THE TAMING OF PLASTIC SLIPS IN VON MISES ELASTO-PLASTICITY

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ABSTRACT. We derive sufficient conditions that prevent the formation of plastic slips in three-dimensional small strain elasto-plasticity when the yield criterion is of the Von Mises type.

Keywords: plasticity, quasi-static evolutions, jump set, space of bounded deformations

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1. INTRODUCTION

Small strain perfect elasto-plasticity is sometimes derided nowadays as over simplistic in its effort to describe creep type behavior in solids and most engineering applications resort to complex models that incorporate some kind of hardening, so as to follow more closely the evolution of the strain in materials that are thought to behave plastically. However, the core elements of plastic behavior should be shared by all models: a yield stress whose attainment signals the onset of plastic flow, and the formation of highly localized shear strains, often called shear bands.

The choice of a proper yield criterion is an important step that has been largely the practitioner's prerogative for lack of a good understanding of the upscaling properties of the various dislocation patterns of most crystalline materials.

On the contrary, the criteria that preside over the formation of shear bands or, more generally, of plastic slips should be exacted from the equations themselves. Yet, to the best of our knowledge, the only evidence of such discontinuities is numerical. Plastic slips are thought to appear as a putative singular limit of numerically computed high strains. As a matter of fact, the abundant literature on plasticity is almost universally silent when it comes to the relationship between plastic flow and plastic slips. To our knowledge, the only acknowledgement of such an intimacy is to be found in [11] (then reiterated in [10, p. 57-58]) where the author(s) derive necessary conditions on the stress tensor on a jump by postulating an ad-hoc flow rule on the jumps through an analogy with the bulk flow rule.

Our goal in this contribution is to offer a more unified insight into that intimacy in the simplest possible elasto-plastic context, that of a Von-Mises type plasticity. To that end, we will lean heavily on the modern mathematical treatment of plasticity which finds its roots in the work of P.-M. Suquet (see e.g. [14],[15]), later completed by various works of R. Temam (see e.g. [16]) and R.V. Kohn and R. Temam (see [12]). That work was revisited, some 20 years later by G. Dal Maso, A. De Simone and M. G. Mora [6] within the framework of the variational theory of rate independent evolutions popularized by A. Mielke (see e.g. [13]).

Those evolutions are quasi-static, that is inertialess. The basic tenet is that the evolution can be viewed as a time-parameterized set of minimization problems for the sum of the elastic energy and of the add-dissipation; see Section 3 for details. The minimizers should also be such that an energy conservation statement, amounting to a kind of first principle in thermodynamics, is satisfied throughout the evolution. Once such an evolution is secured, it remains to show that it satisfies the original system of equations.

In the context of elasto-plasticity, the main hurdle is to recover the so-called flow rule: whenever the (deviatoric part of the) stress reaches the boundary of its admissible set, the plastic strain should flow in the direction normal to that set. The issue at stake is that the plastic strain and its time derivative are measures which may have a Lebesgue-singular part that will not interact well with the stresses because the latter are only defined Lebesgue-almost everywhere. The task is accomplished through rather delicate duality arguments, as recounted in Section 3 below. In particular, the classical flow rule is recovered as recalled in Theorem 3.9.

A boundary flow rule is also obtained in Theorem 3.10. That result, originally derived in [9, Theorem 3.13] is not part and parcel of the classical formulation of elasto-plasticity.

In this paper, we propose to demonstrate that the existence of a variational evolution in the elasto-plastic setting actually implies yet another flow rule, this time on the putative plastic slips. In a Von-Mises setting, that flow rule severely constrains the possible stresses along the plastic slips, as well as the directions of those slips. The ensuing constraints are precisely those that had been postulated in [11]; they are now seen to be a natural outcome of the variational evolution.

As a main consequence, we derive a very simple criterion that, if satisfied on a subdomain and for a time interval, will prevent the onset of additional slips on that subdomain and during that time interval.

The paper is organized as follows.

After a short section (Section 2) devoted to notation and mathematical preliminaries, Section 3 recalls the variational approach to quasi-static elasto-plastic evolutions, specialized to a Von-Mises setting. Details of the derivation of the bulk (and of the boundary) flow rule are also provided. Then, following the approach developed in [6], a flow rule for the singular part of the plastic strain is recovered. It involves a precise representative of the Cauchy stress obtained through an averaging process.

In Section 4, we specialize the flow rule to the case of plastic slips, thereby obtaining the above mentioned restrictions on the form of the precise representative of the deviatoric stress field (Theorem 4.1). Consequently, we formulate in Theorem 4.3 a general result asserting the absence of plastic slips during part of, or the whole evolution. To that effect, the set of Lebesgue points for the Cauchy stress must be large enough.

Such is the case when the external loads and the initial conditions are sufficiently regular as demonstrated in [4]. Under these additional regularity assumptions, we finally propose in Theorem 4.7 a sufficient condition for the application of the general result.

It is remarked that the same arguments would allow one to conclude to the absence of any kind of Lebesgue-singular plastic strain, provided that we knew that the possible diffuse Lebesgue-singular (often called Cantor) part of those strains possesses a rank-one structure. Unfortunately, that result is not available at present (see Remark 4.8) while its counterpart for full gradients is, thanks to a result of G. Alberti [1].

2. Notation and preliminaries

General notation. For $A \subseteq \mathbb{R}^3$, the symbol $\lfloor_A$ stands for "restricted to A".

We will denote by $\mathcal{L}^3$ the Lebesgue volume measure, and by $\mathcal{H}^2$ the two-dimensional Hausdorff measure, which coincides with the usual area measure on sufficiently regular sets (see e.g. [8, Section 2.1] or [3, Section 2.8])

Matrices. We denote by $\mathrm{M}^3_{\mathrm{sym}}$ the set of 3×3-symmetric matrices and by M^3_D the set of trace-free elements of $\mathrm{M}^3_{\mathrm{sym}}$. If M is an element of $\mathrm{M}^3_{\mathrm{sym}}$, then M_D is its deviatoric part, *i.e.*, its projection onto M^3_D with respect to the Frobenius inner product. The symbol $\cdot$ stands for that inner product, as well as for the Euclidean product on $\mathbb{R}^3$, and the symbol $|\cdot|$ for the Frobenius norm, as well as for the Euclidean norm on $\mathbb{R}^3$. The set of symmetric endomorphisms on M^3_D is denoted by $\mathcal{L}_s(\mathrm{M}^3_D)$. For $a, b \in \mathbb{R}^3$, $a \odot b$ stands for the symmetric matrix such that $(a\odot b)_{ij} := (a_i b_j + a_j b_i)/2$.

Functional spaces. Given $E \subseteq \mathbb{R}^3$ Lebesgue measurable, $1 \le p < +\infty$, and M a finite dimensional normed space, $L^p(E;M)$ stands for the space of p-summable functions on E with values in M, with associated norm denoted by $\|\cdot\|_p$. Given $A \subseteq \mathbb{R}^3$ open, $H^1(A;M)$ is the Sobolev space of functions in $L^2(A;M)$ with distributional derivatives in L^2.

Throughout, by "a.e.", we mean "a.e." for the Lebesgue measure on $\mathbb{R}^3$. Otherwise, we will specify the relevant measure.

