A NEW DIFFERENTIATION, SHAPE OF THE UNIT BALL AND PERIMETER MEASURE

VALENTINO MAGNANI

Abstract. We present a new blow-up method that allows for establishing the first general formula to compute the perimeter measure with respect to the spherical Hausdorff measure in noncommutative nilpotent groups. This result leads us to an unexpected relationship between the area formula with respect to a distance and the profile of its corresponding unit ball.

Contents

1. Introduction

In the last decade, the study of sub-Riemannian Geometry, in short SR Geometry, has known a strong impulse in different areas, from PDE and Control Theory to Differential Geometry and Geometric Measure Theory. In particular, a number of Riemannian problems has a sub-Riemannian interpretation in a large framework and this may lead to either foundational questions or to new viewpoints.

The challenging project of developing Geometric Measure Theory on SR manifolds has shown that both of these aspects can happen. With this aim in mind, finding a theory of area in SR Geometry represents the starting point of a demanding program.

Historically, since the seminal works by Carathéodory [12] and Hausdorff [32], many theories grew to study k-dimensional Lebesgue area, smoothness conditions for area

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and coarea formulae, extensions to Finsler spaces, etc. We mention only a few relevant references [1], [2], [3], [9], [10], [13], [15], [16], [18], [19], [20], [21], [22], [29], [31], [34], [41], [43], [46], [47], [52], to give a very small glimpse of the much wider literature in both old and new research lines.

In the modern view, the Riemannian surface area can be computed by Euclidean tools, since smooth subsets have Lipschitz parametrizations, the Rademacher theorem holds and change of variables formulae perfectly fit with the standard density given by the Riemannian metric.

The previous techniques fail completely in the SR case and this is due to two basic difficulties. First of all, we may not have Lipschitz parametrizations, even for smooth subsets of a sub-Riemannian manifold. This forces the use of abstract differentiation theorems for measures, although the second obstacle comes up exactly at this stage. In fact, the Besicovitch covering theorem, shortly B.C.T., fails to hold precisely for two important distances, such as the sub-Riemannian distance and the Cygan-Korányi distance in the Heisenberg group, [36], [48], [51]. Although there exist some special distances such that the B.C.T. holds, [37], a complete development of Geometric Measure Theory in the SR framework has to include all homogeneous distances, or at least the most important ones. To keep this generality, we do not have any general theorem to differentiate an arbitrary Radon measure.

We will show how to overcome these difficulties for hypersurfaces and for finite perimeter sets in special classes of nilpotent Lie groups. Our ambient space is the *homogeneous* stratified group, corresponding to a stratified Lie group equipped with a fixed homogeneous distance, see Section 2. Here the natural problem is to compute the perimeter measure by the Hausdorff measure constructed with the homogeneous distance of the group.

Homogeneous stratified groups are Ahlfors regular and satisfy a Poincaré inequality, hence their metric perimeter measure coincides with the variational perimeter [44], and the general results of [4] give the following formula

$$|\partial_H E| = \beta\, \mathcal{S}_0^{Q-1} \llcorner \mathcal{F}_H E\,, \tag{1}$$

where β is measurable, Q is the Hausdorff dimension of the group, $E \subset \mathbb{G}$ is an h-finite perimeter set, $\mathcal{F}_H E$ is the reduced boundary and $|\partial_H E|$ is the variational perimeter measure on groups, see Section 4 for more details. The $(Q-1)$-dimensional spherical Hausdorff measure with no geometric constant $\mathcal{S}_0^{Q-1}$ is introduced in Definition 3.1.

A variational notion of perimeter measure on SR manifolds has been introduced in [7], where the divergence operator is only defined by the volume measure. This implies that the notion of perimeter measure in stratified groups can be introduced by the standard divergence, see (26).

Finding a geometric expression for β is obviously the crucial question. When the group is the Euclidean space, the classical De Giorgi's theory [15] proves that β is the volume of the unit ball in $\mathbb{R}^{n-1}$. Here we remark the crucial role of the classical area formula, joined with the rectifiability of the reduced boundary. Extensions to the case of Finsler spaces have been also established, [9].

When we consider a noncommutative stratified group, there is a drastic change of the problem, where the classical area formula does not apply. In fact, according to the examples of [35], the reduced boundary $\mathcal{F}_H E$ in general may not be rectifiable in the sense of 3.2.14 of [21], so all of the known methods fail. At present there are no results to find β when the spherical Hausdorff measure is replaced by the Hausdorff measure. On the other hand, some integral representations for Borel measures with respect to the spherical Hausdorff measure can be written.

Next, we will state a general integration formula that works in *diametrically regular* spaces, namely those metric spaces (X,d) such that for each $x \in X$ and $R > 0$ there exists $\delta_{x,R} > 0$ such that $(0, \delta_{x,R}) \ni t \to \operatorname{diam}(B(y,t))$ is continuous for every $y \in X$ such that $d(x,y) \le R$. We have denoted by $B(y,t)$ the open metric ball of center y and radius t. Since we consider metric spaces where all metric balls with positive radius have positive diameter, if $\mu : \mathcal{P}(X) \to [0,+\infty]$ is a measure, then the set

$$\mathcal{S}_{\mu,\zeta_{b,\alpha}} = \mathcal{F}_b \setminus \{S \in \mathcal{F}_b : \zeta_{b,\alpha}(S) = \mu(S) = 0 \text{ or } \zeta_{b,\alpha}(S) = \mu(S) = +\infty\}$$

appearing in Theorem 11 of [40] coincides with the family of closed metric balls $\mathcal{F}_b$, where $\zeta_{b,\alpha} : \mathcal{F}_b \to [0,+\infty)$, $\zeta_{b,\alpha}(S) = c_\alpha \operatorname{diam}(S)^\alpha$ and $c_\alpha > 0$ is an arbitrary constant that defines both the gauge $\zeta_{b,\alpha}$ and the corresponding spherical Hausdorff measure $\mathcal{S}^\alpha$, according to Definition 3.1. Notice that with this definition of $\mathcal{S}^\alpha$ and the definition of $\theta^\alpha(\mu,x)$ in (3), the constant c_α on the right hand side of (2) cancels. In fact, this number essentially plays the role of a geometric constant that can be suitably fixed.

Since $\mathcal{F}_b$ obviously *covers any subset finely*, following the terminology in 2.8.1 of [21], the following result is a slightly simplified version of Theorem 11 in [40].

Theorem 1.1. *Let (X,d) be a diametrically regular metric space, where all balls with positive radius have also positive diameter. Let $\alpha > 0$ and let μ be a Borel regular measure over X such that there exists a countable open covering of X whose elements have μ finite measure. If $B \subset A \subset X$ are Borel sets, then $\theta^\alpha(\mu,\cdot)$ is Borel on A.*

In addition, if $\mathcal{S}^\alpha(A) < +\infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^\alpha \llcorner A$, then we have

$$\mu(B) = \int_B \theta^\alpha(\mu,x)\, d\mathcal{S}^\alpha(x)\,. \tag{2}$$

Since $\operatorname{diam}(B(x,r)) = 2r$ for all $x \in \mathbb{G}$ and $r > 0$, where $\mathbb{G}$ is a homogeneous stratified group and the open ball $B(x,r) \subset \mathbb{G}$ is introduced in Section 2, then the assmptions of Theorem 1.1 are satisfied in $\mathbb{G}$. This is completely independent of the fact that B.C.T. might not hold.

