A SHARP QUANTITATIVE VERSION OF HALES' ISOPERIMETRIC HONEYCOMB THEOREM

M. CAROCCIA AND F. MAGGI

Abstract. We prove a sharp quantitative version of Hales' isoperimetric honeycomb theorem by exploiting a quantitative isoperimetric inequality for polygons and an improved convergence theorem for planar bubble clusters. Further applications include the description of isoperimetric tilings of the torus with respect to almost unit-area constraints or with respect to almost flat Riemannian metrics.

1. Introduction

The isoperimetric nature of the planar "honeycomb tiling" has been apparent since antiquity. Referring to [Mor09, Section 15.1] for a brief historical account on this problem, we just recall here that Hales' isoperimetric theorem, see inequality (1.2) below, gives a precise formulation of this intuitive idea. Our goal here is to strengthen Hales' theorem into a quantitative statement, similarly to what has been done with other isoperimetric theorems in recent years (see, for example, [FMP08, FMP10]).

Following [Mag12, Chapters 29-30], we work in the framework of sets of finite perimeter. A N*-tiling* $\mathcal{E}$ of a two-dimensional torus $\mathcal{T}$ is a family $\mathcal{E}=\{\mathcal{E}(h)\}_{h=1}^N$ of sets of finite perimeter in $\mathcal{T}$ such that $|\mathcal{T}\setminus\bigcup_{h=1}^N\mathcal{E}(h)|=0$ and $|\mathcal{E}(h)\cap\mathcal{E}(k)|=0$ for every $h,k\in\mathbb{N}$, $h\neq k$. The volume of $\mathcal{E}$ is $\mathrm{vol}\,(\mathcal{E})=(|\mathcal{E}(1)|,...,|\mathcal{E}(N)|)$, and the relative perimeter of $\mathcal{E}$ in $A\subset\mathcal{T}$ is given by

$$P(\mathcal{E};A)=\frac{1}{2}\sum_{h=1}^N P(\mathcal{E}(h);A)\,,$$

(where $P(E;A)=\mathcal{H}^1(A\cap\partial E)$ if E is an open set with Lipschitz boundary), while the distance between two tilings $\mathcal{E}$ and $\mathcal{F}$ is defined as

$$\mathrm{d}(\mathcal{E},\mathcal{F})=\frac{1}{2}\sum_{h=1}^N|\mathcal{E}(h)\Delta\mathcal{F}(h)|\,.$$

We say that $\mathcal{E}$ is a *unit-area tiling* of $\mathcal{T}$ if $|\mathcal{E}(h)|=1$ for every $h=1,...,N$. (In particular, in that case, it must be $N=|\mathcal{T}|$). Let $\hat{H}$ denote the reference unit-area hexagon in $\mathbb{R}^2$ depicted in Figure 1, so that $\ell=(12)^{1/4}/3$ is the side-length of $\hat{H}$. Given $\alpha,\beta\in\mathbb{N}$, let us consider the torus $\mathcal{T}=\mathcal{T}_{\alpha,\beta}=\mathbb{R}^2/_\approx$ where

$$(x_1,x_2)\approx(y_1,y_2)\qquad\text{if and only if}\qquad\exists h,k\in\mathbb{N}\text{ s.t.}\quad\begin{cases}x_1=y_1+h\,\beta\,\sqrt{3}\,\ell\,,\\ x_2=y_2+k\,\alpha\,\frac{3}{2}\,\ell\,,\end{cases}$$

and set $H=\hat{H}/_\approx\subset\mathcal{T}$. In order to avoid degenerate situations, *we shall always assume that*

$$\alpha\text{ is even and }\beta\geq 2\,.\tag{1.1}$$

In this way, H is a regular unit-area hexagon (i.e., the vertexes of $\hat{H}$ belong to six different equivalence classes) and one obtains a reference unit-area tiling $\mathcal{H}=\{\mathcal{H}(h)\}_{h=1}^N$ of $\mathcal{T}$ consisting of α rows and β columns of regular hexagons by considering translations of H by $(h\sqrt{3}\ell,3\ell\,k/2)$

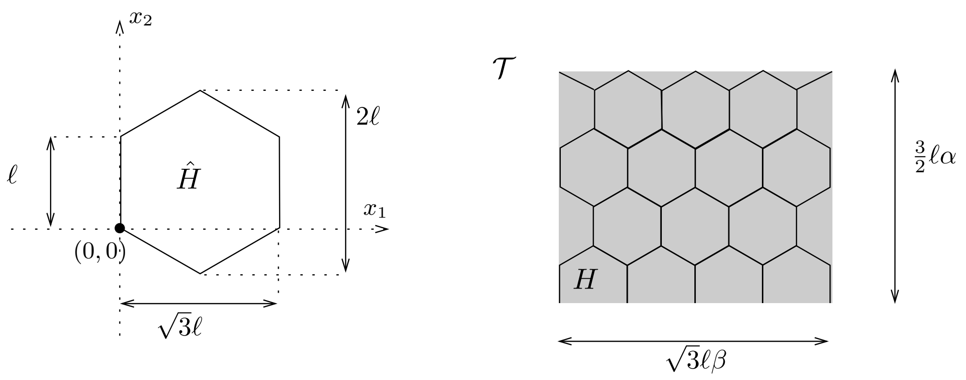

FIGURE 1. Thoroughout the paper $\hat{H}$ denotes the unit-area regular hexagon in $\mathbb{R}^2$ depicted on the left and we set $H = \hat{H}/\approx$. Since $|H| = 1$, one has $P(H) = 2(12)^{1/4}$, and the side-length of H is thus $\ell = (12)^{1/4}/3$. On the right, the torus $\mathcal{T}$ (depicted in gray) and the reference unit-area tiling $\mathcal{H}$ of $\mathcal{T}$ (with $\alpha = \beta = 4$). Notice that $N = |\mathcal{T}| = \alpha\,\beta$. The chambers of $\mathcal{H}$ are enumerated so that $\mathcal{H}(1) = H$, $\{\mathcal{H}(h)\}_{h=1}^{\beta}$ is the bottom row of hexagons in $\mathcal{T}$, and, more generally, if $0 \le k \le \alpha - 1$, then $\{\mathcal{H}(h)\}_{h=1+k\beta}^{(k+1)\beta}$ is the $(k+1)$th row of hexagons in $\mathcal{T}$.

$(h, k \in \mathbb{Z})$; see again Figure 1. Under this assumption, *Hales' isoperimetric honeycomb theorem* asserts that

$$P(\mathcal{E}) \ge P(\mathcal{H})\,, \tag{1.2}$$

whenever $\mathcal{E}$ is a unit-area tiling of $\mathcal{T}$, and that $P(\mathcal{E}) = P(\mathcal{H})$ if and only if (up to a relabeling of the chambers of $\mathcal{E}$) one has $\mathcal{E}(h) = v + \mathcal{H}(h)$ for every $h = 1, ..., N$ and for some $v = (t\sqrt{3}\ell, s\ell)$ with $s, t \in [0, 1]$. Our first main result strengthens this isoperimetric theorem in a sharp quantitative way.

Theorem 1.1. *There exists a positive constant κ depending on $\mathcal{T}$ such that*

$$P(\mathcal{E}) \ge P(\mathcal{H})\left\{1 + \kappa\,\alpha(\mathcal{E})^2\right\}, \tag{1.3}$$

whenever $\mathcal{E}$ is a unit-area tiling of $\mathcal{T}$ and

$$\alpha(\mathcal{E}) = \inf \mathrm{d}(\hat{\mathcal{E}}, v + \mathcal{H})$$

where the minimization takes place among all $v = (t\sqrt{3}\ell, s\ell)$, $s, t \in [0, 1]$, and among all tilings $\hat{\mathcal{E}}$ obtained by setting $\hat{\mathcal{E}}(h) = \mathcal{E}(\sigma(h))$ for a permutation σ of $\{1, ..., N\}$. (Recall that the chambers of the reference honeycomb $\mathcal{H}$ are enumerated in a specific way, see Figure 1.)

Remark 1.2. We notice that (1.3) is sharp in the decay rate of $\alpha(\mathcal{E})$ in terms of $P(\mathcal{E}) - P(\mathcal{H})$. Indeed, if $\omega : (0, \infty) \to (0, \infty)$ is such that $P(\mathcal{E}) \ge P(\mathcal{H})(1 + \omega(\alpha(\mathcal{E})))$ for every unit-area tiling $\mathcal{E}$, then, for some $s_0 > 0$, one must have $\omega(s) \le C\,s^2$ for $s \in (0, s_0)$. Indeed, one can explicitly construct a one-parameter family $\{\mathcal{E}_t\}_{0<t<\varepsilon}$ of unit-area tilings of $\mathcal{T}$ such that $P(\mathcal{E}_t) \le P(\mathcal{H})(1 + C\,\alpha(\mathcal{E}_t)^2)$ and $\{\alpha(\mathcal{E}_t) : t \in (0, \varepsilon)\} = (0, s_0)$, so that $\omega(s) \le C\,s^2$ for every $s \in (0, s_0)$.

In Theorem 3.1 below, inequality (1.3) is proven in much stronger form for $\partial\mathcal{E}$ in a special class of C^1-small $C^{1,1}$-diffeomorphic images of $\partial\mathcal{H}$, see (3.3) and (3.4). The two main ingredients in the proof of Theorem 3.1 are: a quantitative version of the hexagonal isoperimetric inequality, which we deduce from [FRS85, IN14], see Lemma 2.1; and a quantitative version of Hales' hexagonal isoperimetric inequality (the key tool behind Hales' proof of (1.2)), proved in Lemma 3.2. These inequalities allow one to prove that each chamber of the unit-area tiling $\mathcal{E}$ is actually close, in terms of the size of $P(\mathcal{E}) - P(\mathcal{H})$, to some regular unit-area hexagon in $\mathcal{T}$. These hexagons have no reason to fit nicely into an hexagonal honeycomb of $\mathcal{T}$ (that is, a translation of $\mathcal{H}$), therefore we need an additional argument to show that, up to translations and rotations of order $P(\mathcal{E}) - P(\mathcal{H})$, one can achieve this. Having completed the proof of Theorem 3.1, we deduce Theorem 1.1 by a contradiction argument based on an improved convergence theorem for planar bubble clusters that was recently established in [CLM14], and along the lines of the selection principle method proposed in [CL12]. Another consequence of Theorem 3.1, obtained in a similar vein, is the following result, which gives a precise description of isoperimetric tilings of $\mathcal{T}$ subject to an "almost unit-area" constraint.

Theorem 1.3. *There exist positive constants C_0, δ_0 depending on $\mathcal{T}$ with the following property. If $\sum_{h=1}^N m_h = N$ with $m_h > 0$ and $|m_h - 1| < \delta_0$ for every $h = 1, ..., N$, and if $\mathcal{E}_m$ is an N-tiling of $\mathcal{T}$ which is a minimizer in*

$$\inf \left\{ P(\mathcal{E}) : |\mathcal{E}(h)| = m_h \quad \forall h = 1, ..., N \right\} \tag{1.4}$$

then, up to a relabeling of the chambers of $\mathcal{E}_m$, there exists a $C^{1,1}$-diffeomorphism $f_m : \partial\mathcal{H} \to \partial\mathcal{E}_m$ such that

$$\|f_m - (v + \mathrm{Id})\|^2_{C^0(\partial\mathcal{H})} + \|f_m - (v + \mathrm{Id})\|^4_{C^1(\partial\mathcal{H})} \le C_0 \sum_{h=1}^N |m_h - 1| , \tag{1.5}$$

for some $v = (t\sqrt{3}\ell, s\ell)$, $s, t \in [0, 1]$.

Next, let us consider the family X of those $\Phi \in C^0(\mathcal{T} \times S^{n-1}; (0, \infty))$ such that the positive one-homogeneous extension of $\Phi(x, \cdot)$ to $\mathbb{R}^2$ is convex, fix $\psi \in C^0(\mathcal{T}; (0, \infty))$, and consider the isoperimetric problem

$$\lambda(\Phi, \psi) = \inf \left\{ \mathbf{\Phi}(\mathcal{E}) = \frac{1}{2}\sum_{h=1}^N \mathbf{\Phi}(\mathcal{E}(h)) : \int_{\mathcal{E}(h)} \psi = \frac{1}{N}\int_{\mathcal{T}} \psi \quad \forall h = 1, ..., N \right\} , \tag{1.6}$$

where for a set of finite perimeter $E \subset \mathcal{T}$ we have set

$$\mathbf{\Phi}(E; A) = \int_{A \cap \partial^* E} \Phi(x, \nu_E(x))\, d\mathcal{H}^1(x) , \qquad \mathbf{\Phi}(E) = \mathbf{\Phi}(E; \mathbb{R}^n) ,$$

provided $\partial^* E$ and $\nu_E : \partial^* E \to S^1$ denote, respectively, the reduced boundary and the measure-theoretic outer unit normal of E, see [Mag12, Chapter 15]. Notice that although we do not assume Φ to be even, we have nevertheless that $\lambda(\Phi, \psi) = \lambda(\hat\Phi, \psi)$ where $\hat\Phi(x, \nu) = (\Phi(x, \nu) + \Phi(x, -\nu))/2$. An interesting example is obtained when g is a Riemannian metric on $\mathcal{T}$ and

$$\Phi(x, \nu) = \sqrt{g(x)[\nu^\perp, \nu^\perp]} , \qquad \psi = \sqrt{\det(g(x))} ,$$

where $\nu^\perp = (\nu_2, -\nu_1)$ if $\nu = (\nu_1, \nu_2)$. In this case, (1.6) boils down to minimizing the total Riemannian perimeter of a partition of $\mathcal{T}$ into N-regions of equal Riemannian area.

