A SURGERY RESULT FOR THE SPECTRUM OF THE DIRICHLET LAPLACIAN

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Abstract. In this paper we give a method to geometrically modify an open set such that the first k eigenvalues of the Dirichlet Laplacian and its perimeter are not increasing, its measure remains constant, and both perimeter and diameter decrease below a certain threshold. The key point of the analysis relies on the properties of the shape subsolutions for the torsion energy. As well, we apply this result to prove existence of solutions for shape optimization problems of spectral type with both measure and perimeter constraints.

Keywords: shape optimization, eigenvalues, Dirichlet Laplacian

1. Introduction and statement of the main results

The results of this paper are motivated by spectral shape optimization problems for the eigenvalues of the Dirichlet Laplacian, e.g.

$$\min\left\{\lambda_k(\Omega),\ \Omega\subset\mathbb{R}^N,\ |\Omega|=1\right\}, \tag{1.1}$$

where λ_k denotes the k^{th} eigenvalue of the Dirichlet Laplacian and $|\cdot|$ the N dimensional Lebesgue measure ($N\geq 2$).

In order to prove existence of an optimal set Ω for problem (1.1), two different methods were proposed recently. On the one hand, in [19] it is proved a surgery result asserting that one can suitably modify an open set such that the first k eigenvalues of the Dirichlet Laplacian are not increasing, its measure remains constant and its diameter decreases below a certain threshold. The result of [19] together with the Buttazzo-Dal Maso existence theorem [12] (which has a local character) gives a proof of existence of solutions for (1.1). By a different method, based on the so called shape subsolutions (see the definition in Section 2), in [7] is proved the existence of solutions and moreover that all minimizers have finite diameter and finite perimeter.

Recently, Van den Berg has studied in [4] a minimum problem with both a measure *and* a perimeter constraint:

$$\min\left\{\lambda_k(\Omega),\ \Omega\subset\mathbb{R}^N,\ |\Omega|\leq 1,\ \mathrm{Per}(\Omega)\leq P\right\}. \tag{1.2}$$

An existence result for this problem cannot be deduced from the results [7, 19]. The surgery method of [19] can hardly control the perimeter since the procedure generates new pieces of boundary which may have a large surface area. As well, in the presence of two simultaneous constraints, the notion of shape subsolution cannot be used in a direct manner due to the lack of suitable Lagrange multipliers which can take into account *both* geometric constraints.

Motivated by this existence question, in this paper we give a surgery result which follows the main objectives of [19], but with the new requirement on the control of the perimeter. In order to control the perimeter, the "surgery" is done in a new manner, using some of the key ideas of the shape subsolutions. Roughly speaking, we look at the local behavior of the torsion function and prove that if this function is small enough in some region, then one can cut out a piece of the domain controlling simultaneously the variation of the low part of the spectrum, of the measure and of the perimeter.

Throughout the paper, we denote by $\tilde\Omega$ an open set of finite measure. For simplicity, and without restricting the generality, we shall assume that its measure is equal to 1. Here is our main result which, for clarity, is stated in a simplified way:

Theorem 1.1. *For every $K > 0$, there exists $D, C > 0$ depending only on K and the dimension N, such that for every open set $\tilde\Omega \subset \mathbb{R}^N$ with $|\tilde\Omega| = 1$ there exists an open set Ω, with $|\Omega| = 1$ and satisfying*

(1) $\mathrm{diam}\,(\Omega) \le D$ *and* $\mathrm{Per}(\Omega) \le \min\{\mathrm{Per}(\tilde\Omega), C\}$,
(2) *if for some $k \in \mathbb{N}$ it holds $\lambda_k(\tilde\Omega) \le K$, then $\lambda_i(\Omega) \le \lambda_i(\tilde\Omega)$ for all $i = 1, \dots, k$.*

The set Ω is essentially obtained by removing some parts of $\tilde\Omega$ and rescaling it to satisfy the measure constraint. In case the measure of $\tilde\Omega$ is not equal to 1, the constants D and C above depend also on $|\tilde\Omega|$, following the rescaling rules of the eigenvalues, measure and perimeter.

We shall split the result stated above in two distinct parts, Theorems 3.3 and 4.1. The construction of Ω differs depending on which kind of control of the perimeter is desired. If the perimeter of $\tilde\Omega$ is infinite (or larger than C), it is convenient to use an optimization argument related to the shape subsolutions to directly construct the set Ω satisfying all the requirements above on eigenvalues, measure and diameter, but with a perimeter less than C (Theorem 3.3). If the perimeter of $\tilde\Omega$ is finite (for example smaller than C), we produce a different argument, by cutting in a suitable way the set $\tilde\Omega$ with hyperplanes, and removing some strips, decreasing in this way the perimeter (Theorem 4.1) and of course satisfying all the requirements above on eigenvalues, measure and diameter. In this last case, the control of the perimeter is done through a De Giorgi type argument. We point out that the assertions of the two theorems are slightly stronger than the unified formulation stated in Theorem 1.1.

We note that the results of this paper hold true in exactly the same way if instead of "open" sets one works with "quasi-open" or "measurable" sets (see Section 5 and [9, 13, 11]). In general, if $\tilde\Omega$ is quasi-open (or measurable), then the constructed set Ω is quasi-open (respectively measurable).

In the last section of the paper, we point out two consequences of our method. On the one hand, we provide an existence result for (a general version of) problem (1.2). Let us quickly sketch how Theorem 1.1 will be employed for proving existence of a solution in (1.2): for a minimizing sequence $(\tilde\Omega_n)_n$ one can find a positive K such that $\lambda_k(\tilde\Omega_n) \le K$, for all n large enough. Theorem 1.1 will allow to replace the minimizing sequence $(\tilde\Omega_n)_n$ by a new one $(\Omega_n)_n$ which has both diameter and perimeter controlled, getting back to the "local" setting of Buttazzo and Dal Maso.

A second consequence of our result is that for all shape optimization problems

$$\min\{F(\lambda_1(A),\dots,\lambda_k(A)) \ : \ A\subset\mathbb{R}^N,\ \text{measurable},\ |A|=1\},\tag{1.3}$$

where $F\colon \mathbb{R}^k\to\mathbb{R}$ is lower semicontinuous, strictly increasing in one variable and nondecreasing in all the others, any optimal set *has to have* finite perimeter, without any smoothness assumption on F. This is a generalization of [7].

2. The spectrum of the Dirichlet Laplacian and the torsion function

Let $\Omega\subset\mathbb{R}^N$ be an open set of finite measure ($N\ge 2$). Denoting by $H^1_0(\Omega)$ the usual Sobolev space, the eigenvalues of the Dirichlet Laplacian on Ω are defined by

$$\lambda_k(\Omega) := \min_{S_k}\max_{u\in S_k\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2\,dx}\,,\tag{2.1}$$

where the minimum ranges over all k-dimensional subspaces S_k of $H^1_0(\Omega)$.

