THERMAL CONTROL OF A RATE-INDEPENDENT MODEL FOR PERMANENT INELASTIC EFFECTS IN SHAPE MEMORY MATERIALS

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ABSTRACT. We address the thermal control of the quasi-static evolution of a polycrystalline shape memory alloy specimen. The thermomechanical evolution of the body is described by means of an extension of the phenomenological SOUZA-AURICCHIO model [6, 7, 8, 57] accounting also for permanent inelastic effects [9, 11, 27]. By assuming to be able to control the temperature of the body in time we determine the corresponding quasi-static evolution in the *energetic* sense. In a similar way as in [28], using results by RINDLER [49, 50] we prove the existence of optimal controls for a suitably large class of cost functionals.

Shape-memory alloys (SMA) are examples of active materials, showing a remarkable thermo-mechanical behaviour: at suitably high temperatures they are able to completely recover comparably large strains during the loading-unloading cycles (this is the so-called *superelastic effect*) while at lower temperatures permanent deformations appear when unloading, but the material can be forced to recover its original shape by means of a thermal cycle (this is the so-called *shape memory effect*). This characteristic macroscopic behaviour is the result of a solid-solid phase transition at the metallic lattice level between a highly symmetric crystallographic phase, called *austenite*, which is dominant at high temperatures, and less symmetric phases, called *martensites*, which are energetically favorable at lower temperatures or high stresses.

This amazing thermo-mechanical behaviour of the SMA is at the basis of a great variety of innovative applications, going from biomedicine to different branches of engineering. Indeed the engineering literature of SMA models is large and the SMA behaviour has been investigated at all scales and by means of a great number of models (the interested reader can find a lot of references in this respect for instance in our previous paper [27]).

Correspondingly, the mathematical treatment of SMA behaviour is comparably less developed. Some results in this sense refer to the original formulation or modifications of the FRÉMOND model, [33] or in the FALK or FALK-KONOPKA models, [31, 32], see for instance, with no claim of completeness, [1, 2, 19, 22, 35, 38, 48, 62]; we also refer to more recent results concerning phase transitions in shape memory

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alloys, in the framework of the Ginzburg-Landau theory, namely [16, 17] in the isothermal and non-isothermal setting respectively.
Recently, great attention has been paid to the model originally introduced by Souza, Mamiya and Zouain [57], and later refined by Auricchio and Petrini [7] - we shall refer to the SOUZA-AURICCHIO model (SA) in the following.
The SA model shows some advantages in terms of simplicity and robustness with respect to discretization; moreover it is efficient in accommodating modifications and extensions to more general situations, such as for instance deformation effects [9, 11], asymmetric material behavior [10], ferromagnetic effects [4, 3], and finite strains [29, 30].

From the mathematical viewpoint, instead, the isothermal SA model has been studied in [5, 44] and then extended to involve permanent deformation effects in [27, 11] (see also [10] for the derivation of the model); the ferromagnetic model is discussed in [14, 15, 58] and the analysis of the finite strains situation is in [34].

Recently much effort has been done in the direction of including temperature changes in the SA model; these results fall in the more general framework of the analysis of non-isothermal problems in mathematical modeling of complex materials, which is gaining more and more importance in the recent years (see for instance, without aiming at completeness [12, 24, 25, 26, 51, 52] in the setting of thermo-visco-elastic or thermo-elasto-plastic materials). In particular, in the setting of shape memory alloys we can quote the following contributions: [13, 43, 45] (given temperature, 3D), [47] (unknown temperature, but viscous), [40, 39] (unknown temperature, 1D), [28] (given temperature, 3D, where also some optimal control problem is included), [53, 54, 55] (in the direction of the thermodynamics of SMA).

In this paper we focus on the thermal control of a SMA specimen under the Souza-Auricchio model modified to include also permanent inelastic effects. The control of SMA devices is obviously of great importance with respect to applications, as it appears from literature (see for instance [20, 21, 36, 37, 56] and very recently [59, 60], W14 for a selection of results). On the other hand, due to the recent renewed interest towards the SA models with permanent inelasticity, we really believe that the result we present can be, beside its interest on its own in completing a picture the authors started in [27, 28], also a starting point in view of future engineering applications, for instance in the direction of predicting the onset of structural and functional fatigue in the material during cyclic transformations or discussing the possibility of calibrating the constitutive parameters (including the ones appearing in the control problem) by means of simple experiments, with the purpose to evidence the qualitative agreement of the modeling predictions with the outcome of the experimental results.

We assume, also in this contribution, to be able to control the temperature of the specimen in time. This is possible when a SMA body is relatively *thin* in at least one direction and undergoes relatively *low-frequency* loading-unloading cycles, so that the heat produced via deformation and phase-change can be assumed to be (almost) instantaneously dissipated in the environment; for the sake of simplicity, we also focus our attention on a space-homogeneous dependent temperature of the specimen, as this will play the role of the control variable in our setting.

The main results of the paper are, on one hand a novel existence result for the *state problem*, i.e. given the temperature, to determine the quasi-static mechanical evolution of the SMA specimen. We underline that, in the spirit of [28] we require less regularity for the temperature (a fact that will be important with respect to optimal control, as we would like to consider the largest possible set of control temperatures) constructing results that are stronger than the available ones [43, 45, 47]. In particular, the main difficulties come from dealing at the same time with less restrictive assumptions on the temperature (which is non-homogeneous this time) and the original non-regularized formulation of the Souza-Auricchio model with permanent inelasticity, with the extra strain tensor variable $\varepsilon^{\mathrm{pl}}$ coming into play.

On the other hand, the other relevant result we are able to get is the proof of the existence of an optimal control for a suitably large class of cost functionals depending on both mechanics and temperature. The applicative interest in this respect relies on the possibility of activating SMA devices by controlling the temperature of the specimen via Joule's heating, which is one of the basic technological activation mechanisms currently exploited in real applications [23]. Our argument is basically the concrete application of the abstract theory developed by Rindler [49] on existence of optimal controls in the frame of rate-independent systems. We would like to remark that the present existence result represents just a first step in the direction of optimally control the complex thermomechanical behavior of SMA via the Souza-Auricchio model including the permanent inelastic effects. One possible further investigation goes into the direction of possibly computing optimal controls; this however seems to be a quite complex task due to the crucially non-smooth nature of the Souza-Auricchio model, which is conserved when the original SA model is extended in the direction of the description of training and degradation. Some comments on possible future developments of this investigation can be found in Subsection 4.1 of [28].

This is the plan of the paper. In Section 2 we recall the mechanical problem formulation, collect our assumptions, and state the main results. Section 3 reports on the analysis of the state problem whereas the existence proof for optimal controls is developed in Section 4.

1. Mechanical problem formulation

1.1. **Reference configuration and prescribed boundary displacement.** We recall some basic features of our SMA model with permanent inelastic effects. The Reader is referred to [9] for the modelling and to [27] for some mathematical analysis.

We also refer to [11] for some recent developments and to [57, 7] for the original Souza-Auricchio model.

We denote by $\mathbb{R}^{3\times 3}_{\rm sym}$ the space of symmetric 3-tensors in $\mathbb{R}^3$ endowed with the usual scalar product $a{:}b = \mathrm{tr}(ab) := a_{ij}b_{ij}$ (summation convention) and the corresponding norm $|a| = \sqrt{a{:}a}$. The space $\mathbb{R}^{3\times 3}_{\rm sym}$ is orthogonally decomposed as $\mathbb{R}^{3\times 3}_{\rm sym} = \mathbb{R}^{3\times 3}_{\rm dev} \oplus \mathbb{R}1_2$, where $\mathbb{R}1_2$ is the subspace spanned by the identity 2-tensor 1_2, while $\mathbb{R}^{3\times 3}_{\rm dev}$ is the subspace of all *deviatoric* symmetric tensors.

The reference configuration of the body is represented by a non-empty, connected, bounded, and open subset $\Omega \subset \mathbb{R}^3$ with a boundary $\partial\Omega$ that we assume to be Lipschitz; moreover let $\Gamma_{\rm Neu}, \Gamma_{\rm Dir} \subset \partial\Omega$ with $\Gamma_{\rm Neu} \cap \Gamma_{\rm Dir} = \emptyset$ and $\mathcal{H}^2(\Gamma_{\rm Dir}) > 0$, where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure.

Given the displacement $u : \Omega \to \mathbb{R}^3$ from the fixed reference configuration with $u \in H^1_{\rm loc}(\Omega, \mathbb{R}^3)$ we consider the corresponding *symmetric gradient* of u by

$$\varepsilon(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where $u_{i,j}$ means $\partial_j u_i$. In particular, throughout the paper we will make tacit use of the well-known Korn inequality

$$c_{\rm Korn}||u||^2_{H^1(\Omega;\mathbb{R}^3)} \le ||u||^2_{L^2(\Gamma_{\rm Dir};\mathbb{R}^3)} + ||\varepsilon(u)||^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm sym})}$$

for any $u \in H^1(\Omega;\mathbb{R}^3)$ and some constant $c_{\rm Korn} > 0$ depending just on Ω.

