HEIGHT ESTIMATE AND SLICING FORMULAS IN THE HEISENBERG GROUP

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Abstract. We prove a height-estimate (distance from the tangent hyperplane) for Λ-minima of the perimeter in the sub-Riemannian Heisenberg group. The estimate is in terms of a power of the excess (L^2-mean oscillation of the normal) and its proof is based on a new coarea formula for rectifiable sets in the Heisenberg group.

1. Introduction

In this article, we continue the research project started in [20] and [18] on the regularity of H-perimeter minimizing boundaries in the Heisenberg group $\mathbb{H}^n$. Our goal is to prove the so-called *height-estimate* for sets that are Λ-*minima* and have small *excess* inside suitable cylinders, see Theorem 1.3. The proof follows the scheme of the median choice for the measure of the boundary in certain half-cylinders together with a lower dimensional isoperimetric inequality on slices. For minimizing currents in $\mathbb{R}^n$, the principal ideas of the argument go back to Almgren's paper [1] and are carried over by Federer in his Theorem 5.3.4 in [5]. The argument can be also found in the Appendix of [21] and, for Λ-minima of perimeter in $\mathbb{R}^n$, in [13].

Our main technical effort is the proof of a coarea formula (slicing formula) for intrinsic rectifiable sets, see Theorem 1.5. This formula is established in Section 2 and has a nontrivial character because the domain of integration and its slices need not be rectifiable in the standard sense. The relative isoperimetric inequalities that are used in the slices reduce to a single isoperimetric inequality in one slice that is relative to a family of varying domains with uniform isoperimetric constants. This uniformity can be established using the results on regular domains in Carnot groups of step 2 of [19] and the isoperimetric inequality in [9], see Section 3.1.

The $2n+1$-dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group product

$$(z,t) * (\zeta,\tau) = \big(z+\zeta, t+\tau+2\,\mathrm{Im}\langle z,\bar\zeta\rangle\big), \tag{1.1}$$

where $t,\tau \in \mathbb{R}$, $z,\zeta \in \mathbb{C}^n$ and $\langle z,\bar\zeta\rangle = z_1\bar\zeta_1+\ldots+z_n\bar\zeta_n$. The Lie algebra of left-invariant vector fields in $\mathbb{H}^n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}, \tag{1.2}$$

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with $z_j = x_j + iy_j$ and $j = 1, \dots, n$. We denote by H the horizontal sub-bundle of $T\mathbb{H}^n$. Namely, for any $p = (z,t) \in \mathbb{H}^n$ we let

$$H_p = \mathrm{span}\big\{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\big\}.$$

A horizontal section $\varphi \in C^1_c(\Omega; H)$, where $\Omega \subset \mathbb{H}^n$ is an open set, is a vector field of the form

$$\varphi = \sum_{j=1}^n \varphi_j X_j + \varphi_{n+j} Y_j,$$

where $\varphi_j \in C^1_c(\Omega)$.

Let g be the left-invariant Riemannian metric on $\mathbb{H}^n$ that makes orthonormal the vector fields $X_1, \dots, Y_n, T$ in (1.2). For tangent vectors $V, W \in T\mathbb{H}^n$ we let

$$\langle V, W\rangle_g = g(V,W) \quad \text{and} \quad |V|_g = g(V,V)^{1/2}.$$

The sup-norm with respect to g of a horizontal section $\varphi \in C^1_c(\Omega; H)$ is

$$\|\varphi\|_g = \max_{p\in\Omega} |\varphi(p)|_g.$$

The Riemannian divergence of φ is

$$\mathrm{div}_g \varphi = \sum_{j=1}^n X_j\varphi_j + Y_j \varphi_{n+j}.$$

The metric g induces a volume form on $\mathbb{H}^n$ that is left-invariant. Also the Lebesgue measure $\mathscr{L}^{2n+1}$ on $\mathbb{H}^n$ is left-invariant, and by the uniqueness of the Haar measure the volume induced by g is the Lebesgue measure $\mathscr{L}^{2n+1}$. In fact, the proportionality constant is 1.

The *H-perimeter* of a $\mathscr{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ in an open set $\Omega \subset \mathbb{H}^n$ is

$$\mu_E(\Omega) = \sup\left\{ \int_E \mathrm{div}_g\varphi \, d\mathscr{L}^{2n+1} : \varphi \in C^1_c(\Omega; H), \|\varphi\|_g \le 1 \right\}.$$

If $\mu_E(\Omega) < \infty$ we say that E has finite H-perimeter in Ω. If $\mu_E(A) < \infty$ for any open set $A \subset\subset \Omega$, we say that E has locally finite H-perimeter in Ω. In this case, the open sets mapping $A \mapsto \mu_E(A)$ extends to a Radon measure μ_E on Ω that is called *H-perimeter measure* induced by E. Moreover, there exists a μ_E-measurable function $\nu_E : \Omega \to H$ such that $|\nu_E|_g = 1$ μ_E-a.e. and the Gauss-Green integration by parts formula

$$\int_\Omega \langle \varphi, \nu_E\rangle_g \, d\mu_E = -\int_\Omega \mathrm{div}_g \varphi \, d\mathscr{L}^{2n+1}$$

holds for any $\varphi \in C^1_c(\Omega; H)$. The vector ν_E is called *horizontal inner normal* of E in Ω.

The Koràny norm of $p = (z,t) \in \mathbb{H}^n$ is $\|p\|_K = (|z|^4 + t^2)^{1/4}$. For any $r > 0$ and $p \in \mathbb{H}^n$, we define the balls

$$B_r = \big\{q \in \mathbb{H}^n : \|q\|_K < r\big\} \quad \text{and} \quad B_r(p) = \big\{p * q \in \mathbb{H}^n : q \in B_r\big\}.$$

The *measure theoretic boundary* of a measurable set $E \subset \mathbb{H}^n$ is the set

$$\partial E = \big\{ p \in \mathbb{H}^n : \mathscr{L}^{2n+1}(E \cap B_r(p)) > 0 \text{ and } \mathscr{L}^{2n+1}(B_r(p) \setminus E) > 0 \text{ for all } r > 0 \big\}.$$

For a set E with locally finite H-perimeter, the H-perimeter measure μ_E is concentrated on ∂E and, actually, on a subset $\partial^* E$ of ∂E, see below. Moreover, up to modifying E on a Lebesgue negligible set, one can always assume that ∂E coincides with the topological boundary of E, see [22, Proposition 2.5].

Definition 1.1. Let $\Omega \subset \mathbb{H}^n$ be an open set, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$. We say that a set $E \subset \mathbb{H}^n$ with locally finite H-perimeter in Ω is a (Λ, r)*-minimum of H-perimeter in* Ω if, for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$, and $s < r$ such that $E \Delta F \subset\subset B_s(p) \subset\subset \Omega$, there holds

$$\mu_E(B_s(p)) \leq \mu_F(B_s(p)) + \Lambda \mathscr{L}^{2n+1}(E \Delta F),$$

where $E \Delta F = E \setminus F \cup F \setminus E$.

We say that E is *locally H-perimeter minimizing in* Ω if, for any measurable set $F \subset \mathbb{H}^n$ and any open set U such that $E \Delta F \subset\subset U \subset\subset \Omega$, there holds $\mu_E(U) \leq \mu_F(U)$.

We will often use the term Λ-*minimum*, rather than (Λ, r)-minimum, when the role of r is not relevant. In Appendix A, we list without proof some elementary properties of Λ-minima.

We introduce the notion of cylindrical excess. The *height function* $\mathfrak{h} : \mathbb{H}^n \to \mathbb{R}$ is defined by $\mathfrak{h}(p) = p_1$, where p_1 is the first coordinate of $p = (p_1, \dots, p_{2n+1}) \in \mathbb{H}^n$. The set $\mathbb{W} = \{p \in \mathbb{H}^n : \mathfrak{h}(p) = 0\}$ is the vertical hyperplane passing through $0 \in \mathbb{H}^n$ and orthogonal to the left-invariant vector field X_1. The disk in $\mathbb{W}$ of radius $r > 0$ centered at $0 \in \mathbb{W}$ induced by the Korànyi norm is the set $D_r = \big\{ p \in \mathbb{W} : \|p\|_K < r \big\}$. The intrinsic cylinder with central section D_r and height $2r$ is the set

$$C_r = D_r * (-r, r) \subset \mathbb{H}^n.$$

Here and in the sequel, we use the notation $D_r * (-r, r) = \big\{ w * (se_1) \in \mathbb{H}^n : w \in D_r,\ s \in (-r, r) \big\}$, where $se_1 = (s, 0, \dots, 0) \in \mathbb{H}^n$. The cylinder C_r is comparable with the ball $B_r = \{\|p\|_K < r\}$. Namely, there exists a constant $k = k(n) \geq 1$ such that for any $r > 0$ we have

$$B_{r/k} \subset C_r \subset B_{kr}. \tag{1.3}$$

By a rotation of the system of coordinates, it is enough to consider excess in cylinders with basis in $\mathbb{W}$ and axis X_1.

Definition 1.2 (Cylindrical excess)**.** Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter. We define the excess of E in the cylinder C_r oriented by the vector $\nu = -X_1$ as

$$\operatorname{Exc}(E, r, \nu) = \frac{1}{2r^{2n+1}} \int_{C_r} |\nu_E - \nu|_g^2 \, d\mu_E,$$

where μ_E is the H-perimeter measure of E and ν_E is its horizontal inner normal.

Theorem 1.3 (Height estimate)**.** *Let $n \geq 2$. There exist constants $\varepsilon_0 = \varepsilon_0(n) > 0$ and $c_0 = c_0(n) > 0$ with the following property. If $E \subset \mathbb{H}^n$ is a (Λ, r)-minimum of H-perimeter in the cylinder C_{4k^2r}, $\Lambda r \leq 1$, $0 \in \partial E$, and*

$$\mathrm{Exc}(E, 4k^2r, \nu) \leq \varepsilon_0,$$

then

$$\sup\left\{|\mathfrak{h}(p)| \in [0,\infty) : p \in \partial E \cap C_r\right\} \leq c_0\, r\, \mathrm{Exc}(E, 4k^2r, \nu)^{\frac{1}{2(2n+1)}}. \tag{1.4}$$

The constant $k = k(n)$ is the one in (1.3).

The estimate (1.4) does not hold when $n = 1$. In fact, there are sets $E \subset \mathbb{H}^1$ such that $\mathrm{Exc}(E, C_r, \nu) = 0$ but ∂E is not flat in $C_{\varepsilon r}$ for any $\varepsilon > 0$. See the conclusions of Proposition 3.7 in [18]. Theorem 1.3 is proved in Section 3.

Besides local minimizers of H-perimeter, our interest in Λ-minima is also motivated by possible applications to isoperimetric sets. The height estimate is a first step in the regularity theory of Λ-minima of classical perimeter; we refer to [13, Part III] for a detailed account on the subject.

In order to state the slicing formula in its general form, we need the definition of a rectifible set in $\mathbb{H}^n$ of codimension 1. We follow closely [7], where this notion was first introduced.

The Riemannian and horizontal gradients of a function $f \in C^1(\mathbb{H}^n)$ are, respectively,

$$\begin{aligned}
\nabla f &= (X_1 f)X_1 + \cdots + (Y_n f)Y_n + (Tf)T,\\
\nabla_H f &= (X_1 f)X_1 + \cdots + (Y_n f)Y_n.
\end{aligned}$$

We say that a continuous function $f \in C(\Omega)$, with $\Omega \subset \mathbb{H}^n$ open set, is of class $C^1_H(\Omega)$ if the horizontal gradient $\nabla_H f$ exists in the sense of distributions and is represented by continuous functions $X_1 f, \dots, Y_n f$ in Ω. A set $S \subset \mathbb{H}^n$ is an H-regular hypersurface if for all $p \in S$ there exist $r > 0$ and a function $f \in C^1_H(B_r(p))$ such that $S \cap B_r(p) = \left\{q \in B_r(p) : f(q) = 0\right\}$ and $\nabla_H f(p) \neq 0$. Sets with H-regular boundary have locally finite H-perimeter.

For any $p = (z,t) \in \mathbb{H}^n$, let us define the box-norm $\|p\|_\infty = \max\{|z|, |t|^{1/2}\}$ and the balls $U_r = \{q \in \mathbb{H}^n : \|q\|_\infty < r\}$ and $U_r(p) = p * U_r$, with $r > 0$. Let $E \subset \mathbb{H}^n$ be a set. For any $s \geq 0$ define the measure

$$\mathscr{S}^s(E) = \sup_{\delta > 0}\ \inf\Big\{c(n,s)\sum_{i\in\mathbb{N}} r_i^s : E \subset \bigcup_{i\in\mathbb{N}} U_{r_i}(p_i),\ r_i < \delta\Big\}.$$

Above, $c(n,s) > 0$ is a normalization constant that we do not need to specify, here. By Carathèodory's construction, $E \mapsto \mathscr{S}^s(E)$ is a Borel measure in $\mathbb{H}^n$. When $s = 2n+2$, $\mathscr{S}^{2n+2}$ turns out to be the Lebesgue measure $\mathscr{L}^{2n+1}$. Thus, the correct dimension to measure hypersurfaces is $s = 2n+1$. In fact, if E is a set with locally finite H-perimeter in $\mathbb{H}^n$, then we have

$$\mu_E = \mathscr{S}^{2n+1} \llcorner \partial^* E, \tag{1.5}$$

where $\llcorner$ denotes restriction and $\partial^* E$ is the H-reduced boundary of E, namely the set of points $p \in \mathbb{H}^n$ such that $\mu_E(U_r(p)) > 0$ for all $r > 0$, $\fint_{U_r(p)} \nu_E \, d\mu_E \to \nu_E(p)$ as $r \to 0$ and $|\nu_E(p)|_g = 1$. The validity of formula (1.5) depends on the geometry of the balls $U_r(p)$, see [16]. We refer the reader to [7] for more details on the H-reduced boundary.

Definition 1.4. A set $R \subset \mathbb{H}^n$ is $\mathscr{S}^{2n+1}$-rectifiable if there exists a sequence of H-regular hypersurfaces $(S_j)_{j\in\mathbb{N}}$ in $\mathbb{H}^n$ such that

$$\mathscr{S}^{2n+1}\Big(R \setminus \bigcup_{j\in\mathbb{N}} S_j\Big) = 0.$$

By the results of [7], the H-reduced boundary $\partial^* E$ is $\mathscr{S}^{2n+1}$-rectifiable. Definition 1.4 is generalized in [17], where the authors study the notion of an s-rectifiable set in $\mathbb{H}^n$ for any integer $1 \leq s \leq 2n+1$.

