Passing to the limit in maximal slope curves: from a regularized Perona-Malik equation to the total variation flow

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Abstract

We prove that solutions of a mildly regularized Perona-Malik equation converge, in a slow time scale, to solutions of the total variation flow. The convergence result is global-in-time, and holds true in any space dimension.

The proof is based on the general principle that "the limit of gradient-flows is the gradient-flow of the limit". To this end, we exploit a general result relating the Gamma-limit of a sequence of functionals to the limit of the corresponding maximal slope curves.

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1 Introduction

The Perona-Malik equation

$$u_t = \mathrm{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \qquad (x,t) \in \Omega \times (0,+\infty) \tag{1.1}$$

(where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set) is arguably the most celebrated example of forward-backward diffusion process. It was introduced by P. Perona and J. Malik [34] in the context of image denoising. It is formally the gradient-flow of the functional

$$PM(u) := \frac{1}{2}\int_\Omega \log\left(1+|\nabla u(x)|^2\right)\,dx. \tag{1.2}$$

The forward-backward nature of (1.1) depends on the convex-concave behavior of the integrand in (1.2). Equation (1.1) has generated a considerable literature (see [3, 7, 8, 9, 10, 11, 13, 14, 15, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 37, 38]), focussed both on numerical and on analytical aspects. The big challenge is to reconcile the empirical and practical efficacy of the method, supported by several numerical observations, with the expected analytical ill-posedness of a backward parabolic equation. A satisfactory theory would represent a solution to the Perona-Malik paradox, as named after [31], but it still seems to be out of reach.

In a recent paper, P. Guidotti [30] introduced a mild regularization of (1.1). He considered the family of functionals

$$PM_\delta(u) := \frac{1}{2}\int_\Omega \log\left(1+|\nabla u(x)|^2\right)\,dx + \delta\int_\Omega |\nabla u(x)|^2\,dx,$$

where $\delta > 0$ is a parameter. The integrand remains nonconvex, at least in the interesting cases where δ is small, but now it is convex-concave-convex, and it grows quadratically at infinity. This is enough to guarantee that the corresponding gradient-flow equation, which simply reads as

$$u_t = \mathrm{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right) + \delta\,\Delta u \qquad (x,t) \in \Omega \times (0,+\infty), \tag{1.3}$$

has a unique global-in-time solution in the sense of Young measures, according to the theory developed in [32, 19].

Several qualitative properties of these solutions are reported in [30]. In particular, it seems that the well-known staircasing effect of the original Perona-Malik equation is now replaced by a "ramping" effect, namely the tendency of solutions to alternate flat plateaus and bounded growth regions in a piecewise fashion (see all figures in [30]).

In [30] it is also observed that the limit of approximated solutions as $\delta \to 0^+$ is the trivial stationary solution frozen in the initial condition, and "thus the only way to produce a meaningful limit would involve a rescaling of time in the process".

In this paper we follow this path. We take a family of solutions $v_\delta(t)$ of the regularized model (with initial conditions $v_\delta(0)$ converging to some u_0), and then we speed up the evolution by considering the family of functions

$$u_\varepsilon(t) := v_{4^{-1}\varepsilon^2|\log\varepsilon|}\left(\frac{t}{\varepsilon|\log\varepsilon|}\right) \qquad t \geq 0. \tag{1.4}$$

This happens to be the right rescaling factor, in the sense that $u_\varepsilon(t)$ uniformly converges to a nontrivial limit $u(t)$ as $\varepsilon \to 0^+$. Moreover $u(t)$ turns out to be the solution of the total variation flow (see [6, 4])

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \qquad (x,t) \in \Omega \times (0,+\infty), \tag{1.5}$$

with initial condition u_0. Any slower time rescaling leads in the limit to the stationary solution $u(t) \equiv u_0$, and any faster time rescaling produces a limit which for every $t > 0$ coincides with the constant function equal to the average of u_0 in Ω.

All qualitative properties of the total variation flow (see [5]) are consistent with the numerical experiments presented in [30]. Therefore, our convergence result provides a rigorous justification of all these properties.

The proof of the convergence result involves three main steps. First of all, we interpret $u_\varepsilon(t)$ as a gradient-flow. As it comes from [30], $u_\varepsilon(t)$ is the solution in the sense of Young measures of the forward-backward equation associated to the formal gradient-flow of the nonconvex energy

$$E_\varepsilon(u) := \int_\Omega \varphi_\varepsilon\left(|\nabla u(x)|\right)\,dx, \tag{1.6}$$

where

$$\varphi_\varepsilon(\sigma) := \frac{1}{2\varepsilon|\log\varepsilon|}\log\left(1+\sigma^2\right) + \frac{\varepsilon}{4}\sigma^2.$$

The key point is that $u_\varepsilon(t)$ is also the gradient-flow of the relaxed (convex) energy

$$E_\varepsilon^{**}(u) := \int_\Omega \varphi_\varepsilon^{**}\left(|\nabla u(x)|\right)\,dx, \tag{1.7}$$

where φ_ε^{**} is the convexification of φ_ε.

In other words, as a result of the first step we can forget about Young measures and forward-backward equations, and think of $u_\varepsilon(t)$ as the solution of a degenerate forward parabolic equation, or better as the gradient-flow of a convex (although not strictly convex) functional. We stress that this is a general fact. Solutions provided by the theory developed in [32, 19] always coincide with gradient-flows of the corresponding convexified (or relaxed) energies. Quite surprisingly, this has never been observed in the literature up to our knowledge.

In the second step we compute the Gamma-limit of the energies (Theorem 2.1), and we discover that

$$\Gamma\!-\!\lim_{\varepsilon\to 0^+} E_\varepsilon(u) = \Gamma\!-\!\lim_{\varepsilon\to 0^+} E_\varepsilon^{**}(u) = TV(u), \tag{1.8}$$

where $TV(u)$ denotes the total variation of u. Since the total variation flow is the gradient-flow of $TV(u)$, our convergence result is now equivalent to say that the limit of gradient-flows is the gradient-flow of the Gamma-limit.

