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SHARP DIMENSION FREE QUANTITATIVE ESTIMATES FOR THE GAUSSIAN ISOPERIMETRIC INEQUALITY[1]

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We provide a full quantitative version of the Gaussian isoperimetric inequality: the difference between the Gaussian perimeter of a given set and a half-space with the same mass controls the gap between the norms of the corresponding barycenters. In particular, it controls the Gaussian measure of the symmetric difference between the set and the half-space oriented so to have the barycenter in the same direction of the set. Our estimate is independent of the dimension, sharp on the decay rate with respect to the gap and with optimal dependence on the mass.

1. Introduction. The isoperimetric inequality in Gauss space states that among all sets with a given Gaussian measure the half-space has the smallest Gaussian perimeter. This result was first proved by Borell [7] and independently by Sudakov–Tsirelson [25]. Since then many alternative proofs have been proposed, for example, [3, 4, 12], but the issue of completely characterizing the extremals was settled only more recently by Carlen–Kerce [9], establishing that half-spaces are the unique solutions to the Gaussian isoperimetric problem.

The natural issue of proving a quantitative version of the isoperimetric inequality turns out to be a much more delicate task. An estimate in terms of the Fraenkel asymmetry, that is, the Gaussian measure of the symmetric difference between a given set and a half-space, was recently established by Cianchi–Fusco–Maggi–Pratelli [10]. This result provides the sharp decay rate with respect to the Fraenkel asymmetry but with a nonexplicit, dimensionally dependent constant. As for the analogous result in the ground-breaking paper in the Euclidean space [16], the proof is purely geometric and is based on a reflection argument in order to reduce the problem to sets which are $(n-1)$-symmetric. This will cause the constant to blow up at least exponentially with respect to the dimension. However, the fact that in Gauss space most geometric and functional inequalities are independent of the dimension suggests that such a quantitative version of the Gaussian isoperimetric inequality should also be dimension-free. This would also be important for

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possible applications; see [20–22] and the references therein. Indeed, after [10], Mossel–Neeman [20, 21] and Eldan [13] have provided quantitative estimates which are dimension-free but have a sub-optimal decay rate with respect to the Fraenkel asymmetry. It is therefore a natural open problem whether a quantitative estimate holds with a sharp decay rate and, simultaneously, without dimensional dependence.

In this paper, we answer affirmatively to this question. Our result is valid not only for the Fraenkel asymmetry but for a stronger one introduced in [13] which measures the difference of the barycenter of a given set from the barycenter of a half-space. Our quantitative isoperimetric inequality is completely explicit, and it also has the optimal dependence on the mass. The main result is given in terms of the strong asymmetry since in our opinion this is a more natural way to measure the stability of the Gaussian isoperimetric inequality. We will also see that the strong asymmetry appears naturally when one considers an asymmetry which we call the *excess* of the set. This is the Gaussian counterpart of the oscillation asymmetry in the Euclidean setting introduced by Fusco and the third author in [15] (see also [5, 6]).

Subsequent to [16], different proofs in the Euclidean case have been given in [14] (by the optimal transport) and in [1, 11] (using the regularity theory for minimal surfaces and the selection principle). Both of these strategies are rather flexible and have been adopted to prove many other geometric inequalities in a sharp quantitative form. Nevertheless, they do not seem easily implementable for our purpose. Indeed, it is not known if the Gaussian isoperimetric inequality itself can be retrieved from optimal transport (see [28]). On the other hand, the approach via selection principle is by contradiction. Therefore, if it may be adapted to the Gaussian setting, it cannot be used as it is to provide explicit information about the constant in the quantitative isoperimetric inequality. Finally, the proof in [13] is based on stochastic calculus and provides sharp estimates for the Gaussian noise stability inequality. As a corollary, this gives a quantitative estimate for the Gaussian isoperimetric inequality which is, however, not sharp. In order to prove the sharp quantitative estimate we introduce a technique which is based on a direct analysis of the first and the second variation conditions of solutions to a suitable minimization problem. This enables us to obtain the sharp result with a very short proof. We will outline the proof at the end of the Introduction.

In order to describe the problem more precisely, we introduce our setting. Given a Borel set $E \subset \mathbb{R}^n$, $\gamma(E)$ denotes its *Gaussian measure*, defined as

$$\gamma(E) := \frac{1}{(2\pi)^{n/2}} \int_E e^{-|x|^2/2}\, dx.$$

If E is an open set with Lipschitz boundary, $P_\gamma(E)$ denotes its *Gaussian perimeter*, defined as

$$P_\gamma(E) := \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial E} e^{-|x|^2/2}\, d\mathcal{H}^{n-1}(x), \tag{1}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Moreover, given $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$, $H_{\omega,s}$ denotes the half-space of the form

$$H_{\omega,s} := \{x \in \mathbb{R}^n : \langle x, \omega \rangle < s\}.$$

We define also the function $\phi : \mathbb{R} \to (0, 1)$ as

$$\phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-t^2/2}\, dt.$$

Then we have $\gamma(H_{\omega,s}) = \phi(s)$ and $P_\gamma(H_{\omega,s}) = e^{-s^2/2}$. The isoperimetric inequality states that, given an open set E with Lipschitz boundary and mass $\gamma(E) = \phi(s)$, one has

$$P_\gamma(E) \geq e^{-s^2/2}, \tag{2}$$

and the equality holds if and only if $E = H_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.

A natural question is the stability of the inequality (2). Let us denote by $D(E)$ the Gaussian isoperimetric deficit (i.e., the gap between the two sides of the isoperimetric inequality),

$$D(E) := P_\gamma(E) - e^{-s^2/2},$$

and by $\alpha(E)$ the Fraenkel (or the standard) asymmetry,

$$\alpha(E) := \min_{\omega \in \mathbb{S}^{n-1}} \gamma(E \triangle H_{\omega,s}),$$

where $\triangle$ stands for the symmetric difference between sets. As we mentioned, it is proved in [10] that for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the isoperimetric deficit controls the square of the Fraenkel asymmetry, that is,

$$\alpha(E)^2 \leq c(n, s) D(E), \tag{3}$$

and the exponent 2 on the left-hand side is sharp. On the other hand, in [21] a similar estimate is proved (for $s = 0$), with a sub-optimal exponent but with a constant independent of the dimension. The following natural conjecture is stated explicitly in [21], Conjecture 1.8 (see also [20], Open problem 6.1, and the discussion in [13]).

CONJECTURE. *Inequality* (3) *holds for a constant $c(s)$ depending only on the mass s.*

In [13], Eldan introduces a new asymmetry which is equivalent to

$$\beta(E) := \min_{\omega \in \mathbb{S}^{n-1}} |b(E) - b(H_{\omega,s})|, \tag{4}$$

where

$$b(E) := \int_E x\, d\gamma(x)$$

is the (nonrenormalized) barycenter of the set E, and s is chosen such that $\gamma(E) = \phi(s)$. We call this *strong asymmetry* since it controls the standard one as (see Proposition 4)

$$\beta(E) \geq \frac{e^{s^2/2}}{4}\alpha(E)^2. \tag{5}$$

In [13], Corollary 5, it is proved that

$$\beta(E)\left|\log\beta(E)\right|^{-1} \leq c(s)D(E) \tag{6}$$

for an inexplicit constant $c(s)$ depending only on s. Together with (5), this proves the conjecture up to a logarithmic factor. Estimate (6) is derived by the so-called robustness estimate for the Gaussian noise stability, where the presence of the logarithmic term cannot be avoided (see [13], Theorem 2 and discussion in Section 1.1).

In this paper, we fully prove the conjecture. In fact, we prove an even stronger result, since we provide the optimal quantitative estimate in terms of the strong asymmetry. Our main result reads as follows.

MAIN THEOREM. *There exists an absolute constant c such that for every $s \in \mathbb{R}$ and for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the following estimate holds*:

$$\beta(E) \leq c\left(1 + s^2\right)D(E). \tag{7}$$

In Remark 1, we show that the dependence on the mass is optimal. This can be seen by comparing a one-dimensional interval $(-\infty, s)$ with a union of two intervals $(-\infty, -a) \cup (a, \infty)$ with the same Gaussian length. Concerning the numerical value of the constant c, we show that we may consider

$$c = 80\pi^2\sqrt{2\pi},$$

which is not optimal. From (5) and (7), we immediately conclude that for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the following improvement of (3) holds:

$$\alpha(E)^2 \leq 4c\left(1 + s^2\right)e^{-s^2/2}D(E).$$

Finally, since the decay rate with respect to the Fraenkel asymmetry in (3) is sharp this implies that also the linear dependence on $\beta(E)$ in (7) is sharp.

We may state the result of the Main Theorem in a more geometrical way. Define for a given (sufficiently regular) set E its *excess* as

$$\mathcal{E}(E) := \min_{\omega \in \mathbb{S}^{n-1}} \left\{ \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial E} \left|\nu^E - \omega\right|^2 e^{-|x|^2/2}\, d\mathcal{H}^{n-1}(x) \right\},$$

where ν^E is the exterior normal of E. In Corollary 2 at the end of Section 5, we show that for every set E it holds

$$\mathcal{E}(E) = 2D(E) + 2\sqrt{2\pi}\beta(E).$$

Therefore, by the Main Theorem we conclude that the deficit controls also the excess of the set. Roughly speaking, this means that the closer the perimeter of E is to the perimeter of half-space, the flatter its boundary has to be. This is the Gaussian counterpart of the result in [15] for the Euclidean case, and it highlights the importance of the strong asymmetry.

As we already mentioned, the proof of the Main Theorem is based on a direct variational method. The idea is to write the inequality (7) as a minimization problem

$$\min\left\{P_\gamma(E)+\frac{\varepsilon}{2}\left|b(E)\right|^2 : \gamma(E)=\phi(s)\right\}$$

and deduce directly from the first and the second variation conditions that when $\varepsilon>0$ is small enough the only solutions are half-spaces. It is not difficult to see that this is equivalent to the statement of the Main Theorem. In Section 4, we study the regularity of the solutions to the above problem, derive the Euler equation (i.e., the first variation is zero) and the second variation condition. In Section 5, we give the proof of the Main Theorem. The key point of the proof is a careful choice of test functions in the second variation condition, which permits to conclude directly that when ε is sufficiently small every minimizer is a union of parallel stripes. Since this is true in every dimension and the choice of ε does not depend on n, this argument reduces the problem to the one-dimensional case. We give a more detailed overview of the proof in Section 3. Finally, we would like to mention recent works [19, 23] where the authors use the second variation condition to study isoperimetric inequalities in Gauss space.

2. Notation and preliminaries.

In this section, we briefly introduce our basic notation and recall some elementary results from geometric measure theory. For an introduction to the theory of sets of finite perimeter, we refer to [2] and [18].