Theorem 1.4. *Given $L > 0$ and $\gamma \in (0, 1]$, there exist $C_0, \delta_0 > 0$ (depending on $\mathcal{T}$, L and γ) with the following property. If $\mathcal{E}$ is a minimizer in (1.6) for $\Phi \in X \cap \mathrm{Lip}\,(\mathcal{T} \times S^1)$ and $\psi \in C^{1,\gamma}(\mathcal{T})$ such that*

$$\mathrm{Lip}\ \Phi + \|\psi\|_{C^{1,\gamma}(\mathcal{T})} \le L , \tag{1.7}$$
$$\|\Phi - 1\|_{C^0(\mathcal{T} \times S^1)} + \|\psi - 1\|_{C^0(\mathcal{T})} < \delta_0 ,$$

then

$$\inf_{s,t \in [0,1]} \mathrm{hd}(\partial\mathcal{E}, v + \partial\mathcal{H})^4 \le C_0 \left(\|\Phi - \mathrm{Id}\|_{C^0(\mathcal{T} \times S^1)} + \|\psi - 1\|_{C^0(\mathcal{T})} \right) , \tag{1.8}$$

where $v = (t\sqrt{3}\ell, s\ell)$ and $\mathrm{hd}(S, T)$ denote the Hausdorff distance between the closed sets S and T in $\mathcal{T}$.

We deduce Theorem 1.4 from Theorem 1.1 by some comparison arguments and density estimates. Since we are assuming that $\nabla\Phi$ is merely bounded, we do not expect $\partial\mathcal{E}$ to be a C^1-diffeomorphic image of $\partial\mathcal{H}$. From this point of view, (1.8) seems to express a qualitatively sharp control on $\partial\mathcal{E}$. At the same time, when more regular integrands Φ are considered (see, e.g., [DS02] for the kind of assumption one may impose here) one would expect to be able to obtain a control in the spirit of (1.5). However a description of singularities of isoperimetric clusters in this kind of setting, although arguably achievable at least in some special cases, is missing at present. In turn, understanding singularities would be the essential in order to adapt the improved convergence theorem from [CLM14] to this context, and thus to be able to strengthen (1.8) into an estimate analogous to (1.5).

The paper is organized as follows. In section 2 we deduce from [FRS85, IN14] a quantitative isoperimetric inequality for polygons of possible independent interest. In section 3 we prove Theorem 1.1 on small C^1-deformations of $\partial\mathcal{H}$ (actually with the Hausdorff distance between $\partial\mathcal{E}$ and $\partial\mathcal{H}$ in place of $\mathrm{d}(\mathcal{E},\mathcal{H})$ on the right-hand side of (1.3)). In section 4 we exploit the improved convergence theorem from [CLM14] to deduce Theorem 1.1 and Theorem 1.3 from Theorem 3.1, and, finally, to deduce Theorem 1.4 from Theorem 1.1.

Acknowledgement: The work of MC was supported by the project 2010A2TFX2 "Calcolo delle Variazioni" funded by the Italian Ministry of Research and University. The work of FM was supported by NSF Grant DMS-1265910.

2. A quantitative isoperimetric inequality for polygons

Thorough this section we fix $n \ge 3$. We denote by Π a convex unit-area n-gon, and by Π_0 a reference unit-area regular n-gon. If ℓ and r denote, respectively, the side-length and radius of Π_0, then one easily finds that

$$P(\Pi_0) = n\,\ell = 2\sqrt{n\,\tan\Big(\frac{\pi}{n}\Big)}\,,\qquad r^{-1} = \sqrt{n\,\sin\Big(\frac{\pi}{n}\Big)\,\cos\Big(\frac{\pi}{n}\Big)}\,.$$

(Notice that in the other sections of the paper we always assume $n = 6$, so that $\ell = (12)^{1/4}/3$ according to the convention set in the introduction.) The isoperimetric theorem for n-gons asserts that

$$P(\Pi) \ge n\,\ell\,, \tag{2.1}$$

with equality if and only if $\Pi = \rho(\Pi_0)$ for a rigid motion ρ of $\mathbb{R}^2$. A sharp quantitative version of (2.1) is proved in [IN14] starting from the main result in [FRS85]. Precisely, let us now denote by ℓ_i and r_i the lengths of the ith edge and the ith radius of Π (labeled so that $\ell_i = \ell_j$ and $r_i = r_j$ if $i = j$ modulo n), and set

$$\bar\ell = \frac{1}{n}\sum_{i=1}^n \ell_i\,,\qquad \bar r = \frac{1}{n}\sum_{i=1}^n r_i\,.$$

Then [IN14, Corollary 1.3] asserts that

$$C(n)\,\big(P(\Pi)^2 - (n\ell)^2\big) \ge \sum_{i=1}^n (r_i - \bar r)^2 + \sum_{i=1}^n (\ell_i - \bar\ell)^2\,. \tag{2.2}$$

The right-hand side of inequality (2.2) measures the distance of Π from being a unit-area regular n-gon in the sense that if $r_i = \bar r$ and $\ell_i = \bar\ell$, then it must be $\bar r = r$ and $\bar\ell = \ell$ by the area constraint, and thus Π is a regular unit-area n-gon. However, in addressing our problem we shall need (in the case $n = 6$) to control the distance of Π from a specific regular unit-area n-gon by means of $P(\Pi)^2 - (n\ell)^2$. Passing from (2.2) to this kind of control is the subject of the following proposition.

Proposition 2.1. *There exists a positive constant $C(n)$ with the following property: for every convex unit-area n-gon Π there exists a rigid motion ρ of $\mathbb{R}^2$ such that*

$$C(n)\,\big(P(\Pi)^2-(n\ell)^2\big)\ge \mathrm{hd}(\partial\Pi,\partial\rho\Pi_0)^2\,. \tag{2.3}$$

Proof. Up to a translation, we can assume that Π has barycenter at 0. Next, if $P(\Pi)\ge n\ell+\eta\,P(\Pi)$ for some $\eta>0$, then $P(\Pi)^2-(n\ell)^2\ge\eta\,P(\Pi)^2$. Since $\mathrm{hd}(\partial\Pi,\partial\rho\Pi_0)<\mathrm{diam}(\Pi)+\mathrm{diam}(\Pi_0)\le(P(\Pi)+P(\Pi_0))/2\le P(\Pi)$ whenever $\partial\rho\Pi_0$ intersects $\partial\Pi$, we conclude that (2.3) holds with $C(n)=\eta^{-1}$. In other words, in proving (2.3), one can assume without loss of generality that

$$P(\Pi)-n\,\ell<\eta\,P(\Pi) \tag{2.4}$$

for an arbitrarily small constant $\eta=\eta(n)$. By a trivial compactness argument (on the class of convex n-gons with barycenter at 0), one sees that given $\varepsilon>0$ there exists $\eta>0$ such that if (2.4) holds, then, up to rigid motions,

$$\mathrm{hd}(\partial\Pi,\partial\Pi_0)<\varepsilon\,, \tag{2.5}$$

where the reference regular unit-area n-gon Π_0 is assumed to have barycenter at 0.

Now let v_i and w_i denote the positions of the vertexes of Π and Π_0 respectively: by (2.5) and up to a rotation, one can entail that

$$|v_i-w_i|<\varepsilon\,,\qquad\forall i=1,...,n\,,\qquad v_1=\lambda\,w_1\qquad\text{for some }\lambda>0\,.$$

Let ρ_i denote the rotation around the origin such that $\rho_i(v_i)=\lambda_i\,w_i$ for some $\lambda_i>0$ (so that $\rho_1=\mathrm{Id}$ by $v_1=\lambda\,w_1$), and let θ_i denote the angle identifying ρ_i as a counterclockwise rotation; since $\|\rho_i-\mathrm{Id}\|\le|\theta_i|$ and $|\rho_i(v_i)-w_i|=|r_i-r|$, one has

$$\mathrm{hd}(\partial\Pi,\partial\Pi_0)\le C\sum_{i=1}^n|v_i-w_i|\le C\sum_{i=1}^n r_i|\theta_i|+|r_i-r|\,. \tag{2.6}$$

Let us now set $\delta=P(\Pi)-n\ell$: by (2.2) and (2.4) one finds

$$\max_{1\le i\le n}|r_i-\bar r|+|\ell_i-\bar\ell|\le C\,\sqrt{\delta}\,. \tag{2.7}$$

Since $\bar\ell=n^{-1}P(\Pi)$ gives $|\bar\ell-\ell|=n^{-1}\delta$, we deduce from $|\ell_i-\bar\ell|\le C\sqrt{\delta}$ that

$$\max_{1\le i\le n}|\ell_i-\ell|\le C\sqrt{\delta}\,. \tag{2.8}$$

Let now $A(a,b,c)$ denote the area of a triangle with sides of length a, b and c. Since A is a Lipschitz function in an ε-neighborhood of (r,r,ℓ) (where both $(\bar r,\bar r,\ell)$ and (r_i,r_{i+1},ℓ_i) lie by (2.5)), by (2.7), (2.8) and by $|\Pi_0|=|\Pi|$ we find

$$\Big|n\,A(r,r,\ell)-n\,A(\bar r,\bar r,\ell)\Big|=\Big|\sum_{i=1}^n A(r_i,r_{i+1},\ell_i)-n\,A(\bar r,\bar r,\ell)\Big|\le C\,\sqrt{\delta}\,.$$

Since $A(a,a,\ell)=(\ell/4)\,\sqrt{4a^2-\ell^2}$ we immediately see that $|A(r,r,\ell)-A(a,a,\ell)|\ge c\,|a-r|$ whenever $|a-r|<\varepsilon$ and where $c=c(\ell)=c(n)>0$. Thus, $|r-\bar r|\le C\,\sqrt{\delta}$, and (2.7) and (2.8) give

$$\max_{1\le i\le n}|r_i-r|+|\ell_i-\ell|\le C\,\sqrt{\delta}\,. \tag{2.9}$$

If α_i denotes the interior angle between v_i and v_{i+1} (so that $|\alpha_i-2\pi/n|=O(\varepsilon)$ by (2.5)), then

$$\alpha_i=f(r_i,r_{i+1},\ell_i)\,,\qquad\text{where}\quad f(a,b,c)=\arccos\Big(\frac{a^2+b^2-c^2}{2ab}\Big)\,.$$

Since f is a Lipschitz function in an ε-neighborhood of (r,r,ℓ), we conclude from (2.9) that

$$\max_{1\le i\le n}\Big|\alpha_i-\frac{2\pi}{n}\Big|=\max_{1\le i\le n}\big|f(r_i,r_{i+1},\ell_i)-f(r,r,\ell)\big|\le C\,\sqrt{\delta}\,.$$

In particular, since $\theta_1 = 0$ (as $\rho_1 = \mathrm{Id}$), we deduce from this last estimate that $|\theta_i| \le C\sqrt{\delta}$ for $i = 1, ..., n$. We plug this inequality and (2.9) in (2.6) to conclude the proof. □

Coming to the torus $\mathcal{T}$, we shall use the following corollary of Proposition 2.1.

Corollary 2.2. *There exist positive constants η and c, independent from $\mathcal{T}$, with the following property. If Π is a convex hexagon in $\mathcal{T}$ such that $\mathrm{hd}(\partial\Pi, \partial H) \le \eta$, then there exists a regular hexagon H_* in $\mathcal{T}$ with $|\Pi| = |H_*|$*

$$P(\Pi) - P(H)\sqrt{|\Pi|} \ge c\, \mathrm{hd}(\partial\Pi, \partial H_*)^2 . \tag{2.10}$$

Proof. We first notice that by Proposition 2.1 and by scaling, if $\hat{\Pi}$ is a convex hexagon in $\mathbb{R}^2$, then there exists a regular hexagon $\hat{H}_*$ with $|\hat{H}_*| = |\hat{\Pi}|$ and

$$P(\hat{\Pi})^2 - P(\hat{H})^2|\hat{\Pi}| \ge c\, \mathrm{hd}(\partial\hat{\Pi}, \partial\hat{H}_*)^2 . \tag{2.11}$$

Since Π is a convex hexagon in $\mathcal{T}$ with $\mathrm{hd}(\partial\Pi, \partial H) \le \eta$, then there exists a convex hexagon $\hat{\Pi}$ in $\mathbb{R}^2$ isometric to Π with $\mathrm{hd}(\partial\hat{\Pi}, \partial\hat{H}) \le \eta$. In particular, for some constant C independent from $\mathcal{T}$, one has

$$P(\hat{\Pi}) - P(\hat{H})\sqrt{|\hat{\Pi}|} \le C\,\eta\,, \qquad P(\hat{\Pi}) + P(\hat{H})\sqrt{|\hat{\Pi}|} \le C\,,$$

and thus (2.11) gives, up to further decrease the value of c,

$$C\eta \ge P(\hat{\Pi}) - P(\hat{H})\sqrt{|\hat{\Pi}|} \ge c\, \mathrm{hd}(\partial\hat{\Pi}, \partial\hat{H}_*)^2 . \tag{2.12}$$

