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NEW CONTRIBUTIONS TO NONLINEAR CALDERÓN-ZYGMUND THEORY

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CHAPTER 1

Introduction

This thesis deals with basic regularity properties of solutions to nonlinear elliptic

$$-\operatorname{div} a(Du) = \mu \tag{1.0.1}$$

and parabolic equations

$$u_t - \operatorname{div} a(Du) = \mu$$

of quasilinear type. One of the main prototypes in this thesis is given by the familiar p-Laplacian equation

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu \tag{1.0.2}$$

together with its evolutionary analog

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \mu. \tag{1.0.3}$$

The kind of properties and results we are interested in here can be considered a nonlinear analog of the standard, linear Calderón-Zygmund theory for instance available for solutions of the Poisson equation

$$-\Delta u = \mu. \tag{1.0.4}$$

In particular, the basic question we are interested in is the maximal regularity of the gradient of solutions in terms of the regularity of the assigned datum μ, and an instance of the results we are going to discuss here is given, in the case of solutions to (1.0.4), by the well-known implication

$$\mu \in L^\gamma \Longrightarrow D^2u \in L^\gamma \qquad \text{for} \qquad \gamma \in (1,\infty).$$

Another instance, more specifically related to the gradient integrability of solutions, is valid for the equations of the type

$$\Delta u = \operatorname{div} F$$

where, instead, it follows that

$$F \in L^\gamma \Longrightarrow Du \in L^\gamma \qquad \text{for } \gamma \in (1,\infty). \tag{1.0.5}$$

While the classical approach to this problem available for solutions to (1.0.4) is based on an explicit representation formula via fundamental solutions and the analysis of related singular integrals, a different path must be taken in the nonlinear case. In particular, a suitable nonlinear replacement of the Harmonic Analysis tools used in the classical approach must found out, depending both of the kind of regularity one is looking for and on the geometry of the equation considered. Iwaniec [**89**] was the first to get a nonlinear analog of the result in (1.0.5), considering the natural equation

$$\operatorname{div}\left(|Du|^{p-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F\right)$$

and proving that

$$F \in L^\gamma \Longrightarrow Du \in L^\gamma \qquad \text{for } \gamma \geq p. \tag{1.0.6}$$

The fundamental result of Iwaniec marks the beginning of what is nowadays could be called Nonlinear Calderón-Zygmund theory [120] and suggests a possible path to overcome the use of explicit representation formulas: local regularity estimates can be used as a local analog of the fundamental solution while maximal operators replace the use of singular integrals. This technique, which opened the way to several different developments, does not apply in the parabolic case

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F\right) \tag{1.0.7}$$

due to the lack of scaling properties of the equations. The next step has been indeed taken by Acerbi & Mingione [4] who provided a purely PDE approach to nonlinear Calderón-Zygmund estimates, employing no Harmonic Analysis tool but rather providing exit times and covering arguments directly at the level of the equation considered. The final outcome is that (1.0.6) still holds for energy solutions to (1.0.7).

The passage form the linear to the nonlinear case enhances the theory with several additional challenges and new phenomena. For instance, in the nonlinear case we will discuss different notions of solutions leading to analyze situations in which the solutions considered are not in the natural Sobolev spaces and *estimates below the natural growth exponent are needed*. As a matter of fact the extension of (1.0.6) to the whole range $q > p-1$ – which is known for the case $p = 2$ – remains a major open problem, relating to several different areas of modern nonlinear analysis. Indeed, one of the problems we analyze is the degree of regularity of solutions to measure data problems i.e. (1.0.2) where μ is a general Radon measure with finite total mass. These aspects immediately relate these topics with the so called nonlinear potential theory [87], i.e. the study of fine properties of solutions of (1.0.2).

In this respect another classical connection with the classical gradient integrability estimates for solutions to (1.0.4) finds a starting point in the pointwise estimates via Riesz potentials (see Chapter 2 for the definition)

$$|u(x)| \lesssim I_2(|\mu|)(x) \qquad \text{and} \qquad |Du(x)| \lesssim I_1(|\mu|)(x) \tag{1.0.8}$$

that are again consequences of representation formulae via fundamental solutions.