The *spherical Federer density* $\theta^\alpha(\mu,\cdot)$ has been recently introduced in [40] and it has the explicit formula

$$\theta^\alpha(\mu,x)=\inf_{\varepsilon>0}\ \sup\left\{\frac{\mu(\mathbb{B})}{c_\alpha \mathrm{diam}(\mathbb{B})^\alpha}: x\in\mathbb{B}\in\mathcal{F}_b,\ \mathrm{diam}\,\mathbb{B}<\varepsilon\right\}. \tag{3}$$

If we set $c_{Q-1}=2^{1-Q}$ in the previous formula, then (2) holds with $\mathcal{S}^{Q-1}$ replaced by $\mathcal{S}_0^{Q-1}$. Thus, by this choice, the problem of finding β in (1) corresponds to the problem of finding a more explicit formula for $\theta^{Q-1}(|\partial_H E|,\cdot)$.

We wish to mention that the spherical Federer density may differ from the α-density in the sense of 2.10.19 of [21], as shown for instance in [40]. The latter density has been recently related to a new measure theoretic area formula, proved by Franchi, Serapioni and Serra Cassano, [28]. In this formula, the spherical Hausdorff measure of (2) is replaced by the so-called *centered Hausdorff measure*, introduced by Saint Raymond and Tricot in [50], see also [17].

Our point is to show that the explicit formula for the Federer density corresponds to a precise geometric expression. Identifying $\mathbb{G}$ with the direct sum of linear spaces $V_1\oplus V_2\oplus\cdots\oplus V_\iota$, as explained in Section 2, for each $\nu\in V_1\setminus\{0\}$, we define its corresponding *vertical subgroup* $N(\nu)=\nu^\perp\oplus V_2\oplus\cdots\oplus V_\iota$. Here $\nu^\perp$ denotes the subspace of V_1 that is orthogonal to ν. In fact, an auxiliary scalar product is fixed on the linear structure of $\mathbb{G}$. Then we set

$$\beta(d,\nu)=\max_{z\in\mathbb{B}(0,1)}\mathcal{H}^{n-1}\big(\mathbb{B}(z,1)\cap N(\nu)\big), \tag{4}$$

where n is the dimension of $\mathbb{G}$ as linear space and $\mathcal{H}^{n-1}$ is the Euclidean Hausdorff measure in $\mathbb{G}$ arising from the fixed scalar product. The closed metric unit ball $\mathbb{B}(0,1)$ is introduced in Section 2. The number introduced in (4) represents the maximal area of all intersections of $\mathbb{B}(0,1)$ with vertical hyperplanes that are orthogonal to the horizontal direction ν. If $\mathbb{B}(0,1)$ has no symmetries, then the maximal intersection $\beta(d,\tilde\nu)$ with hyperplanes orthogonal to a different direction $\tilde\nu$ might differ. It is also easy to realize that $V_1\setminus\{0\}\ni\nu\to\beta(d,\nu)$ is 0-homogeneous. We are now in the position to state our main result.

Theorem 1.2 (Area formula for the perimeter measure)**.** *Let $\mathbb{G}$ be a stratified group and let $E\subset\mathbb{G}$ be an h-finite perimeter set. If $\mathcal{F}_H E$ is $\mathbb{G}$-rectifiable, then we have*

$$|\partial_H E|=\beta(d,\nu_E)\,\mathcal{S}_0^{Q-1}\llcorner\mathcal{F}_H E\,. \tag{5}$$

To compute the Federer density for the perimeter measure, we consider suitable $\mathbb{G}$-regular sets, see Definition 2.1. From [27], in two step stratified groups the reduced boundary $\mathcal{F}_H E$ can be covered by a countable union of these $\mathbb{G}$-regular sets, up to $\mathcal{S}_0^{Q-1}$-negligible sets, namely it is $\mathbb{G}$-rectifiable. We use this approach since it allows us to differentiate the perimeter measure in a way that can be suitable also for potential extensions to higher codimensional submanifolds.

The $\mathbb{G}$-rectifiability of $\mathcal{F}_H E$ also holds in special classes of higher step groups, [42], and for all C^1 smooth open sets of arbitrary stratified groups, [38]. To include all of these cases, we have stated Theorem 1.2 for all h-finite perimeter sets whose reduced boundary $\mathcal{F}_H E$ is $\mathbb{G}$-rectifiable. We also point out that the $\mathbb{G}$-rectifiability of $\mathcal{F}_H E$ is necessary and it is clearly a crucial fact also in the classical context of Euclidean spaces, when $\mathbb{G}$ is commutative and equipped with a Euclidean norm. However, in a general stratified group it is not yet clear whether all reduced boundaries $\mathcal{F}_H E$ are automatically $\mathbb{G}$-rectifiable. This is still an important open question.

Rather surprisingly, formula (5) holds for an arbitrary homogeneous distance, with no regularity assumption, and it also finds an interesting relationship with the shape of the metric unit ball generated by the distance. As an example of this fact, Theorem 5.2 shows that whenever the unit ball $\mathbb{B}(0,1)$ is convex, then

$$\beta(d,v) = \mathcal{H}^{n-1}\big(N(v) \cap \mathbb{B}(0,1)\big). \tag{6}$$

Thus, it turns out that the Federer density and the $(Q-1)$-density coincide when the metric unit ball is convex. This provides a simpler formula that relates perimeter measure and spherical Hausdorff measure. The key to prove (6) is a concavity-type property for the areas of all parallel one codimensional slices of a convex body, see [11] and Theorem 5.1.

In Section 6 we discuss some classes of symmetries of a homogeneous distance d such that $V_1 \setminus \{0\} \ni \nu \to \beta(d,\nu)$ is a constant function. If we denote by $\omega_{\mathbb{G},Q-1}$ the value of this constant function and we denote by $\mathcal{S}_{\mathbb{G}}^{Q-1}$ the spherical Hausdorff measure constructed by the gauge $\zeta_{b,Q-1}(S) = \omega_{\mathbb{G},Q-1}\mathrm{diam}(S)^{Q-1}/2^{Q-1}$, then we have a simpler representation of the perimeter measure, according to Theorem 1.3.

In simpler terms, since $\beta(d,\cdot)$ is constant, the measure $\beta(d,\nu_E)\,\mathcal{S}_0^{Q-1} \llcorner \mathcal{F}_H E$ of Theorem 1.2 becomes $\omega_{\mathbb{G},Q-1}\mathcal{S}_0^{Q-1} \llcorner \mathcal{F}_H E$, so it is natural to include $\omega_{\mathbb{G},Q-1}$ in the definition of spherical Hausdorff measure, namely $\mathcal{S}_{\mathbb{G}}^{Q-1} = \omega_{\mathbb{G},Q-1}\mathcal{S}_0^{Q-1}$. Taking into account these definitions, we have the following consequence of Theorem 1.2.

Theorem 1.3 (Area formula for symmetric distances)**.** *Let $\mathbb{G}$ be a stratified group equipped with a V_1-vertically symmetric distance and let $E \subset \mathbb{G}$ be set of h-finite perimeter. If $\mathcal{F}_H E$ is $\mathbb{G}$-rectifiable, then*

$$|\partial_H E| = \mathcal{S}_{\mathbb{G}}^{Q-1} \llcorner \mathcal{F}_H E\,. \tag{7}$$

In Definition 6.1, we introduce the class of V_1*-vertically symmetric distances*, whose metric unit ball has a suitable group of symmetries modeled on V_1. Theorem 6.1 shows that all of these distances have the function $V_1 \setminus \{0\} \ni \nu \to \beta(d,\nu)$ constant, hence in Theorem 1.3 we can define the "natural" spherical Hausdorff measure $\mathcal{S}_{\mathbb{G}}^{Q-1}$.

In each stratified group, we can find a homogeneous distance whose unit ball $\mathbb{B}(0,1)$ coincides with a Euclidean ball of sufficiently small radius, see Theorem 2 of [33]. In particular, this distance is V_1-vertically symmetric. Many other important examples of homogeneous distances with this symmetry property are available. We mention for instance the Cygan-Korányi distance, the distance d_∞ of [27] and the sub-Riemannian distance in the Heisenberg group, see Section 5 and Section 6.