By (2.12) and $\mathrm{hd}(\partial\hat{\Pi}, \partial\hat{H}) \le \eta$ we have $\mathrm{hd}(\partial\hat{H}, \partial\hat{H}_*) \le C\sqrt{\eta}$. Now, since $\beta \ge 2$ and α is even one can find $\eta_* > 0$ (independent of α and β) such that $I_{\eta_*}(\hat{H}) = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \hat{H}) \le \eta_*\}$ is compactly contained into a rectangular box of height $3\ell\alpha/2$ and width $\sqrt{3}\ell\beta$. As a consequence, if $\hat{J}$ is a polygon contained in $I_{\eta_*}(\hat{H})$, then $J = \hat{J}/_{\approx} \subset \mathcal{T}$ is isometric to $\hat{J}$. Thus, if $C\sqrt{\eta} < \eta_*$, then $H_* = \hat{H}_*/_{\approx}$ is a regular hexagon in $\mathcal{T}$ with $|H_*| = |\Pi|$ and $\mathrm{hd}(\partial\hat{\Pi}, \partial\hat{H}_*) = \mathrm{hd}(\partial\Pi, \partial H_*)$, and (2.10) follows from (2.12). □

3. Small deformations of the reference honeycomb

The main result of this section is Theorem 3.1, which provides us, on a restricted class of unit-area tilings, with a stronger stability estimate than the one in Theorem 1.1. Before stating this result we need to introduce the following terminology:

Regular and singular sets: Given a N-tiling $\mathcal{E}$ of $\mathcal{T}$ one sets

$$\partial\mathcal{E} = \bigcup_{h=1}^N \partial\mathcal{E}(h)\,, \qquad \partial^*\mathcal{E} = \bigcup_{h=1}^N \partial^*\mathcal{E}(h)\,,$$

$$\Sigma(\mathcal{E}) = \partial\mathcal{E} \setminus \partial^*\mathcal{E}\,, \qquad [\partial\mathcal{E}]_\mu = \{x \in \partial\mathcal{E} : \mathrm{dist}(x, \Sigma(\mathcal{E})) > \mu\}\,, \quad \mu > 0\,,$$

where $\partial^* E$ denotes the reduced boundary of a set of finite perimeter E in $\mathcal{T}$, and where the normalization convention $\partial E = \overline{\partial^* E}$ for sets of finite perimeter is always assumed to be in force, see [Mag12, Section 12.3]. We call $\partial^*\mathcal{E}$ and $\Sigma(\mathcal{E})$ the *regular set* and the *singular set* of $\partial\mathcal{E}$ respectively. In this way, $\partial^*\mathcal{H}$ and $\Sigma(\mathcal{H})$ are, respectively, the union of the open edges and the union of the vertexes of the hexagons $\mathcal{H}(h)$ for $h = 1, ..., N$.

Tilings and maps of class $C^{k,\alpha}$: Given $k \in \mathbb{N}$ and $\alpha \in [0,1]$, one says that a tiling $\mathcal{E}$ of $\mathcal{T}$ is of *class* $C^{k,\alpha}$ if there exist a finite family $\{\gamma_i\}_{i\in I}$ of compact $C^{k,\alpha}$-curves with boundary and a finite family $\{p_j\}_{j\in J}$ of points such that

$$\partial\mathcal{E} = \bigcup_{i\in I} \gamma_i\,, \qquad \partial^*\mathcal{E} = \bigcup_{i\in I} \mathrm{int}\,(\gamma_i)\,, \qquad \Sigma(\mathcal{E}) = \bigcup_{i\in I} \mathrm{bd}\,(\gamma_i) = \bigcup_{j\in J}\{p_j\}\,, \tag{3.1}$$

where int(γ_i) and bd(γ_i) denote the interior and the boundary of γ_i respectively. Moreover, given a function $f:\partial\mathcal{E}\to\mathcal{T}$, one says that $f\in C^{k,\alpha}(\partial\mathcal{E};\mathcal{T})$ if f is continuous on $\partial\mathcal{E}$ and

$$\|f\|_{C^{k,\alpha}(\partial\mathcal{E})}:=\sup_{i\in I}\|f\|_{C^{k,\alpha}(\gamma_i)}<\infty\,.$$

Finally, given two $C^{k,\alpha}$-tilings $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{T}$, one says that f is a $C^{k,\alpha}$-diffeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$ if f is an homeomorphism between $\partial\mathcal{E}$ and $\partial\mathcal{F}$ with $f(\Sigma(\mathcal{E}))=\Sigma(\mathcal{F})$, $f(\partial\mathcal{E}(h))=\partial\mathcal{F}(h)$ for every $h=1,...,N$, $f\in C^{k,\alpha}(\partial\mathcal{E};\mathcal{T})$ and $f^{-1}\in C^{k,\alpha}(\partial\mathcal{E};\mathcal{T})$.

Tangential component of a map and (ε,μ,L)-perturbations of $\mathcal{H}$: Given a tiling $\mathcal{E}$ of $\mathcal{T}$ of class C^1, by taking (3.1) into account one can define $\nu_\mathcal{E}\in C^0(\partial^*\mathcal{E};S^1)$ in such a way that $\nu_\mathcal{E}$ is a unit normal vector to γ_i for every i. Correspondingly, given a map $f:\partial\mathcal{E}\to\mathcal{T}$, we define $\boldsymbol{\tau}_\mathcal{E}f:\partial^*\mathcal{E}\to\mathcal{T}$, the tangential component of f with respect to $\partial\mathcal{E}$, as

$$\boldsymbol{\tau}_\mathcal{E}f(x)=f(x)-(f(x)\cdot\nu_\mathcal{E}(x))\,\nu_\mathcal{E}(x)\,,\qquad x\in\partial^*\mathcal{E}\,.$$

Finally, one says that $\mathcal{E}$ is an *(ε,μ,L)-perturbation of $\mathcal{H}$* if $\mathcal{E}$ is of class $C^{1,1}$ and there exists an homeomorphism f between $\partial\mathcal{H}$ and $\partial\mathcal{E}$ with

$$\begin{aligned}\|f\|_{C^{1,1}(\partial\mathcal{H})}&\le L\,,\\ \|f-\mathrm{Id}\|_{C^1(\partial\mathcal{H})}&\le\varepsilon\,,\\ \|\boldsymbol{\tau}_\mathcal{H}(f-\mathrm{Id})\|_{C^1(\partial^*\mathcal{H})}&\le\frac{L}{\mu}\sup_{\Sigma(\mathcal{H})}|f-\mathrm{Id}|\,,\\ \boldsymbol{\tau}_\mathcal{H}(f-\mathrm{Id})=0\,,&\qquad\text{on }[\partial\mathcal{H}]_\mu\,.\end{aligned}\tag{3.2}$$

Theorem 3.1. *For every $L>0$ there exist positive constants μ_0, ε_0 and c_0 (depending on L and $|\mathcal{T}|$), C depending on $|\mathcal{T}|$ only, and C' depending on L only, with the following property. If $\mathcal{E}$ is a unit-area (ε_0,μ_0,L)-perturbation of $\mathcal{H}$, then there exists $v\in\mathbb{R}^2$ such that*

$$P(\mathcal{E})-P(\mathcal{H})\ge c_0\,\mathrm{hd}(\partial\mathcal{E},v+\partial\mathcal{H})^2\,,\qquad|v|\le C\,\varepsilon_0\,.\tag{3.3}$$

Moreover, there exists a $C^{1,1}$-diffeomorphism f_0 between $v+\partial\mathcal{H}$ and $\partial\mathcal{E}$ such that

$$P(\mathcal{E})-P(\mathcal{H})\ge c_0\Big(\|f_0-\mathrm{Id}\|^2_{C^0(v+\partial\mathcal{H})}+\|f_0-\mathrm{Id}\|^4_{C^1(v+\partial\mathcal{H})}\Big)\,,\tag{3.4}$$

and $\|f_0\|_{C^{1,1}(v+\partial\mathcal{H})}\le C'$.

We premise a lemma to the proof of Theorem 3.1. As said, this lemma provides a quantitative version of (a particular case of) Hales' hexagonal isoperimetric inequality, the key step in the proof of (1.2) in [Hal01].

Lemma 3.2. *There exist positive constants ε_1 and c_1 with the following property. If $\mathcal{E}$ is a unit-area tiling of $\mathcal{T}$ such that there exists an homeomorphism f between $\partial\mathcal{H}$ and $\partial\mathcal{E}$ with $\|f-\mathrm{Id}\|_{C^0(\partial\mathcal{H})}\le\varepsilon_1$, if $E=\mathcal{E}(h)$ for some $h\in\{1,...,N\}$ and Π is the convex envelope of $\Sigma(\mathcal{E})\cap\partial E$ (so that Π is convex hexagon with set of vertexes $\Sigma(\mathcal{E})\cap\partial E$ provided ε_1 is small enough), then there exists a regular hexagon H_* with $|H_*|=|\Pi|$ such that*

$$P(E)\ge P(H)+\frac{P(H)}{2}(|\Pi|-|E|)+c_1\Big(|E\Delta\Pi|^2+\mathrm{hd}(\partial\Pi,\partial H_*)^2\Big)\,.\tag{3.5}$$

Remark 3.3. The constants ε_1 and c_1 will just depend on the metric properties of the unit-area hexagon. In particular they do not depend on $\mathcal{T}$.

Proof of Lemma 3.2. Let $\mathrm{arc}_t(a)$ denote the length of a circular arc that bounds an area $a\ge0$ and whose chord length is $t>0$, and let us set $\mathrm{arc}(a)=\mathrm{arc}_1(a)$. In this way, $\mathrm{arc}:[0,\infty)\to[1,\infty)$ is an increasing function. Since the derivative of arc at a is the curvature of any circular arc bounding an area a above a unit length chord, and since this curvature is increasing as a ranges from 0 to $\pi/8$ (the value $a=\pi/8$ corresponds to the case of an half-disk with unit diameter),

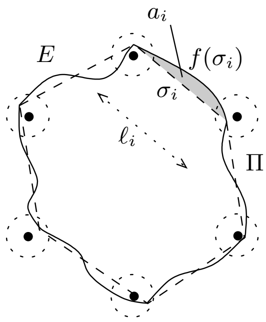

FIGURE 2. The convex hexagon Π spanned by $\Sigma(\mathcal{E})\cap\partial E$. The vertexes of Π are ε_1-close to the vertexes of the unit-area regular hexagon $\mathcal{H}(h)$ (as $E=\mathcal{E}(h)$ and $f(\partial\mathcal{H}(h))=\partial\mathcal{E}(h)$) which are depicted as black dots. The boundaries of Π and E are depicted, respectively, by a dashed line and by a continuous line.

we conclude that arc is convex on $[0,\pi/8]$ (and, in fact, also concave on $[\pi/8,\infty)$). Moreover, a Taylor expansion gives that $\mathrm{arc}''(0^+)>0$: hence there exists $\eta>0$ such that

$$\mathrm{arc}(a)\ge 1+\eta\, a^2\,,\qquad \forall a\in[0,\eta)\,. \tag{3.6}$$

Let ℓ_i denote the length of the ith side of Π, and let a_i denote the total area enclosed between the ith side of Π and the ith side of E; see Figure 2. (If σ_i is the ith side of Π, then the ith side of E is a small C^0-deformation of σ_i with fixed end-points). Noticing that $\mathrm{arc}_t(a)=t\,\mathrm{arc}(a/t^2)$, by Dido's inequality we find that

$$P(E)\ge\sum_{i=1}^6\mathrm{arc}_{\ell_i}(a_i)=\sum_{i=1}^6\ell_i\,\mathrm{arc}\Big(\frac{a_i}{\ell_i^2}\Big)\,.$$

By $\|f-\mathrm{Id}\|_{C^0(\partial\mathcal{H})}\le\varepsilon_1$ and provided $\varepsilon_1\le 1$, one has

$$\mathrm{hd}(\partial\Pi,\partial\mathcal{H}(h))\le\varepsilon_1\,,\qquad \max_{1\le i\le 6}\Big\{a_i,\Big|\ell_i-\frac{P(H)}{6}\Big|\Big\}\le C\,\varepsilon_1\,, \tag{3.7}$$

where a possible value for C in (3.7) is $2(\pi+\ell)$. By (3.7), by further decreasing ε_1, we can assume that $a_i/\ell_i^2\in[0,\pi/8]$ for every $i=1,...,6$. We thus apply Jensen inequality to find that

$$P(E)\ge\sum_{i=1}^6\ell_i\,\mathrm{arc}\Big(\frac{1}{\sum_{i=1}^6\ell_i}\sum_{i=1}^6\frac{a_i}{\ell_i^2}\Big)\,.$$

Since $P(H)/6=(12)^{1/4}/3<1$, by (3.7) we may further assume that $\ell_i\le 1$ for every $i=1,...,6$, and thus conclude by $P(\Pi)=\sum_{i=1}^6\ell_i$, $|E\Delta\Pi|=\sum_{i=1}^6 a_i$, and the monotonicity of arc that

$$P(E)\ge P(\Pi)\,\mathrm{arc}\Big(\frac{|E\Delta\Pi|}{P(\Pi)}\Big)\,. \tag{3.8}$$

(Inequality (3.8) is clearly related to the *chordal isoperimetric inequality* [Hal01, Proposition 6.1-A], see also [Mor09, 15.5].) By (3.6), (3.7) and (3.8),

$$P(E)\ge P(\Pi)+\eta\,\frac{|E\Delta\Pi|^2}{P(\Pi)^2}\ge P(\Pi)+c_1\,|E\Delta\Pi|^2\,, \tag{3.9}$$

where $c_1>0$. Provided ε_1 is small enough, by (3.7) we can apply Corollary 2.2 to find a regular hexagon H_* with $|H_*|=|\Pi|$ and

$$P(\Pi)-P(H)\sqrt{|\Pi|}\ge c\,\mathrm{hd}(\partial\Pi,\partial H_*)^2\,.$$

Thus, up to further decrease the value of c_1, (3.9) gives

$$P(E) \geq P(H)\sqrt{|\Pi|} + c_1\left(\mathrm{hd}(\partial\Pi,\partial H_*)^2 + |E\Delta\Pi|^2\right). \tag{3.10}$$