Estimates in display (1.0.8) allow, for instance, to reduce the growth and integrability issues to the analysis of the corresponding potentials. Moreover, such estimates are at the core of the analysis of the fine properties of solutions to (1.0.4). The nonlinear version of the first estimate in (1.0.8) is a fundamental result of Kilpeläinen & Malý [94] and uses, in a sharp way, suitable nonlinear versions of the classical Riesz potential, namely the so-called Wolff potentials [85, 86] given by

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left[\frac{|\mu|(B_\rho(x))}{\rho^{n-\beta p}}\right]^{1/(p-1)} \frac{d\rho}{\rho} \tag{1.0.9}$$

for $0 < \beta p \leq n$. The estimate of Kilpeläinen & Malý is

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \fint_{B_R(x)} \left(|u| + Rs\right) d\xi \tag{1.0.10}$$

and holds for almost every $x \in \Omega$, and for balls $B_R(x) \subset \Omega$. The previous estimate, up to the additional integral average due to its local character, reduces to the first one in (1.0.8) when $p = 2$. The extension to the gradient level of the estimate (1.0.10) has remained an open issue for some while till Mingione first in the case $p = 2$ but for general quasilinear equations as in (1.0.1), and then Kuusi & Mingione [101, 106] for the case $p \geq 2$ (see also [69, 70] for intermediate results), have shown that the second estimate in (1.0.8) remains true for solutions to (1.0.2)

$$|Du(x)|^{p-1} \lesssim I_1(|\mu|)(x) \tag{1.0.11}$$

thereby unifying the linear and nonlinear potential theories at the gradient level. Eventually, Kuusi & Mingione [**104**] have given suitable extension of the nonlinear potential estimates in the parabolic case. This involves the use of the so called *intrinsic geometry*, a way to look at equations as in (1.0.3) on a suitable space/time cylinders where they behave as the heat equations. Also, a proper notion of nonlinear potential must be considered.

Starting from the above basic results in this thesis we provide new contributions to what might be indeed called *Nonlinear Calderón-Zygmund theory*. Amongst the other things, we shall give new results for quasilinear parabolic equations with measure data, allowing to extend, in the parabolic setting and in cases where equations have measurable or highly discontinuous coefficients, several of the basic consequences of estimate (1.0.11) (see Section 4.2 and Chapter 6). Moreover, again when considering general parabolic equations with measure data, we shall provide new fractional differentiability results extending to the maximal degree of regularity several of the results obtained on the integrability of the gradient, for instance in [**24**] (see Section 4.1 and Chapter 5).

We shall also produce several extension of the potential estimates given in (1.0.11) to solutions to very general class of operators with different kind of degeneracies as for instance those of the type in (1.0.1) where

$$|a(Du)| \lesssim g(|Du|)$$

for a rather general Young function $g(\cdot)$; the standard case is $g(t) = t^{p-1}$ (see Section 4.7 and Chapter 10). Another instance is given by variable exponent operators, i.e those of the type

$$-\operatorname{div}\left(|Du|^{p(x)-2}Du\right) = \mu$$

for which we refer to Sections 4.4, 4.6 and Chapter 9, together with natural evolutionary analogs

$$u_t - \operatorname{div}\left(|Du|^{p(x,t)-2}Du\right) = \operatorname{div}\left(|F|^{p(x,t)-2}F\right).$$

These are treated in Section 4.5 and Chapter 8 where a sharp analog of the result of Acerbi & Mingione [**4**] is given under optimal assumptions of the exponent function $p(x,t)$, i.e.

$$|F|^{p(x,t)} \in L^\gamma \Longrightarrow |Du|^{p(x,t)} \in L^\gamma \qquad \text{for } \gamma \geq 1.$$

A particularly delicate part of this thesis is the one dealing with borderline integrability estimates for solutions to problems of the type (1.0.3), where the right-hand side μ is assumed to be a measure satisfying a density condition of the type

$$|\mu|(Q_R) \lesssim R^\beta. \tag{1.0.12}$$

Here β is a positive number and the previous inequality holds whenever Q_R is a parabolic cylinder. We prove that inequalities of the type in (1.0.12), that roughly speaking quantify the way μ does not concentrate on lower dimensional sets, improve the integrability of the gradient of solutions in a way that precisely relates to the size of β. For this we refer to Section 4.3 and Chapter 7.