Finally, let X be a normed space. We denote by $BV(a,b;X)$ and $AC(a,b;X)$ the space of functions with bounded variation and the space of absolutely continuous functions from $[a,b]$ to X, respectively. We recall that the total variation of $f \in BV(a,b;X)$ is defined as

$$\mathcal{V}_X(f;a,b) := \sup\left\{ \sum_{j=1}^{k} \|f(t_j) - f(t_{j-1})\|_X \, : \, a = t_0 < t_1 < \cdots < t_k = b \right\}.$$

Measures. If E is a locally compact separable metric space, and X a finite dimensional normed space, $\mathcal{M}_b(E;X)$ will denote the space of finite Radon measures on E with values in X. For $\mu \in \mathcal{M}_b(E;X)$, we denote by $|\mu|$ its total variation and by μ^s its singular part with respect to $\mathcal{L}^3$. The space $\mathcal{M}_b(E;X)$ is the topological dual of $C_0^0(E;X^*)$, the set of continuous functions u from E to the vector dual X^* of X which "vanish at the boundary", *i.e.*, such that for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ with $|u(x)| < \varepsilon$ for $x \notin K$.

The (kinematic) space BD. In this paper as in previous works on elasto-plasticity the displacement field u lies in the space of functions of bounded deformations

$$BD(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^3) \,:\, Eu \in \mathcal{M}_b(\Omega;\mathrm{M}^3_{\mathrm{sym}})\}$$

endowed with the norm

$$\|u\|_{BD} := \|u\|_1 + \|Eu\|_{\mathcal{M}_b}.$$

Here and in the remainder of the paper $\Omega \subseteq \mathbb{R}^3$ is open, bounded, with Lipschitz boundary. We refer the reader to e.g. [16, Chapter 2], and [2] for background material.

Besides elementary properties of $BD(\Omega)$, we will only appeal to the structure of Eu as a Radon measure: more precisely, as is the case for functions of bounded variation, the measure Eu decomposes as

$$Eu = E^a u + E^j u + E^c u. \tag{2.1}$$

Here $E^a u$ denotes the part of the measure which is absolutely continuous with respect to $\mathcal{L}^3$, so that

$$E^a u = \mathcal{E}u \, d\mathcal{L}^3, \qquad \text{with } \mathcal{E}u \in L^1(\Omega;\mathrm{M}^3_{\mathrm{sym}}).$$

The singular part is further decomposed into a *jump part* $E^j u$ and a *Cantor part* $E^c u$. Specifically,

$$E^j u = [u] \odot \nu_u \, d\mathcal{H}^2 \lfloor J_u,$$

where J_u stands for the *jump set* of u (see [3, Definition 3.67]), $[u]$ being the jump of u across J_u, while $E^c u$ vanishes on Borel sets which are σ-finite with respect to the area measure $\mathcal{H}^2$ (see [2, Proposition 4.4]).

Finally, we say that

$$u_n \overset{*}{\rightharpoonup} u \qquad \text{weakly}^* \text{ in } BD(\Omega)$$

iff

$$u_n \to u, \text{ strongly in } L^1(\Omega;\mathbb{R}^3) \text{ and } Eu_n \overset{*}{\rightharpoonup} Eu \text{ weakly}^* \text{ in } \mathcal{M}_b(\Omega;\mathrm{M}^3_{\mathrm{sym}}).$$

The (static) space Σ. It is defined as

$$\Sigma := \left\{\sigma \in L^2(\Omega;\mathrm{M}^3_{\mathrm{sym}}) : \operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3) \text{ and } \sigma_D \in L^\infty(\Omega;\mathbb{R}^3)\right\}.$$

It is classical that, if $\sigma \in L^2(\Omega;\mathrm{M}^3_{\mathrm{sym}})$ with $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3)$, $\sigma\nu$ is well defined as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$, ν being the outer normal to $\partial\Omega$.

More generally, consider an arbitrary Lipschitz subdomain $A \subset \overline{\Omega}$ with outer normal ν, and $\Delta \subset \partial A$ open in the relative topology. We can define the restriction of $\sigma\nu$ "on Δ" by testing against functions in $H^{1/2}(\partial A;\mathbb{R}^3)$ with compact support in Δ. This amounts to viewing $\sigma\nu$ as an element of the dual to $H^{1/2}_{00}(\Delta;\mathbb{R}^3)$.

If $\sigma \in \Sigma$, then, in the spirit of [12, Lemma 2.4], we can define a tangential component $[\sigma\nu]_\tau$ of $\sigma\nu$ on Δ such that

$$[\sigma\nu]_\tau \in L^\infty(\Delta;\mathbb{R}^3) \qquad \text{with} \qquad \|[\sigma\nu]_\tau\|_\infty \leq \frac{1}{\sqrt{2}}\|\sigma_D\|_\infty. \tag{2.2}$$

That vector is often referred to in the mechanics literature as the "resolved shear stress". Indeed, consider any regularization $\sigma_n \in C^\infty(\bar{A};\mathrm{M}^3_{\mathrm{sym}})$ of σ on $\bar{A}$ such that

$$\begin{cases} \sigma_n \to \sigma & \text{strongly in } L^2(A;\mathrm{M}^3_{\mathrm{sym}}) \\ \mathrm{div}\sigma_n \to \mathrm{div}\sigma & \text{strongly in } L^2(A;\mathbb{R}^3) \\ \|(\sigma_n)_D\|_\infty \leq \|\sigma_D\|_\infty. \end{cases}$$

Define the tangential component $[\sigma_n\nu]_\tau := (\sigma_n)\nu - ((\sigma_n)\nu\cdot\nu)\nu$. It is readily seen that $[\sigma_n\nu]_\tau = [(\sigma_n)_D\nu]_\tau$ (the tangential component of $(\sigma_n)_D$ is defined analogously). Since $x \mapsto \nu(x)$ is an $L^\infty(\Delta;\mathbb{R}^3)$-mapping, there exists an $L^\infty(\Delta;\mathbb{R}^3)$-function $[\sigma\nu]_\tau$ such that, up to a subsequence,

$$[\sigma_n\nu]_\tau \stackrel{*}{\rightharpoonup} [\sigma\nu]_\tau \text{ weakly}^* \text{ in } L^\infty(\Delta;\mathbb{R}^3).$$

If $\sigma_D \equiv 0$ then, clearly, $[\sigma\nu]_\tau \equiv 0$, so that $[\sigma\nu]_\tau$ is only a function of $(\sigma_n)_D$ which we will denote henceforth by $[\sigma_D\nu]_\tau$. Notice that $[\sigma_D\nu]_\tau$ may depend upon the approximation sequence $\{\sigma_n\}_n$ (or at least upon $\{(\sigma_n)_D\}_n$).

Further, according to Proposition **??** in the appendix, $|[\sigma_n\nu]_\tau| \leq 1/\sqrt{2}|(\sigma_n)_D|$, hence the inequality in (2.2).