We denote by $\{e^{(1)},\dots,e^{(n)}\}$ the canonical base of $\mathbb{R}^n$. For generic point $x\in\mathbb{R}^n$, we denote its j-component by $x_j:=\langle x,e^{(j)}\rangle$ and use the notation $x=(x',x_n)$ when we want to specify the last component. Throughout the paper, $B_R(x)$ denotes the open ball centered at x with radius R. When the ball is centered at the origin we simply write B_R. The family of the Borel sets in $\mathbb{R}^n$ is denoted by $\mathcal{B}$. We denote the $(n-1)$-dimensional Hausdorff measure with Gaussian weight by $\mathcal{H}^{n-1}_\gamma$, that is, for every set $A\in\mathcal{B}$ we define

$$\mathcal{H}^{n-1}_\gamma(A):=\frac{1}{(2\pi)^{(n-1)/2}}\int_A e^{-|x|^2/2}\,d\mathcal{H}^{n-1}(x).$$

A set $E\in\mathcal{B}$ has *locally finite perimeter* if $\chi_E\in BV_{\mathrm{loc}}(\mathbb{R}^n)$, that is, for every ball $B_R\subset\mathbb{R}^n$ it holds

$$\sup\left\{\int_E \operatorname{div}\varphi\,dx : \varphi\in C_0^\infty(B_R;\mathbb{R}^n),\ \sup|\varphi|\le 1\right\}<\infty.$$

If E is a set of locally finite perimeter, we define the *reduced boundary* $\partial^* E$ of E as the set of all points $x \in \mathbb{R}^n$ such that

$$\nu^E(x) := -\lim_{r\to 0^+} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \qquad \text{exists and belongs to } \mathbb{S}^{n-1}.$$

The reduced boundary $\partial^* E$ is a subset of the topological boundary ∂E and coincides, up to a $\mathcal{H}^{n-1}$-negligible set, with the support of $D\chi_E$. When E is an open set with Lipschitz boundary, then $\mathcal{H}^{n-1}(\partial E \Delta \partial^* E) = 0$ [18], Example 12.6. We shall refer to the vector $\nu^E(x)$ as the (*generalized*) *exterior normal* at $x \in \partial^* E$. For more information, we refer to [2], Definition 3.54. When no confusion arises we shall simply write ν and use the notation $\nu_j = \langle \nu, e^{(j)} \rangle$. If E has a locally finite perimeter, then its perimeter in $A \in \mathcal{B}$ is

$$P(E; A) := \mathcal{H}^{n-1}(\partial^* E \cap A).$$

Moreover, by the divergence theorem we have

$$\int_E \operatorname{div} X \, dx = \int_{\partial^* E} \langle X, \nu^E \rangle \, d\mathcal{H}^{n-1}(x)$$

for every Lipschitz continuous vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ with compact support.

In (2), the Gaussian isoperimetric problem was stated for sets with Lipschitz boundary, but this can be extended to more general and more natural class of sets. Indeed, if $E \in \mathcal{B}$ is a set of locally finite perimeter with $\mathcal{H}^{n-1}_\gamma(\partial^* E) < \infty$, then it has a *finite Gaussian perimeter* and we denote its Gaussian perimeter by

$$P_\gamma(E) := \mathcal{H}^{n-1}_\gamma(\partial^* E).$$

Otherwise, we set $P_\gamma(E) := \infty$. It follows from the divergence theorem that

$$\begin{aligned} P_\gamma(E) = \sqrt{2\pi} \sup \bigg\{ \int_E \big(\operatorname{div} \varphi - \langle \varphi, x \rangle\big) \, d\gamma(x) : \varphi \in C_0^\infty\big(B_R; \mathbb{R}^n\big), \\ \sup |\varphi| \le 1 \bigg\} \end{aligned} \tag{8}$$

for every $E \in \mathcal{B}$. If not otherwise specified, throughout we assume that every set has a finite Gaussian perimeter. The above notion of a Gaussian perimeter provides an extension of (1) because, if E is an open set with Lipschitz boundary, then ∂E and $\partial^* E$ coincide up to a $\mathcal{H}^{n-1}$-negligible set.

We recall some notation for calculus on smooth hypersurfaces (see [18], Section 11.3). Let us fix a set $E \subset \mathbb{R}^n$ and assume that there is an open set $U \subset \mathbb{R}^n$ such that $M = \partial E \cap U$ is a C^∞ hypersurface. Assume that we have a vector field $X \in C^\infty(M; \mathbb{R}^n)$. Since the manifold M is smooth we may extend X to U so that $X \in C^\infty(U; \mathbb{R}^n)$. We define the tangential differential of X on M by

$$D_\tau X(x) := DX(x) - \big(DX(x)\nu^E(x)\big) \otimes \nu^E(x) \qquad x \in M,$$

where $\otimes$ denotes the tensor product. It is clear that $D_\tau X$ depends only on the values of X at M, not on the chosen extension. The tangential divergence of X on M is defined by

$$\operatorname{div}_\tau X := \operatorname{Trace}(D_\tau X) = \operatorname{div} X - \langle DX\nu^E, \nu^E\rangle.$$

Similarly, given a function $u \in C^\infty(M)$ we extend it to U and define its tangential gradient by

$$D_\tau u := Du - \langle Du, \nu^E\rangle\nu^E.$$

We define the tangential derivative of u in direction $e^{(i)}$ as

$$\delta_i u := \langle D_\tau u, e^{(i)}\rangle = \partial_{x_i} u - \langle \nabla u, \nu\rangle \nu_i.$$

The tangential Laplacian of u on M is

$$\Delta_\tau u := \operatorname{div}_\tau (D_\tau u) = \sum_{i=1}^n \delta_i(\delta_i u).$$

Since M is smooth, the exterior normal is a smooth vector field $\nu^E \in C^\infty(M; \mathbb{R}^n)$. Then the sum $\mathscr{H}(x)$ of the principal curvatures at $x \in M$ is given by

$$\mathscr{H}(x) = \operatorname{div}_\tau\big(\nu^E(x)\big).$$

We denote by $|B_E|^2$ the sum of the squares of the principal curvatures, which can be written as

$$|B_E|^2 = \operatorname{Trace}\big(D_\tau \nu^E D_\tau \nu^E\big) = \sum_{i,j=1}^n (\delta_i \nu_j)^2.$$

Note that $D_\tau \nu^E$ is symmetric, that is, $\delta_i \nu_j = \delta_j \nu_i$ (see [17], formula (10.11)). Finally, the Gauss–Green theorem, or the divergence theorem, on hypersurfaces states that for every $X \in C_0^\infty(M; \mathbb{R}^n)$ it holds

$$\int_M \operatorname{div}_\tau X \, d\mathcal{H}^{n-1}(x) = \int_M \mathscr{H}\langle X, \nu^E\rangle \, d\mathcal{H}^{n-1}(x).$$

3. Overview of the proof. As we wrote in the Introduction, we will derive our main estimate (7) by a suitable minimization problem. To this aim, given $\varepsilon > 0$ and $s \le 0$, we consider the functional

$$\mathcal{F}(E) = P_\gamma(E) + \frac{\varepsilon}{2}\big|b(E)\big|^2, \qquad \gamma(E) = \phi(s).$$

In fact, in the proof we replace the volume constraint by a volume penalization, but this is of little importance. For simplicity, we will indicate by b_s the norm of the barycenter $b(H_{\omega,s})$, since it does not depend on ω. We have $b(H_{\omega,s}) = -b_s\omega$

and $b_s = e^{-s^2/2}/\sqrt{2\pi}$. It is important to observe that the half-spaces maximize the norm of the barycenter,

$$b_s \ge |b(E)| \tag{9}$$

for every set E such that $\gamma(E) = \phi(s)$. Indeed, if $b(E) \neq 0$, by taking $\omega = -b(E)/|b(E)|$, we have

$$\begin{aligned} |b(E)| - b_s &= \langle b(E) + b_s\omega, -\omega\rangle \\ &= -\int_E \langle x, \omega\rangle \, d\gamma(x) + \int_{H_{\omega,s}} \langle x, \omega\rangle \, d\gamma(x) \\ &= \int_{E\setminus H_{\omega,s}} \big(\langle x, -\omega\rangle + s\big) \, d\gamma(x) + \int_{H_{\omega,s}\setminus E} \big(\langle x, \omega\rangle - s\big) \, d\gamma(x) \\ &\le 0, \end{aligned}$$

because the integrands in the last term are both negative. This enlightens the fact that in minimizing $\mathcal{F}$ the two terms $P_\gamma(E)$ and $|b(E)|$ are in competition. Minimizing $P_\gamma(E)$ means to push the set E at infinity in one direction, so that it becomes closer to a half-space. On the other hand, minimizing $|b(E)|$ means to balance the mass of E with respect to the origin. We will see, and this is the *main point of our analysis*, that for ε small enough the perimeter term overcomes the barycenter, and the only minimizers of $\mathcal{F}$ are the half-spaces $H_{\omega,s}$.

We have observed that the half-spaces maximize the norm of the barycenter. When $b(E) \neq 0$, the minimum in (4) is attained by $\omega = -b(E)/|b(E)|$ and with this choice of ω we have

$$\beta(E) = |b(E) + b_s\omega| = \big|\big(-|b(E)| + b_s\big)\omega\big| = b_s - |b(E)|.$$

Therefore, the strong asymmetry is nothing else than the gap between the maximum b_s and the norm of $b(E)$. If we show that for some ε and Λ (only depending on s) the only minimizers of the functional $\mathcal{F}$ are the half-spaces $H_{\omega,s}$, $\omega \in \mathbb{S}^{n-1}$, we are done, since this implies that for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ it holds

$$\begin{aligned} D(E) &\ge \frac{\varepsilon}{2}\big(b_s^2 - |b(E)|^2\big) = \frac{\varepsilon}{2}\big(b_s + |b(E)|\big)\beta(E) \\ &\ge \frac{\varepsilon}{2\sqrt{2\pi}} e^{-s^2/2}\beta(E). \end{aligned} \tag{10}$$

Since the proof involves many technicalities, we will carry out a sketch of the argument in order to enlighten the core ideas. The proof is divided in two parts. First, we prove standard results concerning the minimizers of $\mathcal{F}$, such as the existence and the regularity of minimizers and derive the Euler equation and the second variation condition. The existence of a minimizer follows directly from a compactness argument using the lower semicontinuity of the Gaussian perimeter.

The regularity is a consequence of the regularity theory for almost minimizers of the perimeter.

The derivation of the Euler equation is standard but we prefer to sketch the argument here. Let E be a minimizer of $\mathcal{F}$ and assume that its boundary is a smooth hypersurface. Given a function $\varphi \in C^\infty(\partial E)$ with zero average, $\int_{\partial E} \varphi \, d\mathcal{H}_\gamma^{n-1}(x) = 0$, we choose a specific vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ such that $X := \varphi \nu^E$ on ∂E. Let $\Phi : \mathbb{R}^n \times (-\delta, \delta) \to \mathbb{R}^n$ be the flow associated with X, that is,

$$\frac{\partial}{\partial t}\Phi(x,t) = X\big(\Phi(x,t)\big), \qquad \Phi(x,0) = x.$$

We perturb E through the flow Φ by defining $E_t := \Phi(E,t)$ for $t \in (-\delta, \delta)$. The zero average condition on φ guarantees that we may choose X in such a way that the flow preserves the volume up to a small error, that is, $\gamma(E_t) = \gamma(E) + o(t^2)$. Then the *first variation* condition for the minimizer

$$\frac{\partial}{\partial t}\mathcal{F}(E_t)\Big|_{t=0} = 0$$

leads to the Euler equation

$$\mathscr{H} - \langle x, \nu\rangle + \varepsilon\langle b, x\rangle = \lambda \qquad \text{on } \partial E,$$

where $b = b(E)$ is the barycenter of E, $\nu = \nu^E$ the exterior normal of ∂E, and λ is the Lagrange multiplier. Furthermore, the *second variation* condition for the minimizer

$$\frac{\partial^2}{\partial t^2}\mathcal{F}(E_t)\Big|_{t=0} \geq 0$$

leads to

$$\begin{aligned} &\int_{\partial E} \big(|D_\tau \varphi|^2 - |B_E|^2\varphi^2 - \varphi^2 + \varepsilon\langle b, \nu\rangle\varphi^2\big)\, d\mathcal{H}_\gamma^{n-1}(x) \\ &\qquad + \varepsilon\left|\int_{\partial E} \varphi x \, d\mathcal{H}_\gamma^{n-1}(x)\right|^2 \geq 0. \end{aligned} \tag{11}$$

