DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2007

Website: http://aimSciences.org
pp. **X–XX**

ALMOST UNIQUENESS RESULT FOR REVERSED VARIATIONAL INEQUALITIES

Dimitri Mugnai

Dipartimento di Matematica e Informatica
Università di Perugia
Via Vanvitelli 1
06123 Perugia - Italy

Abstract. In this note we show that reversed variational inequalities cannot be studied in a general abstract framework as it happens for classical variational inequalities with Stampacchia's Lemma. Indeed, we provide two different situations for reversed variational inequalities which are of the same type from an abstract point of view, but which behave quite differently.

1. **Introduction.** It is now known that bounce problems (see [1], [2]) can be written as *reversed* variational inequalities ([7], [6]), i.e. inequalities in which the verse of the inequality is opposite with respect to the usual verse in inequalities *à la* Stampacchia ([4], [12]).

Inequalities of this kind were introduced in [11] and then studied in [6] and [9] for elliptic operators defined on Hilbert or Banach spaces, and in [8] for the bounce problem: if an open subset B of $\mathbb{R}^N$ represents the "billiard", any bounce trajectory $\gamma : [0,1] \longrightarrow \overline{B}$ between two given points P and Q of B satisfies the following special *reversed* variational inequality:

$$\begin{cases} \displaystyle\int_0^1 \dot{\gamma}\cdot\dot{\delta}\,dt \le 0 \quad \forall\,\delta \in H_0^1([0,1],\mathbb{R}^N) \text{ such that} \\ \delta(t)\cdot\nu(\gamma(t)) \ge 0 \ \forall\, t \text{ in } \Big\{t\in[0,1] \,:\, \gamma(t)\in\partial\Omega\Big\}, \end{cases}$$

where $\nu(x)$ denotes the inward unit normal to B in a point x of $\partial\Omega$. If, in addition, the bounce is perfectly elastic, also a conservation law for the energy holds, i.e.

$$\frac{1}{2}|\dot{\gamma}(t)|^2 = \text{ constant;}$$

see [6], [7] and [8] for more details, also in presence of external nonlinear potential fields.

The nature of inequalities of this type is quite strange, even in the linear case, since it is not possible to find an abstract formulation to study all of them *tout*

2000 *Mathematics Subject Classification.* Primary: 49J40, 35R45; Secondary: 58E35.
Key words and phrases. Reversed variational inequalities, uniqueness.
Research supported by the M.I.U.R. project *Metodi Variazionali ed Equazioni Differenziali Nonlineari.*

court, as the Stampacchia's Lemma for the classical linear variational inequalities. For example, let us consider the two following problems:

$$\begin{cases} u \in H_0^1(0,1) \quad \text{such that } u \geq -1 \text{ a.e. in } (0,1) \text{ and} \\ \displaystyle\int_0^1 u'(v'-u')\,dx \leq 0 \quad \forall\, v \in H_0^1(0,1) \quad \text{such that } v \geq -1 \text{ a.e. in } (0,1), \end{cases} \tag{1}$$

and

$$\begin{cases} u \in H^2(0,1) \cap H_0^1(0,1) \quad \text{such that } u \geq -1 \text{ a.e. in } (0,1) \text{ and} \\ \displaystyle\int_0^1 u''(v''-u'')\,dx \leq 0 \quad \forall\, v \in H^2(0,1) \cap H_0^1(0,1) \\ \qquad\qquad\qquad\qquad\qquad\quad \text{such that } v \geq -1 \text{ a.e. in } (0,1). \end{cases} \tag{2}$$

From an abstract point of view these two problems are formally the same. In fact in both cases one can consider a bilinear form defined in a closed convex subset of a Hilbert space an look for solutions of the associated reversed variational inequality. More precisely, concerning problem (1), set $H = H_0^1(0,1)$ and define the bilinear and continuous form $a : H \times H \to \mathbb{R}$ defined as $a(u,v) = \int_0^1 u'v'\,dx$, which is coercive by Poincaré's inequality, while for (2) set $H = H^2(0,1) \cap H_0^1(0,1)$ and consider the bilinear and continuous form $a : H \times H \to \mathbb{R}$ defined as $a(u,v) = \int_0^1 u''v''\,dx$, which is again coercive by the Open Mapping Theorem (see [11]).