2. Notation, terminology and basic facts

A *stratified group* can be seen as a graded linear space $\mathbb{G} = V_1 \oplus \cdots \oplus V_\iota$ equipped with a polynomial group operation such that its graded Lie algebra $\mathcal{G} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\iota$ satisfies $[\mathcal{V}_1, \mathcal{V}_j] = \mathcal{V}_{j+1}$ for all integers $j \geq 0$, where $\mathcal{V}_i = \{0\}$ for all $i > \iota$ and $\mathcal{V}_\iota \neq \{0\}$. We also point out that identifying $\mathbb{G}$ with the tangent space $T_0\mathbb{G}$ of $\mathbb{G}$ at the origin, we have a canonical isomorphism between V_j and $\mathcal{V}_j$, that associates to each $v \in V_j$ the unique left invariant vector field $V \in \mathcal{V}_j$ such that $V(0) = v$.

The terminology *stratified group* is due to G. B. Folland, [23], see also [24] and [49]. Stratified groups equipped with a sub-Riemannian distance are also well known as Carnot groups, according to the terminology introduced by P. Pansu, [45].

We will denote by n the dimension of $\mathbb{G}$, seen as a linear space. The graded structure of $\mathbb{G}$ allows us to introduce intrinsic dilations $\delta_r : \mathbb{G} \to \mathbb{G}$ as linear mappings such that $\delta_r(y) = r^i y$ for each $y \in V_i$, $r > 0$ and $i = 1, \ldots, \iota$.

A *homogeneous distance* d on $\mathbb{G}$ is a continuous and left invariant distance with $d(\delta_r x, \delta_r y) = r\, d(x, y)$ for all $x, y \in \mathbb{G}$ and $r > 0$. It could be a nice exercise for the reader to show that the latter homogeneity property joined with left invariance imply the continuity of the distance with respect to the unique topology that makes $\mathbb{G}$ a finite dimensional topological vector space. We define the open and closed balls

$$B(y, r) = \Big\{ z \in \mathbb{G} : d(z, y) < r \Big\} \quad \text{and} \quad \mathbb{B}(y, r) = \Big\{ z \in \mathbb{G} : d(z, y) \leq r \Big\} .$$

The corresponding homogeneous norm is denoted by $\|x\| = d(x, 0)$ for all $x \in \mathbb{G}$. The terminology *homogeneous stratified group* may wish to stress that the stratified group is equipped with a homogeneous distance, although such a distance is often understood. The symbol Ω will denote an open subset of a stratified group $\mathbb{G}$. Notice that any homogeneous distance is bi-Lipschitz equivalent to the SR distance.

In our terminology, a C^1_h *smooth function* $f : \Omega \to \mathbb{R}$ on an open set Ω of a stratified group $\mathbb{G}$ has the property that for all $x \in \Omega$ and $X \in \mathcal{V}_1$ the *horizontal derivative*

$$Xf(x) = \lim_{t \to 0} \frac{f\big(\Phi^X_t(x)\big) - f(x)}{t}$$

exists and it is continuous in Ω, where Φ^X denotes the flow of X. We denote by $\mathcal{C}^1_h(\Omega)$ the linear space of all $\mathcal{C}^1_h$ smooth functions on Ω.

We define the linear mapping $d_h f(x) : \mathbb{G} \to \mathbb{R}$ such that

$$d_h f(x)(w) = \begin{cases} 0 & \text{if } w \in V_2 \oplus \cdots \oplus V_\iota \\ \mathcal{W} f(x) & \text{if } w \in V_1 \end{cases}$$

with $w \in V_1$ and $\mathcal{W} \in \mathcal{V}_1$ is the unique left invariant vector field such that $\exp \mathcal{W} = w$. Here $\exp : \mathcal{G} \to \mathbb{G}$ is the standard exponential mapping of $\mathbb{G}$, seen as a Lie group.

In view of Folland and Stein's "stratified mean value theorem", see (1.41) of [24], $f \in \mathcal{C}_h^1(\Omega)$ if and only if $x \to d_h f(x)$ is continuous in Ω and

$$f(xh) - f(x) - d_h f(x)(h) = o(\|h\|) \quad \text{as} \quad \|h\| \to 0. \tag{8}$$

The class of $\mathcal{C}_h^1$ functions has a corresponding implicit function theorem.

Theorem 2.1 (Implicit function theorem)**.** *Let $x \in \Omega$, $X \in \mathcal{V}_1$ and $f \in \mathcal{C}_h^1(\Omega)$ with $Xf(x) \neq 0$. Let N be the kernel of $d_h f(x)$ and let $H = \mathbb{R}\, e_X$, where $e_X = \exp X$. Then we have an open set $V \subset N$ with $0 \in V$, a continuous function $\varphi : V \to H$ and an open neighborhood $U \subset \mathbb{G}$ of x, such that*

$$f^{-1}\big(f(x)\big) \cap U = \big\{x\eta\varphi(\eta) \mid \eta \in V\big\}. \tag{9}$$

This result is an immediate consequence of the Euclidean implicit function theorem, once the proper system of coordinates is fixed. It shows that regular level sets of C_h^1 smooth functions are locally graphs with respect to the group operation, [26]. A more general implicit function theorem for mappings between two stratified groups $\mathbb{G}$ and $\mathbb{M}$ can be also obtained. The new algebraic and topological difficulties of this case are partially overcome using the topological degree and assuming special algebraic factorizations of the source space. Level sets of these mappings define the general class of $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$, see [38] and [39] for more information. For the purposes of this work, the next definition refers to the case $\mathbb{M} = \mathbb{R}$. In this special case, these sets have first appeared in [26] and called $\mathbb{G}$-regular hypersurfaces.

Definition 2.1. We say that a subset $\Sigma \subset \mathbb{G}$ is a *parametrized $\mathbb{G}$-regular hypersurface* if there exists $f \in \mathcal{C}_h^1(\Omega)$ such that $d_h f$ is everywhere nonvanishing and $\Sigma = f^{-1}(0)$.

A *graded basis* $(e_1, \ldots, e_n)$ of $\mathbb{G}$ is defined by assuming that the families of vectors

$$(e_{m_{j-1}+1}, e_{m_{j-1}+2}, \ldots, e_{m_j})$$

are bases of the subspaces V_j and $m_j = \sum_{i=1}^{j} \dim V_i$ for every $j = 1, \ldots, \iota$, where $m_0 = 0$. We also set $m = m_1$.

In the sequel, a graded basis is fixed and the corresponding Lebesgue measure $\mathcal{L}^n$ is automatically defined on $\mathbb{G}$. The left invariance of $\mathcal{L}^n$ with respect to the group operation makes this measure the *Haar measure* μ of $\mathbb{G}$. As mentioned in the introduction, we fix an auxiliary scalar product on $\mathbb{G}$ and we make this choice such that the fixed graded basis is orthonormal. The restriction of this scalar product to V_1 can be translated to the so-called *horizontal fibers*

$$H_x\mathbb{G} = \{X(x) \in T_x\mathbb{G} : X \in \mathcal{V}_1\}$$

as x varies in $\mathbb{G}$, hence defining a left invariant sub-Riemannian metric g on $\mathbb{G}$. We denote by $H\mathbb{G}$ the *horizontal subbundle* of $\mathbb{G}$, whose fibers are $H_x\mathbb{G}$. By a slight abuse of notation, we denote both the norm arising from g and the norm arising from the scalar product of $\mathbb{G}$ by the same symbol $|\cdot|$.