This is the content of the third step, where we deduce it from a general result (Theorem 3.1) which we state and prove in the abstract setting of *maximal slope curves* in metric spaces (see Section 3 for further details and references). We believe that the scope of the general result goes far beyond this simple application.

We conclude by discussing how this problem relates to recent investigations about the "slow time" behavior of approximations of the Perona-Malik equation. The starting point is the observation that the evolution, despite of the different approximation methods, seems to exhibit always three different time scales, named "fast time", "standard time", and "slow time" according to [8].

In a fast time of order $o(1)$ solutions develop microstructures in the concave regime (staircasing). In a time scale of order $O(1)$ (standard time) solutions behave as expected in the original model, with a smoothing effect in the concave regime, and sharpening of regions where the gradient is large. At the "end" of standard time, solutions have an almost piecewise constant structure, and this is consistent with the intuitive idea that piecewise constant functions are stationary points of $PM(u)$. On the other hand, only constant functions (and not piecewise constant functions) are stationary points of the usual approximating models. As a consequence, approximating solutions exhibit a transition from a piecewise constant structure to a constant value (equal to the average of u_0). The transition turns out to be very slow because there is almost no energy left.

This gives rise to the so called "slow time" motion, in which the plateaus of the piecewise constant function move in the vertical direction, with jump points which remain fixed in space. Although the existence of this phase seems to be independent of the approximation method, the law of the vertical motion does depend on it.

All previous results on this problem are limited to the one dimensional case. Let u be a piecewise constant function defined in an interval, and let S_u be the (finite or countable) set of its jump points. Let J_x denote the jump height in a point $x \in S_u$, and let us consider the following energies

$$H_\alpha(u) := \sum_{x\in S_u} |J_x|^\alpha \quad (\text{with } \alpha \in (0,1]), \qquad H_0(u) := \sum_{x\in S_u} \log |J_x|.$$

In the case of a fourth order regularization of (1.1), corresponding to adding a vanishing second order term to (1.2), G. Bellettini and A. Fusco [8] conjectured that the vertical motion is governed by the gradient-flow of $H_{1/2}(u)$. They supported their conjecture by proving the corresponding Gamma-limit result for the energies. The missing step is a rigorous proof that also in that case the limit of gradient-flows is the gradient-flow of the limit. Unfortunately Theorem 3.1 does not apply to their functionals.

In the case of the semidiscrete scheme (see [10, 15]) the vertical motion is governed by the gradient-flow of $H_0(u)$, which in a certain sense represents a limit case. The proof given in [15] exploits a variant of Theorem 3.1, complicated by the fact that $H_0(u)$ is not bounded from below.

What we show in this paper is that in the model proposed in [30] the slow-time vertical motion is governed by the gradient-flow of $H_1(u)$. This is the opposite limit case, and phenomena are completely different. The good news is that the limit energy is convex. This simplifies the analysis, which here can be carried out in any space dimension, and delivers a well known limit problem. On the other hand, the relaxation of $H_1(u)$ is $TV(u)$, hence it is finite in the whole space of bounded variation functions. As a consequence, in this case the slow time motion is not limited to piecewise constant functions. This is hardly surprising after reminding that in this model the motion in standard time is trivial.

This paper is organized as follows. In Section 2 we fix notations and we state our convergence results. Section 3 is devoted to limits of maximal slope curves in metric spaces. In Section 4 we prove our main result.

2 Notations and statements

Let n be a positive integer, and let $\Omega \subseteq \mathbb{R}^n$ be an open set, which for simplicity we assume to be bounded and an extension domain (see Definition 3.20 in [1], satisfied by all bounded open sets with Lipschitz boundary). The more general ambient space we consider is $L^2(\Omega)$. We write $\|u\|_{L^p(\Omega)}$, or simply $\|u\|_p$, to denote the p-norm (with $p \in [1, +\infty])$ of a function $u \in L^p(\Omega)$. All the energies we consider, and in particular $E_\varepsilon(u)$ and $E_\varepsilon^{**}(u)$, are always thought as defined in the whole space $L^2(\Omega)$ by setting them equal to $+\infty$ outside their natural domain. We write $BV(\Omega)$ to denote the space of all functions $u \in L^2(\Omega)$ with finite total variation $TV(u)$. Once again, we think of $TV(u)$ as defined for every $u \in L^2(\Omega)$, with $TV(u) < +\infty$ if and only if $u \in BV(\Omega)$.

For every $\delta > 0$ we consider equation (1.3), with Neumann boundary conditions, and an initial datum. It has been shown in [30] that solutions $v_\delta(t)$ exist in a suitable weak sense. For every $\varepsilon \in (0, 1)$, we define $u_\varepsilon(t)$ by rescaling $v_\delta(t)$ according to (1.4). It turns out that $u_\varepsilon(t)$ is a solution in the same weak sense of equation

$$u_{\varepsilon t} = \frac{1}{\varepsilon|\log \varepsilon|} \operatorname{div}\left(\frac{\nabla u_\varepsilon}{1 + |\nabla u_\varepsilon|^2}\right) + \frac{\varepsilon}{4} \Delta u_\varepsilon \qquad (x, t) \in \Omega \times (0, +\infty), \tag{2.1}$$

with Neumann boundary conditions

$$\frac{\partial u_\varepsilon}{\partial n}(x, t) = 0 \qquad (x, t) \in \partial\Omega \times (0, +\infty), \tag{2.2}$$

and initial condition

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x) \qquad x \in \Omega. \tag{2.3}$$

In the following result we collect properties of $u_\varepsilon(t)$.

Theorem A (Properties of rescaled approximating solutions) *Let n be a positive integer, let $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain, let $\varepsilon \in (0,1)$, and let $u_{0\varepsilon} \in L^2(\Omega)$. Let E_ε^{**} be the functional defined in (1.7).*

Then the following properties hold true.