Finally, in both cases set $K = \{v \in H \,:\, v \geq -1 \quad \text{in } (0,1)\}$, which is easily seen to be a closed and convex subset of H. Then both problems can be written as

$$\begin{cases} u \in K \quad \text{and} \\ a(u, v-u) \leq 0 \quad \forall\, v \in K. \end{cases}$$

Therefore, both problems admit a *reversed* abstract formulation of Stampacchia's variational inequalities, which can be written, in the simplest case, as

$$\begin{cases} u \in K \quad \text{and} \\ a(u, v-u) \geq 0 \quad \forall\, v \in K. \end{cases} \tag{3}$$

Nevertheless, even in these very simple cases given in (1) and (2), it is impossible to find an equivalent formulation of the following simplified version of Stampacchia's Lemma:

Theorem 1 (Stampacchia)**.** *If $a : H \times H \to \mathbb{R}$ is a continuous, bilinear and coercive form on a Hilbert space H and K is a closed and convex subset of H, then there exists a unique solution to problem* (3)*.*

In fact we can prove the following result.

Theorem 2. *Problem* (1) *has an uncountable family of solutions and problem* (2) *has only the trivial solution and another nontrivial one, which is symmetric with respect to $x = 1/2$.*

Here by *solution* of a reversed variational inequality we mean a function u which solves the inequality, but *not* the associated equation. For example a solution of (1) solves the inequality but not the associated equation $u'' = 0$. This means, using the language of the bounce problem, that there is a real bounce on the "wall" -1 and not simply a smooth contact.

However, even if the coercivity assumption fails, one can prove existence results for (1), (2) and also for similar problems. For example in [8] Marino and Saccon consider a bounce problem in a convex domain and prove a multiplicity result for the natural reversed variational inequality which describes this phenomenon.

On the other hand, a natural generalization of (2) is the following. Take Ω smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$, α, $c \in \mathbb{R}$, $\phi : \Omega \longrightarrow \mathbb{R}$ a negative measurable function such that $\sup \phi < 0$ and $K_\phi = \{u \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, u \geq \phi \text{ a.e. in } \Omega\}$ and let us consider the problem

$$(P) \qquad \begin{cases} u \in K_\phi, \\ \displaystyle\int_\Omega \Delta u \Delta(v-u)\, dx - c \int_\Omega Du \cdot D(v-u)\, dx \\ \displaystyle -\alpha \int_\Omega u(v-u)\, dx \leq 0 \quad \forall\, v \in K_\phi. \end{cases}$$

Problem (P) was introduced in [11] as a limit problem for a family of systems describing travelling waves on suspension bridges derived from the original model proposed by Lazer and McKenna in [3], where $\phi \equiv -1$, and then it was studied also in [6].

Due to the strange behaviour of reversed variational inequalities discussed above, there are not many existence results for problem (P), for which an essential role is played by the value $\lambda_1^2 - c\lambda_1$: this is the eigenvalue of $\Delta^2 + c\Delta$ on $H^2(\Omega) \cap H_0^1(\Omega)$ associated to the first eigenfunction e_1 of $-\Delta$ on $H_0^1(\Omega)$. Indeed, due to the boundary conditions, the eigenfunctions of $\Delta^2 + c\Delta$ are the same of $-\Delta$ and the sequence of eigenvalues $(\Lambda_i)_i$ of $\Delta^2 + c\Delta$ is obtained rearranging the sequence of eigenvalues $(\lambda_i)_i$ of $-\Delta$. More precisely, we denote by $(\Lambda_k)_k$ $(\Lambda_1 \leq \Lambda_2 \leq \ldots)$ and by $(E_k)_k$ the eigenvalues and the associated eigenfunctions of $\Delta^2 + c\Delta$ in $H^2(\Omega) \cap H_0^1(\Omega)$, and by $(\lambda_n)_n$ $(\lambda_1 < \lambda_2 \leq \ldots)$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ with eigenfunctions $(e_n)_n$. Then $\{\Lambda_k \,|\, k \in \mathbb{N}\} = \{\lambda_n^2 - c\lambda_n \,|\, n \in \mathbb{N}\}$, and the eigenfunction corresponding to $\lambda_1^2 - c\lambda_1$ is exactly e_1, as already claimed, and it is well known that e_1 can be chosen strictly positive in Ω.