Definition 2.2 (Horizontal subbundle and its sections)**.** Let $O \subset \mathbb{G}$ be an open set. We denote by HO the restriction of the *horizontal subbundle* $H\mathbb{G}$ to the open set O, whose *horizontal fibers* $H_x\mathbb{G}$ are restricted to all points $x \in O$. We denote by $\mathcal{S}_c(HO)$ the linear space of all smooth sections of HO with compact support in O.

For each parametrized $\mathbb{G}$-regular hypersurface $\Sigma \subset \mathbb{G}$ defined by $f : \Omega \to \mathbb{R}$, we may define the *intrinsic measure* of Σ through the *perimeter measure*

$$\sigma_\Sigma(O) = \sup \left\{ \int_{E_f} \operatorname{div} X \, d\mu \;\middle|\; X \in \mathcal{S}_c(HO),\ |X| \le 1 \right\} \tag{10}$$

for any open set $O \subset \Omega$, where $E_f = \{x \in \Omega : f(x) < 0\}$. The symbol div denotes the divergence operator with respect to the Haar measure μ, that in our coordinate system gives the standard divergence operator.

We introduce the *horizontal normal* for a parametrized $\mathbb{G}$-regular hypersurface Σ with defining function f as follows: for each $y \in \Sigma$ we set

$$\nu_\Sigma(y) = \frac{\nabla_H f(y)}{|\nabla_H f(y)|}\,, \tag{11}$$

where $(X_1, \ldots, X_m)$ is an orthonormal basis of $\mathcal{V}_1$ and

$$\nabla_H f = (X_1 f, \ldots, X_m f)$$

is the *horizontal gradient* of f. For our purposes, we do not claim an orientation for Σ, that is a specific choice of $\pm\nu_\Sigma$.

3. Upper blow-up of the perimeter measure

In this section we prove the central result for this work, namely, a new blow-up theorem for the perimeter measure. We recall here our basic notation.

Definition 3.1 (Carathéodory construction)**.** Let $\mathcal{F} \subset \mathcal{P}(\mathbb{G})$ denote a nonempty family of closed subsets of a stratified group $\mathbb{G}$, equipped with a homogeneous distance. Let $\alpha > 0$ and $c_\alpha > 0$ arbitrarily fixed. If $\delta > 0$ and $E \subset \mathbb{G}$, then we define

$$\phi_\delta(E) = \inf \left\{ \sum_{j=0}^{\infty} c_\alpha \operatorname{diam}(B_j)^\alpha : E \subset \bigcup_{j \in \mathbb{N}} B_j,\ \operatorname{diam}(B_j) \le \delta,\ B_j \in \mathcal{F} \right\},$$

where the diameter $\operatorname{diam} B_j$ of B_j is computed by the fixed homogeneous distance. If $\mathcal{F}$ coincides with the family of closed balls $\mathcal{F}_b$, then we set

$$\mathcal{S}^\alpha(E) = \sup_{\delta > 0} \phi_\delta(E).$$

to be the *α-dimensional spherical Hausdorff measure* of E. This defines a Borel regular (outer) measure $\mathcal{S}^\alpha$ on all subsets of $\mathbb{G}$. If in the previous definition of $\mathcal{S}^\alpha$ we choose $c_\alpha = 2^{-\alpha}$, then we use the symbol $\mathcal{S}_0^\alpha$.

In the case $\mathcal{F}$ is the family of all closed sets, $\alpha = k$ is a positive integer less than the linear dimension of $\mathbb{G}$, c_k is the volume of the unit ball in $\mathbb{R}^k$ and the Euclidean distance is fixed on $\mathbb{G}$, then the previous construction yields the Euclidean *k-dimensional Hausdorff measure* on $\mathbb{G}$, that we denote by $\mathcal{H}^k$.

Theorem 3.1 (Upper blow-up)**.** *Let Σ be a parametrized $\mathbb{G}$-regular hypersurface and let $x \in \Sigma$. If σ_Σ is its associated perimeter measure and we have defined $\theta^{Q-1}(\sigma_\Sigma, \cdot)$ as in* (3) *with $c_{Q-1} = 2^{1-Q}$, then we have*

$$\theta^{Q-1}(\sigma_\Sigma, x) = \beta\big(d, \nu_\Sigma(x)\big)\,. \tag{12}$$

Proof. We consider f as the defining function of Σ and select $X_1 \in \mathcal{V}_1$ such that the element $e_{X_1} = \exp X_1 \in V_1$ has unit length and it is orthogonal to the kernel of $d_h f(x)$. Let $X_2, \ldots, X_m \in \mathcal{V}_1$ be such that $(X_2(x), \ldots, X_m(x))$ is an orthonormal basis of this kernel. By Theorem 2.1, we have an open neighborhood $V \subset N$ of the origin, with $N = \ker d_h f(x)$, a continuous function $\varphi : V \to \mathbb{R}$ and an open neighborhood $U \subset \mathbb{G}$ of x, such that

$$\Sigma \cap U = \big\{ x\eta(\varphi(\eta) e_{X_1}) \mid \eta \in V \big\}\,. \tag{13}$$

Up to changing the sign of f and possibly shrinking U, we assume that $X_1 f \geq \alpha > 0$ everywhere on U. To define the intrinsic measure of Σ, we consider the open set

$$E_f = \{ w \in U \mid f(w) < 0 \} = \big\{ x\eta(s e_{X_1}) \in U |\, \eta \in V,\ s < \varphi(\eta) \big\}.$$

From [26], this set has finite perimeter and defining the graph mapping $\Phi : V \to \Sigma$ by $\Phi(\eta) = x\eta\big(\varphi(\eta) e_{X_1}\big)$ for every $\eta \in V$, we also have the formula

$$\sigma_\Sigma\big(\mathbb{B}(y,t)\big) = |\partial E_f|_H\big(\mathbb{B}(y,t)\big) = \int_{\Phi^{-1}(\mathbb{B}(y,t))} \frac{\sqrt{\sum_{j=1}^m X_j f(\Phi(\eta))^2}}{X_1 f(\Phi(\eta))}\, d\mathcal{H}^{n-1}(\eta)$$

for $t > 0$ small and $y \in U$. We make the change of variables $n = \Lambda_t \eta$, where $\Lambda_t : N \to N$ and

$$\Lambda_t \eta = \sum_{j=2}^m t\, \eta_j e_j + \sum_{i=2}^{\iota} \sum_{j=m_{i-1}+1}^{m_i} t^i \eta_j e_j\,.$$

We notice that $\delta_t|_N = \Lambda_t$ and the Jacobian of Λ_t is t^{Q-1}, where Q is the Hausdorff dimension of $\mathbb{G}$. By a change of variables, we get

$$\sigma_\Sigma\big(\mathbb{B}(y,t)\big) = t^{Q-1} \int_{\Lambda_{1/t}\left(\Phi^{-1}(\mathbb{B}(y,t))\right)} \frac{\sqrt{\sum_{j=1}^m X_j f(\Phi(\Lambda_t\eta))^2}}{X_1 f(\Phi(\Lambda_t \eta))}\, d\mathcal{H}^{n-1}(\eta)\,. \tag{14}$$

The general definition of spherical Federer's density, [40], in our setting gives

$$\theta^{Q-1}(\sigma_\Sigma, x) = \inf_{r>0} \sup_{\substack{y \in \mathbb{B}(x,t) \\ 0<t<r}} \frac{\sigma_\Sigma\big(\mathbb{B}(y,t)\big)}{t^{Q-1}}\,. \tag{15}$$

