\ \mathrm{div}\sigma_n \to \mathrm{div}\sigma & \text{strongly in } L^2(A; \mathbb{R}^N) \\ \|(\sigma_n)_D\|_\infty \le \|\sigma_D\|_\infty. \end{cases} \tag{1.2}$$

Define the tangential component $[\sigma_n\nu]_\tau := (\sigma_n)\nu - ((\sigma_n)\nu \cdot \nu)\nu$. It is readily seen that $[\sigma_n\nu]_\tau = [(\sigma_n)_D\nu]_\tau$ (the tangential component of $(\sigma_n)_D$ is defined analogously). Since $x \mapsto \nu(x)$ is an $L^\infty(\Delta; \mathbb{R}^N)$-mapping, there exists an $L^\infty(\Delta; \mathbb{R}^N)$-function $[\sigma\nu]_\tau$ such that, up to a subsequence,

$$[\sigma_n\nu]_\tau \overset{*}{\rightharpoonup} [\sigma\nu]_\tau \text{ weakly}^* \text{ in } L^\infty(\Delta; \mathbb{R}^N). \tag{1.3}$$

If $\sigma_D \equiv 0$ then, clearly, $[\sigma\nu]_\tau \equiv 0$, so that $[\sigma\nu]_\tau$ is only a function of $(\sigma_n)_D$ which we will denote henceforth by $[\sigma_D\nu]_\tau$. Notice that $[\sigma_D\nu]_\tau$ may depend upon the approximation sequence $\{\sigma_n\}_n$ (or at least upon $\{(\sigma_n)_D\}_n$).

Finally, if Δ is a C^2-hypersurface, *i.e.*, a C^2-submanifold of $\mathbb{R}^N$ of dimension $N-1$, then $[\sigma_D\nu]_\tau$ is uniquely determined as an element of $L^\infty(\Delta;\mathbb{R}^N)$. Indeed, for every $\varphi \in H^{1/2}(\partial A;\mathbb{R}^N)$ with support compactly contained in Δ (that is by density $\varphi \in H^{1/2}_{00}(\Delta;\mathbb{R}^N)$),

$$\int_\Delta [\sigma\nu]_\tau \cdot \varphi \, d\mathcal{H}^{N-1} = \langle \sigma\nu, \varphi\rangle - \langle (\sigma\nu)_\nu, \varphi\rangle,$$

where

$$\langle (\sigma\nu)_\nu, \varphi\rangle := \langle \sigma\nu, (\varphi\cdot\nu)\nu\rangle.$$

Since the normal component $(\varphi\cdot\nu)\nu$ of φ with respect to Δ belongs to $H^{1/2}(\partial A;\mathbb{R}^N)$ in view of the regularity of ν on Δ, the definition of $(\sigma\nu)_\nu$ is meaningful.

2 Energetic quasi-static evolutions

In this section we review the variational formulation for a heterogeneous quasi-static evolution in perfect plasticity. When the spatial dependence of the convex set of admissible stresses is continuous, the problem was investigated in [9]. However, in the case where the heterogeneity is made of the juxtaposition of several phases with no particular ordering properties, then the reader should refer to [5]. Of course, both works find their root in the seminal paper [3] in which elasto-plastic evolution was analyzed as a variational evolution.

The reference configuration. In all that follows $\Omega \subset \mathbb{R}^N$ is an open, bounded set with (at least) Lipschitz boundary and exterior normal ν. Further, the Dirichlet part Γ^d of $\partial\Omega$ is a non empty open set in the relative topology of $\partial\Omega$ with boundary $\partial_{\lfloor\partial\Omega}\Gamma^d$ in $\partial\Omega$ and we set $\Gamma_t := \partial\Omega \setminus \bar{\Gamma}^d$. Reproducing the setting of [5, Section 6], we introduce the following

Definition 2.1. We will say that $\partial_{\lfloor\partial\Omega}\Gamma^d$ is admissible iff, for any $\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ with

$$\mathrm{div}\sigma = f \text{ in } \Omega, \quad \sigma\nu = g \text{ on } \Gamma_t, \quad \sigma_D \in L^\infty(\Omega;\mathrm{M}^N_D) \tag{2.1}$$

where $f \in L^N(\Omega;\mathbb{R}^N)$ and $g \in L^\infty(\Gamma_t;\mathbb{R}^N)$, and every $p \in \mathcal{M}_b(\Omega\cup\Gamma^d;\mathrm{M}^N_D)$ such that there exists an associated pair $(u,e) \in BD(\Omega)\times L^{N/N-1}(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ with

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w-u)\odot\nu\,\mathcal{H}^{N-1}\lfloor\Gamma^d \quad \text{on } \Gamma^d,$$

the distribution, defined for all $\varphi \in C^\infty_c(\mathbb{R}^N)$ by

$$\langle \sigma_D, p\rangle(\varphi) := -\int_\Omega \varphi\sigma\cdot(e-Ew)\,dx - \int_\Omega \varphi f\cdot(u-w)\,dx \\ - \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,dx + \int_{\Gamma_t} \varphi g\cdot(u-w)\,d\mathcal{H}^{N-1} \tag{2.2}$$

extends to a bounded Radon measure on $\mathbb{R}^N$ with $|\langle\sigma_D,p\rangle| \le \|\sigma_D\|_\infty |p|$.

Definition 2.1 covers many "practical" settings like those of a hypercube with one of its faces standing for the Dirichlet part Γ^d; see [5, Section 6] for that and other such settings.