Finally, if Δ is a C^2-hypersurface, then $[\sigma_D\nu]_\tau$ is uniquely determined as an element of $L^\infty(\Delta;\mathbb{R}^3)$. Indeed, for every $\varphi \in H^{1/2}(\partial A;\mathbb{R}^3)$ with support compactly contained in Δ (that is by density $\varphi \in H^{1/2}_{00}(\Delta;\mathbb{R}^3)$),

$$\int_\Delta [\sigma_D\nu]_\tau \cdot \varphi \, d\mathcal{H}^2 = \langle \sigma\nu, \varphi\rangle - \langle(\sigma\nu)_\nu, \varphi\rangle,$$

where

$$\langle(\sigma\nu)_\nu, \varphi\rangle := \langle \sigma\nu, (\varphi\cdot\nu)\nu\rangle.$$

Since the normal component $(\varphi\cdot\nu)\nu$ of φ with respect to Δ belongs to $H^{1/2}(\partial A;\mathbb{R}^3)$ in view of the regularity of ν on Δ, the definition of $(\sigma\nu)_\nu$ is meaningful.

If Δ is a countably $\mathcal{H}^2$-rectifiable subset of $\overline{\Omega}$, it admits a well defined normal ν at $\mathcal{H}^2$-a.e. point, so that a construction identical to that detailed above would yield the analogue of (2.2) in that extended setting, namely

$$[(\sigma_n)_D\nu]_\tau \stackrel{*}{\rightharpoonup} [\sigma_D\nu]_\tau \text{ weakly}^* \text{ in } L^\infty_{\mathcal{H}^2}(\Delta;\mathbb{R}^3) \qquad \text{with} \qquad \|[\sigma_D\nu]_\tau\|_\infty \leq \frac{1}{\sqrt{2}}\|\sigma_D\|_\infty.$$

3. Energetic quasi-static evolutions

In this section we investigate the variational formulation of a quasi-static evolution in perfect plasticity introduced in [6]. A first subsection is devoted to a review of the available existence, uniqueness, and regularity results, rewritten in the only case of interest to us in this work namely, three dimensional Von Mises plasticity. In a second subsection, we recall the bulk flow rule that was recovered in [6, Equation (6.14)], as well as the boundary flow rule that was subsequently established in

[9, Equation (3.12)] and specialize them accordingly. We also recall the flow rule on the singular part of the plastic strain derived in [6, Theorems 6.4 and 6.6] (see Theorem 3.11).

3.1. **The setting and the existence result.**

The reference configuration. In all that follows $\Omega \subset \mathbb{R}^3$ is an open, bounded set with (at least) Lipschitz boundary and exterior normal ν. Further, the Dirichlet part Γ_d of $\partial\Omega$ is a non empty open set in the relative topology of $\partial\Omega$ with boundary $\partial\lfloor_{\partial\Omega}\Gamma_d$ in $\partial\Omega$ and we set $\Gamma_t := \partial\Omega \setminus \bar{\Gamma}_d$. Reproducing the setting of [9, Section 6], we introduce the following

Definition 3.1. *We will say that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible iff, for any $\sigma \in L^2(\Omega;\mathrm{M}^3_{\mathrm{sym}})$ with*

$$\mathrm{div}\sigma = f \ \text{ in } \Omega, \qquad \sigma\nu = g \ \text{ on } \Gamma_t, \qquad \sigma_D \in L^\infty(\Omega;\mathrm{M}^3_D)$$

where $f \in L^3(\Omega;\mathbb{R}^3)$ and $g \in L^\infty(\Gamma_t;\mathbb{R}^3)$, and every $p \in \mathcal{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^3_D)$ and $w \in H^1(\Omega;\mathbb{R}^3)$ such that there exists an associated pair $(u,e) \in BD(\Omega)\times L^{3/2}(\Omega;\mathrm{M}^3_{\mathrm{sym}})$ with

$$Eu = e + p \quad \text{ in } \Omega, \qquad p = (w-u)\odot\nu\,\mathcal{H}^2\lfloor\Gamma_d \quad \text{ on } \Gamma_d,$$

the distribution, defined for all $\varphi \in C^\infty_c(\mathbb{R}^3)$ by

$$\begin{aligned}(3.1)\quad \langle\sigma_D,p\rangle(\varphi) := -\int_\Omega \varphi\sigma\cdot(e-Ew)\,dx - \int_\Omega \varphi f\cdot(u-w)\,dx \\ - \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,dx + \int_{\Gamma_t}\varphi g\cdot(u-w)\,d\mathcal{H}^2\end{aligned}$$

extends to a bounded Radon measure on $\mathbb{R}^3$ with $|\langle\sigma_D,p\rangle| \le \|\sigma_D\|_\infty|p|$.

Definition 3.1 covers many "practical" settings like those of a hypercube with one of its faces standing for the Dirichlet part Γ_d; see [9, Section 6] for that and other such settings.

Remark 3.2. Expression (3.1) defines a meaningful distribution on $\mathbb{R}^3$. Indeed, according to [9, Proposition 6.1] if $\sigma \in L^2(\Omega;\mathrm{M}^3_{\mathrm{sym}})$ is such that $\mathrm{div}\sigma \in L^3(\Omega;\mathbb{R}^3)$ and $\sigma_D \in L^\infty(\Omega;\mathrm{M}^3_D)$, then $\sigma \in L^r(\Omega;\mathrm{M}^3_{\mathrm{sym}})$ for every $1 \le r < \infty$ with

$$\|\sigma\|_r \le C_r\left(\|\sigma\|_2 + \|\mathrm{div}\sigma\|_3 + \|\sigma_D\|_\infty\right)$$

for some $C_r > 0$. On the other hand, $u \in L^{3/2}(\Omega;\mathbb{R}^3)$ in view of the embedding of $BD(\Omega)$ into $L^{3/2}(\Omega;\mathbb{R}^3)$. Further, u has a trace on $\partial\Omega$ which belongs to $L^1(\partial\Omega;\mathbb{R}^3)$. Finally note that, if σ is the restriction to Ω of a C^1-function and if $\mathcal{H}^2(\partial\lfloor_{\partial\Omega}\Gamma_d) = 0$, then, an integration by parts in BD (see [16, Chapter 2, Theorem 2.1]) would demonstrate that the right-hand side of (3.1) coincides with the integral of φ with respect to the (well defined) measure $\sigma_D\cdot p$. ¶

Kinematic admissibility. Given the boundary displacement $w \in H^1(\Omega;\mathbb{R}^3)$, we adopt the following

Definition 3.3 (Admissible configurations)**.** *$\mathcal{A}(w)$, the family of admissible configurations relative to w, is the set of triplets (u,e,p) with*

$$u \in BD(\Omega), \qquad e \in L^2(\Omega;\mathrm{M}^3_{\mathrm{sym}}), \qquad p \in \mathcal{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^3_D),$$

and such that

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w-u) \odot \nu\, \mathcal{H}^2 \lfloor \Gamma_d \quad \text{on } \Gamma_d. \tag{3.2}$$

The function u denotes the displacement field on Ω, while e and p are the associated elastic and plastic strains. In view of the additive decomposition (3.2) of Eu and of the general properties of BD functions recalled earlier, p does not charge $\mathcal{H}^2$-negligible sets. Moreover, given a Lipschitz hypersurface $D \subset \Omega$ dividing Ω locally into the subdomains Ω^+ and Ω^-,

$$p\lfloor D = (u^+ - u^-) \odot \nu\, \mathcal{H}^2 \lfloor D,$$

where ν is the normal to D pointing from Ω^- to Ω^+, and $u^\pm$ are the traces on D of the restrictions of u to $\Omega^\pm$. Since p is assumed to take values in the space of deviatoric matrices M^3_D, $u^+ - u^-$ is perpendicular to ν, so that only particular plastic strains are activated along D.

