LIPSCHITZ REGULARITY FOR LOCAL MINIMIZERS OF SOME WIDELY DEGENERATE PROBLEMS

PIERRE BOUSQUET, LORENZO BRASCO, AND VESA JULIN

Abstract. We consider local minimizers of the functional

$$\sum_{i=1}^{N} \int (|u_{x_i}| - \delta_i)_+^p \, dx + \int f \, u \, dx,$$

where $\delta_1, \ldots, \delta_N \geq 0$ and $(\,\cdot\,)_+$ stands for the positive part. Under suitable assumptions on f, we prove that local minimizers are Lipschitz continuous functions if $N = 2$ and $p \geq 2$, or if $N \geq 2$ and $p \geq 4$.

Contents

1. Introduction

1.1. Overview. This paper is devoted to prove Lipschitz continuity for local minimizers of the anisotropic functional

$$\mathfrak{F}(u; \Omega') = \sum_{i=1}^{N} \int_{\Omega'} \frac{(|u_{x_i}| - \delta_i)_+^p}{p} \, dx + \int_{\Omega'} f \, u \, dx, \qquad u \in W^{1,p}_{loc}(\Omega), \ \Omega' \Subset \Omega. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is an open set, $2 \leq p < \infty$, $\delta_i \geq 0$, $(\,\cdot\,)_+$ stands for the positive part and $f \in L^{p'}_{loc}(\Omega)$ where $p' = p/(p-1)$. This functional $\mathfrak{F}$ stands for a model case of a more general class of problems, with specific growth and monotonicity assumptions. For the sake of clarity, the results in this paper

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are only stated for $\mathfrak{F}$. However, their proofs can be easily adapted to embrace general functionals having a similar structure.

The functional $\mathfrak{F}$ naturally arises in problems of Optimal Transport with congestion and anisotropic effects, see for example [6, 7] for some motivations. These two papers contained among others some regularity results for local minimizers of (1.1). For instance [7, Main Theorem] proved that if $f \in L^\infty_{loc}(\Omega)$, then u is "almost Lipschitz", i.e. $u \in W^{1,r}_{loc}(\Omega)$ for every $r \geq 1$. On the other hand, in [6] it is proved that if $f \in W^{1,p'}_{loc}(\Omega)$, then

$$(|u_{x_i}| - \delta_i)_+^{\frac{p}{2}} \frac{u_{x_i}}{|u_{x_i}|} \in W^{1,2}_{loc}(\Omega), \qquad i = 1, \dots, N. \tag{1.2}$$

However, it must be mentioned that to the best of our knowledge, *Lipschitz regularity of local minimizers is still unknown*. More surprisingly, even the case $\delta_1 = \dots = \delta_N = 0$ does not seem to be fully understood.

Observe that local minimizers of (1.1) are local weak solutions of the anisotropic degenerate equation

$$\sum_{i=1}^{N} \left((|u_{x_i}| - \delta_i)_+^{p-1} \frac{u_{x_i}}{|u_{x_i}|} \right)_{x_i} = f, \tag{1.3}$$

which reduces to the Poisson equation for the so-called *pseudo $p-$Laplacian* when $\delta_1 = \dots = \delta_N = 0$, i.e.

$$\sum_{i=1}^{N} \left(|u_{x_i}|^{p-2}\, u_{x_i} \right)_{x_i} = f. \tag{1.4}$$

The terminology "pseudo $p-$Laplacian" appears in [1]. We point out that such an operator already appeared in J.-L. Lions's monograph [17], where existence issues for solutions to evolution equations are tackled.

In order to neatly explain the difficulty of the problem, we now recall some classes of functionals for which the Lipschitz property for local minimizers is known to be true. The first one is given by

$$\int G(\nabla u)\, dx, \tag{1.5}$$

with G enjoying a *$p-$Laplacian type structure at infinity*. This means that there exist $c, C > 0$ and $m \geq 0$ such that G verifies the ellipticity condition

$$\langle D^2 G(z)\, \xi, \xi \rangle \geq c\, |z|^{p-2}\, |\xi|^2, \qquad |z| > m, \tag{1.6}$$

and the growth condition

$$|\nabla G(z)| \leq C\, |z|^{p-1}, \qquad |z| > m. \tag{1.7}$$

We refer the reader to [5, 8, 9, 12] and [13] for example. For completeness, we mention the papers [10, 11] and [19] for related regularity results on the term $\nabla G(\nabla u)$, when $m > 0$.

Another type of well-studied functionals having some similarities with $\mathfrak{F}$ is given by (see for example [2, 3] and [14, Section 4])

$$\int \widetilde{G}(\nabla u)\, dx, \qquad \text{with} \qquad \widetilde{G}(z) = \sum_{i=1}^{N} (\mu + |z_i|^2)^{\frac{p_i}{2}}. \tag{1.8}$$

Here $\mu > 0$ and $1 < p_1 \le p_2 \le \cdots \le p_N$ are possibly different exponents. When the p_i are not equal, such a functional belongs to the class of *problems with non standard growth conditions*, whose systematic study started with the paper [18] by Marcellini. In this case we can infer local Lipschitz continuity if the exponents p_i are not "too far apart" (see the above mentioned references for more details).

However, our functional $\mathfrak{F}$ *does not fall neither in the class of the functional* (1.5) *nor in that of* (1.8). Indeed, observe that in our case

$$F(z) = \sum_{i=1}^{N} \frac{(|z_i| - \delta_i)_+^p}{p},$$

verifies (1.7), *but* (1.6) *crucially fails to hold*, since for every $m > 0$, there always exists z such that $|z| = m$ and the least eigenvalue of $D^2F(z)$ is 0. Observe that this phenomenon already occurs for the pseudo $p-$Laplacian, i.e. when $\delta_1 = \cdots = \delta_N = 0$. Indeed, the main difficulty of the problem is that the region where ellipticity fails *is unbounded.*

For the same reason, $\mathfrak{F}$ is not of the type (1.8), since already in the standard growth case $2 \le p_1 = p_2 = \cdots = p_N$ we have

$$0 < \min_{|\xi|=1} \langle D^2\widetilde{G}(z)\, \xi, \xi\rangle, \qquad z \in \mathbb{R}^N.$$

When one allows $\mu = 0$ in (1.8), the corresponding functional becomes degenerate along the axes $z_i = 0$, like in the case of the pseudo $p-$Laplacian. This case has been considered in the pioneering paper [22] by Uralt'seva and Urdaletova. There the Lipschitz character of minimizers has been shown under some restrictions on the exponents $p_1, \dots, p_N$, by using the so-called *Bernstein method.* Though the growth conditions considered are more general than ours, the type of degeneracy is again weaker than that admitted in $\mathfrak{F}$ (see the next subsection for more comments on the result of [22]).

About the restriction $p \ge 2$ considered in this paper, it is noteworthy to observe that for $1 < p < 2$ our functional has a $p-$Laplacian type structure when $\delta_1 = \cdots = \delta_N = 0$. Indeed, in this case $p - 2 < 0$ and thus (1.6) holds, i.e.

$$\langle D^2F(z)\, \xi, \xi\rangle = (p-1) \sum_{i=1}^{N} |z_i|^{p-2}\, |\xi_i|^2 \ge (p-1)\, |z|^{p-2}\, |\xi|^2,$$

while (1.7) is of course satisfied. Then in this case local minimizers are locally Lipschitz continuous by[1] [13, Theorem 2.7].

1.2. **Main results.** In this paper, we prove the following results. Both results come with a priori estimates, that for ease of readibility we do not detail here. The interested reader could find them in Propositions 4.1 and 5.1.

Theorem A (Two dimensional case)**.** *Let $N = 2$ and $p \ge 2$. Let $f \in W^{1,p'}_{loc}(\Omega)$, where $p' = p/(p-1)$. Then every local minimizer $U \in W^{1,p}_{loc}(\Omega)$ of the functional $\mathfrak{F}$ is a locally Lipschitz continuous function.*

[1] To be more precise, for $1 < p < 2$ the function F is not C^2. However, this is not an issue, since the result of [13, Theorem 2.7] holds for convex functions satisfying a qualified form of uniform convexity for $|z| \ge m$. This coincides with (1.6) if the function is C^2, but it is otherwise more general.

Theorem B (Higher dimensional case)**.** *Let $N\ge 2$ and $p\ge 4$. Let $f\in W^{1,\infty}_{loc}(\Omega)$. Then every local minimizer $U\in W^{1,p}_{loc}(\Omega)$ of the functional $\mathfrak{F}$ is a locally Lipschitz continuous function.*

Let us now spend some words about the methods of proofs. The preliminary step in both cases is a regularization argument. Namely, the functional $\mathfrak{F}$ is replaced by a regularized version $\mathfrak{F}_\varepsilon$, for a small parameter $\varepsilon>0$. This permits to infer the necessary regularity on the solutions u_ε of the regularized problem, in order to justify the manipulations needed to obtain a priori Lipschitz estimates, uniform in ε. Then one aims at taking these estimates to the limit as ε goes to 0. However, one should pay attention to the fact that $\mathfrak{F}$ is not strictly convex when at least one $\delta_i\neq 0$. Thus a sequence of solutions u_ε may not necessarily converge to the desired local minimizer. In [7] a penalization argument was used to fix this issue. Here on the contrary, we use a simpler argument, based on the fact that the lack of strict convexity of $t\mapsto(|t|-\delta_i)_+^p$ is "confined" (see Lemma 2.3).

The core of the proof of Theorem A is the a priori Lipschitz estimate of Proposition 4.1. Such an estimate is achieved by means of a Moser's iteration technique applied to the equation solved by the partial derivatives u_{x_j} of the local minimizer. More precisely, we look at power-type subsolutions of this equation, i.e. quantities like $|u_{x_j}|^s$ for $s\ge 1$. This is a standard strategy for equations having a $p-$Laplacian type structure, but as already said our operator does not have such a structure and this entails several additional difficulties.

As explained in the introduction of [7], the main difficulty of this method is that the Caccioppoli inequality we get for $|u_{x_j}|^s$ is quite involved. Indeed, due to the particular structure of D^2F, in principle we have a control only on a "weighted" norm of $\nabla|u_{x_j}|^s$, the weights being dependent on *all the other components u_{x_i} of the gradient* (see Lemma 3.6 below). Roughly speaking, what we control in the Caccioppoli inequality is a quantity like

$$\sum_{i=1}^N\int |u_{x_i}|^{p-2}\,\left|\left(|u_{x_j}|^{s+1}\right)_{x_i}\right|^2.$$

For the diagonal term, i.e. when $i=j$, we can combine the x_j-derivative of u_{x_j} with the weigth $|u_{x_j}|^{p-2}$ and simply recognize the x_j-derivative of yet another power of u_{x_j}. Since we would like to have a control on the full gradient of such a power of u_{x_j}, we still miss all the x_i-derivatives ($i\neq j$) of this function. To overcome this difficulty, we use in a crucial way the Sobolev property (1.2) together with Hölder's inequality, in order to "cook-up" suitable Caccioppoli inequalities for all these *missing terms*. Surprisingly enough, even if the functional $\mathfrak{F}$ has $p-$growth in every direction, we rely on the *anisotropic Sobolev inequality* due to Troisi (see [21]) in order to produce an iterative scheme of reverse Hölder's inequalities. This procedure works for $N=2$, but it seems to be limited just to the two dimensional case (see Remark 4.2 below).

In contrast Theorem B is valid in every dimension, but we need the restriction $p\ge 4$. This second result partially superposes with the already mentioned [22, Theorem 1] by Uralt'seva and Urdaletova. However, it should be noticed that the monotonicity assumptions on the operator[2] made in [22] does not allow for $\delta_i>0$. Moreover, the result in [22] is stated for $p>3$, but a careful inspection of the proof reveals that the same condition $p\ge 4$ is needed there as well[3].

[2]See equation (8) of the paper [22].

[3]This comes from hypothesis (5) in [22]. Also observe that this condition contains a small typo, m_{i-2} should be replaced by m_i-2.

Both the proofs of Theorem B and that of [22, Theorem 1] are based on a priori Lipschitz bounds, obtained by means of pointwise estimates in the vein of Bernstein method. However, computations are not the same and we believe ours to be slightly simpler. In [22] the first step is to look at the equation solved by a *concave* power of u, given by the function

$$w = (u + \|u\|_{L^\infty} + 1)^\gamma, \qquad 0 < \gamma < 1.$$

Then they consider the equation solved by (some function of) ∇w. There is an extra term in this new equation coming from the concave power which crucially leads to the result.

Here on the contrary we obtain the Lipschitz estimate by directly attacking equation (1.3). The main point is to consider the equation satisfied by the quantity

$$|\nabla u|^2 + \lambda\, u^2,$$

for a suitably large paramater λ. We notice that this is exactly the same test function used to prove classical gradient estimates for *linear* uniformly elliptic equations (see for example [16, Proposition 2.19]).

One of the drawbacks of these two strategies is the assumption on f, which does not seem to be optimal. Indeed, we expect the result to be true under the natural hypothesis $f \in L^q_{loc}(\Omega)$ with $q > N$.

1.3. **Plan of the paper.** In Section 2 we set notations and preliminary results needed throughout the whole paper. In particular, we introduce a regularized version of the problem which will be useful in order to get the desired Lipschitz estimate. Then Section 3 is devoted to prove some Caccioppoli-type inequalities for the gradient of the solution of the regularized problem. The proof of Theorem A is contained in Section 4, while Section 5 contains the proof of Theorem B. Two appendices containing some technical results complement the paper.

