ON A POISSON'S EQUATION ARISING FROM MAGNETISM

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Abstract. We review the proof of existence and uniqueness of the Poisson's equation $\Delta u + \operatorname{div} \mathbf{m} = 0$ whenever $\mathbf{m}$ is a unit L^2-vector field on $\mathbb{R}^3$ with compact support; by standard linear potential theory we deduce also the H^1-regularity of the unique weak solution.

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1. Introduction

In the standard theory of ferromagnetic materials is usually considered an energy, called *magnetostatic*, which is the energy of the magnetostatic field set up by the magnetization vector field $\mathbf{m}$. It turns out that the magnetostatic energy takes the form

$$\int |\nabla u|^2 \, dx$$

where the scalar potential $u \colon \mathbb{R}^3 \to \mathbb{R}$ satisfies the following equation arising from Maxwell's equations:

$$\operatorname{div}(\nabla u + \mathbf{m}\chi_\Omega) = 0, \quad \text{on } \mathbb{R}^3, \tag{1.1}$$

being Ω an open and bounded domain in $\mathbb{R}^3$, which represents the region occupied by a ferromagnetic material, and χ_Ω its characteristic function, that is $\chi_\Omega = 1$ on Ω and 0 otherwise in $\mathbb{R}^3$; for more details on equation (1.1) see [2], [5] and [6]. Without loss of generality, since we will not vary the temperature, which is related with the variation of $|\mathbf{m}|$, we will consider vector fields $\mathbf{m} \colon \Omega \to S^2$, being S^2 the boundary of the unit ball in $\mathbb{R}^3$. Replacing $\mathbf{m}\chi_\Omega$ with $\mathbf{m} \colon \mathbb{R}^3 \to \mathbb{R}^3$, equation (1.1) takes the form

$$\Delta u + \operatorname{div} \mathbf{m} = 0, \text{ on } \mathbb{R}^3, \text{ with } |\mathbf{m}| = \chi_\Omega. \tag{1.2}$$

There is a huge literature on the Poisson's type equation (1.2); we just mention a very recent application in the context of micromagnetics materials: such an equation has been considered in [3] and [4] where an homogenization procedure of a more complete energy functional for polycrystalline magnetic materials has been investigated. In order to solve equation (1.2) we have to introduce its weak formulation, that is

$$\int (\nabla u + \mathbf{m}) \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3). \tag{1.3}$$

In this short note we will explain how the proof of existence and uniqueness of the solution of equation (1.3) in a suitable space of Sobolev-type works; moreover, we will find, exploiting the standard tools coming from the linear potential theory, the explicit form of the solution from which, in particular, it will descends more regularity of such a solution: more precisely, the unique weak solution turns out to be $H^1(\mathbb{R}^3)$, and such a regularity has been stated in [5], but without proof.

2. Some preliminaries of potential theory

We now recall some well known results coming from potential theory; for details we refer to [7]. Let $n \geq 1$ be an integer and, for each $f\colon \mathbb{R}^n \to \mathbb{R}$ measurable and for each $\alpha > 0$, let I_α be the *Riesz potential* given by

$$I_\alpha(f)(x) := c(n,\alpha) \int \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n, \tag{2.1}$$

for a suitable positive constant $c(n,\alpha)$. It turns out that if $\alpha, \beta > 0$ and $\alpha + \beta < n$ then for any $f \in \mathcal{S}(\mathbb{R}^n)$, being $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space on $\mathbb{R}^n$,

$$I_\alpha(I_\beta(f)) = I_{\alpha+\beta}(f). \tag{2.2}$$

Let $\alpha \in (0,n)$ and $1 \leq p < +\infty$ with

$$\frac{1}{p} - \frac{\alpha}{n} < 1.$$

First of all, it turns out that if, more generally, $f \in L^p(\mathbb{R}^n)$ then the integral on the right hand-side of (2.1) converges absolutely for almost any $x \in \mathbb{R}^n$. Moreover, if in particular