Elastic and plastic properties.

The elasticity tensor: The Hooke's law is given by an element $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}_s(\mathrm{M}^3_{\mathrm{sym}}))$ such that

$$c_1 |M|^2 \le \mathbb{C}(x) M \cdot M \le c_2 |M|^2 \text{ for every } M \in \mathrm{M}^3_{\mathrm{sym}}, \tag{3.3}$$

with $c_1, c_2 > 0$.

For every $e \in L^2(\Omega; \mathrm{M}^3_{\mathrm{sym}})$ we set

$$\mathcal{Q}(e) := \frac{1}{2} \int_\Omega \mathbb{C}(x) e \cdot e \, dx.$$

Von Mises dissipation potential: Given $\sigma_c > 0$, the deviatoric part of the stress is constrained to belong to the region

$$K_{\mathrm{vm}} := B\left(0, \sqrt{\frac{2}{3}}\sigma_c\right) \subseteq \mathrm{M}^3_D.$$

The so-called *dissipation potential* $H : \mathrm{M}^3_D \to [0, +\infty[$ given by

$$H(\xi) := \sup\{\tau \cdot \xi : \tau \in K_{\mathrm{vm}}\} = \sqrt{\frac{2}{3}}\sigma_c |\xi|.$$

For every admissible plastic strain p, we define the dissipation functional as

$$\mathcal{H}(p) := \int_{\Omega \cup \Gamma_d} H\left(\frac{p}{|p|}\right) d|p| = \sqrt{\frac{2}{3}}\sigma_c |p|(\Omega \cup \Gamma_d),$$

where $p/|p|$ denotes the Radon-Nikodym derivative of p with respect to its total variation $|p|$.

If $t \mapsto p(t)$ is a map from $[0,T]$ to $\mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^3_D)$, we define, for every $[a,b] \subseteq [0,T]$,

$$\mathcal{D}(0,t;p) := \sqrt{\frac{2}{3}}\sigma_c \mathcal{V}(0,t;p).$$

to be the *total dissipation* over the time interval $[a,b]$.

Body and traction forces: We consider external loads with associated potential

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t,x)\cdot u(x)\,dx + \int_{\Gamma_t} g(t,x)\cdot u(x)\,d\mathcal{H}^2(x),$$

where the body forces $f(t)$ and traction forces $g(t)$ are such that

$$f \in AC(0,T;L^3(\Omega;\mathbb{R}^3)), \quad g \in AC(0,T;L^\infty(\Gamma_t;\mathbb{R}^3)). \tag{3.4}$$

We set

$$\langle \dot{\mathcal{L}}(t), u\rangle := \int_\Omega \dot f(t,x)\cdot u(x)\,dx + \int_{\Gamma_t} \dot g(t,x)\cdot u(x)\,d\mathcal{H}^2(x),$$

and assume the following *uniform safe load condition*:
There exists $\alpha > 0$ and $\rho \in AC(0,T;L^2(\Omega;\mathrm{M}^3_{\mathrm{sym}}))$ with $\rho_D \in AC(0,T;L^\infty(\Omega;\mathrm{M}^N_D))$ such that

$$\begin{cases} -\mathrm{div}\,\rho(t) = f(t) \text{ in } \Omega, \ \rho(t)\nu = g(t) \text{ on } \Gamma_t \\ \rho_D(t,x) \in B\left(0, \sqrt{\tfrac{2}{3}}\sigma_c - \alpha\right), \quad \text{a.e. in } \Omega. \end{cases} \tag{3.5}$$

Prescribed boundary displacements. The boundary displacement w on Γ_d for the time interval $[0,T]$ is given by the trace on Γ_d of some

$$w \in AC(0,T;H^1(\mathbb{R}^3;\mathbb{R}^3)). \tag{3.6}$$

In what follows, the energetic formulation of the quasi-static evolution is detailed in the footstep of [6]: the two ingredients of such evolutions are a stability statement at each time, together with an energy conservation statement that relates the total energy of the system to the work of the loads applied to that system.

Definition 3.4 (Energetic quasi-static evolution). *The mapping*

$$t \mapsto (u(t), e(t), p(t)) \in \mathcal{A}(w(t))$$

is an energetic quasi-static evolution relative to w iff the following conditions hold for every $t \in [0,T]$:

(a) *Global stability: for every $(v,\eta,q) \in \mathcal{A}(w(t))$*

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t), u(t)\rangle \le \mathcal{Q}(\eta) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(q - p(t)). \tag{3.7}$$

(b) *Energy equality: $p \in BV\left(0,T;\mathcal{M}_b(\Omega\cup\Gamma_d;\mathrm{M}^3_D)\right)$ and*

$$\begin{aligned} \mathcal{Q}(e(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{D}(0,t;p) &= \mathcal{Q}(e(0)) - \langle \mathcal{L}(0), u(0)\rangle \\ &+ \int_0^t \left[\int_\Omega \sigma(\tau)\cdot E\dot w(\tau)\,dx - \langle \mathcal{L}(\tau), \dot w(\tau)\rangle\right] d\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\,d\tau, \end{aligned}$$

where $\sigma(t) := \mathbb{C}e(t)$.

The following result has been proved in [6, Theorem 4.5] (see also [9, Theorem 2.7] for an existence theorem which only necessitates Lipschitz regularity for the boundary $\partial\Omega$).

Theorem 3.5 (Existence of quasi-static evolutions). *Assume that* (3.3)*,* (3.4)*,* (3.5)*,* (3.6) *are satisfied, and let $(u_0, e_0, p_0) \in \mathcal{A}(w(0))$ satisfy the global stability condition* (3.7)*.*

Then there exists a quasi-static evolution $\{t \mapsto (u(t), e(t), p(t)), t \in [0,T]\}$ *relative to the boundary displacement* w *such that* $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$. *Finally the Cauchy stress*

$$t \mapsto \sigma(t) := \mathbb{C}e(t)$$

is uniquely determined by the initial conditions.

The following regularity property holds true (see [6, Theorem 5.2]).

Theorem 3.6 (Regularity in time). *Let* $t \in [0,T] \mapsto (u(t), e(t), p(t))$ *be an energetic quasi-static evolution according to Definition 3.4. Then*

$$(u, e, p) \in AC\left(0, T; BD(\Omega) \times L^2(\Omega; \mathrm{M}^3_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^3_D)\right)$$

and for a.e. $t \in [0,T]$ *the following limits exist*

$$\begin{aligned} \dot{u}(t) &:= \lim_{s \to t} \frac{u(s) - u(t)}{s - t} && \textit{weakly}^* \textit{ in } BD(\Omega), \\ \dot{e}(t) &:= \lim_{s \to t} \frac{e(s) - e(t)}{s - t} && \textit{strongly in } L^2(\Omega; \mathrm{M}^3_{\mathrm{sym}}), \\ \dot{p}(t) &:= \lim_{s \to t} \frac{p(s) - p(t)}{s - t} && \textit{strictly in } \mathcal{M}_b(\Omega \cup \Gamma_d; \mathrm{M}^3_D), \end{aligned}$$

with $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}(\dot{w}(t))$. *Moreover, the total dissipation* $\mathcal{D}(0, t; p)$ *is absolutely continuous and*

$$\dot{\mathcal{D}}(0, t; p) = \sqrt{\frac{2}{3}}\sigma_c |\dot{p}(t)|(\Omega \cup \Gamma_d) \qquad \textit{for a.e. } t \in [0,T].$$

Finally there exists a constant $C > 0$ *such that for a.e.* $t \in [0,T]$

$$\|\dot{e}(t)\|_2 + |\dot{p}(t)|(\Omega \cup \Gamma_d; \mathrm{M}^3_D) \leq C\left[\|\dot{\rho}(t)\|_2 + \|\dot{\rho}_D(t)\|_\infty + \|E\dot{w}(t)\|_2\right]. \tag{3.8}$$

3.2. **The flow rule.** The extent to which the afore mentioned energetic quasi-static evolutions are also *classical* evolutions is described in the following results.