Remark 2.2. Expression (2.2) defines a meaningful distribution on $\mathbb{R}^N$. Indeed, according to [5, Proposition 6.1] if $\sigma \in L^2(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ is such that $\mathrm{div}\sigma \in L^N(\Omega;\mathbb{R}^N)$ and $\sigma_D \in L^\infty(\Omega;\mathrm{M}^N_D)$, then $\sigma \in L^r(\Omega;\mathrm{M}^N_{\mathrm{sym}})$ for every $1 \le r < \infty$ with

$$\|\sigma\|_r \le C_r\left(\|\sigma\|_2 + \|\mathrm{div}\sigma\|_N + \|\sigma_D\|_\infty\right)$$

for some $C_r > 0$. On the other hand, $u \in L^{N/N-1}(\Omega;\mathbb{R}^N)$ in view of the embedding of $BD(\Omega)$ into $L^{N/N-1}(\Omega;\mathbb{R}^N)$. Further, u has a trace on $\partial\Omega$ which belongs to $L^1(\partial\Omega;\mathbb{R}^N)$. Finally note that, if σ is the restriction to Ω of a C^1-function and if $\mathcal{H}^{N-1}(\partial_{\lfloor\partial\Omega}\Gamma^d) = 0$, then, an integration by parts in BD (see [12, Chapter 2, Theorem 2.1]) would demonstrate that the right-hand side of (2.2) coincides with the integral of φ with respect to the (well defined) measure $\sigma_D p$. ¶

Further, we assume that Ω is made up of two phases Ω_1, Ω_2, together with the phase interface. Those phases are disjoint open sets with Lipschitz boundary. We have $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and denote by Γ the inner interface, *i.e.*,

$$\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \cap \Omega.$$

We assume the existence of a compact set $S \subset \Gamma$ with $\mathcal{H}^{N-1}(S) = 0$ and such that

$$\Gamma \setminus S \text{ is a } C^2\text{-hypersurface.}$$

Finally, setting

$$S' := \{x \in \partial\Omega : x \in \partial\Omega_1 \cap \partial\Omega_2\},$$

S' is taken to be such that

$$\mathcal{H}^{N-1}(S') = 0,$$

and we set $\Gamma_i^d := (\overline{\Omega}_i \setminus S') \cap \Gamma^d,\ i = 1, 2$.

A domain Ω that satisfies the collection of those (minimal) assumptions will be referred to henceforth as a C^2-*geometrically admissible multiphase domain.*

Kinematic admissibility. Given the boundary displacement $w \in H^1(\Omega; \mathbb{R}^N)$, we adopt the following

Definition 2.3 (Admissible configurations)**.** $\mathcal{A}(w)$, the family of admissible configurations relative to w, is the set of triplets (u, e, p) with

$$u \in BD(\Omega), \qquad e \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}), \qquad p \in \mathcal{M}_b(\Omega \cup \Gamma^d; \mathrm{M}^N_D),$$

and such that

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu\, \mathcal{H}^{N-1} \lfloor \Gamma^d \quad \text{on } \Gamma^d, \tag{2.3}$$

where ν denotes the outer normal to $\partial\Omega$ and $(w - u)$ is to be understood in the sense of traces.

The function u denotes the displacement field on Ω, while e and p are the associated elastic and plastic strains. In view of the additive decomposition (2.3) of Eu and of the general properties of BD functions recalled earlier, p does not charge $\mathcal{H}^{N-1}$-negligible sets. Moreover, given a Lipschitz hypersurface $D \subset \Omega$ dividing Ω locally into the subdomains Ω^+ and Ω^-,

$$p\lfloor D = (u^+ - u^-) \odot \nu\, \mathcal{H}^{N-1} \lfloor D,$$

where ν is the normal to D pointing from Ω^- to Ω^+, and $u^\pm$ are the traces on D of the restrictions of u to $\Omega^\pm$. Since p is assumed to take values in the space of deviatoric matrices M^N_D, $u^+ - u^-$ is perpendicular to ν, so that only particular plastic strains are activated along D.

The elasticity tensor. The Hooke's law is given by an element $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}_s(\mathrm{M}^N_{\mathrm{sym}}))$ such that

$$c_1|M|^2 \le \mathbb{C}(x)M \cdot M \le c_2|M|^2 \text{ for every } M \in \mathrm{M}^N_{\mathrm{sym}}, \tag{2.4}$$

with $c_1, c_2 > 0$.

For every $e \in L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}})$ we set

$$\mathcal{Q}(e) := \frac{1}{2} \int_\Omega \mathbb{C}(x)e \cdot e\, dx.$$

Admissible stresses: In elasto-plasticity, the deviatoric part of the stress σ is assumed to be restricted by the yield condition. For heterogeneous materials, this means that, at a.e. $x \in \Omega$, there exists a convex compact set $K(x) \subset \mathrm{M}^N_D$, the *set of admissible stresses*, such that $\sigma_D(x) \in K(x)$.

We say that the multimap $x \multimap K(x)$ is continuous on $\overline{\Omega}$ if it satisfies the lower semi-continuity condition

$$\forall \varepsilon > 0, \exists U_x \text{ open s.t. } x \in U_x \text{ and } K(x) \subset K(y) + \varepsilon B(0,1) \text{ for every } y \in U_x,$$

together with the upper semi-continuity condition

$$\forall \varepsilon > 0, \exists U_x \text{ open s.t. } x \in U_x \text{ and } K(y) \subset K(x) + \varepsilon B(0,1) \text{ for every } y \in U_x.$$

In that case, we further assume that the sets $K(x)$ cannot be too small or too large, *i.e.*, there exist $c_3, c_4 > 0$ such that

$$B(0,c_3) \subset K(x) \subset B(0,c_4) \qquad \text{for all } x \in \overline{\Omega}. \tag{2.5}$$

In the present setting, the heterogeneity is the result of the assembly of two distinct phases with associated sets of admissible stresses K_1 and K_2 with

$$K_i \text{ closed convex subsets of } \mathrm{M}^N_D, \quad B(0,c_3) \subset K_i \subset B(0,c_4),\ i=1,2. \tag{2.6}$$

Then, the multimap $x \multimap K(x)$ is not a priori defined on the interface Γ, nor on S'. We define it on $\Gamma \setminus S$ as

$$K(x) = \{\sigma_D \in \mathrm{M}^N_D : [\sigma_D \nu(x)]_\tau \in [K_1\nu(x)]_\tau \cap [K_2\nu(x)]_\tau\}, \tag{2.7}$$

where $\nu(x)$ is the associated normal to Γ, and $(\cdot)_\tau$ denotes the orthogonal projection to the hyperplane tangent to Γ at x. Notice that $K(x)$ is a cylinder in M^N_D. On $S \cup S'$, we define $K(x)$ arbitrarily as M^N_D.