We frame our problem within the classical theory of inelastic small strains [41], thus we additionally decompose $\varepsilon(u) = \varepsilon^{\rm el} + \varepsilon^{\rm in}$, where $\varepsilon^{\rm el}$ represents the elastic part of the strain while $\varepsilon^{\rm in}$ refers to the inelastic part due to the martensitic transformations observed in the material. In the particular situation we are going to consider, the inelastic part of the strain $\varepsilon^{\rm in}$ turns to be further decomposed as $\varepsilon^{\rm in} = \varepsilon^{\rm tr} + \varepsilon^{\rm pl}$ where $\varepsilon^{\rm tr}$ is the recoverable part *(or transformation)* of the strain while $e^{\rm pl}$ represents the non-recoverable permanent part of the strain *(the plastic part)*.

The body will be subject to a given surface traction on the part $\Gamma_{\rm Neu}$ of the boundary. On the other hand, non-homogeneous Dirichlet conditions for the displacement will be prescribed on $\Gamma_{\rm Dir}$. More precisely, by letting

$$u^{\rm Dir} \in W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^3)) \tag{1.1}$$

be given, the trace of $u^{\rm Dir}$ on $\Gamma_{\rm Dir}$ plays the role of the prescribed boundary value for the displacement u. In particular, for all given times $t \in [0,T]$, the set of admissible states $(u(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))$ is given by $\mathcal{Y}(u^{\rm Dir}(t))$ where

$$\begin{aligned}(1.2)\qquad & \bar u \in H^1(\Omega;\mathbb{R}^3) \mapsto \mathcal{Y}(\bar u) \\ = \quad & \Big\{(u,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) \in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev})\times L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev}) \ : \ u = \bar u \text{ on } \ \Gamma_{\rm Dir}\Big\}.\end{aligned}$$

For the sake of brevity, from now on we set

$$\mathcal{Y} := H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev})\times L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev}).$$

1.2. **Elastic energy.** We simplify a bit the model by assuming that each phase is isotropic and described by the same elasticity tensor. In particular, we denote the *elastic energy* functional $\mathcal{C} : H^1(\Omega; \mathbb{R}^{3\times 3}_{\rm sym}) \to [0,\infty)$ as

$$\mathcal{C}(a) := \frac{1}{2}\int_\Omega a{:}\mathbb{C}a \,\mathrm{d}x,$$

where $\mathbb{C}$ is the elastic tensor. Therefore, the *elastic contribution* to the stored energy of the material is simply given by $\mathcal{C}(\varepsilon^{\rm el}) = \mathcal{C}(\varepsilon(u) - \varepsilon^{\rm tr} - \varepsilon^{\rm pl})$.

1.3. **Inelastic energy.** The inelastic part of the stored energy of the material is a function of the temperature θ and the inelastic strain $(\varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ only. In particular, the *inelastic energy* density takes the form

$$(\theta, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \mapsto \beta(\theta)|\varepsilon^{\rm tr}| + \frac{1}{2}\varepsilon^{\rm tr}{:}\mathbb{H}^{\rm tr}{:}\varepsilon^{\rm tr} + \frac{1}{2}\varepsilon^{\rm pl}{:}\mathbb{H}^{\rm pl}{:}\varepsilon^{\rm pl} + \varepsilon^{\rm tr}{:}\mathbb{A}{:}\varepsilon^{\rm pl} + I(\varepsilon^{\rm tr} + \varepsilon^{\rm pl}), \tag{1.3}$$

where $\mathbb{H}^{\rm tr}$, $\mathbb{H}^{\rm pl} \in \mathbb{R}^{3\times3\times3\times3}$ are the (symmetric) hardening tensors, $\mathbb{A} \in \mathbb{R}^{3\times3\times3\times3}_{\rm sym}$ is a linear symmetric coupling tensor such that

$$(a, b) \mapsto \frac{1}{2}a{:}\mathbb{H}^{\rm tr}{:}a + \frac{1}{2}b{:}\mathbb{H}^{\rm pl}{:}b + a{:}\mathbb{A}{:}b$$

is positive definite. Moreover, I is the indicator function of the ball $B := \{a \in \mathbb{R}^{3\times3}_{\rm dev} : |a| \le \varepsilon_L\}$ for some $\varepsilon_L > 0$. In particular $I(a) = 0$ if $a \in B$ and $I(a) = \infty$ elsewhere. Finally β is a given Lipschitz continuous nonnegative function describing the temperature dependence of the inelastic response of the medium. In particular, $\beta(\theta)$ corresponds to the austenite-martensite transition critical stress at temperature $\theta > 0$. The original choice $\beta(\theta) = b(\theta - \theta_{\rm m})^+$ of the Souza-Auricchio model ($b > 0$ and $\theta_{\rm m}$ being a critical temperature for the martensite-austenite equilibrium in the stress-free configuration) is included in our frame. Note incidentally that the latter behaviour is not induced by the plastic evolution of $\varepsilon^{\rm pl}$ and the constraining term $I(\varepsilon^{\rm tr} + \varepsilon^{\rm pl})$ refers to the experimental evidence that the inelastic behaviour of the material is confined to some bounded strain proportion. In particular $\varepsilon_L > 0$ measures the maximal inelastic strain which can be obtained via reorientation of the martensitic variants. Other options such that considering two indicator functions $I(\varepsilon^{\rm tr}) + I(\varepsilon^{\rm pl})$ in the energy may also be considered with minor modifications.

In passing, one shall note that the existence and optimal control issues discussed here do not rely on the particular form of the inelastic energy and could be possibly adapted to much more general situations.

A last term has to be introduced in the overall stored energy of the system in order to penalize martensite-martensite interfaces. Indeed, we include an *interfacial energy* term

$$(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \mapsto \nu \int_\Omega |\nabla \varepsilon^{\rm tr}| \mathrm{d}x + \bar{\nu} \int_\Omega |\nabla \varepsilon^{\rm pl}| \mathrm{d}x$$

where $\nu, \bar{\nu} > 0$ are given scale parameters. Note that the latter integrals bear the meaning of a total variation and, as such, will have also a crucial compactifying effect. The occurrence of this interfacial term however does not prevent $(\varepsilon^{\rm tr}, \varepsilon^{\rm pl})$

from possibly exhibiting jumps. This is a particularly desirable feature in connection with shape memory alloys where sharp phase boundaries are usually observed.

1.4. **Stored energy.** Assuming the temperature of the body $\theta \in W^{1,1}(0,T)$ to be (spatially homogeneous and) prescribed, the stored-energy functional $\mathcal{W}(\cdot,\cdot,\cdot;\theta(t)) : \mathcal{Y}(u^{\mathrm{Dir}}(t)) \to [0,+\infty]$ for the body at time $t \in [0,T]$ (where the space $\mathcal{Y}(u^{\mathrm{Dir}}(t))$ has been introduced in (1.2)) will be hence given by

(1.4)

$$\begin{aligned}
\mathcal{W}(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}};\theta(t)) &= \mathcal{C}(\varepsilon(u)-\varepsilon^{\mathrm{tr}}-\varepsilon^{\mathrm{pl}}) \\
&\quad + \int_\Omega \left(\frac{1}{2}\varepsilon^{\mathrm{tr}}{:}\mathbb{H}^{\mathrm{tr}}{:}\varepsilon^{\mathrm{tr}} + \frac{1}{2}\varepsilon^{\mathrm{pl}}{:}\mathbb{H}^{\mathrm{pl}}{:}\varepsilon^{\mathrm{pl}} + \varepsilon^{\mathrm{tr}}{:}\mathbb{A}{:}\varepsilon^{\mathrm{pl}} + I(\varepsilon^{\mathrm{tr}}+e^{\mathrm{pl}})\right)\mathrm{d}x \\
&\quad + \nu\int_\Omega |\nabla\varepsilon^{\mathrm{tr}}|\mathrm{d}x + \bar{\nu}\int_\Omega |\nabla\varepsilon^{\mathrm{pl}}|\mathrm{d}x + \int_\Omega \beta(\theta(t))|\varepsilon^{\mathrm{tr}}|\mathrm{d}x \\
&=: \mathcal{E}(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}})
\end{aligned}$$

In particular, the functional $\mathcal{E}(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})$ collects all terms above which are independent of time (i.e., of the temperature θ) while $\mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}})$ contains the only temperature-driven term. Note that the stored-energy functional $\mathcal{W}(\cdot,\cdot,\cdot;\theta(t))$ is uniformly convex in $H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}})\times L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}})$ even if non-smooth. We underline moreover that $\mathcal{F}$ depends only on the recoverable part of the inelastic strain while it is independent of its plastic part.