An H-regular surface S has a continuous horizontal normal ν_S that is locally defined up to the sign. This normal is given by the formula

$$\nu_S = \frac{\nabla_H f}{|\nabla_H f|_g}, \tag{1.6}$$

where f is a defining function for S. When $S = \partial E$ is the boundary of a smooth set, then ν_S agrees with the horizontal normal ν_E. Then, for an $\mathscr{S}^{2n+1}$-rectifiable set $R \subset \mathbb{H}^n$ there is a unit horizontal normal $\nu_R : R \to H$ that is Borel regular. This normal is uniquely defined $\mathscr{S}^{2n+1}$-a.e. on R up the the sign, see Appendix B. However, formula (1.8) below does not depend on the sign.

In the following theorem, $\Omega \subset \mathbb{H}^n$ is an open set and $u \in C^\infty(\Omega)$ is a smooth function. For any $s \in \mathbb{R}$, we denote by $\Sigma^s = \big\{p \in \Omega : u(p) = s\big\}$ the level sets of u.

Theorem 1.5. *Let $R \subset \Omega$ be an $\mathscr{S}^{2n+1}$-rectifiable set. Then, for a.e. $s \in \mathbb{R}$ there exists a Radon measure μ_R^s on $R \cap \Sigma^s$ such that for any Borel function $h : \Omega \to [0, \infty)$ the function*

$$s \mapsto \int_\Omega h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_R^s \tag{1.7}$$

is $\mathscr{L}^1$-measurable, and we have the coarea formula

$$\int_{\mathbb{R}} \int_\Omega h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_R^s \, ds = \int_R h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u\rangle_g^2} \, d\mathscr{S}^{2n+1}. \tag{1.8}$$

Theorem 1.5 is proved in Section 2. When $R \cap \Sigma^s$ is a regular subset of Σ^s, the measures μ_R^s are natural horizontal perimeters defined in Σ^s.

Coarea formulae in the Heisenberg group are known only for slicing of sets with positive Lebesgue measure, see [14, 15]. Theorem 1.5 is, to our knowledge, the first example of slicing of lower-dimensional sets in a sub-Riemannian framework. Also, Theorem 1.5 is a nontrivial extension of the Riemannian coarea formula, because the set R and the slices $R \cap \Sigma^s$ need not be rectifiable in the standard sense, see [12]. We need the coarea formula (1.8) in the proof of Theorem 1.3, see Section 3.3.

We conclude the introduction by stating a different but equivalent formulation of the coarea formula (1.8) that is closer to standard coarea formulae. This alternative formulation holds only when $n \geq 2$: when $n = 1$, the right hand side in (1.9) might not be well defined, see Remark 2.11.

Theorem 1.6. *Let $\Omega \subset \mathbb{H}^n$, $n \geq 2$, be an open set, $u \in C^\infty(\Omega)$ be a smooth function, and $R \subset \Omega$ be an $\mathscr{S}^{2n+1}$-rectifiable set. Then, for any Borel function $h : \Omega \to [0, \infty)$ there holds*

$$\int_{\mathbb{R}} \int_{\Omega} h \, d\mu_R^s \, ds = \int_R h \, |\nabla u|_g \sqrt{1 - \left\langle \nu_R, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2} \, d\mathscr{S}^{2n+1}, \tag{1.9}$$

where μ_R^s are the measures given by Theorem 1.5.

2. Proof of the coarea formula

2.1. **Horizontal perimeter on submanifolds.** Let $\Sigma \subset \mathbb{H}^n$ be a C^∞ hypersurface. We define the horizontal tangent bundle $H\Sigma$ letting, for any $p \in \Sigma$,

$$H_p\Sigma = H_p \cap T_p\Sigma.$$

In general, the rank of $H\Sigma$ is not constant. This depends on the presence of *characteristic points* on Σ, i.e., points such that $H_p = T_p\Sigma$. For points $p \in \Sigma$ such that $H_p \neq T_p\Sigma$, we have $\dim(H_p\Sigma) = 2n - 1$.

We denote by σ_Σ the surface measure on Σ induced by the Riemannian metric g restricted to the tangent bundle $T\Sigma$.

Definition 2.1. Let $F \subset \Sigma$ be a Borel set and let $\Omega \subset \Sigma$ be an open set. We define the *H-perimeter of F in Ω*

$$\mu_F^\Sigma(\Omega) = \sup \left\{ \int_F \mathrm{div}_g \varphi \, d\sigma_\Sigma : \varphi \in C_c^1(\Omega; H\Sigma), \|\varphi\|_g \leq 1 \right\}. \tag{2.10}$$

We say that the set $F \subset \Sigma$ has locally finite H-perimeter in Ω if $\mu_F^\Sigma(A) < \infty$ for any open set $A \subset\subset \Omega$.

By Riesz' theorem, if $F \subset \Sigma$ has locally finite H-perimeter in Ω, then the open set mapping $A \mapsto \mu_F^\Sigma(A)$ extends to a Radon measure on Ω, called *H-perimeter measure* of F.

Remark 2.2. If $F \subset \Sigma$ is an open set with smooth boundary, then by the divergence theorem we have, for any $\varphi \in C_c^1(\Omega; H\Sigma)$,

$$\int_F \mathrm{div}_g \varphi \, d\sigma_\Sigma = \int_{\partial F} \langle N_{\partial F}, \varphi \rangle_g \, d\lambda_{\partial F}, \tag{2.11}$$

where $N_{\partial F}$ is the Riemannian outer unit normal to ∂F and $d\lambda_{\partial F}$ is the Riemannian $(2n-1)$-dimensional volume form on ∂F induced by g.

From the sup-definition (2.10) and from (2.11), we deduce that the H-perimeter measure of F has the following representation

$$\mu_F^\Sigma = |N_{\partial F}^{H\Sigma}|_g \, \lambda_{\partial F},$$

where $N_{\partial F}^{H\Sigma} \in H\Sigma$ is the g-orthogonal projection of $N_{\partial F} \in T\Sigma$ onto $H\Sigma$.

This formula can be generalized as follows. We denote by $\mathscr{H}_g^{2n-1}$ the $(2n-1)$-dimensional Hausdorff measure in $\mathbb{H}^n$ induced by the metric g.

Lemma 2.3. *Let $F, \Omega \subset \Sigma$ be open sets and assume that there exists a compact set $N \subset \partial F$ such that $\mathscr{H}_g^{2n-1}(N) = 0$ and $(\partial F \setminus N) \cap \Omega$ is a smooth $(2n-1)$-dimensional surface. Then, we have*

$$\mu_F^{\Sigma} \llcorner \Omega = |N_{\partial F}^{H\Sigma}|_g \, \lambda_{\partial F \setminus N} \llcorner \Omega. \tag{2.12}$$

Proof. For any $\varepsilon > 0$ there exist points $p_i \in \mathbb{H}^n$ and radii $r_i \in (0,1)$, $i = 1, \dots, M$, such that

$$N \subset \bigcup_{i=1}^{M} B_g(p_i, r_i) \quad \text{and} \quad \sum_{i=1}^{M} r_i^{2n-1} < \varepsilon,$$

where $B_g(p, r)$ denotes the ball in $\mathbb{H}^n$ with center p and radius r with respect to the metric g. By a partition-of-the-unity argument, there exist functions $f^\varepsilon, g_i^\varepsilon \in C^\infty(\Omega; [0,1])$, $i = 1, \dots, M$, such that

i) $f^\varepsilon + g_1^\varepsilon + \dots + g_M^\varepsilon = \chi_\Omega$;
ii) $f^\varepsilon = 0$ on $\bigcup_{i=1}^{M} B_g(p_i, r_i/2)$;
iii) $\operatorname{spt} g_i^\varepsilon \subset B_g(p_i, r_i)$ for each i;
iv) $|\nabla g_i^\varepsilon|_g \le C r_i^{-1}$ for a constant $C > 0$ independent of ε.

Hence, for any horizontal section $\varphi \in C_c^1(\Omega; H\Sigma)$ we have

$$\begin{aligned} \int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma &= \int_F \operatorname{div}_g (f^\varepsilon \varphi) d\sigma_\Sigma + \sum_{i=1}^{M} \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) d\sigma_\Sigma \\ &= \int_{\partial F \setminus N} \langle f^\varepsilon \varphi, N_{\partial F} \rangle_g d\lambda_{\partial F \setminus N} + \sum_{i=1}^{M} \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) d\sigma_\Sigma, \end{aligned} \tag{2.13}$$

where, by iv),

$$\begin{aligned} \left| \sum_{i=1}^{M} \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) d\sigma_\Sigma \right| &\le \sum_{i=1}^{M} \int_{B_g(p_i, r_i)} \left(\|\operatorname{div}_g \varphi\|_{L^\infty} + C r_i^{-1} \right) d\sigma_\Sigma \\ &\le C' \sum_{i=1}^{M} r_i^{2n-1} \le C' \varepsilon, \end{aligned} \tag{2.14}$$

with a constant $C' > 0$ independent of ε.

Letting $\varepsilon \to 0$, we have $f^\varepsilon \to 1$ pointwise on $\partial F \setminus N$, by i) and iii). Then, from (2.13) and (2.14) we obtain

$$\int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma = \int_{\partial F \setminus N} \langle \varphi, N_{\partial F} \rangle_g d\lambda_{\partial F \setminus N}$$

and the claim (2.12) follows by standard arguments. □

2.2. **Proof of Theorem 1.5.** Let $\Omega \subset \mathbb{H}^n$ be an open set and let $u \in C^\infty(\Omega)$. By Sard's theorem, for a.e. $s \in \mathbb{R}$ the level set

$$\Sigma^s = \big\{ p \in \Omega : u(p) = s \big\}$$

is a smooth hypersurface and, moreover, we have $\nabla u \neq 0$ on Σ^s.

Let $E \subset \mathbb{H}^n$ be a Borel set such that $E \cap \Sigma^s$ has (locally) finite H-perimeter in $\Omega \cap \Sigma^s$, in the sense of Definition 2.1. Then on $\Omega \cap \Sigma^s$ we have the H-perimeter measure $\mu^{\Sigma^s}_{E\cap\Sigma^s}$ induced by $E \cap \Sigma^s$. We shall use the notation

$$\mu^s_E = \mu^{\Sigma^s}_{E\cap\Sigma^s}$$

to denote a measure on Ω that is supported on $\Omega \cap \Sigma^s$.

We start with the following coarea formula in the smooth case, that is deduced from the Riemannan formula.

Lemma 2.4. *Let $\Omega \subset \mathbb{H}^n$ be an open set and $u \in C^\infty(\Omega)$. Let $E \subset \mathbb{H}^n$ be an open set with C^∞ boundary in Ω such that $\mu_E(\Omega) < \infty$. Then we have*

$$\int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu^s_E\, ds = \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2}\, d\mu_E, \tag{2.15}$$

where μ_E is the H-perimeter measure of E and ν_E is its horizontal normal.

Proof. The integral in the left hand side is well defined, because for a.e. $s \in \mathbb{R}$ there holds $\nabla u \neq 0$ on Σ^s. By the coarea formula for Riemannian manifolds, see e.g. [4], for any Borel function $h : \partial E \to [0,\infty]$ we have

$$\int_{\mathbb{R}} \int_{\partial E\cap\Sigma^s} h\, d\lambda_{\partial E\cap\Sigma^s}\, ds = \int_{\partial E} h\, |\nabla^{\partial E} u|_g\, d\sigma_{\partial E}, \tag{2.16}$$

where $\nabla^{\partial E} u$ is the tangential gradient of u on ∂E. Then we have

$$\nabla^{\partial E} u = \nabla u - \langle \nabla u, N_{\partial E}\rangle_g N_{\partial E} \quad \text{and} \quad |\nabla^{\partial E} u|_g = \sqrt{|\nabla u|_g^2 - \langle \nabla u, N_{\partial E}\rangle_g^2}. \tag{2.17}$$

Step 1. Let us define the set

$$C = \left\{ p \in \partial E \cap \Omega : \nabla u(p) \neq 0 \text{ and } N_{\partial E}(p) = \pm \frac{\nabla u(p)}{|\nabla u(p)|_g} \right\}.$$

If $s \in \mathbb{R}$ is such that $\nabla u \neq 0$ on Σ^s, then $C \cap \Sigma^s$ is a closed set in Σ^s. Using the coarea formula (2.16) with the function $h = \chi_C$, we get

$$\int_{\mathbb{R}} \lambda_{\partial E\cap\Sigma^s}(C)\, ds = \int_C |\nabla^{\partial E} u|_g\, d\sigma_{\partial E} = 0,$$

because we have $\nabla^{\partial E} u = 0$ on C. In particular, we deduce that

$$C \cap \Sigma^s \text{ is a closed set in } \Sigma^s \quad \text{and} \quad \lambda_{\partial E\cap\Sigma^s}(C \cap \Sigma^s) = 0 \quad \text{for a.e. } s \in \mathbb{R}. \tag{2.18}$$

If $p \in \Sigma^s$ is a point such that $\nabla u(p) \neq 0$ and $p \notin C$, then Σ^s is a smooth hypersurface in a neighbourhood of p and $E^s = E \cap \Sigma^s$ is a domain in Σ^s with smooth boundary in a neighbourhood of p. Moreover, we have $(\partial E\cap\Sigma^s)\setminus C=\partial E^s\setminus C$. Then, from (2.18) and from Lemma 2.3 we conclude that for a.e. $s\in\mathbb{R}$ we have

$$\mu_E^s=|N_{\partial E^s}^{H\Sigma^s}|_g\lambda_{\partial E^s}. \tag{2.19}$$

By (2.18) and (2.19), there holds

$$\mu_E^s(C\cap\Sigma^s)=\int_{C\cap\Sigma^s}|N_{\partial E^s}^{H\Sigma^s}|_g d\lambda_{\partial E^s}=0 \qquad \text{for a.e. } s\in\mathbb{R}. \tag{2.20}$$