Due to (14), to find $\theta^{Q-1}(\sigma_\Sigma, x)$ we first observe that the sets

$$\Lambda_{1/t}\big(\Phi^{-1}(\mathbb{B}(y,t))\big) = \left\{\eta \in \Lambda_{1/t}V \mid \big(\delta_{1/t}(y^{-1}x)\big)\,\eta\left(\frac{\varphi\big(\delta_t\eta\big)}{t}e_{X_1}\right) \in \mathbb{B}(0,1)\right\} \tag{16}$$

are bounded, uniformly with respect to t, as $y \in \mathbb{B}(x,t)$. To see this fact, since f is $\mathcal{C}^1_h$ smooth, by (8), we first choose

$$0 < \varepsilon_0 < \frac{\alpha}{\|e_{X_1}\|} \le \frac{\inf_U X_1 f}{\|e_{X_1}\|}$$

such that, for V possibly shrinked, containing the origin and depending on ε_0, for all $\eta \in V$ there holds

$$\frac{|f(x\eta)|}{\|\eta\|} = \frac{|f(x\eta) - f(x) - d_h f(x)(\eta)|}{\|\eta\|} \le \varepsilon_0.$$

Let us consider $y \in \mathbb{B}(x,t)$ and choose

$$w \in \Lambda_{1/t}\big(\Phi^{-1}(\mathbb{B}(y,t)\big).$$

Thus, we have

$$\begin{aligned}
\varepsilon_0 \quad &\ge \quad \frac{|f(x\delta_t w)|}{\|\delta_t w\|} = \frac{|f\big(x\delta_t w(\varphi(\delta_t w)e_{X_1})\big) - f(x\delta_t w)|}{\|\delta_t w\|} \\
&= \quad \frac{\left|\int_0^{\varphi(\delta_t w)} X_1 f(x\delta_t w\delta_s e_{X_1})\,ds\right|}{t\,\|w\|} \ge \frac{\alpha}{\|w\|}\,\frac{|\varphi(\delta_t w)|}{t}\,.
\end{aligned}$$

Notice that the previous estimates remain true also in the case $\varphi(\delta_t w) = 0$. Taking into account that $d(x,y) \le t$, we get

$$w\left(\frac{\varphi\big(\delta_t w\big)}{t}e_{X_1}\right) \in \mathbb{B}(0,2). \tag{17}$$

The previous inclusion gives

$$\|w\| \le 2 + \|\big(t^{-1}\varphi(\delta_t w)e_{X_1}\big)^{-1}\| = 2 + \|t^{-1}\varphi(\delta_t w)e_{X_1}\| \le 2 + \|e_{X_1}\|\frac{\varepsilon_0}{\alpha}\|w\|\,.$$

It follows that

$$\Big(1 - \frac{\|e_{X_1}\|\varepsilon_0}{\alpha}\Big)\|w\| \le 2\,.$$

As a consequence, setting $R_0 = 2/(1 - \frac{\|e_{X_1}\|\varepsilon_0}{\alpha})$, we have proved that for $t > 0$ sufficiently small

$$\Lambda_{1/t}\big(\Phi^{-1}(\mathbb{B}(y,t))\big) \subset \mathbb{B}(0,R_0)\,. \tag{18}$$

The first consequence of this inclusion is that $\theta^{Q-1}(\sigma_\Sigma,x)<+\infty$, hence there exist a sequence $\{t_k\}\subset(0,+\infty)$ converging to zero and a sequence of elements $y_k\in\mathbb{B}(x,t_k)$ such that

$$\theta^{Q-1}(\sigma_\Sigma,x)=\lim_{k\to\infty}\int_{\Lambda_{1/t_k}\left(\Phi^{-1}(\mathbb{B}(y_k,t_k))\right)}\frac{\sqrt{\sum_{j=1}^m X_jf(\Phi(\Lambda_{t_k}\eta))^2}}{X_1f(\Phi(\Lambda_{t_k}\eta))}\,d\mathcal{H}^{n-1}(\eta)\,.$$

Possibly extracting a subsequence, there exists $z\in\mathbb{B}(0,1)$ such that

$$\delta_{1/t_k}(y_k^{-1}x)\to z^{-1}\in\mathbb{B}(0,1). \tag{19}$$

Setting $S_z=N\cap\mathbb{B}(z,1)$, we wish to show that for each $w\in N\setminus S_z$ there holds

$$\lim_{k\to\infty}\mathbf{1}_{A_k}(w)=0\,, \tag{20}$$

where $A_k=\Lambda_{1/t_k}\big(\Phi^{-1}(\mathbb{B}(y_k,t_k))\big)$. For this, we have to prove that

$$\lim_{t\to0^+}\frac{\varphi(\delta_tw)}{t}=0. \tag{21}$$

Since φ is only continuous, this makes the proof of this limit more delicate. We define

$$A(w)=\Big\{t\in\mathbb{R}\mid t>0,\ \varphi(\delta_tw)\neq0\Big\}\,.$$

If $\overline{A(w)}$ does not contain zero, the limit (21) becomes obvious. If $0\in\overline{A(w)}$, then we choose an arbitrary infinitesimal sequence $\{\tau_k\}\subset A(w)$. Using the stratified mean value inequality in (1.41) of [24], we notice that

$$\lim_{k\to\infty}\frac{f(x\delta_{\tau_k}w)}{\tau_k}=\lim_{k\to\infty}\frac{f(x\delta_{\tau_k}w)-f(x)}{\tau_k}=0\,.$$

Since $\varphi(\delta_{\tau_k}w)\neq0$, we can multiply and divide by $\varphi(\delta_{\tau_k}w)$, getting

$$0=\lim_{k\to\infty}\frac{f(x\delta_{\tau_k}w)}{\tau_k}=\Big(-X_1f(x)\Big)\lim_{k\to\infty}\frac{\varphi(\delta_{\tau_k}w)}{\tau_k}\,. \tag{22}$$

This proves (21), hence (20) follows. We consider the following integral as the sum

$$\int_{A_k}\frac{\sqrt{\sum_{j=1}^m X_jf(\Phi(\Lambda_{t_k}\eta))^2}}{X_1f(\Phi(\Lambda_{t_k}\eta))}\,d\mathcal{H}^{n-1}(\eta)=I_k+J_k\,,$$

where, introducing the density function

$$\alpha(t,\eta)=\big(X_1f(\Phi(\Lambda_t\eta))\big)^{-1}\sqrt{\sum_{j=1}^m X_jf(\Phi(\Lambda_t\eta))^2},$$

we have set

$$I_k=\int_{A_k\cap S_z}\alpha(t_k,\eta)\,d\mathcal{H}^{n-1}(\eta)\quad\text{and}\quad J_k=\int_{A_k\setminus S_z}\alpha(t_k,\eta)\,d\mathcal{H}^{n-1}(\eta)\,.$$

In principle, when $w \in N \cap \partial\mathbb{B}(z,1)$ we do not have information on the limit of $\mathbf{1}_{A_k \cap S_z}(w)$ as $k \to \infty$. Taking into account (16), this depend on the geometry of $x^{-1}\Sigma \cap \mathbb{B}(0,1)$. However, in this step we wish to prove only one inequality. Then we consider the following inequality

$$I_k \le \int_{S_z} \alpha(t_k,\eta)\, d\mathcal{H}^{n-1}(\eta)\,. \tag{23}$$

Due to (18), we have

$$J_k \le \int_{\mathbb{B}(0,R_0)\setminus S_z} \mathbf{1}_{A_k}(\eta)\, \alpha(t_k,\eta)\, d\mathcal{H}^{n-1}(\eta)\,.$$

The integrand of the previous integral goes to zero as $k \to \infty$, due to (20), and it is uniformly bounded, therefore Lebesgue's convergence theorem implies that $J_k \to 0$. Again Lebesgue's theorem gives

$$\lim_{k\to\infty} \int_{S_z} \alpha(t_k,\eta)\, d\mathcal{H}^{n-1}(\eta) = \mathcal{H}^{n-1}(S_z)\,.$$