The Cauchy stress satisfies the following properties; see [6, Theorem 6.1] or [9, Theorem 3.6].

Theorem 3.7 (Cauchy stress). *Let* $t \in [0,T] \mapsto (u(t), e(t), p(t))$ *be an energetic quasi-static evolution according to Definition 3.4 and let* $\sigma(t) := \mathbb{C}e(t)$ *be the associated Cauchy stress. Then the following conditions hold:*

(a) *Balance equations: For every* $t \in [0,T]$,

$$-\mathrm{div}\, \sigma(t) = f(t) \textit{ in } \Omega, \qquad \sigma(t)\nu = g(t) \textit{ on } \Gamma_t. \tag{3.9}$$

(b) *Stress admissibility condition: For every* $t \in [0,T]$,

$$|\sigma_D(t,x)| \leq \sqrt{\frac{2}{3}}\sigma_c \quad \textit{for a.e. } x \in \Omega. \tag{3.10}$$

As far as the evolution of the plastic deformations is concerned, the following result holds true (see [9, Proposition 3.11]):

Theorem 3.8 (Plastic flow). *Let* $t \in [0,T] \mapsto (u(t), e(t), p(t))$ *be an energetic quasi-static evolution according to Definition 3.4 and let* $\sigma(t) := \mathbb{C}e(t)$ *be the associated Cauchy stress. Assume that* $\partial\lfloor_{\partial\Omega}\Gamma_d$ *is admissible according to Definition 3.1.*

Then, for a.e. $t \in [0,T]$, the dissipation rate and the plastic work rate coincide, i.e.,

$$\sqrt{\frac{2}{3}}\sigma_c|\dot{p}(t)| = \langle \sigma_D(t), \dot{p}(t)\rangle \qquad \textit{as measures on } \Omega \cup \Gamma_d, \tag{3.11}$$

where $\langle \sigma_D(t), \dot{p}(t)\rangle$ denotes the duality between $\sigma_D(t)$ and $\dot{p}(t)$ given through (3.1)*.*

Equality (3.11) should contain all relevant information on the flow of the plastic strains. However, the recovery of more classical Von Mises flow rules is hindered by the low regularity of p.

A flow rule for the abolutely continuous part of the plastic strain can be easily derived.

Theorem 3.9 (Flow rule for the "volumic" plastic deformation)**.** *Assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible according to Definition 3.1. Let $t \in [0,T]$ be such that equality* (3.11) *holds true. If $\dot{p}^a(t) \in L^1(\Omega; \mathrm{M}^N_D)$ denotes the density of the absolutely continuous part of $\dot{p}(t)$, then*

$$|\sigma_D(t,x)| = \sqrt{\frac{2}{3}}\sigma_c \quad \textit{and} \quad \frac{\dot{p}^a(t,x)}{|\dot{p}^a(t,x)|} = \frac{\sigma_D(t,x)}{|\sigma_D(t,x)|} \qquad \textit{for } \mathcal{L}^3\textit{-a.e. } x \in \{|\dot{p}^a(t,x)| > 0\},$$

while

$$\dot{p}^a(t,x) = 0 \qquad \textit{for } \mathcal{L}^3\textit{-a.e. } x \in \left\{ |\sigma_D(t)| < \sqrt{\frac{2}{3}}\sigma_c \right\}.$$

Proof. Since, in view of [9, Theorem 6.2],

$$\langle \sigma_D(t), \dot{p}(t)\rangle^a = \sigma_D(t) \cdot \dot{p}^a(t)\, \mathcal{L}^3,$$

we get that, for a.e. $x \in \Omega$,

$$\sigma_D(t,x) \cdot \dot{p}^a(t,x) = \sqrt{\frac{2}{3}}\sigma_c|\dot{p}^a(t,x)| \qquad \text{and} \qquad |\sigma_D(t,x)| \le \sqrt{\frac{2}{3}}\sigma_c.$$

The conclusion easily follows. □

Following [9], we also obtain a boundary flow rule.

Theorem 3.10 (Boundary flow rule)**.** *Assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible according to Definition 3.1. Let $t \in [0,T]$ be such that equality* (3.11) *holds true, and let $[\sigma_D(t)\nu]_\tau \in L^\infty(\Gamma_d; \mathbb{R}^3)$ be any tangential trace of $\sigma_D(t)\nu$ on Γ_d defined according to* (2.2)*. Then,*

$$|[\sigma_D(t)\nu]_\tau(x)| = \sqrt{\frac{1}{3}}\sigma_c \qquad \textit{and} \tag{3.12}$$

$$\frac{[\sigma_D(t)\nu]_\tau(x)}{|[\sigma_D(t)\nu]_\tau(x)|} = \frac{\dot{w}(t,x) - \dot{u}(t,x)}{|\dot{w}(t,x) - \dot{u}(t,x)|} \quad \textit{for } \mathcal{H}^2\textit{-a.e. } x \textit{ such that } \dot{w}(t,x) \ne \dot{u}(t,x),$$

while

$$\dot{w}(t,x) = \dot{u}(t,x) \qquad \textit{for } \mathcal{H}^2\textit{-a.e. } x \in \left\{ |[\sigma_D(t)\nu]_\tau| < \sqrt{\frac{1}{3}}\sigma_c \right\}.$$

Proof. The proof is similar to that of Theorem 3.9. It suffices to recall that

$$\dot p(t)\lfloor\Gamma_d = [\dot w(t) - \dot u(t)] \odot \nu\, \mathcal{H}^2\lfloor\Gamma_d$$

so that, heeding the choice of the Frobenius norm as matrix norm, we obtain

$$|\dot p(t)|\lfloor\Gamma_d = \frac{1}{\sqrt{2}}|\dot w(t) - \dot u(t)|\, \mathcal{H}^2\lfloor\Gamma_d.$$

Since, according to [9, Lemma 3.8],

$$\langle\sigma_D(t), \dot p(t)\rangle\lfloor\Gamma_d = [\sigma_D(t)\nu]_\tau \cdot [\dot w(t) - \dot u(t)]\, \mathcal{H}^2\lfloor\Gamma_d.$$

equality (3.11) becomes

$$[\sigma_D(t)\nu]_\tau \cdot [\dot w(t) - \dot u(t)]\, \mathcal{H}^2\lfloor\Gamma_d = \sqrt{\frac{1}{3}}\sigma_c|\dot w(t) - \dot u(t)|\, \mathcal{H}^2\lfloor\Gamma_d.$$

By construction of $[\sigma_D(t)\nu]_\tau$, we get, thanks to (2.2),

$$|[\sigma_D(t)\nu]_\tau| \le \sqrt{\frac{1}{3}}\sigma_c,$$

so that the conclusion easily follows. □

In order to obtain a flow rule for the singular part of the plastic deformation, we follow the method introduced in [6, Section 6.2].