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2. Preliminaries

2.1. **Definitions and basic results.** Let $\Omega \subset \mathbb{R}^N$ be an open set and $p \geq 2$. In what follows we set for simplicity

$$g_i(t) = \frac{1}{p}\,(|t| - \delta_i)_+^p, \qquad t \in \mathbb{R},\ i = 1, \dots, N,$$

where $0 \leq \delta_1, \dots, \delta_N$ are given real numbers. We will also define

$$\delta = 1 + \max\{\delta_i \,:\, i = 1, \dots, N\}. \tag{2.1}$$

Remark 2.1 (Smoothness of g_i)**.** When p is an integer and $\delta_i > 0$, g_i is of class $C^{p-1,1}$. When $p \notin \mathbb{N}$, then $g_i \in C^{[p],p-[p]}(\mathbb{R})$ where $[\,\cdot\,]$ denotes the integer part.

Remark 2.2 (The limit case $p=2$)**.** Observe that for $p=2$ and $\delta_i>0$, we have $g_i\in C^{1,1}(\mathbb{R})\cap C^\infty(\mathbb{R}\setminus\{\delta_i,-\delta_i\})$, but $g_i\notin C^2(\mathbb{R})$. In this case, like in [7] a smoothing around $|t|=\delta_i$ would be necessary, notably for the result of Lemma 2.8 below. However, in order not to overburden the presentation, for the sequel we will assume for simplicity $p>2$ (see [7, Section 2] for more details).

We are interested in local minimizers of the following variational integral

$$\mathfrak{F}(u;\Omega')=\sum_{i=1}^N\int_{\Omega'} g_i(u_{x_i})\,dx+\int_{\Omega'} f\,u\,dx,\qquad u\in W^{1,p}_{loc}(\Omega),\tag{2.2}$$

where $f\in L^{p'}_{loc}(\Omega)$ and $\Omega'\Subset\Omega$. We recall that $u\in W^{1,p}_{loc}(\Omega)$ is said to be a *local minimizer* of $\mathfrak{F}$ if for every $\Omega'\Subset\Omega$ we have

$$\mathfrak{F}(u;\Omega')\le\mathfrak{F}(u+\varphi;\Omega'),\qquad \text{for every}\ \ \varphi\in W^{1,p}_0(\Omega').$$

We first observe that $\mathfrak{F}$ is not strictly convex, unless $\delta=1$, i.e. $\delta_1=\dots=\delta_N=0$. Thus minimizers are not unique in general. The following result guarantees that it will be sufficient to prove the desired result for one minimizer.

Lemma 2.3 (Propagation of regularity)**.** *Let $B\Subset\Omega$ be a ball and $V\in W^{1,p}(B)$. Let $u_1,u_2\in W^{1,p}(\Omega)$ be two solutions of*

$$\min\left\{\mathfrak{F}(v;B)\,:\,v-V\in W^{1,p}_0(B)\right\}.\tag{2.3}$$

Then it holds

$$\Big||(u_1)_{x_i}|-|(u_2)_{x_i}|\Big|\le 2\,\delta_i,\qquad a.\,e.\ in\ B,\ i=1,\dots,N.\tag{2.4}$$

In particular, if a minimizer of (2.3) is (locally) Lipschitz, then this remains true for all the other minimizers.

Proof. Let us suppose that (2.4) is not true. Then there exists $i_0\in\{1,\dots,N\}$ such that

$$E_{i_0}:=\left\{x\in B\,:\,\Big||(u_1)_{x_{i_0}}|-|(u_2)_{x_{i_0}}|\Big|>2\,\delta_{i_0}\right\},$$

has strictly positive measure. We then set $u_s=(1-s)\,u_1+s\,u_2$ for some $s\in(0,1)$ and observe that this is admissible in (2.3). In view of Lemma A.1 in Appendix A,

$$g_{i_0}\Big((1-s)\,(u_1)_{x_{i_0}}+s\,(u_2)_{x_{i_0}}\Big)<(1-s)\,g_{i_0}((u_1)_{x_{i_0}})+s\,g_{i_0}((u_2)_{x_{i_0}}),\qquad \text{a. e. in } E_{i_0}.$$

Thus we get

$$\mathfrak{F}(u_s)<(1-s)\,\mathfrak{F}(u_1)+s\,\mathfrak{F}(u_2)=\mathfrak{F}(u_1)=\mathfrak{F}(u_2),$$

which gives the desired contradiction. $\square$

We will also need the following regularity result, which is essentially contained in [20, Theorem 9.2]. A more general result of this type can be found in [4, Main Theorem].

Theorem 2.4. *Let $B\subset\mathbb{R}^N$ be a ball, $V\in C^2(\overline{B})$ and $f\in L^\infty(B)$. Let us consider the problem*

$$\min\left\{\int_B H(\nabla v)\,dx+\int_B f\,v\,dx\,:\,v-V\in W^{1,1}_0(B)\right\},\tag{2.5}$$

where $H:\mathbb{R}^N\to[0,\infty)$ is a C^2 convex function such that for some $\mu>0$

$$\langle D^2\,H(z)\,\xi,\xi\rangle\ge\mu\,|\xi|^2,\qquad \xi,z\in\mathbb{R}^N.\tag{2.6}$$

Then there exists a unique solution u. Moreover, $u \in W^{1,\infty}(B)$.

Proof. By [20, Theorem 9.2], we have that the problem

$$\min\left\{\int_B H(\nabla v)\,dx + \int_B f\,v\,dx \,:\, v - V \in W_0^{1,\infty}(B)\right\},$$

admits a solution $u \in W^{1,\infty}(B)$, which is also unique by strict convexity of the Lagrangian in the gradient variable. Then u satisfies the Euler-Lagrange equation

$$\int_\Omega \langle \nabla H(\nabla u), \nabla\varphi\rangle\,dx + \int_\Omega f\,\varphi\,dx = 0, \qquad \text{for every } \varphi \in W_0^{1,\infty}(B). \tag{2.7}$$

Thanks to the hypotheses on H and f and to the fact that $\nabla u \in L^\infty(B)$, equation (2.7) still holds with test functions $\varphi \in W_0^{1,1}(B)$. Thus by convexity, u solves (2.5) as well. □

2.2. **Approximation scheme.** We now introduce a regularized version of the original problem. We set

$$g_{i,\varepsilon}(t) = g_i(t) + \frac{\varepsilon}{2}\,t^2 = \frac{1}{p}\,(|t| - \delta_i)_+^p + \frac{\varepsilon}{2}\,t^2, \qquad t \in \mathbb{R}. \tag{2.8}$$

From now on, *we fix U a local minimizer of $\mathfrak{F}$*. We also fix a ball

$$B \Subset \Omega \qquad \text{such that} \qquad 2\,B \Subset \Omega \text{ as well.}$$

Here $\lambda\,B$ denotes the ball having the same center as B, scaled by a factor $\lambda > 0$.

For every $0 < \varepsilon \ll 1$ and every $x \in \overline{B}$, we set $U_\varepsilon(x) = U * \varrho_\varepsilon(x)$, where ϱ_ε is a smooth convolution kernel, supported in a ball of radius ε centered at the origin.

Then by definition of U_ε there exists $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon \le \varepsilon_0$

$$\|U_\varepsilon\|_{W^{1,p}(B)} = \|\nabla U_\varepsilon\|_{L^p(B)} + \|U_\varepsilon\|_{L^p(B)} \le \|\nabla U\|_{L^p(2\,B)} + \|U\|_{L^p(2\,B)}. \tag{2.9}$$

Finally, we define

$$\mathfrak{F}_\varepsilon(v; B) = \sum_{i=1}^N \int_B g_{i,\varepsilon}(v_{x_i})\,dx + \int_B f_\varepsilon\,v\,dx,$$

where $f_\varepsilon = f * \varrho_\varepsilon$. The following preliminary result is standard.

Lemma 2.5 (Basic energy estimate)**.** *For $0 < \varepsilon \le \varepsilon_0 < 1$, there exists a unique solution u_ε to the problem*

$$\min\left\{\mathfrak{F}_\varepsilon(v; B) \,:\, v - U_\varepsilon \in W_0^{1,p}(B)\right\}. \tag{2.10}$$

Moreover, there exists a constant $C = C(N, p) > 0$ such that the following uniform estimate holds

$$\int_B |\nabla u_\varepsilon|^p\,dx \le C\left[\int_{2\,B} |\nabla U|^p\,dx + |B|^{\frac{p'}{N}} \int_{2\,B} |f|^{p'}\,dx + (\varepsilon_0 + (\delta - 1)^p)|B|\right] =: C_1. \tag{2.11}$$

Proof. We start by observing that existence and uniqueness of u_ε follow from Theorem 2.4.

In order to prove (2.11), we use the minimality of u_ε, which implies $\mathfrak{F}_\varepsilon(u_\varepsilon; B) \le \mathfrak{F}_\varepsilon(U_\varepsilon; B)$. This gives

$$\sum_{i=1}^N \int_B g_{i,\varepsilon}((u_\varepsilon)_{x_i})\,dx \le \sum_{i=1}^N \int_B g_{i,\varepsilon}((U_\varepsilon)_{x_i})\,dx + \int_B |f_\varepsilon|\,|u_\varepsilon - U_\varepsilon|\,dx.$$

By recalling the definition (2.8) of $g_{i,\varepsilon}$, we have

$$\frac{1}{p}\left(\frac{|t|^p}{2^{p-1}}-(\delta-1)^p\right)\le g_{i,\varepsilon}(t)\le\frac{2}{p}\,|t|^p+\varepsilon_0\,\frac{p-2}{2p}, \tag{2.12}$$

The lower bound in (2.12) follows from

$$|t|^p\le 2^{p-1}\,((|t|-\delta_i)_+^p+\delta_i^p),$$

and the definition (2.1) of δ, while the upper bound is a consequence of Young's inequality. This implies

$$\sum_{i=1}^N\int_B|(u_\varepsilon)_{x_i}|^p\,dx\le C\sum_{i=1}^N\int_B|(U_\varepsilon)_{x_i}|^p+C\int_B|f_\varepsilon|\,|u_\varepsilon-U_\varepsilon|\,dx+C\,(\varepsilon_0+(\delta-1)^p)\,|B|,$$

where $C=C(N,p)>0$. By using $\|f_\varepsilon\|_{L^{p'}(B)}\le\|f\|_{L^{p'}(2\,B)}$ and (2.9), standard computations involving Poincaré inequality lead to the desired conclusion. □

Lemma 2.6 (Regularity of the minimizer I)**.** *If $f\in L^\infty_{loc}(\Omega)$, then $U\in L^\infty_{loc}(\Omega)$ and there exists a constant $C=C(N,p)>0$ such that for every $B_{2\,\varrho_0}\Subset\Omega$ we have*

$$\|U\|_{L^\infty(B_{\varrho_0/2})}\le C\left[\left(\fint_{B_{\varrho_0}}|U|^p\,dx\right)^{\frac{1}{p}}+\left(\delta+\varrho_0^{\frac{1}{p-1}}\,\|f\|^{\frac{1}{p-1}}_{L^\infty(B_{2\,\varrho_0})}\right)\,\varrho_0\right]. \tag{2.13}$$

Moreover, if u_ε still denotes the unique minimizer of (2.10), then there exists a constant $C=C(N,p)>0$ such that for every $0<\varepsilon\le\varepsilon_0$ we have

$$\|u_\varepsilon\|_{L^\infty(B)}\le C\left[\|U\|_{L^\infty(2\,B)}+\left(\left(\frac{C_1}{|B|}\right)^{\frac{1}{p}}+\delta+|B|^{\frac{1}{N\,(p-1)}}\,\|f\|^{\frac{1}{p-1}}_{L^\infty(2\,B)}\right)\,|B|^{\frac{1}{N}}\right]. \tag{2.14}$$

Here C_1 is the same quantity appearing in (2.11).

Proof. In order to prove the uniform L^∞ estimate, let us introduce the Lagragian

$$L_\varepsilon(x,u,z)=\sum_{i=1}^N g_{i,\varepsilon}(z_i)+f_\varepsilon(x)\,u. \tag{2.15}$$

Then we use again (2.12). This implies that for every $\Omega'\Subset\Omega$ and every $0\le\varepsilon\le\operatorname{dist}(\Omega',\partial\Omega)/2$

$$c\,|z|^p-\|f_\varepsilon\|_{L^\infty(\Omega')}\,|u|-C'\,\delta^p\le L_\varepsilon(x,u,z)\le\frac{1}{c}\,|z|^p+\|f_\varepsilon\|_{L^\infty(\Omega')}\,|u|+C'\delta^p. \tag{2.16}$$

with $0<c=c(N,p)<1$ and $C'=C'(N,p)>0$. In the previous inequality we also used that $\delta\ge1$. By definition of local minimizer, we have that if we choose $B_{2\,\varrho_0}\Subset\Omega$ the function U solves

$$\min\left\{\int_{B_{2\,\varrho_0}}L_0(x,v,\nabla v)\,dx\,:\,v-U\in W^{1,p}_0(B_{2\,\varrho_0})\right\}.$$

By using (2.16), we can appeal to the local a priori estimate [15, Theorem 7.5] and get for U

$$\|U\|_{L^\infty(B_{\varrho_0/2})}\le C\left[\left(\fint_{B_{\varrho_0}}|U|^p\,dx\right)^{\frac{1}{p}}+\left(\delta+\varrho_0^{\frac{1}{p-1}}\,\|f\|^{\frac{1}{p-1}}_{L^\infty(B_{2\,\varrho_0})}\right)\,\varrho_0\right],$$

with a constant $C=C(N,p)>0$.