1.5. **Load and traction.** In addition to the above-prescribed boundary displacement conditions on Γ_{Dir}, we shall consider some imposed body force f and surface traction g, as well. We assume to be given

$$f \in W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^3)), \quad g \in W^{1,1}(0,T;L^2(\Gamma_{\mathrm{Neu}};\mathbb{R}^3)) \tag{1.5}$$

and define the *total load* $\ell \in W^{1,1}(0,T;(H^1(\Omega;\mathbb{R}^3))')$ as

$$\langle \ell(t),u\rangle := \int_\Omega f(t)\cdot u\,\mathrm{d}x + \int_{\Gamma_{\mathrm{Neu}}} g(t)\cdot u\,\mathrm{d}\mathcal{H}^2 \quad \forall u\in H^1(\Omega;\mathbb{R}^3),\ t\in[0,T]$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega;\mathbb{R}^3))'$ and $H^1(\Omega;\mathbb{R}^3)$. We would like to point out that in (1.5) other choices could have been possible, for instance taking $L^{6/5}(\Omega;\mathbb{R}^3)$ or $L^{4/3}(\Omega;\mathbb{R}^3)$ (and correspondingly $L^{6/5}(\Gamma_{\mathrm{Neu}};\mathbb{R}^3)$ or $L^{4/3}(\Gamma_{\mathrm{Neu}};\mathbb{R}^3)$); we anyway prefer to keep the choice of L^2 for the sake of simplicity.

1.6. **Dissipation potential.** In view of the application of the framework of the energetic solutions, started in [42, 46], we introduce a *dissipation (pseudo-)potential* $\mathcal{D}\colon L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}}\times\mathbb{R}^{3\times3}_{\mathrm{dev}})\to[0,\infty)$, governing the evolution of the quasi-static system, starting from a suitable initial state $(u_0,\varepsilon_0^{\mathrm{tr}},\varepsilon_0^{\mathrm{pl}})$ as

$$\mathcal{D}(a,b) := \int_\Omega D(a,b)\,\mathrm{d}x,$$

where the dissipation (density) function $D : \mathbb{R}^{3\times 3}_{\mathrm{dev}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}} \to [0, +\infty)$ is continuous, positively 1-homogeneous and fulfills the triangle inequality

$$D(a_1 + a_2, b_1 + b_2) \le D(a_1, b_1) + D(a_2, b_2) \tag{1.6}$$

for all $a_1, a_2, b_1, b_2 \in \mathbb{R}^{3\times 3}_{\mathrm{dev}}$. Just for fixing the ideas, we can think of

$$D(\dot\varepsilon^{\mathrm{tr}}, \dot\varepsilon^{\mathrm{pl}}) = ((R^{\mathrm{tr}})^p |\dot\varepsilon^{\mathrm{tr}}|^p + (R^{\mathrm{pl}})^p |\dot\varepsilon^{\mathrm{pl}}|^p)^{1/p}, \qquad p \in [1, \infty)$$

as a possible expression for D, where $R^{\mathrm{tr}}, R^{\mathrm{pl}}$ represent positive transformation radii and with the convention that, in the case $p = \infty$ we have

$$D(\dot\varepsilon^{\mathrm{tr}}, \dot\varepsilon^{\mathrm{tr}}) = \max(R^{\mathrm{tr}}|\dot\varepsilon^{\mathrm{tr}}| + R^{\mathrm{pl}}|\dot\varepsilon^{\mathrm{pl}}|).$$

Moreover, for any $(\varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) : [0, T] \to \mathbb{R}^{3\times 3}_{\mathrm{dev}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}}$ we let the *total dissipation* of the process on the time interval $[s, t] \subseteq [0, T]$ be given by

$$\begin{aligned} &\mathrm{Diss}_D(\varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}; [s, t]) \\ := \quad &\sup\left\{ \sum_{i=1}^{N} \mathcal{D}(\varepsilon^{\mathrm{tr}}(t_i) - \varepsilon^{\mathrm{pl}}(t_{i-1}), \varepsilon^{\mathrm{tr}}(t_i) - \varepsilon^{\mathrm{pl}}(t_{i-1})) : \{s = t_0 < t_1 < \cdots < t_N = t\} \right\}, \end{aligned}$$

the supremum is taken over the set of all finite partitions of the interval $[s, t]$. An analogue notion of $\mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}; [s, t])$ based on the functional $\mathcal{D}$ for functions of time taking values in $L^1(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{dev}} \times \mathbb{R}^{3\times 3}_{\mathrm{dev}})$ will also be considered.

1.7. **Energetic formulation.** First of all we define the set of *stable states* $\mathcal{S}(t, \theta)$ at time t and for the temperature θ as

$$\begin{aligned} \mathcal{S}(t, \theta) := \Big\{ &(u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) \in \mathcal{Y}(u^{\mathrm{Dir}}(t)) : \mathcal{E}(u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) < \infty \text{ and } \forall (\bar u, \bar\varepsilon^{\mathrm{tr}}, \bar\varepsilon^{\mathrm{pl}}) \in \\ &\mathcal{Y}(u^{\mathrm{Dir}}(t)),\ \mathcal{E}(u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta, \varepsilon^{\mathrm{tr}}) - \langle \ell(t), u \rangle \le \mathcal{E}(\bar u, \bar\varepsilon^{\mathrm{tr}}, \bar\varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta, \bar\varepsilon^{\mathrm{tr}}) \\ &- \langle \ell(t), \bar u \rangle + \mathcal{D}(\varepsilon^{\mathrm{tr}} - \bar\varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}} - \bar\varepsilon^{\mathrm{pl}}) \Big\}. \end{aligned} \tag{1.7}$$

This definition can be interpreted as follows: given a stable state $(u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}})$, no competitor state $(\bar u, \bar\varepsilon^{\mathrm{tr}}, \bar\varepsilon^{\mathrm{pl}})$ can be preferred in terms of balance between total energy and dissipation. This global minimality requirement is here completely justified by the convexity of the total energy of the body and by the fact that the dissipation potential depends only on rates but it is independent of the state itself.

Now, *energetic solutions*, according to the definition given in [42, 46], are everywhere defined functions $t : [0, T] \mapsto (u(t), \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) \in \mathcal{Y}(u^{\mathrm{Dir}}(t))$ such that

$$(u(0), \varepsilon^{\mathrm{tr}}(0), \varepsilon^{\mathrm{pl}}(0)) = (u_0, \varepsilon^{\mathrm{tr}}_0, \varepsilon^{\mathrm{pl}}_0)$$

for some give initial datum $(u_0, \varepsilon^{\mathrm{tr}}_0, \varepsilon^{\mathrm{pl}}_0) \in \mathcal{Y}(u^{\mathrm{Dir}}(0))$; moreover the functions $t \mapsto \langle \dot\ell(t), u(t) \rangle$ and $t \mapsto \beta'(\theta(t))\dot\theta(t)|\varepsilon^{\mathrm{tr}}|$ are integrable and for all $t \in [0, T]$ the following two conditions are satisfied:

Global stability:

$$(u(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t))\in\mathcal{S}(t,\theta(t)). \tag{1.8}$$

Energy balance:

$$\begin{aligned}&\mathcal{E}(u(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t))+\mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}}(t))-\langle\ell(t),u(t)\rangle+\mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}};[0,t])\\ =&\mathcal{E}(u(0),\varepsilon^{\mathrm{tr}}(0),\varepsilon^{\mathrm{pl}}(0))+\mathcal{F}(\theta(0),\varepsilon^{\mathrm{tr}}(0))-\langle\ell(0),u(0)\rangle\\ &+\int_0^t\int_\Omega\beta'(\theta(s))\dot\theta(s)|\varepsilon^{\mathrm{tr}}|\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\langle\dot\ell(s),u(s)\rangle\,\mathrm{d}s.\end{aligned} \tag{1.9}$$

2. The state problem

Suppose that the space-homogeneous temperature $\theta\in W^{1,1}(0,T)$ is given. We would like to prove the existence of a suitably weak solution to the quasi-static *state problem*

$$\begin{aligned}\mathbb{C}(\varepsilon(u)-\varepsilon^{\mathrm{tr}}-\varepsilon^{\mathrm{pl}})&=\sigma\quad\text{in }\Omega\times(0,T),\\ \nabla\cdot\sigma+f&=0\quad\text{in }\Omega\times(0,T),\\ u&=u^{\mathrm{Dir}}\quad\text{on }\Gamma_{\mathrm{Dir}}\times(0,T),\\ \sigma n&=g\quad\text{on }\Gamma_{\mathrm{Neu}}\times(0,T),\end{aligned}$$

$$\partial_{\dot\varepsilon^{\mathrm{tr}}}\mathcal{D}(\dot\varepsilon^{\mathrm{tr}},\dot\varepsilon^{\mathrm{pl}})+\partial_{\varepsilon^{\mathrm{tr}}}\mathcal{W}(u(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t);\theta(t))\ni 0\text{ in }L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}}\times\mathbb{R}^{3\times3}_{\mathrm{dev}}),\forall t\in(0,T),$$
$$\partial_{\dot\varepsilon^{\mathrm{pl}}}\mathcal{D}(\dot\varepsilon^{\mathrm{tr}},\dot\varepsilon^{\mathrm{pl}})+\partial_{\varepsilon^{\mathrm{pl}}}\mathcal{W}(u(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t);\theta(t))\ni 0\text{ in }L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}}\times\mathbb{R}^{3\times3}_{\mathrm{dev}}),\forall t\in(0,T),$$
$$u(0)=u_0,\quad\varepsilon^{\mathrm{tr}}(0)=\varepsilon^{\mathrm{tr}}_0\quad\varepsilon^{\mathrm{pl}}(0)=\varepsilon^{\mathrm{pl}}_0\quad\text{in }\Omega$$

where σ stands for the stress, n is the outward normal to Γ_{Neu}, and ∂ is the subdifferential in the sense of convex analysis [18].