In the second part, we effectively use the Euler equation and the second variation condition to prove that half spaces are the unique minimizers of $\mathcal{F}$. Given a minimizer E, assume (without loss of generality) that its barycenter is in direction $-e^{(n)}$, that is, $b(E) = -|b|e^{(n)}$. As we said, we have to show that $E = H_{e^n,s}$. In order to understand how the profile of the set E varies in the directions perpendicular to e^n, the *key idea* is to use as φ the functions ν_j, $j \in \{1, \dots, n-1\}$, where $\nu_j = \langle \nu, e^{(j)}\rangle$. We are allowed to do this because ν_j has zero average [see (38)]. From the Euler equation, we get

$$\left|\int_{\partial E} \nu_j x \, d\mathcal{H}_\gamma^{n-1}(x)\right|^2 \leq C \int_{\partial E} \nu_j^2 \, d\mathcal{H}_\gamma^{n-1}(x)$$

for some C depending on s (but not on n). Therefore, when ε is small enough the second variation condition (11) provides the inequality

$$\int_{\partial E}\left(|D_\tau \nu_j|^2 - |B_E|^2\nu_j^2 - \varepsilon|b|\nu_n\nu_j^2 - \frac{1}{2}\nu_j^2\right) d\mathcal{H}_\gamma^{n-1}(x) \geq 0. \tag{12}$$

Let δ_j be the tangential derivative in $e^{(j)}$-direction and Δ_τ the tangential Laplacian. By differentiating the Euler equation with respect to δ_j and by using the geometric equality

$$\Delta_\tau \nu_j = -|B_E|^2\nu_j + \delta_j \mathscr{H} \qquad \text{on } \partial E$$

we deduce

$$\Delta_\tau \nu_j - \langle D_\tau \nu_j, x\rangle = -|B_E|^2\nu_j - \varepsilon|b|\nu_n\nu_j \qquad \text{on } \partial E.$$

We multiply the above equation by ν_j, integrate it over ∂E and use the divergence theorem on hypersurfaces to get

$$\int_{\partial E}\left(|D_\tau \nu_j|^2 - |B_E|^2\nu_j^2 - \varepsilon|b|\nu_n\nu_j^2\right) d\mathcal{H}_\gamma^{n-1}(x) = 0. \tag{13}$$

By comparing (12) and (13), we conclude that necessarily $\nu_j \equiv 0$ on ∂E, that is, E is constituted by strips perpendicular to e^n. To complete the proof, we show that ∂E is connected, which implies that E is the half-space $H_{e^n,s}$.

4. Minimization problem. In this section, we study the functional $\mathcal{F}: \mathcal{B} \to \mathbb{R}^+$ defined by

$$\mathcal{F}(E) = P_\gamma(E) + \frac{\varepsilon}{2}\left|b(E)\right|^2 + \Lambda\left|\gamma(E) - \phi(s)\right|, \tag{14}$$

where $\varepsilon > 0$, $\Lambda > 0$, and $s \leq 0$ are given. The last term is a volume penalization that forces (for Λ large enough) the minimizers of $\mathcal{F}$ to have Gaussian measure $\phi(s)$. We first prove the existence of minimizers and then study their regularity. We calculate also the Euler equation and the second variation of $\mathcal{F}$. All these results are nowadays standard, but for the reader's convenience we prefer to give the proofs. Specific properties of the minimizers will be analyzed in the next section, along the proof of our Main Theorem.

PROPOSITION 1. *The functional $\mathcal{F}$ has a minimizer.*

PROOF. Consider a sequence E_h in $\mathcal{B}$ such that

$$\lim_{h\to\infty} \mathcal{F}(E_h) = \inf\{\mathcal{F}(F) : F \in \mathcal{B}\}.$$

Since for any bounded open set $A \subset \mathbb{R}^n$, one has that $\sup_h P(E_h; A)$ is finite, the compactness theorem for BV functions (see [2], Theorem 3.23) ensures the existence of a Borel set $E \subset \mathbb{R}^n$ such that, up to a subsequence, $\chi_{E_h} \to \chi_E$ strongly in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Given $R > 0$, let r_h and r be such that

$$\phi(r_h) = \gamma(E_h \setminus B_R) \quad \text{and} \quad \phi(r) = \gamma(\mathbb{R}^n \setminus B_R).$$

From inequality (9), we get

$$\left|\int_{E_h \setminus B_R} x \, d\gamma(x)\right| \le \frac{e^{-r_h^2/2}}{\sqrt{2\pi}} \le \frac{e^{-r^2/2}}{\sqrt{2\pi}}.$$

A similar estimate holds also for the set $F \setminus B_R$. Therefore, since

$$\left|\int_{E_h} x \, d\gamma(x) - \int_E x \, d\gamma(x)\right| \le \left|\int_{\mathbb{R}^n} (\chi_{E_h} - \chi_E)\chi_{B_R} x \, d\gamma(x)\right| + \frac{2e^{-r^2/2}}{\sqrt{2\pi}},$$

we have that $b(E) = \lim_{h\to\infty} b(E_h)$. Equation (8) implies that the Gaussian perimeter is lower semicontinuous with respect to L^1_{loc} convergence of sets, namely $P_\gamma(E) \le \liminf_{h\to\infty} P_\gamma(E_h)$, so that $\mathcal{F}(E) \le \mathcal{F}(F)$ for every set $F \in \mathcal{B}$. $\square$

The regularity of the minimizers of $\mathcal{F}$ follows from the regularity theory for almost minimizers of the perimeter [26]. From the regularity point of view the advantage of having the strong asymmetry in the functional (14) instead of the standard one is that the minimizers are smooth outside the singular set. The fact that one may gain regularity by replacing the standard asymmetry by a stronger one is also observed in a different context in [8].

PROPOSITION 2. *Let E be a minimizer of $\mathcal{F}$ defined in* (14). *Then the reduced boundary $\partial^* E$ is a relatively open, smooth hypersurface and satisfies the Euler equation*

$$\mathscr{H} - \langle x, \nu\rangle + \varepsilon\langle b, x\rangle = \lambda \qquad \text{on } \partial^* E, \tag{15}$$

where $b = b(E)$ and $\nu = \nu^E$. Here, λ is the Lagrange multiplier which can be estimated by

$$|\lambda| \le \Lambda.$$

The singular part of the boundary $\partial E \setminus \partial^* E$ is empty when $n < 8$, while for $n \ge 8$ its Hausdorff dimension can be estimated by $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \le n - 8$.

PROOF. First of all, we note that ∂E is the topological boundary of a properly chosen representative of the set (see [18], Proposition 12.19).

Let us fix $x_0 \in \partial E$ and $r \in (0, 1)$. From the minimality we deduce that for every set $F \subset \mathbb{R}^n$ with locally finite perimeter such that $F \triangle E \subset B_{2r}(x_0)$ it holds

$$P_\gamma(E) \le P_\gamma(F) + C\gamma(F \triangle E) \tag{16}$$

for some constant C depending on $|x_0|$. If we choose $F = E \cup B_r(x_0)$ we get from (16) that

$$P_\gamma(E) \le P_\gamma\big(E \cup B_r(x_0)\big) + C\gamma\big(B_r(x_0)\big).$$

On the other hand, arguing as in [18], Lemma 12.22, we obtain

$$P_\gamma\big(E \cup B_r(x_0)\big) + P_\gamma\big(E \cap B_r(x_0)\big) \le P_\gamma(E) + P_\gamma\big(B_r(x_0)\big).$$

The previous two inequalities yield

$$P_\gamma\big(E \cap B_r(x_0)\big) \le P_\gamma\big(B_r(x_0)\big) + C\gamma\big(B_r(x_0)\big) \le Cr^{n-1}.$$

The left-hand side can be estimated simply by

$$P_\gamma\big(E \cap B_r(x_0)\big) \ge ce^{-|x_0|^2} P\big(E; B_r(x_0)\big).$$

Therefore, we obtain

$$P\big(E; B_r(x_0)\big) \le C_0 r^{n-1} \tag{17}$$

for some constant $C_0 = C_0(|x_0|)$. Note that for every $x \in B_r(x_0)$ and $r \in (0, 1)$ it holds

$$\big|e^{-|x|^2/2} - e^{-|x_0|^2/2}\big| \le Cr$$

for some constant C. Therefore, (16) and (17) imply that for all sets F with $F \triangle E \subset\subset B_r(x_0)$ and $r \le 1$ it holds

$$P\big(E; B_r(x_0)\big) \le P\big(F; B_r(x_0)\big) + Cr^n$$

for some constant C depending on $|x_0|$. It follows from [26], Theorem 1.9 (see also [18], Theorem 21.8) that $\partial^* E$ is a relatively open (in ∂E) $C^{1,\sigma}$ hypersurface for every $\sigma < 1/2$, and that the singular set $\partial E \setminus \partial^* E$ is empty when $n < 8$, while $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \le n - 8$ when $n \ge 8$.

Let us next prove that $\partial^* E$ satisfies the Euler equation (15). Since $\partial^* E$ is relatively open we find an open set $U \subset \mathbb{R}^n$ such that $\partial E \cap U = \partial^* E$. Let us first prove that for every $X \in C_0^1(U; \mathbb{R}^n)$ with $\int_{\partial^* E} \langle X, \nu\rangle \, d\mathcal{H}_\gamma^{n-1}(x) = 0$ we have

$$\int_{\partial^* E} \operatorname{div}_\tau X - \langle X, x\rangle \, d\mathcal{H}_\gamma^{n-1}(x) + \varepsilon \int_{\partial^* E} \langle b, x\rangle \langle X, \nu\rangle \, d\mathcal{H}_\gamma^{n-1}(x) = 0. \tag{18}$$

To this aim let $\Phi : U \times (-\delta, \delta) \to U$ be the flow associated with X, that is,

$$\frac{\partial}{\partial t}\Phi(x, t) = X\big(\Phi(x, t)\big), \qquad \Phi(x, 0) = x.$$

There exists a time interval $(-\delta, \delta)$ such that the flow Φ is defined in $U \times (-\delta, \delta)$, it is C^1 regular and for every $t \in (-\delta, \delta)$ the map $x \mapsto \Phi(x, t)$ is a local C^1 diffeomorphism [27], Theorem 6.1. Because X vanishes near the boundary of U, $\Phi(x, t) = x$ for every point x near ∂U. With this in mind, we extend the flow to every $t \in (-\delta, \delta)$ and $x \in \mathbb{R}^n \setminus U$ by $\Phi(x, t) = x$. Then for small values of t the map $x \mapsto \Phi(x,t)$ is a C^1 diffeomorphism. We define $E_t := \Phi(E,t)$. Let us denote the Jacobian of $\Phi(\cdot,t)$ by $J\Phi(x,t)$ and the tangential Jacobian on $\partial^* E$ by $J_\tau \Phi(x,t)$. We recall the formulas (see [24])

$$\frac{\partial}{\partial t}\Big|_{t=0} J\Phi(x,t) = \operatorname{div} X \quad \text{and} \quad \frac{\partial}{\partial t}\Big|_{t=0} J_\tau \Phi(\cdot,t) = \operatorname{div}_\tau X. \tag{19}$$

Note also that by definition $\frac{\partial}{\partial t}|_{t=0}\Phi(x,t) = X(x)$ and $\Phi(x,0) = x$. Then we have by change of variables