We now come to u_ε, which solves

$$\min\left\{\int_B L_\varepsilon(x,v,\nabla v)\,dx\,:\,v-U_\varepsilon\in W^{1,p}_0(B)\right\}.$$

Again thanks to (2.16), if we now use [15, Remark 7.6] we get $u_\varepsilon\in L^\infty(B)$ with the global estimate

$$\|u_\varepsilon\|_{L^\infty(B)}\le C\,\left[\left(\fint_B |u_\varepsilon|^p\,dx\right)^{\frac{1}{p}}+\|U_\varepsilon\|_{L^\infty(\partial B)}+\left(\delta+|B|^{\frac{1}{p-1}}\,\|f_\varepsilon\|^{\frac{1}{p-1}}_{L^\infty(B)}\right)\,|B|^{\frac{1}{N}}\right],$$

where $C=C(N,p)>0$. We then observe that

$$\left(\fint_B |u_\varepsilon|^p\,dx\right)^{\frac{1}{p}}\le C\,\left(\fint_B |U_\varepsilon|^p\,dx\right)^{\frac{1}{p}}+C\,|B|^{\frac{1}{N}-\frac{1}{p}}\,C_1^{\frac{1}{p}},$$

thanks to the triangle inequality, Poincaré inequality and (2.11). If we now use

$$\|U_\varepsilon\|_{L^p(B)}\le\|U\|_{L^p(2\,B)}\le|2\,B|^{\frac{1}{p}}\,\|U\|_{L^\infty(2\,B)},\qquad \|U_\varepsilon\|_{L^\infty(\partial B)}\le\|U\|_{L^\infty(2\,B)}$$

and

$$\|f_\varepsilon\|_{L^\infty(B)}\le\|f\|_{L^\infty(2\,B)},$$

we get the desired conclusion. □

Remark 2.7. The uniform L^∞ estimate (2.14) will be needed in the proof of Theorem B.

The following result is not optimal, but it is suitable to our needs.

Lemma 2.8 (Regularity of the minimizer II)**.** *Let u_ε still denote the unique minimizer of* (2.10)*. We have $u_\varepsilon\in C^k_{loc}(B)$, where*

$$k=\begin{cases}2, & \textit{if } 2<p\le 3,\\ 3, & \textit{if } p>3.\end{cases}$$

Proof. By Theorem 2.4 we already know that $u_\varepsilon\in W^{1,\infty}(B)$. Thus the quantity $\ell=\|\nabla u_\varepsilon\|_{L^\infty(B)}$ is finite. By optimality, we have that u_ε solves the elliptic equation

$$\operatorname{div}(\nabla F_\varepsilon(\nabla u_\varepsilon))=f_\varepsilon,\qquad \text{in } B,\tag{2.17}$$

where F_ε is given by

$$F_\varepsilon(z)=\sum_{i=1}^N g_i(z_i)+\frac{\varepsilon}{2}\,|z|^2,\qquad z\in\mathbb{R}^N.$$

Since we have

$$\varepsilon\,|\xi|^2\le\langle D^2F_\varepsilon(\nabla u_\varepsilon)\,\xi,\xi\rangle\le\left(\varepsilon+(p-1)\,\ell^{p-2}\right)\,|\xi|^2,\qquad \text{on } B,$$

we can infer $u_\varepsilon\in W^{2,2}_{loc}(B)$ by a standard differential quotients argument (see for example [15, Theorem 8.1]). This in turn permits to find the equation locally solved by ∇u_ε, by differentiating (2.17). Thus $\nabla u_\varepsilon\in C^{0,\sigma}_{loc}(B)$ by the celebrated De Giorgi–Moser–Nash Theorem, for some $\sigma>0$. It remains to observe that $F_\varepsilon\in C^{k,\alpha}$, where k is as in the statement and

$$\alpha=\begin{cases}\min\{p-2,1\}, & \text{if } 2<p\le 3,\\ \min\{p-3,1\}, & \text{if } p>3.\end{cases}$$

Then [15, Theorem 10.18] implies that u_ε has the claimed regularity properties. □

Lemma 2.9 (Convergence to a minimizer)**.** *With the same notation as before, there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\varepsilon_0)$ converging to 0, such that*

$$\lim_{k\to\infty} \|u_{\varepsilon_k} - \widetilde{u}\|_{L^p(B)} = 0,$$

where $\widetilde{u}$ is a solution of

$$\min\left\{\mathfrak{F}(v;B)\,:\, v - U \in W^{1,p}_0(B)\right\}. \tag{2.18}$$

Proof. By (2.11), there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ converging to 0 as k goes to ∞ and a function $\widetilde{u} \in W^{1,p}(B)$ such that $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ converges weakly to $\widetilde{u}$ in $W^{1,p}(B)$ and strongly in $L^p(B)$. Let us prove that $\widetilde{u}$ is actually a solution of (2.18).

The function $U_{\varepsilon_k} = U * \varrho_{\varepsilon_k}$ is of course admissible for the regularized problem (2.10). By using this, the minimality of u_{ε_k}, the definition of $\mathfrak{F}_{\varepsilon_k}$ and the strong convergence of f_{ε_k} to f, we get

$$\liminf_{k\to\infty} \mathfrak{F}_{\varepsilon_k}(U_{\varepsilon_k};B) \ge \liminf_{k\to\infty} \mathfrak{F}_{\varepsilon_k}(u_{\varepsilon_k};B) \ge \liminf_{k\to\infty} \mathfrak{F}(u_{\varepsilon_k};B) \ge \mathfrak{F}(\widetilde{u};B).$$

In the last inequality we also used the weak lower semicontinuity of $\mathfrak{F}$. We then observe that by using the strong convergence of U_{ε_k} to U and inequality (A.2) in Appendix A, we get

$$\begin{aligned}\lim_{k\to\infty} |\mathfrak{F}_{\varepsilon_k}(U_{\varepsilon_k};B) - \mathfrak{F}(U;B)| &\le \lim_{k\to\infty} \sum_{i=1}^N \int_B |g_i((U_{\varepsilon_k})_{x_i}) - g_i(U_{x_i})|\,dx \\ &\quad + \lim_{k\to\infty} \frac{\varepsilon_k}{2}\int_B |\nabla U_{\varepsilon_k}|^2 + \lim_{k\to\infty}\int_B |f_{\varepsilon_k}\,U_{\varepsilon_k} - f\,U|\,dx = 0,\end{aligned}$$

and thus

$$\mathfrak{F}(U;B) = \lim_{k\to\infty} \mathfrak{F}_{\varepsilon_k}(U_{\varepsilon_k};B) \ge \mathfrak{F}(\widetilde{u};B).$$

Since by construction U is a local minimizer of $\mathfrak{F}$, the function U itself is a solution of (2.18). Then the previous inequality implies that $\widetilde{u}$ is a minimizer. □

3. Local energy estimates for the regularized problem

For the ball $B \Subset \Omega$ we consider the regularized problem (2.10). We still denote by u_ε its unique solution, which verifies the Euler-Lagrange equation

$$\sum_{i=1}^N \int g'_{i,\varepsilon}((u_\varepsilon)_{x_i})\,\varphi_{x_i}\,dx + \int f_\varepsilon\,\varphi\,dx = 0, \qquad \varphi \in W^{1,p}_0(B). \tag{3.1}$$

From now on, in order to simplify the notation, we will systematically forget the subscript ε on u_ε and simply write u.

We now insert a test function of the form $\varphi = \psi_{x_j} \in W^{1,p}_0(B)$ in (3.1), compactly supported in B. Then an integration by parts leads us to

$$\sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i})\,u_{x_i\,x_j}\,\psi_{x_i}\,dx - \int f_\varepsilon\,\psi_{x_j}\,dx = 0, \tag{3.2}$$

for $j = 1,\dots,N$. This is the equation solved by u_{x_j}.

3.1. **Caccioppoli-type inequalities.** In what follows we use the parameter δ defined in (2.1). The general Caccioppoli inequality for an important class of subsolutions is given by the following result.

Lemma 3.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a C^2 convex function such that*

$$\Phi'(t) \equiv 0 \qquad \text{for } |t| \le \delta. \tag{3.3}$$

Then there exists a constant $C_2 = C_2(p) > 0$ such that for every Lipschitz function η with compact support in B and every $j = 1, \dots, N$, we have

$$\begin{aligned}
&\sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left|\big(\Phi(u_{x_j})\big)_{x_i}\right|^2 \eta^2\, dx \\
&\qquad \le C_2 \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i})\, |\Phi(u_{x_j})|^2\, |\eta_{x_i}|^2\, dx \\
&\qquad + C_2 \int_{A_j} |f_\varepsilon|^2 \Big[\Phi'(u_{x_j})^2 + \Phi''(u_{x_j})\, \Phi(u_{x_j})\Big]\, \eta^2\, dx + C_2 \int_{A_j} \Phi(u_{x_j})^2\, |\eta_{x_j}|^2\, dx,
\end{aligned} \tag{3.4}$$

where we set $A_j = \{x \in B \,:\, |u_{x_j}| \ge \delta\}$.

Proof. In (3.2) we take the test function[4] $\psi = \zeta\, \Phi'(u_{x_j})$, with $\Phi : \mathbb{R} \to \mathbb{R}^+$ as in the statement and ζ a nonnegative Lipschitz function with support in B. We thus obtain

$$\sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i})\, \big(\Phi(u_{x_j})\big)_{x_i}\, \zeta_{x_i}\, dx + \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i})\, u^2_{x_i\, x_j}\, \Phi''(u_{x_j})\, \zeta\, dx = \int_{A_j} f_\varepsilon\, \big(\zeta\, \Phi'(u_{x_j})\big)_{x_j}\, dx.$$

Finally, we test the previous equation against $\zeta = \eta^2\, \Phi(u_{x_j})$, where η is again a Lipschitz function with support in B. Then we get

$$\begin{aligned}
&\sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left|\big(\Phi(u_{x_j})\big)_{x_i}\right|^2 \eta^2\, dx + \mathcal{S}(\eta) \\
&\qquad \le 2 \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left|\big(\Phi(u_{x_j})\big)_{x_i}\right| \Phi(u_{x_j})\, |\eta|\, |\eta_{x_i}| dx + \int_{A_j} |f_\varepsilon| \left|\big(\eta^2\, \Phi(u_{x_j})\, \Phi'(u_{x_j})\big)_{x_j}\right| dx,
\end{aligned}$$

where we have introduced the *sponge term*

$$\mathcal{S}(\eta) = \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i})\, u^2_{x_i\, x_j}\, \Phi''(u_{x_j})\, \Phi(u_{x_j})\, \eta^2\, dx,$$

which is indeed positive. From the previous inequality, by Young's inequality in the first term on the right-hand side

$$\begin{aligned}
&\sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left|\big(\Phi(u_{x_j})\big)_{x_i}\right|^2 \eta^2\, dx + 2\, \mathcal{S}(\eta) \\
&\qquad \le 4 \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i})\, |\Phi(u_{x_j})|^2\, |\eta_{x_i}|^2\, dx + 2 \int_{A_j} |f_\varepsilon| \left|\big(\eta^2\, \Phi(u_{x_j})\, \Phi'(u_{x_j})\big)_{x_j}\right| dx.
\end{aligned} \tag{3.5}$$

[4] Observe that this is a legitimate test function by Lemma 2.8.

We now estimate the term containing f_ε. We first observe

$$\int_{A_j} |f_\varepsilon| \left| \left(\eta^2\, \Phi(u_{x_j})\, \Phi'(u_{x_j})\right)_{x_j} \right| dx \le \int_{A_j} |f_\varepsilon|\, |\Phi'(u_{x_j})| \left| \left(\Phi(u_{x_j})\right)_{x_j} \right| \eta^2\, dx$$
$$+ 2 \int_{A_j} |f_\varepsilon|\, |\Phi'(u_{x_j})|\, \Phi(u_{x_j})\, |\eta|\, |\eta_{x_j}|\, dx + \int_{A_j} |f_\varepsilon| \left| \left(\Phi'(u_{x_j})\right)_{x_j} \right| \Phi(u_{x_j})\, \eta^2\, dx.$$

On the set A_j we have

$$g''_{j,\varepsilon}(u_{x_j}) \ge (p-1), \tag{3.6}$$

since $\delta \ge 1$ by definition (2.1). Let us consider the first term above containing f_ε:

$$\begin{aligned}\int_{A_j} |f_\varepsilon|\, |\Phi'(u_{x_j})| \left| \left(\Phi(u_{x_j})\right)_{x_j} \right| \eta^2\, dx &\le \frac{1}{2\,\tau} \int_{A_j} |f_\varepsilon|^2\, |\Phi'(u_{x_j})|^2\, \eta^2\, dx \\ &\quad + \frac{\tau}{2} \int_{A_j} \left| \left(\Phi(u_{x_j})\right)_{x_j} \right|^2 \eta^2\, dx \\ &\le \frac{1}{2\,\tau} \int_{A_j} |f_\varepsilon|^2\, |\Phi'(u_{x_j})|^2\, \eta^2\, dx \\ &\quad + \frac{\tau}{2\,(p-1)} \int_{A_j} g''_{j,\varepsilon}(u_{x_j}) \left| \left(\Phi(u_{x_j})\right)_{x_j} \right|^2 \eta^2\, dx.\end{aligned}$$

In the last estimate we used (3.6). Then the last integral can be absorbed in the left-hand side of (3.5), by taking $\tau = (p-1)/2$. The second term containing f_ε is simply estimated by Young's inequality

$$\int_{A_j} |f_\varepsilon|\, |\Phi'(u_{x_j})|\, \Phi(u_{x_j})\, \eta\, |\eta_{x_j}|\, dx \le \frac{1}{2} \int_{A_j} |f_\varepsilon|^2\, |\Phi'(u_{x_j})|^2\, \eta^2\, dx + \frac{1}{2} \int_{A_j} \Phi(u_{x_j})^2\, |\eta_{x_j}|^2\, dx,$$

while for the last one we use the sponge term $\mathcal{S}(\eta)$ to absorb the Hessian of u. Namely, we have

$$\begin{aligned}\int_{A_j} |f_\varepsilon| \left| \left(\Phi'(u_{x_j})\right)_{x_j} \right| \Phi(u_{x_j})\, \eta^2\, dx &= \int_{A_j} |f_\varepsilon|\, |u_{x_j\, x_j}|\, \Phi''(u_{x_j})\, \Phi(u_{x_j})\, \eta^2\, dx \\ &\le \frac{\tau}{2} \int_{A_j} u^2_{x_j\, x_j}\, \Phi''(u_{x_j})\, \Phi(u_{x_j})\, \eta^2\, dx \\ &\quad + \frac{1}{2\,\tau} \int_{A_j} |f_\varepsilon|^2\, \Phi''(u_{x_j})\, \Phi(u_{x_j})\, \eta^2\, dx \\ &\le \frac{\tau}{2\,(p-1)}\, \mathcal{S}(\eta) + \frac{1}{2\,\tau} \int_{A_j} |f_\varepsilon|^2\, \Phi''(u_{x_j})\, \Phi(u_{x_j})\, \eta^2\, dx.\end{aligned}$$

In the last estimate we used again (3.6). The term $\tau/(p-1)\, \mathcal{S}(\eta)$ can then be absorbed in the left-hand side of (3.5). This concludes the proof. □

If we allow for derivatives of f_ε on the right-hand side of (3.4), the previous estimate is simpler to get. In this case we can allow for more general subsolutions.