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

then

$$I_\alpha\colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \tag{2.3}$$

is linear and continuous. Strictly related with Riesz potentials is the notion of *Riesz transform*: for any $f \in L^p(\mathbb{R}^n)$, with $1 \leq p < +\infty$, and for any $j = 1, \dots, n$, we let

$$R_j(f)(x) := c(n) \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) \, dy, \quad x \in \mathbb{R}^n,$$

whenever the limit exists; $c(n)$ is a suitable positive constant. It turns out that

$$R_j\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is linear and continuous; furthermore, we have the following fundamental relation between the first order Riesz potential I_1 and the Riesz transform:

$$R_j(f) = -\partial_j I_1(f), \tag{2.4}$$

for any $j = 1, \dots, n$.

3. Existence and uniqueness

Let $\mathbf{f} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. We first investigate existence and uniqueness of weak solutions of $\Delta u + \operatorname{div} \mathbf{f} = 0$ on $\mathbb{R}^3$, following, for instance, [1]. Let $\mathcal{E} \colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be the fundamental solution of the Laplace operator on $\mathbb{R}^3$, i.e.

$$\mathcal{E}(x) := -\frac{1}{4\pi|x|}.$$

Moreover, let

$$\mathcal{P}(\mathbf{f})(x) := -\int \nabla\mathcal{E}(x-y) \cdot \mathbf{f}(y)\, dy, \quad x \in \mathbb{R}^3.$$

Theorem 3.1. *Let $H := \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3; \mathbb{R}^3)\}$. Then $\mathcal{P}(\mathbf{f})$ is the unique weak solution in H of the equation $\Delta u + \operatorname{div} \mathbf{f} = 0$ on $\mathbb{R}^3$.*

Proof. We divide the proof in some steps.

Step 1. First of all we claim that $\mathcal{P}(\mathbf{f}) \in H$. For, let $g \in L^2(\mathbb{R}^3)$ and, for any $i = 1, 2, 3$ we let

$$\mathcal{P}_i(g)(x) := -\int \partial_i\mathcal{E}(x-y) g(y)\, dy, \quad x \in \mathbb{R}^3.$$

Since

$$\partial_i \mathcal{E} = c\frac{x_i}{|x|^3}$$

then

$$|\mathcal{P}_i(g)(x)| \le c\int \frac{|g(y)|}{|x-y|^2}\, dy = I_1(|g|)$$

and $\mathcal{P}_i(g) \in L^6(\mathbb{R}^3)$ from (2.3), being $g \in L^2(\mathbb{R}^3)$. In order to prove that $\partial_j \mathcal{P}_i(g) \in L^2(\mathbb{R}^3)$, let us choose a sequence $(g_h)_{h\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^3)$ with $g_h \to g$ strongly in $L^2(\mathbb{R}^3)$. Using the explicit form of $\mathcal{E}$, we immediately get

$$|\mathcal{P}_i(g_h) - \mathcal{P}(g)| \le c|I_1(g_h - g)|, \quad \text{on } \mathbb{R}^3.$$

Hence, by the continuity of $I_1 \colon L^2(\mathbb{R}^3) \to L^6(\mathbb{R}^3)$, we get $\mathcal{P}_i(g_h) \to \mathcal{P}_i(g)$, strongly in $L^6(\mathbb{R}^3)$. Now, for any $h \in \mathbb{N}$ we have

$$\mathcal{P}_i(g_h)(x) = \int \mathcal{E}(x-y)\partial_i g_h(y)\, dy = \tilde{c} I_2(\partial_i g_h)(x), \quad \tilde{c} \neq 0.$$

Therefore, using (2.4) and (2.2) we easily get

$$R_j R_i(g_h) = \partial_j I_2(\partial_i g_h) = \frac{1}{\tilde{c}} \partial_j \mathcal{P}_i(g_h).$$