The *torsion* function of Ω, denoted by w_Ω, is the function which minimizes the *torsion* energy

$$E(\Omega) := \min_{u\in H^1_0(\Omega)} \frac12\int_{\mathbb{R}^N}|Du|^2dx - \int_{\mathbb{R}^N} u dx,$$

and satisfies in a weak sense

$$-\Delta w_\Omega = 1 \quad \text{in } \Omega,\qquad w_\Omega\in H^1_0(\Omega).$$

Note that the torsion energy is negative if $\Omega\neq\emptyset$ and

$$E(\Omega) = -\frac12\int_{\mathbb{R}^N} w_\Omega dx < 0.$$

A fundamental property of the torsion function is the Saint-Venant inequality, which states that among all open sets of equal volume, the ball maximizes the L^1-norm of the torsion function, which is often called *torsional rigidity* of a set and denoted $T(\cdot)$. This leads to the following inequality

$$T(\Omega) = \int_\Omega w_\Omega dx \le |\Omega|^{\frac{N+2}{N}} \frac{\omega_N^{-2/N}}{N(N+2)},\tag{2.2}$$

where ω_N is the volume of the ball of *radius* 1 in $\mathbb{R}^N$. By direct computation, the torsional rigidity of a ball of radius R centered in the origin is

$$T(B_R) = \int_{B_R} \frac{R^2-|x|^2}{2N} = R^{N+2}\frac{\omega_N}{N(N+2)}.$$

A similar inequality between the L^∞ norms is a consequence of the rearrangement result proved by Talenti in [20] and reads:

$$\|w_\Omega\|_\infty \le \Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{2}{N}}\frac{1}{2N}.\tag{2.3}$$

We recall as well the inequality of Kohler-Jobin [17]

$$\lambda_1(\Omega)^{\frac{(N+2)}{2}}T(\Omega) \ge \lambda_1(B_1)^{\frac{(N+2)}{2}}T(B_1).\tag{2.4}$$

We denote $C_{KJ} = \lambda_1(B_R)^{\frac{(N+2)}{2}}T(B_R)$, where B_R is a ball of radius $R > 0$. Since the quantity on the right hand side is scale invariant, then it is clear that the constant C_{KJ} depends only on the dimension N, and we can rewrite the Kohler-Jobin inequality as

$$T(\Omega) \geq \frac{C_{KJ}}{\lambda_1(\Omega)^{\frac{N+2}{2}}}.$$

We recall the following bound on the ratio between eigenvalues of the Dirichlet Laplacian, which can be found in [2]. For all $k \in \mathbb{N}$ there exists a constant M_k, depending *only* on k and the dimension N, such that

$$1 \leq \frac{\lambda_k(\Omega)}{\lambda_1(\Omega)} \leq M_k. \tag{2.5}$$

Another fundamental inequality, proved in [3] (see also [5]), relates the L^∞ norm of the torsion function with the first eigenvalue and reads

$$\frac{1}{\lambda_1(\Omega)} \leq \|w_\Omega\|_\infty \leq \frac{4 + 3N\log 2}{\lambda_1(\Omega)}. \tag{2.6}$$

We also recall the following inequality due to Berezin, Li and Yau (see [18]), which asserts that for some constant C_{BLY} depending only on the dimension of the space N, we have

$$\forall k \in \mathbb{N} \quad \lambda_k(\Omega) \geq C_{BLY}\Big(\frac{k}{|\Omega|}\Big)^{\frac{2}{N}}. \tag{2.7}$$

The way we shall use this inequality is the following: if one fixes $K > 0$, then the number of eigenvalues of Ω below K, is at most of $\left(\frac{K}{C_{BLY}}\right)^{\frac{N}{2}}|\Omega|$.

For sets satisfying $\Omega_1 \subseteq \Omega_2$, the following inequality was proved in [7] : for every $k \in \mathbb{N}$

$$\Big|\frac{1}{\lambda_k(\Omega_1)} - \frac{1}{\lambda_k(\Omega_2)}\Big| \leq 4k^2e^{1/4\pi}\lambda_k(\Omega_2)^{N/2}(E(\Omega_1) - E(\Omega_2)). \tag{2.8}$$

We recall the definition of subsolutions.

Definition 2.1. *Let $c > 0$. We say that $\tilde\Omega \subset \mathbb{R}^N$ is a shape subsolution for the energy if for all $\Omega \subset \tilde\Omega$*

$$E(\tilde\Omega) + c|\tilde\Omega| \leq E(\Omega) + c|\Omega|.$$

It is proved in [7] that, if $\tilde\Omega$ is a shape subsolution for the energy, then $\tilde\Omega$ is bounded (with controlled diameter) and has finite perimeter.

We conclude this Section with a result relating the value of the torsion function at one point to its sum on some neighborhood.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $w = w_\Omega$ be its torsion function. For every $\theta > 0$, there exists $\delta_0 > 0$ depending only on N, θ such that if $w(x_0) \geq \theta$ for some $x_0 \in \mathbb{R}^N$, then*

$$\int_{B_\delta(x_0)} w\,dx \geq \frac{\theta\omega_N}{2}\delta^N, \qquad \forall \delta \in (0, \delta_0).$$

Proof. We recall (see for instance [12]) that if one extends the torsion function by zero on $\mathbb{R}^N \setminus \Omega$, then it satisfies $-\Delta w_\Omega \leq 1$ in the sense of distributions in $\mathbb{R}^N$. Consequently, for every $x_0 \in \mathbb{R}^N$ the function $x \mapsto w(x) + \frac{|x-x_0|^2}{2N}$ is subharmonic in $\mathbb{R}^N$, and we have that for all $\delta > 0$

$$\theta \leq w(x_0) \leq \frac{1}{|B_\delta|} \int_{B_\delta(x_0)} (w(x) + \frac{|x-x_0|^2}{2N})\, dx = \frac{1}{|B_\delta|} \int_{B_\delta(x_0)} w dx + \frac{\delta^2}{2(N+2)}.$$

For some δ_0 sufficiently small (e.g. equal to $\sqrt{\theta(N+2)}$), we have $\forall\, 0 < \delta \leq \delta_0$

$$\int_{B_\delta(x_0)} w dx \geq \frac{\theta \omega_N}{2} \delta^N.$$

□

3. Control of the spectrum by subsolutions

Before stating our first result, we outline the main ideas. Let $\tilde{\Omega} \subset \mathbb{R}^N$ be a given open set of finite measure. Assume that for *some* set $\Omega \subset \tilde{\Omega}$ and for *some* constant $c > 0$ we have

$$E(\Omega) + c|\Omega| \leq E(\tilde{\Omega}) + c|\tilde{\Omega}|. \tag{3.1}$$

Then, we shall observe that a *certain* number of low eigenvalues of the rescaled set $\left(\frac{|\tilde{\Omega}|}{|\Omega|}\right)^{\frac{1}{N}} \Omega$ are not larger than the corresponding eigenvalues on $\tilde{\Omega}$. In practice, given $k \in \mathbb{N}$, we shall provide a suitable constant c, such that for every open set Ω satisfying (3.1), the first k- eigenvalues of the set $\left(\frac{|\tilde{\Omega}|}{|\Omega|}\right)^{\frac{1}{N}} \Omega$ are not larger than the corresponding eigenvalues on $\tilde{\Omega}$. The larger is k, the smaller is the constant c.