Lemma 3.2 (Right-hand side in a Sobolev space)**.** *Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a C^1 convex function. Then there exists a constant $C_3 = C_3(p) > 0$ such that for every Lipschitz function η with compact* *support in B and every $j = 1, \dots, N$, we have*

$$
\begin{aligned}
&\sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \left| \left(\Phi(u_{x_j})\right)_{x_i} \right|^2 \eta^2 \, dx \\
&\leq C_3 \sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \, |\Phi(u_{x_j})|^2 \, |\eta_{x_i}|^2 \, dx + C_3 \int |(f_\varepsilon)_{x_j}| \, |\Phi'(u_{x_j})| \, |\Phi(u_{x_j})| \, \eta^2 \, dx.
\end{aligned}
\tag{3.7}
$$

Proof. Let us suppose for simplicity that $\Phi \in C^2$. If this were not the case, a standard smoothing argument would be needed, we leave the details to the reader.

We start by observing that equation (3.2) can also be written as

$$
\sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \, u_{x_i \, x_j} \, \psi_{x_i} \, dx + \int (f_\varepsilon)_{x_j} \, \psi \, dx = 0, \quad j = 1, \dots, N.
\tag{3.8}
$$

Then we take in (3.8) the test function $\psi = \zeta \, \Phi'(u_{x_j})$ as before, with Φ as in the statement and ζ a nonnegative Lipschitz function supported in B. We obtain

$$
\sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \, \left(\Phi(u_{x_j})\right)_{x_i} \, \zeta_{x_i} \, dx \leq - \int (f_\varepsilon)_{x_j} \, \Phi'(u_{x_j}) \, \zeta \, dx,
$$

thanks to the fact that $\zeta \, \Phi'' \geq 0$. Finally, we take again $\zeta = \eta^2 \, \Phi(u_{x_j})$, to get

$$
\begin{aligned}
\sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \left| \left(\Phi(u_{x_j})\right)_{x_i} \right|^2 \eta^2 \, dx &\leq 2 \sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \left| \left(\Phi(u_{x_j})\right)_{x_i} \right| \Phi(u_{x_j}) \, |\eta| \, |\eta_{x_i}| \, dx \\
&\quad + \int \left|(f_\varepsilon)_{x_j}\right| \, |\Phi'(u_{x_j})| \, |\Phi(u_{x_j})| \, \eta^2 \, dx.
\end{aligned}
$$

By using Young's inequality as before, we get

$$
\begin{aligned}
\sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \left| \left(\Phi(u_{x_j})\right)_{x_i} \right|^2 \eta^2 \, dx &\leq 4 \sum_{i=1}^{N} \int g''_{i,\varepsilon}(u_{x_i}) \, |\Phi(u_{x_j})|^2 \, |\eta_{x_i}|^2 \, dx \\
&\quad + 2 \int |(f_\varepsilon)_{x_j}| \, |\Phi'(u_{x_j})| \, |\Phi(u_{x_j})| \, \eta^2 \, dx.
\end{aligned}
$$

This concludes the proof. □

3.2. **A Sobolev estimate.** In what follows we set

$$
W_j = \delta^2 + (|u_{x_j}| - \delta)_+^2, \qquad j = 1, \dots, N.
$$

Lemma 3.3. *There exists a constant $C_4 = C_4(p) > 0$ such that for every Lipschitz function η with compact support in B and every $j = 1, \dots, N$, we have*

$$
\sum_{i=1}^{N} \int \left| \nabla W_i^{\frac{p}{4}} \right|^2 \eta^2 \, dx \leq C_4 \, \delta^{p-2} \sum_{i,j=1}^{N} \int W_i^{\frac{p-2}{2}} \, W_j \, |\eta_{x_i}|^2 \, dx + C_4 \, \delta^{p-2} \sum_{j=1}^{N} \int |(f_\varepsilon)_{x_j}| \, \sqrt{W_j} \, \eta^2 \, dx.
\tag{3.9}
$$

Proof. We insert the test function $\psi = \eta^2\, u_{x_j}$ in (3.8). With computations similar to that of Lemma 3.2, we now get for every $j = 1, \dots, N$

$$\sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i})\, |u_{x_i\,x_j}|^2\, \eta^2\, dx \le 4 \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i})\, |u_{x_j}|^2\, |\eta_{x_i}|^2\, dx + 2 \int |(f_\varepsilon)_{x_j}|\, |u_{x_j}|\, \eta^2\, dx. \tag{3.10}$$

Then we observe that by Lemma A.3 for $|t| \ge \delta$

$$\begin{aligned} g''_{i,\varepsilon}(t) &\ge (p-1) \left(\frac{\delta - \delta_i}{\delta}\right)^{p-2} \left(\delta^2 + (|t| - \delta)_+^2\right)^{\frac{p-2}{2}} \\ &\ge \frac{p-1}{\delta^{p-2}} \left(\delta^2 + (|t| - \delta)_+^2\right)^{\frac{p-4}{2}} (|t| - \delta)_+^2. \end{aligned} \tag{3.11}$$

In the second inequality above we also used that $p \ge 2$ and $\delta - \delta_i \ge 1$. Then, by (3.11) we have[5]

$$g''_{i,\varepsilon}(u_{x_i})\, |u_{x_i\,x_j}|^2 \ge \frac{p-1}{\delta^{p-2}}\, W_i^{\frac{p-4}{2}}\, (|u_{x_i}| - \delta)_+^2\, |u_{x_i\,x_j}|^2 = \frac{c}{\delta^{p-2}} \left| \left(W_i^{\frac{p}{4}}\right)_{x_j} \right|^2,$$

where $c = c(p) > 0$. We further observe that

$$|u_{x_j}| \le \sqrt{2}\, \sqrt{W_j},$$

which implies as well

$$g''_{i,\varepsilon}(u_{x_i}) = (p-1)\, (|u_{x_i}| - \delta_i)_+^{p-2} + \varepsilon \le c\, W_i^{\frac{p-2}{2}}, \tag{3.12}$$

where $c = c(p) > 0$. Then we get the desired result from (3.10). $\square$

In what follows, we will use for simplicity the notation

$$\fint_E \varphi\, dx := \frac{1}{|E|} \int_E \varphi\, dx.$$

Corollary 3.4. *There exists a constant $C_5 = C_5(p, N) > 0$ such that for every pair of concentric balls $B_{R_1} \Subset B_{R_0} \Subset B$, we have*

$$\begin{aligned} \sum_{j=1}^N \frac{1}{R_1^{N-2}} \int_{B_{R_1}} \left|\nabla W_j^{\frac{p}{4}}\right|^2 dx &\le C_5\, \delta^{p-2} \left(\frac{R_0}{R_1}\right)^{N-2} \left(\frac{R_0}{R_0 - R_1}\right)^2 \sum_{j=1}^N \fint_{B_{R_0}} W_j^{\frac{p}{2}}\, dx \\ &\quad + C_5\, \delta^{p-2} \left(\frac{R_0}{R_1}\right)^{N-2} R_0^{\frac{2}{p-1} - N + 2} \int_{B_{R_0}} |\nabla f_\varepsilon|^{p'}\, dx. \end{aligned} \tag{3.13}$$

Proof. Let us assume for simplicity that the balls are centered at the origin. It is sufficient to insert the test function

$$\eta(x) = \min\left\{1, \frac{(R_0 - |x|)_+}{R_0 - R_1}\right\},$$

in (3.9) and then use Hölder's and Young's inequalities in the right-hand side. These give

$$\sum_{i,j=1}^N \int W_i^{\frac{p-2}{2}}\, W_j\, |\eta_{x_i}|^2\, dx \le \frac{1}{(R_0 - R_1)^2} \sum_{i,j=1}^N \left(\int_{B_{R_0}} W_i^{\frac{p}{2}}\, dx\right)^{\frac{p-2}{p}} \left(\int_{B_{R_0}} W_j^{\frac{p}{2}}\, dx\right)^{\frac{2}{p}},$$

[5] Observe that the inequality holds true everywhere, not only on A_i, since W_i is constant outside A_i.

and

$$\sum_{j=1}^{N}\int |(f_\varepsilon)_{x_j}|\,\sqrt{W_j}\,\eta^2\,dx \le \frac{p-1}{p}\sum_{j=1}^{N} R_0^{\frac{2}{p-1}}\int_{B_{R_0}} |(f_\varepsilon)_{x_j}|^{p'}\,dx + \frac{1}{p\,R_0^2}\sum_{j=1}^{N}\int_{B_{R_0}} W_j^{\frac{p}{2}}\,dx,$$

which concludes the proof. □

Remark 3.5 (Uniform Sobolev estimate)**.** From the previous result, we obtain that if $f \in W^{1,p'}_{loc}(\Omega)$, then for every $i = 1,\dots,N$ the function $W_i^{p/4}$ enjoys a $W^{1,2}_{loc}(B)$ estimate independent of ε, thanks to (2.11) and

$$\|f_\varepsilon\|_{W^{1,p'}(B_{R_0})} \le \|f\|_{W^{1,p'}(2\,B)}.$$

3.3. **Power-type subsolutions.** We still use the notation

$$W_j = \delta^2 + (|u_{x_j}| - \delta)_+^2, \qquad j = 1,\dots,N.$$

Then we have the following result.

Lemma 3.6. *There exists a constant $C_6 = C_6(p) > 0$ such that for every $s \ge 0$, every Lipschitz function η with compact support in B and every $j = 1,\dots,N$, we have*

$$\sum_{i=1}^{N}\int g''_{i,\varepsilon}(u_{x_i})\left|\left(W_j^{\frac{s+1}{2}}\right)_{x_i}\right|^2\eta^2\,dx \le C_6\sum_{i=1}^{N}\int W_i^{\frac{p-2}{2}}\,W_j^{s+1}\,|\nabla\eta|^2\,dx + C_6\,(s+1)^2\int |f_\varepsilon|^2\,W_j^s\,\eta^2\,dx. \tag{3.14}$$

Proof. In equation (3.4) we make the choice[6]

$$\Phi(t) = \left(\delta^2 + (|t| - \delta)_+^2\right)^{\frac{s+1}{2}},$$

for $s \ge 0$ which satisfies hypothesis (3.3). Observe that by definition we have

$$\Phi(u_{x_j}) = W_j^{\frac{s+1}{2}},$$

so that

$$\left|\left(\Phi(u_{x_j})\right)_{x_i}\right|^2 = \left|\left(W_j^{\frac{s+1}{2}}\right)_{x_i}\right|^2.$$

Thus the left-hand side of (3.4) coincides with

$$\sum_{i=1}^{N}\int g''_{i,\varepsilon}(u_{x_i})\left|\left(W_j^{\frac{s+1}{2}}\right)_{x_i}\right|^2\eta^2\,dx.$$

We now come to the right-hand side:

$$\begin{aligned}\sum_{i=1}^{N}\int g''_{i,\varepsilon}(u_{x_i})\,|\Phi(u_{x_j})|^2\,|\eta_{x_i}|^2\,dx &= \sum_{i=1}^{N}\int g''_{i,\varepsilon}(u_{x_i})\,W_j^{s+1}\,|\eta_{x_i}|^2\,dx\\ &\le C\sum_{i=1}^{N}\int W_i^{\frac{p-2}{2}}\,W_j^{s+1}\,|\eta_{x_i}|^2\,dx,\end{aligned}$$

[6] Observe that this function is not C^2, but only $C^{1,1}$ near $t = \delta$ or $t = -\delta$. This is not a big issue, since in any case Φ'' stays bounded as $|t| \to \delta$, thus we can use (3.4) for a regularization of Φ and then pass to the limit at the end.

thanks to (3.12). For the other two terms, by using the definition of Φ we simply have

$$\begin{aligned}\int_{A_j} |f_\varepsilon|^2 \left[\Phi'(u_{x_j})^2 + \Phi''(u_{x_j})\,\Phi(u_{x_j})\right]\eta^2\,dx + \int_{A_j} \Phi(u_{x_j})^2\,|\eta_{x_j}|^2\,dx \\ \le C\,(s+1)^2 \int |f_\varepsilon|^2\,W_j^s\,\eta^2\,dx \\ + \sum_{i=1}^N \int W_i^{\frac{p-2}{2}}\,W_j^{s+1}\,|\eta_{x_j}|^2\,dx,\end{aligned}$$

where we used that

$$\left[\Phi'(t)^2 + \Phi''(t)\,\Phi(t)\right] \le C\,(s+1)^2\left(\delta^2 + (|t|-\delta)_+^2\right)^s,$$

for some $C > 0$ independent of s and $W_j^{s+1} \le \sum_{i=1}^N W_i^{\frac{p-2}{2}}\,W_j^{s+1}$, which follows from $W_i \ge 1$. □

In particular, we get an estimate for the *diagonal terms*, corresponding to $i = j$.