Finally, given $\tau > 0$ let $\lambda > 0$ be such that $\sqrt{1-s} \geq 1-(s/2)-\tau\, s^2$ for $|s| < \lambda$: up to further decrease ε_1, by $\|f-\mathrm{Id}\|_{C^0(\partial\mathcal{H})} \leq \varepsilon_1$ we entail $|\sigma| < \lambda$ for $\sigma = |E| - |\Pi|$, and thus deduce with the aid of (3.10) and $|E| = 1$ that

$$P(E) \geq P(H) - \frac{P(H)}{2}\,\sigma - P(H)\tau\,\sigma^2 + c_1\left(\mathrm{hd}(\partial\Pi,\partial H_*)^2 + |E\Delta\Pi|^2\right). \tag{3.11}$$

Since $|\sigma| = ||E| - |\Pi|| \leq |E\Delta\Pi|$, for τ small enough depending from c_1, we prove (3.5). □

Proof of Theorem 3.1. Step one: The reflection of $\mathbb{R}^2$ with respect to a generic line does not induce a map on $\mathcal{T}$. However, by (1.1), one has that if $R_\theta\hat{H}$ denotes the counterclockwise rotation of $\hat{H}$ by an angle θ around the origin, then $R_\theta\hat{H}$ is compactly contained in a box of height $3\ell\alpha/2 \geq 3\ell$ and width $\sqrt{3}\ell\beta \geq 2\sqrt{3}\ell$ for every θ. As a consequence, given a unit-area regular hexagon K in $\mathcal{T}$, all the rotations of K are well-defined as unit-area regular hexagons in $\mathcal{T}$; in particular, it always makes sense to define the reflection $g_\sigma(K)$ of K with respect to an edge σ of K. Taking this into account, we notice that there exist positive constants η and C (independent of $\mathcal{T}$) such that, if K and K' are unit-area regular hexagons in $\mathcal{T}$, and if σ and σ' are edges of K and K' respectively, then

$$\begin{cases} \mathrm{hd}(\sigma,\sigma') \leq \eta\,, \\ |K\Delta K'| \geq 2-\eta\,, \end{cases} \qquad \Rightarrow \qquad \mathrm{hd}(\partial g_\sigma(K),\partial K') \leq C\,\mathrm{hd}(\sigma,\sigma')\,.$$

This geometric remark is going to be repeatedly used in the following arguments, where we shall denote by ε_1 and c_1 the constants of Lemma 3.2 and set $\delta = P(\mathcal{E}) - P(\mathcal{H})$. We notice that, by the area formula and since $\|f-\mathrm{Id}\|_{C^1(\partial\mathcal{H})} \leq \varepsilon_0$, one has

$$\delta \leq C\,P(\mathcal{H})\,\varepsilon_0^2\,, \tag{3.12}$$

where C is independent from $\mathcal{T}$ and where $P(\mathcal{H}) = |\mathcal{T}|\,P(H)/2$.

Step two: We claim that, if ε_0 is small enough depending only from $|\mathcal{T}|$, and if Π_h denotes the convex envelope of $\partial\mathcal{E}(h)\cap\Sigma(\mathcal{E})$ (so that Π_h is a convex hexagon, not necessarily with unit-area), then for every $h = 1,...,N$ there exists a regular unit-area hexagon K_h such that

$$\mathrm{hd}(\partial\Pi_h,\partial K_h) \leq C\sqrt{\delta}\,, \tag{3.13}$$

$$|K_h\Delta K_{h+1}| \geq 2 - C\sqrt{\delta}\,, \tag{3.14}$$

where here and in the rest of this step, C denotes a constant depending from $|\mathcal{T}|$ only. Indeed, since $\{\Pi_h\}_{h=1}^N$ is a partition of $\mathcal{T}$, one has $\sum_{h=1}^N |\Pi_h| = |\mathcal{T}| = \sum_{h=1}^N |\mathcal{E}(h)|$. By requiring $\varepsilon_0 \leq \varepsilon_1$ we can apply Lemma 3.2 to each $\mathcal{E}(h)$ in order to find regular hexagons H_h^* with $|H_h^*| = |\Pi_h|$ such that, by adding up (3.5) on h, one finds

$$2\,\delta = \sum_{h=1}^N (P(\mathcal{E}(h)) - P(H)) \geq c_1 \sum_{h=1}^N \left(|\mathcal{E}(h)\Delta\Pi_h|^2 + \mathrm{hd}(\partial\Pi_h,\partial H_h^*)^2\right). \tag{3.15}$$

By (3.15),

$$||\Pi_h| - 1| \leq |\mathcal{E}(h)\Delta\Pi_h| \leq \sqrt{\frac{2\delta}{c_1}}\,. \tag{3.16}$$

By (1.1), we may further decrease the value of η introduced in step one so to have that if J is a regular hexagon in $\mathcal{T}$ with $||J|-1| \leq \eta$, then it makes sense to scale J with respect to its barycenter in order to obtain a unit-area regular hexagon J' with $\mathrm{hd}(\partial J,\partial J') \leq C\,||J|-1|$. In particular, by (3.12) and (3.16), up to decrease the value of ε_0 we can define unit-area hexagons K_h in $\mathcal{T}$ with the property that

$$\mathrm{hd}(\partial K_h, \partial H_h^*) \le C\,||H_h^*| - 1| = C\,||\Pi_h| - 1| \le C\,\sqrt{\delta}\,.$$

By combining this estimate with (3.15) we prove (3.13). By (3.13), $|K_j \Delta \Pi_j| \le C\sqrt{\delta}$ for every j, and thus

$$|K_h \Delta K_{h+1}| \ge |\mathcal{E}(h)\Delta\mathcal{E}(h+1)| - \sum_{j=h}^{h+1} |\mathcal{E}(j)\Delta K_j| \ge 2 - C\,\sqrt{\delta} - \sum_{j=h}^{h+1} |\mathcal{E}(j)\Delta \Pi_j|\,.$$

In particular, (3.14) follows from (3.15).

Step three: We claim the existence of a tiling $\mathcal{H}_0 = v + \mathcal{H}$ of $\mathcal{T}$ such that

$$\mathrm{hd}(\Sigma(\mathcal{E}), \Sigma(\mathcal{H}_0)) \le C\,\sqrt{\delta}\,, \qquad |v| \le C\,\varepsilon_0\,, \tag{3.17}$$

where here and in the rest of this step, C denotes a constant depending from $|\mathcal{T}|$ only. Let us recall from Figure 1 that the chambers of $\mathcal{H}$ are ordered so that $\{\mathcal{H}(h)\}_{h=1}^{\beta}$ is the "bottom row" in the grid defined by $\mathcal{H}$ and that $\mathcal{H}(1) = H$. Since $\mathcal{E}$ is an $(\varepsilon_0, \mu_0, L)$-perturbation of $\mathcal{H}$ one has

$$\max\Big\{\mathrm{hd}(\partial\mathcal{E}(h), \partial\mathcal{H}(h)), \mathrm{hd}(\partial\Pi_h, \partial\mathcal{H}(h))\Big\} \le \varepsilon_0\,, \qquad \forall h = 1, ..., N\,, \tag{3.18}$$

so that (3.13) implies $\mathrm{hd}(\partial H, \partial K_1) \le C\,\varepsilon_0$. In particular, there exists $|\theta|, |s|, |t| \le C\varepsilon_0$ such that

$$K_1 = (t\sqrt{3}\ell, s\ell) + R_\theta H\,,$$

where, with a slight abuse of notation, $R_\theta H$ denotes the counterclockwise rotation of H by an angle θ around its left-bottom vertex (see step one). Of course, there is no reason to get a better estimate than $|s|, |t| \le C\,\varepsilon_0$ here (indeed, $\mathcal{E}$ itself could just be an ε_0-size translation of $\mathcal{H}$). Nevertheless, if $\theta \ne 0$, then we cannot fit K_1 into an hexagonal honeycomb of $\mathcal{T}$: therefore one expects

$$|\theta| \le C\sqrt{\delta}\,. \tag{3.19}$$

We prove (3.19): set $J_1 = K_1$, let τ_1 be the common edge between Π_1 and Π_2, and let σ_1 and σ_1' be the edges of K_1 and K_2 respectively such that, thanks to (3.13), $\mathrm{hd}(\tau_1, \sigma_1) + \mathrm{hd}(\tau_1, \sigma_1') \le C\,\sqrt{\delta}$. In this way $\mathrm{hd}(\sigma_1, \sigma_1') \le C\,\sqrt{\delta}$, and by (3.14) we can apply step one to deduce

$$\mathrm{hd}(\partial J_2, \partial K_2) \le C\,\mathrm{hd}(\sigma_1, \sigma_1') \le C\sqrt{\delta}\,, \qquad |J_2 \Delta K_2| \le C\,\sqrt{\delta}\,, \tag{3.20}$$

where J_2 is the reflection of J_1 with respect to σ_1. Let now τ_2 be common side between Π_2 and Π_3. By (3.13) and (3.20) we have $\mathrm{hd}(\partial J_2, \partial\Pi_2) + \mathrm{hd}(\partial K_3, \partial\Pi_3) \le C\sqrt{\delta}$, thus there exist edges σ_2 and σ_2' of J_2 and K_3 respectively such that $\mathrm{hd}(\tau_2, \sigma) + \mathrm{hd}(\tau_2, \sigma') \le C\,\sqrt{\delta}$. By (3.14) and (3.20) one has $|J_2 \Delta K_3| \ge 2 - C\sqrt{\delta}$, so that by step one $\mathrm{hd}(\partial J_3, \partial K_3) \le C\sqrt{\delta}$ where J_3 is the reflection of J_2 with respect to σ_2. If we repeat this argument β-times, then we find regular unit-area hexagons $J_1, ..., J_\beta$ such that $J_1 = K_1$, J_h is obtained by reflecting J_{h-1} with respect to its "vertical" right edge, and $\mathrm{hd}(\partial J_h, \partial K_h) \le C\,\sqrt{\delta}$ for $h = 1, ..., \beta$. By construction, Π_β and Π_1 also share a common edge τ, and correspondingly J_β and K_1 have edges σ and σ' respectively with $\mathrm{hd}(\tau, \sigma) + \mathrm{hd}(\tau, \sigma') \le C\,\sqrt{\delta}$. By reflecting J_β with respect to σ we thus find a regular unit area hexagon J_* with

$$\mathrm{hd}(\partial J_*, \partial K_1) \le C\,\sqrt{\delta}\,.$$

At the same time, since J_* has been obtained by iteratively reflecting $J_1 = K_1$ with respect to its "vertical" right edge, we find that

$$\mathrm{hd}(\partial J_*, \partial J_1) \ge \frac{|\theta|}{C}\,.$$

Thus (3.19) holds. As a consequence, up to apply to K_1 a rotation of size $C\sqrt{\delta}$, one can assume that

$$K_1 = (t\sqrt{3}\ell, s\ell) + H\,, \qquad \text{for some } |t|,|s| \le C\,\varepsilon_0\,. \tag{3.21}$$

In particular, if we set $\mathcal{H}_0(h) = (t\sqrt{3}\ell, s\ell) + \mathcal{H}(h)$, then $\mathcal{H}_0$ defines a unit-area tiling of $\mathcal{T}$ by regular hexagons. By arguing as in the proof of (3.19), one easily sees that

$$\mathrm{hd}(\partial\Pi_h, \partial\mathcal{H}_0(h)) \le C\sqrt{\delta}\,, \qquad \forall h = 1,...,N\,. \tag{3.22}$$

In particular, the set of vertexes of Π_h and $\mathcal{H}_0(h)$ lie at distance $C\sqrt{\delta}$. Since $\Sigma(\mathcal{E})$ is the set of all the vertexes of the Π_hs, we complete the proof of (3.17).

Step four: We show that if μ_0 is small enough with respect to L, and ε_0 is small enough with respect to μ_0 and $|\mathcal{T}|$, then there exists a $C^{1,1}$-diffeomorphism f_0 between $\partial\mathcal{H}_0$ and $\partial\mathcal{E}$ such that

$$\|f_0\|_{C^{1,1}(\partial\mathcal{H}_0)} \le C\,, \qquad \|f_0 - \mathrm{Id}\|_{C^1(\partial\mathcal{H}_0)} \le C\,\mu_0\,, \tag{3.23}$$

$$\|(f_0 - \mathrm{Id})\cdot\tau_0\|_{C^1(\partial\mathcal{H}_0)} \le C \sup_{\Sigma(\mathcal{H}_0)} |f_0 - \mathrm{Id}|\,. \tag{3.24}$$

where C depends on L only. The map f_0 is more useful than the map f appearing in (3.2) because the best estimate for $f - \mathrm{Id}$ on $\Sigma(\mathcal{H})$ is of order ε_0, while, thanks to (3.17), we have a much more precise information about $f_0 - \mathrm{Id}$ on $\Sigma(\mathcal{H}_0)$, namely

$$\sup_{\Sigma(\mathcal{H}_0)} |f_0 - \mathrm{Id}| \le C\sqrt{\delta}\,. \tag{3.25}$$

(In (3.25), C depends on $|\mathcal{T}|$.) Let us also notice that we cannot just define f_0 by composing f with the translation bringing $\partial\mathcal{H}_0$ onto $\partial\mathcal{H}$, because this translation is $O(\varepsilon_0)$, and thus the resulting map f_0 would still have tangential displacement $O(\varepsilon_0)$. We thus need a more precise construction, directly relating $\partial\mathcal{H}_0$ and $\partial\mathcal{E}$.