Indeed, assume (3.1) for some c. From (2.8), we get

$$\lambda_k(\Omega) - \lambda_k(\tilde{\Omega}) \leq 4k^2 e^{1/4\pi} \lambda_k(\Omega) \lambda_k(\tilde{\Omega})^{(N+2)/2} [E(\Omega) - E(\tilde{\Omega})]. \tag{3.2}$$

Using hypothesis (3.1) and the elementary inequality $|\tilde{\Omega}| - |\Omega| \leq |\tilde{\Omega}|^{\frac{N-2}{N}} \frac{N}{2} (|\tilde{\Omega}|^{\frac{2}{N}} - |\Omega|^{\frac{2}{N}})$, we get

$$\lambda_k(\Omega) - \lambda_k(\tilde{\Omega}) \leq c\Big(4k^2 e^{1/4\pi} \lambda_k(\Omega) \lambda_k(\tilde{\Omega})^{(N+2)/2}\Big) |\tilde{\Omega}|^{\frac{N-2}{N}} \frac{N}{2} (|\tilde{\Omega}|^{\frac{2}{N}} - |\Omega|^{\frac{2}{N}}). \tag{3.3}$$

Then, for every Λ such that

$$c\Big(2k^2 e^{1/4\pi} \lambda_k(\Omega) \lambda_k(\tilde{\Omega})^{(N+2)/2}\Big) |\tilde{\Omega}|^{\frac{N-2}{N}} N \leq \Lambda \tag{3.4}$$

we get

$$\lambda_k(\Omega) - \lambda_k(\tilde{\Omega}) \leq \Lambda (|\tilde{\Omega}|^{\frac{2}{N}} - |\Omega|^{\frac{2}{N}}),$$

so

$$\lambda_k(\Omega) + \Lambda |\Omega|^{\frac{2}{N}} \leq \lambda_k(\tilde{\Omega}) + \Lambda |\tilde{\Omega}|^{\frac{2}{N}}. \tag{3.5}$$

If

$$c\Big(2k^2 e^{1/4\pi} \lambda_k(\Omega) \lambda_k(\tilde{\Omega})^{(N+2)/2}\Big) |\tilde{\Omega}|^{\frac{N-2}{N}} N \leq \frac{\lambda_k(\tilde{\Omega})}{|\tilde{\Omega}|^{\frac{2}{N}}}, \tag{3.6}$$

following (3.4) we can choose $\Lambda = \frac{\lambda_k(\tilde\Omega)}{|\tilde\Omega|^{\frac{2}{N}}}$ in (3.5). Thanks to the arithmetic-geometric mean inequality and the choice of Λ, inequality (3.5) leads to

$$2\sqrt{\lambda_k(\Omega)\Lambda|\Omega|^{\frac{2}{N}}} \le \lambda_k(\Omega) + \Lambda|\Omega|^{\frac{2}{N}} \le \lambda_k(\tilde\Omega) + \Lambda|\tilde\Omega|^{\frac{2}{N}} = 2\sqrt{\lambda_k(\tilde\Omega)\Lambda|\tilde\Omega|^{\frac{2}{N}}} \tag{3.7}$$

hence

$$\lambda_k(\Omega)|\Omega|^{\frac{2}{N}} \le \lambda_k(\tilde\Omega)|\tilde\Omega|^{\frac{2}{N}}. \tag{3.8}$$

In view of (3.6), provided c satisifies

$$c \le \frac{1}{2Nk^2 e^{\frac{1}{4\pi}}\lambda_k(\Omega)\lambda_k(\tilde\Omega)^{\frac{N}{2}}|\tilde\Omega|} \tag{3.9}$$

the choice of $\Lambda = \frac{\lambda_k(\tilde\Omega)}{|\tilde\Omega|^{\frac{2}{N}}}$ can be done. Note that if c satisfies (3.9) leading to (3.5) for λ_k, then the same c will lead to similar inequalities for λ_i, with $i \le k$ since

$$\frac{1}{2Nk^2 e^{\frac{1}{4\pi}}\lambda_k(\Omega)\lambda_k(\tilde\Omega)^{\frac{N}{2}}|\tilde\Omega|} \le \frac{1}{2Ni^2 e^{\frac{1}{4\pi}}\lambda_i(\Omega)\lambda_i(\tilde\Omega)^{\frac{N}{2}}|\tilde\Omega|}.$$

The only unknown quantity influencing the value of c in (3.5) is $\lambda_k(\Omega)$. This is one of the fundamental issues of our analysis. Roughly speaking, we shall control the upper bound of $\lambda_k(\Omega)$ by a multiple of $\lambda_k(\tilde\Omega)$, provided that c is small enough.

Let us now give the precise statement.

Lemma 3.1. *Let $K > 0$. Let $\tilde\Omega \subset \mathbb{R}^N$ be an open set of unit measure, such that for some $k \in \mathbb{N}$ $\lambda_k(\tilde\Omega) \le K$. There exist two constants $c, \beta > 0$ depending only on K and N (and not on $\tilde\Omega$) such that for all $\Omega \subset \tilde\Omega$ satisfying*

$$E(\Omega) + c|\Omega| \le E(\tilde\Omega) + c|\tilde\Omega| \tag{3.10}$$

we have

$$\lambda_i(\Omega)|\Omega|^{2/N} \le \lambda_i(\tilde\Omega)|\tilde\Omega|^{2/N}, \qquad \forall\, i = 1, \dots, k \tag{3.11}$$

and $|\Omega| \ge \beta|\tilde\Omega|$. Moreover, if inequality (3.10) *is strict, then also inequality* (3.11) *becomes strict.*

Proof. We divide the proof in several steps.
Step 1. The constant c can be chosen such that $E(\tilde\Omega) + c|\tilde\Omega|$ is negative (and is controlled only by $\lambda_1(\tilde\Omega)$; in particular this implies that Ω is not the empty set). Indeed, thanks to the Kohler-Jobin inequality (2.4), we have a control of the torsional rigidity of $\tilde\Omega$ from below:

$$\int_{\tilde\Omega} w_{\tilde\Omega} dx \ge C_{KJ}\frac{1}{\lambda_1(\tilde\Omega)^{\frac{N+2}{2}}}. \tag{3.12}$$

Then

$$E(\tilde\Omega) + c|\tilde\Omega| = -\frac{1}{2}\int_{\tilde\Omega} w_{\tilde\Omega}\, dx + c|\tilde\Omega| \le -\frac{C_{KJ}}{2\lambda_1(\tilde\Omega)^{\frac{N+2}{2}}} + c|\tilde\Omega|.$$