For every $r > 0$ and $x \in \Omega$ we define the stress averages

$$\sigma^r(t,x) := \frac{1}{|B_r(x) \cap \Omega|}\int_{B_r(x)\cap\Omega} \sigma(t,y)\, dy.$$

The following result holds true (compare with [6, Theorems 6.4 and 6.6]).

Theorem 3.11 (Flow rule on the singular support of $\dot p(t)$)**.** *Assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible according to Definition 3.1. Let $t \in [0,T]$ be such that equality (3.11) holds true, and let $\dot p^s(t)$ denote the singular part of $\dot p(t)$. Then for $r \to 0^+$*

$$\sigma^r_D(t) \to \hat\sigma_D(t) \qquad \text{strongly in } L^1_{|\dot p^s(t)|}(\Omega; \mathrm{M}^3_D),$$

where

(3.13)

$$|\hat\sigma_D(t,x)| = \sqrt{\frac{2}{3}}\sigma_c \quad \text{and} \quad \frac{\dot p^s(t)}{|\dot p^s(t)|}(x) = \frac{\hat\sigma_D(t,x)}{|\hat\sigma_D(t,x)|} \qquad \text{for } |\dot p^s(t)|\text{-a.e. } x \in \Omega.$$

In (3.13), $\dot p^s(t)/|\dot p^s(t)|$ denotes the Radon-Nikodym derivative of $\dot p^s(t)$ with respect to its total variation.

Proof. Let us consider $A \subset\subset \Omega$. Since for r small enough, $\sigma^r(t)$ is continuous with a continuous divergence on A (thanks to the equilibrium condition (3.9)), we have that

$$\langle\sigma^r_D(t), \dot p(t)\rangle = \sigma^r_D(t) \cdot \frac{\dot p(t)}{|\dot p(t)|}|\dot p(t)| \qquad \text{on } A. \tag{3.14}$$

Moreover, since

$$\sigma^r(t) \to \sigma(t) \qquad \text{strongly in } L^2(A; \mathrm{M}^3_{\mathrm{sym}})$$

and

$$\mathrm{div}\ \sigma^r(t) \to \mathrm{div}\ \sigma(t) \qquad \text{strongly in } L^3(A; \mathbb{R}^3)$$

we deduce that

$$\langle\sigma^r_D(t), \dot p(t)\rangle \overset{*}{\rightharpoonup} \langle\sigma_D(t), \dot p(t)\rangle \qquad \text{weakly* in } \mathcal{M}_b(A).$$

In view of the stress admissibility condition (3.10),

$$|\sigma^r_D(t)| \le \sqrt{\frac{2}{3}}\sigma_c \qquad \text{on } A, \tag{3.15}$$

so that, up to a subsequence in r,

$$\sigma^r_D(t) \xrightharpoonup{*} \hat\sigma_D(t) \qquad \text{weakly* in } L^\infty_{|\dot p^s(t)|}(A;\mathrm{M}^3_D). \tag{3.16}$$

In the light of (3.14) and (3.11) we deduce the equality

$$\hat\sigma_D(t)\cdot\frac{\dot p^s(t)}{|\dot p^s(t)|} = \sqrt{\frac{2}{3}}\sigma_c\frac{\dot p^s(t)}{|\dot p^s(t)|} \qquad |\dot p^s(t)|\text{-a.e. on } A.$$

The previous equality and (3.15) entail (3.13) on A: in particular, $\hat\sigma_D(t)$ is uniquely determined and there is no need to pass to a subsequence. Moreover, (3.15) and the fact that $|\hat\sigma_D(t,x)| = \sqrt{2/3}\,\sigma_c$ $|\dot p^s(t)|$-a.e. on A, imply that the weak* convergence in (3.16) can be improved to strong convergence in $L^1_{|\dot p^s(t)|}(A)$.

Since A is arbitrary, and σ^r_D is uniformly bounded on Ω, the previous results can be extended to Ω, which completes the proof. □

4. Prohibiting plastic slips

4.1. **Flow rule for plastic slips.** Let

$$t \mapsto (u(t), e(t), p(t)) \in \mathcal{A}(w(t))$$

be a quasi-static evolution according to Definition 3.4 with associated Cauchy stress $\sigma(t) := \mathbb{C}e(t)$.

Theorem 4.1 (Flow rule on a slip). *Assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible according to Definition 3.1. Let $t\in[0,T]$ be such that equality (3.11) holds true. Then $\hat\sigma_D(t)$ defined in Theorem 3.11 satisfies*

$$|\hat\sigma_D(t,x)| = \sqrt{\frac{2}{3}}\sigma_c \qquad \textit{and} \qquad \frac{[\dot u(t,x)]\odot\nu_{\dot u(t)}}{|[\dot u(t,x)]\odot\nu_{\dot u(t)}|} = \frac{\hat\sigma_D(t,x)}{|\hat\sigma_D(t,x)|} \tag{4.1}$$

$$\textit{for } \mathcal{H}^2\textit{-a.e. } x\in J_{\dot u(t)}.$$

In particular for $\mathcal{H}^2$-a.e. $x\in J_{\dot u(t)}$, there exists a basis (e'_1, e'_2, e'_3) such that

$$\hat\sigma_D(t,x) = \mathrm{diag}\left(-\frac{\sigma_c}{\sqrt 3}, 0, \frac{\sigma_c}{\sqrt 3}\right). \tag{4.2}$$

Moreover the orthogonal lines determined by $[\dot u(t,x)]$ and $\nu_{\dot u(t)}(x)$ are bisected by e'_1 and $\pm e'_3$ (and viceversa).

Proof. Because $\dot p^s(t)\lfloor J_{\dot u(t)} = [\dot u(t)]\odot\nu_{\dot u(t)}\,\mathcal{H}^2\lfloor J_{\dot u(t)}$, (4.1) is a direct consequence of the flow rule (3.13) for the singular part of the plastic deformation. Finally, property (4.2) follows by Proposition A.1 in the Appendix. □

Remark 4.2. The previous result shows that the part of yield surface $\partial_{slip}K_{\mathrm{vm}}$ for which plastic slips can be activated is a three dimensional sub-manifold of the four dimensional manifold ∂K_{vm}. If the normal to the slip plane is given, then the admissible stresses form a one dimensional manifold (parameterized by the direction of the slip in the plane). ¶

4.2. **On the formation of plastic slips.** We now have at our disposal the various ingredients for the formulation of a condition which will prevent the formation of plastic slips. Defining $S_{\sigma(t)}$ to be the complement of the set of Lebesgue points for $\sigma(t)$, we obtain the following

Theorem 4.3 (Absence of plastic slips). *Assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible according to Definition 3.1. Let $A \subseteq \Omega$ be open, and let the Cauchy stress satisfies the following assumptions for a.e. $t \in [t_1, t_2] \subseteq [0, T]$:*

(a) $\mathcal{H}^2(S_{\sigma(t)} \cap A) = 0$;
(b) *The Lebesgue values $\tilde{\sigma}(t, x)$ for $x \in A$ do not satisfy* (4.2).