To this end, we fix an edge σ of $\mathcal{H}$, and set $\sigma_0 = v+\sigma$, so that σ_0 is an edge of $\mathcal{H}_0$. We denote by τ_0 and $\nu_0 = \tau_0^\perp$ the constant tangent and normal unit-vector fields to σ_0 (and, obviously, to σ). We let $\gamma = f(\sigma)$ and set $\tau(x) = \nabla^\sigma f(f^{-1}(x))[\tau_0]$ and $\nu(x) = \tau(x)^\perp$, where $\nabla^\sigma f$ denotes the tangential gradient of f with respect to σ. Finally, we set $[\sigma_0]_t = \{x \in \sigma_0 : \mathrm{dist}(x, \mathrm{bd}\,(\sigma_0)) > t\}$ for $t > 0$. By [CLM14, Theorem 2.6, Proposition B.2], given $M > 0$ there exist positive constants C_1 and μ_1 (depending on M and σ_0) such that if, for some $\rho \le \mu_1^2$, γ satisfies the following properties

(a) $\mathrm{hd}(\sigma_0, \gamma) + \mathrm{hd}(\mathrm{bd}\,(\sigma_0), \mathrm{bd}\,(\gamma)) \le \rho$;
(b) $|\tau(p) - \tau_0| + |\tau(q) - \tau_0| \le \rho$ where $\{p, q\} = f(\mathrm{bd}\,(\sigma))$;
(c) there exists a map $\psi_0 \in C^{1,1}([\sigma_0]_\rho)$ such that

$$[\gamma]_{3\rho} \subset (\mathrm{Id} + \psi_0\nu_0)\big([\sigma_0]_\rho\big) \subset \gamma\,,$$

$$\|\psi_0\|_{C^{1,1}([\sigma_0]_\rho)} \le M\,, \qquad \|\psi_0\|_{C^1([\sigma_0]_\rho)} \le \rho\,;$$

(d) $|\nu(x) - \nu(y)| \le M\,|x - y|$ and $|\nu(x)\cdot(y - x)| \le M\,|x - y|^2$ for every $x, y \in \gamma$,

then, there exists a $C^{1,1}$-diffeomorphism f_0 between σ_0 and γ such that $f_0(\mathrm{bd}\,(\sigma_0)) = \mathrm{bd}\,(\gamma)$ and

$$\|f_0\|_{C^{1,1}(\sigma_0)} \le C_1\,, \qquad \|f_0 - \mathrm{Id}\|_{C^1(\sigma_0)} \le \frac{C_1}{\mu_1}\,\rho\,,$$

$$\|(f_0 - \mathrm{Id})\cdot\tau_0\|_{C^1(\sigma_0)} \le \frac{C_1}{\mu_1} \sup_{\mathrm{bd}\,(\sigma_0)} |f_0 - \mathrm{Id}|\,.$$

(Since σ_0 is just a segment of fixed length $\ell = (12)^{1/4}/3$, we shall not stress the dependence of C_1 and μ_1 on σ_0.) We notice that property (a) holds provided $\rho \ge C\varepsilon_0$ for some C depending on $|\mathcal{T}|$ only: indeed, by $\|f - \mathrm{Id}\|_{C^0(\sigma)} \le \varepsilon_0$ one finds $\mathrm{hd}(\sigma, \gamma) + \mathrm{hd}(\mathrm{bd}\,(\sigma), \mathrm{bd}\,(\gamma)) \le \varepsilon_0$, while $|v| \le C\,\varepsilon_0$ (recall (3.17)) gives $\mathrm{hd}(\sigma, \sigma_0) \le C\,\varepsilon_0$. Similarly, property (b) holds if $\rho \ge \varepsilon_0$, as $\tau(x) = \nabla^\sigma f(f^{-1}(x))[\tau_0]$ and $\|f - \mathrm{Id}\|_{C^1(\sigma)} \le \varepsilon_0$. Property (d) follows easily from $\|f\|_{C^{1,1}(\sigma)} \le L$

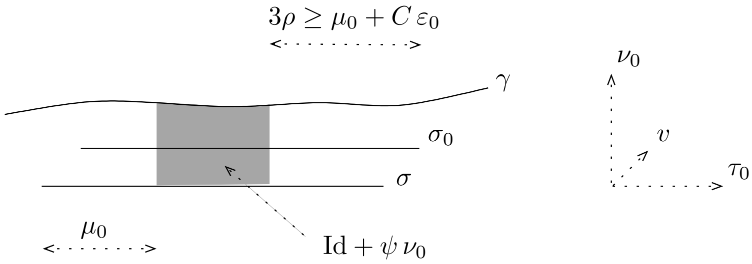

FIGURE 3. The function ψ_0 is defined by computing the values of ψ after a projection of σ_0 onto σ.

and $\|f - \mathrm{Id}\|_{C^1(\sigma)} \le \varepsilon_0$ with $M = M(L)$. Finally, concerning property (c), we notice that by exploiting the fact that $\mathcal{E}$ is an $(\varepsilon_0, \mu_0, L)$-perturbation of $\mathcal{H}$ and setting $\psi = (f - \mathrm{Id}) \cdot \nu_0$, one has $\psi \in C^{1,1}([\sigma]_{\mu_0})$ with

$$[\gamma]_{\mu_0+2\varepsilon_0} \subset (\mathrm{Id} + \psi\nu_0)\big([\sigma]_{\mu_0}\big) \subset \gamma\,, \tag{3.26}$$

$$\|\psi\|_{C^{1,1}([\sigma]_{\mu_0})} \le L\,, \qquad \|\psi\|_{C^1([\sigma]_{\mu_0})} \le \varepsilon_0\,, \tag{3.27}$$

where the first inclusion in (3.26) follows from $\|f - \mathrm{Id}\|_{C^0(\sigma_0)} \le \varepsilon_0$ and $\gamma = f(\sigma)$. By exploiting (3.26), (3.27), and the fact that $\sigma_0 = v + \sigma$ with $|v| \le C\,\varepsilon_0$ by (3.17), one can find two constants $C_2 \le C_3$ (both depending just on $|\mathcal{T}|$) and $\psi_0 \in C^{1,1}([\sigma]_{\mu_0+C_2\,\varepsilon_0})$ such that properties (a), (b) and (d) hold with $\rho = \mu_0 + C_2\,\varepsilon_0$, and

$$[\gamma]_{\mu_0+C_3\,\varepsilon_0} \subset (\mathrm{Id} + \psi_0\nu_0)\big([\sigma_0]_{\mu_0+C_2\,\varepsilon_0}\big) \subset \gamma\,, \tag{3.28}$$

$$\|\psi_0\|_{C^{1,1}([\sigma_0]_{\mu_0+C_2\,\varepsilon_0})} \le L\,, \qquad \|\psi_0\|_{C^1([\sigma_0]_{\mu_0+C_2\,\varepsilon_0})} \le \varepsilon_0\,, \tag{3.29}$$

see Figure 3. Of course one can entail $3\rho > \mu_0 + C_3\,\varepsilon_0$ by requiring ε_0 small enough with respect to μ_0: in this way, property (c) follows from (3.28) and (3.29). Summarizing, we have shown that if μ_0 is small enough depending on L (that is, depending on $M = M(L)$), and if ε_0 is small enough with respect to μ_0 and $|\mathcal{T}|$, then properties (a)–(d) hold with $\rho = \mu_0 + C_2\,\varepsilon_0$. Up to further decrease the values of μ_0 and ε_0 we may entail $\rho \le \mu_1^2$, and thus, thanks to [CLM14, Theorem 2.6, Proposition B.2], find a $C^{1,1}$-diffeomorphism f_0 between σ_0 and γ such that $f_0(\mathrm{bd}\,(\sigma_0)) = \mathrm{bd}\,(\gamma)$ and

$$\|f_0\|_{C^{1,1}(\sigma_0)} \le C\,, \qquad \|f_0 - \mathrm{Id}\|_{C^1(\sigma_0)} \le C\,\mu_0\,,$$
$$\|(f_0 - \mathrm{Id}) \cdot \tau_0\|_{C^{1,1}(\sigma_0)} \le C \sup_{\mathrm{bd}\,(\sigma_0)} |f_0 - \mathrm{Id}|\,,$$

where C depends on L only. By repeating this construction on every edge σ_0 of $\partial\mathcal{H}_0$ we complete the proof of (3.23) and (3.24).

Step four: With a little abuse of notation, let us denote by $\{\sigma_i\}_{i=1}^{3N}$ the family of segments such that $\partial\mathcal{H}_0 = \bigcup_{i=1}^{3N} \sigma_i$. For every i let τ_i denote a constant tangent unit vector to σ_i. If we set $g = f_0 - \mathrm{Id}$, then we have

$$P(\mathcal{E}) - P(\mathcal{H}) = \sum_{i=1}^{3N} \int_{\sigma_i} \big(|\nabla^{\sigma_i} g[\tau_i] + \tau_i| - 1\big)\, d\mathcal{H}^1\,,$$

where, by $\|g\|_{C^1(\partial\mathcal{H}_0)} \le \mu_0$, $\sqrt{1+t} \ge 1 + t/2 - t^2/8 - C\,|t|^3$ ($t \ge -1$), and provided μ_0 is small enough,

$$\begin{aligned} |\nabla^{\sigma_i} g[\tau_i] + \tau_i| - 1 &= \sqrt{1 + 2\tau_i \cdot \nabla^{\sigma_i} g[\tau_i] + |\nabla^{\sigma_i} g[\tau_i]|^2} - 1 \\ &\ge \tau_i \cdot \nabla^{\sigma_i} g[\tau_i] + \frac{|\nabla^{\sigma_i} g[\tau_i]|^2}{2} - \frac{|2\,\tau_i \cdot \nabla^{\sigma_i} g[\tau_i]|^2}{8} - C\,\mu_0\,|\nabla^{\sigma_i} g[\tau_i]|^2\,. \end{aligned}$$

Let $\Sigma(\mathcal{H}_0)=\{p_j\}_{j=1}^{2N}$, and for $p_j\in\mathrm{bd}\,(\sigma_i)$ denote by v_j^i the tangent unit vector to σ_i at p_j pointing outside σ_i. In this way,

$$\sum_{i=1}^{3N}\int_{\sigma_i}\tau_i\cdot\nabla^{\sigma_i}g[\tau_i]\,d\mathcal{H}^1=\sum_{j=1}^{2N}\sum_{\{i:p_j\in\mathrm{bd}\,(\sigma_i)\}}v_j^i\,g(p_j)=0\,,$$

since $\{i:p_j\in\mathrm{bd}\,(\sigma_i)\}=\{i_1,i_2,i_3\}$ with $v_j^{i_2}$ and $v_j^{i_3}$ obtained from $v_j^{i_1}$ by counterclockwise rotations of $2\pi/3$ and $4\pi/3$ respectively. Hence, if we set $\nu_i=\tau_i^\perp$, then

$$P(\mathcal{E})-P(\mathcal{H})\ge\sum_{i=1}^{3N}\int_{\sigma_i}\frac{|\nu_i\cdot\nabla^{\sigma_i}g[\tau_i]|^2}{2}\,d\mathcal{H}^1-C\,\mu_0\int_{\sigma_i}|\nabla^{\sigma_i}g[\tau_i]|^2\,d\mathcal{H}^1\,. \tag{3.30}$$

By (3.24) and (3.25) we find that

$$\sup_{1\le i\le 3N}\|\tau_i\cdot\nabla^{\sigma_i}g[\tau_i]\|_{C^0(\sigma_i)}\le C\sqrt{\delta}\,,$$

where C depends on L and $|\mathcal{T}|$. By combining this last inequality with (3.30), and provided μ_0 is small enough with respect to L and $|\mathcal{T}|$, we find

$$C\sqrt{\delta}\ge\sum_{i=1}^{3N}\int_{\sigma_i}|\nabla^{\sigma_i}g[\tau_i]|\ge\sum_{i=1}^{3N}\|g-g(p_{j(i)})\|_{C^0(\sigma_i)}\,, \tag{3.31}$$

where for each $i=1,...,3N$ we have picked $p_{j(i)}\in\mathrm{bd}\,(\sigma_i)$. By (3.25) we have $|g(p_{j(i)})|\le C\sqrt{\delta}$, so that (3.31) implies

$$C\sqrt{\delta}\ge\sum_{i=1}^{3N}\|g\|_{C^0(\sigma_i)}=\|f_0-\mathrm{Id}\|_{C^0(\partial\mathcal{H}_0)}\,. \tag{3.32}$$

Since f_0 is a bijection between $\partial\mathcal{H}_0$ and $\partial\mathcal{E}$, we find that $\|f_0-\mathrm{Id}\|_{C^0(\partial\mathcal{H}_0)}\ge\mathrm{hd}(\partial\mathcal{H}_0,\partial\mathcal{E})$ and thus prove (3.3). We now notice that if $u:(a,b)\to\mathbb{R}$ is a Lipschitz function, then

$$\|u\|^2_{C^0(a,b)}\le 8\,\max\Big\{\mathrm{Lip}\,(u),\frac{1}{b-a}\Big\}\,\|u\|_{L^1(a,b)}\,. \tag{3.33}$$

Indeed, let $x_0\in(a,b)$ be such that $u(x_0)=\|u\|_{C^0(a,b)}$ and set $L=\mathrm{Lip}\,(u)$, $r=|u(x_0)|/4L$. If $(x_0,x_0+r)\subset(a,b)$ or $(x_0-r,x_0)\subset(a,b)$, then by integrating $|u(y)|\ge|u(x_0)|-L|x_0-y|$ in y over (x_0,x_0+r) or over (x_0-r,x_0) respectively, we find

$$\int_{(a,b)}|u|\ge r\,|u(x_0)|-L\frac{r^2}{2}\ge\frac{|u(x_0)|^2}{8L}\,;$$

otherwise one has $b-a\le 2r$ and thus $|u(y)|\ge|u(x_0)|/2$ for every $y\in(a,b)$. In order to complete the proof of (3.4) we just need to use (3.32) and to combine the first inequality in (3.31) with $\|f_0\|_{C^{1,1}(\partial\mathcal{H})}\le C$ and with (3.33) (applied to the components of $\nabla^{\partial^*\mathcal{H}_0}(f_0-\mathrm{Id})$). □

4. Proof of Theorem 1.1, Theorem 1.3 and Theorem 1.4

We start by introducing the following fundamental tool in the study of isoperimetric problems with multiple volume constraints. This kind of construction is originally found in [Alm76], and it is fully detailed in our setting in [Mag12, Sections 29.5-29.6], see also [CLM14, Theorem C.1]. Since the version of this lemma needed here does not seem to appear elsewhere, we give some details of the proof.