Step 2. We prove (2.15) by plugging into (2.16) the Borel function $h:\partial E\to[0,\infty]$

$$h=\begin{cases}\dfrac{|N_{\partial E}^H|_g\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}}{|\nabla u|_g\sqrt{1-\langle N_{\partial E},\frac{\nabla u}{|\nabla u|_g}\rangle_g^2}} & \text{on } \partial E\setminus(C\cup\{\nabla u=0\})\\ 0 & \text{on } C\cup\{\nabla u=0\}.\end{cases}$$

Above, $N_{\partial E}^H$ is the projection of the Riemannian normal $N_{\partial E}$ onto H and ν_E is the horizontal normal. Namely, we have

$$N_{\partial E}^H=N_{\partial E}-\langle N_{\partial E},T\rangle_g T \quad\text{and}\quad \nu_E=\frac{N_{\partial E}^H}{|N_{\partial E}^H|_g}.$$

The H-perimeter measure of E is

$$\mu_E=|N_{\partial E}^H|_g\sigma_{\partial E}. \tag{2.21}$$

Using (2.17) and (2.21), we find

$$\begin{aligned}\int_{\partial E}h\,|\nabla^{\partial E}u|\,d\sigma_{\partial E}&=\int_{\partial E\setminus(C\cup\{\nabla u=0\})}|N_{\partial E}^H|_g\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\sigma_{\partial E}\\&=\int_{\partial E\setminus(C\cup\{\nabla u=0\})}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E\\&=\int_{\partial E}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E,\end{aligned} \tag{2.22}$$

where the last equality is justified by the fact that if $p\in C\cup\{\nabla u=0\}$ then

$$\sqrt{|\nabla_H u(p)|_g^2-\langle\nu_E(p),\nabla_H u(p)\rangle_g^2}=0.$$

For a.e. $s\in\mathbb{R}$, we have $\nabla u\neq 0$ on Σ^s. Using (2.21) and the fact that $h=0$ on $C\cup\{\nabla_H u=0\}$, letting $\Lambda^s=(\partial E\cap\Sigma^s)\setminus(C\cup\{\nabla_H u=0\})$ we obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\partial E\cap\Sigma^s}h\,d\lambda_{\partial E^s}\,ds&=\int_{\mathbb{R}}\int_{\Lambda^s}\frac{|N_{\partial E}^H|_g\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}}{|\nabla u|_g\sqrt{1-\langle N_{\partial E},\frac{\nabla u}{|\nabla u|_g}\rangle_g^2}}\,d\lambda_{\partial E^s}\,ds\\&=\int_{\mathbb{R}}\int_{\Lambda^s}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,\vartheta^s\,d\lambda_{\partial E^s}\,ds,\end{aligned} \tag{2.23}$$

where we let

$$\vartheta^s = \frac{\sqrt{|N^H_{\partial E}|_g^2 - \langle N^H_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\rangle_g^2}}{\sqrt{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g^2}}.$$

We will prove in *Step 3* that, for any $s \in \mathbb{R}$ such that $\nabla u \neq 0$ on Σ^s, there holds

$$\vartheta^s = |N^{H\Sigma^s}_{\partial E^s}|_g \quad \text{on } \Lambda^s. \tag{2.24}$$

Using (2.24), (2.19), and (2.20) formula (2.23) becomes

$$\begin{aligned} \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} h \, d\lambda_{\partial E \cap \Sigma^s} \, ds &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} |N^{H\Sigma^s}_{\partial E^s}|_g \, d\lambda_{\partial E^s} \, ds \\ &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu^s_E \, ds \\ &= \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu^s_E \, ds. \end{aligned} \tag{2.25}$$

The proof is complete, because (2.15) follows from (2.16), (2.22) and (2.25).

Step 3. We prove claim (2.24). Let us introduce the vector field W in $\Omega \backslash \{\nabla_H u = 0\}$

$$W = \frac{Tu}{|\nabla u|_g} \frac{\nabla_H u}{|\nabla_H u|_g} - \frac{|\nabla_H u|_g}{|\nabla u|_g} T.$$

It can be checked that $|W|_g = 1$ and $Wu = 0$. In particular, for a.e. s we have $W \in T\Sigma^s$. Moreover, W is g-orthogonal to $H\Sigma^s$ because any vector in $H\Sigma^s$ is orthogonal both to $\nabla_H u$ and to T. It follows that

$$N^{H\Sigma^s}_{\partial E^s} = N_{\partial E^s} - \langle N_{\partial E^s}, W\rangle_g$$

and, in particular,

$$|N^{H\Sigma^s}_{\partial E^s}|_g^2 = 1 - \langle N_{\partial E^s}, W\rangle_g^2.$$

Starting from the formula

$$N_{\partial E^s} = \frac{N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g \frac{\nabla u}{|\nabla u|_g}}{|N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g \frac{\nabla u}{|\nabla u|_g}|_g} = \frac{N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g \frac{\nabla u}{|\nabla u|_g}}{\sqrt{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g^2}},$$

we find

$$|N^{H\Sigma^s}_{\partial E^s}|_g^2 = \frac{M}{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g^2},$$

where we let

$$M = 1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g^2 - \left\langle N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g \frac{\nabla u}{|\nabla u|_g}, W \right\rangle_g^2.$$

We claim that on the open set $\{\nabla_H u \neq 0\}$ there holds

$$M = |N^H_{\partial E}|_g^2 - \langle N^H_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\rangle_g^2, \tag{2.26}$$

and formula (2.24) follows from (2.26). Using the identity $\nabla u = \nabla_H u + (Tu)T$ and the orthogonality

$$\left\langle N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g}\rangle_g \frac{\nabla u}{|\nabla u|_g}, \nabla u \right\rangle_g = 0,$$

we find

$$\begin{aligned} M &= 1 - \left\langle N_{\partial E}, \frac{\nabla_H u + (Tu)T}{|\nabla u|_g} \right\rangle_g^2 - \Big(\frac{Tu}{|\nabla u|_g} \langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\rangle_g - \frac{|\nabla_H u|_g}{|\nabla u|_g} \langle N_{\partial E}, T\rangle_g \Big)^2 \\ &= 1 - \langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\rangle_g^2 \frac{|\nabla_H u|_g^2 + (Tu)^2}{|\nabla u|_g^2} - \langle N_{\partial E}, T\rangle_g^2 \frac{|\nabla_H u|_g^2 + (Tu)^2}{|\nabla u|_g^2} \\ &= 1 - \langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\rangle_g^2 - \langle N_{\partial E}, T\rangle_g^2 \\ &= 1 - \langle N_{\partial E}, T\rangle_g^2 - \Big(\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\rangle_g - \Big\langle \langle N_{\partial E}, T\rangle_g T, \frac{\nabla_H u}{|\nabla_H u|_g} \Big\rangle_g \Big)^2 \\ &= |N_{\partial E}^H|_g^2 - \langle N_{\partial E}^H, \frac{\nabla_H u}{|\nabla_H u|_g}\rangle_g^2. \end{aligned} \tag{2.27}$$

This ends the proof. □

We prove a coarea inequality.

Proposition 2.5. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ a set with finite H-perimeter in Ω, and let $h : \partial E \to [0, \infty]$ be a Borel function. Then we have*

$$\int_{\mathbb{R}} \int_\Omega h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \le \int_\Omega h \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E. \tag{2.28}$$

Proof. The coarea inequality (2.28) follows from the smooth case of Lemma 2.4 by an approximation and lower semicontinuity argument.

Step 1. By [6, Theorem 2.2.2], there exists a sequence of smooth sets $(E_j)_{j\in\mathbb{N}}$ in Ω such that

$$\chi_{E_j} \overset{L^1(\Omega)}{\to} \chi_E \quad \text{as } j \to \infty \qquad \text{and} \qquad \lim_{j\to\infty} \mu_{E_j}(\Omega) = \mu(\Omega).$$

By a straightforward adaptation of the proof of [2, Proposition 3.13], we also have that $\nu_{E_j}\mu_{E_j} \to \nu_E \mu_E$ weakly* in Ω. Namely, for any $\psi \in C_c(\Omega; H)$ there holds

$$\lim_{j\to\infty} \int_\Omega \langle \psi, \nu_{E_j}\rangle_g \, d\mu_{E_j} = \int_\Omega \langle \psi, \nu_E\rangle_g \, d\mu_E.$$

Let $A \subset\subset \Omega$ be an open set such that $\lim_{j\to\infty} \mu_{E_j}(A) = \mu_E(A)$. By Reshetnyak's continuity theorem (see e.g. [2, Theorem 2.39]), we have

$$\lim_{j\to\infty} \int_A f(p, \nu_{E_j}(p)) \, d\mu_{E_j} = \int_A f(p, \nu_E(p)) \, d\mu_E$$

for any continuous and bounded function f. In particular,

$$\lim_{j\to\infty}\int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_{E_j}, \nabla_H u\rangle_g^2}\, d\mu_{E_j} = \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2}\, d\mu_E. \tag{2.29}$$

Step 2. Let $(E_j)_{j\in\mathbb{N}}$ be the sequence introduced in *Step 1.* Then, for a.e. $s \in \mathbb{R}$ we have

$$\nabla u \neq 0 \text{ on } \Sigma^s \qquad \text{and} \qquad \chi_{E_j} \to \chi_E \text{ in } L^1(\Sigma^s, \sigma_{\Sigma^s}) \text{ as } j \to \infty.$$

In particular, for any such s and for any open set $A \subset \Sigma^s \cap \Omega$ there holds

$$\mu_E^s(A) \le \liminf_{j\to\infty} \mu_{E_j}^s(A).$$

From Fatou's Lemma and from the continuity of $\frac{|\nabla_H u|_g}{|\nabla u|_g}$ on Σ^s, it follows that

$$\begin{aligned}
\int_A \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_E^s &= \int_0^\infty \mu_E^s\left(\left\{p \in A : \tfrac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t\right\}\right) dt \\
&\le \int_0^\infty \liminf_{j\to\infty} \mu_{E_j}^s\left(\left\{p \in A : \tfrac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t\right\}\right) dt \\
&\le \liminf_{j\to\infty} \int_0^\infty \mu_{E_j}^s\left(\left\{p \in A : \tfrac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t\right\}\right) dt \\
&= \liminf_{j\to\infty} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_{E_j}^s.
\end{aligned}$$

Using again Fatou's Lemma and Lemma 2.4,

$$\begin{aligned}
\int_{\mathbb{R}}\int_A \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_E^s\, ds &\le \int_{\mathbb{R}} \liminf_{j\to\infty} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_{E_j}^s\, ds \\
&\le \liminf_{j\to\infty} \int_{\mathbb{R}}\int_A \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_{E_j}^s\, ds \\
&= \liminf_{j\to\infty} \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_{E_j}, \nabla_H u\rangle_g^2}\, d\mu_{E_j}.
\end{aligned}$$

This, together with (2.29), gives

$$\int_{\mathbb{R}}\int_A \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_E^s\, ds \le \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2}\, d\mu_E.$$

Step 3. Any open set $A \subset \Omega$ can be approximated by a sequence $(A_k)_{k\in\mathbb{N}}$ of open sets such that

$$A_k \subset\subset \Omega, \quad A_k \subset A_{k+1}, \quad \bigcup_{k=1}^\infty A_k = A \quad \text{and} \quad \mu_E(\partial A_k) = 0.$$

In particular, for each $k \in \mathbb{N}$ we have

$$\begin{aligned}
\liminf_{j\to\infty} \mu_{E_j}(A_k) &\le \limsup_{j\to\infty} \mu_{E_j}(\overline{A}_k) \le \mu_E(\overline{A}_k) \\
&= \mu_E(A_k) \le \liminf_{j\to\infty} \mu_{E_j}(A_k).
\end{aligned}$$

Hence, the inequalities are equalities, i.e., $\mu_E(A_k) = \lim_{j\to\infty} \mu_{E_j}(A_k)$. By *Step 2*, for any $k \in \mathbb{N}$ there holds

$$\int_{\mathbb{R}} \int_{A_k} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \le \int_{A_k} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E.$$

By monotone convergence, letting $k \to \infty$ we obtain for any open set $A \subset \Omega$

$$\int_{\mathbb{R}} \int_{A} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \le \int_{A} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E.$$

By a standard approximation argument, it is enough to prove (2.28) for the characteristic function $h = \chi_B$ of a Borel set $B \subset \partial E$. Since the measure $\sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2}\, \mu_E$ is a Radon measure on ∂E, there exists a sequence of open sets B_j such that $B \subset B_j$ for any $j \in \mathbb{N}$ and

$$\lim_{j\to\infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E = \int_{B} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E.$$

Therefore, we have

$$\begin{aligned}
\int_{\mathbb{R}} \int_{B} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds &\le \liminf_{j\to\infty} \int_{\mathbb{R}} \int_{B_j} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \\
&\le \lim_{j\to\infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E \\
&= \int_{B} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E,
\end{aligned}$$

and this concludes the proof. □

In the next step, we prove an approximate coarea formula for sets E such that the boundary ∂E is an H-regular surface.

Lemma 2.6. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ an open set such that $\partial E \cap \Omega$ is an H-regular hypersurface, and $\bar p \in \partial E \cap \Omega$ a point such that*

$$\nabla_H u(\bar p) \neq 0 \quad \text{and} \quad \nu_E(\bar p) \neq \pm \frac{\nabla_H u(\bar p)}{|\nabla_H u(\bar p)|_g}.$$

Then, for any $\varepsilon > 0$ there exists $\bar r = \bar r(\bar p, \varepsilon) > 0$ such that $B_{\bar r}(\bar p) \subset \Omega$ and, for any $r \in (0, \bar r)$,

$$\begin{aligned}
(1-\varepsilon) &\int_{B_r(\bar p)} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E \\
&\le \int_{\mathbb{R}} \int_{B_r(\bar p)} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \\
&\le (1+\varepsilon) \int_{B_r(\bar p)} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2} \, d\mu_E.
\end{aligned}$$

Proof. We can without loss of generality assume that $\bar{p} = 0$ and $u(0) = 0$. We divide the proof into several steps.

Step 1: preliminary considerations. The horizontal vector field $V_{2n} = \frac{\nabla_H u}{|\nabla_H u|_g}$ is well defined in a neighbourhood $\Omega_\varepsilon \subset \mathbb{H}^n$ of 0. For any $s \in \mathbb{R}$, the hypersurface $\Sigma^s = \{p \in \Omega : u(p) = s\}$ is smooth in Ω_ε because $\nabla_H u \neq 0$ on Ω_ε.