Henceforth, we refer to this case as the *pure two-phase case*.

The dissipation potential: The Legendre transform of $\mathbb{I}_{K(x)}$ yields the dissipation potential $H : (\Omega \cup \Gamma^d) \times \mathrm{M}^N_D \to [0,+\infty[$ given, for *every* $x \in \Omega \times \Gamma^d$ and every $\xi \in \mathrm{M}^N_D$, by

$$H(x,\xi) := \sup\{\tau \cdot \xi : \tau \in K(x)\}.$$

It is easily seen that, in the continuous as well as in the pure two-phase cases, the map $\xi \mapsto H(x,\xi)$ is convex and positively one-homogeneous, while H is Borel and lower semi-continuous on $(\Omega \cup \Gamma^d) \times \mathrm{M}^N_D$ (see [9, Proposition 2.4] and [5, Section 2]).

In the pure two-phase case, note that, for $x \in \Gamma \setminus S$, H reads as

$$H(x, a \odot \nu(x)) := \inf\{H_i(a_i \odot \nu(x)) + H_j(-a_j \odot \nu(x)) : \\ a = a_i - a_j,\ a_i, a_j \in \mathbb{R}^N,\ a_i \perp \nu(x), a_j \perp \nu(x)\},$$

if $\xi = a \odot \nu(x) \in \mathrm{M}^N_D$, $a \perp \nu(x)$, and

$$H(x,\xi) = +\infty \qquad \text{otherwise on } \mathrm{M}^N_D.$$

Above and throughout the rest of this paper

$$H_i(\xi) := \sup\{\tau \cdot \xi : \tau \in K_i\}.$$

Finally,

$$c_3|\xi| \le H(x,\xi) \le c_4|\xi|, \qquad x \in (\Omega \cup \Gamma^d) \setminus (\Gamma \cup S'),\ \xi \in \mathrm{M}^N_D$$
$$c_3|a \odot \nu(x)| \le H(x, a \odot \nu(x)) \le c_4|a \odot \nu(x)|,\ \ x \in \Gamma \setminus S,\ a \in \mathbb{R}^N, a \perp \nu(x).$$

Remark 2.4. In the two-phase case, we can decide that the admissible stress set on the interface is not that described through (2.7), but rather that is is associated with some compact convex set K_3 containing 0. Then, mimicking (2.7), we define

$$K(x) = \{\sigma_D \in \mathrm{M}^N_D : [\sigma_D \nu(x)]_\tau \in [K_3\nu(x)]_\tau\} \tag{2.8}$$

on $\Gamma \setminus S$ and complete the definition of K by M^N_D on $S \cup S'$.

The resulting dissipation potential H, defined on $\Gamma \setminus S'$ as

$$H(x, a \odot \nu(x)) := H_3(a \odot \nu(x)),\ a \in \mathbb{R}^N,\ a \perp \nu(x),$$

and

$$H(x,\xi) = +\infty \qquad \text{otherwise on } \mathrm{M}^N_D,$$

can then be seen to enjoy the same properties as in the pure two-phase case, provided that

$$B(0,c_3) \subset K_3 \subset K_1 \cap K_2.$$

We call this latter setting the *two-phase + interface case.* ¶

For every admissible plastic strain $p \in \mathcal{P}$, we define the dissipation functional as

$$\mathcal{H}(p) := \int_{\Omega\cup\Gamma^d} H\left(x, \frac{p}{|p|}\right) d|p|, \tag{2.9}$$

where $p/|p|$ denotes the Radon-Nikodym derivative of p with respect to its variation $|p|$.

If $t \mapsto p(t)$ is a map from $[0,T]$ to $\mathcal{M}_b(\Omega\cup\Gamma^d; \mathrm{M}^N_D)$, we also define, for every $[a,b] \subseteq [0,T]$,

$$\mathcal{D}(a,b;p) := \sup\left\{\sum_{j=1}^k \mathcal{H}\left(p(t_j) - p(t_{j-1})\right) \,:\, a = t_0 < t_1 < \dots < t_k = b\right\}$$

to be the *total dissipation* over the time interval $[a,b]$.

Body and traction forces: For simplicity, we do not consider any kind of force loads in this study. Adding those would only render the argument less legible. The results would be identical, provided that suitable *safe loads conditions* are satisfied (see [3, Section 2.2]).

Prescribed boundary displacements. The boundary displacement w on Γ^d for the time interval $[0,T]$ is given by the trace on Γ^d of some

$$w \in AC(0,T; H^1(\mathbb{R}^N;\mathbb{R}^N)). \tag{2.10}$$

In what follows, the energetic formulation of the quasi-static evolution is detailed in the footstep of [3]: the two ingredients of such evolutions are a stability statement at each time, together with an energy conservation statement that relates the total energy of the system to the work of the loads applied to that system.

Definition 2.5 (Energetic quasi-static evolution)**.** The mapping

$$t \mapsto (u(t), e(t), p(t)) \in \mathcal{A}(w(t))$$

is an energetic quasi-static evolution relative to w iff the following conditions hold for every $t \in [0,T]$:

(a) Global stability: for every $(v,\eta,q) \in \mathcal{A}(w(t))$

$$\mathcal{Q}(e(t)) \le \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)). \tag{2.11}$$

(b) Energy equality: $p \in BV\left(0,T; \mathcal{M}_b(\Omega\cup\Gamma^d;\mathrm{M}^N_D)\right)$ and

$$\mathcal{Q}(e(t)) + \mathcal{D}(0,t;p) = \mathcal{Q}(e(0)) + \int_0^t \int_\Omega \sigma(\tau)\cdot E\dot{w}(\tau)\,dx,$$

where $\sigma(t) := \mathbb{C}e(t)$.