Lemma 4.1 (Volume-fixing variations)**.** *If $\mathcal{E}_0$ is a N-tiling of $\mathcal{T}$, $\gamma\in(0,1]$ and $L>0$, then there exist positive constants r_0, σ_0, ε_0, and C_0 (depending on $\mathcal{E}_0$, L and γ only) with the following property: if $\eta\in\mathbb{R}^N$ with $\sum_{h=1}^N\eta_h=0$, $\Phi\in\mathrm{Lip}(\mathcal{T}\times S^1;(0,\infty))$, $\psi\in C^{1,\gamma}(\mathcal{T};(0,\infty))$, $x\in\mathcal{T}$, and $\mathcal{E}$ and $\mathcal{F}$ are N-tilings of $\mathcal{T}$ with*

$$\|\Phi\|_{C^{0,1}(\mathcal{T}\times S^1)}+\|\psi\|_{C^{1,\gamma}(\mathcal{T})}\le L\,, \tag{4.1}$$
$$\mathrm{d}(\mathcal{E},\mathcal{E}_0)\le\varepsilon_0\,, \tag{4.2}$$
$$\mathcal{F}\Delta\mathcal{E}\subset\subset B_{x,r_0}\,,\qquad |\eta|<\sigma_0\,, \tag{4.3}$$

then there exists a N-cluster $\mathcal{F}'$ such that

$$\mathcal{F}'\Delta\mathcal{F}\quad\subset\subset\quad\mathcal{T}\setminus\overline{B}_{x,r_0}\,, \tag{4.4}$$
$$\int_{\mathcal{F}'(h)}\psi\quad=\quad\eta_h+\int_{\mathcal{E}(h)}\psi\,, \tag{4.5}$$
$$|\mathbf{\Phi}(\mathcal{F}')-\mathbf{\Phi}(\mathcal{F})|\quad\le\quad C_0\,P(\mathcal{E})\Big(\sum_{h=1}^N\Big|\int_{\mathcal{F}(h)}\psi-\int_{\mathcal{E}(h)}\psi\Big|+|\eta|\Big)\,, \tag{4.6}$$
$$|\mathrm{d}(\mathcal{F}',\mathcal{E})-\mathrm{d}(\mathcal{F},\mathcal{E})|\quad\le\quad C_0\,P(\mathcal{E})\Big(\sum_{h=1}^N\Big|\int_{\mathcal{F}(h)}\psi-\int_{\mathcal{E}(h)}\psi\Big|+|\eta|\Big)\,. \tag{4.7}$$

Remark 4.2. In practice we are going to apply this lemma either with $\eta=0$ and $\mathcal{F}\Delta\mathcal{E}\ne\emptyset$, or with $\eta\ne0$ and $\mathcal{F}=\mathcal{E}$. In the first case, we are given a compactly supported variation $\mathcal{F}$ of $\mathcal{E}$, and we want to modify $\mathcal{F}$ outside of B_{x,r_0} into a new N-tiling $\mathcal{F}'$ so that $\int_{\mathcal{F}'(h)}\psi=\int_{\mathcal{E}(h)}\psi$ for every $h=1,...,N$. In the second case we want to modify $\mathcal{E}$ so that $\int_{\mathcal{E}(h)}\psi$ is changed into $\eta_h+\int_{\mathcal{E}(h)}\psi$ for every $h=1,...,N$. In both cases, we want to control the change in $\mathbf{\Phi}$-energy and the change in distance from $\mathcal{E}$ needed to pass from $\mathcal{F}$ to $\mathcal{F}'$. The name attached to the lemma is motivated by the fact that one usually takes $\psi\equiv1$.

Proof of Lemma 4.1. The basic step consists in picking up a ball $B_{z,\varepsilon}$ and notice that if $T\in C^\infty_c(B_{z,\varepsilon};\mathbb{R}^2)$ and $f_t(x)=x+t\,T(x)$ for $x\in\mathcal{T}$, then for every Borel set $E\subset\mathcal{T}$ the function $\mathbf{\Psi}_E(t)=\int_{f_t(E)}\psi=\int_E\psi(f_t)Jf_t$ is of class $C^{1,\gamma}(-t_0,t_0)$ with

$$\|\mathbf{\Psi}_E\|_{C^{1,\gamma}(-t_0,t_0)}\le C\,,\qquad\Big|\int_{f_t(E)}\psi-\int_E\psi-t\int_E\mathrm{div}\,(\psi\,T)\Big|\le C\,|t|^{1+\gamma}\,, \tag{4.8}$$

where t_0 and C denote positive constants depending only on γ, L, $|\mathcal{T}|$, and $\|T\|_{C^1(\mathcal{T})}$. Next, one considers two families of balls $\{B_{z_i,\varepsilon}\}_{i=1}^M$ and $\{B_{y_i,\varepsilon}\}_{i=1}^M$ with $z_i\,,y_i\in\partial^*\mathcal{E}_0(h(i))\cap\partial^*\mathcal{E}_0(k(i))$ (for $1\le h(i)\ne k(i)\le N$ to be properly chosen – see condition (4.13) below) and with $|z_i-z_j|>2\varepsilon$ and $|y_i-y_j|>2\varepsilon$ for $1\le i<j\le M$ and $|y_i-z_j|>2\varepsilon$ for $1\le i\le j\le M$. For each i we can find $T_i\in C^\infty_c(B_{z_i,\varepsilon};\mathbb{R}^2)$ such that

$$\int_{\mathcal{E}_0(h(i))}\mathrm{div}\,(\psi\,T_i)=1=-\int_{\mathcal{E}_0(k(i))}\mathrm{div}\,(\psi\,T_i)\,, \tag{4.9}$$
$$\int_{\mathcal{E}_0(j)}\mathrm{div}\,(\psi\,T_i)=0\,,\qquad j\ne h(i),k(i)\,. \tag{4.10}$$

Let us consider the smooth map $f:(-t_0,t_0)^M\times\mathcal{T}\to\mathcal{T}$ defined by $f(\mathbf{t},x)=x+\sum_{i=1}^M t_i\,T_i(x)$, $\mathbf{t}=(t_1,...,t_M)$, so that for $t_0>0$ small enough $f(\mathbf{t},\cdot)$ is a smooth diffeomorphism of $\mathcal{T}$ with

$$\mathrm{spt}(f(\mathbf{t},\cdot)-\mathrm{Id})\subset\subset\bigcup_{i=1}^M B_{z_i,\varepsilon}\,. \tag{4.11}$$

If we let $\alpha=(\alpha_1,...,\alpha_N)\in C^{1,\gamma}((-t_0,t_0)^M;\mathbb{R}^N)$ be defined by

$$\alpha_h(\mathbf{t})=\int_{f(\mathbf{t},\mathcal{E}(h))}\psi-\int_{\mathcal{E}(h)}\psi\,,\qquad h=1,...,N\,,$$

then $\alpha((-t_0,t_0)^M)\subset V=\{\eta\in\mathbb{R}^N:\sum_{h=1}^N\eta_h=0\}$, $\|\alpha\|_{C^{1,\gamma}((-t_0,t_0)^M)}\le C$, and, by (4.1), (4.2), (4.8), (4.9) and (4.10), one finds

$$\Big|\frac{\partial\alpha_{h(i)}}{\partial t_i}(\mathbf{t})-1\Big|+\Big|\frac{\partial\alpha_{k(i)}}{\partial t_i}(\mathbf{t})+1\Big|+\max_{j\ne h(i),k(i)}\Big|\frac{\partial\alpha_j}{\partial t_i}(\mathbf{t})\Big|\le C\,\varepsilon_0\,,\tag{4.12}$$

where, from now on, C denotes a constant depending only on L, γ, $|\mathcal{T}|$, and $\mathcal{E}_0$ (through $\|T_i\|_{C^1(\mathcal{T})}$). Provided $h(i)$ and $k(i)$ are suitable defined (see [Mag12, Step one, Proof of Theorem 29.14]) one can entail from (4.12) that

$$\dim\nabla\alpha(\mathbf{0})=N-1\,.\tag{4.13}$$

By the implicit function theorem there exists $\sigma_1>0$ and an open neighborhood U of $\mathbf{0}\in\mathbb{R}^M$ such that $\alpha^{-1}\in C^{1,\gamma}(V_{\sigma_1};U)$ with $V_{\sigma_1}=\{\eta\in V:|\eta|<\sigma_1\}$, and

$$|\alpha^{-1}(\eta)|\le C\,|\eta|\,,\qquad\forall\eta\in V_{\sigma_1}\,.\tag{4.14}$$

Similarly, we may construct functions g and β, analogous to f and α, starting from the family of balls $\{B_{y_i,\varepsilon}\}_{i=1}^M$. Now let $\mathcal{F}$ be as in (4.3), and assume that

$$\sigma_0+\|\psi\|_{C^0(\mathcal{T})}\pi\,r_0^2<\sigma_1\,.\tag{4.15}$$

Up to further decrease the value of r_0 with respect to ε, we may also assume that $\overline{B}_{x,r_0}\cap\overline{B}_{z_i,\varepsilon}=\emptyset$ for every $i=1,...,M$, or that $\overline{B}_{x,r_0}\cap\overline{B}_{y_i,\varepsilon}=\emptyset$ for every $i=1,...,M$. Without loss of generality we may assume to be in the former case, and set

$$\mathcal{F}'(h)=\big(\mathcal{F}(h)\cap B_{x,r_0}\big)\cup\big(f(\alpha^{-1}(w),\mathcal{E}(h))\setminus B_{x,r_0}\big)\,,\qquad 1\le h\le N\,,$$

where w_h is defined by the identity

$$\int_{\mathcal{F}(h)\cap B_{x,r_0}}\psi=\eta_h-w_h-\int_{\mathcal{E}(h)\cap B_{x,r_0}}\psi\,,\qquad 1\le h\le N\,.$$

By construction one has (4.4). Moreover, by definition of w_h, by (4.11) and since $\overline{B}_{x,r_0}\cap\overline{B}_{z_i,\varepsilon}=\emptyset$ for every $i=1,...,M$, one has

$$\begin{aligned}\int_{\mathcal{F}'(h)}\psi-\int_{\mathcal{E}(h)}\psi&=\int_{\mathcal{F}(h)\cap B_{x,r_0}}\psi+\int_{f(\alpha^{-1}(w),\mathcal{E}(h))\setminus B_{x,r_0}}\psi-\int_{\mathcal{E}(h)}\psi\\&=\eta_h-w_h+\int_{f(\alpha^{-1}(w),\mathcal{E}(h))\setminus B_{x,r_0}}\psi-\int_{\mathcal{E}(h)\setminus B_{x,r_0}}\psi\\&=\eta_h-w_h+\int_{f(\alpha^{-1}(w),\mathcal{E}(h))}\psi-\int_{\mathcal{E}(h)}\psi=\eta_h-w_h+\alpha_h(\alpha^{-1}(w))\,.\end{aligned}$$

By (4.3) and (4.15) one has $|w|<\sigma_1$, so that (4.5) is proved. We now notice that by [DPM14, Equation (2.9)]

$$\mathbf{\Phi}(f(\mathbf{t},E))=\int_{f(\mathbf{t},\partial^*E)}\Phi(y,\nu_{f_t(E)}(y))\,d\mathcal{H}^1(y)=\int_{\partial^*E}\Phi\Big(f_t(x),\mathrm{cof}\nabla f_t(x)[\nu_E(x)]\Big)\,d\mathcal{H}^1(x)\,,$$

so that, by (4.1), $|\mathbf{\Phi}(f(\mathbf{t},E))-\mathbf{\Phi}(E)|\le C\,|t|\,P(E)$. By (4.14) we immediately deduce (4.6). Finally (4.7) is obtained by exploiting [CLM14, Lemma C.2]. □

We now translate the improved convergence theorem for planar bubble clusters from [CLM14] in the case of tilings of $\mathcal{T}$. One says that a N-tiling $\mathcal{E}$ of $\mathcal{T}$ is (Λ, r_0)-*minimizing* if

$$P(\mathcal{E}) \le P(\mathcal{F}) + \Lambda\, \mathrm{d}(\mathcal{E}, \mathcal{F}) ,$$

whenever $\mathcal{F}$ is a N-tiling of $\mathcal{T}$ and $\mathcal{E}\Delta\mathcal{F} = \bigcup_{h=1}^N \mathcal{E}(h)\Delta\mathcal{F}(h) \subset\subset B_{x,r_0}$ for some $x \in \mathcal{T}$. If $\mathcal{E}$ is a (Λ, r_0)-minimizing tiling of $\mathcal{T}$, then (by a trivial adaptation of, say, [CLM14, Theorem 3.16]) $\mathcal{E}$ is of class $C^{1,1}$. Moreover, the curves γ_i and the points p_j in (3.1) are such that each γ_i has distributional curvature bounded by Λ, and for every p_j there exists exactly three curves from $\{\gamma_i\}_{i\in I}$ which share p_j as a common boundary point, and meet at p_j by forming three 120 degrees angles.