There are horizontal vector fields $V_1, \dots, V_{2n-1}$ on Ω_ε such that $V_1, \dots, V_{2n}$ is a g-orthonormal frame. In particular, we have $V_j u = 0$ for all $j = 1, \dots, 2n-1$, i.e.,

$$H_p\Sigma^s = \operatorname{span}\{V_1(p), \dots, V_{2n-1}(p)\} \quad \text{for all } p \in \Sigma^s \cap \Omega_\varepsilon. \tag{2.30}$$

Possibly shirinking Ω_ε, reordering $\{V_j\}_{j=1,\dots,2n-1}$, and changing the sign of V_1, we can assume (see [23, Lemma 4.3 and Lemma 4.4]) that there exist a function $f : \Omega_\varepsilon \to \mathbb{R}$ and a number $\delta > 0$ such that:

a) $f \in C^1_H(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon \setminus \partial E)$;
b) $E \cap \Omega_\varepsilon = \{p \in \Omega_\varepsilon : f(p) > 0\}$;
c) $V_1 f \geq \delta > 0$ on Ω_ε.

By [23, Remark 4.7], we have also $\nu_E = \frac{\nabla_H f}{|\nabla_H f|_g}$ on $\partial E \cap \Omega_\epsilon$.

Step 2: change of coordinates. Let $S \subset \mathbb{H}^n$ be a $(2n-1)$-dimensional smooth submanifold such that:

i) $0 \in S$;
ii) $S \subset \Sigma^0 \cap \Omega_\varepsilon$; in particular, ∇u is g-orthogonal to S;
iii) $V_1(0)$ is g-orthogonal to S at 0;
iv) there exists a diffeomorphism $H : U \to \mathbb{H}^n$, where $U \subset \mathbb{R}^{2n-1}$ is an open set with $0 \in U$, such that $H(0) = 0$ and $H(U) = S \cap \Omega_\varepsilon$;
v) the area element JH of H satisfies $JH(0) = 1$. Namely, there holds

$$JH(0) = \lim_{r \to 0} \frac{\lambda_S(H(B^E_r))}{\mathscr{L}^{2n-1}(B^E_r)} = 1,$$

where $B^E_r = \{p \in \mathbb{R}^{2n-1} : |p| < r\}$ is a Euclidean ball and λ_S is the Riemannian $(2n-1)$-volume measure on S induced by g.

For small enough $a, b > 0$ and possibly shirinking U and Ω_ε, the mapping $G : (-a, a) \times (-b, b) \times U \to \mathbb{H}^n$

$$G(v, z, w) = \exp(vV_1) \exp\left(z \tfrac{\nabla u}{|\nabla u|^2_g}\right)(H(w))$$

is a diffeomorphism from $\widetilde{\Omega}_\varepsilon = (-a, a) \times (-b, b) \times U$ onto Ω_ε. The differential of G satisfies

$$dG\Big(\frac{\partial}{\partial v}\Big) = V_1 \quad \text{and} \quad dG(0)\Big(\frac{\partial}{\partial z}\Big) = \frac{\nabla u(0)}{|\nabla u(0)|^2_g}.$$

Moreover, the tangent space $T_0 S = \operatorname{Im} dH(0)$ is g-orthogonal to $V_1(0)$ and $\frac{\nabla u(0)}{|\nabla u(0)|^2_g}$. We denote by G_z the restriction of G to $(-a, a) \times \{z\} \times U$, i.e., $G_z(v, w) = G(v, z, w)$.

From the above considerations, we deduce that the area elements of G and of G_0 satisfy

$$JG(0) = \frac{1}{|\nabla u(0)|_g} \quad \text{and} \quad JG_0(0) = 1.$$

Then, possibly shrinking further $\widetilde{\Omega}_\varepsilon$, we have

$$(1-\varepsilon)JG(v,z,w) \le \frac{JG_z(v,w)}{\big|\nabla u \circ G(v,z,w)\big|_g} \le (1+\varepsilon)JG(v,z,w), \tag{2.31}$$

for all $(v,z,w) \in \widetilde{\Omega}_\varepsilon$.

For $j = 1,\dots,2n$, we define on $\widetilde{\Omega}_\varepsilon$ the vector fields $\widetilde{V}_j = (dG)^{-1}(V_j)$. By the definition of G, we have $\widetilde{V}_1 = \partial/\partial v$. We also define $\widetilde{u} = u \circ G \in C^\infty(\widetilde{\Omega}_\varepsilon)$, $\widetilde{f} = f \circ G : \widetilde{\Omega}_\varepsilon \to \mathbb{R}$, and $\widetilde{E} = G^{-1}(E)$. Then:

1) we have $\widetilde{E} = \{q \in \widetilde{\Omega}_\varepsilon : \widetilde{f}(q) > 0\}$;
2) we have $\widetilde{f} \in C^\infty(\widetilde{\Omega}_\varepsilon \setminus \partial\widetilde{E})$;
3) the derivative $\widetilde{V}_j\widetilde{f}$ is defined in the sense of distributions with respect to the measure $\mu = JG\mathscr{L}^{2n+1}$. Namely, for all $\psi \in C_c^\infty(\widetilde{\Omega}_\varepsilon)$ we have

$$\int_{\widetilde{\Omega}_\varepsilon} (\widetilde{V}_j\widetilde{f})\,\psi\,d\mu = -\int_{\widetilde{\Omega}_\varepsilon} \widetilde{f}\,\widetilde{V}_j^*\psi\,d\mu,$$

where $\widetilde{V}_j^*$ is the adjoint operator of $\widetilde{V}_j$ with respect to μ. Then we have $\widetilde{V}_j\widetilde{f} = (V_jf)\circ G$ and so $\widetilde{V}_j\widetilde{f}$ is a continuous function for any $j = 1,\dots,2n$. In particular, we have $\widetilde{V}_1\widetilde{f} = \partial_v\widetilde{f} \ge \delta > 0$.

Step 3: approximate coarea formula. We follow the argument of [23, Propositions 4.1 and 4.5], see also Remark 4.7 therein.

Possibly shrinking $\widetilde{\Omega}_\varepsilon$ and Ω_ε, there exists a continuous function $\phi : (-b,b)\times U \to (-a,a)$ such that:

A) $\partial\widetilde{E}\cap\widetilde{\Omega}_\varepsilon$ is the graph of ϕ. Namely, letting $\Phi : (-b,b)\times U \to \mathbb{R}^{2n+1}$, $\Phi(z,w) = (\phi(z,w),z,w)$, we have:

$$\partial\widetilde{E}\cap\widetilde{\Omega}_\varepsilon = \Phi((-b,b)\times U).$$

B) The measure μ_E is

$$\mu_E \llcorner \Omega_\varepsilon = (G\circ\Phi)_\#\left(\left(\frac{|\widetilde{V}\widetilde{f}|}{\widetilde{V}_1\tilde{f}}JG\right)\circ\Phi\ \mathscr{L}^{2n}\llcorner((-b,b)\times U)\right), \tag{2.32}$$

where $(G\circ\Phi)_\#$ denotes the push-forward and

$$|\widetilde{V}\tilde{f}| = \Big(\sum_{j=1}^{2n}\big(\widetilde{V}_j\widetilde{f}\big)^2\Big)^{1/2}.$$

Using $V_1 u = 0$ and $u \circ H = 0$ (this follows from $H(U) = S \cap \Omega_\varepsilon \subset \Sigma^0 \cap \Omega_\varepsilon$), we obtain

$$\begin{aligned} \widetilde{u}(v,z,w) &= u(G(v,z,w)) = u\Big(\exp(vV_1)\exp\big(z\tfrac{\nabla u}{|\nabla u|_g^2}\big)(H(w))\Big) \\ &= u\Big(\exp\big(z\tfrac{\nabla u}{|\nabla u|_g^2}\big)(H(w))\Big) \\ &= z + u(H(w)) = z. \end{aligned}$$

In particular, from $\widetilde{u} = u \circ G$ we deduce that

$$G^{-1}(\Sigma^s \cap \Omega_\varepsilon) = (-a,a) \times \{s\} \times U.$$

We denote by JG_s the Jacobian (area element) of G_s. We also define the restriction $\Phi_s : U \to \mathbb{R}^{2n+1}$, $\Phi_s(w) = \Phi(s,w)$, for any $s \in (-b,b)$.

By (2.30), for any $s \in \mathbb{R}$ the measure $\mu_E^s = \mu_{E\cap\Sigma_s}^{\Sigma^s}$ is the horizontal perimeter of $E \cap \Sigma^s$ with respect to the Carnot-Carathéodory structure induced by the family $V_1, \dots, V_{2n-1}$ on Σ^s. We can repeat the argument that lead to (2.32) to obtain

$$\mu_E^s \llcorner \Omega_\varepsilon = (G \circ \Phi_s)_\# \left(\left(\frac{|\widetilde{V}'\widetilde{f}|}{\widetilde{V}_1\widetilde{f}}\, JG_s \right) \circ \Phi_s \mathscr{L}^{2n-1} \llcorner U \right), \tag{2.33}$$

where $\widetilde{V}'\widetilde{f} = (\widetilde{V}_1\widetilde{f}, \dots, \widetilde{V}_{2n-1}\widetilde{f})$. We omit details of the proof of (2.33). The proof is a line-by-line repetition of Proposition 4.5 in [23] with the unique difference that now the horizontal perimeter is defined in a curved manifold.

Let us fix $\bar{r} > 0$ such that $B_{\bar{r}} \subset \Omega_\varepsilon$, and for any $r \in (0,\bar{r})$ let

$$\begin{aligned} A_{s,r} &= \big\{ w \in U : G(0,s,w) \in B_r \big\}, \\ A_r &= \big\{ (s,w) \in (-b,b) \times U : w \in A_{s,r} \big\}. \end{aligned}$$

By Fubini-Tonelli theorem and by (2.33), the function

$$s \mapsto \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_E^s = \int_{A_{s,r}} \left(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \right) \left(\frac{|\widetilde{V}'\widetilde{f}|}{\widetilde{V}_1\widetilde{f}}\, JG_s \right) \circ \Phi_s \, d\mathscr{L}^{2n-1} \tag{2.34}$$

is $\mathscr{L}^1$-measurable. Here and hereafter, the composition $\circ\Phi_s$ acts on the product. Thus, from Fubini-Tonelli theorem and (2.31) we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_E^s\, ds &= \int_{\mathbb{R}} \int_{A_{s,r}} \left(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \right) \left(\frac{|\widetilde{V}'\widetilde{f}|}{\widetilde{V}_1\widetilde{f}}\, JG_s \right) \circ \Phi_s(w)\, d\mathscr{L}^{2n-1}(w)\, ds \\ &= \int_{A_r} (|\nabla_H u|_g \circ G) \left(\frac{|\widetilde{V}'\widetilde{f}|}{\widetilde{V}_1\widetilde{f}}\, \frac{JG_s}{|\nabla u|_g \circ G} \right) \circ \Phi(s,w)\, d\mathscr{L}^{2n}(s,w) \\ &\le (1+\varepsilon) \int_{A_r} (|\nabla_H u|_g \circ G) \left(\frac{|\widetilde{V}\widetilde{f}|}{\widetilde{V}_1\widetilde{f}} \sqrt{1 - \tfrac{(\widetilde{V}_{2n}\widetilde{f})^2}{|\widetilde{V}\widetilde{f}|^2}}\, JG \right) \circ \Phi(s,w)\, d\mathscr{L}^{2n}(s,w). \end{aligned} \tag{2.35}$$

From the identity

$$\frac{\widetilde{V}_{2n}\widetilde{f}}{|\widetilde{V}\widetilde{f}|} = \frac{V_{2n}f}{|\nabla_H f|_g} \circ G = \left\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \frac{\nabla_H f}{|\nabla_H f|_g} \right\rangle_g \circ G = \left\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \nu_E \right\rangle_g \circ G, \tag{2.36}$$

and from (2.32) we deduce that

$$\begin{aligned}\int_{\mathbb{R}}\int_{B_r}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds &\le (1+\varepsilon)\int_{B_r}|\nabla_H u|_g\sqrt{1-\langle\tfrac{\nabla_H u}{|\nabla_H u|_g},\nu_E\rangle_g^2}\,d\mu_E\\ &= (1+\varepsilon)\int_{B_r}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.\end{aligned}\tag{2.37}$$

In a similar way, we obtain

$$\int_{\mathbb{R}}\int_{B_r}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds \ge (1-\varepsilon)\int_{B_r}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.$$

This concludes the proof. □

We can now prove the coarea formula for H-regular boundaries.

Proposition 2.7. *Let $\Omega\subset\mathbb{H}^n$ be an open set, $u\in C^\infty(\Omega)$, and $E\subset\mathbb{H}^n$ be an open domain such that $\partial E\cap\Omega$ is an H-regular hypersurface. Then*

$$\int_{\mathbb{R}}\int_{\Omega}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds = \int_{\Omega}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.\tag{2.38}$$

Proof. Let us define the set

$$A=\Big\{p\in\partial E\cap\Omega:\nabla_H u(p)\neq 0 \text{ and } \nu_E(p)\neq\pm\frac{\nabla_H u(p)}{|\nabla_H u(p)|_g}\Big\}.$$

The set A is relatively open in $\partial E\cap\Omega$. Let $\varepsilon>0$ be fixed. Since the measure μ_E is locally doubling on $\partial E\cap\Omega$ (see e.g. [23, Corollary 4.13]), by Lemma 2.6 and Vitali covering Theorem (see e.g. [11, Theorem 1.6]) there exists a countable (or finite) collection of balls $B_{r_i}(p_i)$, $i\in\mathbb{N}$, such that:

i) for any $i\in\mathbb{N}$ we have $p_i\in A$ and $0<r_i<\bar r(p_i,\varepsilon)$, where $\bar r$ is as in the statement of Lemma 2.6;
ii) the balls $B_{r_i}(p_i)$ are contained in A and pairwise disjoint;
iii) $\mu_E\big(A\setminus\bigcup_{i\in\mathbb{N}}B_{r_i}(p_i)\big)=0$.