In fact, $X_j f(0) = 0$ for $j = 2, \ldots, m$, hence $\alpha(t_k,\eta) \to 1$ as $k \to \infty$. This gives

$$\theta^{Q-1}(\sigma_\Sigma, x) = \lim_{k\to\infty} \int_{A_k} \frac{\sqrt{\sum_{j=1}^m X_j f(\Phi(\Lambda_{t_k}\eta))^2}}{X_1 f(\Phi(\Lambda_{t_k}\eta))}\, d\mathcal{H}^{n-1}(\eta) \le \mathcal{H}^{n-1}(S_z) \le \mathcal{H}^{n-1}(S_{z_0})\,,$$

where $z_0 \in \mathbb{B}(0,1)$ is such that

$$\mathcal{H}^{n-1}(S_{z_0}) = \beta(d, \nu_\Sigma(x))\,. \tag{24}$$

To prove the opposite inequality, we select $y_t^0 = x\delta_t z_0 \in \mathbb{B}(x,t)$ and fix $\lambda > 1$. We observe that

$$\sup_{0<t<r} \frac{\sigma_\Sigma\big(\mathbb{B}(y_t^0, \lambda t)\big)}{(\lambda t)^{Q-1}} \le \sup_{\substack{y\in\mathbb{B}(x,t)\\ 0<t<\lambda r}} \frac{\sigma_\Sigma\big(\mathbb{B}(y,t)\big)}{t^{Q-1}}$$

for every $r > 0$, therefore the definition of spherical Federer density (15) yields

$$\limsup_{t\to 0^+} \frac{\sigma_\Sigma\big(\mathbb{B}(y_t^0, \lambda t)\big)}{(\lambda t)^{Q-1}} \le \theta^{Q-1}(\sigma_\Sigma, x)\,.$$

Taking into account (16), we set

$$A_t^0 = \Lambda_{1/\lambda t}\big(\Phi^{-1}(\mathbb{B}(y_t^0,\lambda t))\big) = \left\{ \eta \in \Lambda_{1/\lambda t} V \mid \eta \left(\frac{\varphi\big(\delta_{\lambda t}\eta\big)}{\lambda t} e_{X_1} \right) \in \mathbb{B}\big(\delta_{1/\lambda} z_0, 1\big) \right\}, \tag{25}$$

that implies

$$\frac{\sigma_\Sigma\big(\mathbb{B}(y_t^0,\lambda t)\big)}{(\lambda t)^{Q-1}} = \int_{A_t^0} \alpha(\lambda t,\eta)\, d\mathcal{H}^{n-1}(\eta) = \frac{1}{\lambda^{Q-1}} \int_{\delta_\lambda A_t^0} \alpha(\lambda t, \delta_{1/\lambda}\eta)\, d\mathcal{H}^{n-1}(\eta)\,.$$

We have $\delta_\lambda A_t^0 = \{\eta \in \Lambda_{1/t}V \mid \eta\big(\frac{\varphi(\delta_t\eta)}{t}e_{X_1}\big) \in \mathbb{B}(z_0,\lambda)\}$ and observe that $\alpha(t,\eta)$ is well defined and bounded on $[0,\bar\varepsilon]\times\big(N\cap B(z_0,\lambda)\big)$, for $\bar\varepsilon$ sufficiently small. Taking into account that

$$\lim_{t\to 0^+} \mathbf{1}_{\delta_\lambda A_t^0}(w) = 1$$

for all $w\in N\cap B(z_0,\lambda)$, along with the inequality

$$\lambda^{1-Q}\int_{N\cap B(z_0,\lambda)} \mathbf{1}_{\delta_\lambda A_t^0}(w)\,\alpha(\lambda t,\delta_{1/\lambda}\eta)\,d\mathcal{H}^{n-1}(\eta) \le \frac{\sigma_\Sigma\big(\mathbb{B}(y_t^0,\lambda t)\big)}{(\lambda t)^{Q-1}}\,,$$

it follows that

$$\lambda^{1-Q}\mathcal{H}^{n-1}\big(N\cap B(z_0,\lambda)\big)\le \theta^{Q-1}(\sigma_\Sigma,x).$$

As $\lambda\to 1^+$ the opposite inequality follows, hence concluding the proof. □

4. Area formulae for the perimeter measure

This section is devoted to establish the general relationship between perimeter measure and spherical Hausdorff measure. Joining Theorem 3.1 with Theorem 1.1, we obtain the following result.

Theorem 4.1 (Area formula)**.** *Let Σ be a parametrized $\mathbb{G}$-regular hypersurface. If σ_Σ is its associated perimeter measure* (10)*, then*

$$\sigma_\Sigma = \beta(d,\nu_\Sigma)\,\mathcal{S}_0^{Q-1}\llcorner\Sigma\,.$$

This theorem also yields an area formula for the perimeter measure. To present this result we introduce a few more definitions. A subset $S\subset\mathbb{G}$ is $\mathbb{G}$-*rectifiable* if there exists a countable family $\{\Sigma_j \mid j\in\mathbb{N}\}$ of parametrized $\mathbb{G}$-regular hypersurfaces Σ_j such that

$$\mathcal{S}_0^{Q-1}\big(S\setminus\bigcup\Sigma_j\big)=0.$$

A measurable set $E\subset\mathbb{G}$ has h-finite perimeter if

$$\sup\left\{\int_E \operatorname{div}X\,d\mu \,\middle|\, X\in C_c^1(\mathbb{G},H\mathbb{G}),\ |X|\le 1\right\}<+\infty. \tag{26}$$

This allows for defining a finite Radon measure $|\partial_H E|$ on $\mathbb{G}$, see for instance [7]. The *reduced boundary* $\mathcal{F}_H E$ is the set of points $x\in\mathbb{G}$ such that there exists $\nu_E(x)\in V_1$ with $|\nu_E(x)|=1$ and

$$\lim_{r\to 0^+}\frac{1}{|\partial_H E|\big(\mathbb{B}(x,r)\big)}\int_{\mathbb{B}(x,r)}\nu_E(y)\,d|\partial_H E|(y)=\nu_E(x)\,,$$

where ν_E *is the generalized inward normal of* E, see [27] for more information.

Proof of Theorem 1.2. Since the perimeter measure is asymptotically doubling, the perimeter measure $|\partial_H E|$ is concentrated on the reduced boundary $\mathcal{F}_H E$, see [4]. Moreover, the $\mathbb{G}$-rectifiability of $\mathcal{F}_H E$ implies the existence of a countable family $\{\Sigma_j \mid j \in \mathbb{N}\}$ of parametrized $\mathbb{G}$-regular hypersurfaces Σ_j such that

$$\mathcal{S}_0^{Q-1}\Big(\mathcal{F}_H E \setminus \bigcup \Sigma_j\Big) = 0 \quad \text{and} \quad \nu_E(y) = \pm \nu_{\Sigma_j}(y) \text{ for } |\partial_H E|\text{-a.e. } y \in \Sigma_j \cap \mathcal{F}_H E.$$