The following result has been proved in [9, Theorem 3.14] for the continuous setting, or [5, Theorem 2.7] for the pure two-phase setting; note that, in either case, more general domains are admissible than those considered here:

Theorem 2.6 (Existence of quasi-static evolutions)**.** *Suppose that Ω is C^2-geometrically admissible multiphase domain. Assume that (2.4), (2.10) are satisfied, and let $(u_0, e_0, p_0) \in \mathcal{A}(w(0))$ satisfy the global stability condition (2.11). Finally, assume that the multi-map $x \multimap K(x)$ is either continuous, or corresponds to a pure two-phase case.*

Then there exists a quasi-static evolution $\{t \mapsto (u(t), e(t), p(t)), t \in [0, T]\}$ relative to the boundary displacement w such that $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$. Finally the Cauchy stress

$$t \mapsto \sigma(t) := \mathbb{C}e(t)$$

is uniquely determined by the initial conditions.

Remark 2.7. The following regularity property holds true (see [3, Theorem 5.2] and [5, Proposition 2.11]):

$$(u, e, p) \in AC\left(0, T; BD(\Omega) \times L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma^d; \mathrm{M}^N_D)\right)$$

with $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}(\dot{w}(t))$. Also, the total dissipation $\mathcal{D}(0, t; p)$ is absolutely continuous. ¶

The extent to which the afore mentioned energetic quasi-static evolutions are also *classical* evolutions is described in details in e.g. [5, Section 3]. For our purpose it suffices to note that

Remark 2.8. Any quasi-static evolution in the sense of Definition 2.5 satisfies the balance equations

$$\begin{cases} \mathrm{div}\,\sigma(t) = 0 \text{ in } \Omega, \\ \sigma(t)\nu = 0 \text{ on } \partial\Omega \setminus \overline{\Gamma}^d, \end{cases}$$

and the admissibility constraint in the phases

$$\sigma_D(t, x) \in K(x) \text{ for a.e. } x \in \Omega.$$ ¶

3 A model with a vanishing interfacial layer

In this section, we wish to view the two-phase behavior as the limit of a smoothly varying multi map $x \multimap K^\varepsilon(x)$ as the smoothing parameter ε tends to 0. To this effect we consider the following two continuously increasing multi-maps:

$$\tau \in [0,1] \multimap K_i(\tau) = \begin{cases} K_i, \ \tau = 1 \\ K_3, \ \tau = 0 \end{cases} \quad i = 1, 2, \tag{3.1}$$

where

$$K_3 \text{ is a closed convex subset of } K_1 \cap K_2, \ B(0, c_3) \subset K_3. \tag{3.2}$$

Remark 3.1. For example one could take $K_i(\tau) = \tau K_1 + (1 - \tau) K_3$. ¶

We then consider

$$\varphi^\varepsilon \in C^\infty(\mathbb{R}^N; [0,1]); \ \varphi^\varepsilon(x) := \begin{cases} 1, \ x \in \overline{\Omega}_1, & \mathrm{dist(x}, \Gamma) \ge \varepsilon \\ 1, \ x \in \overline{\Omega}_2, & \mathrm{dist(x}, \Gamma) \ge \varepsilon \\ 0, \ x \in \overline{\Omega}_1 \cup \overline{\Omega}_2, & \mathrm{dist(x}, \Gamma) \le \varepsilon/2. \end{cases} \tag{3.3}$$

and define

$$K^\varepsilon(x) := \chi_{\Omega_1 \cup \Gamma^d_1}(x) K_1(\varphi^\varepsilon(x)) + \chi_\Gamma(x) K_3 + \chi_{\Omega_2 \cup \Gamma^d_2}(x) K_2(\varphi^\varepsilon(x)). \tag{3.4}$$

The associated elasto-plastic model may be viewed as a two-phase model with a continuous transition to a smaller admissible set of stresses, namely K_3, near the boundary Γ.

Since the associated multi-map $x \multimap K^\varepsilon(x)$ is obviously continuous and satisfies (2.5), Theorem 2.6 applies and delivers an energetic quasi-static evolution

$$t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t)) \in \mathcal{A}(w(t))$$

with associated dissipation potential $H^\varepsilon(x,\xi) := \sup\{\tau \cdot \xi : \tau \in K^\varepsilon(x)\}$ and associated total dissipation $\mathcal{D}^\varepsilon(0,t;p^\varepsilon)$. Remark that, for $i = 1, 2$,

$$H^\varepsilon(x,\xi) = H_i(\xi),\ x \in \Omega_i \cup \Gamma_i^d,\ \mathrm{dist(x}, \Gamma) \geq \varepsilon, \tag{3.5}$$

whereas, since the maps $K_i(\tau)$ are increasing,

$$H^\varepsilon(x,\xi) \geq H_3(\xi),\ x \in \Omega \cup \Gamma^d. \tag{3.6}$$

Further, for simplicity sake, we assume that

$$w(0) = 0, \quad (u_0, e_0, p_0) = (0,0,0), \tag{3.7}$$

so that the initial minimizing state of the ε-problem is always $(0,0,0)$.

Define

$$K(x) := \begin{cases} K_1,\ x \in \Omega_1 \cup \Gamma_1^d \\ K_2,\ x \in \Omega_2 \cup \Gamma_2^d \\ \{\sigma_D \in \mathrm{M}_D^N : [\sigma_D \nu(x)]_\tau \in [K_3 \nu(x)]_\tau\},\ x \in \Gamma \setminus S \\ \mathrm{M}_D^N,\ \ x \in S \cup S'. \end{cases} \tag{3.8}$$

and the associated dissipation potential

$$H(x,\xi) = \begin{cases} H_1(\xi),\ x \in \Omega_1 \cup \Gamma_1^d \\ H_2(\xi),\ x \in \Omega_2 \cup \Gamma_2^d \\ H_3(a \otimes \nu(x)),\ x \in \Gamma \setminus S,\ \xi = a \otimes \nu(x), a \in \mathbb{R}^N, a \perp \nu(x) \\ \infty,\ \text{else}. \end{cases} \tag{3.9}$$

We also define, with obvious definitions, the dissipation potential $\mathcal{H}$ and the total dissipation $\mathcal{D}$.