The first existence results for problem (P) appear in [11] and are the following: if

- if $\alpha < \lambda_1^2 - c\lambda_1$ and $N \leq 3$, or
- if $\alpha > \lambda_1^2 - c\lambda_1$ and $N = 2, 3$,

then there exists a non trivial solution of problem (P). As already remarked, the eigenvalue $\Lambda = \lambda_1^2 - c\lambda_1$ corresponds to a critical case, in which there is a strong lack of compactness and it seems hard to exhibit any solution (see [11]). Moreover, starting from [11], in [6] the authors provide a multiplicity result which is more general than the one proved in [11], and for this purpose they use an abstract theory introduced in [7] and which is extremely useful, for example, in non smooth cases. In order to recall the multiplicity result, and also for further uses, we finally set the following notation: if $u \in H^2(\Omega) \cap H_0^1(\Omega)$ we define

$$f_\alpha(u) = \frac{1}{2} \int_\Omega |\Delta u|^2\, dx - \frac{c}{2} \int_\Omega |Du|^2\, dx - \frac{\alpha}{2} \int_\Omega u^2\, dx.$$

Theorem 3 ([11], [6]). *Assume that there exists $s \in \mathbb{N}$ such that*

- $\Lambda_s < \Lambda_{s+1} \leq \lambda_1^2 - c\lambda_1$ *and* $N \leq 3$; *or*
- $\lambda_1^2 - c\lambda_1 < \Lambda_s < \Lambda_{s+1}$ *and* $N = 2$ *or* $N = 3$.

Then $\exists\, \delta_s > 0$ such that, if $\Lambda_s - \delta_s < \alpha < \Lambda_s$, problem (P) has at least 3 nontrivial solutions $u_{\alpha,i}$ with $f_\alpha(u_{\alpha,i}) > 0$ for $i = 1, 2, 3$. Moreover, two of such solutions, say $u_{\alpha,1}$ and $u_{\alpha,2}$, are such that

$$\lim_{\alpha \to \Lambda_s^-} f_\alpha(u_{\alpha,i}) = 0, \qquad i = 1, 2. \tag{4}$$

As before, all these functions solve the inequality, but not the associated problem

$$\begin{cases} \Delta^2 u + c\Delta u - \alpha u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

so that they are real *bounce* solutions. In fact, since α is not an eigenvalue of $\Delta^2 + c\Delta$, the unique solution of (5) is the trivial one. On the other hand, adapting to a non smooth setting the technique used in [10], one can show that also nonlinear classical variational inequalities in presence of a nonlinearity which behaves like $|u|^{p-2}u$, $p > 2$ and subcritical in the usual sense, have three nontrivial solutions near each eigenvalue of the principal part (see [5], which extends to variational inequalities the result found in [10] for the associated equations).

Therefore, Theorem 3 shows that when the bilinear form associated to the problem is not coercive, multiplicity results are possible and reasonable. On the other hand, contrary to Stampacchia's Lemma, also in the coercive case we have multiplicity, though we recover uniqueness if we do not consider the trivial solution, as Theorem 2 states.

2. **Proof of Theorem 2.**

First of all, let us remark that both problems (1) and (2) admit the trivial function as solution. We now show that the first former problem has infinitely many other solutions, while the latter has only *one* nontrivial solution.

Proof of Theorem 2. Concerning problem (1), we explicitly exhibit an uncountable family of solutions: take $\alpha \in (0,1)$ and consider the function

$$u_\alpha(x) = \begin{cases} -\dfrac{1}{\alpha}\, x & \text{if } x \in [0,\alpha], \\ -\dfrac{1}{\alpha - 1}(x-1) & \text{if } x \in (\alpha, 1] \end{cases}$$

It is clear that u_α belongs to $H = H_0^1(0,1)$ and that $u_\alpha \geq -1$. Moreover, computing $a(u_\alpha, v - u_\alpha)$ for any v in H greater than -1, we get

$$\begin{aligned} \int_0^1 u_\alpha'(v' - u_\alpha')\, dx &= -\frac{1}{\alpha}\int_0^\alpha \left(v' + \frac{1}{\alpha}\right) dx - \frac{1}{\alpha - 1}\int_\alpha^1 \left(v' + \frac{1}{\alpha-1}\right) dx \\ &= \frac{v(\alpha)+1}{\alpha(\alpha-1)} \leq 0, \end{aligned}$$

since $\alpha \in (0,1)$ and $v(\alpha) \geq -1$, for $v \in K$. Therefore, for any $\alpha \in (0,1)$ the function u_α solves (1), so that the first part of the Theorem is proved.