In fact, locality of the perimeter measure, see Corollary 2.6 of [6], shows that for $|\partial_H E|$-a.e. $z \in \Sigma_j \cap \mathcal{F}_H E$ there holds $\nu_E(z) = \pm \nu_{\Sigma_j}(z)$. On each Σ_j we define the Radon measure

$$\mu_j = \beta(d, \nu_{\Sigma_j})\, \mathcal{S}_0^{Q-1} \llcorner \Sigma_j\,,$$

hence the area formula of Theorem 4.1 gives $\mu_j = \sigma_{\Sigma_j}$. The argument of the upper blow-up theorem clearly simplifies in the case of the following centered blow-up, giving

$$\lim_{r \to 0^+} \frac{\sigma_{\Sigma_j}\big(\mathbb{B}(y,r)\big)}{r^{Q-1}} = \mathcal{H}^{n-1}\Big(N\big(\nu_{\Sigma_j}(y)\big) \cap \mathbb{B}(0,1)\Big) \tag{27}$$

for all $y \in \Sigma_j$. From Theorem 1.2 of [5], there exists $\mathcal{R}_E \subset \mathcal{F}_H E$ such that

$$|\partial_H E|\big(\mathcal{F}_H E \setminus \mathcal{R}_E\big) = 0$$

and for every $x \in \mathcal{R}_E$ the following vertical halfspace

$$Z(\nu_E(x)) = \big\{ v \in \mathbb{G} \mid v = v_1 + v_0,\, v_1 \in V_1,\, v_0 \in V_2 \oplus \cdots \oplus V_\iota,\, \langle v_1, \nu_E(x) \rangle > 0 \big\}$$

belongs to $\mathrm{Tan}(E,x)$, see Definition 6.3 of [5]. Since $|\partial_H E|$ is asymptotically doubling, [4], the perimeter measure $|\partial_H E|$ can differentiate μ_j, hence we can define

$$\delta_E(x) = \lim_{r \to 0^+} \frac{\mu_j\big(\mathbb{B}(x,r)\big)}{|\partial_H E|\big(\mathbb{B}(x,r)\big)} = \lim_{r \to 0^+} \frac{\sigma_{\Sigma_j}\big(\mathbb{B}(x,r)\big)}{|\partial_H E|\big(\mathbb{B}(x,r)\big)} \tag{28}$$

for each $x \in \mathcal{R}_E^1 \cap \Sigma_j$, where we have chosen $\mathcal{R}_E^1 \subset \mathcal{R}_E$ such that

$$|\partial_H E|(\mathcal{R}_E \setminus \mathcal{R}_E^1) = 0 \quad \text{and} \quad \nu_E(x) = \pm \nu_{\Sigma_j}(x) \tag{29}$$

for each $j \in \mathbb{N}$ and each $x \in \mathcal{R}_E^1 \cap \Sigma_j$. Moreover, for the same x the vertical half-space $Z(\nu_E(x))$ belongs to $\mathrm{Tan}(E,x)$ and there exists an infinitesimal positive sequence (r_k) of radii, possibly depending on x, such that

$$|\partial_H E_{x,r_k}|\big(\mathbb{B}(0,1)\big) \longrightarrow |\partial_H Z(\nu_E(x))|\big(\mathbb{B}(0,1)\big) \quad \text{as} \quad k \to \infty\,, \tag{30}$$

where we have defined the translated and rescaled sets $E_{x,r_k} = \delta_{1/r_k}(x^{-1}E)$. By standard formulae on the translated and rescaled perimeter measure, it follows that

$$|\partial_H E|\big(\mathbb{B}(x,r_k)\big) = r_k^{Q-1}\, |\partial_H E_{x,r_k}|\big(\mathbb{B}(0,1)\big) \tag{31}$$

The boundary of $Z(\nu_E(x))$ is precisely $N\big(\nu_E(x)\big)$, that is a parametrized $\mathbb{G}$-regular hypersurface. We also have

$$|\partial_H Z(\nu_E(x))|\big(\mathbb{B}(0,1)\big) = \mathcal{H}^{n-1}\big(N(\nu_E(x)) \cap \mathbb{B}(0,1)\big)\,. \tag{32}$$

The previous equality could be obtained directly from the definition of perimeter measure, since we have a smooth boundary. We may also use the formula of [26] for the representation of the perimeter measure. In this case, we consider the linear defining function $f_0 : \mathbb{G} \to \mathbb{R}$, where

$$f_0(y) = \langle \pi_1(y), \nu_E(x) \rangle$$

and $\pi_1 : \mathbb{G} \to V_1$ is the projection with respect to the direct sum $\mathbb{G} = V_1 \oplus \cdots \oplus V_\iota$. We can select an orthonormal basis of $\mathcal{V}_1$ such that $X_1(0) = \nu_E(x)$, hence

$$\begin{aligned} |\partial_H Z(\nu_E(x))|\big(\mathbb{B}(0,1)\big) &= \int_{\Phi_0^{-1}(\mathbb{B}(0,1))} \frac{\sqrt{\sum_{j=1}^m X_j f_0(\Phi_0(\eta))^2}}{X_1 f_0(\Phi_0(\eta))} \, d\mathcal{H}^{n-1}(\eta) \\ &= \mathcal{H}^{n-1}\big(N(\nu_E(x)) \cap \mathbb{B}(0,1)\big) \,, \end{aligned}$$

since $\Phi_0(\eta) = \eta$. Joining (27) with $y = x$, (29), (30), (31) and (32), we obtain that $\delta_E(x) = 1$, hence (28) immediately leads us to the conclusion. □

5. Vertical sections of convex homogeneous balls

In this section we study those homogeneous distances with convex unit ball. The following classical result of convex geometry will play a key role.

Theorem 5.1 ([11]). *Let H be an n-dimensional Hilbert space with $n \geq 2$ and let C be a compact convex set with nonempty interior that contains the origin. Let $v \in H \setminus \{0\}$ and let N denote the orthogonal space to v. Then the function*

$$\psi(t) = \big[\mathcal{H}^{n-1}\big(C \cap (tv + N)\big)\big]^{1/(n-1)}$$

is concave on the interval $\{t \in \mathbb{R} : C \cap (tv + N) \neq \emptyset\}$.

Thus, we are in the position to establish the result of this section.

Theorem 5.2. *If d is a homogeneous distance such that the corresponding unit ball $\mathbb{B}(0,1)$ is convex and $v \in V_1 \setminus \{0\}$, then we have*

$$\beta(d, v) = \mathcal{H}^{n-1}\big(N(v) \cap \mathbb{B}(0,1)\big) \,. \tag{33}$$

Proof. Let us set $N = N(v)$, where

$$N(v) = v^\perp \oplus V_2 \oplus \cdots \oplus V_\iota$$

is the vertical subgroup orthogonal to the horizontal direction v, with respect to the fixed scalar product. The Euclidean Jacobian of the translation $\tau_z : N \to zN$ is one for all $z \in \mathbb{G}$, hence

$$\mathcal{H}^{n-1}\big(N \cap \mathbb{B}(z,1)\big) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap z^{-1}N\big) \,.$$

To study the previous function with respect to z, we introduce

$$a(z) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap zN\big) \,.$$

Defining $H = \mathbb{R}v$, we have two canonical projections $\pi_1 : \mathbb{G} \to H$ and $\pi_2 : \mathbb{G} \to N$ such that $y = \pi_1(y)\pi_2(y)$ for all $y \in \mathbb{G}$, see Proposition 7.6 of [39]. Since H is a horizontal subspace, one can also check that $\pi_1 : \mathbb{G} \to H$ is precisely the linear projection onto H with respect to the direct sum of linear spaces $H \oplus N = \mathbb{G}$. As a consequence,

$$a(z) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap zN\big) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap \pi_1(z)N\big)\,. \tag{34}$$

Furthermore, $\pi_1(z)N = \pi_1(z) + N$ and $\mathbb{B}(0,1)^{-1} = -\mathbb{B}(0,1)$, therefore

$$a(z) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap \big(\pi_1(z) + N\big)\big) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap \big(-\pi_1(z) + N\big)\big)$$

and the property $-\pi_1(z) = \pi_1(z^{-1})$ yields

$$a(z) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap \big(\pi_1(z^{-1}) + N\big)\big) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap \big(\pi_1(z^{-1})N\big)\big).$$