(1) (Weak Young measure solution and regularity) *There exists a unique function $u_\varepsilon(t)$ and a (not necessarily unique) gradient Young measure ν_ε in $\Omega \times [0,+\infty)$ such that the pair $(u_\varepsilon, \nu_\varepsilon)$ is a weak Young measure valued solution of problem (2.1) through (2.3) in the sense of [32, 19].*

Moreover, we have that

$$u_\varepsilon \in C^0\left([0,+\infty); L^2(\Omega)\right) \cap C^1\left((0,+\infty); L^2(\Omega)\right),$$

and for every $t > 0$ we have that $u_\varepsilon(t)$ is regular enough so that the right-hand side of (2.4) lies in $L^2(\Omega)$.

(2) (Degenerate forward parabolic equation) *The function u_ε of statement (1) is the unique solution in $\Omega \times (0,+\infty)$ of the partial differential equation*

$$u_{\varepsilon t} = -\nabla E_\varepsilon^{**}(u_\varepsilon) = \operatorname{div}\left[(\varphi_\varepsilon^{**})'\left(|\nabla u_\varepsilon|\right)\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right], \tag{2.4}$$

with Neumann boundary conditions (2.2), and initial condition (2.3).

(3) (Gradient-flow integral inequality) *The function u_ε of statement (1) is the unique function satisfying (2.3) and the inequality*

$$E_\varepsilon^{**}(u_\varepsilon(s)) - E_\varepsilon^{**}(u_\varepsilon(t)) \geq \frac{1}{2}\int_s^t \|u_\varepsilon'(\tau)\|_2^2\,d\tau + \frac{1}{2}\int_s^t \|\nabla E_\varepsilon^{**}(u_\varepsilon(\tau))\|_2^2\,d\tau \tag{2.5}$$

for every $0 \leq s \leq t$.

(4) (L^p estimate) *If $u_{0\varepsilon} \in L^p(\Omega)$ for some $p \in [1,+\infty]$, then $u_\varepsilon(t) \in L^p(\Omega)$ for every $t \geq 0$, and the function $t \to \|u_\varepsilon(t)\|_{L^p(\Omega)}$ is nonincreasing.*

Theorem A above shows that the function $u_\varepsilon(t)$ can be characterized in at least three different ways, either as the solution in the sense of Young measures of a forward-backward equation, or as the solution of a forward degenerate parabolic equation (for example in the sense of [12]), or as a maximal slope curve (gradient-flow inequalities).

What we need in this paper is only the last one. Solutions generate a contraction semigroup in $L^2(\Omega)$.

As far as we know this equivalence has never been stated explicitly. Nevertheless, it follows from some general facts which nowadays are quite well known, and which we now recall briefly.

First of all, the three approaches lead to a *unique* solution. Uniqueness follows in the first case from the strong requirements imposed on the structure of the corresponding Young measure, in the second case from the contraction property of the semigroup generated by a forward parabolic equation, in the third case from the convexity of the energy $E_\varepsilon^{**}(u)$.

Secondly, in all three approaches the solution is usually obtained (or at least it can be obtained) as the limit of approximated solutions constructed via an iterated minimization process, known as minimizing movement (see [17]). As already observed in [32, 19, 30], the minimization procedure gives the same result when applied to $E_\varepsilon(u)$ or $E_\varepsilon^{**}(u)$.

In conclusion, all three approaches define a unique solution through an analogous procedure, hence the solution is the same.

In this paper we are interested in the behavior of $u_\varepsilon(t)$ as $\varepsilon \to 0^+$. Following the gradient-flow approach, the first thing to do is understanding the limit behavior of the energies. This is the content of next result.

Theorem 2.1 (Gamma-convergence and compactness) *Let n be a positive integer, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain. For every $\varepsilon > 0$, let $E_\varepsilon : L^2(\Omega) \to [0, +\infty]$ and $E_\varepsilon^{**} : L^2(\Omega) \to [0, +\infty]$ be defined as in (1.6) and (1.7), respectively, if $u \in H^1(\Omega)$, and $+\infty$ otherwise.*

Then we have the following conclusions.

(1) (Gamma-convergence) *We have that (1.8) holds true with respect to the topology of $L^2(\Omega)$.*

(2) (Compactness) *Let $\{u_\varepsilon\}_{\varepsilon \in (0,1)} \subseteq L^2(\Omega)$ be a family of functions such that*

$$\sup_{\varepsilon \in (0,1)} \left\{ \|u_\varepsilon\|_\infty + E_\varepsilon^{**}(u_\varepsilon) \right\} < +\infty, \tag{2.6}$$

Then the family $\{u_\varepsilon\}$ is relatively compact in $L^2(\Omega)$.

The gradient-flow of the limit functional $TV(u)$ is the so called *total variation flow*, and the corresponding partial differential equation is (1.5). The right-hand side of (1.5) needs to be interpreted in a suitable weak sense when the gradient vanishes, and this happens in large regions because solutions tend to develop flat plateaus. A general existence and uniqueness result was proved by F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón [4] (see also [5]) using the theory of accretive operators in Banach spaces. In [4] the operator in the right-hand side of (1.5) is interpreted as the limit of the p-Laplacian as $p \to 1^+$.

In this paper we limit ourselves to initial data in $L^2(\Omega)$, in which case existence of a unique solution is provided also by the theory of maximal monotone operators [12], as explained in [6]. In this context the right-hand side of (1.5) is the subdifferential of the convex functional $TV(u)$. As in the case of approximating problems, this formulation is equivalent to the gradient-flow integral inequality

$$TV(u(s)) - TV(u(t)) \geq \frac{1}{2}\int_s^t \|u'(\tau)\|_2^2\,d\tau + \frac{1}{2}\int_s^t \|\nabla TV(u(\tau))\|_2^2\,d\tau$$

for every $0 \leq s \leq t$, where $\|\nabla TV(u(\tau))\|_2$ is the minimal norm of an element in the subdifferential of the functional TV in the point $u(\tau)$, which in turn coincides with the slope of TV in $u(\tau)$ as defined in Section 3 in an abstract metric setting. This is the characterization of the total variation flow which we need in this paper.

Our main result is the convergence of $u_\varepsilon(t)$ to the solution of the total variation flow with the same boundary conditions, and the limit initial datum. We point out that we do not assume initial data to be a recovery sequence, and we do not ask their energy to be bounded.