Now consider problem (2). As already remarked, the zero function is a solution, so let us look for a nontrivial solution u. Since u is of class C^1 (recall that $H^2([0,1]) \cap H_0^1([0,1]) \hookrightarrow C^1([0,1])$), the set $\mathcal{C} = \{x \in (0,1) : u(x) > -1\}$ is open. Moreover, as in the case of classical variational inequalities, it is easily seen that in $\mathcal{C}$ the solution u satisfies the associated equation $u^{(4)} = 0$.

Since u is continuous, there is a first point $\alpha \in (0,1)$ such that $u(\alpha) = -1$. But u is also of class C^1, and α is a minimum point for u, thus $u'(\alpha) = 0$. Moreover, it is standard to see that $u''(0) = 0$, since $u \in H^2([0,1]) \cap H_0^1([0,1])$. In this way we get

$$u(x) = \frac{1}{2\alpha^3}x^3 - \frac{3}{2\alpha}x \qquad \forall\, x \in [0,\alpha].$$

Assume by contradiction that there exists $\beta \in (\alpha, 1)$ such that $u(x) = -1\ \forall x \in [\alpha, \beta]$. Now take $v \in H^2([0,1]) \cap H_0^1([0,1])$ with support contained in $[0, \beta]$, so that, taking in mind the boundary conditions,

$$\begin{aligned}\int_0^1 u''(v''-u'')\,dx &= \int_0^\alpha u''(v''-u'')\,dx \\ &= u''(\alpha^-)v'(\alpha) - u'''(\alpha^-)v(\alpha) - \int_0^\alpha (u'')^2\,dx - \int_\beta^1 (u'')^2\,dx \\ &= \frac{3}{\alpha^2}v'(\alpha) - \frac{3}{\alpha^3}v(\alpha) - \frac{3}{\alpha^3} - \int_\beta^1 (u'')^2\,dx.\end{aligned}$$

Of course it is not possible that the last quantity is non positive for any v chosen as above, and so u cannot be a solution of (2). Then u cannot be equal to -1 in a right neighborhood of α.

We can also exclude that there exists $\beta \in (\alpha, 1)$ such that $u(x) > -1\ \forall x \in (\alpha, \beta)$ and $u(\beta) = -1$. In fact, there should be a maximum point $\gamma \in (\alpha, \beta)$ and so $u'(\gamma) = 0$ and $u''(\gamma) \leq 0$. But, as before, u is a polynomial of degree 3 in (α, β) and therefore it cannot become convex twice, near α and β, otherwise its second derivative should have two zeroes in (α, β), which is impossible, since u'' is a polynomial of degree 1.

Of course, this procedure lets us actually exclude that there is another contact point except α, even if α is not the first contact point.

Therefore we can conclude that $u(x) > -1\ \forall x \in (\alpha, 1]$. Proceeding as before we can find

$$u(x) = \frac{(x-1)^3}{2(\alpha-1)^3} - \frac{3(x-1)}{2(\alpha-1)} \qquad \forall x \in (\alpha, 1].$$

Summing up, we have

$$u(x) = \begin{cases} \dfrac{1}{2\alpha^3}x^3 - \dfrac{3}{2\alpha}x & \text{if } x \in [0, \alpha] \\ \dfrac{(x-1)^3}{2(\alpha-1)^3} - \dfrac{3(x-1)}{2(\alpha-1)} & \text{if } x \in (\alpha, 1]. \end{cases}$$

Now, for any $v \in H^2([0,1]) \cap H_0^1([0,1])$ with $v \geq -1$, let us compute

$$\begin{aligned}\int_0^1 u''(v''-u'')\,dx &= \int_0^\alpha u''v''\,dx + \int_\alpha^1 u''v''\,dx - \int_0^1 (u'')^2\,dx \\ &= v'(\alpha)\big(u''(\alpha^-) - u''(\alpha^+)\big) + v(\alpha)\big(u'''(\alpha^+) - u'''(\alpha^-)\big) \\ &\quad - 3\left(\frac{1}{\alpha^3} - \frac{1}{(\alpha-1)^3}\right) \\ &= 3v'(\alpha)\left(\frac{1}{\alpha^2} - \frac{1}{(\alpha-1)^2}\right) - 3\big(v(\alpha)+1\big)\left(\frac{1}{\alpha^3} - \frac{1}{(\alpha-1)^3}\right).\end{aligned}$$