The main result of the section will be the following:

Theorem 2.1 (Existence for the state problem)**.** *Assume* (1.1) *and* (1.5)*. Given $\theta\in W^{1,1}(0,T)$ and an initial value $(u_0,\varepsilon^{\mathrm{tr}}_0,\varepsilon^{\mathrm{pl}}_0)\in\mathcal{S}(0,\theta(0))$ there exists an energetic solution $(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})$ of the state problem in the sense of* (1.8)-(1.9)*. Moreover, all energetic solutions belong to the space*

$$\mathcal{K}:=W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}})\times L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}})). \tag{2.1}$$

2.1. **Proof.** The proof of Theorem 2.1 follows the by-now classical approach of convergence of time discretization for rate-independent evolution problems, see [42]. In particular, in the spirit of [49] some specific care is devoted to the prove the convergence of the powers of external actions. In particular, as already observed in our previous result [28], these powers read here

$$\int_0^t\int_\Omega\beta'(\theta(s))\dot\theta(s)|\varepsilon^{\mathrm{tr}}|\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\langle\dot\ell(s),u(s)\rangle\,\mathrm{d}s \tag{2.2}$$

which doesn't satisfy the classical absolute continuity requirement [42, Assumption (A5)] of the general energetic solvability theory as θ and ℓ are just absolutely continuous here. While the former existence results for given temperature in [43, 45] by-pass this problem by requiring the temperature to be C^1, this would not be enough here as we are interested in establishing an optimal control result via θ. Therefore we would like to better consider the least possible time-regularity for the temperature θ entailing the solvability of the state problem. We try to avoid the problem of the lack of absolute continuity of the above term (2.2) by considering their concrete form. For the sake of completeness we present here a sketch of the proof, in which the difficulties related to the particular form of the external powers sum up with the problems linked to the presence of permanent inelastic effects.

A change of variables. We perform here a change of variables in order to reduce to a time-independent state space by considering the case of homogeneous Dirichlet boundary conditions. In particular, we let $v = u - u^{\mathrm{Dir}}$ and focus on the triplet $(v, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}})$ taking values in the space $\mathcal{Y}_0 := \mathcal{Y}(0)$. We easily compute that

$$\begin{aligned}&\mathcal{E}(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) - \langle \ell(t), u\rangle = \mathcal{E}(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})\\ &+\int_\Omega \varepsilon(u^{\mathrm{Dir}}){:}\mathbb{C}(\varepsilon(v){-}\varepsilon^{\mathrm{tr}}{-}\varepsilon^{\mathrm{pl}})\,\mathrm{d}x - \langle \ell(t), v\rangle + \mathcal{C}(\varepsilon(u^{\mathrm{Dir}})) - \langle \ell(t), u^{\mathrm{Dir}}\rangle.\end{aligned}$$

Let now $L : [0,T] \to \mathcal{Y}_0'$ be given by

$$\begin{aligned}&\langle L(t), (v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})\rangle\\ := \quad &-\int_\Omega \varepsilon(u^{\mathrm{Dir}}(t)){:}\mathbb{C}(\varepsilon(v){-}\varepsilon^{\mathrm{tr}}{-}\varepsilon^{\mathrm{pl}})\,\mathrm{d}x + \langle \ell(t), v\rangle \quad \forall (v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) \in \mathcal{Y}_0,\ t\in[0,T]\end{aligned}$$

and notice that $L \in W^{1,1}(0,T;\mathcal{Y}_0')$. It turns out that $(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})$ is an energetic solution of the quasi-static evolution problem (1.8)-(1.9) if and only if $(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) : t \mapsto \mathcal{Y}_0$ is such that $(v(0),\varepsilon^{\mathrm{tr}}(0),\varepsilon^{\mathrm{pl}}(0)) = (v_0,\varepsilon^{\mathrm{tr}}_0,\varepsilon^{\mathrm{pl}}_0) := (u^0 - u^{\mathrm{Dir}}(0),\varepsilon^{\mathrm{tr}}_0,\varepsilon^{\mathrm{pl}}_0)$, the functions $t \mapsto \langle \dot\ell(t), v(t)\rangle$ and $t \mapsto \beta'(\theta(t))\dot\theta(t)|\varepsilon^{\mathrm{tr}}(t)|$ are integrable, and we have, for all $t\in[0,T]$,

Stability (in the v variable):

$$\begin{aligned}&(v(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t)) \in \tilde{\mathcal{S}}(t,\theta(t)) := \{(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})\in\mathcal{Y}_0 : \forall (\bar v,\bar\varepsilon^{\mathrm{tr}},\bar\varepsilon^{\mathrm{pl}})\in\mathcal{Y}_0,\\ &\mathcal{E}(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}}) - \langle L(t),(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})\rangle \le \mathcal{E}(\bar v,\bar\varepsilon^{\mathrm{tr}},\bar\varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta(t),\bar\varepsilon^{\mathrm{tr}})\\ &- \langle L(t),(\bar v,\bar\varepsilon^{\mathrm{tr}},\bar\varepsilon^{\mathrm{pl}})\rangle + \mathcal{D}(\varepsilon^{\mathrm{tr}}{-}\bar\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}{-}\bar\varepsilon^{\mathrm{pl}})\},\end{aligned}\tag{2.3}$$

Energy balance (in the v variable):

$$\begin{aligned}&\mathcal{E}(v(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t)) + \mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}}(t)) - \langle L(t),(v(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t))\rangle\\ &+ \mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}};[0,t])\\ &= \mathcal{E}(v(0),\varepsilon^{\mathrm{tr}}(0),\varepsilon^{\mathrm{pl}}(0)) + \mathcal{F}(\theta(0),\varepsilon^{\mathrm{tr}}(0)) - \langle L(0),(v(0),\varepsilon^{\mathrm{tr}}(0),\varepsilon^{\mathrm{pl}}(0))\rangle\\ &+\int_0^t\int_\Omega \beta'(\theta(s))\dot\theta(s)|\varepsilon^{\mathrm{tr}}|\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \langle \dot\ell(s), v(s)\rangle\,\mathrm{d}s.\end{aligned}\tag{2.4}$$

Note that this change of variables does not affect the part of the energy $\mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}})$ depending only on θ and $\varepsilon^{\mathrm{tr}}$.

Time discretization. Assume to be given a sequence of partitions $\{0 = t_0^n < t_1^n < \cdots < t_{N^n-1}^n < t_{N^n}^n = T\}$ with diameter $\tau^n = \max_{i=1,\dots,N^n}(t_i^n - t_{i-1}^n)$ going to 0 as $n \to \infty$. We inductively define (a sequence of) unique solutions $\{(v_i^n, \varepsilon_i^{\mathrm{tr,n}}, \varepsilon_i^{\mathrm{pl,n}})\}_{i=0}^{N^n}$ of the incremental problems

$$
\begin{aligned}
&(v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n}) = \operatorname{Arg\,Min}_{(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})\in\mathcal{Y}_0}\big(\mathcal{E}(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta(t_i^n),\varepsilon^{\mathrm{tr}}) \\
&-\langle L(t_i^n),(v,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})\rangle + \mathcal{D}(\varepsilon^{\mathrm{tr}} - \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon^{\mathrm{pl}} - \varepsilon_{i-1}^{\mathrm{pl},n})\big)
\end{aligned}
\tag{2.5}
$$

for $i = 1,\dots,N^n$ with $(v_0^n, \varepsilon_0^{\mathrm{tr},n}, \varepsilon_0^{\mathrm{pl},n}) = (v_0, \varepsilon_0^{\mathrm{tr}}, \varepsilon_0^{\mathrm{pl}})$. The latter minimum problems are uniquely solvable due to the fact that the map $(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) \mapsto \mathcal{E}(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}}) - \langle L(t),(u,\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}})\rangle + \mathcal{D}(\varepsilon^{\mathrm{tr}} - \bar\varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}} - \bar\varepsilon^{\mathrm{pl}})$ is uniformly convex and lower semicontinuous in $\mathcal{Y}_0$ for any given $\theta(t) \in \mathbb{R}$ and $(\bar\varepsilon^{\mathrm{tr}}, \bar\varepsilon^{\mathrm{pl}}) \in L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}} \times \mathbb{R}^{3\times3}_{\mathrm{dev}})$.