Since $\lambda_1(\tilde\Omega) \le K$, the right hand side above is negative, as soon as we choose

$$c \le \frac{C_{KJ}}{2|\tilde\Omega|K^{\frac{N+2}{2}}}. \tag{3.13}$$

Step 2. The constant c can be chosen such that for every Ω satisfying (3.10), we have $\|w_\Omega\|_\infty \geq \frac{1}{2}\|w_{\tilde\Omega}\|_\infty$. Indeed, denote $h := \|w_{\tilde\Omega}\|_\infty$ and assume that $\|w_\Omega\|_\infty < \frac{\|w_{\tilde\Omega}\|_\infty}{2}$. Then, from the inequality $w_\Omega \leq \min\{w_{\tilde\Omega}, \frac{h}{2}\}$ we get

$$\begin{aligned}
&\frac{1}{2}\int w_\Omega dx - \frac{1}{2}\int w_{\tilde\Omega} dx \\
&= \frac{1}{2}\int w_\Omega dx - \frac{1}{2}\int \min\{w_{\tilde\Omega}, h/2\}dx - \frac{1}{2}\int_{\{w_{\tilde\Omega}>h/2\}} (w_{\tilde\Omega} - h/2)^+ dx \\
&\leq -\frac{1}{2}\int_{\{w_{\tilde\Omega}>h/2\}} (w_{\tilde\Omega} - h/2)^+ dx,
\end{aligned}$$

and using $E(\tilde\Omega) - E(\Omega) = \frac{1}{2}\int w_\Omega dx - \frac{1}{2}\int w_{\tilde\Omega} dx$, inequality (3.10) leads to

$$\frac{1}{2}\int_{\{w_{\tilde\Omega}>h/2\}} (w_{\tilde\Omega} - h/2)^+ dx \leq c|\tilde\Omega| - c|\Omega|.$$

Thanks to the fact that $(w_{\tilde\Omega} - h/2)^+ = w_{\{w_{\tilde\Omega}>h/2\}}$, using the Kohler-Jobin inequality and (2.6) on the set $\{w_{\tilde\Omega} > h/2\}$, we get

$$\frac{C_{KJ}}{2(4+3N\log 2)^{\frac{N+2}{2}}}\left(\frac{h}{2}\right)^{\frac{N+2}{2}} \leq \frac{1}{2}\int_{\{w_{\tilde\Omega}>h/2\}} (w_{\tilde\Omega} - h/2)^+ < c|\tilde\Omega|. \tag{3.14}$$

Consequently, if

$$c \leq \frac{C_{KJ}}{2(4+3N\log 2)^{\frac{N+2}{2}}}\left(\frac{\|w_{\tilde\Omega}\|_\infty}{2}\right)^{\frac{N+2}{2}}\frac{1}{|\tilde\Omega|}$$

inequality (3.14) can not be satisfied, hence $\|w_\Omega\|_\infty \geq \frac{\|w_{\tilde\Omega}\|_\infty}{2}$.

We note that, using the inequalities (2.5) and (2.6) together with the fact that $\|w_\Omega\|_\infty \geq \frac{\|w_{\tilde\Omega}\|_\infty}{2}$, one gets that for every $i \in \mathbb{N}$ the corresponding eigenvalues on Ω and $\tilde\Omega$ are comparable

$$\lambda_i(\tilde\Omega) \leq \lambda_i(\Omega) \leq (8+6N\log 2)M_i\lambda_i(\tilde\Omega). \tag{3.15}$$

Step 3. Proof of inequality (3.11). Choosing c satisfying Steps 1 and 2, and

$$c \leq \frac{1}{2Nk_0^2 M_{k_0} e^{1/4\pi}(8+6N\log 2)K^{\frac{N+2}{2}}|\tilde\Omega|}, \tag{3.16}$$

where k_0 is the smallest integer, larger than $\left(\frac{K}{C_{BLY}}\right)^{\frac{N}{2}}|\tilde\Omega|$ (see (2.7)), we get that inequality (3.9) is satisfied for every k such that $\lambda_k(\tilde\Omega) \leq K$, as $k \leq k_0$.

From the argument developed in (3.2)-(3.7), we get

$$\lambda_i(\Omega)|\Omega|^{2/N} \leq \lambda_i(\tilde\Omega)|\tilde\Omega|^{2/N}, \qquad \forall\, i = 1, \dots, k. \tag{3.17}$$

We remark that if we have the strict inequality in hypothesis (3.10), that is, $E(\Omega) + c|\Omega| < E(\tilde\Omega) + c|\tilde\Omega|$, then we have the strict inequality in (3.5) and thus also in (3.17)

Step 4. In order to control the diameter of the rescaled set, we justify the existence of $\beta > 0$ such that $|\Omega| \geq \beta|\tilde\Omega|$. Indeed, we have the chain of inequalities (the last one being a consequence of (2.3))

$$\frac{1}{2K} \le \frac{1}{2\lambda_1(\tilde\Omega)} \le \frac{\|w_{\tilde\Omega}\|_\infty}{2} \le \|w_\Omega\|_\infty \le \Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{2}{N}} \frac{1}{2N},$$

which gives the estimate for β, depending only on K, N. □

Remark 3.2. *Inequality* (3.15) *in Step 2 holds true for every index* $i \in \mathbb{N}$. *If one focuses only to the first* k *eigenvalues, the constants in the right hand side of* (3.15) *can be improved and made explicit.*

Since by the hypothesis of Lemma 3.1 we have

$$E(\Omega) - E(\tilde\Omega) \le c(|\tilde\Omega| - |\Omega|) \le c|\tilde\Omega| = c,$$

inequality (2.8) *leads to*

$$\frac{1}{\lambda_k(\tilde\Omega)} - \frac{1}{\lambda_k(\Omega)} \le 4k^2 e^{1/4\pi} \lambda_k(\tilde\Omega)^{N/2}(E(\Omega) - E(\tilde\Omega)) \le 4ck^2 e^{1/4\pi}\lambda_k(\tilde\Omega)^{N/2}. \tag{3.18}$$

If we choose

$$c \le \frac{1}{8k^2 e^{1/4\pi}\lambda_k(\tilde\Omega)^{N/2+1}},$$

we get

$$4ck^2 e^{1/4\pi}\lambda_k(\tilde\Omega)^{N/2} \le \frac{1}{2\lambda_k(\tilde\Omega)},$$

hence from (3.18)

$$\frac{1}{2\lambda_k(\tilde\Omega)} \le \frac{1}{\lambda_k(\Omega)} \quad\Longrightarrow\quad \lambda_k(\tilde\Omega) \le \lambda_k(\Omega) \le 2\lambda_k(\tilde\Omega).$$

Below, the diameter of an open, disconnected set is referred as the sum of the diameters of each connected component. For a measurable set $E \subset \mathbb{R}^N$, we denote

$$\mathrm{Per}(E) := \sup\{\int_E \mathrm{div}\, V dx : V \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N), \|V\|_\infty \le 1\}$$

the generalized perimeter in the sense of De Giorgi. We prefer to formulate Theorem 3.3 below and Theorem 4.1 in Section 4 using the De Giorgi perimeter rather than the Hausdorff measure of the topological boundary, since it is a more general notion that applies as well to non-smooth sets (as quasi-open or measurable). Nevertheless, if $\tilde\Omega$ is open and smooth, then the modified set Ω is smooth enough, so that $\mathrm{Per}(\tilde\Omega) = \mathcal{H}^{N-1}(\partial\tilde\Omega)$ and $\mathrm{Per}(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)$.