Then, no plastic slip can occur on A in the time interval $[t_1, t_2]$, i.e., for every $t \in [t_1, t_2]$,

$$[u(t)] \odot \nu_{u(t)}\, \mathcal{H}^2\lfloor(J_{u(t)} \cap A) = [u(t_1)] \odot \nu_{u(t_1)}\, \mathcal{H}^2\lfloor(J_{u(t_1)} \cap A). \tag{4.3}$$

In particular, if $\mathcal{H}^2(J_{u(t_1)} \cap A) = 0$, then, for $t \in [t_1, t_2]$,

$$p(t) = p^a(t, x)\, \mathcal{L}^3 + E^c u(t) \qquad \textit{on } A, \qquad p^a(t) \in L^1(\Omega; \mathrm{M}^3_D). \tag{4.4}$$

Proof. Thanks to Theorem 4.1, we get

$$\mathcal{H}^2(J_{\dot{u}(t)} \cap A) = 0 \qquad \text{for a.e. } t \in [t_1, t_2]. \tag{4.5}$$

Indeed, the representation (4.2) cannot hold true in view of the assumptions on the stress, since

$$\hat{\sigma}_D(t, x) = \tilde{\sigma}_D(t, x) \qquad \text{for } \mathcal{H}^2\text{-a.e. } x \in J_{\dot{u}(t)}.$$

Recall that $t \mapsto u(t)$ is absolutely continuous in $BD(\Omega)$. However, since $BD(\Omega)$ is not reflexive, we cannot in general express the measure $Eu(t)$ as a Bochner integral of its derivative. Nevertheless, for every $\varphi \in C_c(\Omega; \mathrm{M}^3_{\mathrm{sym}})$, and for $t \in [t_1, t_2]$, we may write

$$\langle Eu(t), \varphi\rangle - \langle Eu(t_1), \varphi\rangle = \int_{t_1}^{t} \langle E\dot{u}(\tau), \varphi\rangle\, d\tau.$$

Let $K \subseteq A$ be compact and contained in a C^1-hypersurface, and let $\psi \in C(K; \mathrm{M}^3_D)$. Consider a sequence $(\varphi_n)_{n\in\mathbb{N}}$ converging pointwise to $\psi 1_K$ with $\|\varphi_n\|_\infty \le \|\psi\|_\infty$. In view of (3.8) we deduce by dominated convergence that

$$\langle Eu(t), \psi 1_K\rangle - \langle Eu(t_1), \psi 1_K\rangle = \int_{t_1}^{t} \langle E\dot{u}(\tau), \psi 1_K\rangle\, d\tau = 0,$$

where the last equality is obtained in view of (4.5). Since K is arbitrary, the countably $\mathcal{H}^2$-rectifiability of $J_{u(t)}$, together with the basic decomposition (2.1) for $Eu(t)$, yields that

$$E^j u(t) = E^j u(t_1) \quad \text{on } A,$$

which entails (4.3). Finally (4.4) is a consequence of (4.3) and of the admissibility condition $Eu(t) = e(t) + p(t)$. □

Remark 4.4. Let $A \subseteq \Omega \cup \Gamma_d$ be open in the relative topology. Require additionally that any tangential trace $[\sigma_D(t)\nu]_\tau$ on Γ_d be such that

$$|[\sigma_D(t)\nu]_\tau| < \sqrt{\frac{1}{3}}\sigma_c, \ \mathcal{H}^2\text{-a.e. on } A \cap \Gamma_d.$$

In view of the flow rule (3.12) on the boundary, we conclude that no plastic slips occur on A (boundary included) in the time interval $[t_1, t_2]$. ¶

Item (a) in Theorem 4.3 will be implied by suitable regularity on the Cauchy stress. Assume that, besides (3.4), the external body force f satisfies

$$\begin{cases} Df \in L^\infty(0,T; L^3_{loc}(\Omega; M^{3\times 3})) \\ \Delta f \in L^\infty(0,T; L^3_{loc}(\Omega; \mathbb{R}^3)), \end{cases} \tag{4.6}$$

while the initial configuration $(u_0, e_0, p_0) \in \mathcal{A}(w(0))$ satisfies

$$\sigma_0 := \mathbb{C}e_0 \in H^1_{loc}(\Omega; \mathrm{M}^3_{\mathrm{sym}}). \tag{4.7}$$

Then the following result holds true.

Proposition 4.5 (Higher regularity for the stress). *Assume that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is admissible according to Definition 3.1 and that the additional smoothness assumptions* (4.6) *and* (4.7) *hold true. Then,*

$$\sigma \in L^\infty(0,T; H^1_{loc}(\Omega; \mathrm{M}^3_{\mathrm{sym}})). \tag{4.8}$$

In particular, the Lebesgue points of $\sigma(t)$ for a.e. $t \in [0,T]$ have full $\mathcal{H}^2$-measure in Ω.

Proof. With the assumptions of Theorem 3.5, together with (4.6)-(4.7), at our disposal, the regularity (4.8) for the stress has been proved in e.g. [7, Theorem 2.1], provided that the boundary $\partial\Omega$ is C^2 and that $\partial\lfloor_{\partial\Omega}\Gamma_d$ is also C^2. See also similar results in [4]. The seemingly more stringent assumption on the regularity of the boundary found in [7] is only there to ensure existence of a quasistatic evolution in the sense of Theorem 3.5. Since we appeal to more recent results which only require Lipschitz regularity of the boundary $\partial\Omega$ [9, Theorem 2], the regularity result extends verbatim to that setting.

Finally, because $\sigma(t) \in H^1_{loc}(\Omega; \mathrm{M}^3_{\mathrm{sym}})$ for a.e. $t \in [0,T]$, it admits a precise representative cap_2-a.e., hence $\mathcal{H}^\alpha$-a.e. in Ω for $\alpha > 1$ (see e.g. [8, Sections 4.7, 4.8]). In particular, $\mathcal{H}^2$-a.e. point in Ω is a Lebesgue point for $\sigma(t)$. $\square$

Remark 4.6. For general Lipschitz domains, any additional regularity of $\sigma(t)$ up to the boundary is unclear; however, see [5] for possible extensions of the regularity up to the boundary. ¶

Using the regularity properties of the stress, we can formulate the following result.

Theorem 4.7 (A sufficient condition for the absence of plastic slips). *Let $\partial\lfloor_{\partial\Omega}\Gamma_d$ be admissible according to Definition 3.1 and assume that the additional smoothness assumptions* (4.6) *and* (4.7) *hold true. Let $A \subseteq \Omega$ be open, and let $[t_1,t_2] \subseteq [0,T]$ be such that there exists $\eta > 0$ with*

$$\left|\sigma^1_D(t,x) + \frac{\sigma_c}{\sqrt{3}}\right| + |\sigma^2_D(t,x)| + \left|\sigma^3_D(t,x) - \frac{\sigma_c}{\sqrt{3}}\right| \geq \eta \tag{4.9}$$

for a.e. $x \in A$ and a.e. $t \in [t_1,t_2]$,

where $\sigma^1_D(t,x) \leq \sigma^2_D(t,x) \leq \sigma^3_D(t,x)$ are the eigenstresses of $\sigma_D(t,x)$.