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} \gamma(E_t) &= \frac{\partial}{\partial t}\Big|_{t=0} \left(\int_E e^{-|\Phi(x,t)|^2/2} J\Phi(x,t)\, dx \right) \\ &= \int_E \big(\operatorname{div} X - \langle X, x\rangle\big) e^{-|x|^2/2}\, dx \\ &= \int_E \operatorname{div}\big(e^{-|x|^2/2} X\big)\, dx \\ &= \int_{\partial^* E} \langle X, \nu\rangle \, d\mathcal{H}^{n-1}_\gamma(x) = 0. \end{aligned} \tag{20}$$

This means that X produces a zero first-order volume variation of E and, therefore,

$$\frac{\partial}{\partial t}\Big|_{t=0} \big|\gamma(E_t) - \phi(s)\big| = 0.$$

We obtain the formula (18) by the minimality of E and by change of variables

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} P_\gamma(E_t) &= \frac{\partial}{\partial t}\Big|_{t=0} \left(\int_{\partial^* E} \big(e^{-|\Phi(x,t)|^2/2} J_\tau \Phi(x,t)\big)\, d\mathcal{H}^{n-1}(x) \right) \\ &= \int_{\partial^* E} \operatorname{div}_\tau X - \langle X, x\rangle \, d\mathcal{H}^{n-1}_\gamma(x) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} |b(E_t)|^2 &= \frac{\partial}{\partial t}\Big|_{t=0} \left| \int_E \big(\Phi(x,t) e^{-|\Phi(x,t)|^2/2} J\Phi(x,t)\big)\, dx \right|^2 \\ &= 2\int_E \big(\langle b, X\rangle - \langle b, x\rangle\langle X, x\rangle + \langle b, x\rangle \operatorname{div} X\big) e^{-|x|^2/2}\, dx \\ &= 2\int_E \operatorname{div}\big(\langle b, x\rangle e^{-|x|^2/2} X\big)\, dx \\ &= 2\int_{\partial^* E} \langle b, x\rangle\langle X, \nu\rangle \, d\mathcal{H}^{n-1}_\gamma(x). \end{aligned}$$

We use (18) to show that $\partial^* E$ satisfies the Euler equation (15) in a weak sense, that is, there exists a number $\lambda \in \mathbb{R}$ such that for every $X \in C^1_0(U;\mathbb{R}^n)$ we have

$$\begin{aligned} &\int_{\partial^* E} \operatorname{div}_\tau X - \langle X, x\rangle \, d\mathcal{H}^{n-1}_\gamma(x) + \varepsilon \int_{\partial^* E} \langle b, x\rangle\langle X, \nu\rangle \, d\mathcal{H}^{n-1}_\gamma(x) \\ &\quad = \lambda \int_{\partial^* E} \langle X, \nu\rangle \, d\mathcal{H}^{n-1}_\gamma(x). \end{aligned} \tag{21}$$

Let $X_1, X_2 \in C_0^1(U;\mathbb{R}^n)$ be such that $\int_{\partial^* E}\langle X_i, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x) \neq 0$, $i=1,2$. Denote $\alpha_1 = \int_{\partial^* E}\langle X_1, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)$ and $\alpha_2 = \int_{\partial^* E}\langle X_2, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)$, and define

$$X := X_1 - \frac{\alpha_1}{\alpha_2} X_2.$$

Then $X \in C_0^1(U;\mathbb{R}^n)$ satisfies $\int_{\partial^* E}\langle X, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x) = 0$ and (18) implies

$$\begin{aligned}
&\frac{1}{\alpha_1}\Big(\int_{\partial^* E} \operatorname{div}_\tau X_1 - \langle X_1, x\rangle\, d\mathcal{H}_\gamma^{n-1}(x) + \varepsilon \int_{\partial^* E}\langle b, x\rangle\langle X_1, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)\Big)\\
&\quad = \frac{1}{\alpha_2}\Big(\int_{\partial^* E} \operatorname{div}_\tau X_2 - \langle X_2, x\rangle\, d\mathcal{H}_\gamma^{n-1}(x) + \varepsilon \int_{\partial^* E}\langle b, x\rangle\langle X_2, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)\Big).
\end{aligned}$$

Therefore, there exists $\lambda \in \mathbb{R}$ such that (21) holds.

Since the reduced boundary $\partial^* E$ is a $C^{1,\sigma}$ manifold and since it satisfies the Euler equation (15) in a weak sense, from classical Schauder estimates we deduce that $\partial^* E$ is in fact a C^∞ hypersurface. In particular, we conclude that the Euler equation (15) holds pointwise on $\partial^* E$.

Finally, in order to bound the Lagrange multiplier λ, let $X \in C_0^1(U;\mathbb{R}^n)$ be any vector field, and let $\Phi(x,t)$, $E_t = \Phi(E,t)$ be as above. Then by the above calculations we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_{t=0}\Big(P_\gamma(E_t) + \frac{\varepsilon}{2}|b(E_t)|^2\Big)\\
&\qquad = \int_{\partial^* E} \operatorname{div}_\tau X - \langle X, x\rangle + \varepsilon\langle b, x\rangle\langle X, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)\\
&\qquad = \int_{\partial^* E} \big(\mathscr{H} - \langle x, \nu\rangle + \varepsilon\langle b, x\rangle\big)\langle X, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)\\
&\qquad = \lambda \int_{\partial^* E}\langle X, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{t\to 0} \frac{|\gamma(E_t) - \phi(s)| - |\gamma(E) - \phi(s)|}{t} &\le \Big|\frac{\partial}{\partial t}\Big|_{t=0}\gamma(E_t)\Big|\\
&= \Big|\int_{\partial^* E}\langle X, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)\Big|.
\end{aligned}$$

Therefore, by the minimality of E we have

$$\lambda \int_{\partial^* E}\langle X, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x) + \Lambda\Big|\int_{\partial^* E}\langle X, \nu\rangle\, d\mathcal{H}_\gamma^{n-1}(x)\Big| \ge 0$$

for every $X \in C_0^1(U;\mathbb{R}^n)$. This proves the claim. $\square$

Next, we derive the second-order condition for minimizers of the functional $\mathcal{F}$, that is, the quadratic form associated with the second variation is nonnegative. Let us briefly explain what we mean by this. Let $\varphi : \partial^* E \to \mathbb{R}$ be a smooth function with compact support such that it has zero average, that is, $\int_{\partial^* E} \varphi \, d\mathcal{H}^{n-1}_\gamma(x) = 0$. We choose a specific vector field $X : \mathbb{R}^n \to \mathbb{R}^n$, such that $X := \varphi \nu^E$ on $\partial^* E$. We denote the associated flow by Φ and define $E_t := \Phi(E, t)$. We note that since φ has zero average then by (20) X produces a zero first-order volume variation of E. This enables us to define X in such a way that the volume variation produced by X is zero up to second order, that is, $\gamma(E_t) = \gamma(E) + o(t^2)$ [see (22) and (24)]. Therefore, under the condition that φ has zero average the volume penalization term in the functional $\mathcal{F}$ is negligible. The second variation of the functional $\mathcal{F}$ at E in the direction φ is then defined to be the value

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{F}(E_t).$$

It turns out that the choice of the vector field X ensures that the second derivative exists and it follows from the minimality of E that this value is non-negative. Moreover, the second variation at E defines a quadratic form over all functions $\varphi \in C_0^\infty(\partial^* E)$ with zero average.

The calculations of the second variation are standard (see [1, 18, 19, 24] for similar cases) but since they are technically challenging we include them for the reader's convenience. We note that since ∂E is not necessarily smooth we may only perturb the regular part of the boundary. We write $u \in C_0^\infty(\partial^* E)$ when $u : \partial^* E \to \mathbb{R}$ is a smooth function with compact support.

PROPOSITION 3. *Let E be a minimizer of $\mathcal{F}$. The quadratic form associated with the second variation is nonnegative*

$$\begin{aligned} J[\varphi] := &\int_{\partial^* E} \big(|D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \varphi^2 + \varepsilon \langle b, \nu \rangle \varphi^2\big) \, d\mathcal{H}^{n-1}_\gamma(x) \\ &+ \varepsilon \left| \int_{\partial^* E} \varphi x \, d\mathcal{H}^{n-1}_\gamma(x) \right|^2 \geq 0 \end{aligned}$$

for every $\varphi \in C_0^\infty(\partial^* E)$ which satisfies

$$\int_{\partial^* E} \varphi \, d\mathcal{H}^{n-1}_\gamma(x) = 0.$$

Here, $b = b(E)$ and $\nu = \nu^E$, while $|B_E|^2$ is the sum of the squares of the curvatures.

PROOF. Assume that $\varphi \in C_0^\infty(\partial^* E)$ satisfies $\int_{\partial^* E} \varphi \, d\mathcal{H}^{n-1}_\gamma(x) = 0$. Let $d_E : \mathbb{R}^n \to \mathbb{R}$ be the signed distance function of E

$$d_E(x) := \begin{cases} \operatorname{dist}(x, \partial E), & \text{for } x \in \mathbb{R}^n \setminus E, \\ -\operatorname{dist}(x, \partial E), & \text{for } x \in E. \end{cases}$$

It follows from Proposition 2 that there is an open set $U \subset \mathbb{R}^n$ such that d_E is smooth in U and the support of φ is in $\partial^* E \cap U$. We extend φ to U, and call the extension simply by φ, so that $\varphi \in C_0^\infty(U)$ and

$$\partial_\nu \varphi = \big(\langle x, \nu\rangle - \mathscr{H}\big)\varphi \qquad \text{on } \partial^* E. \tag{22}$$

Finally, we define the vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ by $X := \varphi \nabla d_E$ in U and $X := 0$ in $\mathbb{R}^n \setminus U$. Note that X is smooth and $X = \varphi\nu$ on $\partial^* E$.

Let $\Phi : \mathbb{R}^n \times (-\delta, \delta) \to \mathbb{R}^n$ be the flow associated with X, that is,

$$\frac{\partial}{\partial t}\Phi(x,t) = X\big(\Phi(x,t)\big), \qquad \Phi(x,0) = x$$

and define $E_t = \Phi(E,t)$. Let us denote the Jacobian of $\Phi(\cdot,t)$ by $J\Phi(x,t)$ and the tangential Jacobian on $\partial^* E$ by $J_\tau \Phi(x,t)$. We recall the formulas (19) and also (see again [24]) the formulas

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\bigg|_{t=0} J\Phi(x,t) = \operatorname{div}\big((\operatorname{div} X) X\big), \\ &\frac{\partial^2}{\partial t^2}\bigg|_{t=0} J_\tau \Phi(\cdot,t) = \big|(D_\tau X)^T \nu\big|^2 + (\operatorname{div}_\tau X)^2 + \operatorname{div}_\tau Z - \operatorname{Tr}(D_\tau X)^2, \end{aligned} \tag{23}$$

where $Z := \frac{\partial^2 \Phi(x,t)}{\partial t^2}|_{t=0}$ is the acceleration field. Recall also that by definition $\Phi(x,0) = x$ and $\frac{\partial}{\partial t}|_{t=0}\Phi(x,t) = X$.