Theorem 2.2 (Global-in-time convergence) *Let n be a positive integer, let $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain, let $u_0 \in L^2(\Omega)$, and let $\{u_{0\varepsilon}\}_{\varepsilon\in(0,1)} \subseteq L^2(\Omega)$ be a family of functions such that*

$$\lim_{\varepsilon\to 0^+} u_{0\varepsilon} = u_0 \qquad \text{in } L^2(\Omega). \tag{2.7}$$

For every $\varepsilon \in (0,1)$, let u_ε be the solution of the rescaled approximating problem with initial condition $u_{0\varepsilon}$, in the sense of Theorem A. Let $u(t)$ be the solution of the total variation flow with Neumann boundary conditions and initial datum u_0.

Then we have that $u_\varepsilon(t) \to u(t)$ in $C^0([0,+\infty); L^2(\Omega))$, namely

$$\lim_{\varepsilon\to 0^+} \sup_{t\geq 0} \|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} = 0. \tag{2.8}$$

We conclude this section with a heuristic argument which justifies the rescaling leading to (1.6).

Let us consider the one dimensional case where $\Omega = (-1,1)$, let $J > 0$, and let $v \in BV((-1,1))$ be the piecewise constant function equal to $-J/2$ in $(-1,0)$, and equal to $J/2$ in $(0,1)$. Let $\eta > 0$, and let $h \in H^1(\mathbb{R})$ be a function such that $h(x) = J/2$ for every $x \geq \eta$, $h(x) = -J/2$ for every $x \leq -\eta$, and $h'(x) > 0$ for almost every $x \in (-\eta,\eta)$. For every $\varepsilon > 0$, let $v_\varepsilon(x)$ be the approximation of v defined as $v_\varepsilon(x) := h(x/\varepsilon)$ for every $x \in (-1,1)$.

For every $\varepsilon \leq \eta^{-1}$, plugging v_ε into (1.6), with a variable change we obtain that

$$E_\varepsilon(v_\varepsilon) = \frac{1}{2|\log\varepsilon|}\int_{-\eta}^{\eta} \log\left(1 + \frac{1}{\varepsilon^2}\left[h'(x)\right]^2\right)dx + \frac{1}{4}\int_{-\eta}^{\eta}\left[h'(x)\right]^2 dx.$$

Letting $\varepsilon \to 0^+$ we find that

$$\liminf_{\varepsilon \to 0^+} E_\varepsilon(v_\varepsilon) \geq 2\eta + \frac{1}{4}\int_{-\eta}^{\eta} [h'(x)]^2 \, dx. \tag{2.9}$$

In order to estimate the right-hand side, we first minimize with respect to h, and we discover that the optimal choice is the function $h(x)$ defined as $J(2\eta)^{-1}x$ for every $x \in (-\eta, \eta)$. Then we compute the integral, and finally we apply the inequality between arithmetic and geometric mean to deduce that

$$2\eta + \frac{1}{4}\int_{-\eta}^{\eta} [h'(x)]^2 \, dx \geq 2\eta + \frac{J^2}{8\eta} \geq J. \tag{2.10}$$

Estimates (2.9) and (2.10) suggest that the cost of a jump could be the jump height, which leads to conjecture that the Gamma-limit is $TV(u)$ for a general u in any dimension.

3 Passing to the limit in maximal slope curves

The abstract theory of gradient-flows in metric spaces was introduced in [16], and then developed by the same authors and collaborators in a series of papers (see [18, 33] and the references quoted therein). For a modern presentation we refer to [2]. Here we just recall some basic definitions.

Let (X, d) be a metric space, let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended real line, and let $F : X \to \overline{\mathbb{R}}$ be any function. The (descending) *slope* $|\nabla F|(x)$ of F in x is defined to be $+\infty$ if $F(x) \notin \mathbb{R}$, and otherwise

$$|\nabla F|(x) := \limsup_{y \to x} \frac{\max\{F(x) - F(y), 0\}}{d(x,y)} \in [0, +\infty].$$

For every $T > 0$, the space $AC^2([0,T]; X)$ is the set of all functions $v : [0,T] \to X$ for which there exists $g \in L^2((0,T))$ such that

$$d(v(t), v(s)) \leq \int_s^t g(\tau)\, d\tau \qquad \forall\, 0 \leq s \leq t \leq T. \tag{3.1}$$

It can be seen that there exists a smallest function $g(t)$ satisfying (3.1). This function is called the *metric derivative* of v, and it is denoted by $|v'|(t)$.

A *maximal slope curve* for F in $[0,T]$ is a triple (u, ψ, E) where

- $u \in AC^2([0,T]; X)$,

- $\psi : [0,T] \to \mathbb{R}$ is a nonincreasing function such that for every $0 \leq s \leq t \leq T$ we have that

$$\psi(s) - \psi(t) \geq \frac{1}{2}\int_s^t |u'|^2(\tau)\,d\tau + \frac{1}{2}\int_s^t |\nabla F|^2\,(u(\tau))\,d\tau, \tag{3.2}$$

- $E \subseteq [0,T]$ is a set with Lebesgue measure equal to 0 such that

$$\psi(t) = F(u(t)) \qquad \forall t \in [0,T] \setminus E. \tag{3.3}$$

To be more precise, the second integral in the right-hand side of (3.2) should be an upper integral, since at this level of generality there is no reason for the function $t \to |\nabla F|(u(t))$ to be measurable. On the other hand, it can be easily proved that actually it is always true that $|u'|(t) = |\nabla F|(u(t))$ for almost every $t \in [0,T]$, which implies the required measurability.

When F is a C^1 function in a Hilbert space X, this weak formulation is equivalent to the classical one, namely to asking that $u'(t) = -\nabla F(u(t))$ for every $t \in [0,T]$.

Besides generality, the advantage of this weak formulation is that inequalities and integrals are more stable than equalities and derivatives. It follows that maximal slope curves exist under general assumptions on F (see Theorem 2.3.1 in [2]), and are quite stable when passing to the limit, both with respect to initial conditions, and with respect to functionals. Results in this direction are contained in [18] and [35] in a Hilbert setting, and in [36] in a metric setting, but assuming that initial data are a recovery sequence.