Next, we denote by $(v_n, \varepsilon_n^{\mathrm{tr}}, \varepsilon_n^{\mathrm{pl}})$ the right-continuous and piecewise constant interpolant of the values $\{(v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n})\}_{i=0}^{N^n}$ on the partition. Moreover, we let $t_n\colon [0,T] \to [0,T]$ be given by $t_n(t) = t_{i-1}^n$ for $t \in [t_{i-1}^n, t_i^n)$ for $i \in 1,\dots,N^n$.

Stability at the discrete level. The minimality in (2.5) entails that $(v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n})$ is stable at time t_i^n, that is $(v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n}) \in \tilde{\mathcal{S}}(t_i^n, \theta(t_i^n))$, for all $i = 1,\dots,N$. Indeed, for any $(\bar v, \bar\varepsilon^{\mathrm{tr}}, \bar\varepsilon^{\mathrm{pl}}) \in \mathcal{Y}_0$, we get

$$
\begin{aligned}
&\mathcal{E}(v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n}) + \mathcal{F}(\theta(t_i^n), \varepsilon_i^{\mathrm{tr},n}) - \langle L(t_i^n), (v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n})\rangle \\
&+ \mathcal{D}(\varepsilon_i^{\mathrm{tr},n} - \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n} - \varepsilon_{i-1}^{\mathrm{pl},n}) \le \mathcal{E}(\bar v, \bar\varepsilon^{\mathrm{tr}}, \bar\varepsilon^{\mathrm{pl}}) + \mathcal{F}(\theta(t_i^n), \bar\varepsilon^{\mathrm{tr}}) - \langle L(t_i^n), (\bar v, \bar\varepsilon^{\mathrm{tr}}, \bar\varepsilon^{\mathrm{pl}})\rangle \\
&+ \mathcal{D}(\bar\varepsilon^{\mathrm{tr}} - \varepsilon_i^{\mathrm{tr},n}, \bar\varepsilon^{\mathrm{pl}} - \varepsilon_i^{\mathrm{pl},n}) + \mathcal{D}(\varepsilon_i^{\mathrm{tr},n} - \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n} - \varepsilon_{i-1}^{\mathrm{pl},n})
\end{aligned}
$$

and the term $\mathcal{D}(\varepsilon_i^{\mathrm{tr},n} - \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n} - \varepsilon_{i-1}^{\mathrm{pl},n})$ cancels out.

Convergence to a time-continuous evolution. Due to the minimality (2.5) of $(v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n})$ we deduce that

$$
\begin{aligned}
&\mathcal{E}(v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n}) - \mathcal{E}(v_{i-1}^n, \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon_{i-1}^{\mathrm{pl},n}) + \mathcal{F}(\theta(t_i^n), \varepsilon_i^{\mathrm{tr},n}) - \mathcal{F}(\theta(t_{i-1}^n), \varepsilon_{i-1}^{\mathrm{tr},n}) \\
&- \langle L(t_i^n), (v_i^n, \varepsilon_i^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n})\rangle + \langle L(t_{i-1}^n), (v_{i-1}^n, \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon_{i-1}^{\mathrm{pl},n})\rangle \\
&+ \mathcal{D}(\varepsilon_i^{\mathrm{tr},n} - \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n} - \varepsilon_{i-1}^{\mathrm{pl},n}) \\
&\le \mathcal{F}(\theta(t_i^n), \varepsilon_{i-1}^{\mathrm{tr},n}) - \mathcal{F}(\theta(t_{i-1}^n), \varepsilon_{i-1}^{\mathrm{tr},n}) - \langle L(t_i^n) - L(t_{i-1}^n), v_{i-1}^n\rangle.
\end{aligned}
$$

Summing up for i from 1 to $m \le N^n$, we get

$$
\begin{aligned}
&\mathcal{E}(v_m^n, \varepsilon_m^{\mathrm{tr},n}, \varepsilon_m^{\mathrm{pl},n}) - \mathcal{E}(v_0, \varepsilon_0^{\mathrm{tr}}, \varepsilon_0^{\mathrm{pl}}) + \mathcal{F}(\theta(t_m^n), \varepsilon_m^{\mathrm{tr},n}) - \mathcal{F}(\theta(0), \varepsilon_0^{\mathrm{tr}}) \\
&- \langle L(t_m^n), (v_m^n, \varepsilon_m^{\mathrm{tr},n}, \varepsilon_m^{\mathrm{pl},n})\rangle + \langle L(0), (v_0, \varepsilon_0^{\mathrm{tr}}, \varepsilon_0^{\mathrm{pl}})\rangle + \sum_{i=1}^m \mathcal{D}(\varepsilon_i^{\mathrm{tr},n} - \varepsilon_{i-1}^{\mathrm{tr},n}, \varepsilon_i^{\mathrm{pl},n} - \varepsilon_{i-1}^{\mathrm{pl},n}) \\
&\le \int_0^{t_m^n}\int_\Omega \beta'(\theta(s))\dot\theta(s)|\varepsilon_n^{\mathrm{tr}}|\,\mathrm{d}x\,\mathrm{d}s - \int_0^{t_m^n}\langle \dot\ell(s), v_n(s)\rangle\,\mathrm{d}s.
\end{aligned}
\tag{2.6}
$$

By exploiting the discrete Lemma and the coercivity of $\mathcal{E}$ we deduce that

$$\sup_{t\in[0,T]}\big(\mathcal{E}(v_n(t),\varepsilon_n^{\rm tr}(t),\varepsilon_n^{\rm pl}(t))+\mathcal{F}(\theta(t),\varepsilon_n^{\rm tr}(t))\big)\ \text{ and }\ \mathrm{Diss}_{\mathcal{D}}(\varepsilon_n^{\rm tr},\varepsilon_n^{\rm pl};[0,T])$$

$$\text{are bounded independently of } n. \tag{2.7}$$

Now Helly's selection principle entails the possibility of finding a (not relabeled) subsequence of partitions and a non-decreasing function $\phi:[0,T]\to[0,\infty)$ such that

$$(\varepsilon_n^{\rm tr}(t),\varepsilon_n^{\rm pl}(t))\to(\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t))\ \text{ weakly in }\ L^2(\Omega;\mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}^{3\times3}_{\rm dev})\ \ \forall t\in[0,T]$$

$$\mathrm{Diss}_{\mathcal{D}}(\varepsilon_n^{\rm tr},\varepsilon_n^{\rm pl};[0,t])\to\phi(t)\ \ \forall t\in[0,T]$$

$$\mathrm{Diss}_{\mathcal{D}}(\varepsilon_n^{\rm tr},\varepsilon_n^{\rm pl};[s,t])\le\ \phi(t)-\phi(s)\ \ \forall[s,t]\subset[0,T]. \tag{2.8}$$

On the other hand, due to the quadratic character of $\mathcal{E}(\cdot,\varepsilon^{\rm tr},\varepsilon^{\rm pl})$ we have from minimality (2.5) that $v_n=\mathcal{L}(\varepsilon_n^{\rm tr},\varepsilon_n^{\rm pl},L\circ t_n)$ where $\mathcal{L}:L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}^{3\times3}_{\rm dev})\times\mathcal{Y}_0'\to H^1(\Omega;\mathbb{R}^3)$ is a linear and continuous operator. In particular $v_n=\mathcal{L}(\varepsilon_n^{\rm tr},\varepsilon_n^{\rm pl},L\circ t_n)\to\mathcal{L}(\varepsilon^{\rm tr},\varepsilon^{\rm pl},L)=:v$. Moreover the convergence of energy and dissipation can be also achieved.

Stability of the limit trajectory. We now prove that the set

$$\mathbb{S}:=\bigcup_{t\in[0,T]}(t,\tilde{\mathcal{S}}(t,\theta(t)))$$

is closed with respect to the weak topology of $\mathbb{R}\times\mathcal{Y}_0$. Let $(t_k,v_k,\varepsilon_k^{\rm tr},\varepsilon_k^{\rm pl})\in\mathbb{S}$ with $t_k\to t$ and $(v_k,\varepsilon_k^{\rm tr},\varepsilon_k^{\rm pl})\to(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})$ weakly in $\mathcal{Y}_0$. By the lower semicontinuity of $\mathcal{E}$ and $\mathcal{F}$ and accounting of the strong continuity of $\mathcal{D}$ in $L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}^{3\times3}_{\rm dev})$ and the continuity of θ and L we get

$$\begin{aligned}
&\mathcal{E}(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})+\mathcal{F}(\theta(t),\varepsilon^{\rm tr})-\langle L(t),(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\rangle\\
&\le\liminf_{k\to+\infty}\big(\mathcal{E}(v_k,\varepsilon_k^{\rm tr},\varepsilon_k^{\rm pl})+\mathcal{F}(\theta(t_k),\varepsilon_k^{\rm tr})-\langle L(t_k),(v_k,\varepsilon_k^{\rm tr},\varepsilon_k^{\rm pl})\rangle\big)\\
&\le\liminf_{k\to+\infty}\big(\mathcal{E}(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})+\mathcal{F}(\theta(t_k),\bar\varepsilon^{\rm tr})-\langle L(t_k),(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})\rangle+\mathcal{D}(\varepsilon^{\rm tr}-\bar\varepsilon^{\rm tr},\varepsilon^{\rm pl}-\bar\varepsilon^{\rm pl})\big)\\
&=\mathcal{E}(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})+\mathcal{F}(\theta(t),\bar\varepsilon^{\rm tr})-\langle L(t),(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})\rangle+\mathcal{D}(\varepsilon^{\rm tr}-\bar\varepsilon^{\rm tr},\varepsilon^{\rm pl}-\bar\varepsilon^{\rm pl}),
\end{aligned}$$

for any $(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})\in\mathcal{Y}_0$. Then $(t,v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\in\mathbb{S}$.