We begin by differentiating the Gaussian volume. Similarly to (20), by a change of variables we use (19) and (23) to calculate

$$\frac{\partial}{\partial t}\bigg|_{t=0} \gamma(E_t) = \int_{\partial^* E} \varphi \, d\mathcal{H}_\gamma^{n-1}(x) = 0$$

and

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \gamma(E_t) &= \int_E \operatorname{div}\big(\operatorname{div}\big(X e^{-|x|^2/2}\big) X\big)\, dx \\ &= \int_{\partial^* E} \varphi \partial_\nu \varphi + \big(\mathscr{H} - \langle x, \nu\rangle\big)\varphi^2 \, d\mathcal{H}_\gamma^{n-1}(x) = 0, \end{aligned} \tag{24}$$

where the last equality comes from (22). Hence, $\gamma(E_t) = \gamma(E) + o(t^2)$ and

$$\frac{\partial^2}{\partial t^2}\bigg|_{t=0} \big|\gamma(E_t) - \phi(s)\big| = 0.$$

Since $t \mapsto P_\gamma(E_t)$ and $t \mapsto |b(E_t)|^2$ are smooth with respect to t we have by the minimality of E that

$$0 \le \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \mathcal{F}(E_t) = \frac{\partial^2}{\partial t^2}\bigg|_{t=0} P_\gamma(E_t) + \frac{\varepsilon}{2}\frac{\partial^2}{\partial t^2}\bigg|_{t=0} \big|b(E_t)\big|^2. \tag{25}$$

Thus, we need to differentiate the perimeter and the barycenter.

To differentiate the perimeter, we write

$$P_\gamma(E_t) = \int_{\partial^* E} e^{-|\Phi(x,t)|^2/2} J_\tau \Phi(x,t)\, d\mathcal{H}^{n-1}(x).$$

We differentiate this twice and use (23) to get

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\bigg|_{t=0} P_\gamma(E_t) \\
&\quad = \int_{\partial^* E} \big(|(D_\tau X)^T \nu|^2 + (\operatorname{div}_\tau X)^2 + \operatorname{div}_\tau Z - \operatorname{Tr}(D_\tau X)^2\big)\, d\mathcal{H}^{n-1}_\gamma(x) \\
&\qquad + \int_{\partial^* E} \big(-2 \operatorname{div}_\tau X \langle X, x\rangle - \langle Z, x\rangle - |X|^2 + \langle X, x\rangle^2\big)\, d\mathcal{H}^{n-1}_\gamma(x) \\
&\quad = \int_{\partial^* E} \big(|D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \varphi^2\big)\, d\mathcal{H}^{n-1}_\gamma(x) \\
&\qquad + \int_{\partial^* E} \big(\mathscr{H} - \langle x, \nu\rangle\big)\big(\varphi \partial_\nu \varphi + \big(\mathscr{H} - \langle x, \nu\rangle\big)\varphi^2\big)\, d\mathcal{H}^{n-1}_\gamma(x).
\end{aligned} \tag{26}$$

Let us denote $b_t = b(E_t)$, $\dot{b} = \frac{\partial}{\partial t}|_{t=0} b_t$ and $\ddot{b} = \frac{\partial^2}{\partial t^2}|_{t=0} b_t$. Then

$$\frac{\partial^2}{\partial t^2}\bigg|_{t=0} |b_t|^2 = 2\langle b, \ddot{b}\rangle + 2|\dot{b}|^2.$$

To differentiate the barycenter we write

$$b_t = \int_E \Phi(x,t) e^{-|\Phi(x,t)|^2/2} J\Phi(x,t)\, dx.$$

We use (19) and (23), and get after differentiating once that

$$\dot{b} = \int_{\partial^* E} \varphi x \, d\mathcal{H}^{n-1}_\gamma(x) \tag{27}$$

and after differentiating twice that

$$\begin{aligned}
\ddot{b} &= \int_E \big(x \operatorname{div}\big((\operatorname{div} X) X\big) + 2X(\operatorname{div} X) - 2x\langle X, x\rangle(\operatorname{div} X) \\
&\quad - 2X\langle X, x\rangle\big) e^{-|x|^2/2}\, dx \\
&\quad + \int_E \big((DX)X + x\langle X, x\rangle^2 - x\langle DXX, x\rangle - x|X|^2\big) e^{-|x|^2/2}\, dx \\
&= \int_E \big((DX)X e^{-|x|^2/2} + 2X \operatorname{div}\big(X e^{-|x|^2/2}\big) \\
&\quad + x \operatorname{div}\big(\operatorname{div}\big(X e^{-|x|^2/2}\big) X\big)\big)\, dx.
\end{aligned}$$

Thus, we obtain by the divergence theorem that

$$
\begin{aligned}
\langle b, \ddot{b}\rangle &= \int_E \operatorname{div}\big(\langle X, b\rangle X e^{-|x|^2/2}\big) + \operatorname{div}\big(\langle x, b\rangle \big(\operatorname{div}\big(X e^{-|x|^2/2}\big) X\big)\big)\, dx \\
&= \int_{\partial^* E} \langle X, b\rangle \langle X, \nu\rangle \, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad + \int_{\partial^* E} \langle b, x\rangle \langle X, \nu\rangle \big(\operatorname{div}\big(X e^{-|x|^2/2}\big)\big)\, d\mathcal{H}^{n-1}(x) \\
&= \int_{\partial^* E} \langle b, \nu\rangle \varphi^2 \, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad + \int_{\partial^* E} \langle b, x\rangle \big(\varphi \partial_\nu \varphi + \big(\mathscr{H} - \langle x, \nu\rangle\big)\varphi^2\big)\, d\mathcal{H}_\gamma^{n-1}(x).
\end{aligned}
\tag{28}
$$

Therefore, (25), (26), (27) and (28) imply

$$
\begin{aligned}
&\int_{\partial^* E} \big(|D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \varphi^2 + \varepsilon \langle b, \nu\rangle \varphi^2\big)\, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad + \varepsilon \left| \int_{\partial^* E} \varphi x \, d\mathcal{H}_\gamma^{n-1}(x) \right|^2 \\
&\quad + \int_{\partial^* E} \big(\mathscr{H} - \langle x, \nu\rangle + \varepsilon \langle b, x\rangle\big)\big(\varphi \partial_\nu \varphi + \big(\mathscr{H} - \langle x, \nu\rangle\big)\varphi^2\big)\, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad = \frac{\partial^2}{\partial t^2}\Big|_{t=0} P_\gamma(E_t) + \varepsilon\big(\langle b, \ddot{b}\rangle + |\dot{b}|^2\big) \\
&\quad = \frac{\partial^2}{\partial t^2}\Big|_{t=0} \mathcal{F}(E_t) \geq 0.
\end{aligned}
\tag{29}
$$

We use the Euler equation (15) and (22) to conclude that

$$
\begin{aligned}
&\int_{\partial^* E} \big(\mathscr{H} - \langle x, \nu\rangle + \varepsilon \langle b, x\rangle\big)\big(\varphi \partial_\nu \varphi + \big(\mathscr{H} - \langle x, \nu\rangle\big)\varphi^2\big)\, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad = \lambda \int_{\partial^* E} \varphi \partial_\nu \varphi + \big(\mathscr{H} - \langle x, \nu\rangle\big)\varphi^2 \, d\mathcal{H}_\gamma^{n-1}(x) = 0.
\end{aligned}
$$

Hence, the claim follows from (29). □

We would like to extend the quadratic form in Proposition 3 to more general functions than $\varphi \in C_0^\infty(\partial^* E)$. To this aim, we define the function space $H_\gamma^1(\partial^* E)$ as the closure of $C_0^\infty(\partial^* E)$ with respect to the norm $\|u\|_{H_\gamma^1(\partial^* E)} = \|u\|_{L_\gamma^2(\partial^* E)} + \|D_\tau u\|_{L_\gamma^2(\partial^* E, \mathbb{R}^n)}$. Here, $L_\gamma^2(\partial^* E)$ is the set of square integrable functions on $\partial^* E$ with respect to the measure γ. A priori the definition of $H_\gamma^1(\partial^* E)$ seems rather restrictive since it is not clear if even constant functions belong to $H_\gamma^1(\partial^* E)$. However, the information on the singular set $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq n - 8$

from Proposition 2 ensures that the singular set has capacity zero and it is therefore negligible. It follows that every smooth function $u \in C^\infty(\partial^* E)$ which has finite H^1_γ-norm is in $H^1_\gamma(\partial^* E)$. Recall that $\partial^* E$ is a relatively open, C^∞ hypersurface. In particular, if $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that the $H^1_\gamma(\partial^* E)$ norm of its restriction on $\partial^* E$ is bounded, then the restriction is in $H^1_\gamma(\partial^* E)$.

LEMMA 1. *Let E be a minimizer of $\mathcal{F}$. If $u \in C^\infty(\partial^* E)$ is such that $\|u\|_{H^1_\gamma(\partial^* E)} < \infty$, then $u \in H^1_\gamma(\partial^* E)$.*

PROOF. By truncation, we may assume that u is bounded and by a standard mollification argument it is enough to find Lipschitz continuous functions u_k with a compact support on $\partial^* E$ such that $\lim_{k\to\infty} \|u - u_k\|_{H^1_\gamma(\partial^* E)} = 0$. We will show that there exist Lipschitz continuous functions $\zeta_k : \partial^* E \to \mathbb{R}$ with compact support such that $0 \le \zeta_k \le 1$, $\zeta_k \to 1$ in $H^1_\gamma(\partial^* E)$ and $\zeta_k(x) \to 1$ pointwise on $\partial^* E$. We may then choose $u_k = u\zeta_k$ and the claim follows.

Let us fix $k \in \mathbb{N}$. First of all let us choose a large radius R_k such that the Gaussian perimeter of E outside the ball B_{R_k} is small, that is, $P_\gamma(E; \mathbb{R}^n \setminus B_{R_k}) \le 1/k$. We choose a cut-off function $\eta_k \in C_0^\infty(B_{2R_k})$ such that $|D\eta_k(x)| \le 1$ for every $x \in \mathbb{R}^n$ and $\zeta \equiv 1$ in B_{R_k}.

Denote the singular set by $\Sigma := \partial E \setminus \partial^* E$. Proposition 2 implies that Σ is a closed set with $\mathcal{H}^{n-3}(\Sigma) = 0$. Therefore, we may cover $\Sigma \cap \overline{B}_{2R_k}$ with balls $B_{r_i} := B_{r_i}(x_i)$, $i = 1, \dots, N_k$, with radii $r_i \le 1/2$ such that

$$\sum_{i=1}^{N_k} r_i^{n-3} \le \frac{1}{C_0} \frac{1}{k},$$

where $C_0 = C_0(2R_k)$ is the constant from the estimate (17) for the radius $2R_k$. For every ball B_{2r_i}, we define a cut-off function $\psi_i \in C_0^\infty(B_{2r_i})$ such that $\psi_i \equiv 1$ in B_{r_i}, $0 \le \psi_i \le 1$ and $|D\psi_i| \le \frac{2}{r_i}$. Define

$$\theta_k(x) := \max_i \psi_i(x), \qquad x \in \mathbb{R}^n.$$

Then $\theta_k(x) = 1$ for $x \in \bigcup_i B_{r_i}$, $\theta_k(x) = 0$ for $x \neq \bigcup_i B_{2r_i}$ and it is Lipschitz continuous. We may estimate its weak tangential gradient on $\partial^* E$ by

$$|D_\tau \theta_k(x)| \le \max_i |D_\tau \psi_i(x)| \le \left(\sum_{i=1}^{N_k} |D\psi_i(x)|^2 \right)^{1/2}$$

for $\mathcal{H}^{n-1}$-almost every $x \in \partial^* E$. Since $\Sigma \cap B_{2R_k} \subset \bigcup_i B_{r_i}$ the function

$$\zeta_k = (1 - \theta_k)\eta_k$$

has compact support on $\partial^* E$. Note that by (17) it holds $P(E; B_{2r_i}) \le C_0 r_i^{n-1}$. Hence, we have that

$$\begin{aligned}\|D_\tau \zeta_k\|^2_{L^2_\gamma(\partial^* E)} &\le 2\int_{\partial^* E} \big(|D_\tau \eta_k|^2 + |D_\tau \theta_k|^2\big)\, d\gamma(x)\\ &\le 2P_\gamma\big(E; \mathbb{R}^n \setminus B_{R_k}\big) + 2\sum_{i=1}^{N_k} \int_{\partial^* E \cap B_{2r_i}} |D\psi_i|^2\, d\mathcal{H}^{n-1}\\ &\le \frac{2}{k} + 8\sum_{i=1}^{N_k} r_i^{-2} P(E; B_{2r_i})\\ &\le \frac{2}{k} + 8C_0 \sum_{i=1}^{N_k} r_i^{n-3} \le \frac{10}{k}.\end{aligned}$$

Similarly, we conclude that $\|\zeta_k - 1\|^2_{L^2_\gamma(\partial^* E)} \to 0$ as $k \to \infty$. $\square$

5. Quantitative estimates. In this section, we focus on the proof of our main result, as well as on some of its direct consequences. The proof of the Main Theorem is divided in several steps. The core of the proof is step 3 where we prove that any minimizer of the functional $\mathcal{F}$ is a half-space. In the final part of the proof (step 4), we only need to prove that every minimizer has the right volume.