In the context of Remark 2.4, the definitions above correspond to a two-phase + interface case. We propose to prove the following

Theorem 3.2 (An evolution for the two-phase + interface case)**.** *Assume that* Ω *is a* C^2-geometrically admissible multiphase domain *and that assumptions* (2.4)*,* (2.10)*,* (3.1)-(3.7) *are satisfied. Also assume the admissibility of* $\partial_{\lfloor \partial\Omega}\Gamma^d$ *(see Definition 2.1).*

There exists a subsequence of $\{\varepsilon\}$ *(that we do not relabel) and a quasi-static evolution* $t \mapsto (u(t), e(t), p(t))$ *relative to* w *in the sense of Definition 2.5 with*

$$(u(0), e(0), p(0)) = (0,0,0)$$

and H *defined through* (3.9) *such that*

$$\begin{aligned} u^\varepsilon(t) &\overset{*}{\rightharpoonup} u(t) \text{ weakly}^* \text{ in } BD(\Omega) \\ e^\varepsilon(t) &\to e(t) \text{ strongly in } L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}) \\ p^\varepsilon(t) &\overset{*}{\rightharpoonup} p(t) \text{ weakly}^* \text{ in } \mathcal{M}_b(\Omega \cup \Gamma^d; \mathrm{M}_D^N), \end{aligned}$$

for every $t \in [0,T]$*. Finally, for every* $t \in [0,T]$*,*

$$\lim_\varepsilon \mathcal{D}^\varepsilon(0,t;p^\varepsilon) = \mathcal{D}(0,t;p).$$

Remark 3.3. The theorem above implies in particular the existence of a quasi-static evolution for the dissipation potential associated to $K(x)$. The ensuing evolution is different from that obtained in the absence of the vanishing interface characterized by the admissible set K_3. Indeed the latter would correspond to an identical $K(x)$ except on $\Gamma \setminus S$ where it would be given through (2.7) whereas it is given here through (2.8). See further remarks in that direction in Section 4 below. ¶

Proof.
Step 1 - Bounds. The energy equality immediately implies that, for some $C > 0$ and every $t \in [0, T]$,

$$\|e^\varepsilon(t)\|_2 + \mathcal{V}_{\mathcal{M}_b(\Omega\cup\Gamma^d;\mathrm{M}^N_D)}(p^\varepsilon; 0, t) \le C.$$

Let $\Omega' \subseteq \mathbb{R}^N$ be open, bounded and such that $\Omega \cup \Gamma^d = \overline{\Omega} \cap \Omega'$. We extend $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ to Ω' by setting

$$u^\varepsilon(t) = w(t), \qquad e^\varepsilon(t) = Ew(t), \qquad p^\varepsilon(t) = 0 \qquad \text{on } \Omega' \setminus \Omega.$$

Clearly

$$Eu^\varepsilon(t) = e^\varepsilon(t) + p^\varepsilon(t) \qquad \text{on } \Omega'.$$

By a generalized version of Helly's theorem (see [7, Theorem 3.2]), there exists a subsequence, not relabeled, such that, for every $t \in [0, T]$,

$$p^\varepsilon(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega'; \mathrm{M}^N_D),$$

for some $p \in BV(0, T; \mathcal{M}_b(\Omega'; \mathrm{M}^N_D))$. For every $t \in [0, T]$, there exists a further subsequence $\{\varepsilon_t\}$ such that

$$e^{\varepsilon_t}(t) \rightharpoonup e(t) \qquad \text{weakly in } L^2(\Omega'; \mathrm{M}^N_{\mathrm{sym}}),$$

and, appealing to Korn's inequality in BD,

$$u^{\varepsilon_t}(t) \rightharpoonup u(t) \qquad \text{weakly}^* \text{ in } BD(\Omega'),$$

for some $u(t) \in BD(\Omega')$ with

$$Eu(t) = e(t) + p(t) \qquad \text{on } \Omega'.$$

Clearly $u(t) = w(t)$, $e(t) = Ew(t)$ and $p(t) = 0$ on $\Omega' \setminus \overline{\Omega}$, so that we deduce

$$p(t)\lfloor\Gamma^d = (w(t) - u(t)) \odot \nu\, \mathcal{H}^{N-1}\lfloor\Gamma^d.$$

By restricting $(u(t), e(t))$ to Ω and $p(t)$ to $\Omega \cup \Gamma^d$, we get

$$(u(t), e(t), p(t)) \in \mathcal{A}(w(t))$$

with

$$\begin{cases} u^{\varepsilon_t}(t) \overset{*}{\rightharpoonup} u(t) & \text{weakly}^* \text{ in } BD(\Omega), \\ e^{\varepsilon_t}(t) \rightharpoonup e(t) & \text{weakly in } L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}), \\ p^\varepsilon(t) \overset{*}{\rightharpoonup} p(t) & \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega \cup \Gamma^d; \mathrm{M}^N_D). \end{cases} \tag{3.10}$$

Step 2 - Stresses. Set $t \in [0, T]$. Since

$$\sigma^{\varepsilon_t}(t) = \mathbb{C}e^{\varepsilon_t}(t) \rightharpoonup \sigma(t) := \mathbb{C}e(t) \qquad \text{weakly in } L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}),$$

we deduce, using the balance equations in Remark 2.8, that

$$\begin{cases} \mathrm{div}\sigma(t) = 0 & \text{in } \Omega \\ \sigma(t)\nu = 0 & \text{on } \partial\Omega \setminus \bar{\Gamma}^d. \end{cases}$$

Concerning the stress constraint, the stress constraint in Remark 2.8 implies that

$$(\sigma^{\varepsilon_t})_D(t,x) \in K^\varepsilon(x) \qquad \text{for a.e. } x \in \Omega.$$

Since $\varphi^\varepsilon \equiv 1$ if $\mathrm{dist}(\mathrm{x}, \Gamma) \geq \varepsilon$, for ε small enough the previous constraint reduces to

$$(\sigma^{\varepsilon_t})_D(t,x) \in K_i \qquad \text{for a.e. } x \in A$$

on any $A \subset\subset \Omega_i$. Since K_i is convex and closed, we conclude that

$$\sigma_D(t,x) \in K_i \qquad \text{for a.e. } x \in \Omega_i,\ i = 1,2. \tag{3.11}$$

Now, on $\Gamma \setminus S$, $[\sigma_D^{\varepsilon_t}\nu]_\tau \in [K_3\nu]_\tau$ by the very definition of K^ε, so that, in particular,

$$\|[\sigma_D^{\varepsilon_t}(t)\nu]_\tau\|_\infty \leq C, \tag{3.12}$$

for some constant $C > 0$. But, as detailed earlier in Section 1.2, since $\Gamma \setminus S$ is a C^2-hypersurface, $[\sigma_D^{\varepsilon_t}\nu]_\tau$ is uniquely defined as the distribution $\sigma_D^{\varepsilon_t}(t)\nu - (\sigma_D^{\varepsilon_t}(t)\nu \cdot \nu)\nu$ on $\Gamma \setminus S$. That distribution converges to $\sigma_D(t)\nu - (\sigma_D(t)\nu \cdot \nu)\nu$ on on $\Gamma \setminus S$. But the latter is precisely $[\sigma_D(t)\nu]_\tau$.