Corollary 3.7. *There exists a constant $C_7 = C_7(p) > 0$ such that for every $s \ge 0$, every Lipschitz function η with compact support in Ω and every $j = 1,\dots,N$, we have*

$$\begin{aligned}\int \left|\left(W_j^{\frac{p}{4}+\frac{s}{2}}\right)_{x_j}\right|^2 \eta^2\,dx \le C_7\,\delta^{p-2}\sum_{i=1}^N \int W_i^{\frac{p-2}{2}}\,W_j^{s+1}\,|\nabla\eta|^2\,dx \\ + C_7\,\delta^{p-2}\,(s+1)^2 \int |f_\varepsilon|^2\,W_j^s\,\eta^2\,dx.\end{aligned} \tag{3.15}$$

Proof. We fix j, by keeping only the term $i = j$ and dropping all the others in the left-hand side of (3.14), we get

$$\begin{aligned}\int_{A_j} g''_{j,\varepsilon}(u_{x_j})\left|\left(W_j^{\frac{s+1}{2}}\right)_{x_j}\right|^2 \eta^2\,dx \le C_6 \sum_{i=1}^N \int W_i^{\frac{p-2}{2}}\,W_j^{s+1}\,|\nabla\eta|^2\,dx \\ + C_6\,(s+1)^2 \int |f_\varepsilon|^2\,W_j^s\,\eta^2\,dx,\end{aligned}$$

where we recall that $A_j = \{x \in B : |u_{x_j}| \ge \delta\}$. We now observe that again by Lemma A.3 on A_j we have

$$g''_{j,\varepsilon}(u_{x_j}) \ge \frac{p-1}{\delta^{p-2}}\,\left[\delta^2 + (|u_{x_j}|-\delta)_+^2\right]^{\frac{p-2}{2}} = \frac{p-1}{\delta^{p-2}}\,W_j^{\frac{p-2}{2}},$$

and that

$$W_j^{\frac{p-2}{2}}\left|\left(W_j^{\frac{s+1}{2}}\right)_{x_j}\right|^2 = \left(\frac{2+2\,s}{p+2\,s}\right)^2\left|\left(W_j^{\frac{p}{4}+\frac{s}{2}}\right)_{x_j}\right|^2 \ge \left(\frac{2}{p}\right)^2\left|\left(W_j^{\frac{p}{4}+\frac{s}{2}}\right)_{x_j}\right|^2,$$

so that the conclusion follows. □

4. Proof of Theorem A

The core of the proof of Theorem A is the a priori estimate of Proposition 4.1 below. We postpone it and proceed with the proof of Theorem A.

Proof. Let $\Omega' \Subset \Omega$ and set $d = \mathrm{dist}(\Omega', \partial\Omega)$. We take $0 < r_0 \le d/100$, then Ω' can be covered by a finite number of balls centered at points in Ω' and having radius r_0. Let $B_{r_0} := B_{r_0}(x_0) \Subset \Omega$ be one of these balls, it is clearly sufficient to show that

$$\|\nabla U\|_{L^\infty(B_{r_0})} < +\infty.$$

To this aim we take the solution u_ε of the regularized problem (2.10) in the ball $B := B_{4\,r_0}(x_0)$. Observe that by construction we have $2\,B = B_{8\,r_0}(x_0) \Subset \Omega$. Then there exists $\varepsilon_0 = \varepsilon_0(r_0) > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$

$$\|f_\varepsilon\|_{W^{1,p'}(B_{2\,r_0})} + \|f_\varepsilon\|_{L^{2\,p'}(B_{2\,r_0})} \le \|f\|_{W^{1,p'}(2\,B)} + \|f\|_{L^{2\,p'}(2\,B)} \le C\,\|f\|_{W^{1,p'}(2\,B)}, \tag{4.1}$$

for some $C = C(p, |B|) > 0$. In the second estimate we used Poincaré-Sobolev inequality, indeed for $N = 2$ we have[7]

$$W^{1,p'}(2\,B) \hookrightarrow L^{2\,p'}(2\,B), \qquad \text{since } 2\,p' < \frac{2\,p'}{2 - p'}.$$

By using (4.1) and (2.11) in estimate (4.3) below with $R_0 = 2\,r_0$ we get

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{r_0})} \le C, \qquad \text{for every } 0 < \varepsilon \le \varepsilon_0, \tag{4.2}$$

where $C > 0$ depends only on p, δ, r_0, $\|f\|_{W^{1,p'}(2\,B)}$ and the constant C_1 in (2.11). We then observe that by Lemma 2.9, we can find a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ converging to 0 and such that $\{u_{\varepsilon_k}\}$ converges strongly in $L^p(B)$ and weakly in $W^{1,p}(B)$ to a solution $\widetilde{u}$ of

$$\min\{\mathfrak{F}(v; B) \, : \, v - U \in W^{1,p}_0(B)\}.$$

By lower semicontinuity we have that $\widetilde{u}$ still satisfies (4.2). It is now sufficient to use Lemma 2.3 in order to transfer this Lipschitz estimate from $\widetilde{u}$ to the original local minimizer U. This concludes the proof. □

Proposition 4.1 (Uniform Lipschitz estimate, $N = 2$)**.** *Let $N = 2$ and $p \ge 2$. Then for every pair of concentric balls $B_{r_0} \Subset B_{R_0} \Subset B$ and $i = 1, 2$ we have*

$$\Big\|(u_\varepsilon)_{x_i}\Big\|_{L^\infty(B_{r_0})} \le C_8\,\delta^{p-2}\left(\frac{R_0}{R_0 - r_0}\right)^4 \mathcal{J}(u_\varepsilon, f_\varepsilon; R_0, r_0)^2 \left[\left(\fint_{B_{R_0}} |(u_\varepsilon)_{x_i}|^p\,dx\right)^{\frac{1}{p}} + \delta\right], \tag{4.3}$$

where $C_8 = C_8(p) > 0$ is a constant that only depends on p and

$$\begin{aligned}\mathcal{J}(u_\varepsilon, f_\varepsilon; R_0, r_0) &= \delta^{p-2}\left(\frac{R_0}{R_0 - r_0}\right)^2 \left[\fint_{B_{R_0}} |\nabla u_\varepsilon|^p\,dx + \delta^p\right] \\ &\quad + \delta^{p-2}\,R_0^{\frac{2}{p-1}} \int_{B_{R_0}} |\nabla f_\varepsilon|^{p'}\,dx + \delta^{p-2}\,R_0^{\frac{2}{p}} \left(\int_{B_{R_0}} |f_\varepsilon|^{2\,p'}\,dx\right)^{\frac{1}{p'}}.\end{aligned}$$

[7] In contrast with p^*, the exponent $2\,p'$ has the advantage of being well-defined even in the case $p = p' = 2$.

Proof. For notational simplicity, we write again u in place of u_ε. We still use the notation

$$W_j = \delta^2 + \big(|u_{x_j}| - \delta\big)_+^2, \qquad j = 1, 2.$$

We give the proof for u_{x_1}, the one for u_{x_2} being exactly the same. By (3.15) we already know that

$$\int \left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\right)_{x_1}\right|^2 \eta^2\,dx \le C_7\,\delta^{p-2} \sum_{i=1}^2 \int W_i^{\frac{p-2}{2}}\, W_1^{s+1}\, |\nabla\eta|^2\,dx + C_7\,\delta^{p-2}\,(s+1)^2 \int |f_\varepsilon|^2\, W_1^s\, \eta^2\,dx,$$

where η is any Lipschitz function supported on B and such that $0 \le \eta \le 1$. We add the term

$$\int |\eta_{x_1}|^2\; W_1^{\frac{p}{2}+s}\,dx,$$

on both sides of the previous inequality and observe that

$$\int \left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\right)_{x_1}\right|^2 \eta^2\,dx + \int W_1^{\frac{p}{2}+s}\, |\eta_{x_1}|^2\,dx \ge \frac{1}{2} \int \left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_1}\right|^2 dx.$$

We thus obtain

$$\begin{aligned} \int \left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_1}\right|^2 dx &\le C\,\delta^{p-2} \sum_{i=1}^2 \int W_i^{\frac{p-2}{2}}\, W_1^{s+1}\, |\nabla\eta|^2\,dx \\ &\quad + C\,\delta^{p-2}\,(s+1)^2 \int |f_\varepsilon|^2\, W_1^s\,\eta^2\,dx, \end{aligned} \tag{4.4}$$

with $C = C(p) > 0$, where we used that $\delta \ge 1$.

The main problem of the Caccioppoli inequality (3.14) is that apparently we can not use it to control the *missing term*

$$\left(W_1^{\frac{p}{4}+\frac{s}{2}}\right)_{x_2}.$$

Thus there is an obstruction to derive estimates for $\nabla W_1^{\frac{p}{4}+\frac{s}{2}}$ which could lead to an interative scheme of reverse Hölder's inequalities. In order to overcome this problem, we observe that

$$\left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\right)_{x_2}\right| = \frac{p+2\,s}{p}\, \left|\left(W_1^{\frac{p}{4}}\right)_{x_2}\right|\, W_1^{\frac{s}{2}}.$$

Then if we fix $1 < q < 2$, by Hölder's inequality with exponents $2/q$ and $2/(2-q)$, we have

$$\left(\int \left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\right)_{x_2}\right|^q \eta^q\,dx\right)^{\frac{2}{q}} \le \left(\frac{p+2\,s}{p}\right)^2 \left(\int \left|\left(W_1^{\frac{p}{4}}\right)_{x_2}\right|^2 \eta^2\,dx\right) \left(\int_{\mathrm{spt}(\eta)} W_1^{\frac{q}{2-q}\,s}\,dx\right)^{\frac{2-q}{q}}.$$

The precise value of q will be specified later. We now add the term

$$\left(\int W_1^{\frac{p\,q}{4}+\frac{s\,q}{2}}\, |\eta_{x_2}|^q\,dx\right)^{\frac{2}{q}},$$

on both sides of the previous inequality and observe that by triangle inequality

$$\left(\int \left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\right)_{x_2}\right|^q \eta^q\,dx\right)^{\frac{2}{q}} + \left(\int W_1^{\frac{p\,q}{4}+\frac{s\,q}{2}}\, |\eta_{x_2}|^q\,dx\right)^{\frac{2}{q}} \ge \frac{1}{2}\left(\int \left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_2}\right|^q dx\right)^{\frac{2}{q}}.$$

Thus we get

$$\left(\int\left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_2}\right|^q dx\right)^{\frac{2}{q}} \le C\,(1+s)^2\left(\int\left|\left(W_1^{\frac{p}{4}}\right)_{x_2}\right|^2\eta^2\,dx\right)\left(\int_{\mathrm{spt}(\eta)} W_1^{\frac{q}{2-q}\,s}\,dx\right)^{\frac{2-q}{q}} + C\left(\int W_1^{\frac{p\,q}{4}+\frac{s\,q}{2}}\,|\eta_{x_2}|^q\,dx\right)^{\frac{2}{q}}, \tag{4.5}$$

with $C = C(p) > 0$. We assume again for simplicity that all the balls are centered at the origin. We then fix the two radii $R_0 > r_0 > 0$ of the statement and we set

$$R_1 := \frac{R_0 + r_0}{2}. \tag{4.6}$$

For $r_0 < r < R < R_1$, we take $\eta \in W_0^{1,\infty}(B_R)$ to be the standard cut-off function

$$\eta(x) = \min\left\{1, \frac{(R-|x|)_+}{R-r}\right\}.$$

By multiplying (4.4) and (4.5) we get

$$\begin{aligned}&\left(\int\left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_1}\right|^2 dx\right)\left(\int\left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_2}\right|^q dx\right)^{\frac{2}{q}}\\ &\le C\,\delta^{p-2}\left[\frac{1}{(R-r)^2}\sum_{i=1}^2\int_{B_R} W_i^{\frac{p-2}{2}}\,W_1^{s+1}\,dx + (s+1)^2\int_{B_R}|f_\varepsilon|^2\,W_1^s\,dx\right]\\ &\times\left[(s+1)^2\left(\int_{B_R}\left|\left(W_1^{\frac{p}{4}}\right)_{x_2}\right|^2 dx\right)\left(\int_{B_R} W_1^{\frac{q}{2-q}\,s}\,dx\right)^{\frac{2-q}{q}}\right.\\ &\left.+\frac{1}{(R-r)^2}\left(\int_{B_R} W_1^{\frac{p\,q}{4}+\frac{s\,q}{2}}\,dx\right)^{\frac{2}{q}}\right].\end{aligned} \tag{4.7}$$