PROOF OF THE MAIN THEOREM. Since $\beta(E) = \beta(\mathbb{R}^n \setminus E)$, we may restrict ourselves to the case $s \le 0$. As explained in Section 3, we have to prove that the for some ε and Λ (only depending on s) the only minimizers of the functional $\mathcal{F}$ are the half-spaces $H_{\omega,s}$, $\omega \in \mathbb{S}^{n-1}$. We will show that this is indeed the case when we choose ε and Λ as

$$\varepsilon = \frac{e^{s^2/2}}{40\pi^2(1+s^2)} \quad \text{and} \quad \Lambda = \frac{\sqrt{2}e^{-s^2/2}}{\phi(s)}. \tag{30}$$

With this choice in (10), we have (7) with the constant

$$c = 80\pi^2\sqrt{2\pi}.$$

Assume now that E is a minimizer of $\mathcal{F}$ and, without loss of generality, that its barycenter is in the direction of $-e^{(n)}$, that is, $b(E) = -|b|e^{(n)}$. We will denote $H_s = H_{e^n,s}$ and show that $E = H_s$. We divide the proof into four steps.

Step 1. As a first step, we prove an upper bound for $\int_{\partial^* E} \langle x, \omega\rangle^2\, d\gamma(x)$, that is, for every $\omega \in \mathbb{S}^{n-1}$ it holds

$$\int_{\partial^* E} \langle x, \omega\rangle^2\, d\mathcal{H}^{n-1}_\gamma(x) \le 20\pi^2(1+s^2)e^{-s^2/2}.$$

The proof is similar to the classical Caccioppoli inequality in the theory of elliptic equations.

We begin with few observations. Using H_s as a competitor, the minimality of E implies

$$P_\gamma(E) \le \mathcal{F}(H_s) = P_\gamma(H_s) + \frac{\varepsilon}{2}\big|b(H_s)\big|^2 \le \frac{10}{9}e^{-s^2/2}. \tag{31}$$

Let r be such that $\phi(r) = \gamma(E)$. Since H_r maximizes the length of the barycenter we have by the Gaussian isoperimetric inequality and by (31) that

$$|b| \le \big|b(H_r)\big| = \frac{1}{\sqrt{2\pi}}P_\gamma(H_r) \le \frac{1}{\sqrt{2\pi}}P_\gamma(E) \le \frac{10}{9\sqrt{2\pi}}e^{-s^2/2}.$$

From our choice of ε in (30), it follows that

$$\varepsilon|b| \le \tfrac{1}{4}. \tag{32}$$

By second-order analysis, it is easy to check that the function

$$g(s) := e^{-s^2/2} + (\sqrt{2\pi}s - \pi)\phi(s)$$

is nonpositive in $(-\infty, 0]$. Indeed, g' is nonpositive and $\lim_{s\to-\infty} g(s) = 0$. Therefore,

$$\Lambda^2 + 1 = 2\frac{e^{-s^2}}{\phi(s)^2} + 1 \le 2(\pi - \sqrt{2\pi}s)^2 + 1 \le \frac{9}{2}\pi^2\big(1 + s^2\big). \tag{33}$$

Since $\partial^* E$ is smooth, we deduce from the Euler equation (15) that for every Lipschitz continuous vector field $X : \partial^* E \to \mathbb{R}^n$ with compact support it holds

$$\begin{aligned}\int_{\partial^* E}\big(\operatorname{div}_\tau X - \langle X, x\rangle\big)\, d\mathcal{H}^{n-1}_\gamma(x) - \varepsilon|b| \int_{\partial^* E} x_n\langle X, \nu\rangle\, d\mathcal{H}^{n-1}_\gamma(x)\\ = \lambda \int_{\partial^* E}\langle X, \nu\rangle\, d\mathcal{H}^{n-1}_\gamma(x).\end{aligned} \tag{34}$$

To obtain (34), simply multiply the Euler equation (15) by $\langle X, \nu\rangle$ and use the divergence theorem on hypersurfaces.

Let $\zeta_k : \partial^* E \to \mathbb{R}$ be the sequence of Lipschitz continuous functions from the proof of Lemma 1 which have compact support, $0 \le \zeta_k \le 1$ and $\zeta_k \to 1$ in $H^1_\gamma(\partial^* E)$. Let us fix $\omega \in \mathbb{S}^{n-1}$ and choose $X = -\zeta_k^2 x_\omega \omega$ in (34), where $x_\omega = \langle x, \omega\rangle$. We use (32), (34) and Young's inequality to get

$$\begin{aligned}\int_{\partial^* E}&(x_\omega^2 - \big(1 - \langle \nu, \omega\rangle^2\big))\zeta_k^2\, d\gamma(x) - \frac{1}{8}\int_{\partial^* E}(x_\omega^2 + x_n^2)\zeta_k^2\, d\gamma(x)\\ &\le |\lambda| \int_{\partial^* E}|x_\omega|\zeta_k^2\, d\gamma(x) + 2\int_{\partial^* E}\zeta_k|x_\omega||D_\tau\zeta_k|\, d\gamma(x)\\ &\le \lambda^2 P_\gamma(E) + \frac{1}{2}\int_{\partial^* E}x_\omega^2\zeta_k^2\, d\gamma(x) + 4\int_{\partial^* E}|D_\tau\zeta_k|^2\, d\gamma(x).\end{aligned}$$

This yields

$$\frac{3}{8}\int_{\partial^* E} x_\omega^2 \zeta_k^2 \, d\gamma(x) - \frac{1}{8}\int_{\partial^* E} x_n^2 \zeta_k^2 \, d\gamma(x)$$
$$\leq (\lambda^2+1) P_\gamma(E) + 4\int_{\partial^* E} |D_\tau \zeta_k|^2 \, d\gamma(x).$$

Maximizing over $\omega \in \mathbb{S}^{n-1}$ gives

$$\max_{\omega\in\mathbb{S}^{n-1}} \left(\frac{1}{4}\int_{\partial^* E} x_\omega^2 \zeta_k^2 \, d\gamma(x)\right) \leq (\lambda^2+1) P_\gamma(E) + 4\int_{\partial^* E} |D_\tau \zeta_k|^2 \, d\gamma(x).$$

By letting $k \to \infty$, from the bound $|\lambda| \leq \Lambda$ proved in Proposition 2, and from (31) and (33) we deduce

$$\max_{\omega\in\mathbb{S}^{n-1}} \int_{\partial^* E} \langle x, \omega\rangle^2 \, d\mathcal{H}_\gamma^{n-1}(x) \leq 4(\Lambda^2+1) P_\gamma(E) \leq 20\pi^2 (1+s^2) e^{-s^2/2}.$$

Step 2. In this step, we use the previous step and Proposition 3 to conclude that for every $\varphi \in H_\gamma^1(\partial^* E)$ with $\int_{\partial^* E} \varphi \, d\mathcal{H}_\gamma^{n-1}(x) = 0$ it holds

$$\int_{\partial^* E} \left(|D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \frac{1}{2}\varphi^2 - \varepsilon |b| \nu_n \varphi^2\right) d\mathcal{H}_\gamma^{n-1}(x) \geq 0. \tag{35}$$

Recall that $H_\gamma^1(\partial^* E)$ is the closure of $C_0^\infty(\partial^* E)$ with respect to H_γ^1-norm.

Let $\varphi \in H_\gamma^1(\partial^* E)$ with $\int_{\partial^* E} \varphi \, d\mathcal{H}_\gamma^{n-1}(x) = 0$. Then there exists $\varphi_k \in C_0^\infty(\partial^* E)$ such that $\varphi_k \to \varphi$ in $H_\gamma^1(\partial^* E)$. In particular, since $\int_{\partial^* E} \varphi_k \, d\mathcal{H}_\gamma^{n-1}(x)$ vanishes as k goes to infinity, by slightly changing the functions φ_k we may assume that they satisfy $\int_{\partial^* E} \varphi_k \, d\mathcal{H}_\gamma^{n-1}(x) = 0$ and still converge to φ in $H_\gamma^1(\partial^* E)$. Let $\omega_k \in \mathbb{S}^{n-1}$ be vectors such that

$$\left|\int_{\partial^* E} \varphi_k x \, d\mathcal{H}_\gamma^{n-1}(x)\right| = \left\langle \int_{\partial^* E} \varphi_k x \, d\mathcal{H}_\gamma^{n-1}(x), \omega_k \right\rangle = \int_{\partial^* E} \langle x, \omega_k\rangle \varphi_k \, d\mathcal{H}_\gamma^{n-1}(x).$$

We use Proposition 3 and step 1 to conclude

$$\int_{\partial^* E} (|D_\tau \varphi_k|^2 - |B_E|^2 \varphi_k^2 - \varphi_k^2 - \varepsilon |b| \nu_n \varphi_k^2) \, d\mathcal{H}_\gamma^{n-1}(x)$$
$$\geq -\varepsilon \left(\int_{\partial^* E} \langle x, \omega_k\rangle^2 \, d\mathcal{H}_\gamma^{n-1}(x)\right)\left(\int_{\partial^* E} \varphi_k^2 \, d\mathcal{H}_\gamma^{n-1}(x)\right)$$
$$\geq -\varepsilon 20\pi^2 (1+s^2) e^{-s^2/2} \left(\int_{\partial^* E} \varphi_k^2 \, d\mathcal{H}_\gamma^{n-1}(x)\right).$$

From our choice of ε in (30), we conclude that (35) holds for every φ_k. Since $\varphi_k \to \varphi$ in $H_\gamma^1(\partial^* E)$, (35) follows by letting $k \to \infty$ and by noticing that Fatou's lemma implies

$$\liminf_{k\to\infty} \int_{\partial^* E} |B_E|^2 \varphi_k^2 \, d\mathcal{H}_\gamma^{n-1}(x) \geq \int_{\partial^* E} |B_E|^2 \varphi^2 \, d\mathcal{H}_\gamma^{n-1}(x).$$

Before the next step, we remark that by (35) we have

$$\int_{\partial^* E} |B_E|^2 \varphi^2 \, d\mathcal{H}_\gamma^{n-1}(x) \le C \|\varphi\|^2_{H^1_\gamma(\partial^* E)}$$

for every $\varphi \in H^1_\gamma(\partial^* E)$ with zero average. Recalling Lemma 1, it is not difficult to see that this implies

$$\int_{\partial^* E} |B_E|^2 \, d\mathcal{H}_\gamma^{n-1}(x) < \infty. \tag{36}$$

We leave the proof of this estimate to the reader.