By assumption $v(\alpha) + 1 \geq 0$ and $\dfrac{1}{\alpha^3} - \dfrac{1}{(\alpha-1)^3} > 0$, since $\alpha \in (0,1)$, so that

$$-3\big(v(\alpha)+1\big)\left(\frac{1}{\alpha^3} - \frac{1}{(\alpha-1)^3}\right) \leq 0.$$

Therefore, if u is a solution it is necessary and sufficient that the coefficient of $v'(\alpha)$ in the right hand side of the previous inequality is 0. Indeed, assume by contradiction that $\dfrac{1}{\alpha^2} - \dfrac{1}{(\alpha-1)^2} < 0$. Then one can choose $v \in H^2([0,1]) \cap H^1_0([0,1])$ with $v \geq -1$ such that $v'(\alpha) < 0$ and, more precisely, so negatively large that

$$v'(\alpha)\left(\frac{1}{\alpha^2} - \frac{1}{(\alpha-1)^2}\right) > \big(v(\alpha)+1\big)\left(\frac{1}{\alpha^3} - \frac{1}{(\alpha-1)^3}\right);$$

for example one can take v such that $v(\alpha) = 0$ and

$$v'(\alpha) = 2\,\frac{\dfrac{1}{\alpha^3} - \dfrac{1}{(\alpha-1)^3}}{\dfrac{1}{\alpha^2} - \dfrac{1}{(\alpha-1)^2}} < 0.$$

Thus, for such a v, we would have

$$\int_0^1 u''(v'' - u'')\,dx > 0,$$

that is u does *not* satisfy the reversed variational inequality.

The case $\dfrac{1}{\alpha^2} - \dfrac{1}{(\alpha-1)^2}) > 0$ can be treated in the same way, and so we can conclude that

$$\frac{1}{\alpha^2} - \frac{1}{(\alpha-1)^2} = 0,$$

i.e. $\alpha = 1/2$. In this way we have shown that the *unique* nontrivial solution of problem (2) is

$$u(x) = \begin{cases} 4x^3 - 3x & \text{if } x \in [0,1/2] \\ \\ -4(x-1)^3 + 3(x-1) & \text{if } x \in (1/2,1]. \end{cases}$$

Of course, this solution is symmetric with respect to $x = 1/2$, and Theorem 2 is thus completely proved. □

Acknowledgements. The author wishes to thank Antonio Marino for many interesting discussions about reversed variational inequalities.

REFERENCES

[1] G. Buttazzo and D. Percivale, *On the approximation of the elastic bounce problem on Riemannian manifolds*, J. Differential Equations, **47** (1983), 227–245.
[2] M. Degiovanni, *Multiplicity of solutions for the bounce problem*, J. Differential Equations, **54** (1984), 414–428.
[3] A. C. Lazer and P. J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Review, **32** (1990), 537–578.
[4] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., **20** (1967), 493–519.
[5] P. Magrone, D. Mugnai and R. Servadei, *Multiplicity of solutions for semilinear variational inequalities via linking and ∇–theorems*, J. Differential Equations, **228** (2006), 191–225.
[6] A. Marino and D. Mugnai, *Asymptotical multiplicity and some reversed variational inequalities*, Topol. Methods Nonlinear Anal., **20** (2002), 43-62.
[7] A. Marino and D. Mugnai, *Asymptotically critical points and their multiplicity*, Topol. Methods Nonlinear Anal., **19** (2002), 29–38.
[8] A. Marino and C. Saccon, *Asymptotically critical points and multiple solutions in the elastic bounce problem*, in "Variational Analysis and Applications," Nonconvex Optim. Appl., **79**, Springer, New York, 2005, 651–663.
[9] D. Mugnai, *Bounce on a p–Laplacian*, Comm. Pure Appl. Anal., **2** (2003), 363–371.

[10] D. Mugnai, *Multiplicity of critical points in presence of a linking: application to a superlinear boundary value problem*, NoDEA. Nonlinear Differential Equations Appl., **11** (2004), 379–391.

[11] D. Mugnai, *On a "reversed" variational inequality*, Topol. Methods Nonlinear Anal., **17** (2001), 321–358.

[12] G. Stampacchia, *Formes bilinaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris, **258** (1964), 4413–4416.

Received September 2006; revised March 2007.

E-mail address: mugnai@dipmat.unipg.it