It follows that we have:

$$\begin{aligned}\int_{\mathbb{R}}\int_{\bigcup_{i\in\mathbb{N}}B_{r_i}(p_i)}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds &\le (1+\varepsilon)\int_{\bigcup_{i\in\mathbb{N}}B_{r_i}(p_i)}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E\\ &=(1+\varepsilon)\int_A\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E\\ &=(1+\varepsilon)\int_\Omega\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.\end{aligned}\tag{2.39}$$

The last equality follows from the fact that $\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}=0$ outside A. In the same way one also obtains

$$\int_{\mathbb{R}}\int_{\bigcup_{i\in\mathbb{N}}B_{r_i}(p_i)}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds \ge (1-\varepsilon)\int_\Omega\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.\tag{2.40}$$

Moreover, by Proposition 2.5, there holds

$$\int_{\mathbb{R}}\int_{\Omega\setminus\bigcup_{i\in\mathbb{N}}B_{r_i}(p_i)}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds\le\int_{\Omega\setminus\bigcup_{i\in\mathbb{N}}B_{r_i}(p_i)}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E=0.$$

In particular, the integral on the left hand side of the last inequality is 0 and, by (2.39) and (2.40), we obtain

$$\begin{aligned}(1-\varepsilon)\int_\Omega\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E\\ &\le\int_{\mathbb{R}}\int_\Omega\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds\\ &\le(1+\varepsilon)\int_\Omega\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, this concludes the proof. □

By a standard approximation argument, we also have the following extension of the coarea formula (2.38).

Proposition 2.8. *Let $\Omega\subset\mathbb{H}^n$ be an open set, $u\in C^\infty(\Omega)$, and E be an open domain such that $\partial E\cap\Omega$ is an H-regular hypersurface. Then, for any Borel function $h:\partial E\to[0,\infty)$ there holds*

$$\int_{\mathbb{R}}\int_\Omega h\,\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds=\int_\Omega h\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.$$

Our next step is to prove the coarea formula for $\mathscr{S}^{2n+1}$-rectifiable sets.

Lemma 2.9. *Let $R\subset\mathbb{H}^n$ be an $\mathscr{S}^{2n+1}$-rectifiable set. Then, there exists a Borel $\mathscr{S}^{2n+1}$-rectifiable set $R'\subset\mathbb{H}^n$ such that $\mathscr{S}^{2n+1}(R\Delta R')=0$.*

Proof. By assumption, there exist a $\mathscr{S}^{2n+1}$-negligible set N and H-regular hypersurfaces $S_j\subset\mathbb{H}^n$, $j\in\mathbb{N}$, such that

$$R\subset N\cup\bigcup_{j=1}^\infty S_j.$$

It is proved in [7, 3] that (up to a localization argument), for any $j\in\mathbb{N}$, there exist an open set $U_j\subset\mathbb{R}^{2n}$, an omeomorphism $\Phi_j:U_j\to S_j$, and a continuous function $\rho_j:U_j\to[1,\infty)$ such that $\mathscr{S}^{2n+1}\llcorner S_j=\Phi_{j\#}(\rho_j\,\mathscr{L}^{2n}\llcorner U_j)$. Since the Lebesgue measure $\mathscr{L}^{2n}$ is a complete Borel measure, for any $j\in\mathbb{N}$ there exists a Borel set $T_j\subset U_j$ such that

$$\mathscr{L}^{2n}(T_j\Delta\Phi_j^{-1}(R\cap S_j))=0.$$

In particular, the Borel set

$$R'=\bigcup_{j=1}^\infty\Phi_j(T_j)$$

is $\mathscr{S}^{2n+1}$-equivalent to R. □

Proof of Theorem 1.5. Step 1. We prove (1.8) when R is an H-regular hypersurface. Then, R is locally the boundary of an open set $E \subset \mathbb{H}^n$ with H-regular boundary. Moreover, we have (locally) $\mu_E = \mathscr{S}^{2n+1} \llcorner R$ and $\nu_E = \nu_R$, up to the sign.

We define the measures $\mu_R^s = \mu_E^s$ for any s such that $\nabla u \neq 0$ on Σ^s. The measurability of the function in (1.7) follows from the argument (2.34). Formula (1.8) follows from Proposition 2.8.

Step 2. We prove (1.8) when R is an $\mathscr{S}^{2n+1}$-rectifiable Borel set. There exist a $\mathscr{S}^{2n+1}$-negligible set N and H-regular hypersurfaces $S_j \subset \mathbb{H}^n$, $j \in \mathbb{N}$ such that

$$R \subset N \cup \bigcup_{j=1}^{\infty} S_j.$$

Each S_j is (locally) the boundary of an open set E_j with H-regular boundary. We denote by $\mu_{E_j}^s$ the perimeter measure on $\partial E_j \cap \Sigma^s$ induced by E_j.

We define the pairwise disjoint Borel sets $R_j = (R \cap S_j) \setminus \cup_{h=1}^{j-1} S_h$ and we let

$$\mu_R^s = \sum_{j=1}^{\infty} \mu_{E_j}^s \llcorner R_j.$$

The definition is well posed for any s such that $\nabla u \neq 0$ on Σ^s. We have $\nu_R = \pm \nu_{E_j}$ $\mathscr{S}^{2n+1}$-a.e. on R_j and the sign of ν_R does not affect formula (1.8). From the *Step 1*, for each $j \in \mathbb{N}$ the function

$$s \mapsto \int_{R_j} h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_{E_j}^s$$

is $\mathscr{L}^1$-measurable; here, we were allowed to utilize *Step 1* because χ_{R_j} is Borel regular. Thus also the function

$$s \mapsto \int_{\Omega} h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_R^s = \sum_{j=1}^{\infty} \int_{R_j} h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_{E_j}^s$$

is measurable. Moreover, we have

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\Omega} h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_R^s \, ds &= \sum_{j=1}^{\infty} \int_{\mathbb{R}} \int_{R_j} h \, \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_{E_j}^s \, ds \\
&= \sum_{j=1}^{\infty} \int_{R_j} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} \, d\mathscr{S}^{2n+1} \\
&= \int_{R} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} \, d\mathscr{S}^{2n+1}.
\end{aligned}$$

Step 3. Finally, if R is $\mathscr{S}^{2n+1}$-rectifiable but not Borel, we set $\mu_R^s = \mu_{R'}^s$, where R' is a Borel set as in Lemma 2.9. Again, this definition is well posed for a.e. $s \in \mathbb{R}$. This concludes the proof. $\square$

2.3. **Proof of Theorem 1.6.** In this subsection we assume $n \geq 2$.

Lemma 2.10. *For $n \geq 2$, let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $R \subset \Omega$ an $\mathscr{S}^{2n+1}$-rectifiable set. Then*

$$\mathscr{S}^{2n+1}(\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\}) = 0.$$

Proof. It is enough to prove the lemma when R is an H-regular hypersurface. Let

$$A = \big\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\big\}.$$

We claim that $\mathscr{S}^{2n+1}(A) = 0$.

Let $p \in A$ be a fixed point and let $\nu_R(p)$ be the horizontal normal to R at p. Since $n \geq 2$, we have

$$\dim\{V(p) \in H_p : \langle V(p), \nu_R(p)\rangle_g = 0\} = 2n - 1 \geq n + 1.$$

Thus there exist left invariant horizontal vector fields V, W such that

$$\langle V(p), \nu_R(p)\rangle_g = \langle W(p), \nu_R(p)\rangle_g = 0 \quad \text{and} \quad [V, W] = T.$$

From $\nabla_H u(p) = 0$ and $\nabla u(p) \neq 0$ we deduce that $Tu(p) \neq 0$. It follows that

$$VWu(p) - WVu(p) = Tu(p) \neq 0,$$

and, in particular, we have either $VWu(p) \neq 0$ or $WVu(p) \neq 0$. Without loss of generality, we assume that $VWu(p) \neq 0$. Then the set $S = \{q \in \Omega : Wu(q) = 0\}$ is an H-regular hypersurface near the point $p \in S$. Since we have

$$\langle V(p), \nu_R(p)\rangle_g = 0 \quad \text{and} \quad \langle V(p), \nu_S(p)\rangle_g = \frac{VWu(p)}{|\nabla_H Wu(p)|_g} \neq 0,$$

we deduce that $\nu_R(p)$ and $\nu_S(p)$ are linearly independent. Then there exists $r > 0$ such that the set $R \cap S \cap B_r(p)$ is a 2-codimensional H-regular surface (see [8]). Therefore, by [8, Corollary 4.4], the Hausdorff dimension in the Carnot-Carathéodory metric of $A \cap B_r(p) \subset R \cap S \cap B_r(p)$ is not greater than $2n$. This is enough to conclude. □

Remark 2.11. Lemma 2.10 is not valid in the case $n = 1$. Consider the smooth surface $R = \{(x, y, t) \in \mathbb{H}^1 : x = 0\}$ and the function $u(x, y, t) = t - 2xy$. We have

$$\nabla u = -4xY + T \quad \text{and} \quad \nabla_H u = -4xY.$$

Then we have

$$\big\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\big\} = R$$

and $\mathscr{S}^3(R) = \infty$.

If $n \geq 2$ and Ω, u, and R are as in Lemma 2.10, then the function

$$|\nabla u|_g \sqrt{1 - \left\langle \nu_E, \tfrac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2}$$

is defined $\mathscr{S}^{2n+1}$-almost everywhere on R. We agree that its value is 0 when $|\nabla u|_g = 0$. Notice that, in this case, $\frac{\nabla_H u}{|\nabla_H u|_g}$ is not defined.

Proof of Theorem 1.6. Let $\varepsilon > 0$ be fixed. Then (1.9) can be obtained by plugging the function $\frac{|\nabla u|_g}{\varepsilon+|\nabla_H u|_g} h$ into (1.8), letting $\varepsilon \to 0$ and using the monotone convergence theorem. □

3. Height estimate

In this section, we prove Theorem 1.3. We discuss first a relative isoperimetric inequality on slices. Then we list some elementary properties of excess, and finally we proceed with the proof.

We assume throughout this section that $n \geq 2$.

3.1. **Relative isoperimetric inequalities.** For each $s \in \mathbb{R}$, we define the level sets of the height function

$$\mathbb{H}^n_s = \{p \in \mathbb{H}^n : \mathscr{h}(p) = s\}.$$

Let H^s be the g-orthogonal projection of H onto the tangent space of $\mathbb{H}^n_s$. Since the vector field X_1 is orthogonal to $\mathbb{H}^n_s$, while the vector fields $X_2, \dots, X_n, Y_1, \dots, Y_n$ are tangent to $\mathbb{H}^n_s$, then at any point $p \in \mathbb{H}^n_s$ we have

$$H^s_p = \operatorname{span}\{X_2(p), \dots, X_n(p), Y_1^s(p), Y_2(p), \dots, Y_n(p)\},$$

where $X_2, Y_2, \dots, X_n, Y_n$ are as in (1.2) and

$$Y_1^s = \frac{\partial}{\partial y_1} - 2s\frac{\partial}{\partial t}.$$

The natural volume in $\mathbb{H}^n_s$ is the Lebesgue measure $\mathscr{L}^{2n}$. For any measurable set $F \subset \mathbb{H}^n_s$ and any open set $\Omega \subset \mathbb{H}^n_s$, we define

$$\mu^s_F(\Omega) = \sup\Big\{ \int_F \operatorname{div}^s_g \varphi \, d\mathscr{L}^{2n} : \varphi \in C^1_c(\Omega; H^s), \ \|\varphi\|_g \leq 1\Big\},$$

where $\operatorname{div}^s_g \varphi = X_2\varphi_2 + \ldots + X_n\varphi_n + Y_1^s\varphi_{n+1} + \ldots + Y_n\varphi_{2n}$. If $\mu^s_F(\Omega) < \infty$ then μ^s_F is a Radon measure in Ω.

By Theorem 1.6, for any Borel function $h : \mathbb{H}^n \to [0,\infty)$ and any set E with locally finite H-perimeter in $\mathbb{H}^n$, we have the following coarea formula

$$\int_{\mathbb{R}} \int_{\mathbb{H}^n_s} h \, d\mu^s_{E^s} \, ds = \int_{\mathbb{H}^n} h \sqrt{1 - \langle \nu_E, X_1 \rangle_g^2} \, d\mu_E, \tag{3.41}$$

where $E^s = E \cap \mathbb{H}^n_s$ is the section of E with $\mathbb{H}^n_s$. Notice that $\nabla_H \mathscr{h} = X_1$.

In the proof of Theorem 1.3, we need a relative isoperimetric inequality in each slice $\mathbb{H}^n_s$ for $s \in (-1,1)$. These slices are cosets of $\mathbb{W} = \mathbb{H}^n_0$ and the isoperimetric inequalities in $\mathbb{H}^n_s$ can be reduced to an isoperimetric inequality in the central slice $\mathbb{W} = \mathbb{H}^n_0$ relative to a family of varying domains.

For any $s \in (-1,1)$ let $\Omega_s \subset \mathbb{W}$ be the set $\Omega_s = (-se_1) * D_1 * (se_1)$. This is the left translation by $-se_1$ of the section $C_1 \cap \mathbb{H}^n_s$. See the introduction for the definition of D_1 and C_1. With the coordinates $(y_1, \widehat{z}, t) \in \mathbb{W} = \mathbb{R} \times \mathbb{C}^{n-1} \times \mathbb{R}$, we have

$$\Omega_s = \{(y_1, \widehat{z}, t) \in \mathbb{W} : (y_1^2 + |\widehat{z}|^2)^2 + (t - 4sy_1)^2 < 1\}.$$

The sets $\Omega_s \subset \mathbb{W}$ are open and convex in the standard sense. The boundary $\partial\Omega_s$ is a $(2n-1)$-dimensional C^∞ embedded surface with the following property. There are $4n$ open convex sets $U_1, \dots, U_{4n} \subset \mathbb{W}$ such that $\partial\Omega_s \subset \bigcup_{i=1}^{4n} U_i$ and for each i the portion of boundary $\partial\Omega_s \cap U_i$ is a graph of the form $p_j = f_i^s(\widehat{p}_j)$ with $j = 2, \dots, 2n+1$ and $\widehat{p}_j = (p_2, \dots, p_{j-1}, p_{j+1}, \dots, p_{2n+1}) \in V_i$, where $V_i \subset \mathbb{R}^{2n-1}$ is an open convex set and $f_i^s \in C^\infty(V_i)$ is a function such that

$$|\nabla f_i^s(\widehat{p}_j) - \nabla f_i^s(\widehat{q}_j)| \leq K|\widehat{p}_j - \widehat{q}_j| \quad \text{for all } \widehat{p}_j, \widehat{q}_j \in V_i, \tag{3.42}$$

where $K > 0$ is a constant independent of $i = 1, \dots, 4n$ and independent of $s \in (-1, 1)$. In other words, the boundary $\partial\Omega_s$ is of class $C^{1,1}$ uniformly in $s \in (-1, 1)$.