Now we would like to exploit the latter closure property in order to prove that $(v(t),\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t))$ is a stable state, i.e. (2.3) holds. First of all we have that $t\mapsto t_n(t)$ converges uniformly to the identity and $(v_n(t),\varepsilon_n^{\rm tr}(t),\varepsilon_n^{\rm pl}(t))=(v_n(t_n(t)),\varepsilon_n^{\rm tr}(t_n(t)),\varepsilon_n^{\rm pl}(t_n(t)))$ converges to $(v(t),\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t))$. Since

$$(t_n(t),v_n(t_n(t)),(\varepsilon_n^{\rm tr}(t_n(t)),\varepsilon_n^{\rm pl}(t_n(t)))\in\mathbb{S}$$

then the stability (2.3) follows and the set $\mathbb{S}$ is closed.

Upper energy estimate. We can rewrite the inequality (2.6) in the following way

$$\begin{aligned}&\mathcal{E}(v_n(t),\varepsilon_n^{\mathrm{tr}}(t),\varepsilon_n^{\mathrm{pl}}(t))+\mathcal{F}(\theta(t_n(t)),\varepsilon_n^{\mathrm{tr}}(t))-\langle L(t_n(t)),(v_n(t),\varepsilon_n^{\mathrm{tr}}(t),\varepsilon_n^{\mathrm{pl}}(t))\rangle\\&+\mathrm{Diss}_{\mathcal{D}}(\varepsilon_n^{\mathrm{tr}},\varepsilon_n^{\mathrm{pl}};[0,t_n(t)])\le\mathcal{E}(v_0,\varepsilon_0^{\mathrm{tr}},\varepsilon_0^{\mathrm{pl}})+\mathcal{F}(\theta(0),\varepsilon_0^{\mathrm{tr}})-\langle L(0),(v_0,\varepsilon_0^{\mathrm{tr}},\varepsilon_0^{\mathrm{pl}})\rangle\\&+\int_0^{t_n(t)}\int_\Omega\beta'(\theta(s))\dot\theta(s)|\varepsilon_n^{\mathrm{tr}}|\,\mathrm{d}x\,\mathrm{d}s-\int_0^{t_n(t)}\langle\dot\ell(s),v_n(s)\rangle\,\mathrm{d}s.\end{aligned}$$

We now pass to the lim inf in the latter relation by exploiting the lower semicontinuity of $\mathcal{E}$ and $\mathcal{F}$, the integrability of $\dot\ell$ and of $(\beta\circ\theta)^{\dot{}}$, and the boundedness of v_n from (2.7)-(2.8). By the Lebesgue Dominated Convergence Theorem, we deduce that

$$\begin{aligned}&\mathcal{E}(v(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t))+\mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}}(t))-\langle L(t),(v(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t))\rangle\\&+\mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}};[0,t])\le\ \mathcal{E}(v_0,\varepsilon_0^{\mathrm{tr}},\varepsilon_0^{\mathrm{pl}})+\mathcal{F}(\theta(0),\varepsilon_0^{\mathrm{tr}})-\langle L(0),(v_0,\varepsilon_0^{\mathrm{tr}},\varepsilon_0^{\mathrm{pl}})\rangle\\&+\int_0^{t}\int_\Omega\beta'(\theta(s))\dot\theta(s)|\varepsilon^{\mathrm{tr}}|\,\mathrm{d}x\,\mathrm{d}s-\int_0^{t}\langle\dot\ell(s),v(s)\rangle\,\mathrm{d}s,\end{aligned}\tag{2.9}$$

for all $t\in[0,T]$, i.e., the upper energy estimate.

Lower energy estimate. Let us now check the converse inequality with respect to (2.9). Fix $t\in[0,T]$ and assume to be given a sequence of partitions $\{0=s_0^m<s_1^m<\dots<s_{M^m-1}^m<s_{M^m}^m=t\}$ such that $\max_{j=1,\dots,M^m}(s_j^m-s_{j-1}^m)\to0$. We shall let $s_m(s):=s_j^m$ for $s\in(s_{j-1}^m,s_j^m]$, $j=1,\dots,M^m$, $v_m:=v\circ s_m$, $\varepsilon_m^{\mathrm{tr}}:=\varepsilon^{\mathrm{tr}}\circ s_m$ and $\varepsilon_m^{\mathrm{pl}}:=\varepsilon^{\mathrm{pl}}\circ s_m$. From the stability condition $(v(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_{j-1}^m),\varepsilon^{\mathrm{pl}}(s_{j-1}^m))\in\tilde{\mathcal{S}}(s_{j-1}^m,\theta(s_{j-1}^m))$ we have

$$\begin{aligned}&\mathcal{E}(v(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_{j-1}^m),\varepsilon^{\mathrm{pl}}(s_{j-1}^m))+\mathcal{F}(\theta(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_{j-1}^m))\\&-\langle L(s_{j-1}^m),(v(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_{j-1}^m),\varepsilon^{\mathrm{pl}}(s_{j-1}^m))\rangle\\&\le\mathcal{E}(v(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m),\varepsilon^{\mathrm{pl}}(s_j^m))+\mathcal{F}(\theta(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_j^m))\\&-\langle L(s_{j-1}^m),(v(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m),\varepsilon^{\mathrm{pl}}(s_j^m))\rangle+\mathcal{D}(\varepsilon^{\mathrm{tr}}(s_j^m)-\varepsilon^{\mathrm{tr}}(s_{j-1}^m),\varepsilon^{\mathrm{pl}}(s_j^m)-\varepsilon^{\mathrm{pl}}(s_{j-1}^m)).\end{aligned}$$

We now add

$$\mathcal{F}(\theta(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m))-\mathcal{F}(\theta(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_j^m))-\langle L(s_j^m)-L(s_{j-1}^m),(v(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m),\varepsilon^{\mathrm{pl}}(s_j^m))\rangle$$

to both sides and rearrange the terms in order to obtain

$$\begin{aligned}&\mathcal{E}(v(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m),\varepsilon^{\mathrm{pl}}(s_j^m))+\mathcal{F}(\theta(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m))-\langle L(s_j^m),(v(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m),\varepsilon^{\mathrm{pl}}(s_j^m))\rangle\\&+\mathcal{D}(\varepsilon^{\mathrm{tr}}(s_{j-1}^m)-\varepsilon^{\mathrm{tr}}(s_j^m),\varepsilon^{\mathrm{pl}}(s_{j-1}^m)-\varepsilon^{\mathrm{pl}}(s_j^m))\ge\mathcal{E}(v(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_{j-1}^m),\varepsilon^{\mathrm{pl}}(s_{j-1}^m))\\&+\mathcal{F}(\theta(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_{j-1}^m))-\langle L(s_{j-1}^m),(v(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_{j-1}^m),\varepsilon^{\mathrm{pl}}(s_{j-1}^m))\rangle\\&+\mathcal{F}(\theta(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m))-\mathcal{F}(\theta(s_{j-1}^m),\varepsilon^{\mathrm{tr}}(s_j^m))\\&-\langle L(s_j^m)-L(s_{j-1}^m),(v(s_j^m),\varepsilon^{\mathrm{tr}}(s_j^m),\varepsilon^{\mathrm{pl}}(s_j^m))\rangle.\end{aligned}\tag{2.10}$$

Summing up for $j = 0, \dots, M^m$ we deduce that

$$\begin{aligned}&\mathcal{E}(v(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) + \mathcal{F}(\theta(t), \varepsilon^{\rm tr}(t)) - \langle L(t), (v(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))\rangle \\ &+ \mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}; [0,t]) \geq \mathcal{E}(v_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0) + \mathcal{F}(\theta(0), \varepsilon^{\rm tr}(0)) - \langle L(0), (v_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0)\rangle \\ &+ \sum_{j=1}^{M^m} \int_\Omega \big(\beta(\theta(s^m_j)) - \beta(\theta(s^m_{j-1}))\big)|\varepsilon^{\rm tr}(s^m_j)|\,\mathrm{d}x - \int_0^t \langle \dot{\ell}(s), v_m(s)\rangle\,\mathrm{d}s.\end{aligned} \tag{2.11}$$