We notice that, by (1.2), the reference honeycomb $\mathcal{H}$ is a $(0, \infty)$-minimizing unit-area tiling of $\mathcal{T}$. The following result is what we call an *improved convergence theorem*.

Theorem 4.3. *Given $\Lambda \ge 0$, there exist positive constants L and $\mu_* > 0$ (depending on Λ and $\mathcal{H}$) with the following property. If $N = |\mathcal{T}|$, $\mu < \mu_*$ and $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ is a sequence of (Λ, r_0)-minimizing N-tilings of $\mathcal{T}$ (for some $r_0 > 0$) with $\mathrm{d}(\mathcal{E}_k, \mathcal{H}) \to 0$ as $k \to \infty$, then there exist $k(\mu) \in \mathbb{N}$ and, for every $k \ge k(\mu)$, a $C^{1,1}$-diffeomorphism f_k with*

$$\sup_{k\ge k(\mu)} \|f_k\|_{C^{1,1}(\partial\mathcal{H})} \le L , \qquad \lim_{k\to\infty} \|f_k - \mathrm{Id}\|_{C^1(\partial\mathcal{H})} = 0 , \tag{4.16}$$

$$\boldsymbol{\tau}_{\mathcal{H}}(f_k - \mathrm{Id}) = 0 \quad \text{on } [\partial\mathcal{H}]_\mu , \qquad \|\boldsymbol{\tau}_{\mathcal{H}}(f_k - \mathrm{Id})\|_{C^1(\partial^*\mathcal{H})} \le \frac{L}{\mu} \sup_{\Sigma(\mathcal{H})} |f_k - \mathrm{Id}| . \tag{4.17}$$

In particular, $\mathcal{E}_k$ is a (ε_k, μ, L)-perturbation of $\mathcal{H}$ whenever $k \ge k(\mu)$.

Proof. This is a simple variant of [CLM14, Theorem 1.5], and therefore we omit the details. □

Let us now set

$$\kappa = \kappa(\mathcal{T}) = \inf \liminf_{k\to\infty} \frac{P(\mathcal{F}_k) - P(\mathcal{H})}{\alpha(\mathcal{F}_k)^2} , \tag{4.18}$$

where the infimum is taken among all sequences $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ of unit-area tilings of $\mathcal{T}$ such that $\alpha(\mathcal{F}_k) > 0$ for every $k \in \mathbb{N}$ and $\alpha(\mathcal{F}_k) \to 0$ as $k \to \infty$. By a compactness argument, Theorem 1.1 is equivalent in saying that $\kappa > 0$.

Lemma 4.4. *If $\kappa = 0$, then there exists a sequence of (Λ, r_0)-minimizing unit-area tilings $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ such that $\alpha(\mathcal{E}_k) > 0$ for every $k \in \mathbb{N}$, $\alpha(\mathcal{E}_k) \to 0$ as $k \to \infty$, and*

$$P(\mathcal{E}_k) = P(\mathcal{H}) + o(\alpha(\mathcal{E}_k)^2) , \qquad \text{as } k \to \infty . \tag{4.19}$$

Proof. By definition of κ, and since we are assuming $\kappa = 0$, there exist unit-area tilings $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ of $\mathcal{T}$ such that $\alpha(\mathcal{F}_k) > 0$ for every $k \in \mathbb{N}$, and

$$\alpha(\mathcal{F}_k) \to 0 , \qquad P(\mathcal{F}_k) = P(\mathcal{H}) + o(\alpha(\mathcal{F}_k)^2) , \qquad \text{as } k \to \infty . \tag{4.20}$$

For every $k \in \mathbb{N}$, let $\mathcal{E}_k$ be a minimizer in the variational problem

$$\inf \Big\{ P(\mathcal{E}) + \mathrm{d}(\mathcal{E}, \mathcal{F}_k)^2 \mid \mathcal{E} \text{ unit-area tiling of } \mathcal{T} \text{ with } \alpha(\mathcal{E}) > 0 \Big\} .$$

By comparing $\mathcal{E}_k$ with $\mathcal{F}_k$ and then subtracting $P(\mathcal{H})$ one has

$$P(\mathcal{E}_k) - P(\mathcal{H}) + \mathrm{d}(\mathcal{E}_k, \mathcal{F}_k)^2 \le P(\mathcal{F}_k) - P(\mathcal{H}) = o(\alpha(\mathcal{F}_k)^2) . \tag{4.21}$$

Since $|\alpha(\mathcal{E}_k) - \alpha(\mathcal{F}_k)| \le \mathrm{d}(\mathcal{E}_k, \mathcal{F}_k)$ and $P(\mathcal{E}_k) \ge P(\mathcal{H})$, we conclude that

$$\lim_{k\to\infty} \frac{\alpha(\mathcal{E}_k)}{\alpha(\mathcal{F}_k)} = 1 , \tag{4.22}$$

so that, in particular, $\alpha(\mathcal{E}_k) \to 0$ as $k \to \infty$. Dividing by $\alpha(\mathcal{E}_k)^2$ in (4.21) and using (4.22), we complete the proof of (4.19). We now show that each $\mathcal{E}_k$ is (Λ, r_0)-minimizing in $\mathcal{T}$. Indeed, let r_0, ε_0, σ_0 and C_0 be the constants associated by Lemma 4.1 to $\mathcal{E}_0 = \mathcal{H}$, $\mathbf{\Phi} = P$ and $\psi \equiv 1$. Since $\alpha(\mathcal{E}_k) \to 0$, up to translations we have $\mathrm{d}(\mathcal{E}_k, \mathcal{H}) \le \varepsilon_0$ for k large. We apply Lemma 4.1 with $\mathcal{E} = \mathcal{E}_k$, $\mathcal{F}$ a N-tiling with $\mathcal{E}_k \Delta \mathcal{F} \subset\subset B_{x,r_0}$ for some $x \in \mathcal{T}$, and $\eta = 0$, to find a unit-area tiling $\mathcal{F}'$ such that

$$\begin{aligned} & P(\mathcal{E}_k) \le P(\mathcal{E}_k) + \mathrm{d}(\mathcal{E}_k, \mathcal{F}_k)^2 \le P(\mathcal{F}') + \mathrm{d}(\mathcal{F}', \mathcal{E}_k)^2 \\ \le \;\; & P(\mathcal{F}) + C_0\, P(\mathcal{E}_k)\, |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E}_k)| + \Big(\mathrm{d}(\mathcal{F}, \mathcal{E}_k) + C_0\, P(\mathcal{E}_k)\, |\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E}_k)|\Big)^2 . \end{aligned}$$

Hence $P(\mathcal{E}_k) \le P(\mathcal{F}) + \Lambda\, \mathrm{d}(\mathcal{E}_k, \mathcal{F})$ thanks to $|\mathrm{vol}\,(\mathcal{F}) - \mathrm{vol}\,(\mathcal{E}_k)| \le \mathrm{d}(\mathcal{F}, \mathcal{E}_k)$ and since, for k large enough, $P(\mathcal{E}_k) \le 2\, P(\mathcal{H})$. □

Proof of Theorem 1.1. We argue by contradiction. If the theorem is false, then $\kappa = 0$ and thus by Lemma 4.4 there exists a sequence $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ of (Λ, r_0)-minimizing unit-area tilings of $\mathcal{T}$ such that $\alpha(\mathcal{E}_k) > 0$, $\alpha(\mathcal{E}_k) \to 0$ as $k \to \infty$ and

$$P(\mathcal{E}_k) = P(\mathcal{H}) + o(\alpha(\mathcal{E}_k)^2)\,, \qquad \text{as } k \to \infty\,.$$

Up to translation we may assume that $\alpha(\mathcal{E}_k) = \mathrm{d}(\mathcal{E}_k, \mathcal{H}) \to 0$ as $k \to \infty$. Let L and μ_* be the constants of Theorem 4.3 (which depends on Λ and $\mathcal{H}$) so that for every $\mu < \mu_*$ there exists $k(\mu) \in \mathbb{N}$ such that $\mathcal{E}_k$ is a (ε_k, μ, L)-perturbation of $\mathcal{H}$ for every $k \ge k(\mu)$, with $\varepsilon_k \to 0$ as $k \to \infty$. Let ε_0 and μ_0 be determined as in Theorem 3.1 depending on L and $|\mathcal{T}|$. If we set $\mu = \min\{\mu_*, \mu_0\}$ and increase $k(\mu)$ so that $\varepsilon_k \le \varepsilon_0$ for $k \ge k(\mu)$, then by Theorem 3.1, one finds $v_k \in \mathbb{R}^2$ with $|v_k| \le C\, \varepsilon_k$ such that

$$P(\mathcal{E}_k) - P(\mathcal{H}) \ge c_0\, \mathrm{hd}(\partial\mathcal{E}_k, v_k + \partial\mathcal{H})^2 \ge c\, \mathrm{d}(\mathcal{E}_k, v_k + \mathcal{H})^2 \ge c\, \alpha(\mathcal{E}_k)^2\,,$$

for some positive constant c. We have thus reached a contradiction, and proved the theorem. □

Proof of Theorem 1.3. Let $\mathcal{E}_j = \mathcal{E}_{m^j}$ be minimizers in (1.4) for a sequence $\{m^j\}_{j\in\mathbb{N}}$ such that $\sum_{h=1}^N m_h^j = N$, $m_h^j > 0$ and $m_h^j \to 1$ as $j \to \infty$. By an explicit construction, for every j large enough we can construct a small deformation $\mathcal{H}_j$ of $\mathcal{H}$ such that $|\mathcal{H}_j(h)| = m_h^j$ and $P(\mathcal{H}_j) \le P(\mathcal{H}) + C\, \max_{1\le h\le N} |m_h^j - 1|$, with C independent from j. (Alternatively, one can apply Lemma 4.1 with $\mathcal{E}_0 = \mathcal{E} = \mathcal{F} = \mathcal{H}$, $\mathbf{\Phi} = P$, $\psi \equiv 1$ and $\eta_h = m_h^j - 1$.) As a consequence, $\sup_{j\in\mathbb{N}} P(\mathcal{E}_j) < \infty$, and thus, up to extracting subsequences, $\mathrm{d}(\mathcal{E}_j, \mathcal{E}_0) \to 0$ where $\mathcal{E}_0$ is a unit-area tiling of $\mathcal{T}$. In particular,

$$P(\mathcal{H}) \le P(\mathcal{E}_0) \le \liminf_{j\to\infty} P(\mathcal{E}_j) \le \liminf_{j\to\infty} P(\mathcal{H}) + C \max_{1\le h\le N} |m_h^j - 1| = P(\mathcal{H})\,.$$

By Hales' theorem, up to a relabeling of $\mathcal{E}_0$, $\mathcal{E}_0 = v + \mathcal{H}$ for $v = (t\sqrt{3}\ell, s\ell)$ and $t, s \in [0,1]$. By performing the same relabeling on each $\mathcal{E}_j$, we have $\mathrm{d}(\mathcal{E}_j, v + \mathcal{H}) \to 0$. By exploiting Lemma 4.1 as in the proof of Lemma 4.4 one sees that each $\mathcal{E}_j$ is a (Λ, r_0)-minimizing tiling in $\mathcal{T}$, and then by arguing as in the proof of Theorem 1.1 we find a constant L (depending on Λ and $\mathcal{H}$) such that $\mathcal{E}_j - v$ is an $(\varepsilon_j, \mu_0, L)$-perturbation of $\mathcal{H}$ for μ_0 as in Theorem 3.1 and for $\varepsilon_j \to 0$ as $j \to \infty$. By Theorem 3.1, for j large enough there exist $v_j \to 0$ and $C^{1,1}$-diffeomorphism f_j between $v_j + \partial\mathcal{H}$ and $\partial\mathcal{E}_j - v$, with

$$C \max_{1\le h\le N} |m_h^j - 1| \ge P(\mathcal{E}_j) - P(\mathcal{H}) \ge c\left(\|f_j - \mathrm{Id}\|^2_{C^0(v_j+\partial\mathcal{H})} + \|f_j - \mathrm{Id}\|^4_{C^1(v_j+\partial\mathcal{H})}\right).$$

Theorem 1.3 is then deduced by a contradiction argument. □

Proof of Theorem 1.4. In the following we denote by $\mathcal{E}_\delta$ a minimizer in (1.6), and set

$$\delta = \delta(\Phi, \psi) = \|\Phi - 1\|_{C^0(\mathcal{T}\times S^1)} + \|\psi - 1\|_{C^0(\mathcal{T})}\,,$$

so that $\delta < \delta_0$. We notice that for every $E \subset \mathcal{T}$ of finite perimeter one has

$$
\begin{aligned}
\Big| \int_E \psi - |E| \Big| &\le C\,|E|\,\|\psi - 1\|_{C^0(\mathcal{T})}\,, && (4.23)\\
|\mathbf{\Phi}(E) - P(E)| &\le C\,\min\{P(E)\,,\mathbf{\Phi}(E)\}\,\|\Phi - 1\|_{C^0(\mathcal{T}\times S^1)}\,, && (4.24)
\end{aligned}
$$

where in (4.24) we have also used the fact that $P(E) \le 2\,\mathbf{\Phi}(E)$ provided $\delta_0 \le 1$.