We now estimate the terms appearing in the right-hand side of (4.7). To this aim, it will be useful to introduce the quantity

$$\begin{aligned}\mathcal{I}(W_1, W_2, f_\varepsilon; R_0, R_1) = \sum_{i=1}^2\left[\left(\frac{R_0}{R_1}\right)^2 \fint_{B_{R_0}} W_i^{\frac{p}{2}}\,dx + \int_{B_{R_1}}\left|\nabla W_i^{\frac{p}{4}}\right|^2 dx\right]\\ + R_0^{\frac{2}{p}}\left(\int_{B_{R_0}}|f_\varepsilon|^{2\,p'}\,dx\right)^{\frac{1}{p'}}.\end{aligned} \tag{4.8}$$

Then we start with the first term on the right-hand side of (4.7). Observe that

$$\sum_{i=1}^2\int_{B_R} W_i^{\frac{p-2}{2}}\,W_1^{s+1}\,dx = \int_{B_R} W_1^{\frac{p}{2}}\,W_1^s\,dx + \int_{B_R} W_2^{\frac{p-2}{2}}\,W_1\,W_1^s\,dx.$$

We use Hölder's inequality in conjunction with Sobolev-Poincaré inequality[8], to get

$$\begin{aligned}\int_{B_R} W_1^{\frac{p}{2}}\,W_1^s\,dx &\le C\left[\fint_{B_{R_1}} W_1^{\frac{p}{2}}\,dx+\int_{B_{R_1}}\left|\nabla W_1^{\frac{p}{4}}\right|^2 dx\right] R_0^{\frac{2}{p'}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\\ &\le C\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\,R_0^{\frac{2}{p'}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}},\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}\int_{B_R} W_2^{\frac{p-2}{2}}\,W_1\,W_1^s\,dx &\le C\left[\sum_{i=1}^2\left(\int_{B_{R_1}}\left(W_i^{\frac{p}{4}}\right)^{2\,p'}dx\right)^{\frac{1}{p'}}\right]\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\\ &\le C\left\{\sum_{i=1}^2\left[\fint_{B_{R_1}} W_i^{\frac{p}{2}}\,dx+\int_{B_{R_1}}\left|\nabla W_i^{\frac{p}{4}}\right|^2 dx\right]\right\} R_0^{\frac{2}{p'}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\\ &\le C\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\,R_0^{\frac{2}{p'}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}},\end{aligned}\tag{4.10}$$

for some constant $C=C(p)>0$ depening only on p.

The term containing f_ε in (4.7) is estimated as follows. By Hölder's inequality and the definition of $\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)$

$$\begin{aligned}\int_{B_R}|f_\varepsilon|^2\,W_1^s\,dx &\le \left(\int_{B_{R_0}}|f_\varepsilon|^{2\,p'}\,dx\right)^{\frac{1}{p'}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\\ &\le \mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\,R_0^{-\frac{2}{p}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\end{aligned}\tag{4.11}$$

For the last term on the right-hand side of (4.7), by Hölder's inequality and estimate (4.9) we have

$$\begin{aligned}\left(\int_{B_R} W_1^{\frac{p\,q}{4}+\frac{s\,q}{2}}\,dx\right)^{\frac{2}{q}} &\le C\,R_0^{2\left(\frac{2}{q}-1\right)}\int_{B_R} W_1^{\frac{p}{2}}\,W_1^s\,dx\\ &\le C\,R_0^{2\left(\frac{2}{q}-\frac{1}{p}\right)}\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}},\end{aligned}\tag{4.12}$$

where $C=C(q)>0$.

Finally, for the left-hand side of (4.7), we have

$$\left(\int\left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_1}\right|^2 dx\right)\left(\int\left|\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)_{x_2}\right|^q dx\right)^{\frac{2}{q}} \ge \mathcal{T}_q^2\left(\int\left(W_1^{\frac{p}{4}+\frac{s}{2}}\,\eta\right)^{\overline{q}^*} dx\right)^{\frac{4}{\overline{q}^*}}.\tag{4.13}$$

[8] Since we are in dimension $N=2$, we have $W^{1,2}(B_{R_1})\hookrightarrow L^{2\,p'}(B_{R_1})$ and

$$\left(\int_{B_{R_1}}\left(W_i^{\frac{p}{4}}\right)^{2\,p'}dx\right)^{\frac{1}{p'}}dx \le C\,R_1^{\frac{2}{p'}}\left[\fint_{B_{R_1}}\left(W_i^{\frac{p}{4}}\right)^2 dx+\int_{B_{R_1}}\left|\nabla W_i^{\frac{p}{4}}\right|^2 dx\right],$$

with a constant $C=C(p)>0$.

Here we used the *anisotropic Sobolev-Troisi inequality* (see Appendix B) for the compactly supported function $W_1^{(p+2\,s)/4}\,\eta$. The exponent $\overline{q}^*$ is defined by

$$\overline{q}^* = \frac{2\,\overline{q}}{2-\overline{q}}, \qquad \text{where} \quad \frac{1}{\overline{q}} = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{q}\right),$$

so that

$$\overline{q} = \frac{4\,q}{2+q} \qquad \text{and} \qquad \overline{q}^* = \frac{4\,q}{2-q},$$

the constant $\mathcal{T}_q$ only depends on q and it degenerates to 0 as q goes to 2.

By using (4.9), (4.10), (4.11), (4.12) and (4.13) in (4.7), we then arrive at

(4.14)

$$\begin{aligned}\left[\int_{B_r}\left(W_1^{\frac{p}{2}+s}\right)^{\frac{2\,q}{2-q}}\,dx\right]^{\frac{2-q}{q}} &\le C\,\delta^{p-2}\left[\left(\frac{R_0}{R-r}\right)^2 \mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\,R_0^{-\frac{2}{p}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\right.\\ &\quad\left.+(s+1)^2\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\,R_0^{-\frac{2}{p}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\right]\\ &\quad\times\left[(s+1)^2\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\left(\int_{B_R} W_1^{\frac{q}{2-q}\,s}\,dx\right)^{\frac{2-q}{q}}\right.\\ &\quad\left.+\left(\frac{R_0}{R-r}\right)^2 R_0^{2\left(\frac{2}{q}-\frac{1}{p}-1\right)}\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{1}{p}}\right],\end{aligned}$$

for a constant $C = C(p,q) > 0$. We now choose $1 < q < 2$ as follows

$$q = \frac{2\,p}{p+1}. \tag{4.15}$$

Observe that with such a choice, we have

$$\frac{q}{2-q} = p \qquad \text{and} \qquad \frac{2}{q} - \frac{1}{p} - 1 = 0.$$

We further observe that

$$\left(W_1^{\frac{p}{2}+s}\right)^{2\,p} \ge W_1^{2\,s\,p},$$

since $W_i \ge 1$. Then (4.14) becomes

$$\left(\int_{B_r} W_1^{2\,p\,s}\,dx\right)^{\frac{1}{p}} \le C\,\delta^{p-2}\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)^2\left[\left(\frac{R_0}{R-r}\right)^2+(s+1)^2\right]^2 R_0^{-\frac{2}{p}}\left(\int_{B_R} W_1^{s\,p}\,dx\right)^{\frac{2}{p}}.$$

By using that $R_0/(R-r) \ge 1$ and $(s+1) \ge 1$ and introducing the notation $\vartheta = p\,s$, then the previous estimate finally gives

$$\|W_1\|_{L^{2\,\vartheta}(B_r)} \le \left[C\,\delta^{\frac{p-2}{2}}\,\mathcal{I}(W_1,W_2,f_\varepsilon;R_0,R_1)\left(\frac{R_0}{R-r}\right)^2\left(\frac{\vartheta}{p}+1\right)^2\right]^{\frac{p}{\vartheta}} R_0^{-\frac{1}{\vartheta}}\,\|W_1\|_{L^{\vartheta}(B_R)}, \tag{4.16}$$

possibly for a different constant $C = C(p) > 0$. *This is the iterative scheme of reverse Hölder's inequalities needed to launch a Moser's iteration.*

We now recall the definition (4.6) of R_1 and consider the sequences

$$r_k = r_0 + \frac{R_1 - r_0}{2^k} \qquad \text{and} \qquad \vartheta_k = 2\,\vartheta_{k-1} = 2^k\,\vartheta_0 = 2^k\,\frac{p}{2}.$$

Then iterating (4.16) infinitely many times with $R = r_k$ and $r = r_{k+1}$, we get

$$\|W_1\|_{L^\infty(B_{r_0})} \le C\,\delta^{2\,(p-2)} \left(\frac{R_0}{R_0 - r_0}\right)^8 \mathcal{I}(W_1, W_2, f_\varepsilon; R_0)^4\, R_0^{-\frac{4}{p}} \left(\int_{B_{R_1}} W_1^{\frac{p}{2}}\,dx\right)^{\frac{2}{p}},$$

for some constant $C = C(p) > 0$. We notice that $u_{x_1}^2 \le W_1 \le u_{x_1}^2 + \delta^2$, by definition of W_1. Then we obtain with simple manipulations

$$\|u_{x_1}\|_{L^\infty(B_{r_0})} \le C\,\delta^{p-2} \left(\frac{R_0}{R_0 - r_0}\right)^4 \mathcal{I}(W_1, W_2, f_\varepsilon; R_0, R_1)^2 \left[\left(\fint_{B_{R_0}} |u_{x_1}|^p\,dx\right)^{\frac{1}{p}} + \delta\right],$$

for a possibly different constant $C = C(p) > 0$. By recalling that R_1 is defined in (4.6) and using Corollary 3.4, the term $\mathcal{I}(W_1, W_2, f_\varepsilon; R_0, R_1)$ defined in (4.8) can be estimated as follows

$$\begin{aligned}\mathcal{I}(W_1, W_2, f_\varepsilon; R_0, R_1) \le C\,\delta^{p-2} &\left(\frac{R_0}{R_0 - r_0}\right)^2 \left[\sum_{j=1}^{2} \fint_{B_{R_0}} |u_{x_j}|^p\,dx + \delta^p\right] \\ &+ C\,\delta^{p-2}\,R_0^{\frac{2}{p-1}} \int_{B_{R_0}} |\nabla f_\varepsilon|^{p'}\,dx + \delta^{p-2}\,R_0^{\frac{2}{p}} \left(\int_{B_{R_0}} |f_\varepsilon|^{2\,p'}\,dx\right)^{\frac{1}{p'}}.\end{aligned}$$

This concludes the proof. □

Remark 4.2. Observe that the previous strategy does not seem to work for $N \ge 3$. Indeed, in this case we would have $N - 1$ missing terms, i.e.

$$\partial_{x_i} W_1^{\frac{p}{4} + \frac{s}{2}}, \qquad i = 2, \dots, N.$$

By proceeding as before for each of these terms, i.e. combining (3.9) and Hölder's inequality, one would have on the left-hand side the term

$$\left(\int \left|\left(W_1^{\frac{p}{4} + \frac{s}{2}}\,\eta\right)_{x_1}\right|^2 dx\right) \prod_{i=2}^{N} \left(\int \left|\left(W_1^{\frac{p}{4} + \frac{s}{2}}\,\eta\right)_{x_i}\right|^q dx\right)^{\frac{2}{q}},$$

which in turn can be estimated from below by Sobolev-Troisi inequality by

$$\left(\int_{B_r} \left(W_1^{\frac{p}{4} + \frac{s}{2}}\right)^{\overline{q}^*} dx\right)^{\frac{2}{\overline{q}^*}}.$$

The right-hand side would still contain the term

$$\left(\int_{B_R} W_1^{\frac{q}{2-q}\,s}\,dx\right)^{\frac{2-q}{q}}.$$

The exponent $\overline{q}^*$ is now defined by

$$\overline{q}^* = \frac{N\,\overline{q}}{N - \overline{q}}, \qquad \text{where} \qquad \frac{1}{\overline{q}} = \frac{1}{N}\left(\frac{1}{2} + \frac{N-1}{q}\right),$$

so that

$$\overline{q} = \frac{2\,N\,q}{2\,N + q - 2} \qquad \text{and} \qquad \overline{q}^* = \frac{2\,N\,q}{2\,N - q - 2}.$$

Then Moser's iteration would work if

$$\frac{\overline{q}^*}{2}\,s > \frac{q}{2-q}\,s \qquad \Longleftrightarrow \qquad q < \frac{2}{N-1}.$$

Of course, when $N \geq 3$ the last condition does not fit with the requirement $q > 1$.

5. Proof of Theorem B

Proof. The proof follows the same lines as that of Theorem A described at the beginning of Section 4. The essential difference is the uniform Lipschitz estimate of Proposition 5.1 below, which replaces that of Proposition 4.1. More precisely, with the notation already introduced at the beginning of Section 4, we only need to replace (4.1) by

$$\|f_\varepsilon\|_{W^{1,\infty}(B_{2\,r_0})} \leq \|f\|_{W^{1,\infty}(2\,B)}.$$

Then we observe that by (2.14) and (2.13) of Lemma 2.6

$$\|u_\varepsilon\|_{L^\infty(B_{R_0})} \leq C\,\left[\left(\fint_{4\,B} |U|^p\,dx\right)^{\frac{1}{p}} + \left(\left(\frac{C_1}{|B|}\right)^{\frac{1}{p}} + \delta + |B|^{\frac{1}{N\,(p-1)}}\,\|f\|^{\frac{1}{p-1}}_{L^\infty(8\,B)}\right)\,|B|^{\frac{1}{N}}\right],$$

for some $C > 0$ independent of ε. The quantity C_1 is the same appearing in (2.11) and the previous estimate is obtained by choosing $B_{\varrho_0} = 4\,B$ in (2.13). Then from (5.1) below we obtain a local uniform Lipschitz estimate. We leave the details to the reader. □

Proposition 5.1 (Uniform Lipschitz estimate, $p \geq 4$)**.** *Let $N \geq 2$ and $p \geq 4$. For every pair of concentric balls $B_{r_0} \Subset B_{R_0} \Subset B$, we have*

$$\begin{aligned}\|\nabla u_\varepsilon\|_{L^\infty(B_{r_0})} \leq C_9\,&\left[\left(1 + \frac{1}{(R_0 - r_0)^{2/p}}\right)\,\delta + \|f_\varepsilon\|^{\frac{1}{p-1}}_{W^{1,\infty}(B_{R_0})}\right. \\ &\left. + \left(1 + \frac{1}{(R_0 - r_0)^2}\right)\,\|u_\varepsilon\|_{L^\infty(B_{R_0})}\right],\end{aligned} \tag{5.1}$$

where $C_9 = C_9(N,p) > 0$ does not depend on ε.