Step 3. In this step, we will prove that our minimizer E is a half-space

$$E = H_t = \{x \in \mathbb{R}^n : x_n < t\} \qquad \text{for some } t \in \mathbb{R}. \tag{37}$$

This is the main step of the proof.

Let $j \in \{1, \dots, n-1\}$. Since we assumed that the barycenter $b(E)$ is in $-e^{(n)}$ direction, the divergence theorem yields

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\partial^* E} \nu_j \, d\mathcal{H}_\gamma^{n-1}(x) &= \frac{1}{(2\pi)^{n/2}} \int_E \operatorname{div}\big(e^{(j)} e^{-|x|^2/2}\big) \, dx \\ &= -\int_E x_j \, d\gamma(x) = -\langle b(E), e^{(j)} \rangle = 0. \end{aligned} \tag{38}$$

In other words, the function ν_j has zero average. Moreover, (36) implies

$$\int_{\partial^* E} |D_\tau \nu_j|^2 \, d\mathcal{H}_\gamma^{n-1}(x) \le \int_{\partial^* E} |B_E|^2 \, d\mathcal{H}_\gamma^{n-1}(x) < \infty.$$

From Lemma 1, we deduce that $\nu_j \in H^1_\gamma(\partial^* E)$ and we may thus use (35) to conclude

$$\int_{\partial^* E} \Big(|D_\tau \nu_j|^2 - |B_E|^2 \nu_j^2 - \frac{1}{2} \nu_j^2 - \varepsilon |b| \nu_n \nu_j^2 \Big) \, d\mathcal{H}_\gamma^{n-1}(x) \ge 0. \tag{39}$$

Recall the notion of tangential derivative δ_i, tangential gradient D_τ and tangential Laplacian Δ_τ defined in Section 2. We recall the well-known equation (see, e.g., [17], Lemma 10.7)

$$\Delta_\tau \nu_j = -|B_E|^2 \nu_j + \delta_j \mathscr{H} \qquad \text{on } \partial^* E.$$

Note also that

$$\delta_j \langle x, \nu \rangle = \sum_{i=1}^n (\delta_j x_i) \nu_i + (\delta_j \nu_i) x_i = \nu_j - \sum_{i=1}^n \nu_j \nu_i^2 + (\delta_i \nu_j) x_i = \langle D_\tau \nu_j, x \rangle,$$

where in the second equality we used $\delta_j \nu_i = \delta_i \nu_j$ and in the last equality we used $\sum_{i=1}^n \nu_i^2 = |\nu|^2 = 1$. We differentiate the Euler equation (15) with respect to δ_j and by the two above equations we deduce that

$$\Delta_\tau \nu_j - \langle D_\tau \nu_j, x \rangle = -|B_E|^2 \nu_j - \varepsilon |b| \nu_n \nu_j \qquad \text{on } \partial^* E.$$

The last term follows from $\delta_j x_n = -\nu_j \nu_n$, since $j \neq n$. Let $\zeta_k : \partial^* E \to \mathbb{R}$ be as in step 1. We multiply the previous equation by $\zeta_k \nu_j$, integrate over $\partial^* E$ and use the divergence theorem on hypersurfaces to conclude

$$\begin{aligned}
&\int_{\partial^* E} \zeta_k \big(|B_E|^2 \nu_j^2 + \varepsilon |b| \nu_n \nu_j^2\big)\, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad = -\int_{\partial^* E} \zeta_k \nu_j \big(\Delta_\tau \nu_j - \langle D_\tau \nu_j, x\rangle\big)\, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad = -\int_{\partial^* E} \zeta_k \nu_j \operatorname{div}_\tau\big(D_\tau \nu_j e^{-|x|^2/2}\big)\, d\mathcal{H}^{n-1}(x) \\
&\quad = -\int_{\partial^* E} \operatorname{div}_\tau\big(\zeta_k \nu_j D_\tau \nu_j e^{-|x|^2/2}\big)\, d\mathcal{H}^{n-1}(x) \\
&\qquad + \int_{\partial^* E} \big\langle D_\tau(\zeta_k \nu_j), D_\tau \nu_j\big\rangle\, d\mathcal{H}_\gamma^{n-1}(x) \\
&\quad = \int_{\partial^* E} \zeta_k |D_\tau \nu_j|^2\, d\mathcal{H}_\gamma^{n-1}(x) + \int_{\partial^* E} \nu_j \langle D_\tau \zeta_k, D_\tau \nu_j\rangle\, d\mathcal{H}_\gamma^{n-1}(x).
\end{aligned}$$

Since $\|D_\tau \zeta_k\|_{L^2(\partial^* E)} \to 0$ as $k \to \infty$, we deduce from the previous equation that

$$\int_{\partial^* E} \big(|B_E|^2 \nu_j^2 + \varepsilon |b| \nu_n \nu_j^2\big)\, d\mathcal{H}_\gamma^{n-1}(x) = \int_{\partial^* E} |D_\tau \nu_j|^2\, d\mathcal{H}_\gamma^{n-1}(x).$$

Thus, we get from (39) that

$$-\frac{1}{2} \int_{\partial^* E} \nu_j^2\, d\mathcal{H}_\gamma^{n-1}(x) \geq 0.$$

This implies $\nu_j \equiv 0$ on $\partial^* E$. Since E has locally finite perimeter in $\mathbb{R}^n$, De Giorgi's structure theorem [18], Theorem 15.9, yields

$$D\chi_E = -\nu \mathcal{H}^{n-1} \llcorner \partial^* E.$$

Therefore, the distributional partial derivatives $D_j \chi_E$, $j = 1, \dots, n-1$, are all zero and necessarily $E = \mathbb{R}^{n-1} \times F$ for some set F of locally finite perimeter in $\mathbb{R}$. In particular, the topological boundary of E is smooth and $\partial^* E = \partial E$.

We will show that the boundary of E is connected, which will imply that E is a half-space. To this aim, we use the argument from [24]. We argue by contradiction and assume that there are two disjoint closed sets $\Gamma_1, \Gamma_2 \subset \partial E$ such that $\partial E = \Gamma_1 \cup \Gamma_2$. Let $a_1 < 0 < a_2$ be two numbers such that the function $\varphi : \partial E \to \mathbb{R}$

$$\varphi := \begin{cases} a_1, & \text{on } \Gamma_1, \\ a_2, & \text{on } \Gamma_2 \end{cases}$$

has zero average. Then clearly $\varphi \in H_\gamma^1(\partial E)$ and, therefore, (35) implies

$$\int_{\partial E} \left(|B_E|^2 \varphi^2 + \frac{1}{2} \varphi^2 + \varepsilon |b| \nu_n \varphi^2 \right) d\mathcal{H}_\gamma^{n-1}(x) \leq 0.$$

From (32), we deduce

$$\int_{\partial E}\left(|B_E|^2\varphi^2+\frac{1}{4}\varphi^2\right)d\mathcal{H}^{n-1}_\gamma(x)\le 0$$

which is obviously impossible. Hence, ∂E is connected.

Step 4. We need yet to show that E has the correct volume, that is, $\gamma(E)=\phi(s)$. Since we have proved (37), we only need to show that the function $f:\mathbb{R}\to(0,\infty)$

$$f(t):=\mathcal{F}(H_t)=e^{-t^2/2}+\frac{\varepsilon}{4\pi}e^{-t^2}+\Lambda\left|\phi(t)-\phi(s)\right|$$

attains its minimum at $t=s\le 0$.

Note that for every $t<0$ it holds $f(t)<f(|t|)$. Moreover, the function f is clearly increasing on $(s,0)$. Hence, we only need to show that $f(s)<f(t)$ for every $t<s$. In $(-\infty,s)$ we have

$$f'(t)=-te^{-t^2/2}-\frac{\varepsilon}{2\pi}te^{-t^2}-\frac{\Lambda}{\sqrt{2\pi}}e^{-t^2/2}.$$

In particular, f increases, reaches its maximum and decreases to $f(s)$. From our choices of Λ and ε in (30), we have

$$\lim_{t\to-\infty}f(t)=\Lambda\phi(s)\ge\sqrt{2}e^{-s^2/2}>f(s).$$

Thus, the function f attains its minimum at $t=s$ which implies

$$\gamma(E)=\phi(s).$$

This completes the proof. □

REMARK 1. We remark that the dependence on the mass in (7) is optimal. This can be verified by considering the one-dimensional set $E_s=(-\infty,a(s))\cup(-a(s),\infty)$, where $s<0$, and $a(s)<s$ is a number such that

$$\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{a(s)}e^{-t^2/2}\,dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s}e^{-t^2/2}\,dt, \tag{40}$$

that is, $\gamma(E_s)=\phi(s)$. Then $b(E_s)=0$ and $\beta(E_s)=\frac{1}{\sqrt{2\pi}}e^{-s^2/2}$. The sharp mass dependence follows from

$$\liminf_{s\to-\infty}\frac{D(E_s)}{s^{-2}\beta(E_s)}=\sqrt{2\pi}\liminf_{s\to-\infty}\frac{2e^{-a(s)^2/2}-e^{-s^2/2}}{s^{-2}e^{-s^2/2}}\le 2\sqrt{2\pi}. \tag{41}$$

For the reader's convenience, we will give the calculations below.

To show (41), we write $a(s) = s - \varepsilon(s)$. From (40), it follows that $\varepsilon(s) \to 0$ as $s \to -\infty$. We claim that

$$\liminf_{s\to-\infty} \frac{\varepsilon'(s)}{s^{-2}} \le 1.$$

Indeed, if this were not true then we would have $\varepsilon(s) \ge \frac{1}{|s|}$ when $|s|$ is large. Then it follows from (40) that

$$\frac{1}{2} \le \lim_{s\to-\infty} \frac{\int_{-\infty}^{s+1/s} e^{-t^2/2}\,dt}{\int_{-\infty}^{s} e^{-t^2/2}\,dt} = \lim_{s\to-\infty} \frac{(1-1/s^2)e^{-(s+1/s)^2/2}}{e^{-s^2/2}} = \frac{1}{e}$$

which is a contradiction. By differentiating (40) with respect to s and substituting in the left-hand side of (41) we obtain

$$\liminf_{s\to-\infty} \frac{2e^{-(s-\varepsilon(s))^2/2} - e^{-s^2/2}}{s^{-2}e^{-s^2/2}} = \liminf_{s\to-\infty} \frac{2\varepsilon'(s)e^{-(s-\varepsilon(s))^2/2}}{s^{-2}e^{-s^2/2}} \le 2.$$

We proceed by proving that the strong asymmetry controls the square of the standard one. Let us introduce a variant of the Fraenkel asymmetry. Given a Borel set E with $\gamma(E) = \phi(s)$ we define

$$\hat{\alpha}(E) := \begin{cases} 2\phi(-|s|), & \text{if } b(E) = 0, \\ \gamma(E \triangle H_{\omega,s}), & \text{if } b(E) \ne 0, \end{cases}$$

where $\omega = -b(E)/|b(E)|$. Since $\alpha(E) \le 2\phi(-|s|)$, then trivially $\hat{\alpha}(E) \ge \alpha(E)$. Compared to the asymmetry α, the asymmetry $\hat{\alpha}$ has the advantage that the half-space is chosen to be in the direction of the barycenter. The following estimate can be found in [13] but without explicit constant. We give a proof where we obtain the optimal dependence on the mass.