Now, we are in position to prove the first result.

Theorem 3.3. *For every* $K > 0$, *there exists* $D, C > 0$ *depending only on* K *and the dimension* N *such that for every open set* $\tilde\Omega \subset \mathbb{R}^N$ *with* $|\tilde\Omega| = 1$ *there exists an open set* Ω *with* $|\Omega| = 1$, $\mathrm{diam}\,(\Omega) \le D$, $\mathrm{Per}(\Omega) \le C$ *and, if for some* $k \in \mathbb{N}$ *it holds* $\lambda_k(\tilde\Omega) \le K$, *then* $\lambda_i(\Omega) \le \lambda_i(\tilde\Omega)$, *for all* $1 \le i \le k$.

Proof. Recall that from the Berezin-Li-Yau inequality, the maximal index k_0 for which it is possible that $\lambda_{k_0}(\tilde\Omega) \le K$ is lower than $\Big[\Big(\frac{K}{C_{BLY}}\Big)^{\frac{N}{2}} |\tilde\Omega|\Big]$.

Let us consider the minimum problem

$$\min_{\Omega \subset \tilde\Omega} \{E(\Omega) + c|\Omega|\},$$

with c the constant given by Lemma 3.1. This problem has at least one solution, denoted Ω^*, which is an open set (see for instance [15]) and it is also a shape subsolution of the energy. The results from [7] give that $\operatorname{diam}(\Omega^*) \le D(c)$ and that $\operatorname{Per}(\Omega^*) \le C(c)$. We remind that c depends only on K and the dimension N. Using Step 4 of Lemma 3.1, we have that the set $\Omega := |\Omega^*|^{-1/N}\Omega^*$ has still diameter and perimeter bounded by constants depending only on K, N, thanks to the fact that $|\Omega^*| \ge \beta|\tilde{\Omega}|$. Moreover we have that if k is an index such that $\lambda_k(\tilde{\Omega}) \le K$ then

$$\forall i = 1, \dots, k, \ \ \lambda_i(\Omega) \le \lambda_i(\tilde{\Omega}).$$

□

4. Control of the perimeter

Here is the main result of the Section.

Theorem 4.1. *For every $K, P > 0$, there exist $D > 0$ depending only on K, P and the dimension N, such that for every open set $\tilde{\Omega} \subset \mathbb{R}^N$ with $|\tilde{\Omega}| = 1$, $\operatorname{Per}(\tilde{\Omega}) \le P$, there exists an open set Ω of unit measure with $\operatorname{diam}(\Omega) \le D$, $\operatorname{Per}(\Omega) \le \operatorname{Per}(\tilde{\Omega})$ such that if for some $k \in \mathbb{N}$ it holds $\lambda_k(\tilde{\Omega}) \le K$, then $\lambda_i(\Omega) \le \lambda_i(\tilde{\Omega})$ for all $1 \le i \le k$.*

For technical purposes, we introduce a suitable notion of diameter in a prescribed direction. In the coordinate direction $e_1 \in \mathbb{R}^N$ we set

$$\operatorname{diam}_{e_1}(\Omega) := \mathcal{H}^1\left(t \in \mathbb{R} \ : \ \mathcal{H}^{N-1}(\Omega \cap \{x_1 = t\}) > 0\right).$$

For every $x_1 \in \mathbb{R}$ and $r > 0$, we define the *strip* centered in x_1 of width $2r$ orthogonal to $\mathbb{R}e_1$ by

$$S_r(x_1) := [-r + x_1, r + x_1] \times \mathbb{R}^{N-1}.$$

Its topological boundary is $\partial S_r(x_1) := \{-r + x_1, r + x_1\} \times \mathbb{R}^{N-1}$. If $x_1 = 0$, we simply denote S_r instead of $S_r(0)$.

The main idea of the following lemma is inspired from [1] and was also used in [13], [7], and [10] under different settings. We point out that here we do not use optimality, but only an inequality between two fixed domains.

Lemma 4.2. *Let $\tilde{\Omega}$ be an open set of measure 1. For all $c > 0$, there exist $C_0, r_0 > 0$, with $C_0 r_0 \le \min\left\{\frac{c}{2}, \frac{1}{2}\right\}$ such that if for some $r \le r_0$ the function $w_{\tilde{\Omega}}$ is not identically zero in S_r and*

$$E(\tilde{\Omega}) + c|\tilde{\Omega}| \le E(\tilde{\Omega} \setminus S_r) + c|\tilde{\Omega} \setminus S_r|, \tag{4.1}$$

then

$$\max_{S_{2r}} w_{\tilde{\Omega}} \ge C_0 r. \tag{4.2}$$

Proof. Below, we denote $w := w_{\tilde\Omega}$ and $\varepsilon := \max_{S_{2r}} w$ and introduce the function $\eta\colon \mathbb{R}^N \to \mathbb{R}^+ :$

$$\begin{cases} \eta = 0 \quad \text{in } S_r, \qquad \eta = \varepsilon \quad \text{in } \mathbb{R}^N \setminus S_{2r}, \\ -\Delta\eta = 1 \quad \text{in } S_{2r} \setminus S_r, \\ \eta = 0 \quad \text{on } \partial S_r, \\ \eta = \varepsilon \quad \text{on } \partial S_{2r}. \end{cases} \tag{4.3}$$

By abuse of notation, we set $w \wedge \eta = w \cdot 1_{\mathbb{R}^N \setminus S_{2r}} + \min\{w, \eta\} \cdot 1_{S_{2r}}$. Then, the function $w \wedge \eta$ belongs to $H^1_0(\tilde\Omega \setminus S_r)$, and using it as test on $\tilde\Omega \setminus S_r$ we get

$$E(\tilde\Omega \setminus S_r) \le \frac{1}{2}\int |D(w \wedge \eta)|^2 dx - \int w \wedge \eta dx.$$

Hypothesis (4.1) gives

$$\frac{1}{2}\int |Dw|^2 dx - \int w dx + c|S_r \cap \tilde\Omega| \le \frac{1}{2}\int |D(w \wedge \eta)|^2 dx - \int w \wedge \eta dx.$$

Since $\varepsilon \le C_0 r_0 \le \frac{c}{2}$ we get $w \wedge \eta = w$ in $\tilde\Omega \setminus S_{2r}$. Denoting the outer unit normal to a set by ν,

$$\begin{aligned} \frac{1}{2}\int_{S_r} |Dw|^2 dx + \frac{c}{2}|S_r \cap \tilde\Omega| &\le \frac{1}{2}\int_{S_r} |Dw|^2 dx - \int_{S_r} w dx + c|S_r \cap \tilde\Omega| \\ &\le \frac{1}{2}\int_{S_{2r}\setminus S_r} (|D(w \wedge \eta)|^2 - |Dw|^2) dx - \int_{S_{2r}\setminus S_r} (w \wedge \eta - w) dx \\ &= \frac{1}{2}\int_{S_{2r}\setminus S_r \cap \{w>\eta\}} (|D\eta|^2 - |Dw|^2) dx - \int_{S_{2r}\setminus S_r} (w-\eta)^+ dx \\ &\le \int_{S_{2r}\setminus S_r \cap \{w>\eta\}} -D\eta \cdot D(w-\eta) dx - \int_{S_{2r}\setminus S_r} (w-\eta)^+ dx \\ &= -\int_{\partial S_r} \frac{\partial \eta}{\partial \nu}(w-\eta)^+ dx = |\eta'(r)| \int_{\partial S_r} w \, d\mathcal{H}^{N-1}. \end{aligned}$$