Proof. As usual, for notational simplicity we simply write u in place of u_ε. By Lemma 2.8, we get that u is indeed a local C^3 solution of the equation (3.1) in B, i.e. it verifies

$$\sum_{i=1}^N \left(g'_{i,\varepsilon}(u_{x_i})\right)_{x_i} = f_\varepsilon, \qquad \text{in } B',$$

for every $B' \Subset B$. This means that pointwise we have

$$\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,u_{x_i\,x_i} = f_\varepsilon \qquad \text{in } B'. \tag{5.2}$$

We now derive the previous equation with respect to x_j and obtain

$$\sum_{i=1}^N \left[g'''_{i,\varepsilon}(u_{x_i})\,u_{x_i\,x_i}\,u_{x_i\,x_j} + g''_{i,\varepsilon}(u_{x_i})\,u_{x_i\,x_i\,x_j}\right] = (f_\varepsilon)_{x_j}, \qquad \text{in } B'. \tag{5.3}$$

We introduce the following linear differential operator

$$L[\psi]=\sum_{i=1}^N \Big[g'''_{i,\varepsilon}(u_{x_i})\,u_{x_i\,x_i}\,\psi_{x_i}+g''_{i,\varepsilon}(u_{x_i})\,\psi_{x_i\,x_i}\Big], \tag{5.4}$$

then (5.3) can be simply written as $L[u_{x_j}]=(f_\varepsilon)_{x_j}$. Also observe that

$$L[\varphi\,\psi]=\varphi\,L[\psi]+\psi\,L[\varphi]+2\,\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,\varphi_{x_i}\,\psi_{x_i}.$$

Thus for $\varphi=\psi=u_{x_j}$ we obtain

$$L[u_{x_j}^2]=2\,u_{x_j}\,L[u_{x_j}]+2\,\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,\Big[\big(u_{x_j}\big)_{x_i}\Big]^2=2\,u_{x_j}\,(f_\varepsilon)_{x_j}+2\,\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_j\,x_i}.$$

By linearity of L we thus get

$$L\left[|\nabla u|^2\right]=2\,\sum_{j=1}^N u_{x_j}\,(f_\varepsilon)_{x_j}+2\,\sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_j\,x_i}.$$

We now fix a pair of concentric balls $B_{r_0}\Subset B_{R_0}\Subset B$ as in the statement of Proposition 5.1. Let $\zeta\in C^2_0(B_{R_0})$ be a function such that $0\le\zeta\le 1$ and

$$\zeta=1 \text{ on } B_{r_0},\qquad |\nabla\zeta|^2\le\frac{C}{(R_0-r_0)^2}\,\zeta\qquad\text{and}\qquad |D^2\zeta|\le\frac{C}{(R_0-r_0)^2}, \tag{5.5}$$

for some universal constant $C>0$ and consider in B_{R_0} the equation for the function $\zeta\,|\nabla u|^2+\lambda\,u^2$. The parameter λ will play a crucial role and will be chosen later. By using the product rule for L and its linearity, we get

$$\begin{aligned}
L\left[\zeta\,|\nabla u|^2+\lambda\,u^2\right]&=\zeta\,L\left[|\nabla u|^2\right]+|\nabla u|^2\,L[\zeta]+2\,\sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,(u^2_{x_j})_{x_i}\,\zeta_{x_i}+\lambda\,L[u^2]\\
&=2\,\sum_{j=1}^N u_{x_j}\,(f_\varepsilon)_{x_j}\,\zeta+2\,\sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_j\,x_i}\,\zeta\\
&\quad+|\nabla u|^2\,L[\zeta]+2\,\sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,(u^2_{x_j})_{x_i}\,\zeta_{x_i}\\
&\quad+2\,\lambda\,u\,L[u]+2\,\lambda\,\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_i}.
\end{aligned}$$

By using the expression (5.4) of L and the equation (5.2), we can rewrite the previous identity as follows

$$L\left[\zeta\,|\nabla u|^2+\lambda\,u^2\right]=2\,\mathcal{F}+2\,\zeta\,\mathcal{G}_1+2\,\mathcal{G}_2+2\,\lambda\,\mathcal{G}_3+\mathcal{G}_4, \tag{5.6}$$

where we used the notation

$$\mathcal{F}=\lambda\,u\,f_\varepsilon+\zeta\,\sum_{j=1}^N u_{x_j}\,(f_\varepsilon)_{x_j},\qquad \mathcal{G}_1=\sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_j\,x_i},$$

$$\mathcal{G}_2 = \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,(u^2_{x_j})_{x_i}\,\zeta_{x_i} + \frac{|\nabla u|^2}{2}\,\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,\zeta_{x_i\,x_i}, \qquad \mathcal{G}_3 = \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_i},$$

and

$$\mathcal{G}_4 = \sum_{i=1}^N g'''_{i,\varepsilon}(u_{x_i})\,u_{x_i\,x_i}\,\left[2\,\lambda\,u\,u_{x_i} + |\nabla u|^2\,\zeta_{x_i}\right].$$

We proceed to estimate separately each term on the right-hand side of (5.6).

The term $\mathcal{F}$.

We set for simplicity

$$M := \|u\|_{L^\infty(B_{R_0})}. \tag{5.7}$$

We can suppose that $M > 0$, otherwise there is nothing to prove. By Cauchy-Schwarz inequality, Young's inequality and the fact that $0 \le \zeta \le 1$ we get

$$\begin{aligned}\mathcal{F} &\ge -\lambda\,\|u\|_{L^\infty(B_{R_0})}\,\|f_\varepsilon\|_{L^\infty(B_{R_0})} - \zeta\,|\nabla u|\,|\nabla f_\varepsilon| \\ &\ge -\frac{\lambda^p\,M^p}{p} - \frac{1}{p}\,|\nabla u|^p - c\,\|f_\varepsilon\|^{p'}_{W^{1,\infty}(B_{R_0})},\end{aligned} \tag{5.8}$$

where $c = c(p) > 0$.

The term $\mathcal{G}_1$.

This is a positive term and for the moment we simply keep it. It will act as a sponge term, in order to absorb (negative) terms containing the Hessian of u.

The term $\mathcal{G}_2$.

This can be estimated by Young's inequality and (5.5) as follows

$$\begin{aligned}\mathcal{G}_2 \ge& -\tau\,\sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_j\,x_i}\,\zeta^2_{x_i} - \frac{1}{\tau}\,\sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})\,u^2_{x_j} \\ &- \frac{|\nabla u|^2}{2}\,\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,|\zeta_{x_i\,x_i}| \\ \ge& -\tau\,\frac{C}{(R_0-r_0)^2}\,\zeta\,\mathcal{G}_1 - \frac{|\nabla u|^2}{2}\,\left(\frac{C}{(R_0-r_0)^2} + \frac{2}{\tau}\right)\,\sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}),\end{aligned}$$

where $\tau > 0$ is a positive parameter. We then observe that the last term can be further estimated by using

$$g''_{i,\varepsilon}(u_{x_i}) \le (p-1)\,|\nabla u|^{p-2} + 1, \tag{5.9}$$

so that

$$|\nabla u|^2\,\sum_{j=1}^N g''_{i,\varepsilon}(u_{x_i}) \le N((p-1)\,|\nabla u|^p + |\nabla u|^2) \le N\,\left(p-1+\frac{2}{p}\right)\,|\nabla u|^p + N\,\frac{p-2}{p}.$$

In the end we get

$$\mathcal{G}_2 \geq -\tau\,\frac{C}{(R_0-r_0)^2}\,\zeta\,\mathcal{G}_1 - \frac{C_1'\,(\tau+(R_0-r_0)^2)}{\tau\,(R_0-r_0)^2}\,|\nabla u|^p - \frac{C_1'\,(\tau+(R_0-r_0)^2)}{\tau\,(R_0-r_0)^2}, \tag{5.10}$$

where $C_1' = C_1'(p,N,C) > 0$.

The term $\mathcal{G}_3$.

By using the form of $g_{i,\varepsilon}$, the convexity of the function $m \mapsto m^{p-2}$ and recalling the definition (2.1) of δ, we have

$$\mathcal{G}_3 = \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i})\,|u_{x_i}|^2 \geq (p-1)\,\left(\frac{1}{2^{p-3}}\,\sum_{i=1}^N |u_{x_i}|^p - \delta^{p-2}\,|\nabla u|^2\right).$$

By further applying Young's inequality to estimate the term $\delta^{p-2}\,|\nabla u|^2$ and using that

$$\sum_{i=1}^N |u_{x_i}|^p \geq N^{\frac{2-p}{2}}\,|\nabla u|^p,$$

we end up with

$$\mathcal{G}_3 \geq C_1''\,|\nabla u|^p - C_2''\,\delta^p, \tag{5.11}$$

where $C_1'' = C_1''(p,N) > 0$ and $C_2'' = C_2''(p,N) > 0$.

The term $\mathcal{G}_4$

This is the most delicate term and *it is precisely here that the condition $p \geq 4$ becomes vital.* First we have

$$\mathcal{G}_4 \geq -\sum_{i=1}^N |g'''_{i,\varepsilon}(u_{x_i})|\,|u_{x_i\,x_i}|\,\left[2\,\lambda\,|u|\,|u_{x_i}| + |\nabla u|^2\,|\zeta_{x_i}|\right].$$

Then we observe that by Cauchy-Schwarz inequality (recall the definition (2.8) of $g_{i,\varepsilon}$) we have

$$\begin{aligned}
|\nabla u|^2\,\sum_{i=1}^N |g'''_{i,\varepsilon}(u_{x_i})|\,|u_{x_i\,x_i}|\,|\zeta_{x_i}| &\leq c\,|\nabla u|^2\,\left(\sum_{i=1}^N (|u_{x_i}|-\delta_i)_+^{p-2}\,u^2_{x_i\,x_i}\,|\zeta_{x_i}|^2\right)^{\frac12}\\
&\qquad\times\left(\sum_{i=1}^N (|u_{x_i}|-\delta_i)_+^{p-4}\right)^{\frac12}\\
&\leq c\,\left(\sum_{i=1}^N (|u_{x_i}|-\delta_i)_+^{p-2}\,u^2_{x_i\,x_i}\,|\zeta_{x_i}|^2\right)^{\frac12}\,|\nabla u|^{\frac{p}{2}}
\end{aligned}$$

for some constant $c = c(N,p) > 0$. In the last inequality we used that

$$(|u_{x_i}|-\delta_i)_+^{p-4} \leq |u_{x_i}|^{p-4},$$

which is true since $p \geq 4$. By further using (5.5), the definition of $g_{i,\varepsilon}$ and Young's inequality, from the previous inequality we get

$$|\nabla u|^2\,\sum_{i=1}^N |g'''_{i,\varepsilon}(u_{x_i})|\,|u_{x_i\,x_i}|\,|\zeta_{x_i}| \leq \frac{c\,\sqrt{C}}{R_0-r_0}\,(\zeta\,\mathcal{G}_1)^{\frac12}\,|\nabla u|^{\frac{p}{2}} \leq \tau\,\frac{C_1'''}{(R_0-r_0)^2}\,\zeta\,\mathcal{G}_1 + \frac{C_1'''}{\tau}\,|\nabla u|^p,$$

for some constant $C_1''' = C_1'''(C,N,p) > 0$. Similarly, by recalling (5.7) we have

$$2\,\lambda \sum_{i=1}^N |g_{i,\varepsilon}'''(u_{x_i})|\,|u_{x_i\,x_i}|\,|u|\,|u_{x_i}| \le c\,M\,\lambda \left(\sum_{i=1}^N g_{i,\varepsilon}''(u_{x_i})\,u_{x_i\,x_i}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^N (|u_{x_i}|-\delta_i)_+^{p-4}\,|u_{x_i}|^2\right)^{\frac{1}{2}}$$
$$\le c\,M\,\lambda\,\mathcal{G}_1^{\frac{1}{2}}\,|\nabla u|^{\frac{p-2}{2}} \le C_2'''\,\left(M^2\,\lambda^2\,\mathcal{G}_1\right)^{\frac{p}{p+2}} + C_2'''\,|\nabla u|^p,$$

for some constant $C_2''' = C_2'''(N,p) > 0$. By keeping everything together, we get

$$\mathcal{G}_4 \ge -\tau\,\frac{C_1'''}{(R_0-r_0)^2}\,\zeta\,\mathcal{G}_1 - \left(\frac{C_1'''}{\tau} + C_2'''\right)\,|\nabla u|^p - C_2'''\,\left(M^2\,\lambda^2\,\mathcal{G}_1\right)^{\frac{p}{p+2}}. \tag{5.12}$$

Collecting all the estimates.