Because of the bound (3.12), we conclude that

$$[\sigma_D^{\varepsilon_t}(t)\nu]_\tau \xrightharpoonup{*} [\sigma_D(t)\nu]_\tau, \text{weakly}^* \text{ in } L^\infty(\Gamma \setminus S).$$

Since the weak-$*$ limits of sequences of elements with values in $[K_3\nu]_\tau$ remain there in view of the convex and closed character of that set, we finally obtain that

$$[\sigma_D(t)\nu]_\tau \in [K_3\nu]_\tau \text{ on } \Gamma \setminus S. \tag{3.13}$$

Step 3 - Global Stability. Set $t \in [0,T]$. In view of (3.11),(3.13), an argument identical to that of [5, Proposition 3.9] would demonstrate that, for every $(v,\eta,q) \in \mathcal{A}(0)$,

$$H\left(x, \frac{q}{|q|}\right)|q| \geq \langle \sigma_D(t), q\rangle \text{ as measures on } \Omega \cup \Gamma^d. \tag{3.14}$$

Thanks to the admissibility of $\partial_{\lfloor \partial\Omega}\Gamma^d$, we can compute the masses and we obtain, in view of (2.2) (with $f \equiv g \equiv 0$),

$$\mathcal{H}(q) \geq -\int_\Omega \sigma(t) \cdot \eta\, dx.$$

The previous inequality immediately implies global stability by convexity of the quadratic form $\mathcal{Q}(e)$. In particular, as demonstrated in [5, Remark 2.6], $(u(t), e(t))$ is uniquely determined by $p(t)$, so that the convergences in (3.10) hold without passing to a t-dependent subsequence.

Step 4 - Lower semi-continuity of the dissipations. We argue at fixed $t \in [0,T]$. Set

$$p^\varepsilon(t) = p_1^\varepsilon + p_2^\varepsilon + p_3^\varepsilon,$$

where, for $i = 1,2$,

$$p_i^\varepsilon := p^\varepsilon(t)\lfloor(\Omega_i \cup \Gamma_i^d)$$

and

$$p_3^\varepsilon := p^\varepsilon(t)\lfloor\Gamma.$$

We can assume that, up to a (t-dependent) subsequence,

$$p_i^\varepsilon \xrightharpoonup{*} p_i \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega \cup \Gamma^d; \mathrm{M}_D^N), \tag{3.15}$$

for $i = 1,3$. Clearly,

$$p(t) = p_1 + p_2 + p_3$$

with $\operatorname{supp}(p_i) \subseteq \Omega_i \cup \Gamma \cup \Gamma_i^d \cup S'$ and $\operatorname{supp}(p_3) \subseteq \Gamma$, so that, in particular

$$p(t)\lfloor(\Omega_i \cup \Gamma_i^d) = p_i\lfloor(\Omega_i \cup \Gamma_i^d). \tag{3.16}$$

Further, according to [5, Lemma 5.1], for $i = 1, 2$,

$$p_i\lfloor(\Gamma \setminus S) = \mp a_i \odot \nu\, \lambda_i, \tag{3.17}$$

where ν is the normal to Γ pointing towards Ω_2, λ_i is a finite positive measure supported on $\Gamma \setminus S$ and a_i is a Borel function on $\Gamma \setminus S$ with $a_i \perp \nu$ λ_i-a.e. on $\Gamma \setminus S$.

Now, as far as p_3^ε is concerned, we have

$$p_3^\varepsilon = (u_2^\varepsilon - u_1^\varepsilon) \odot \nu\, \mathcal{H}^{N-1}\lfloor\Gamma,$$

where u_2^ε and u_1^ε are the traces of u^ε on Γ coming from Ω_2 and Ω_1 respectively. Since p_3^ε is a bounded measure on Γ, we immediately conclude that, for some $C > 0$,

$$\int_\Gamma |u_2^\varepsilon - u_1^\varepsilon| d\mathcal{H}^{N-1} \leq C,$$

so that, up to a subsequence that will not be relabeled,

$$(u_2^\varepsilon - u_1^\varepsilon)\mathcal{H}^{N-1}\lfloor\Gamma \overset{*}{\rightharpoonup} \eta = b|\eta|, \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Gamma; \mathbb{R}^N)$$

where b is the Borel Radon-Nikodym derivative of η with respect to its variation measure $|\eta|$. Since $x \mapsto \nu(x)$ is continuous on $\Gamma \setminus S$, we deduce that

$$p_3 = b \odot \nu|\eta|, \quad \text{on } \Gamma \setminus S. \tag{3.18}$$

Recalling (3.17), (3.18) and taking into account that p does not charge sets of $\mathcal{H}^{N-1}$-measure 0 while $\mathcal{H}^{N-1}(S) = 0$, we conclude that, in particular,

$$p(t)\lfloor\Gamma = -a_1 \odot \nu\lambda_1 + a_2 \odot \nu\lambda_2 + b \odot \nu|\eta| = c \odot \nu\, \zeta, \tag{3.19}$$

where $\zeta := \lambda_1 + \lambda_2 + |\eta|$ and c is a suitable Borel function on Γ.