Here we state a quite general result, used in a special case also in [15].

Theorem 3.1 (Limits of maximal slope curves) *Let X be a metric space, let $F : X \to \overline{\mathbb{R}}$ be a function, and let $F_n : X \to \overline{\mathbb{R}}$ be a sequence of functions.*

Let us assume that for every $x \in X$, and every pair of sequences $\{n_k\} \subseteq \mathbb{N}$ and $\{x_k\} \subseteq X$, we have the implication

$$\boxed{\begin{array}{r} n_k \to +\infty,\ x_k \to x \\ \sup\limits_{k\in\mathbb{N}} \left\{|F_{n_k}(x_k)| + |\nabla F_{n_k}|(x_k)\right\} < +\infty \end{array}} \Longrightarrow \boxed{\begin{array}{l} \lim\limits_{k\to+\infty} F_{n_k}(x_k) = F(x). \\ \liminf\limits_{k\to+\infty} |\nabla F_{n_k}|(x_k) \geq |\nabla F|(x). \end{array}} \tag{3.4}$$

Let $T > 0$, and for every $n \in \mathbb{N}$ let (u_n, ψ_n, E_n) be a maximal slope curve for F_n in $[0,T]$. Let us assume that

$$\sup_{n\in\mathbb{N}} \max_{t\in[0,T]} |\psi_n(t)| < +\infty, \tag{3.5}$$

and that $\{u_n\}$ has a pointwise limit, namely there exists $u : [0,T] \to X$ such that

$$\lim_{n\to+\infty} u_n(t) = u(t) \qquad \forall t \in [0,T].$$

Then there exist ψ and E such that (u, ψ, E) is a maximal slope curve for F.

Proof From the definition of maximal slope curve, for every $n \in \mathbb{N}$ we have that

$$\psi_n(s) - \psi_n(t) \geq \frac{1}{2} \int_s^t |u_n'|^2(\tau)\, d\tau + \frac{1}{2} \int_s^t |\nabla F_n|^2 (u_n(\tau))\, d\tau \tag{3.6}$$

for every $0 \leq s \leq t \leq T$, and

$$\psi_n(t) = F_n(u_n(t)) \qquad \forall t \in [0, T] \setminus E_n. \tag{3.7}$$

The functions $\psi_n(t)$ are nonincreasing, and equi-bounded because of (3.5). Thus the usual compactness result for monotone functions (known as Helly's Lemma, see for example [2, Lemma 3.3.3]) implies the existence of a nonincreasing function $\psi : [0, T] \to [0, +\infty)$ such that (up to subsequences, not relabeled)

$$\lim_{n \to +\infty} \psi_n(t) = \psi(t) \qquad \forall t \in [0, T]. \tag{3.8}$$

Let us consider now the right-hand side of (3.6). Setting $s = 0$ and $t = T$, and using once more assumption (3.5), we obtain that

$$\sup_{n \in \mathbb{N}} \int_0^T |u_n'|^2(\tau)\, d\tau < +\infty, \tag{3.9}$$

$$\sup_{n \in \mathbb{N}} \int_0^T |\nabla F_n|^2 (u_n(\tau))\, d\tau < +\infty. \tag{3.10}$$

From (3.9) we easily deduce that $u \in AC^2([0, T]; X)$, and

$$\liminf_{n \to +\infty} \int_s^t |u_n'|^2(\tau)\, d\tau \geq \int_s^t |u'|^2(\tau)\, d\tau \qquad \forall\, 0 \leq s \leq t \leq T. \tag{3.11}$$

From (3.10) and Fatou's Lemma we obtain that

$$\int_0^T \left(\liminf_{n \to +\infty} |\nabla F_n|^2 (u_n(\tau)) \right) d\tau \leq \liminf_{n \to +\infty} \int_0^T |\nabla F_n|^2 (u_n(\tau))\, d\tau < +\infty,$$

hence there exists a set $E' \subseteq [0, T]$, with Lebesgue measure equal to 0, such that

$$\liminf_{n \to +\infty} |\nabla F_n| (u_n(t)) < +\infty \qquad \forall t \in [0, T] \setminus E'. \tag{3.12}$$

Let us introduce the set

$$E := E' \cup \left(\bigcup_{n \in \mathbb{N}} E_n \right),$$

which has clearly Lebesgue measure equal to 0. Let us consider any $t \in [0, T] \setminus E$. Since $t \notin E_n$, from (3.7) and (3.5) we have that

$$\sup_{n \in \mathbb{N}} |F_n(u_n(t))| = \sup_{n \in \mathbb{N}} |\psi_n(t)| < +\infty. \tag{3.13}$$

Moreover, due to (3.12) there exists a (t-dependent) sequence $n_k \to +\infty$ such that

$$\lim_{k\to+\infty} |\nabla F_{n_k}|(u_{n_k}(t)) = \liminf_{n\to+\infty} |\nabla F_n|(u_n(t)) < +\infty. \tag{3.14}$$

Thanks to (3.13) and (3.14), sequences $\{n_k\}$ and $\{u_{n_k}(t)\}$ satisfy the assumptions in the left-hand side of (3.4). If follows that for every $t \in [0,T] \setminus E$ we have that

$$\psi(t) = \lim_{n\to+\infty} \psi_n(t) = \lim_{k\to+\infty} \psi_{n_k}(t) = \lim_{k\to+\infty} F_{n_k}(u_{n_k}(t)) = F(u(t)),$$

which proves (3.3), and

$$|\nabla F|(u(t)) \le \liminf_{k\to+\infty} |\nabla F_{n_k}|(u_{n_k}(t)) = \lim_{k\to+\infty} |\nabla F_{n_k}|(u_{n_k}(t)) = \liminf_{n\to+\infty} |\nabla F_n|(u_n(t)),$$

so that one more application of Fatou's Lemma gives that

$$\int_s^t |\nabla F|^2 (u(\tau))\, d\tau \le \liminf_{n\to+\infty} \int_s^t |\nabla F_n|^2 (u_n(\tau))\, d\tau \qquad \forall\, 0 \le s \le t \le T. \tag{3.15}$$

We can now take the lim inf of both sides of (3.6). Thanks to (3.8), (3.11), and (3.15) we obtain that (3.2) holds true. This completes the proof that (u, ψ, E) is a maximal slope curve for F. $\square$

Thanks to Theorem 3.1 above, any convergence result for maximal slope curves is reduced to verifying three assumptions. The first one is the existence of a pointwise limit, namely a compactness result. The second one is estimate (3.5), which in general follows from suitable assumptions on the sequence of initial data and some boundedness from below of the functionals. The third and more important assumption is (3.4). In the last part of this section we show that (3.4) follows from Gamma-convergence in a class of functionals which contains all convex functionals in Banach spaces.