By the continuity of the Riesz transform we deduce that $||\partial_j \mathcal{P}_i(g_h)||_2 \le \bar{c}||g_h||_2$. Thus $\partial_j \mathcal{P}_i(g_h) \rightharpoonup u_{ij}$, for some $u_{ij} \in L^2(\mathbb{R}^3)$. Passing to the limit as $h \to +\infty$ in

$$\int \partial_j \mathcal{P}_i(g_h) \varphi \, dx = -\int \mathcal{P}_i(g_h) \partial_j \varphi \, dx$$

we deduce that $u_{ij} = \partial_j \mathcal{P}_i(g)$ which means that $\nabla \mathcal{P}_i(g) \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. In order to conclude it is sufficient to notice that

$$\mathcal{P}(\mathbf{f}) = \sum_{i=1}^{3} \mathcal{P}_i(f^{(i)})$$

being $f^{(i)}$, for $i = 1, 2, 3$, the components of $\mathbf{f}$.

Step 2. Now we prove that $\mathcal{P}(\mathbf{f})$ is a weak solution of the equation $\Delta u + \operatorname{div} \mathbf{f} = 0$, that is

$$\int (\nabla \mathcal{P}(\mathbf{f}) + \mathbf{f}) \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3). \tag{3.1}$$

Let $(\mathbf{f}_h)_{h \in \mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with $\mathbf{f}_h \to \mathbf{f}$ strongly in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. First of all we have, integrating by parts,

$$-\mathcal{P}(\mathbf{f}_h)(x) = \int \mathcal{E}(x-y) \operatorname{div} \mathbf{f}_h(y) \, dy$$

and therefore, since $\mathcal{E}$ is the fundamental solution of the Laplace operator on $\mathbb{R}^3$,

$$-\int \mathcal{P}(\mathbf{f}_h) \Delta \varphi \, dx = \int \operatorname{div} \mathbf{f}_h \varphi \, dx = -\int \mathbf{f}_h \cdot \nabla \varphi \, dx.$$

Passing to the limit as $h \to +\infty$ we obtain

$$-\int \mathcal{P}(\mathbf{f}) \Delta \varphi \, dx = -\int \mathbf{f} \cdot \nabla \varphi \, dx$$

which means, since $\mathcal{P}(\mathbf{f}) \in H$,

$$\int \nabla \mathcal{P}(\mathbf{f}) \cdot \nabla \varphi \, dx = -\int \mathbf{f} \cdot \nabla \varphi \, dx.$$

Therefore we get (3.1).

Step 3. In order to conclude the proof, we have to show that the weak solution in H is unique. If $u_1, u_2 \in H$ satisfy

$$\int (\nabla u + \mathbf{f}) \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3)$$

then $w := u_2 - u_1$ satisfies

$$\int \nabla w \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3).$$

Now, if we choose $\varphi_h \to w$ in H then passing to the limit as $h \to +\infty$,

$$0 = \int \nabla w \cdot \nabla \varphi_h \, dx \to \int |\nabla w|^2 \, dx$$

from which we get w constant, and since $w \in H$, we deduce that $w = 0$, and thus $u_1 = u_2$, which yields the conclusion. □

We are ready to prove the existence and uniqueness result for the equation (1.3).

Corollary 3.2. *Let Ω be an open and bounded domain in $\mathbb{R}^3$ and let $\mathbf{m} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ with $|\mathbf{m}| = \chi_\Omega$. Then the equation $\Delta u + \operatorname{div} \mathbf{m} = 0$ on $\mathbb{R}^3$ admits a unique weak solution $u \in H^1(\mathbb{R}^3)$.*

Proof. Taking into account Theorem 3.1, it is sufficient to prove that $\mathcal{P}(\mathbf{m}) \in L^2(\mathbb{R}^3)$. Using the very definition of $\mathcal{P}(\mathbf{m})$ and $\mathcal{E}$, we have, since $|\mathbf{m}| = \chi_\Omega$, $|\mathcal{P}(\mathbf{m})| \leq c I_1(\chi_\Omega)$ and $I_1(\chi_\Omega) \in L^2(\mathbb{R}^3)$ since $\chi_\Omega \in L^\infty(\Omega)$ and Ω is bounded; this yields the conclusion. □

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