Fix $\eta > 0$. In view of (3.5),(3.6), a direct application of Reshetnyak's lower-semi-continuity theorem (see e.g. [2, Theorem 2.38]) yields, for $i = 1, 2$,

$$\begin{aligned}
\liminf_\varepsilon \int_{\Omega\cup\Gamma^d} H^\varepsilon\left(x, \frac{p_i^\varepsilon}{|p_i^\varepsilon|}\right) d|p_i^\varepsilon| &= \liminf_\varepsilon \int_{\Omega'} H^\varepsilon\left(x, \frac{p_i^\varepsilon}{|p_i^\varepsilon|}\right) d|p_i^\varepsilon| \\
\geq \liminf_\varepsilon \int_{\{x\in\Omega' :\, \mathrm{dist(x},\Gamma)>\eta\}} & H_i\left(\frac{p_i^\varepsilon}{|p_i^\varepsilon|}\right) d|p_i^\varepsilon| + \liminf_\varepsilon \int_{\{x\in\Omega' :\, \mathrm{dist(x},\Gamma)<\eta/2\}} H_3\left(\frac{p_i^\varepsilon}{|p_i^\varepsilon|}\right) d|p_i^\varepsilon| \\
&\geq \int_{\{x\in\Omega' :\, \mathrm{dist(x},\Gamma)>\eta\}} H_i\left(\frac{p_i}{|p_i|}\right) d|p_i| + \int_{\{x\in\Omega' :\, \mathrm{dist(x},\Gamma)<\eta/2\}} H_3\left(\frac{p_i}{|p_i|}\right) d|p_i|.
\end{aligned}$$

Letting $\eta \searrow 0$ in the previous inequality and recalling (3.17) permits to conclude that, for $i = 1, 2$,

$$\begin{aligned}
\liminf_\varepsilon \int_{\Omega\cup\Gamma^d} H^\varepsilon\left(x, \frac{p_i^\varepsilon}{|p_i^\varepsilon|}\right) d|p_i^\varepsilon| &\geq \int_{\Omega_i\cup\Gamma_i^d} H_i\left(\frac{p_i}{|p_i|}\right) d|p_i| + \int_\Gamma H_3\left(\frac{p_i}{|p_i|}\right) d|p_i| \\
&\geq \int_{\Omega_i\cup\Gamma_i^d} H_i\left(\frac{p_i}{|p_i|}\right) d|p_i| + \int_{\Gamma\setminus S} H_3\left(\mp a_i \odot \nu\right) d\lambda_i.
\end{aligned} \tag{3.20}$$

Further, a second application of Reshetnyak's lower-semi-continuity theorem and (3.18) imply that

$$\begin{aligned}
\liminf_\varepsilon \int_\Gamma H^\varepsilon\left(x, \frac{p_3^\varepsilon}{|p_3^\varepsilon|}\right) d|p_3^\varepsilon| = \liminf_\varepsilon \int_\Gamma H_3\left(\frac{p_3^\varepsilon}{|p_3^\varepsilon|}\right) d|p_3^\varepsilon| &\geq \int_\Gamma H_3\left(\frac{p_3}{|p_3|}\right) d|p_3| \\
&\geq \int_{\Gamma\setminus S} H_3\left(b \odot \nu\right) d|\eta|.
\end{aligned} \tag{3.21}$$

Collecting (3.20) and (3.21), we obtain

$$\liminf_{\varepsilon} \mathcal{H}^{\varepsilon}(p^{\varepsilon}(t)) \geq \sum_{i=1,2} \left\{ \int_{\Omega_i \cup \Gamma_i^d} H_i \left(\frac{p_i}{|p_i|} \right) d|p_i| + \int_{\Gamma \setminus S} H_3 \left(\mp a_i \odot \nu \right) d\lambda_i \right\} + \int_{\Gamma \setminus S} H_3 \left(b \odot \nu \right) d|\eta|.$$

The sub-additive character of H_3, (3.16), (3.19) finally imply that

$$\liminf_{\varepsilon} \mathcal{H}^{\varepsilon}(p^{\varepsilon}(t)) \geq \sum_{i=1,2} \int_{\Omega_i \cup \Gamma_i^d} H_i \left(\frac{p_i}{|p_i|} \right) d|p_i| + \int_{\Gamma \setminus S} H_3(c \odot \nu) d\zeta = \sum_{i=1,2} \int_{\Omega_i \cup \Gamma_i^d} H_i \left(\frac{p(t)}{|p(t)|} \right) d|p(t)| + \int_{\Gamma} H_3 \left(\frac{p(t)}{|p(t)|} \right) d|p(t)|,$$

which establishes that

$$\liminf_{\varepsilon} \mathcal{H}^{\varepsilon}(p^{\varepsilon}(t)) \geq \mathcal{H}(p(t)). \tag{3.22}$$

Step 5 - Energy equality. For every $t \in [0, T]$, using (3.22), we get

$$\begin{aligned} \mathcal{Q}(e(t)) + \mathcal{D}(0, t; p) &\leq \liminf_{\varepsilon} \mathcal{Q}(e^{\varepsilon}(t)) + \liminf_{\varepsilon} \mathcal{D}(0, t; p^{\varepsilon}) \\ &\leq \liminf_{\varepsilon} \left[\mathcal{Q}(e^{\varepsilon}(t)) + \mathcal{D}(0, t; p^{\varepsilon}) \right] \\ &\leq \limsup_{\varepsilon} \left[\mathcal{Q}(e^{\varepsilon}(t)) + \mathcal{D}(0, t; p^{\varepsilon}) \right] \\ &= \int_0^t \int_{\Omega} \sigma(\tau) \cdot E\dot{w}(\tau) \, dx \, d\tau \leq \mathcal{Q}(e(t)) + \mathcal{D}(0, t; p). \end{aligned}$$

Above, the last equality is obtained by dominated convergence and the last inequality is a consequence of the global stability of $(u(t), e(t), p(t)) \in \mathcal{A}(w(t))$ proved in step 3; see the end of the proof of [5, Theorem 2.7, after equation (2.29)].