Definition 3.2 (Slope Cone Property) Let (X, d) be a metric space. A function $F : X \to \overline{\mathbb{R}}$ satisfies the *Slope Cone Property* if

$$F(y) \ge F(x) - |\nabla F|(x) \cdot d(x, y)$$

for every $y \in X$ and every $x \in X$ such that $F(x) \in \mathbb{R}$ and $|\nabla F|(x) < +\infty$.

Remark 3.3 Let X be a Banach space. Then every *convex* function $F : X \to [0, +\infty]$ fulfils the Slope Cone Property; moreover, if F is also lower semicontinuous, its slope $|\nabla F|(x)$ is lower semicontinuous.

On the other hand, also in Banach spaces there do exist nonconvex functions satisfying the Slope Cone Property. An example is $F(x) = -|x|$ in $\mathbb{R}$.

Proposition 3.4 *Let X be a metric space, and let $F_n : X \to \overline{\mathbb{R}}$ be a sequence of functions satisfying the Slope Cone Property. Let us assume that there exists*

$$F(x) := \Gamma - \lim_{n\to+\infty} F_n(x). \tag{3.16}$$

Then the sequence $\{F_n\}$ satisfies assumption (3.4) of Theorem 3.1.

Proof Let $n_k \to +\infty$ and $x_k \to x$ be two sequences as in assumption (3.4). Let M be the supremum in the left-hand side of (3.4). Let $z_n \to x$ be a recovery sequence for x, namely a sequence such that $F_n(z_n) \to F(x)$.

From the Slope Cone Property, and the uniform bound on slopes, it follows that

$$F_{n_k}(z_{n_k}) \geq F_{n_k}(x_k) - |\nabla F_{n_k}|(x_k) \cdot d(x_k, z_{n_k}) \geq F_{n_k}(x_k) - M\, d(x_k, z_{n_k}).$$

Taking the lim sup of both sides we obtain that

$$F(x) = \limsup_{k \to +\infty} F_{n_k}(z_{n_k}) \geq \limsup_{k \to +\infty} F_{n_k}(x_k).$$

The opposite inequality with the lim inf follows from assumption (3.16). This proves the first limit in the right-hand side of (3.4).

Let us prove now the lim inf inequality for slopes. Let L denote the lim inf in the left-hand side of (3.4). Let us take any $y \in X$, and let $y_n \to y$ be a corresponding recovery sequence. Due to the Slope Cone Property we have that

$$F_{n_k}(y_{n_k}) \geq F_{n_k}(x_k) - |\nabla F_{n_k}|(x_k) \cdot d(x_k, y_{n_k}).$$

We already proved that the first term in the right-hand side tends to $F(x)$. Therefore, taking the lim sup of both sides we obtain that $F(y) \geq F(x) - L\, d(x, y)$. Since y is arbitrary, this easily implies that $L \geq |\nabla F|(x)$, which completes the proof. □

We conclude by mentioning a straightforward extension of Definition 3.2 and Proposition 3.4 in the same spirit of [18, 33].

Remark 3.5 One can weaken Definition 3.2 by asking that $F : X \to \mathbb{R} \cup \{+\infty\}$ (so we exclude the value $-\infty$), and there exists a continuous function $\Phi : X^2 \times \mathbb{R}^3 \to \mathbb{R}$ such that $\Phi(x, x, u, v, w) = 0$ for every $(x, u, v, w) \in X \times \mathbb{R}^3$, and

$$F(y) \geq F(x) - |\nabla F|(x) \cdot d(x, y) - \Phi\big(x, y, F(x), F(y), |\nabla F|(x)\big) \cdot d(x, y)$$

for every $(x, y) \in X^2$ such that $F(x) \in \mathbb{R}$, $F(y) \in \mathbb{R}$, and $|\nabla F|(x) \in \mathbb{R}$.

We call this property "Φ Slope Cone Property". In a Banach space it is fulfilled, for example, by the sum of a C^1 function and a convex function.

It can be easily proved that Proposition 3.4 holds true also if all functions F_n satisfy the Φ Slope Cone Property with respect to the same function Φ.

4 Proofs

4.1 Gamma-convergence and compactness

The first equality in (1.8) is a general property of Gamma-convergence. So we can concentrate on the second one.

A standard approach, suggested by the heuristic argument at the end of Section 2, involves a blow up argument in order to reduce the Gamma-liminf inequality to minimizing the right-hand side of (2.9) with respect to η and h, and a proof of the Gamma-limsup inequality via the density of piecewise constant functions with smooth level sets, for which a recovery sequence can be constructed by adapting the optimal profile in the direction orthogonal to level sets.

In both cases we follow a different and more elementary approach, which exploits that our functionals depend only on the gradient.

Gamma-liminf inequality We claim that for every $a \in (0,1)$ and $b \in (0,1)$ there exists $\varepsilon_1 \in (0,1)$ such that

$$\varphi_\varepsilon^{**}(\sigma) \geq a|\sigma| - b \qquad \forall \sigma \in \mathbb{R},\ \forall \varepsilon \in (0, \varepsilon_1). \tag{4.1}$$

If we prove this claim, then for every $u \in L^2(\Omega)$ and every $\varepsilon \in (0, \varepsilon_1)$ we have that

$$E_\varepsilon^{**}(u) = \int_\Omega \varphi_\varepsilon^{**}\left(|\nabla u(x)|\right)\, dx \geq \int_\Omega \big(a|\nabla u(x)| - b\big)\, dx = a\, TV(u) - b|\Omega|.$$

Since the functional $TV(u)$ is lower semicontinuous, this proves that

$$\Gamma\!-\!\liminf_{\varepsilon \to 0^+} E_\varepsilon^{**}(u) \geq a\, TV(u) - b|\Omega|.$$

Letting $a \to 1^-$ and $b \to 0^+$, we obtain the required inequality.