By Theorem 3.2 in [19], the domain $\Omega_s \subset \mathbb{W}$ is a non-tangentially accessible (NTA) domain in the metric space $(\mathbb{W}, d_{CC})$ where d_{CC} is the Carnot-Carathéodory metric induced by the horizontal distribution H_p^0. In particular, Ω_s is a (weak) John domain in the sense of [10]. Namely, there exist a point $p_0 \in \Omega_s$, e.g. $p_0 = 0$, and a constant $C_J > 0$ such that for any point $p \in \Omega_s$ there exists a continuous curve $\gamma : [0, 1] \to \Omega_s$ such that $\gamma(1) = p_0$, $\gamma(0) = p$, and

$$\mathrm{dist}_{CC}(\gamma(\sigma), \partial\Omega_s) \geq C_J d_{CC}(\gamma(\sigma), p), \quad \sigma \in [0, 1]. \tag{3.43}$$

By Theorem 3.2 in [19], the John constant C_J depends only on the constant $K > 0$ in (3.42). This claim is not stated explicitly in Theorem 3.2 of [19] but it is evident from the proof. In particular, the John constant C_J is independent of $s \in (-1, 1)$. Then, by Theorem 1.22 in [9] we have the following result.

Theorem 3.1. *Let $n \geq 2$. There exists a constant $C(n) > 0$ such that for any $s \in (-1, 1)$ and for any measurable set $F \subset \mathbb{W}$ there holds*

$$\min\{\mathscr{L}^{2n}(F \cap \Omega_s), \mathscr{L}^{2n}(\Omega_s \setminus F)\}^{\frac{2n}{2n+1}} \leq C(n) \frac{\mathrm{diam}_{CC}(\Omega_s)}{\mathscr{L}^{2n}(\Omega_s)^{\frac{1}{2n+1}}} \mu_F^0(\Omega_s). \tag{3.44}$$

An alternative proof of Theorem 3.1 can be obtained using the Sobolev-Poincaré inequalities proved in [10] in the general setting of metric spaces.

The diameter $\mathrm{diam}_{CC}(\Omega_s)$ is bounded for $s \in (-1, 1)$ and $\mathscr{L}^{2n}(\Omega_s) > 0$ is a constant independent of s. Then we obtain the following version of (3.44).

Corollary 3.2. *Let $n \geq 2$. For any $\tau \in (0, 1)$ there exists a constant $C(n, \tau) > 0$ such that for $s \in (-1, 1)$ and for any measurable set $F \subset \mathbb{W}$ satisfying*

$$\mathscr{L}^{2n}(F \cap \Omega_s) \leq \tau\, \mathscr{L}^{2n}(\Omega_s)$$

there holds

$$\mu_F^0(\Omega_s) \geq C(n, \tau) \mathscr{L}^{2n}(F \cap \Omega_s)^{\frac{2n}{2n+1}}.$$

3.2. **Elementary properties of excess.** We list here, without proof, the most basic properties of the cylindrical excess introduced in Definition 1.2. Their proofs are easy adaptations of those for the classical excess, see e.g. [13, Chapter 22]. Note that, except for property 3), they hold also in the case $n = 1$.

1) For all $0 < r < s$ we have

$$\mathrm{Exc}(E, r, \nu) \le \left(\frac{s}{r}\right)^{2n+1} \mathrm{Exc}(E, s, \nu). \tag{3.45}$$

2) If $(E_j)_{j\in\mathbb{N}}$ is a sequence of sets with locally finite H-perimeter such that $E_j \to E$ as $j \to \infty$ in $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, then we have for any $r > 0$

$$\mathrm{Exc}(E, r, \nu) \le \liminf_{j\to\infty} \mathrm{Exc}(E_j, r, \nu). \tag{3.46}$$

3) Let $n \ge 2$. If $E \subset \mathbb{H}^n$ is a set such that $\mathrm{Exc}(E, r, \nu) = 0$ and $0 \in \partial^* E$, then

$$E \cap C_r = \big\{p \in C_r : \mathscr{h}(p) < 0\big\}. \tag{3.47}$$

In particular, we have $\nu_E = \nu$ in $C_r \cap \partial E$. See also [18, Proposition 3.6].

4) For any $\lambda > 0$ and $r > 0$ we have

$$\mathrm{Exc}(\lambda E, \lambda r, \nu) = \mathrm{Exc}(E, r, \nu), \tag{3.48}$$

where $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$.

3.3. **Proof of Theorem 1.3.** The following result is a first suboptimal version of Theorem 1.3.

Lemma 3.3. *Let $n \ge 2$. For any $s \in (0, 1)$, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$ with $\Lambda r \le 1$, there exists a constant $\omega(n, s, \Lambda, r) > 0$ such that if $E \subset \mathbb{H}^n$ is a (Λ, r)-minimum of H-perimeter in the cylinder C_2, $0 \in \partial E$, and $\mathrm{Exc}(E, 2, \nu) \le \omega(n, s, \Lambda, r)$, then*

$$\begin{aligned}
&|\mathscr{h}(p)| < s \text{ for any } p \in \partial E \cap C_1,\\
&\mathscr{L}^{2n+1}\big(\{p \in E \cap C_1 : \mathscr{h}(p) > s\}\big) = 0,\\
&\mathscr{L}^{2n+1}\big(\{p \in C_1 \setminus E : \mathscr{h}(p) < -s\}\big) = 0.
\end{aligned}$$

Proof. By contradiction, assume that there exist $s \in (0, 1)$ and a sequence of sets $(E_j)_{j\in\mathbb{N}}$ that are (Λ, r)-minima in C_2 and such such that

$$\lim_{j\to\infty} \mathrm{Exc}(E_j, 2, \nu) = 0$$

and at least one of the following facts holds:

$$\begin{aligned}
&\text{either} \quad \text{there exists } p \in \partial E_j \cap C_1 \text{ such that } s \le |\mathscr{h}(p)| \le 1, && (3.49)\\
&\text{or} \quad \mathscr{L}^{2n+1}\big(\{p \in E_j \cap C_1 : \mathscr{h}(p) > s\}\big) > 0, && (3.50)\\
&\text{or} \quad \mathscr{L}^{2n+1}\big(\{p \in C_1 \setminus E_j : \mathscr{h}(p) < -s\}\big) > 0. && (3.51)
\end{aligned}$$

By Theorem 4.3 in the Appendix A, there exists a measurable set $F \subset C_{5/3}$ such that F is a (Λ, r)-minimum in $C_{5/3}$, $0 \in \partial F$ and (possibly up to subsequences) $E_j \cap C_{5/3} \to F$ in $L^1(C_{5/3})$. By (3.46) and (3.45), we obtain

$$\mathrm{Exc}(F, 4/3, \nu) \le \liminf_{j\to\infty} \mathrm{Exc}(E_j, 4/3, \nu) \le \left(\tfrac{3}{2}\right)^{2n+1} \lim_{j\to\infty} \mathrm{Exc}(E_j, 2, \nu) = 0.$$

Since $0 \in \partial F$, by (3.47) the set $F \cap C_{4/3}$ is (equivalent to) a halfspace with horizontal inner normal $\nu = -X_1$, and, namely,

$$F \cap C_{4/3} = \{p \in C_{4/3} : \mathfrak{h}(p) < 0\}.$$

Assume that (3.49) holds for infinitely many j. Then, up to a subsequence, there are points $(p_j)_{j\in\mathbb{N}}$ and p_0 such that

$$p_j \in \partial E_j \cap C_1, \quad |\mathfrak{h}(p_j)| \in \ (s, 1] \quad \text{and} \quad p_j \to p_0 \in \partial F \cap \bar{C}_1.$$

We used again Theorem 4.3 in the Appendix A. This is a contradiction because $\partial F \cap \bar{C}_1 = \{p \in \bar{C}_1 : \mathfrak{h}(p) = 0\}$. Here, we used $n \geq 2$. Therefore, there exists $j_0 \in \mathbb{N}$ such that

$$\{p \in \partial E_j \cap C_1 : s \leq |\mathfrak{h}(p)| \leq 1\} = \emptyset \quad \text{for all } j \geq j_0,$$

and hence

$$\mu_{E_j}(C_1 \setminus \{p \in \mathbb{H}^n : |\mathfrak{h}(p)| \leq s\}) = 0.$$

This implies that, for $j \geq j_0$, χ_{E_j} is constant on the two connected components $C_1 \cap \{p : \mathfrak{h}(p) > s\}$ and $C_1 \cap \{p : \mathfrak{h}(p) < -s\}$. Since the sequence $(E_j)_{j\in\mathbb{N}}$ converges in $L^1(C_1)$ to the halfspace F, then for any $j \geq j_0$ we have

$$\begin{aligned} \chi_{E_j} &= 0 \quad \mathscr{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \mathfrak{h}(p) > s\}, \quad \text{and} \\ \chi_{E_j} &= 1 \quad \mathscr{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \mathfrak{h}(p) < -s\}. \end{aligned}$$

This contradicts both (3.50) and (3.51) and concludes the proof. □

Let $\pi : \mathbb{H}^n \to \mathbb{W}$ be the group projection defined, for any $p \in \mathbb{H}^n$, by the formula

$$p = \pi(p) * (\mathfrak{h}(p)\mathrm{e}_1).$$

For any set $E \subset \mathbb{H}^n$ and for any $s \in \mathbb{R}$, we let $E^s = E \cap \mathbb{H}^n_s$ and we define the projection

$$E_s = \pi(E^s) = \big\{w \in \mathbb{W} : w * (s\mathrm{e}_1) \in E\big\}.$$

Lemma 3.4. *Let $n \geq 2$, let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter and $0 \in \partial E$, and let $s_0 \in (0, 1)$ be such that*

$$|\mathfrak{h}(p)| < s_0 \text{ for any } p \in \partial E \cap C_1, \tag{3.52}$$

$$\mathscr{L}^{2n+1}\big(\{p \in E \cap C_1 : \mathfrak{h}(p) > s_0\}\big) = 0, \tag{3.53}$$

$$\mathscr{L}^{2n+1}\big(\{p \in C_1 \setminus E : \mathfrak{h}(p) < -s_0\}\big) = 0. \tag{3.54}$$

Then, for a.e. $s \in (-1, 1)$ and for any continuous function $\varphi \in C_c(D_1)$ we have, with $M = \partial^* E \cap C_1$ and $M_s = M \cap \{\mathfrak{h} > s\}$,

$$\int_{E_s \cap D_1} \varphi \, d\mathscr{L}^{2n} = -\int_{M_s} \varphi \circ \pi \, \langle \nu_E, X_1 \rangle_g \, d\mathscr{S}^{2n+1}. \tag{3.55}$$

In particular, for any Borel set $G \subset D_1$, we have

$$\mathscr{L}^{2n}(G) = -\int_{M\cap\pi^{-1}(G)} \langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1}, \tag{3.56}$$

$$\mathscr{L}^{2n}(G) \le \mathscr{S}^{2n+1}(M\cap \pi^{-1}(G)). \tag{3.57}$$

Proof. It is enough to prove (3.55). Indeed, taking $s < -s_0$ in (3.55) and recalling (3.52) and (3.54), we obtain

$$\int_{D_1} \varphi \, d\mathscr{L}^{2n} = -\int_M \varphi\circ\pi \,\langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1}. \tag{3.58}$$

Formula (3.56) follows from (3.58) by considering smooth approximations of χ_G. Formula (3.57) is immediate from (3.56) and $|\langle \nu_E, X_1\rangle_g| \le 1$.

We prove (3.55) for a.e. $s \in (-1,1)$ and, namely, for those s satisfying the property (3.61) below. Up to an approximation argument, we may assume that $\varphi \in C^1_c(D_1)$. Let $r\in(0,1)$ and $\sigma \in (\max\{s_0, s\}, 1)$ be fixed. We define

$$F = E\cap (D_r * (s,\sigma)) = E \cap \big\{ w*(\varrho \mathrm{e}_1)\in \mathbb{H}^n : w\in D_r,\ \varrho\in(s,\sigma)\big\}.$$

We claim that for a.e. $r\in(0,1)$ and any s satisfying (3.61) we have

$$\langle \nu_F, X_1\rangle_g \mu_F = \langle \nu_E, X_1\rangle_g \mathscr{S}^{2n+1} \llcorner \partial^* E\cap (D_r*(s,\sigma)) + \mathscr{L}^{2n}\llcorner E\cap D_r^s. \tag{3.59}$$

Above, we let $D_r^s = \{w*(s\mathrm{e}_1)\in\mathbb{H}^n : w\in D_r\}$. We postpone the proof of (3.59). Let Z be a horizontal vector field of the form $Z = (\varphi\circ\pi)X_1$. We have $\mathrm{div}_g Z = 0$ because $X_1(\varphi\circ\pi) = 0$. Hence, we obtain

$$0 = \int_F \mathrm{div}_g\, Z\, d\mathscr{L}^{2n+1} = -\int_{\mathbb{H}^n} \varphi\circ\pi\,\langle \nu_F, X_1\rangle_g d\mu_F,$$

i.e., by Fubini-Tonelli theorem and by (3.59),

$$-\int_{E_s\cap D_r} \varphi\, d\mathscr{L}^{2n} = -\int_{E\cap D_r^s} \varphi\circ\pi \, d\mathscr{L}^{2n} = \int_{\partial^*E\cap(D_r*(s,\sigma))} \varphi\circ\pi\,\langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1}.$$

Formula (3.55) follows on letting first $r\nearrow 1$ and then $\sigma \nearrow 1$.