We can handle the first term in the last line of (2.11) as follows

$$\sum_{j=1}^{M^m} \int_\Omega \big(\beta(\theta(s^m_j)) - \beta(\theta(s^m_{j-1}))\big)|\varepsilon^{\rm tr}(s^m_j)|\,\mathrm{d}x = \int_0^t \int_\Omega \left(\fint_m \frac{\mathrm{d}}{\mathrm{d}t}(\beta\circ\theta)\right) |\varepsilon^{\rm tr} \circ s_m|\mathrm{d}x\,\mathrm{d}s$$

where we used some obvious notation for the piecewise mean on the partition. As $\mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}; [0,t]) < \infty$ we have that $(\varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ is continuous in $L^1(\Omega, \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev})$ with the exception of at most a countable number of times. This in particular entails that $\varepsilon^{\rm tr}_m \to \varepsilon^{\rm tr}$ and $\varepsilon^{\rm pl}_m \to \varepsilon^{\rm pl}$ pointwise almost everywhere in $[0,t]$. Moreover, $\beta \circ \theta \in W^{1,1}(0,T)$ and one has that

$$\fint_m \frac{\mathrm{d}}{\mathrm{d}t}(\beta\circ\theta) \to \beta'(\theta)\dot{\theta} \quad \text{a.e. in } [0,t].$$

Hence, by Dominated Convegence we can conclude that

$$\sum_{j=1}^{M^m} \int_\Omega \big(\beta(\theta(s^m_j)) - \beta(\theta(s^m_{j-1}))\big)|\varepsilon^{\rm tr}(s^m_j)|\,\mathrm{d}x \to \int_0^t \int_\Omega \beta'(\theta(s))\dot{\theta}(s)|\varepsilon^{\rm tr}|\,\mathrm{d}x\,\mathrm{d}s.$$

From the fact that $(v(s), \varepsilon^{\rm tr}(s), \varepsilon^{\rm pl}(s)) \in \tilde{\mathcal{S}}(s, \theta(s))$ for all $s \in [0,t]$ we readily deduce that $v = \mathcal{L}(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}, L)$. In particular, v has at most a countable number of discontinuity points in time and $v_m \to v$ pointwise almost everywhere. Eventually, we can pass to the limit into inequality (2.11) and conclude that

$$\begin{aligned}&\mathcal{E}(v(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) + \mathcal{F}(\theta(t), \varepsilon^{\rm tr}(t)) - \langle L(t), (v(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))\rangle \\ &+ \mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}; [0,t]) \geq \mathcal{E}(v_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0) + \mathcal{F}(\theta(0), \varepsilon^{\rm tr}(0)) - \langle L(0), (v_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0)\rangle \\ &+ \int_0^t \int_\Omega \beta'(\theta(s))\dot{\theta}(s)|\varepsilon^{\rm tr}|\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \langle \dot{\ell}(s), v(s)\rangle\,\mathrm{d}s.\end{aligned}$$

Absolute continuity of the evolution. Let us now prove that indeed $t \mapsto (v(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))$ is absolutely continuous in $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^{3\times3}_{\rm dev}) \times L^2(\Omega;\mathbb{R}^{3\times3}_{\rm dev})$. To this end, we come back to the original variables $(u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) = (v + u^{\rm Dir}, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$. From the stability (1.8) at time s and the uniform convexity of $\mathcal{E} + \mathcal{F}$ of constant

$\alpha > 0$ we get that

$$
\begin{aligned}
&\alpha\|u(t)-u(s)\|^2_{H^1(\Omega;\mathbb{R}^3)} + \alpha\|\varepsilon^{\mathrm{tr}}(t)-\varepsilon^{\mathrm{tr}}(s)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})} + \alpha\|\varepsilon^{\mathrm{pl}}(t)-\varepsilon^{\mathrm{pl}}(s)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})} \\
&\leq \mathcal{E}(u(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t)) + \mathcal{F}(\theta(s),\varepsilon^{\mathrm{tr}}(t)) - \langle \ell(s), u(t)\rangle - \mathcal{E}(u(s),\varepsilon^{\mathrm{tr}}(s),\varepsilon^{\mathrm{pl}}(s)) \\
&- \mathcal{F}(\theta(s),\varepsilon^{\mathrm{tr}}(s)) + \langle \ell(s), u(s)\rangle + \mathcal{D}(\varepsilon^{\mathrm{tr}}(t)-\varepsilon^{\mathrm{tr}}(s), \varepsilon^{\mathrm{pl}}(t)-\varepsilon^{\mathrm{pl}}(s)) \\
&\leq \mathcal{E}(u(t),\varepsilon^{\mathrm{tr}}(t),\varepsilon^{\mathrm{pl}}(t)) + \mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}}(t)) - \langle \ell(t), u(t)\rangle - \mathcal{E}(u(s),\varepsilon^{\mathrm{tr}}(s),\varepsilon^{\mathrm{pl}}(s)) \\
&- \mathcal{F}(\theta(s),\varepsilon^{\mathrm{tr}}(s)) + \langle \ell(s), u(s)\rangle + \mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\mathrm{tr}},\varepsilon^{\mathrm{pl}};[s,t]) - \mathcal{F}(\theta(t),\varepsilon^{\mathrm{tr}}(t)) \\
&+ \mathcal{F}(\theta(s),\varepsilon^{\mathrm{tr}}(t)) + \langle \ell(t)-\ell(s), u(t)\rangle \overset{(1.9)}{=} \int_s^t \int_\Omega \beta'(\theta(r))\dot\theta(r)|\varepsilon^{\mathrm{tr}}|\,\mathrm{d}x\,\mathrm{d}r \\
&- \int_\Omega \big(\beta(\theta(t)) - \beta(\theta(s))\big)|\varepsilon^{\mathrm{tr}}(t)|\,\mathrm{d}x - \int_s^t \langle \dot\ell(r), u(r)\rangle\,\mathrm{d}r + \langle \ell(t)-\ell(s), u(t)\rangle \\
&= \int_s^t \int_\Omega \beta'(\theta(r))\dot\theta(r)\big(|\varepsilon^{\mathrm{tr}}(r)|-|\varepsilon^{\mathrm{tr}}(t)|\big)\,\mathrm{d}x\,\mathrm{d}r - \int_s^t \langle \dot\ell(r), u(r)-u(t)\rangle\,\mathrm{d}r \\
&\leq \int_s^t \int_\Omega |\beta'(\theta(r))\dot\theta(r)|\,|\varepsilon^{\mathrm{tr}}(r)-\varepsilon^{\mathrm{tr}}(t)|\,\mathrm{d}x\,\mathrm{d}r \\
&+ \int_s^t \|\dot\ell(r)\|_{(H^1(\Omega;\mathbb{R}^3))'}\|u(r)-u(t)\|_{H^1(\Omega;\mathbb{R}^3)}\mathrm{d}r.
\end{aligned}
$$

Hence, by means of Gronwall's Lemma one checks that

$$
\begin{aligned}
&\|u(t)-u(s)\|_{H^1(\Omega;\mathbb{R}^3)} + \|\varepsilon^{\mathrm{tr}}(t)-\varepsilon^{\mathrm{tr}}(s)\|_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})} + \|\varepsilon^{\mathrm{pl}}(t)-\varepsilon^{\mathrm{pl}}(s)\|_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})} \\
\leq\quad & c\int_s^t \int_\Omega \big(|\beta'(\theta(r))\dot\theta(r)|+|\dot\ell(r)|\big)\mathrm{d}x\,\mathrm{d}r
\end{aligned}
$$

for some suitable constant c depending just on α and T. In particular, the absolute continuity of $t \mapsto (v(t), \varepsilon^{\mathrm{tr}}(t), \varepsilon^{\mathrm{pl}}(t))$ ensues.

3. The optimal control problem

As already mentioned in the introduction, in this section we are going to present a result of existence of optimal controls. Our focus here is on the situation of a SMA specimen which deforms under given mechanical loading under the influence of a controlled space-homogeneous time-dependent temperature θ. In particular, given the temperature $\theta \in W^{1,1}(0,T)$ we denote by

$$\mathrm{Sol}(\theta) \subset \mathcal{K}$$

the set of energetic solutions of the state problem from Theorem 2.1, where we recall that the set $\mathcal{K}$ has been introduced in (2.1). Let the set of admissible temperatures (controls) be denoted by $\Theta \subset W^{1,1}(0,T)$. Then, the optimal control problem consists in the minimization of a given cost functional

$$\mathcal{J} : \mathcal{K} \times \Theta \to (-\infty, \infty]$$

which is depending on both the energetic solution and the control. Our problem is to find an *optimal control* $\theta_* \in \Theta$ and a corresponding *optimal energetic solution*

$(u_*, \varepsilon_*^{\rm tr}, \varepsilon_*^{\rm pl}) \in {\rm Sol}(\theta_*)$ such that

$$(u_*, \varepsilon_*^{\rm tr}, \varepsilon_*^{\rm pl}) \in \operatorname{Arg\,Min}\left\{\mathcal{J}(u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}, \theta) \mid (u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \in {\rm Sol}(\theta),\ \theta \in \Theta\right\}.$$

For brevity, from now on we set

$$\overline{\mathcal{K}} := L^\infty(0,T; H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev}) \times L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev})). \tag{3.1}$$

In order to possibly find optimal controls we shall consider the following standard requirements.