Thus, by (34) we get $a(z) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap \big(z^{-1}N\big)\big) = a(z^{-1}) = a(-z)$, hence a is an even function. For every $t \in \mathbb{R}$ we may define the function

$$b(t) = \Big[\mathcal{H}^{n-1}\Big(\mathbb{B}(0,1) \cap \big(tv + N\big)\Big)\Big]^{1/(n-1)}$$

By Theorem 5.1, the function $b(t) = \sqrt[n-1]{a(tv)}$ is concave and even on the compact interval

$$I = \{t \in \mathbb{R} : \mathbb{B}(0,1) \cap (tv + N) \neq \emptyset\}\,,$$

hence we get

$$\beta(d,v) = \max_{z \in \mathbb{B}(0,1)} \mathcal{H}^{n-1}\big((\mathbb{B}(z,1) \cap N(v)\big) = \mathcal{H}^{n-1}\big(N(v) \cap \mathbb{B}(0,1)\big)\,.$$

Being $N(v) \cap \partial\mathbb{B}(0,1)$ locally parametrized by Lipschitz mappings on an $(n-2)$ dimensional open set, then we obviously have $\mathcal{H}^{n-1}\big(N(v) \cap \partial\mathbb{B}(0,1)\big) = 0$, concluding the proof. □

In any general homogeneous group we can always find a homogeneous distance with convex unit ball, [33]. The next examples provide other distances with this property.

Example 5.3. Let $N = V_1 \oplus V_2$ be an H-type group. We have an explicit formula for a homogeneous distance $d(x,y) = \|x^{-1}y\|$ such that

$$\|x\| = \sqrt[4]{|x_1|^4 + 16|x_2|^2}$$

where $x, y \in N$, $x = x_1 + x_2$ and $x_i \in V_i$ for $i = 1, 2$, see [14]. The unit ball with respect to this distance is clearly a convex set.

Example 5.4. Let $\mathbb{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_\iota$ be any stratified group. From the Baker-Campbell-Hausdorff formula it is easy to see the existence of constants $\varepsilon_j > 0$, with $j = 1, \ldots, \iota$ and $\varepsilon_1 = 1$, such that setting

$$\|x\| = \max\{\varepsilon_j |x_j|^{1/j}\}$$

with $x_i \in V_i$ for all $i = 1, \dots, \iota$, we have actually defined the homogeneous distance $d_\infty(x, y) = \|x^{-1}y\|$, as it was observed in [27]. The unit ball $\mathbb{B}(0,1)$ with respect to d_∞ is clearly a convex set.

6. Vertically symmetric distances

The next definition introduces those distances whose symmetries allows for having a precise geometric constant in the definition of the spherical Hausdorff measure $\mathcal{S}_{\mathbb{G}}^{Q-1}$, as discussed in the introduction. Theorem 6.1 below will prove this fact.

Definition 6.1. Let $\mathbb{G}$ be a stratified group of topological dimension n, with direct decomposition $\mathbb{G} = V_1 \oplus W$ and $W = V_2 \oplus \cdots \oplus V_\iota$. We equip $\mathbb{G}$ by a scalar product that makes V_1 and W orthogonal. We consider a family $\mathcal{F}_1 \subset O(V_1)$ that acts transitively on V_1, where $O(V_1)$ denotes the group of isometries of V_1. We set

$$\mathcal{O}_1 = \{T \in GL_n(\mathbb{G}) : T_W = \mathrm{Id}_W,\ T|_{V_1} \in \mathcal{F}_1\}.$$

We denote by $\mathbf{p}_1 : \mathbb{G} \to V_1$ the orthogonal projection onto V_1 and set

$$\mathbb{B}(0,1) = \{y \in \mathbb{G} : d(y,0) \le 1\},$$

where d is a homogeneous distance of $\mathbb{G}$. We say that d is V_1-*vertically symmetric* if

(1) $\mathbf{p}_1\big(\mathbb{B}(0,1)\big) = \mathbb{B}(0,1) \cap V_1 = \{h \in V_1 : |h| \le r_0\}$ for some $r_0 > 0$,
(2) $T\big(\mathbb{B}(0,1)\big) = \mathbb{B}(0,1)$ for all $T \in \mathcal{F}_1$.

Remark 6.1. It is not difficult to observe that the distances of Example 5.3 and Example 5.4 are both V_1-vertically symmetric. The sub-Riemannian distance of the Heisenberg group is also V_1-vertically symmetric. This can be also checked by the explicit formula for the profile of its sub-Riemannian unit ball.

Theorem 6.1. *If a homogeneous distance d is V_1-vertically symmetric, then $\beta(d, \cdot)$ is a constant function.*

Proof. Let $z \in \mathbb{B}(0,1)$ and choose $\nu_1, \nu_2 \in V_1$. Since $\mathcal{F}_1$ is transitive on V_1 there exists $T \in \mathcal{F}_1$ such that $T(\nu_1) = \nu_2$. We set $H = \mathbb{R}\nu_1$ and see $\mathbb{G}$ as the inner semidirect product between H and $N(\nu_1)$. In fact, we have the two canonical projections

$$\pi_1 : \mathbb{G} \to H \quad \text{and} \quad \pi_2 : \mathbb{G} \to N(\nu_1)$$

such that $y = \pi_1(y)\pi_2(y)$ for all $y \in \mathbb{G}$. We set $\pi_1(z^{-1}) = h$ and $\pi_2(z^{-1}) = n$, therefore

$$\mathcal{H}^{n-1}\big(\mathbb{B}(z,1) \cap N(\nu_1)\big) = \mathcal{H}^{n-1}\big(\mathbb{B}(0,1) \cap \big(h + N(\nu_1)\big)\big).$$

By property (2) of Definition 6.1, it follows that

$$\mathcal{H}^{n-1}\big(\mathbb{B}(z,1) \cap N(\nu_1)\big) = \mathcal{H}^{n-1}\Big(\mathbb{B}(0,1) \cap \Big(Th + T\big(N(\nu_1)\big)\Big)\Big).$$

Since π_1 is the linear projection onto H with respect to the decomposition $H \oplus N(\nu_1)$ and $H \perp N(\nu_1)$, we can write z^{-1} as the following sum of orthogonal vectors

$$w + h' + h,$$

where $w \in W$ and $h, h' \in V_1$. By property (1) of Definition 6.1, we get

$$\mathbf{p}_1(z^{-1}) = h' + h \in \mathbb{B}(0,1) \cap V_1 = \{y \in V_1 : |y| \le r_0\}.$$

Since h and h' are orthogonal, we get $h \in \{y \in V_1 : |y| \le r_0\} = \mathbb{B}(0,1) \cap V_1$, hence

$$T(h) \in \mathbb{B}(0,1) \cap V_1.$$

Since T is orthogonal, $T\big(N(\nu_1)\big) = N(\nu_2)$ and we obtain

$$\mathcal{H}^{n-1}\big(\mathbb{B}(z,1) \cap N(\nu_1)\big) = \mathcal{H}^{n-1}\Big(\mathbb{B}(0,1) \cap \Big(\big(T(h)\big)N(\nu_2)\Big)\Big).$$

It follows that

$$\mathcal{H}^{n-1}\big(\mathbb{B}(z,1) \cap N(\nu_1)\big) = \mathcal{H}^{n-1}\big(\mathbb{B}\big(T(h)^{-1},1\big) \cap N(\nu_2)\big) \le \beta(d,\nu_2).$$

The arbitrary choice of $z \in \mathbb{B}(0,1)$ yields $\beta(d,\nu_1) \le \beta(d,\nu_2)$. Exchanging the role of ν_1 for that of ν_2, we conclude the proof. □

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Valentino Magnani, Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I-56127, Pisa

E-mail address: magnani@dm.unipi.it