We are left with the proof of (3.59). Let $\psi\in C^1_c(\mathbb{H}^n)$ be a test function. For any $w\in\mathbb{W}$ we let

$$E_w = \{\varrho\in\mathbb{R} : w*(\varrho\mathrm{e}_1)\in E\},\quad \psi_w(\varrho) = \psi(w*(\varrho\mathrm{e}_1)).$$

Then we have $\psi_w\in C^1_c(\mathbb{R})$ and, by Fubini-Tonelli theorem,

$$\begin{aligned} -\int_F X_1\psi\, d\mathscr{L}^{2n+1} &= -\int_{D_r}\int_s^\sigma \chi_E(w*(\varrho\mathrm{e}_1)) X_1\psi(w*(\varrho\mathrm{e}_1))\, d\varrho\, d\mathscr{L}^{2n}(w) \\ &= -\int_{D_r}\int_s^\sigma \chi_{E_w}(\varrho)\psi_w'(\varrho)\, d\varrho\, d\mathscr{L}^{2n}(w) \\ &= \int_{D_r}\left[\int_s^\sigma \psi_w\, dD\chi_{E_w} - \psi_w(\sigma)\chi_{E_w}(\sigma^-) + \psi_w(s)\chi_{E_w}(s^+)\right] d\mathscr{L}^{2n}(w), \end{aligned} \tag{3.60}$$

where $D\chi_{E_w}$ is the derivative of χ_{E_w} in the sense of distributions and $\chi_{E_w}(\sigma^-), \chi_{E_w}(s^+)$ are the classical trace values of χ_{E_w} at the endpoints of the interval (s,σ). We used the fact that the function χ_{E_w} is of bounded variation for $\mathscr{L}^{2n}$-a.e. $w \in \mathbb{W}$, which in turn is a consequence of the fact that $X_1\chi_E$ is a signed Radon measure. For any such w, the trace of χ_{E_w} satisfies

$$\chi_{E_w}(s^+) = \chi_{E_w}(s) = \chi_E(w * (s\mathrm{e}_1)) \quad \text{for a.e. } s,$$

so that, by Fubini's Theorem, for a.e. $s \in \mathbb{R}$ there holds

$$\chi_{E_w}(s^+) = \chi_E(w * (s\mathrm{e}_1)) \quad \text{for } \mathscr{L}^{2n}\text{-a.e. } w \in D_1. \tag{3.61}$$

With a similar argument, using (3.53) and the fact that $\sigma > s_0$ one can see that

$$\chi_{E_w}(\sigma^-) = \chi_E(w * (\sigma\mathrm{e}_1)) = 0 \quad \text{for } \mathscr{L}^{2n}\text{-a.e. } w \in D_1. \tag{3.62}$$

We refer the reader to [2] for an extensive account on BV functions and traces. By (3.60), (3.61) and (3.62) we obtain

$$\begin{aligned} -\int_F X_1\psi \, d\mathscr{L}^{2n+1} &= \int_{D_r}\int_s^\sigma \psi_w \, dD\chi_{E_w} \, d\mathscr{L}^{2n}(w) + \int_{D_r} \psi_w(s)\chi_{E_w}(s) \, d\mathscr{L}^{2n}(w) \\ &= \int_{D_r*(s,\sigma)} \psi \, \langle \nu_E, X_1\rangle_g d\mu_E + \int_{E\cap D_r^s} \psi \, d\mathscr{L}^{2n} \\ &= \int_{\partial^* E\cap(D_r*(s,\sigma))} \psi \, \langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1} + \int_{E\cap D_r^s} \psi \, d\mathscr{L}^{2n}, \end{aligned}$$

and (3.59) follows. □

Corollary 3.5. *Under the same assumptions and notation of Lemma 3.4, for a.e. $s \in (-1,1)$ there holds*

$$0 \le \mathscr{S}^{2n+1}(M_s) - \mathscr{L}^{2n}(E_s \cap D_1) \le \mathrm{Exc}(E,1,\nu). \tag{3.63}$$

Moreover, we have

$$\mathscr{S}^{2n+1}(M) - \mathscr{L}^{2n}(D_1) = \mathrm{Exc}(E,1,\nu). \tag{3.64}$$

Proof. On approximating χ_{D_1} with functions $\varphi \in C_c(D_1)$, by (3.55) we get

$$\mathscr{L}^{2n}(E_s \cap D_1) = -\int_{M_s} \langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1},$$

and the first inequality in (3.63) follows. The second inequality follows from

$$\begin{aligned} \mathscr{S}^{2n+1}(M_s) - \mathscr{L}^{2n}(E_s \cap D_1) &= \int_{M_s} \big(1 + \langle \nu_E, X_1\rangle_g\big) \, d\mathscr{S}^{2n+1} \\ &= \int_{M_s} \frac{|\nu_E - \nu|_g^2}{2} \, d\mathscr{S}^{2n+1} \\ &\le \mathrm{Exc}(E,1,\nu). \end{aligned} \tag{3.65}$$

Notice that $\nu = -X_1$. Finally, (3.64) follows on choosing a suitable $s < -s_0$ and recalling (3.52) and (3.54). In this case, the inequality in (3.65) becomes an equality and the proof is concluded. □

Proof of Theorem 1.3. Step 1. Up to replacing E with the rescaled set $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z,t) \in E\}$ with $\lambda = 1/2k^2 r$ and recalling (3.48), we can without loss of generality assume that E is a $(\Lambda', \frac{1}{2k^2})$-minimum of H-perimeter in C_2 with

$$\frac{\Lambda'}{2k^2} \le 1, \qquad 0 \in \partial E, \qquad \mathrm{Exc}(E, 2, \nu) \le \varepsilon_0(n). \tag{3.66}$$

Our goal is to find $\varepsilon_0(n)$ and $c_1(n) > 0$ such that, if (3.66) holds, then

$$\sup\left\{|\mathfrak{h}(p)| : p \in \partial E \cap C_{1/2k^2}\right\} \le c_1(n)\,\mathrm{Exc}(E,2,\nu)^{\frac{1}{2(2n+1)}}. \tag{3.67}$$

We require

$$\varepsilon_0(n) \le \omega\big(n, \tfrac{1}{4k}, 2k^2, \tfrac{1}{2k^2}\big), \tag{3.68}$$

where ω is given by Lemma 3.3. Two further assumptions on $\varepsilon_0(n)$ will be made later in (3.80) and (3.85). By (3.66), E is a $(2k^2, \frac{1}{2k^2})$-minimum in C_2. Letting $M = \partial E \cap C_1$, by Lemma 3.3 and (3.68) we have

$$|\mathfrak{h}(p)| < \tfrac{1}{4k} \text{ for any } p \in M, \tag{3.69}$$

$$\mathscr{L}^{2n+1}\big(\{p \in E \cap C_1 : \mathfrak{h}(p) > \tfrac{1}{4k}\}\big) = 0, \tag{3.70}$$

$$\mathscr{L}^{2n+1}\big(\{p \in C_1 \setminus E : \mathfrak{h}(p) < -\tfrac{1}{4k}\}\big) = 0. \tag{3.71}$$

By (3.64) and (3.45) we get

$$0 \le \mathscr{S}^{2n+1}(M) - \mathscr{L}^{2n}(D_1) \le \mathrm{Exc}(E,1,\nu) \le 2^{2n+1}\mathrm{Exc}(E,2,\nu). \tag{3.72}$$

Corollary 3.5 implies that, for a.e. $s \in (-1,1)$,

$$0 \le \mathscr{S}^{2n+1}(M_s) - \mathscr{L}^{2n}(E_s \cap D_1) \le \mathrm{Exc}(E,1,\nu) \le 2^{2n+1}\mathrm{Exc}(E,2,\nu) \tag{3.73}$$

where, as before, $M_s = M \cap \{\mathfrak{h} > s\}$.

Step 2. Consider the function $f : (-1,1) \to [0, \mathscr{S}^{2n+1}(M)]$ defined by

$$f(s) = \mathscr{S}^{2n+1}(M_s), \quad s \in (-1,1).$$

The function f is nonincreasing, right-continuous and, by (3.69), it satisfies

$$f(s) = \mathscr{S}^{2n+1}(M) \text{ for any } s \in (-1, -\tfrac{1}{4k}],$$
$$f(s) = 0 \text{ for any } s \in (\tfrac{1}{4k}, 1].$$

In particular, there exists $s_0 \in (-\frac{1}{4k}, \frac{1}{4k})$ such that

$$\begin{aligned} f(s) &\ge \mathscr{S}^{2n+1}(M)/2 \text{ for any } s < s_0,\\ f(s) &\le \mathscr{S}^{2n+1}(M)/2 \text{ for any } s \ge s_0. \end{aligned} \tag{3.74}$$

Let $s_1 \in (s_0, \frac{1}{4k})$ be such that

$$\begin{aligned} f(s) &\ge \sqrt{\mathrm{Exc}(E,2,\nu)} \text{ for any } s < s_1,\\ f(s) = \mathscr{S}^{2n+1}(M_s) &\le \sqrt{\mathrm{Exc}(E,2,\nu)} \text{ for any } s \ge s_1. \end{aligned} \tag{3.75}$$

We claim that there exists $c_2(n) > 0$ such that

$$\mathfrak{h}(p) \le s_1 + c_2(n)\mathrm{Exc}(E,2,\nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3.76}$$

Inequality (3.76) is trivial for any $p \in \partial E \cap C_{1/2k^2}$ with $\mathfrak{h}(p) \le s_1$. If $p \in \partial E \cap C_{1/2k^2}$ is such that $\mathfrak{h}(p) > s_1$, then

$$B_{\mathfrak{h}(p)-s_1}(p) \subset B_{1/2k}(p) \subset B_{1/k} \subset C_1.$$

We used the fact that $\|p\|_K \le \frac{1}{2k}$ whenever $p \in C_{1/2k^2}$, see (1.3). Therefore

$$B_{\mathfrak{h}(p)-s_1}(p) \subset C_1 \cap \{\mathfrak{h} > s_1\}$$

and, by the density estimate (4.91) of Theorem 4.1 in Appendix A,

$$\begin{aligned} k_3(n)(\mathfrak{h}(p) - s_1)^{2n+1} &\le \mu_E(B_{\mathfrak{h}(p)-s_1}(p)) \le \mu_E(C_1 \cap \{\mathfrak{h} > s_1\}) \\ &= \mathscr{S}^{2n+1}(M_{s_1}) = f(s_1) \le \sqrt{\mathrm{Exc}(E,2,\nu)}. \end{aligned}$$

This proves (3.76).

Step 3. We claim that there exists $c_3(n) > 0$ such that

$$s_1 - s_0 \le c_3(n)\mathrm{Exc}(E,2,\nu)^{\frac{1}{2(2n+1)}}. \tag{3.77}$$

By the coarea formula (3.41) with $h = \chi_{C_1}$, $D_1^s = \{p \in C_1 : \mathfrak{h}(p) = s\}$, and $E^s = \{p \in E : \mathfrak{h}(p) = s\}$, we have

$$\int_{-1}^{1} \int_{D_1^s} d\mu_{E^s}^s \, ds = \int_{C_1} \sqrt{1 - \langle \nu_E, X_1 \rangle_g^2}\, d\mu_E \le \sqrt{2} \int_M \sqrt{1 + \langle \nu_E, X_1 \rangle_g}\, d\mathscr{S}^{2n+1}.$$

By Hölder inequality, (4.91), (3.56), and (3.72), we deduce that

$$\begin{aligned} \int_{-1}^{1} \int_{D_1^s} d\mu_{E^s}^s \, ds \le& \sqrt{2\mathscr{S}^{2n+1}(M)} \left(\int_M (1 + \langle \nu_E, X_1 \rangle_g)\, d\mathscr{S}^{2n+1} \right)^{1/2} \\ \le& c_4(n)(\mathscr{S}^{2n+1}(M) - \mathscr{L}^{2n}(D_1))^{1/2} \\ \le& c_5(n)\sqrt{\mathrm{Exc}(E,2,\nu)}. \end{aligned} \tag{3.78}$$

By Corollary 3.5 and (3.72), we obtain, for a.e. $s \in [s_0, s_1)$,

$$\begin{aligned} \mathscr{L}^{2n}(E_s \cap D_1) &\le \mathscr{S}^{2n+1}(M_s) = f(s) \le f(s_0) \\ &\le \frac{\mathscr{S}^{2n+1}(M)}{2} \le \frac{\mathscr{L}^{2n}(D_1) + 2^{2n+1}\mathrm{Exc}(E,2,\nu)}{2} \le \frac{3}{4}\mathscr{L}^{2n}(D_1). \end{aligned} \tag{3.79}$$

The last inequality holds provided that

$$2^{2n+1}\varepsilon_0(n) \le \frac{\mathscr{L}^{2n}(D_1)}{4}. \tag{3.80}$$

Let $\Omega_s = (-se_1) * D_1^s = (-se_1) * D_1 * (se_1)$ and $F_s = (-se_1) * E^s$. We have

$$\mathscr{L}^{2n}(\Omega_s) = \mathscr{L}^{2n}(D_1^s) = \mathscr{L}^{2n}(D_1), \tag{3.81}$$

and, by (3.79),

$$\mathscr{L}^{2n}(F_s \cap \Omega_s) = \mathscr{L}^{2n}(E^s \cap D_1^s) = \mathscr{L}^{2n}(E_s \cap D_1) \le \frac{3}{4}\mathscr{L}^{2n}(D_1). \tag{3.82}$$

Moreover, by left invariance we also have

$$\mu_{E^s}^s(D_1^s) = \mu_{F_s}^0(\Omega_s). \tag{3.83}$$

By (3.81)–(3.83) and Corollary 3.2, there exists a constant $k(n) > 0$ independent of $s \in (-1, 1)$ such that

$$\mu_{E^s}(D_1^s) = \mu^0_{F_s}(\Omega_s) \geq k(n)\, \mathscr{L}^{2n}(F_s \cap \Omega_s)^{\frac{2n}{2n+1}} = k(n)\, \mathscr{L}^{2n}(E^s \cap D_1^s)^{\frac{2n}{2n+1}}. \tag{3.84}$$

This, together with (3.78), gives

$$\begin{aligned} c_6(n)\sqrt{\mathrm{Exc}(E, 2, \nu)} \; &\geq \int_{s_0}^{s_1} \mathscr{L}^{2n}(E^s \cap D_1^s)^{\frac{2n}{2n+1}}\, ds \\ &\overset{(3.73)}{\geq} \int_{s_0}^{s_1} \Big(\mathscr{S}^{2n+1}(M_s) - 2^{2n+1}\mathrm{Exc}(E, 2, \nu))\Big)^{\frac{2n}{2n+1}} ds \\ &\overset{(3.75)}{\geq} \int_{s_0}^{s_1} \Big(\sqrt{\mathrm{Exc}(E, 2, \nu)} - 2^{2n+1}\mathrm{Exc}(E, 2, \nu))\Big)^{\frac{2n}{2n+1}} \; ds \\ &\geq \frac{1}{2}\int_{s_0}^{s_1} \mathrm{Exc}(E, 2, \nu)^{\frac{n}{2n+1}}\, ds. \end{aligned}$$

In the last inequality, we require that $\varepsilon_0(n)$ satisfies

$$\sqrt{z} - 2^{2n+1} z \geq \tfrac{1}{2}\sqrt{z} \quad \text{for all } z \in [0, \varepsilon_0(n)]. \tag{3.85}$$

It follows that

$$c_6(n)\sqrt{\mathrm{Exc}(E, 2, \nu)} \geq \frac{1}{2}\, \mathrm{Exc}(E, 2, \nu)^{\frac{n}{2n+1}}(s_1 - s_0),$$

and (3.77) follows.