So we are left to prove (4.1). Since the right-hand side is convex, it is enough to prove (4.1) with φ_ε instead of φ_ε^{**}. Moreover, without loss of generality we can assume that $\sigma \geq 0$. Now we distinguish four cases.

If $\sigma \in [0, b]$, then we have that

$$\varphi_\varepsilon(\sigma) \geq 0 \geq ab - b \geq a\sigma - b.$$

If $\sigma \in [b, (e^2 - 1)^{1/2}]$, then we have that

$$\varphi_\varepsilon(\sigma) - a\sigma \geq \frac{1}{2\varepsilon|\log \varepsilon|} \log(1 + b^2) - a\sqrt{e^2 - 1}.$$

The right-hand side tends to $+\infty$ as $\varepsilon \to 0^+$, so it is greater than $-b$ when ε is small enough.

If $\sigma \in [(e^2 - 1)^{1/2}, (\varepsilon|\log \varepsilon|)^{-1}]$, then we have that

$$\varphi_\varepsilon(\sigma) - a\sigma \geq \frac{1}{2\varepsilon|\log \varepsilon|} \log e^2 - \frac{a}{\varepsilon|\log \varepsilon|} = \frac{1 - a}{\varepsilon|\log \varepsilon|},$$

so that the conclusion follows as in the previous case.

Finally, when $\sigma \geq (\varepsilon|\log\varepsilon|)^{-1}$ we apply the inequality between arithmetic mean and geometric mean, and we obtain that

$$\varphi_\varepsilon(\sigma) \geq \frac{\log\sigma}{\varepsilon|\log\varepsilon|} + \frac{\varepsilon}{4}\sigma^2 \geq \sigma \cdot \left(\frac{\log\sigma}{|\log\varepsilon|}\right)^{1/2} \geq \sigma \cdot \left\{\frac{1}{|\log\varepsilon|}\log\left(\frac{1}{\varepsilon|\log\varepsilon|}\right)\right\}^{1/2}.$$

It is not difficult to see that the coefficient of σ tends to 1 as $\varepsilon \to 0^+$, hence it is greater than a when ε is small enough (in this point it is essential that $a < 1$).

This completes the proof of (4.1), hence also of the Gamma-liminf inequality.

Gamma-limsup inequality By a classical density argument it is enough to find a recovery sequence for all functions $u \in C^1(\Omega)$ whose gradient is bounded in Ω. To this end, it is enough to show that for any such function we have that

$$\limsup_{\varepsilon\to 0^+} \int_\Omega \varphi_\varepsilon^{**}\left(|\nabla u(x)|\right)\,dx \leq TV(u). \tag{4.2}$$

In turn, (4.2) is proved if we show that for every $M > 0$ we have that

$$\varphi_\varepsilon^{**}(\sigma) \leq a_\varepsilon|\sigma| \qquad \forall\sigma \in [-M, M]$$

for a suitable coefficient a_ε which tends to 1 as $\varepsilon \to 0^+$. Let us assume, without loss of generality, that $\sigma \in [0, M]$. Since φ_ε^{**} is the convexification of φ_ε, we can estimate $\varphi_\varepsilon^{**}(\sigma)$ from above with the linear function interpolating the values of φ_ε in 0 and $2\varepsilon^{-1}$. As soon as $M \geq 2\varepsilon^{-1}$ we obtain that

$$\varphi_\varepsilon^{**}(\sigma) \leq \frac{\varepsilon}{2}\varphi_\varepsilon\left(\frac{2}{\varepsilon}\right) \cdot \sigma \leq \frac{1}{2}\left(\frac{\log(1+4\varepsilon^{-2})}{2|\log\varepsilon|} + 1\right) \cdot \sigma \qquad \forall\sigma \in [0, M].$$

As required, the coefficient of σ in the right-hand side tends to 1. This completes the proof of (4.2).

Compactness From (4.1) with $a = b = 1/2$ we obtain that

$$E_\varepsilon^{**}(u_\varepsilon) \geq \frac{1}{2}\int_\Omega \left(|\nabla u_\varepsilon(x)| - 1\right)dx = \frac{1}{2}TV(u_\varepsilon) - \frac{1}{2}|\Omega|.$$

This estimate and assumption (2.6) yield a uniform bound on the L^∞-norm and on the total variation of u_ε. Thanks to well known embedding theorems in $BV(\Omega)$ (see [1], this is the point where we need Ω to be an extension domain), this implies that the family $\{u_\varepsilon\}$ is relatively compact in $L^p(\Omega)$ for every $p < +\infty$, and in particular in $L^2(\Omega)$. □

4.2 Convergence of approximating solutions

In the first three paragraphs we prove the result with the further assumption that

$$\sup_{\varepsilon\in(0,1)} \left\{\|u_{0\varepsilon}\|_\infty + E^{**}_\varepsilon(u_{0\varepsilon})\right\} < +\infty. \tag{4.3}$$

Then in the last paragraph we prove it for general data.

Compactness on bounded time intervals We show that, for every $T > 0$, the family $\{u_\varepsilon\}$ is relatively compact in $C^0([0,T]; L^2(\Omega))$. This follows from Ascoli's theorem provided that we show that solutions are 1/2-Hölder continuous with equi-bounded Hölder constants, and that for every fixed $t \geq 0$ the family $\{u_\varepsilon(t)\} \subseteq L^2(\Omega)$ is relatively compact.

From (2.5) and Hölder's inequality we have that

$$\begin{aligned}\|u_\varepsilon(t) - u_\varepsilon(s)\|_2 \;\leq\; \int_s^t \|u'_\varepsilon(\tau)\|_2\,d\tau \;&\leq\; |t-s|^{1/2}\left\{\int_0^t \|u'_\varepsilon(\tau)\|_2^2\,d\tau\right\}^{1/2}\\ &\leq\; |t-s|^{1/2}\left\{2E^{**}_\varepsilon(u_{0\varepsilon})\right\}^{1/2},\end{aligned}$$

so that the uniform bound on Hölder constants follows from assumption (4.3).