Step one: We claim that, provided δ_0 is small enough, then

$$
\begin{aligned}
\mathbf{\Phi}(\mathcal{E}_\delta) &\le 2\,P(\mathcal{H})\,, && (4.25)\\
P(\mathcal{E}_\delta) &\le P(\mathcal{H}) + C\,\delta\,. && (4.26)
\end{aligned}
$$

Indeed, by considering an explicit small modification of $\mathcal{H}$ (or by applying Lemma 4.1 with $\mathcal{E} = \mathcal{E}_0 = \mathcal{F} = \mathcal{H}$ and $\eta \ne 0$) we can construct a N-tiling $\mathcal{H}'$ of $\mathcal{T}$ such that $\int_{\mathcal{H}'(h)} \psi = N^{-1} \int_{\mathcal{T}} \psi$ for every $h = 1, ..., N$ and $\mathbf{\Phi}(\mathcal{H}') \le \mathbf{\Phi}(\mathcal{H}) + C\,\delta$. By $\mathbf{\Phi}(\mathcal{E}_\delta) \le \mathbf{\Phi}(\mathcal{H}')$ and by (4.24)

$$
\mathbf{\Phi}(\mathcal{E}_\delta) \le \mathbf{\Phi}(\mathcal{H}) + C\,\delta \le P(\mathcal{H}) + C\,\delta\,, \tag{4.27}
$$

which implies (4.25). Again by (4.24), $P(\mathcal{E}_\delta) \le \mathbf{\Phi}(\mathcal{E}_\delta) + C\,\delta$, and (4.27) gives (4.26).

Step two: We now show that if $\delta_j = \delta(\Phi_j, \psi_j) \to 0$ and $\mathcal{E}_j$ is a minimizer in (1.6) associated to Φ_j and ψ_j, then (and up to subsequences and to relabeling the chambers of $\mathcal{E}_j$) $\mathrm{d}(\mathcal{E}_j, v + \mathcal{H}) \to 0$ for some $v = (t\sqrt{3}\ell, s\ell)$, $s, t \in [0,1]$. By (4.25) and since $\mathbf{\Phi}_j(E) \ge P(E)/2$ for every $E \subset \mathcal{T}$ we find that $\sup_{j\in\mathbb{N}} P(\mathcal{E}_j) \le 4\,P(\mathcal{H})$. By compactness, there exists a N-tiling $\mathcal{E}_*$ of $\mathcal{T}$ such that $\mathrm{d}(\mathcal{E}_j, \mathcal{E}_*) \to 0$ (up to subsequences). By (4.23), $\int_{\mathcal{E}_j(h)} \psi_j = N^{-1} \int_{\mathcal{T}} \psi_j$ implies $m_j(h) = |\mathcal{E}_j(h)| \to 1$ for every $h = 1, ..., N$. In particular, $\mathcal{E}_*$ is a unit-area tiling of $\mathcal{T}$, and thus by (1.2), by lower semicontinuity and by (4.26)

$$
P(\mathcal{H}) \le P(\mathcal{E}_*) \le \liminf_{j\to\infty} P(\mathcal{E}_j) \le P(\mathcal{H})\,. \tag{4.28}
$$

By Hales' theorem, up a relabeling, $\mathcal{E}_* = v + \mathcal{H}$.

Step three: Let ε_0, r_0, σ_0 and C_0 be the constants associated to $\mathcal{E}_0 = \mathcal{H}$, Φ and ψ by Lemma 4.1. (Notice that the same constants will work on any translation of $\mathcal{H}$, and that these constants ultimately depend on L and γ only.) By step two we can assume that δ_0 is small enough to entail $\mathrm{d}(\mathcal{E}_\delta, v_\delta + \mathcal{H}) \le \varepsilon_0$ for some translation v_δ. We now claim that there exist positive constants $r_1\,, c_0 > 0$ such that

$$
|\mathcal{E}_\delta(h) \cap B_{x,r}| \ge c_0\,r^2\,, \qquad \forall x \in \partial\mathcal{E}_\delta(h)\,, r < r_1\,, h = 1, ..., N\,. \tag{4.29}
$$

This is a classical argument, see for example [Mag12, Lemma 30.2], and we include some details just for the sake of completeness. Without loss of generality let us set $h = 1$ and fix $x \in \partial\mathcal{E}_\delta(1)$ and $r < r_1 \le r_0$ such that $P(\mathcal{E}_\delta; \partial B_{x,r}) = 0$. There exists $j \in \{1, ..., N\}$ such that

$$
\mathcal{H}^1(\partial^*\mathcal{E}_\delta(1) \cap \partial^*\mathcal{E}_\delta(j) \cap B_{x,r}) \ge \mathcal{H}^1(\partial^*\mathcal{E}_\delta(1) \cap \partial^*\mathcal{E}_\delta(h) \cap B_{x,r})\,, \qquad \forall h \ne 1, j\,. \tag{4.30}
$$

If we set $\mathcal{F}(1) = \mathcal{E}_\delta(1) \setminus B_{x,r}$, $\mathcal{F}(j) = \mathcal{E}_\delta(j) \cup (\mathcal{E}_\delta(1) \cap B_{x,r})$ and $\mathcal{F}(h) = \mathcal{E}_\delta(h)$ for $h \ne 1, j$, then by applying Lemma 4.1 with $\mathcal{E}_0 = v_\delta + \mathcal{H}$, $\mathcal{E} = \mathcal{E}_\delta$, and $\eta = 0$ and setting $u(r) = |\mathcal{E}_\delta(1) \cap B_{x,r}|$, we find that, if $\varepsilon < r_0 - r$, then

$$
\begin{aligned}
\mathbf{\Phi}(\mathcal{E}_\delta; B_{x,r+\varepsilon}) &\le \mathbf{\Phi}(\mathcal{F}; B_{x,r+\varepsilon}) + C_0\,P(\mathcal{E}_\delta)\Big|\int_{\mathcal{E}_\delta(1)\cap B_{x,r}} \psi\Big|\\
&\le \mathbf{\Phi}(\mathcal{E}_\delta; B_{x,r+\varepsilon}) + \hat{\mathbf{\Phi}}(B_{x,r}; \mathcal{E}_\delta(1))\\
&\quad - \int_{\partial^*\mathcal{E}_\delta(1)\cap\partial^*\mathcal{E}_\delta(j)\cap B_{x,r}} \hat{\Phi}(y, \nu_{\mathcal{E}_\delta(1)}(y))\,d\mathcal{H}^1 + C\,u(r)\,,
\end{aligned}
$$

where we have set $\hat\Phi(x,\nu)=(\Phi(x,\nu)+\Phi(x,-\nu))/2$. In particular, by (4.30) and by $2\ge\Phi\ge 1/2$, for every $h\ne 1$ one finds

$$\mathcal{H}^1(\partial^*\mathcal{E}_\delta(1)\cap\partial^*\mathcal{E}_\delta(h)\cap B_{x,r})\le C\big(\mathcal{H}^1(\mathcal{E}_\delta(1)\cap\partial B_{x,r})+u(r)\big)\,,$$

i.e.

$$P(\mathcal{E}_\delta(1);B_{x,r})\le C(u'(r)+u(r))\,,\qquad\text{for a.e. } r<r_1\,.$$

By adding $u'(r)=\mathcal{H}^1(\mathcal{E}_\delta(1)\cap\partial B_{x,r})$ to both sides we find that

$$C(u'(r)+u(r))\ge P(\mathcal{E}_\delta(1)\cap B_{x,r})\ge 2\sqrt{\pi\,u(r)}\,.$$

In particular if r_1 is small enough to give $C\,u(r)\le C\sqrt{\pi r_1^2\,u(r)}\le\sqrt{\pi\,u(r)}$, then we find $\sqrt{u(r)}\le C\,u'(r)$ for a.e. $r<r_1$. This proves (4.29).

Step four: We now conclude the proof. Again by step two and by Lemma 4.1, one can find a unit-area tiling $\mathcal{E}_\delta'$ of $\mathcal{T}$ such that $P(\mathcal{E}_\delta')\le P(\mathcal{E}_\delta)+C\,\delta$ and $\mathrm{d}(\mathcal{E}_\delta',\mathcal{E}_\delta)\le C\,\delta$. By Theorem 1.1 and up to permutations of the chambers of $\mathcal{E}_\delta$, we find a translation v_δ such that

$$c\,\mathrm{d}(\mathcal{E}_\delta',v_\delta+\mathcal{H})^2\le P(\mathcal{E}_\delta')-P(\mathcal{H})\le P(\mathcal{E}_\delta)-P(\mathcal{H})+C\,\delta\le C\,\delta\,,$$

where in the last inequality we have used (4.26). Since $\mathrm{d}(\mathcal{E}_\delta',v_\delta+\mathcal{H})\ge\mathrm{d}(\mathcal{E}_\delta,v_\delta+\mathcal{H})-\mathrm{d}(\mathcal{E}_\delta',\mathcal{E}_\delta)$ we conclude

$$\mathrm{d}(\mathcal{E}_\delta,v_\delta+\mathcal{H})^2\le C\,\delta\,.$$

Setting for the sake of brevity $v_\delta=0$, we now pick $x\in\partial\mathcal{E}_\delta(1)$ such that $\mathrm{dist}(x,\partial\mathcal{H}(1))\ge\mathrm{dist}(y,\partial\mathcal{H}(1))$ for every $y\in\partial\mathcal{E}_\delta(1)$. Let $r=\min\{r_1,\mathrm{dist}(x,\partial\mathcal{H}(1))\}$, so that either $B_{x,r}\subset\mathcal{T}\setminus\mathcal{H}(1)$ or $B_{x,r}\subset\mathcal{H}(1)$. In particular, provided δ_0 is small enough with respect to c_0, either

$$\mathrm{d}(\mathcal{E}_\delta,\mathcal{H})\ge|\mathcal{E}_\delta(1)\setminus\mathcal{H}(1)|\ge|\mathcal{E}_\delta(1)\cap B_{x,r}|\ge c_0\,r^2\ge c_0\ \mathrm{dist}(x,\partial\mathcal{H}(1))^2\,,$$

or

$$\begin{aligned}\mathrm{d}(\mathcal{E}_\delta,\mathcal{H})\ &\ge\ |\mathcal{H}(1)\setminus\mathcal{E}_\delta(1)|\ge|B_{x,r}\setminus\mathcal{E}_\delta(1)|=\Big|\bigcup_{h=2}^N B_{x,r}\cap\mathcal{E}_\delta(h)\Big|\\ &\ge\ (N-1)c_0\,r^2\ge c_0\ \mathrm{dist}(x,\partial\mathcal{H}(1))^2\,;\end{aligned}$$

in both cases, $\partial\mathcal{E}_\delta(1)\subset I_\varepsilon(\partial\mathcal{H}(1))$ for $\varepsilon=C\,\sqrt{\mathrm{d}(\mathcal{E}_\delta,\mathcal{H})}$. By the same argument (based on area density estimates for $\mathcal{H}$, which hold trivially) one finds that $\partial\mathcal{H}(1)\subset I_\varepsilon(\partial\mathcal{E}_\delta(1))$. □

Bibliography

[Alm76] F. J. Jr. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.*, 4(165):viii+199 pp, 1976.

[CL12] M. Cicalese and G. P. Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Rat. Mech. Anal.*, 206(2):617–643, 2012.

[CLM14] M. Cicalese, G. P. Leonardi, and F. Maggi. Improved convergence theorems for bubble clusters. I. The planar case. 2014. preprint arXiv:1409.6652.

[DPM14] G. De Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. 2014. preprint arXiv:1402.0549, accepted on Arch. Rat. Mech. Anal.

[DS02] F. Duzaar and K. Steffen. Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. *J. reine angew. Math.*, 564:73–138, 2002.

[FMP08] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. Math.*, 168:941–980, 2008.

[FMP10] A Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Inv. Math.*, 182(1):167–211, 2010.

[FRS85] J. C. Fisher, D. Ruoff, and J. Shilleto. Perpendicular polygons. *Amer. Math. Monthly*, 92(1):23–37, 1985.

[Hal01] T. C. Hales. The honeycomb conjecture. *Discrete Comput. Geom.*, 25(1):1–22, 2001.

[IN14] E. Indrei and L. Nurbekyan. On the stability of the polygonal isoperimetric inequality. 2014. arXiv:1402.4460.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2012.

[Mor09] F. Morgan. *Geometric measure theory. A beginner's guide. Fourth edition.* Elsevier/Academic Press, Amsterdam, 2009. viii+249 pp.

Dipartmento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
E-mail address: caroccia.marco@gmail.com

Department of Mathematics, University of Texas at Austin, Austin, TX, USA
E-mail address: maggi@math.utexas.edu