Compatibility of the initial value and the controls:

$$(u_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl}) \in \mathcal{S}(0, \theta(0)) \quad \forall \theta \in \Theta. \tag{3.2}$$

Compactness of controls:

$$\Theta \text{ is weakly compact in } W^{1,r}(0,T) \text{ for some } r > 1. \tag{3.3}$$

Lower semicontinuity of the cost functional:

$$\left.\begin{array}{l} \theta_n \to \theta \text{ weakly in } W^{1,r}(0,T) \text{ for some } r>1 \\ (u_n, \varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}) \in {\rm Sol}(\theta_n), \\ (u_n, \varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}) \to (u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \text{ weakly-star in } \overline{\mathcal{K}} \end{array}\right\}$$
$$\Rightarrow\ \mathcal{J}(u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}, \theta) \le \liminf_{n\to\infty} \mathcal{J}(u_n, \varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}, \theta_n). \tag{3.4}$$

The compatibility condition in (3.2) was already presented in [49] and is important to ensure that the initial values are stable independently of the choice of the control. The compactness of Θ from (3.3) is here chosen just for the sake of simplicity. In particular it can be relaxed by asking for some extra coercivity with respect to θ on the functional J.

The lower semicontinuity requirement in (3.4) is standard. We remark that a possible *quadratic* cost functional covered by our theory is

$$\begin{aligned} & J(u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}, \theta) \\ = \ & \int_0^T\!\!\int_\Omega |u - u_{\rm d}|^2 {\rm d}x\,{\rm d}t + \int_0^T\!\!\int_\Omega |\varepsilon^{\rm tr} - \varepsilon^{\rm tr}_{\rm d}|^2 {\rm d}x\,{\rm d}t + \int_0^T\!\!\int_\Omega |\varepsilon^{\rm pl} - \varepsilon^{\rm pl}_{\rm d}|^2 {\rm d}x\,{\rm d}t \\ & + \int_\Omega |u(T) - u_{\rm f}|^2\,{\rm d}x + \int_\Omega |\varepsilon^{\rm tr}(T) - \varepsilon^{\rm tr}_{\rm f}|^2\,{\rm d}x + \int_\Omega |\varepsilon^{\rm pl}(T) - \varepsilon^{\rm pl}_{\rm f}|^2\,{\rm d}x \end{aligned}$$

where $u_{\rm d} : [0,T] \to L^2(\Omega;\mathbb{R}^3)$, $\varepsilon^{\rm tr}_{\rm d}, \varepsilon^{\rm pl}_{\rm d} : [0,T] \to L^2(\Omega;\mathbb{R}^{3\times3})$ are given desired displacement and inelastic strain profiles whereas $u_{\rm f} \in L^2(\Omega;\mathbb{R}^3)$ and $\varepsilon^{\rm tr}_{\rm f}, \varepsilon^{\rm pl}_{\rm f} \in L^2(\Omega;\mathbb{R}^{3\times3})$ are desired target states. Note that the latter functional is not lower semicontinuous with respect to the weak-star topology of $\overline{\mathcal{K}}$, where we remind that $\overline{\mathcal{K}}$ has been introduced in (3.1). Still, it fulfills (3.4) due to the fact that the required extra compactness is provided by the condition $(u_n, \varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}) \in {\rm Sol}(\theta_n)$.

Our main result is the following.

Theorem 3.1 (Existence of optimal controls)**.** *Under assumptions* (1.1), (1.5), *and* (3.2)-(3.4) *there exists an optimal control* θ_* *and a corresponding optimal energetic solution* $(u_*, \varepsilon_*^{\mathrm{tr}}, \varepsilon_*^{\mathrm{pl}}) \in \mathrm{Sol}(\theta_*)$.

Remark 3.2. We would like to point out that we interpret our result as follows: the existence of an optimal control θ in the sense that it minimizes $\mathcal{J}$ together with some (possibly suitable selected) response $(u_*, \varepsilon_*^{\mathrm{tr}}, \varepsilon_*^{\mathrm{pl}}) \in \mathrm{Sol}(\theta_*)$.

3.1. **Proof.** We shall finally turn to the proof of Theorem 3.1. Let $(u_n, \varepsilon_n^{\mathrm{tr}}, \varepsilon_n^{\mathrm{pl}}, \theta_n)$ with $(u_n, \varepsilon_n^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) \in \mathrm{Sol}(\theta_n)$ be a minimizing sequence for $\mathcal{J}$, namely

$$\mathcal{J}(u_n, \varepsilon_n^{\mathrm{tr}}, \varepsilon_n^{\mathrm{pl}}, \theta_n) \to \inf \left\{ \mathcal{J}(u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}, \theta) \ : \ (u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) \in \mathrm{Sol}(\theta), \ \theta \in \Theta \right\}.$$

Owing to the compactness (3.3), we can extract a not relabeled subsequence in such a way that both $\theta_n \to \theta$ and $\beta \circ \theta_n \to \beta \circ \theta$ weakly in $W^{1,r}(0,T)$ and uniformly. By exploiting the energy balance (1.9) we readily get that

$$\sup_{t \in [0,T]} \left(\mathcal{E}(u_n(t), \varepsilon_n^{\mathrm{tr}}(t), \varepsilon_n^{\mathrm{pl}}(t)) + \mathcal{F}(\theta_n(t), \varepsilon_n^{\mathrm{tr}}(t)) \right); \ \mathrm{Diss}_{\mathcal{D}}(\varepsilon_n^{\mathrm{tr}}, \varepsilon_n^{\mathrm{pl}}; [0,T])$$

are bounded independently of n.

We can hence extract again (still not relabeling) in order to get that:
$\varepsilon_n^{\mathrm{tr}} \to \varepsilon^{\mathrm{tr}}$ pointwise in $L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{dev}})$
$\varepsilon_n^{\mathrm{pl}} \to \varepsilon^{\mathrm{pl}}$ pointwise in $L^2(\Omega; \mathbb{R}^{3\times3}_{\mathrm{dev}})$
$u_n \to u$ pointwise in $H^1(\Omega; \mathbb{R}^3)$ (by linearity)
$(u_n, \varepsilon_n^{\mathrm{tr}}, \varepsilon_n^{\mathrm{pl}}) \to (u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}})$ weakly-star in $\overline{\mathcal{K}}$.

The proof of Theorem 2.1 can be adapted to the present situation in order to ensure that $(u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) \in \mathrm{Sol}(\theta)$. To this aim, the differences arise solely in the treatment of those terms containing θ_n. In particular, the above-mentioned convergences of θ_n and $\varepsilon_n^{\mathrm{tr}}$ entail directly the convergence

$$\int_\Omega \beta(\theta_n(t)) |\varepsilon_n^{\mathrm{tr}}(t)| \,\mathrm{d}x \to \int_\Omega \beta(\theta(t)) |\varepsilon^{\mathrm{tr}}(t)| \,\mathrm{d}x \quad \forall t \in [0,T].$$

Moreover, one can also check for the limit

$$\int_0^t \int_\Omega \beta'(\theta_n(s)) \dot{\theta}_n(s) |\varepsilon_n^{\mathrm{tr}}| \,\mathrm{d}x \,\mathrm{d}s \to \int_0^t \int_\Omega \beta'(\theta(s)) \dot{\theta}(s) |\varepsilon^{\mathrm{tr}}| \,\mathrm{d}x \,\mathrm{d}s$$

as we have that $\beta'(\theta_n)\dot{\theta}_n \to \beta'(\theta)\dot{\theta}$ weakly in $L^r(0,T)$ and (by possibly extracting again) $\varepsilon_n^{\mathrm{tr}} \to \varepsilon^{\mathrm{tr}}$ strongly in $L^p(\Omega \times [0,T]; \mathbb{R}^{3\times3}_{\mathrm{dev}})$ for all $p \in [1, \infty)$ (recall that $\varepsilon_n^{\mathrm{tr}}$ are uniformly bounded in $\mathbb{R}^{3\times3}_{\mathrm{dev}}$). As we now have that $(u, \varepsilon^{\mathrm{tr}}, \varepsilon^{\mathrm{pl}}) \in \mathrm{Sol}(\theta)$, the assertion follows directly from the lower semicontinuity assumption (3.4).

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