The following trace inequality holds:

$$\int_{\partial S_r} w \, d\mathcal{H}^{N-1} \le C(N)\left(\frac{1}{r}\int_{S_r} w dx + \int_{S_r} |Dw| dx\right).$$

Assume for contradiction that (4.2) does not hold, that is $\max_{S_{2r}} w \ < C_0 r$. From this inequality and the arithmetic-geometric mean inequality on the gradient term, we get

$$\int_{\partial S_r} w \, d\mathcal{H}^{N-1} \le C(N)\left((C_0 + \frac{1}{2})|S_r \cap \tilde\Omega| + \frac{1}{2}\int_{S_r \cap \tilde\Omega} |Dw|^2 dx\right).$$

If $\int_{S_r} |Dw|^2 dx + |S_r \cap \tilde\Omega| = 0$ then $w = 0$ in the "strip" S_r. Otherwise $\int_{S_r} |Dw|^2 dx + |S_r \cap \tilde\Omega| > 0$ and from the previous inequality we get

$$\min\left\{\frac{1}{2}, \frac{c}{2}\right\} \le |\eta'(r)| C(N)(C_0 + 1).$$

Since $|\eta'(r)| = |C_0 - r/2|$, choosing C_0 and r_0 small enough we get a contradiction. In particular it is enough to choose C_0, r_0 satisfying

$$(C_0 + 1)\max\{C_0, r_0/2\} < \frac{\min\{1, c\}}{2C(N)},$$

and we note that they depend only on N and c.

□

The following corollary can be proved in the very same way as Lemma 4.2.

Corollary 4.3. *For all $c > 0$ there exist $C_0, r_0 > 0$ with $C_0 r_0 \le \min\left\{\frac{c}{2}, \frac{1}{2}\right\}$ such that if for some $r \le r_0$ and $x_1, \dots, x_n \in \mathbb{R}$ it holds*

$$\max\left\{w_{\tilde\Omega}(x) \ : \ x \in \cup_{i=1}^n S_{2r}(x_i)\right\} \le C_0 r,$$

and

$$S_{2r}(x_i) \cap S_{2r}(x_j) = \emptyset \text{ for all } i, j = 1, \dots, n, \ i \ne j,$$

then we have that

$$E(\tilde\Omega \setminus \cup_{i=1}^n S_r(x_i)) + c|\tilde\Omega \setminus \cup_{i=1}^n S_r(x_i)| \le E(\tilde\Omega) + c|\tilde\Omega|. \tag{4.4}$$

Here we outline the main idea for proving Theorem 4.1. Let c be as in Lemma 3.1 and r_0, C_0 be the constants from Lemma 4.2, for that particular choice of c. We shall remove a finite number of strips from the region where $w_{\tilde\Omega}$ is no larger than $C_0 r_0$. Following inequality (4.4) and Lemma 3.1, we shall be able to control the eigenvalues after rescaling. The control of the perimeter, will be done by a suitable choice of the position of the strips. Contrary to the construction in [19], the new perimeter introduced by sectioning with hyperplanes does not depend on the $\mathcal{H}^{N-2}$ measure of the boundary of the sections.

Let $l_0 > 0$ and $n \in \mathbb{N}$. The value of l_0 will be precised below, in Lemma 4.4. Assume $x^i \in \mathbb{R}^N$ and $L_i > 2l_0$, $i = 1, \dots, n$ are such that $S_{2L_i}(x_1^i) \cap S_{2L_j}(x_1^j) = \emptyset$ if $i \ne j$. For every $t \in [0, l_0]$ we define:

$$S(t) := \cup_{i=1}^n S_{L_i - t}(x_1^i).$$

For an open set of unit measure $\tilde\Omega$, we denote $m(t) := |S(t) \cap \tilde\Omega|$ the mass of the union of strips in $\tilde\Omega$ and

$$\sigma(t) := \sum_{i=1}^n \mathcal{H}^{N-1}(\tilde\Omega \cap \{L_i - t, L_i + t\} \times \mathbb{R}^{N-1}),$$

the new perimeter introduced by the sections with the hyperplanes and

$$p(t) = \sum_{i=1}^n \operatorname{Per}(\tilde\Omega \cap (L_i - t, L_i + t) \times \mathbb{R}^{N-1}) - \sigma(t),$$

the perimeter of $\tilde\Omega$ inside the strips. We denote the rescaled set,

$$\Omega(t) := (1 - m(t))^{-1/N}(\tilde\Omega \setminus S(t)).$$

Lemma 4.4. *Given $P > 0$ and an open set $\tilde\Omega$ of unit measure, with $\operatorname{Per}(\tilde\Omega) \le P$, there exist two constants l_0 and $\widehat{m}$, depending* only *on P and the dimension N, such that if $m(l_0) \le \widehat{m}$ then there exists $t \in [0, l_0]$ such that $\operatorname{Per}(\Omega(t)) \le \operatorname{Per}(\tilde\Omega)$.*

Proof. First of all, we notice that, by definition, $t \mapsto m(t)$ is a nonincreasing function and for a.e. $t \in (0, l_0)$, we have that $\sigma(t) = -m'(t)$. If for every $t \in [0, l_0]$ we would have $\mathrm{Per}(\Omega(t)) > \mathrm{Per}(\tilde\Omega)$, we get:

$$\mathrm{Per}(\tilde\Omega) - p(t) + \sigma(t) \geq \mathrm{Per}(\tilde\Omega)(1 - m(t))^{\frac{N-1}{N}}.$$

It is possible to choose $\widehat{m}$ small enough (depending only on P, N: every $\widehat{m} \leq (1/2P)^N$ works), such that if $m(t) \leq \widehat{m}$, then

$$(1 - m(t))^{\frac{N-1}{N}} \geq 1 - \frac{m(t)^{\frac{N-1}{N}}}{2P} \geq 1 - \frac{m(t)^{\frac{N-1}{N}}}{2\mathrm{Per}(\tilde\Omega)}.$$

Putting the above inequalities together and using the isoperimetric inequality for the set $S(t)$,

$$\mathrm{Per}(\tilde\Omega) + 2\sigma(t) \geq \mathrm{Per}(\tilde\Omega) - \frac{m(t)^{\frac{N-1}{N}}}{2} + p(t) + \sigma(t) \geq \mathrm{Per}(\tilde\Omega) - \frac{m(t)^{\frac{N-1}{N}}}{2} + N\omega_N^{1/N} m(t)^{\frac{N-1}{N}}.$$