PROPOSITION 4. *Let $E \subset \mathbb{R}^n$ be a set with $\gamma(E) = \phi(s)$. Then*

$$\beta(E) \ge \frac{e^{s^2/2}}{4}\hat{\alpha}(E)^2. \tag{42}$$

PROOF. Since $\hat{\alpha}(E) = \hat{\alpha}(\mathbb{R}^n \setminus E)$ we may restrict ourselves to the case $s \le 0$. By first-order analysis, it is easy to check that the function

$$f(s) := e^{-s^2/2} - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{s} e^{-x_n^2/2}\,dx_n$$

is nonnegative in $(-\infty, 0]$ or, equivalently, that $e^{-s^2/2} \ge 2\phi(s)$. Therefore, if $b(E) = 0$ we immediately have

$$\beta(E) = b_s = \frac{e^{-s^2/2}}{\sqrt{2\pi}} \ge \frac{e^{s^2/2}}{\sqrt{2\pi}}\hat{\alpha}(E)^2.$$

Assume now that $b(E) \neq 0$ and, without loss of generality, that $e^{(n)} = -b(E)/|b(E)|$. For simplicity we write $H = H_{e^{(n)},s}$. Let a_1 and a_2 be positive numbers such that

$$\gamma(E \setminus H) = \frac{1}{\sqrt{2\pi}} \int_{s-a_1}^{s} e^{-x_n^2/2}\, dx_n = \frac{1}{\sqrt{2\pi}} \int_{s}^{s+a_2} e^{-x_n^2/2}\, dx_n.$$

Consider the sets $E^+ := E \setminus H$, $E^- := E \cap H$, $F^+ := \mathbb{R}^{n-1} \times [s, s + a_2)$, $F^- := \mathbb{R}^{n-1} \times (-\infty, s - a_1)$, and $F := F^+ \cup F^-$. By construction $\gamma(F) = \phi(s)$, $\gamma(F^+) = \gamma(E^+)$, and $\gamma(F^-) = \gamma(E^-)$. We have

$$\begin{aligned}
\beta(E) - \beta(F) &= \int_E x_n\, d\gamma(x) - \int_F x_n\, d\gamma(x) \\
&= \int_{E^+\setminus F^+} (x_n - s - a_2)\, d\gamma(x) + \int_{F^+\setminus E^+} (-x_n + s + a_2)\, d\gamma(x) \\
&\quad + \int_{E^-\setminus F^-} (x_n - s + a_1)\, d\gamma(x) + \int_{F^-\setminus E^-} (-x_n + s - a_1)\, d\gamma(x) \\
&\geq 0,
\end{aligned}$$

because the integrands in the last term are all positive.

Since $\gamma(E \setminus H) = \gamma(H \setminus E)$, it is sufficient to show that $\beta(F) \geq e^{s^2/2}\gamma(E \setminus H)^2$. By first-order analysis, it is easy to check that for a fixed $s \leq 0$ the function

$$g(t) := \int_{s-t}^{s} (-x_n + s) e^{-x_n^2/2}\, dx_n - \frac{e^{s^2/2}}{2} \left(\int_{s-t}^{s} e^{-x_n^2/2}\, dx_n \right)^2$$

is nonnegative in $[0, \infty)$. Indeed, g' is nonnegative and $g(0) = 0$. By rearranging terms as above, we deduce

$$\begin{aligned}
\beta(F) &= \int_F x_n\, d\gamma(x) - \int_H x_n\, d\gamma(x) \\
&= \int_{F\setminus H} (x_n - s)\, d\gamma(x) + \int_{H\setminus F} (-x_n + s)\, d\gamma(x) \\
&\geq \frac{1}{\sqrt{2\pi}} \int_{s-a_1}^{s} (-x_n + s) e^{-x_n^2/2}\, dx_n \\
&\geq \frac{e^{s^2/2}}{2\sqrt{2\pi}} \left(\int_{s-a_1}^{s} e^{-x_n^2/2}\, dx_n \right)^2 \\
&= \sqrt{\frac{\pi}{2}} e^{s^2/2} \gamma(E \setminus H)^2.
\end{aligned}$$

□

By the Main Theorem and Proposition 4, we immediately conclude that the deficit controls the Fraenkel asymmetry.

COROLLARY 1. *There exists an absolute constant c such that for every $s \in \mathbb{R}$ and for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the following estimate holds*:

$$\hat{\alpha}(E)^2 \le c(1+s^2)e^{-s^2/2} D(E). \tag{43}$$

REMARK 2. The reduction to the set F in Proposition 4 gives in particular that the dependence on the mass in (42) is optimal. We note that even though the dependence on the mass in (7) and in (42) are optimal, we do not know if these together provide the optimal mass dependence for (43).

Given a set E of finite Gaussian perimeter, the *excess* of E is defined as

$$\mathcal{E}(E) := \min_{\omega \in \mathbb{S}^{n-1}} \left\{ \int_{\partial^* E} |\nu^E - \omega|^2 \, d\mathcal{H}^{n-1}_\gamma(x) \right\}. \tag{44}$$

We conclude by proving that the isoperimetric deficit controls the excess of the set.

COROLLARY 2. *There exists an absolute constant c such that for every $s \in \mathbb{R}$ and for every set of finite Gaussian perimeter $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the following estimate holds*:

$$\mathcal{E}(E) \le c(1+s^2) D(E). \tag{45}$$

Moreover, if $b(E) \neq 0$, the minimum in (44) *is attained by $\omega = -b(E)/|b(E)|$.*

PROOF. By the divergence theorem,

$$\begin{aligned}
\langle b(E), \omega \rangle &= \frac{1}{(2\pi)^{n/2}} \int_E \langle x, \omega \rangle e^{-|x|^2/2} \, dx \\
&= -\frac{1}{(2\pi)^{n/2}} \int_E \operatorname{div}(e^{-|x|^2/2} \omega) \, dx \\
&= -\frac{1}{(2\pi)^{n/2}} \int_{\partial^* E} \langle \omega, \nu^E \rangle e^{-|x|^2/2} \, d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{2\sqrt{2\pi}} \int_{\partial^* E} |\omega - \nu^E|^2 \, d\mathcal{H}^{n-1}_\gamma(x) - \frac{1}{\sqrt{2\pi}} \int_{\partial^* E} d\mathcal{H}^{n-1}_\gamma(x).
\end{aligned}$$

By minimizing over $\omega \in \mathbb{S}^{n-1}$, we get

$$\mathcal{E}(E) = 2P_\gamma(E) - 2\sqrt{2\pi}|b(E)| = 2D(E) + 2\sqrt{2\pi}\beta(E).$$

Finally, thanks to the estimate (7), we obtain (45). $\square$

REFERENCES

[1] ACERBI, E., FUSCO, N. and MORINI, M. (2013). Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.* **322** 515–557. MR3077924

[2] AMBROSIO, L., FUSCO, N. and PALLARA, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. The Clarendon Press, New York. MR1857292

[3] BAKRY, D. and LEDOUX, M. (1996). Lévy–Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator. *Invent. Math.* **123** 259–281. MR1374200

[4] BOBKOV, S. G. (1997). An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space. *Ann. Probab.* **25** 206–214. MR1428506

[5] BÖGELEIN, V., DUZAAR, F. and FUSCO, N. (2015). A quantitative isoperimetric inequality on the sphere. *Adv. Calc. Var.* To apppear. DOI:10.1515/acv-2015-0042.

[6] BÖGELEIN, V., DUZAAR, F. and SCHEVEN, C. (2015). A sharp quantitative isoperimetric inequality in hyperbolic n-space. *Calc. Var. Partial Differential Equations* **54** 3967–4017. MR3426101

[7] BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216. MR0399402

[8] BRASCO, L., DE PHILIPPIS, G. and VELICHKOV, B. (2015). Faber–Krahn inequalities in sharp quantitative form. *Duke Math. J.* **164** 1777–1831. MR3357184

[9] CARLEN, E. A. and KERCE, C. (2001). On the cases of equality in Bobkov's inequality and Gaussian rearrangement. *Calc. Var. Partial Differential Equations* **13** 1–18. MR1854254

[10] CIANCHI, A., FUSCO, N., MAGGI, F. and PRATELLI, A. (2011). On the isoperimetric deficit in Gauss space. *Amer. J. Math.* **133** 131–186. MR2752937

[11] CICALESE, M. and LEONARDI, G. P. (2012). A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Ration. Mech. Anal.* **206** 617–643. MR2980529

[12] EHRHARD, A. (1983). Symétrisation dans l'espace de Gauss. *Math. Scand.* **53** 281–301. MR0745081

[13] ELDAN, R. (2015). A two-sided estimate for the Gaussian noise stability deficit. *Invent. Math.* **201** 561–624. MR3370621

[14] FIGALLI, A., MAGGI, F. and PRATELLI, A. (2010). A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.* **182** 167–211. MR2672283

[15] FUSCO, N. and JULIN, V. (2014). A strong form of the quantitative isoperimetric inequality. *Calc. Var. Partial Differential Equations* **50** 925–937. MR3216839

[16] FUSCO, N., MAGGI, F. and PRATELLI, A. (2008). The sharp quantitative isoperimetric inequality. *Ann. of Math.* (2) **168** 941–980. MR2456887

[17] GIUSTI, E. (1984). *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser, Basel. MR0775682

[18] MAGGI, F. (2012). *Sets of Finite Perimeter and Geometric Variational Problems*: *An Introduction to Geometric Measure Theory*. *Cambridge Studies in Advanced Mathematics* **135**. Cambridge Univ. Press, Cambridge. MR2976521

[19] MCGONAGLE, M. and ROSS, J. (2015). The hyperplane is the only stable, smooth solution to the isoperimetric problem in Gaussian space. *Geom. Dedicata* **178** 277–296. MR3397495

[20] MOSSEL, E. and NEEMAN, J. (2015). Robust dimension free isoperimetry in Gaussian space. *Ann. Probab.* **43** 971–991. MR3342656

[21] MOSSEL, E. and NEEMAN, J. (2015). Robust optimality of Gaussian noise stability. *J. Eur. Math. Soc.* (*JEMS*) **17** 433–482. MR3317748

[22] MOSSEL, E., O'DONNELL, R. and OLESZKIEWICZ, K. (2010). Noise stability of functions with low influences: Invariance and optimality. *Ann. of Math.* (2) **171** 295–341. MR2630040

[23] ROSALES, C. (2014). Isoperimetric and stable sets for log-concave perturbations of Gaussian measures. *Anal. Geom. Metr. Spaces* **2** 328–358. MR3290382

[24] STERNBERG, P. and ZUMBRUN, K. (1998). A Poincaré inequality with applications to volume-constrained area-minimizing surfaces. *J. Reine Angew. Math.* **503** 63–85. MR1650327

[25] SUDAKOV, V. N. and TSIRELSON, B. S. (1974). Extremal properties of half-spaces for spherically invariant measures: Problems in the theory of probability distributions, II. *Zap. NauČn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **41** 14–24, 165. MR0365680

[26] TAMANINI, I. (1984). Regularity results for almost minimal oriented hypersurfaces in $\mathbb{R}^n$. Quaderni del Dipartimento di Matematica Dell'Università di Lecce, Lecce. Available at cvgmt.sns.it/paper/1807/.

[27] TESCHL, G. (2012). *Ordinary Differential Equations and Dynamical Systems*. *Graduate Studies in Mathematics* **140**. Amer. Math. Soc., Providence, RI. MR2961944

[28] VILLANI, C. (2009). *Optimal Transport*. *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454

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