Moreover, from (2.5) and statement (4) of Theorem A we have also that the functions $t \to E^{**}_\varepsilon(u_\varepsilon(t))$ and $t \to \|u_\varepsilon(t)\|_\infty$ are nonincreasing, hence

$$\|u_\varepsilon(t)\|_\infty + E^{**}_\varepsilon(u_\varepsilon(t)) \leq \|u_{0\varepsilon}\|_\infty + E^{**}_\varepsilon(u_{0\varepsilon}) \qquad \forall t \geq 0. \tag{4.4}$$

Thanks to assumption (4.3), the right-hand side is bounded independently of ε. Therefore the compactness result in statement (2) of Theorem 2.1 implies that the family $\{u_\varepsilon(t)\}$ is relatively compact in $L^2(\Omega)$ for every fixed $t \geq 0$.

Characterization of the limit We prove that, for every $T > 0$, any limit point in the interval $[0,T]$ of the family $\{u_\varepsilon\}$ of approximating solutions is the solution $u(t)$ of the total variation flow in the same interval. Since the solution of the limit problem is unique, this is enough to prove the convergence of the whole family.

Let $\varepsilon_n \to 0^+$ be any sequence such that $u_{\varepsilon_n}(t)$ uniformly converges to some $v(t)$ in $[0,T]$. By (2.7) we have that $v(0) = u_0$. So it is enough to show that $v(t)$ is a maximal slope curve for the functional $TV(u)$ in $[0,T]$.

To this end, we apply Theorem 3.1 to the sequence of functionals $\{E^{**}_{\varepsilon_n}(u)\}$. Indeed they are convex functionals, and they Gamma-converge to $TV(u)$ because of Theorem 2.1. Thus Remark 3.3 and Proposition 3.4 prove that assumption (3.4) of Theorem 3.1 is satisfied. Also (3.5) holds true because of (4.4), and the fact that the functionals are nonnegative.

Since all the assumptions are satisfied, Theorem 3.1 implies that $v(t)$ is a maximal slope curve for the functional $TV(u)$.

Uniform convergence for all positive times It remains to prove that the convergence is global-in-time, as stated in (2.8). This follows from two general facts.

The first one is that $u(t)$ tends, as $t \to +\infty$, to the constant function u_∞ equal to the average of u_0 in Ω (see [5]). The second fact is that $u_\varepsilon(t) - u_\infty$ is the solution of the approximating problem with initial datum $u_{0\varepsilon} - u_\infty$, hence statement (4) of Theorem A (with $p = 2$) implies that $t \to \|u_\varepsilon(t) - u_\infty\|_2$ is a nonincreasing function.

Therefore for every $T > 0$ we have that

$$\begin{aligned}\sup_{t\geq T}\|u_\varepsilon(t) - u(t)\|_2 &\leq \sup_{t\geq T}\|u_\varepsilon(t) - u_\infty\|_2 + \sup_{t\geq T}\|u(t) - u_\infty\|_2\\ &= \|u_\varepsilon(T) - u_\infty\|_2 + \sup_{t\geq T}\|u(t) - u_\infty\|_2\\ &\leq \|u_\varepsilon(T) - u(T)\|_2 + 2\sup_{t\geq T}\|u(t) - u_\infty\|_2,\end{aligned}$$

hence

$$\begin{aligned}\sup_{t\geq 0}\|u_\varepsilon(t) - u(t)\|_2 &= \max\left\{\sup_{t\in[0,T]}\|u_\varepsilon(t) - u(t)\|_2,\ \sup_{t\geq T}\|u_\varepsilon(t) - u(t)\|_2\right\}\\ &\leq \sup_{t\in[0,T]}\|u_\varepsilon(t) - u(t)\|_2 + 2\sup_{t\geq T}\|u(t) - u_\infty\|_2.\end{aligned}$$

Letting $\varepsilon \to 0^+$, the first term tends to 0 because of the convergence result in $[0, T]$. Letting $T \to +\infty$, the second term tends to 0 because $u(t) \to u_\infty$. This proves (2.8) provided that initial data satisfy (4.3).

Convergence for general data Let now $u_{0\varepsilon} \to u_0$ be any sequence satisfying (2.7). Let us choose a sequence $\{u_{0n}\} \subseteq L^\infty(\Omega) \cap W^{1,\infty}(\Omega)$ with $u_{0n} \to u_0$. For every $n \in \mathbb{N}$, let $u_{\varepsilon,n}(t)$ be the solution of the approximating problem with $u_{\varepsilon,n}(0) = u_{0n}$, and let $u_n(t)$ be the solution of the limit problem with $u_n(0) = u_{0n}$. For every $n \in \mathbb{N}$ we already know that $u_{\varepsilon,n} \to u_n$ in $C^0([0, +\infty); L^2(\Omega))$, because in this case the sequence of initial data does not depend on ε and satisfies (4.3).

Since both the approximating problems and the limit problem generate a contraction semigroup in $L^2(\Omega)$, we have that

$$\begin{aligned}\|u_\varepsilon(t) - u(t)\|_2 &\leq \|u_\varepsilon(t) - u_{\varepsilon,n}(t)\|_2 + \|u_{\varepsilon,n}(t) - u_n(t)\|_2 + \|u_n(t) - u(t)\|_2\\ &\leq \|u_{0\varepsilon} - u_{0n}\|_2 + \|u_{\varepsilon,n}(t) - u_n(t)\|_2 + \|u_{0n} - u_0\|_2.\end{aligned}$$

Taking the supremum over all $t \geq 0$, and letting $\varepsilon \to 0^+$, we obtain that

$$\limsup_{\varepsilon\to 0^+}\sup_{t\geq 0}\|u_\varepsilon(t) - u(t)\|_2 \leq 2\|u_{0n} - u_0\|_2.$$

Letting $n \to +\infty$, we finally obtain (2.8) for general data. $\square$

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