Since $2N\omega_N^{1/N} - 1 > 0$, we obtain:

$$-m'(t) \geq (2N\omega_N^{1/N} - 1)\frac{m(t)^{\frac{N-1}{N}}}{4}.$$

By integrating on $[0, l_0]$ we get

$$m^{1/N}(0) - m^{1/N}(l_0) \geq (2\omega_N^{1/N} - 1)\frac{l_0}{4N}.$$

Since $m(0) = \widehat{m}$ and $m(l_0) \geq 0$, choosing $l_0 > \frac{4N}{2\omega_N^{1/N} - 1}\widehat{m}^{1/N}$ we get a contradiction. □

Remark 4.5. *Thanks to the choice of C_0, r_0 done in Lemma 4.2, we deduce that if $A \subset \tilde\Omega$ is such that $\max_A w_{\tilde\Omega} \leq C_0 r_0$, then $E(A) + c|A| \geq 0$. Indeed, using the monotonicity of the torsion function:*

$$E(A) + c|A| = -\frac{1}{2}\int w_A dx + c|A| \geq -\frac{1}{2}\int_A w_{\tilde\Omega} dx + c|A| \geq -\frac{C_0 r_0 |A|}{2} + c|A| \geq 0,$$

since $C_0 r_0 \leq 2c$ from the hypotheses of Lemma 4.2.

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. We fix the constant c such that Lemma 3.1 is satisfied, we get C_0, r_0 from Lemma 4.2. Without loss of generality, we can assume that C_0 and r_0 are small enough, such that

$$\frac{1}{2C_0 r_0} \geq K.$$

If we denote by A a subset of $\tilde\Omega$ with $\max_A w_{\tilde\Omega} \leq C_0 r_0$ then, having in mind (2.6), we get the following chain of inequalities

$$\lambda_1(A)(1 - \widehat{m})^{2/N} \geq \frac{1}{\|w_A\|_\infty}(1 - \widehat{m})^{2/N} \geq \frac{(1 - \widehat{m})^{2/N}}{C_0 r_0}.$$

Choosing $\widehat{m}$ such that $(1 - \widehat{m})^{2/N} \geq \frac{1}{2}$ we get

$$\lambda_1(A)(1 - \widehat{m})^{2/N} \geq \frac{1}{2C_0 r_0} \geq K.$$

Possibly decreasing $\widehat{m}$ such that it is $\widehat{m} \leq (1/2P)^N$, we can assume that it works as well for Lemma 4.4.

For simplicity we rename $w = w_{\tilde{\Omega}}$. The region where $w(x) \geq C_0 r_0$ is contained in a finite union of strips with width $4r_0$. Indeed, we define

$$X_0 := \left\{ x_1 \in \mathbb{R} \, : \, \max_{S_{2r_0}(x_1)} w \geq C_0 r_0 \right\}, \qquad \widetilde{X} := \bigcup \left\{ S_{2r_0}(t) : t \in X_0 \right\}.$$

From Lemma 2.2 and the Saint-Venant inequality (2.2) the set $\widetilde{X}$ is contained in the union of at most $n = n(r_0, N)$ of disjoint strips (each of width at least $4r_0$), for example it is enough to choose

$$n \leq \frac{2\omega_N^{-(N+2)/N}}{C_0 r_0 \min\left\{ \sqrt{C_0 r_0 (N+2)}, r_0 \right\}}.$$

Let us call X the projection of $\widetilde{X}$ on $\mathbb{R}e_1$.

The set $\mathbb{R} \setminus X$ is a finite union of disjoint segments and of the infinite intervals at $\pm\infty$, say

$$\mathbb{R} \setminus X = (-\infty, b_0) \cup \left[\bigcup_{i=1}^{n} (a_i, b_i) \right] \cup (a_{n+1}, \infty).$$

If a segment (a_i, b_i) has a length less than or equal to $8r_0 + 2l_0$, we shall ignore it in our further construction and just add the corresponding strip to the set $\tilde{X}$ and renumber the index i if necessary. The total length of those such segments is at most $n(8r_0 + 2l_0)$.

Therefore, we shall assume in the sequel that all segments (a_i, b_i) have a length greater than $8r_0 + 2l_0$. We denote $\overline{a}_i = a_i + (4r_0 + l_0)$, $\overline{b}_i = b_i - (4r_0 + l_0)$ and

$$Y = \left[\bigcup_{i=1}^{n+1} (a_i, \overline{a}_i) \right] \cup \left[\bigcup_{i=0}^{n} (\overline{b}_i, b_i) \right].$$

In order to highlight the main idea, let us assume in a first instance that

$$|\left(Y \times \mathbb{R}^{N-1} \right) \cap \tilde{\Omega}| \leq \widehat{m}. \tag{4.5}$$

If we are in this situation, we perform a simultaneous "cut" as in Lemma 4.2, removing the following union of strips:

$$S_t := S_{r_0}(b_0 - 2r_0 - t) \bigcup S_{r_0}(a_i + 2r_0 + t) \bigcup S_{r_0}(b_i - 2r_0 - t) \bigcup S_{r_0}(a_{n+1} + 2r_0 + t),$$

for every $t \in [0, l_0]$.

Following the assumption (4.5) and Lemma 4.4, there exists a value t such that the perimeter of the rescaled set $|\tilde{\Omega} \setminus \overline{S}_t|^{-\frac{1}{N}} (\tilde{\Omega} \setminus \overline{S}_t)$ is at most $\mathrm{Per}(\tilde{\Omega})$. Moreover, from the choice of c and Lemma 4.2, all the eigenvalues less than K of the rescaled set are not greater than the ones on $\tilde{\Omega}$.

In order to handle the diameter of the rescaled set, we replace all the connected components having a projection on $\mathbb{R}e_1$ disjoint from X by one ball, such that the volume remains unchanged. In this way, the perimeter does not increase, while the low part of the spectrum (below K) can only decrease, since the first eigenvalue of every such a connected component is not smaller than $1/(C_0 r_0) \geq 2K$.

In view of the definition of $\operatorname{diam}_{e_1}(\Omega)$ given at the beginning of this Section, we get the diameter bound:

$$\operatorname{diam}_{e_1}(\Omega) \leq \operatorname{diam}_{e_1}(\widehat{\Omega})(1-\widehat{m})^{-1/N} \leq 2\Big(\mathcal{H}^1(X) + n(8r_0+2l_0) + 2r_0(n+2)\Big) + 2\omega_N^{-\frac{1}{N}}.$$

If assumption (4.5) does not hold, we cannot apply directly Lemma 4.4. Let $p \in \mathbb{N}$ depending only on P and the dimension, be such that $\frac{1}{p} \leq \widehat{m} < \frac{1}{p-1}$. If

$$a_i + p(4r_0+l_0) > b_i - p(4r_0+l_0),$$

we ignore this strip and add it to X, renumbering the index i if necessary. There exists $s \in [0, p-1]$ such that replacing simultaneously all a_i with $a_i + s(4r_0+l_0)$ and b_i with $b_i - s(4r_0+l_0)$ the assumption (4.5) is satisfied and so we finish the proof, adding at worst $4np(4r_0+l_0)$ to the diameter.