We conclude that the energy equality holds, so that $t \mapsto (u(t), e(t), p(t))$ is a quasi-static evolution for the two-phase + interface case according to Definition 2.5 and Remark 2.4. Moreover, the previous inequalities entail that

$$\lim_{\varepsilon} \left[\mathcal{Q}(e^{\varepsilon}(t)) + \mathcal{D}^{\varepsilon}(0, t; p^{\varepsilon}) \right] = \mathcal{Q}(e(t)) + \mathcal{D}(0, t; p)$$

from which we infer

$$\lim_{\varepsilon} \mathcal{Q}(e^{\varepsilon}(t)) = \mathcal{Q}(e(t)) \qquad \text{and} \qquad \lim_{\varepsilon} \mathcal{D}^{\varepsilon}(0, t; p^{\varepsilon}) = \mathcal{D}(0, t; p).$$

Thus in particular

$$e^{\varepsilon}(t) \to e(t) \qquad \text{strongly in } L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}),$$

which concludes the proof.

□

4 Remarks

In this last section, we put forth various short remarks concerning the evolution obtained in Theorem 2.6.

Interfacial stress admissibility. In the course of proving Theorem 2.6, we have established (see (3.13)) that

$$[\sigma_D(t)\nu]_{\tau} \in [K_3 \nu]_{\tau}, \ \mathcal{H}^{N-1} \text{a.e. on } \Gamma. \tag{4.1}$$

Flow Rule. Any solution of the quasi-static evolution given in Theorem 2.6 satisfies a flow rule as detailed in the following

Theorem 4.1 (Flow rule). *Consider a* C^2-geometrically admissible multiphase domain. *Also assume the admissibility of* $\partial_{\lfloor \partial\Omega}\Gamma^d$ *(see Definition 2.1). For a.e.* $t \in [0,T]$,

$$\frac{\dot p(t,x)}{|\dot p(t,x)|} \in N_{K(x)}(\sigma_D(t,x)) \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \{|\dot p(t| > 0\}.$$

Moroever,

$$\frac{\dot u_2(t,x) - \dot u_1(t,x)}{|\dot u_2(t,x) - \dot u_1(t,x)|} \in \vec{N}_{[K_3\nu(x)]_\tau}([\sigma_D(t)\nu]_\tau(x)) \text{ for } \mathcal{H}^{N-1} \text{ a.e. } x \in \{\dot u_1(t) \neq \dot u_2(t)\}, \tag{4.2}$$

where $\dot u_1(t)$ *and* $\dot u_2(t)$ *are the traces on* Γ *of the restrictions of* $\dot u(t)$ *to* Ω_1 *and* Ω_2 *respectively, assuming that* ν *points from* Ω_1 *to* Ω_2*, and where* $\vec{N}_{[K_3\nu(x)]_\tau}(\zeta)$ *denotes the normal cone – a cone of vectors – to* $[K_3\nu(x)]_\tau$ *at a vector* $\zeta \perp \nu(x)$.

Finally, there exists $[\sigma_D(t)\nu]_\tau$ *such that, for* $\mathcal{H}^{N-1}$*-a.e.* $x \in \Gamma_d \cap \overline{\Omega}_i$ *with* $\dot w(t,x) \neq \dot u(t,x)$,

$$\frac{\dot w(t,x) - \dot u(t,x)}{|\dot w(t,x) - \dot u(t,x)|} \in \vec{N}_{[K_i(x)\nu(x)]_\tau}([\sigma_D(t)\nu]_\tau(x)).$$

The proof will not be given here. It follows verbatim that of [5, Propositions 3.9, 3.11, and Theorem 3.13].

Note that, in the pure two-phase case, the interfacial flow rule is different; in lieu of (4.2), one has, according to [5, Theorem 3.13],

$$\frac{\dot u_2(t,x) - \dot u_1(t,x)}{|\dot u_2(t,x) - \dot u_1(t,x)|} \in \vec{N}_{[K_1\nu(x)]_\tau \cap [K_2\nu(x)]_\tau}([\sigma_D(t)\nu]_\tau(x)) \text{ for } \mathcal{H}^{N-1} \text{ a.e. } x \in \{\dot u_1(t) \neq \dot u_2(t)\}.$$

So the interfacial effect due to the presence of the vanishing layer is felt in the admissibility rule (4.1), as well as in the flow rule (4.2).

Uniqueness of the Stress. It can be established (see e.g. [5, Remark 2.6]) that the Cauchy stress

$$t \mapsto \sigma(t) := \mathbb{C}e(t)$$

is uniquely determined by the initial conditions. Consequently, *any* quasi-static evolution for the two-phase + interface case will be such that

$$[\sigma_D(x,t)\nu(x)]_\tau \in [K_3\nu(x)]_\tau,\ \mathcal{H}^{N-1} \text{ - a.e. on } \Gamma,\ t \in [0,T],$$

whereas the balance equations (see Remark 2.8) and the stress admissibility constraints on each phase only permit *a priori* to assert that

$$[\sigma_D(x,t)\nu(x)]_\tau \in [(K_1 \cap K_2)\nu(x)]_\tau,\ \mathcal{H}^{N-1} \text{ - a.e. on } \Gamma,\ t \in [0,T].$$

Dissipation. In order to secure the lower semi-continuity of the dissipations in the fourth step of the proof of Theorem 2.6, we had to assume that $K_3 \subset K_1 \cap K_2$, so that, correspondingly, $H_3(\xi) \le H_i(\xi),\ i = 1,2$. Barring this, the limit process fails.

A direct proof of the existence of an energetic quasi-static evolution for a two phase + interface evolution could be produced in the spirit of that of [5, Theorem 2.7]. The main hurdle, that is the lower semi-continuous character of the dissipation H defined in (3.9) would become impossible to prove whenever K_3 is not a subset of $K_1 \cap K_2$.

Although, as stated above, one can prove directly the existence of an energetic quasi-static evolution for a two phase + interface evolution, two results cannot be achieved through such a direct proof: the interfacial stress condition (4.1) and the interfacial flow rule (4.2). The approximation process devised in Section 3 is instrumental in deriving (4.1) from which (4.2) can then be obtained as in the proof of Theorem 4.1.

So, any elasto-plastic model for a two-phase + interface model will have a dissipation on the interface Γ which is less than that of the pure two-phase case, and correspondingly a set of admissible stresses on the interface that is smaller than $K_1 \cap K_2$. Thus, the pure two-phase case can be seen as the maximally dissipative interfacial model compatible with the bulk dissipations.

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