Then, the conclusion of Theorem 4.3 still holds true, i.e., no plastic slip can occur on A in the time interval $[t_1,t_2]$.

Proof. Thanks to Proposition 4.5, $\mathcal{H}^2$-a.e. point in A is a Lebesgue point for $\sigma(t)$. We claim that condition (4.9) is satisfied at every Lebesgue point of $\sigma(t)$ in A, i.e., if $\tilde{\sigma}(t,x)$ denotes the Lebesgue value of $\sigma(t)$ at x, that

$$\left|\tilde{\sigma}^1_D(t,x) + \frac{\sigma_c}{\sqrt{3}}\right| + |\tilde{\sigma}^2_D(t,x)| + \left|\tilde{\sigma}^3_D(t,x) - \frac{\sigma_c}{\sqrt{3}}\right| \geq \eta. \tag{4.10}$$

Then $\tilde{\sigma}_D(t,x)$ cannot have the critical structure (4.2), and the conclusion follows by Theorem 4.3.

In order to prove (4.10), let $x \in A$ be a Lebesgue point for $\sigma(t)$. Recall that Lebesgue point are points of approximate continuity (see e.g [8, Section 1.7]), so that, for every $\varepsilon > 0$,

$$\lim_{r\to 0^+} \frac{1}{r^N}\mathcal{L}^N\left(\{y \in B_r(x)\,:\,|\sigma(t,y) - \tilde{\sigma}(t,x)| > \varepsilon\}\right) = 0.$$

Since a.e. $y \in B_r(x)$ satisfies (4.9), a diagonal argument (in r and ε) yields a sequence $x_n \to x$ satisfying (4.9) and such that

$$\sigma(t,x_n) \to \tilde{\sigma}(t,x).$$

A continuity argument on the eigenvalues of a matrix entails in turn that

$$\sigma^i_D(t,x_n) \to \tilde{\sigma}^i_D(t,x) \qquad i = 1,2,3$$

so that (4.10) follows. □

Remark 4.8 (On the Cantor part of the plastic strain)**.** The structure for $\sigma_D(t)$ at a plastic slip is a consequence of the symmetrized rank-one structure of $\dot{p}(t)/|\dot{p}(t)|$ on $J_{\dot{u}(t)}$. If such a structure was also available for the Cantor part of $\dot{p}(t)$, then the analogue of (4.5) would hold true for the whole $\mathcal{L}^3$-singular part of $E\dot{u}(t)$. In turn, this would entail that plasticity can only develop in an absolutely continuous way, i.e., that the measure $\dot{p}(t)$ would be absolutely continuous w.r.t. $\mathcal{L}^3$. Since $\dot{p}^c(t) = E^c\dot{u}(t)$, the symmetrized rank-one structure would be implied by an extension of Alberti's rank one theorem [1] from the BV to the BD setting.

Similarly, such an extension would also permit to obtain the precise representation (4.2) $|\dot{p}^s(t)|$-a.e. on $\Omega \cup \Gamma_d$. ¶

Appendix A

The following result in linear algebra proves useful for our work.

Proposition A.1. *Let $a, b \in \mathbb{R}^3$ be non zero vectors with $a \cdot b = 0$. There exists an orthonormal basis (e'_1, e'_2, e'_3) such that*

$$a \odot b = \operatorname{diag}\left(-\frac{|a||b|}{2}, 0, \frac{|a||b|}{2}\right).$$

Moreover, the orthogonal lines with directors a, b are bisected by e'_1 and $\pm e'_3$ (and viceversa).

Proof. Note that 0 is an eigenvalue with eigenvector $a \times b$. We can thus choose a basis (e'_1, e'_2, e'_3) of eigenvectors for $a \odot b$ such that e'_2 is parallel to $a \times b$, so that in particular

$$span\{a,b\} = span\{e'_1, e'_3\}. \tag{A.1}$$

Since $a \odot b$ has zero trace,

$$a \odot b = \mathrm{diag}(-\lambda, 0, \lambda)$$

for some $\lambda \geq 0$ in the basis (e_1', e_2', e_3') (upon possible permutation of the vectors e_1' and e_3'). Taking into account that

$$|a \odot b| = \frac{|a||b|}{\sqrt{2}},$$

the diagonal representation easily follows.

Thanks to (A.1), the orthogonality of the vectors a and b leads to

$$\begin{cases} e_a := \dfrac{a}{|a|} = \alpha e_1' + \beta e_3' \\ e_b := \dfrac{b}{|b|} = \mp\beta e_1' \pm \alpha e_3'. \end{cases}$$

Since

$$(e_a \odot e_b)e_a = \frac{1}{2}e_b \qquad \text{and} \qquad (e_a \odot e_b)e_b = \frac{1}{2}e_a,$$

while

$$e_a \odot e_b = \mathrm{diag}\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$$

we get

$$|\alpha| = |\beta| = \frac{1}{\sqrt{2}},$$

and the geometric property thus follows. □

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References

[1] Giovanni Alberti. Rank one property for derivatives of functions with bounded variation. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(2):239–274, 1993.

[2] L. Ambrosio, A. Coscia, and G. Dal Maso. Fine properties of functions with bounded deformations. *Arch. Rat. Mech. Anal.*, 139:201–238, 1997.

[3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[4] A. Bensoussan and J. Frehse. Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and H^1 regularity. *Comment. Math. Univ. Carolin.*, 37(2):285–304, 1996.

[5] M. Bulíček, J. Frehse, and J. Málek. On boundary regularity for the stress in problems of linearized elasto-plasticity. *Int. J. Adv. Sci. Appl. Math.*, 1:141–156, 2009.

[6] G. Dal Maso, A. DeSimone, and M. G. Mora. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Ration. Mech. Anal.*, 180(2):237–291, 2006.

[7] A. Demyanov. Regularity of stresses in Prandtl-Reuss perfect plasticity. *Calc. Var. Partial Differential Equations*, 34(1):23–72, 2009.

[8] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[9] G. A. Francfort and A. Giacomini. Small-strain heterogeneous elastoplasticity revisited. *Comm. Pure Appl. Math.*, 65(9):1185–1241, 2012.

[10] B. Halphen and J. Salençon. *Elastoplasticité*. Presses de l'Ecole Nationale des Ponts et Chaussées, 1987.

[11] Bernard Halphen. Sur les discontinuités de vitesse en élastoplasticité. *C. R. Acad. Sci. Paris Sér. A-B*, 287(7):A569–A572, 1978.
[12] R.V. Kohn and R. Temam. Dual spaces of stresses and strains, with applications to Hencky plasticity. *Appl. Math. Optim.*, 10(1):1–35, 1983.
[13] Alexander Mielke. Evolution of rate-independent systems. In A. Dafermos and E. Feireisl, editors, *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 461–559. Elsevier/North-Holland, Amsterdam, 2005.
[14] Pierre-M. Suquet. Un espace fonctionnel pour les équations de la plasticité. *Ann. Fac. Sci. Toulouse Math. (5)*, 1(1):77–87, 1979.
[15] Pierre-M. Suquet. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mécanique*, 20(1):3–39, 1981.
[16] Roger Temam. *Problèmes mathématiques en plasticité*, volume 12 of *Méthodes Mathématiques de l'Informatique*. Gauthier-Villars, Montrouge, 1983.

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