Since the choice of the direction e_1 was arbitrary, we can repeat all the process of the proof of Theorem 4.1 for all the coordinate direction, finding a set which has diameter bounded in all directions, unit measure, better eigenvalues than $\tilde{\Omega}$ up to level k and perimeter lower than $\tilde{\Omega}$. □

5. Further remarks and applications

The results of the paper hold true if instead of open sets, one works with quasi-open sets or measurable sets. For a measurable set $\Omega \subseteq \mathbb{R}^N$ we define the Sobolev space $\widetilde{H}^1_0(\Omega)$ as a closed subspace in $H^1(\mathbb{R}^N)$, by

$$\widetilde{H}^1_0(\Omega) = \Big\{u \in H^1(\mathbb{R}^N) \ : \ |\{u \neq 0\} \setminus \Omega| = 0\Big\}.$$

In order to introduce the notion of quasi-open set, we need to define the capacity. For every $E \subseteq \mathbb{R}^N$ the *capacity* of E is defined as

$$\operatorname{cap}(E) = \min\Big\{\|v\|^2_{H^1(\mathbb{R}^N)} \ : \ v \in H^1(\mathbb{R}^N),\ v \geq 1 \text{ a.e. in a neighborhood of } E\Big\}.$$

A set $\Omega \subset \mathbb{R}^N$ is then called *quasi-open* if for every $\varepsilon > 0$ there exists an open set E_ε with $\operatorname{cap}(E_\varepsilon) < \varepsilon$ such that $\Omega \cup E_\varepsilon$ is open. For every function $u \in H^1(\mathbb{R}^N)$ the following limit

$$\lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} u \, dy$$

exists for all points $x \in \mathbb{R}^N$, except a set of capacity zero. In the sequel, we identify u with this limit.

For a quasi-open set Ω, one defines the Sobolev space

$$\widehat{H}^1_0(\Omega) := \Big\{u \in H^1(\mathbb{R}^N) : \ \operatorname{cap}(\{u \neq 0\} \setminus \Omega) = 0\Big\}. \tag{5.1}$$

If Ω is open, then $\widehat{H}^1_0(\Omega)$ coincides with the usual Sobolev space $H^1_0(\Omega)$, while the spaces $\widetilde{H}^1_0(\Omega)$ and $H^1_0(\Omega)$ do not coincide in general (for example one can think to a disk minus a radius). It holds that for all measurable sets $E \subset \mathbb{R}^N$, there exists a quasi-open set $\omega \subset E$ *a.e.* such that $\widehat{H}^1_0(\omega) = \widetilde{H}^1_0(E)$.

Let us point out that in any of the frameworks: *open sets, quasi-open sets, measurable sets*, the eigenvalues of the Dirichlet-Laplacian are defined following the min-max formula (2.1), where S_k runs in the corresponding Sobolev space (see [9, 13]). In both the frameworks of quasi-open and measurable sets, Theorems 3.3 and 4.1 hold true, by just replacing in their statements the word *open* by *quasi-open* and *measurable*, respectively. If the set $\tilde{\Omega}$ is quasi-open (measurable), the new build set Ω is quasi-open (measurable, respectively) as well. The proofs are completely identical, using the De Giorgi perimeter instead of the $(N-1)$-Hausdorff measure of the sections in Lemma 4.4.

Following [9, 13], the natural framework to prove existence of a result for problems involving measure and perimeter, is to consider the Sobolev spaces on measurable sets. We give the proof of an existence result for a more general version of problem (1.2). For simplicity, we formally set in the sequel $\lambda_k(\emptyset) = +\infty$.

Theorem 5.1. *Let $k, N \in \mathbb{N}$, $F \colon \overline{\mathbb{R}}_+^k \to \mathbb{R}$ nondecreasing in each variable and lower semicontinuous. Then for every $P > 0$ there exists a solution for the problem*

$$\min\big\{F(\lambda_1(E), \ldots, \lambda_k(E)),\ E \subset \mathbb{R}^N,\ \textit{measurable},\ |E| \le 1,\ \mathrm{Per}(E) \le P\big\}. \tag{5.2}$$

Proof. It is enough to take a minimizing sequence (of measurable sets) $(\tilde{E}_n)_n$ for (5.2). There are two possibilities: either for a subsequence (still denoted using the same index) $\lambda_k(\tilde{E}_n) \to +\infty$, and so as well $\lambda_1(\tilde{E}_n) \to +\infty$ from (2.5): in this case the empty set formally solves the problem. Alternatively, there exists K such that $\limsup_{n\to\infty} \lambda_k(\tilde{E}_n) \le K$. If this is the case, we get $\liminf_{n\to\infty} |\tilde{E}_n| > 0$ and we apply Theorem 4.1 in order to get a new minimizing sequence $(E_n)_n$ consisting of sets with uniformly bounded diameter. Then it is enough to use the result by Buttazzo and Dal Maso [12], rephrased in [9, Theorem 3.5] in the context of equi-bounded measurable sets, in order to achieve existence. □

Remark 5.2. *The two dimensional case is particular. One could achieve a direct proof of Theorem 5.1, using the existence result of Buttazzo-Dal Maso. Roughly speaking, in two dimensions, controlled perimeter leads to controlled diameter in the sense introduced in Theorem 4.1.*

In order to achieve an existence result in the family of open sets for problem (5.2)*, one should prove a regularity result. Such a result is definitely not straightforward and does not make the object of this paper.*

We conclude the section by highlighting a surprising consequence of Lemma 3.1.

Corollary 5.3. *Let $F \colon \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous functional nondecreasing in each variable and strictly increasing in at least one variable. Then all the optimal sets for the problem*

$$\min\{F(\lambda_1(A), \ldots, \lambda_k(A))\ :\ A \subset \mathbb{R}^N,\ \textit{measurable},\ |A| = 1\}, \tag{5.3}$$

are quasi-open and have finite perimeter.

Proof. Let us consider an optimal set A_{opt} for (5.3). If the corresponding quasi-open set which supports the same Sobolev space has strictly lower measure, then A_{opt} can not be optimal, from rescaling and strict monotonicity of F. So A_{opt} is quasi-open. Assume for contradiction that it has not finite perimeter. Then for all $c > 0$, one can find a set $A_c \subset A_{opt}$ such that

$$E(A_c) + c|A_c| < E(A_{opt}) + c|A_{opt}|,$$

otherwise A_{opt} would be an energy subsolution, which has finite perimeter thanks to [7, Theorem 2.2]. It is then enough to choose $c > 0$ such that we can apply Lemma 3.1 for $K = \lambda_k(A_{opt})$ and obtain that for all $i = 1, \dots, k$,

$$\lambda_i(A_c)|A_c|^{2/N} < \lambda_i(A_{opt})|A_{opt}|^{2/N}.$$

Since the functional F is strictly increasing in at least one variable, this inequality contradicts the optimality of A_{opt} for (5.3). Indeed, after suitably rescaling A_c such that its measure becomes equal to 1 one gets $F(\lambda_1(A_c), \dots, \lambda_k(A_c)) < F(\lambda_1(A_{opt}), \dots, \lambda_k(A_{opt}))$. □

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