We now go back to (5.6) and use (5.8), (5.10), (5.11) and (5.12). Then we get

$$L[\zeta\,|\nabla u|^2 + \lambda\,u^2] \ge \left[2 - \frac{\tau}{(R_0-r_0)^2}\,(2\,C + C_1''')\right]\zeta\,\mathcal{G}_1 - C_2'''\,(M\,\lambda)^{\frac{2\,p}{p+2}}\,\mathcal{G}_1^{\frac{p}{p+2}}$$
$$+\left(2\,\lambda\,C_1'' - 2\,\frac{C_1'\,(\tau+(R_0-r_0)^2)}{\tau\,(R_0-r_0)^2} - \frac{2}{p} - \frac{C_1'''}{\tau} - C_2'''\right)\,|\nabla u|^p$$
$$-\,2\,\left(\frac{C_1'\,(\tau+(R_0-r_0)^2)}{\tau\,(R_0-r_0)^2} + \lambda\,C_2''\,\delta^p\right) - \frac{2}{p}\,(M\,\lambda)^p - 2\,c\,\|f_\varepsilon\|_{W^{1,\infty}(B_{R_0})}^{p'}.$$

We now choose τ and λ as follows

$$\tau = \frac{(R_0-r_0)^2}{2\,C + C_1'''} \qquad \text{and} \qquad \lambda = \frac{1}{2\,C_1''}\left(1 + 2\,\frac{C_1'\,(\tau+(R_0-r_0)^2)}{\tau\,(R_0-r_0)^2} + \frac{2}{p} + \frac{C_1'''}{\tau} + C_2'''\right). \tag{5.13}$$

Observe that the choices of τ and λ only depend on N, p and $(R_0-r_0)^2$ and are in particular independent of ε. Thus we obtain

$$L[\zeta\,|\nabla u|^2 + \lambda\,u^2] \ge \Big(\zeta\,\mathcal{G}_1 + |\nabla u|^p\Big) - C_2'''\,\left(M^2\,\lambda^2\,\mathcal{G}_1\right)^{\frac{p}{p+2}} - \widetilde{c}. \tag{5.14}$$

where we set for simplicity

$$\widetilde{c} := 2\,\frac{C_1'\,(\tau+(R_0-r_0)^2)}{\tau\,(R_0-r_0)^2} + 2\,\lambda\,C_2''\,\delta^p + \frac{2\,\lambda^p}{p}\,M^p + 2\,c\,\|f_\varepsilon\|_{W^{1,\infty}(B_{R_0})}^{p'}. \tag{5.15}$$

Let us now consider the maximum of the function $\zeta\,|\nabla u|^2 + \lambda\,u^2$ in $\overline{B_{R_0}}$. Let $x_0 \in \overline{B_{R_0}}$ be such a maximum point, we first prove

$$\zeta(x_0)^{\frac{1}{2}}\,|\nabla u(x_0)| \le \widetilde{C}, \tag{5.16}$$

for some constant $\widetilde{C} > 0$ depending on the data of the problem.

If $x_0 \in \partial B_{R_0}$, then (5.16) trivially holds true, since $\zeta(x_0) = 0$. Thus, let us assume that $x_0 \in B_{R_0}$. In this case we get

$$\nabla\left(\zeta\,|\nabla u|^2 + \lambda\,u^2\right) = 0 \qquad \text{at } x = x_0,$$

and

$$D^2\left(\zeta\,|\nabla u|^2 + \lambda\,u^2\right) \le 0 \qquad \text{at } x = x_0.$$

Thus at the maximum point x_0 we have

$$L[\zeta\,|\nabla u|^2 + \lambda\,u^2] = \sum_{i=1}^N g_{i,\varepsilon}''(u_{x_i})\,\left(\zeta\,|\nabla u|^2 + \lambda\,u^2\right)_{x_i\,x_i} \le 0.$$

By combining this with (5.14), we then get

$$\widetilde{c} \geq \zeta\, \mathcal{G}_1 - C_2'''\, \left(M^2\, \lambda^2\, \mathcal{G}_1\right)^{\frac{p}{p+2}} + |\nabla u|^p.$$

We multiply the previous by $\zeta(x_0)^{p/2} > 0$, then by Young's inequality once again we get

$$\begin{aligned}\widetilde{c}\,\zeta(x_0)^{\frac{p}{2}} &\geq \zeta(x_0)^{\frac{p+2}{2}}\, \mathcal{G}_1 - C_2'''\, (M\,\lambda)^{\frac{2\,p}{p+2}}\, \left(\zeta(x_0)^{\frac{p+2}{2}}\, \mathcal{G}_1\right)^{\frac{p}{p+2}} + \zeta(x_0)^{\frac{p}{2}}\, |\nabla u(x_0)|^p \\ &\geq \left(1 - C_2'''\, \frac{p}{p+2}\, \alpha\right)\, \zeta(x_0)^{\frac{p+2}{2}}\, \mathcal{G}_1 \\ &- \frac{2\,\alpha^{-\frac{p}{2}}}{p+2}\, C_2'''\, (M\,\lambda)^p + \zeta(x_0)^{\frac{p}{2}}\, |\nabla u(x_0)|^p.\end{aligned}$$

If we choose

$$\alpha = \frac{p+2}{p}\, \frac{1}{C_2'''},$$

and use that $\zeta \leq 1$, from the previous estimate we get (5.16), with

$$\widetilde{C} := \left[\widetilde{c} + \frac{2\,\lambda^p\, M^p}{p+2}\, \left(\frac{p}{p+2}\right)^{\frac{p}{2}}\, (C_2''')^{\frac{p+2}{2}}\right]^{\frac{1}{p}}.$$

By using the bound (5.16), we can now conclude. Indeed, we get (recall that $\zeta = 1$ on B_{r_0})

$$\begin{aligned}\max_{B_{r_0}} |\nabla u| \leq \left(\max_{B_{r_0}} \left[\zeta\, |\nabla u|^2 + \lambda\, u^2\right]\right)^{\frac{1}{2}} &\leq \zeta(x_0)^{\frac{1}{2}}\, |\nabla u(x_0)| + \lambda^{\frac{1}{2}}\, u(x_0) \\ &\leq \widetilde{C} + \lambda^{\frac{1}{2}}\, u(x_0) \leq \widetilde{C} + \lambda^{\frac{1}{2}}\, M.\end{aligned} \tag{5.17}$$

In order to obtain the claimed estimate (5.1), we first observe that by recalling the choices (5.13) of τ and λ, we have

$$\lambda \leq C\, \left(1 + \frac{1}{(R_0 - r_0)^2}\right),$$

for some $C = C(N,p) > 0$. Then from (5.15) we get

$$\widetilde{c} \leq C\, \left(1 + \frac{1}{(R_0 - r_0)^2}\right)\, (1 + \delta^p) + C\, \left(1 + \frac{1}{(R_0 - r_0)^2}\right)^p\, M^p + C\, \|f_\varepsilon\|^{p'}_{W^{1,\infty}(B_{R_0})},$$

possibly for a different constant $C = C(N,p) > 0$. We use this to estimate the constant $\widetilde{C}$ above: subaddivitiy of $m \mapsto m^{1/p}$ and the fact that $\delta \geq 1$ imply

$$\widetilde{C} \leq C\, \left(1 + \frac{1}{(R_0 - r_0)^2}\right)^{\frac{1}{p}}\, \delta + C\, \left(1 + \frac{1}{(R_0 - r_0)^2}\right)\, M + C\, \|f_\varepsilon\|^{\frac{1}{p-1}}_{W^{1,\infty}(B_{R_0})},$$

still for some $C = C(N,p) > 0$. With some simple manipulations, from (5.17) we now get the desired estimate. □

Appendix A. Some properties of the functions g_i

The functions g_i have the following convexity property.

Lemma A.1. *For every $t_1, t_2 \in \mathbb{R}$ such that $|t_1 - t_2| > 2\,\delta_i$ we have*

$$g_i((1-s)\,t_1 + s\,t_2) < (1-s)\,g_i(t_1) + s\,g_i(t_2), \qquad s \in (0,1),\ i = 1, \dots, N. \tag{A.1}$$

Proof. The function g_i is convex so that the inequality "$\le$" holds true for every t_1, t_2. If $g_i((1-s)\,t_1 + s\,t_2) = (1-s)\,g_i(t_1) + s\,g_i(t_2)$, then g_i is affine on the segment $[t_1, t_2]$. This can only happen when $t_1, t_2 \in [-\delta_i, \delta_i]$, in which case $|t_1 - t_2| \le 2\delta_i$. □

They also satisfy the following Lipschitz-type estimate.

Lemma A.2. *Let $p \ge 2$. For every $t_1, t_2 \in \mathbb{R}$ and $i = 1, \dots, N$, we have*

$$|g_i(t_1) - g_i(t_2)| \le \left(|t_1|^{p-1} + |t_2|^{p-1}\right)\,|t_1 - t_2|. \tag{A.2}$$

Proof. By basic calculus we have

$$|g_i(t_1) - g_i(t_2)| = |g_i'((1-s)\,t_1 + s\,t_2)|\,|t_1 - t_2|,$$

for some $s \in [0,1]$. For $p \ge 2$, the function $t \mapsto |g_i'(t)|$ is convex and

$$|g_i'(t)| \le |t|^{p-1}, \qquad t \in \mathbb{R}.$$

Thus we get the conclusion. □

The following basic estimate has been used various times.

Lemma A.3. *Let $p \ge 2$. For every $i = 1, \dots, N$ and every $T \ge \delta_i$, we have*

$$g_i''(t) \ge (p-1)\left(\frac{T-\delta_i}{T}\right)^{p-2}\left(T^2 + (|t| - T)_+^2\right)^{\frac{p-2}{2}}, \qquad \textit{for every } |t| \ge T.$$

Proof. For $T = \delta_i$ there is nothing to prove, thus we can suppose that $T > \delta_i$. We use the elementary inequality

$$\frac{T}{T-\delta_i}\,(|t| - \delta_i) \ge |t|, \qquad \text{for every } |t| \ge T.$$

This implies that for every $|t| \ge T$, we have

$$\frac{T}{T-\delta_i}\,(|t| - \delta_i)_+ \ge T + (|t| - T)_+ \ge \left(T^2 + (|t| - T)_+^2\right)^{\frac{1}{2}}.$$

By multiplying everything by $(T - \delta_i)/T$ and raising to the power $p-2$, we get the desired conclusion. □

Appendix B. An anisotropic Sobolev inequality in dimension 2

In the proof of Proposition 4.1 we used Sobolev-Troisi inequality. For the reader's convenience, we give a proof of the particular case we needed.

Lemma B.1. *Let $1 < q < 2$, then for every $u \in C_0^\infty(\mathbb{R}^2)$ we have*

$$\mathcal{T}_q \left(\int_{\mathbb{R}^2} |u|^{\frac{4\,q}{2-q}}\,dx\right)^{\frac{2-q}{2\,q}} \le \left(\int_{\mathbb{R}^2} |u_{x_1}|^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |u_{x_2}|^q\,dx\right)^{\frac{1}{q}}, \tag{B.1}$$

where the constant $\mathcal{T}_q$ is given by

$$\mathcal{T}_q=\frac{(2-q)^2}{4\,q^2-(2-q)^2}>0.$$

Proof. We first observe that for every $\alpha,\beta>1$, by basic calculus we have

$$|u(x_1,x_2)|^\alpha=\alpha\int_{-\infty}^{x_1}u_{x_1}(t,x_2)\,|u(t,x_2)|^{\alpha-2}\,u(t,x_2)\,dt,$$

and

$$|u(x_1,x_2)|^\beta=\beta\int_{-\infty}^{x_2}u_{x_2}(x_1,s)\,|u(x_1,s)|^{\beta-2}\,u(x_1,s)\,ds.$$

Thus

$$|u(x_1,x_2)|^{\alpha+\beta}\le\alpha\,\beta\left(\int_{\mathbb{R}}|u_{x_1}(t,x_2)|\,|u(t,x_2)|^{\alpha-1}\,dt\right)\left(\int_{\mathbb{R}}|u_{x_2}(x_1,s)|\,|u(x_1,s)|^{\beta-1}\,ds\right).$$

If we now integrate over $\mathbb{R}^2$ and use Fubini Theorem on the right-hand side, we get

$$\int_{\mathbb{R}^2}|u|^{\alpha+\beta}\,dx\le\alpha\,\beta\left(\int_{\mathbb{R}^2}|u_{x_1}|\,|u|^{\alpha-1}\,dx\right)\left(\int_{\mathbb{R}^2}|u_{x_2}|\,|u|^{\beta-1}dx\right).\tag{B.2}$$

By Hölder's inequality we then have

$$\begin{aligned}\left(\int_{\mathbb{R}^2}|u_{x_1}|\,|u|^{\alpha-1}\,dx\right)&\left(\int_{\mathbb{R}^2}|u_{x_2}|\,|u|^{\beta-1}dx\right)\\&\le\left(\int_{\mathbb{R}^2}|u_{x_1}|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|u_{x_2}|^q\,dx\right)^{\frac{1}{q}}\\&\times\left(\int_{\mathbb{R}^2}|u|^{2\,(\alpha-1)}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|u|^{\frac{q}{q-1}\,(\beta-1)}dx\right)^{\frac{q-1}{q}}.\end{aligned}$$

We now choose α and β in such a way that

$$2\,(\alpha-1)=\alpha+\beta\qquad\text{and}\qquad\frac{q}{q-1}\,(\beta-1)=\alpha+\beta,$$

that is

$$\alpha=\frac{2+q}{2-q}\qquad\text{and}\qquad\beta=\frac{3\,q-2}{2-q}.$$

Observe that with these choices we have $\alpha+\beta=4\,q/(2-q)$. Thus from (B.2) we get (B.1), with

$$\mathcal{T}_q=\frac{1}{\alpha}\,\frac{1}{\beta}=\frac{(2-q)^2}{4\,q^2-(2-q)^2},$$

as desired. □

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