Step 4. Recalling (3.76) and (3.77), we proved that there exist $\varepsilon_0(n)$ and $c_6(n)$ such that the following holds. If E is a $(2k^2, \frac{1}{2k^2})$-minimum of H-perimeter in C_2 such that

$$0 \in \partial E, \qquad \mathrm{Exc}(E, 2, \nu) \leq \varepsilon_0(n)$$

and $s_0 = s_0(E)$ satisfies (3.74), then

$$\mathscr{h}(p) - s_0 \leq c_7(n)\mathrm{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3.86}$$

Let us introduce the mapping $\Psi : \mathbb{H}^n \to \mathbb{H}^n$

$$\Psi(x_1, x_2 \ldots, x_n, y_1, \ldots, y_n, t) = (-x_1, -x_2, \ldots, -x_n, y_1, \ldots, y_n, -t)\,.$$

Then we have $\Psi^{-1} = \Psi$, $\Psi(C_2) = C_2$, $\langle X_j, \nu_{\Psi(F)}\rangle_g = -\langle X_j, \nu_F\rangle_g \circ \Psi$, $\langle Y_j, \nu_{\Psi(F)}\rangle_g = \langle Y_j, \nu_F\rangle_g \circ \Psi$, and $\mu_{\Psi(F)} = \Psi_\# \mu_F$, for any set F with locally finite H-perimeter; here, $\Psi_\#$ denotes the standard push-forward of measures. Therefore, the set $\widetilde{E} = \Psi(\mathbb{H}^n \setminus E)$ satisfies the following properties:

i) $\widetilde{E}$ is a $(2k^2, \frac{1}{2k^2})$-minimum of H-perimeter in C_2;

ii) $0 \in \partial\widetilde{E}$ and

$$\mathrm{Exc}(\widetilde{E}, 2, \nu) = \frac{1}{2^Q}\int_{\partial^*\widetilde{E} \cap C_2} |\nu_{\widetilde{E}} - \nu|_g^2\, d\mathscr{S}^{2n+1} = \mathrm{Exc}(E, 2, \nu) \leq \varepsilon_0(n);$$

iii) setting $\widetilde{M} = \partial^* \widetilde{E} \cap C_1 = \Psi(M)$ and $\widetilde{f}(s) = \mathscr{S}^{2n+1}(\widetilde{M} \cap \{\hbar > s\})$, we have

$$\widetilde{f}(s) \geq \mathscr{S}^{2n+1}(\widetilde{M})/2 = \mathscr{S}^{2n+1}(M)/2 \text{ for any } s < -s_0,$$

$$\widetilde{f}(s) \leq \mathscr{S}^{2n+1}(M)/2 \text{ for any } s \geq -s_0.$$

Formula (3.86) for the set $\widetilde{E}$ gives

$$\hbar(p) + s_0 \leq c_7(n)\mathrm{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial \widetilde{E} \cap C_{1/2k^2}.$$

Notice that we have $p \in \partial \widetilde{E}$ if and only if $\Psi(p) \in \partial E$ and, moreover, $\hbar(\Psi(p)) = -\hbar(p)$. Hence, we have

$$-\hbar(p) + s_0 \leq c_7(n)\mathrm{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3.87}$$

By (3.86) and (3.87) we obtain

$$|\hbar(p) - s_0| \leq c_7(n)\mathrm{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}, \tag{3.88}$$

and, in particular,

$$|s_0| \leq c_7(n)\mathrm{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}}, \tag{3.89}$$

because $0 \in \partial E \cap C_{1/2k^2}$. Inequalities (3.88) and (3.89) give (3.67). This completes the proof. □

4. Appendix A

We list some basic properties of Λ-minima of H-perimeter in $\mathbb{H}^n$. The proofs are straightforward adaptations of the proofs for Λ-minima of perimeter in $\mathbb{R}^n$.

Theorem 4.1 (Density estimates)**.** *There exist constants $k_1(n), k_2(n), k_3(n), k_4(n) > 0$ with the following property. If E is a (Λ, r)-minimum of H-perimeter in $\Omega \subset \mathbb{H}^n$, $p \in \partial E \cap \Omega$, $B_r(p) \subset \Omega$ and $s < r$, then*

$$k_1(n) \leq \frac{\mathscr{L}^{2n+1}(E \cap B_s(p))}{s^{2n+2}} \leq k_2(n) \tag{4.90}$$

$$k_3(n) \leq \frac{\mu_E(B_s(p))}{s^{2n+1}} \leq k_4(n). \tag{4.91}$$

For a proof see [13, Theorem 21.11]. By standard arguments Theorem 4.1 implies the following corollary.

Corollary 4.2. *If E is a (Λ, r)-minimum of H-perimeter in $\Omega \subset \mathbb{H}^n$, then*

$$\mathscr{S}^{2n+1}((\partial E \setminus \partial^* E) \cap \Omega) = 0.$$

Theorem 4.3. *Let $(E_j)_{j \in \mathbb{N}}$ be a sequence of (Λ, r)-minima of H-perimeter in an open set $\Omega \subset \mathbb{H}^n$, $\Lambda r \leq 1$. Then there exists a (Λ, r)-minimum E of H-perimeter in Ω and a subsequence $(E_{j_k})_{k \in \mathbb{N}}$ such that*

$$E_{j_k} \to E \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \nu_{E_{j_k}} \mu_{E_{j_k}} \overset{*}{\rightharpoonup} \nu_E \mu_E$$

as $k \to \infty$. Moreover, the measure theoretic boundaries ∂E_{j_k} converge to ∂E in the sense of Kuratowski, i.e.,

i) *if $p_{j_k} \in \partial E_j \cap \Omega$ and $p_{j_k} \to p \in \Omega$, then $p \in \partial E$;*
ii) *if $p \in \partial E \cap \Omega$, then there exists a sequence $(p_{j_k})_{k\in\mathbb{N}}$ such that $p_{j_k} \in \partial E_{j_k} \cap \Omega$ and $p_{j_k} \to p$.*

For a proof in the case of the perimeter in $\mathbb{R}^n$, see [13, Chapter 21].

5. Appendix B

We define a Borel unit normal ν_R to an $\mathscr{S}^{2n+1}$-rectifiable set $R \subset \mathbb{H}^n$ and we show that the definition is well posed $\mathscr{S}^{2n+1}$-almost everywhere, up to the sign. The normal ν_S to an H-regular hypersurface $S \subset \mathbb{H}^n$ is defined in (1.6).

Definition 5.1. Let $R \subset \mathbb{H}^n$ be an $\mathscr{S}^{2n+1}$-rectifiable set such that

$$\mathscr{S}^{2n+1}\Big(R \setminus \bigcup_{j\in\mathbb{N}} S_j\Big) = 0 \tag{5.92}$$

for a sequence of H-regular hypersurfaces $(S_j)_{j\in\mathbb{N}}$ in $\mathbb{H}^n$. For any $p \in R \cap \bigcup_{j\in\mathbb{N}} S_j$ we define

$$\nu_R(p) = \nu_{S_{\bar{j}}}(p),$$

where $\bar{j}$ is the unique integer such that $p \in S_{\bar{j}} \setminus \bigcup_{j<\bar{j}} S_j$.

We show that Definition 5.1 is well posed, up to a sign, for $\mathscr{S}^{2n+1}$-a.e. p. Namely, let $(S^1_j)_{j\in\mathbb{N}}$ and $(S^2_j)_{j\in\mathbb{N}}$ be two sequences of H-regular hypersurfaces in $\mathbb{H}^n$ for which (5.92) holds and denote by ν^1_R and ν^2_R, respectively, the associated normals to R according to Definition 5.1. We show that $\nu^1_R = \nu^2_R$ $\mathscr{S}^{2n+1}$-a.e. on R, up the the sign.

Let $A \subset R$ be the set of points such that either $\nu^1_R(p)$ is not defined, or $\nu^2_R(p)$ is not defined, or they are both defined and $\nu^1_R(p) \neq \pm\nu^2_R(p)$. It is enough to show that $\mathscr{S}^{2n+1}(A) = 0$. This is a consequence of the following lemma.

Lemma 5.2. *Let S_1, S_2 be two H-regular hypersurfaces in $\mathbb{H}^n$ and let*

$$A = \{p \in S_1 \cap S_2 : \nu_{S_1}(p) \neq \pm\nu_{S_2}(p)\}.$$

Then, the Hausdorff dimension of A in the Carnot-Carathéodory metric is at most $2n$, $\dim_{CC}(A) \leq 2n$, and, in particular, $\mathscr{S}^{2n+1}(A) = 0$.

Proof. The blow-up of S_i, $i = 1, 2$, at a point $p \in A$ is a vertical hyperplane $\Pi_i \times \mathbb{R} \subset \mathbb{R}^{2n} \times \mathbb{R} \equiv \mathbb{H}^n$, see e.g. [7], where:

i) by blow-up of S_i at p we mean the limit

$$\lim_{\lambda\to\infty} \lambda(p^{-1} * S_i)$$

in the Gromov-Hausdorff sense. Recall that, for $E \subset \mathbb{H}^n$, we define $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$).
ii) For $i = 1, 2$, $\Pi_i \subset \mathbb{R}^{2n}$ is the normal hyperplane to $\nu_{S_i}(p) \in H_p \equiv \mathbb{R}^{2n}$.

It follows that the blow-up of A at p is contained in the blow-up of $S_1 \cap S_2$ at p, i.e., in $(\Pi_1 \cap \Pi_2) \times \mathbb{R}$. Since $\nu_{S_1}(p) \neq \pm \nu_{S_2}(p)$, $\Pi_1 \cap \Pi_2$ is a $(2n-2)$-dimensional plane in $\mathbb{R}^{2n}$, and we conclude thanks to the following lemma. $\square$

Lemma 5.3. *Let $k = 0, 1, \dots, 2n$ and $A \subset \mathbb{H}^n$ be such that for any $p \in A$, the blow-up of A at p is contained in $\Pi_p \times \mathbb{R}$ for some plane $\Pi_p \subset \mathbb{R}^{2n}$ of dimension k. Then we have $\dim_{CC}(A) \leq k + 2$.*

Proof. We claim that for any $\eta > 0$ we have

$$\mathscr{S}^{k+2+\eta}(A) = 0. \tag{5.93}$$

Let $\varepsilon \in (0, 1/2)$ be such that $C\varepsilon^\eta \leq 1/2$, where $C = C(n)$ is a constant that will be fixed later in the proof. By the definition of blow-up, for any $p \in A$ there exists $r_p > 0$ such that for all $r \in (0, r_p)$ we have

$$(p^{-1} * A) \cap U_r \subset (\Pi_p)_{\varepsilon r} \times \mathbb{R},$$

where $(\Pi_p)_{\varepsilon r}$ denotes the (εr)-neighbourhood of Π_p in $\mathbb{R}^{2n}$. For any $j \in \mathbb{N}$ set

$$A_j = \{p \in A \cap B_j : r_p > 1/j\}.$$

To prove (5.93), it is enough to prove that

$$\mathscr{S}^{k+2+\eta}(A_j) = 0$$

for any fixed $j \geq 1$. This, in turn, will follow if we show that, for any fixed $\delta \in (0, \frac{1}{2j})$, one has

$$\begin{aligned} \inf \Big\{ \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i),\ r_i < 2\varepsilon\delta \Big\} \leq \\ \leq \frac{1}{2} \inf \Big\{ \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i),\ r_i < \delta \Big\}. \end{aligned} \tag{5.94}$$

Let $(U_{r_i}(p_i))_{i \in \mathbb{N}}$ be a covering of A_j with balls of radius smaller than δ. There exist points $\bar{p}_i \in A_j$ such that $(U_{2r_i}(\bar{p}_i))_{i \in \mathbb{N}}$ is a covering of A_j with balls of radius smaller than $2\delta < 1/j$. By definition of A_j, we have

$$(\bar{p}_i^{-1} * A_j) \cap U_{2r_i} \subset ((\Pi_{\bar{p}_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}.$$

The set $((\Pi_{\bar{p}_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}$ can be covered by a family of balls $(U_{\varepsilon r_i}(p_h^i))_{h \in H_i}$ of radius $\varepsilon r_i < 2\varepsilon\delta$ in such a way that the cardinality of H_i is bounded by $C\varepsilon^{-k-2}$, where the constant C depends only on n and not on ε. In particular, the family of balls $(U_{\varepsilon r_i}(\bar{p}_i * p_h^i))_{i \in \mathbb{N}, h \in H_i}$ is a covering of A_j and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \sum_{h \in H_i} (\text{radius}\, U_{\varepsilon r_i}(\bar{p}_i * p_h^i))^{k+2+\eta} &= \sum_{i \in \mathbb{N}} \sum_{h \in H_i} (\varepsilon r_i)^{k+2+\eta} \leq C\varepsilon^{-k-2} \sum_{i \in \mathbb{N}} (\varepsilon r_i)^{k+2+\eta} \\ &= C\varepsilon^\eta \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} \leq \frac{1}{2} \sum_i r_i^{k+2+\eta}. \end{aligned}$$

This proves (5.94) and concludes the proof. $\square$

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