More precisely, the thesis is now structured as follows. Chapters 2 and 3 are of introductory character. In the first one we shall give a brief outlook at the existing results, in order to put the next pages into a correct perspective, while in the second one we shall fix the notation and the technical preliminaries that will be necessary for the following developments. In Chapter 4 we shall describe in detail the new results obtained here, while the remaining ones contain the detailed proofs of the statements of Chapter 4.

CHAPTER 2

Linear and nonlinear Calderón-Zygmund theories

The main purpose of the classical Calderón-Zygmund theory is to give integrability and/or regularity results for solution to linear and elliptic and parabolic partial differential equations, possibly in a sharp way, in terms of integrability/regularity of the data. This definition is very rough, but nevertheless it is really concrete, since it points out the principal issue we are dealing with: elliptic (or parabolic) partial differential equations with data enjoining some regularity properties. However we warn the reader that by "regular" we will, from now on, mean almost every property less general than mere "measurability", at least when not dealing with measure data. For example, an interesting part of this theory is given by the case where the datum is a L^1 function. In this case solutions are far from being regular in the classical sense i.e. continuous, for instance. They turn out to be differentiable in suitably weak sense, and this already means they enjoy a few regularity properties.

In this chapter we will first briefly outline the basic elements of the classical, linear Calderón-Zygmund theory, eventually turning to describe the available results in the nonlinear setting, which is the main our main concern in this thesis. In this case the theory matches with what is nowadays called Nonlinear Potential Theory and one of our aims is to show how our topic lies therefore as a cornerstone between the two theories. Moreover, when treating the parabolic case we shall deal with the delicate theory of evolutionary p-parabolic type equations, where classical concepts have to be suitably reformulated in order to fit the natural geometry of the equations considered.

2.1. The linear case

Most of the features and the classical Calderón-Zygmund theory in the linear case are already contained in the simplest PDE example, the Poisson equation

$$-\Delta u = \mu. \tag{2.1.1}$$

Here, and throughout the whole Chapter, we shall consider (2.1.1) on the whole $\mathbb{R}^n$ with $n \geq 2$. Since we are here mainly interested in a priori estimates, we shall initially assume that μ is smooth with compact support, while u is the unique solution which decays to zero at infinity, at least in this first Sections. The case when μ is a general Borel measure with finite total mass can be then reached via approximation procedures, that turn out to be particularly simple in the linear case (2.1.1).

It is well known that the solution to (2.1.1) is given via the representation formula, i.e. convolution with the fundamental solution Green's function:

$$u(x) = G\big(|x-\cdot|\big) * \mu \quad \text{with} \quad G\big(|x-y|\big) \approx \begin{cases} \dfrac{1}{|x-y|^{n-2}} & \text{if } n \geq 3, \\ -\ln\big(|x-y|\big) & \text{if } n = 2. \end{cases} \tag{2.1.2}$$

The symbol $\approx$ means that quantities are equal up to constants inessential in our discussion; in this case just a normalization constant depending on n. This representation of the solution via the convolution with Green's function clearly allows to give sharp estimates of Calderón-Zygmund type: indeed consider, with $\beta \in (0, n]$ and μ Borel measure on $\mathbb{R}^n$, the linear Riesz operator

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}},$$

also called the β-Riesz potential of μ. Using (2.1.2) one obtains in a straightforward way the pointwise estimates

$$|u(x)| \lesssim I_2(|\mu|)(x) \qquad \text{and} \qquad |Du(x)| \lesssim I_1(|\mu|)(x),$$

the second one obtained by differentiation of (2.1.2) while the first one is valid in the case $n > 2$. For this reason, without loss of generality, in the following we shall restrict to this case, keeping in mind that the two-dimensional one $n = 2$ can be easily obtained by similar arguments.

Using the latter estimate and the classic regularizing property of the Riesz potential

$$I_\beta : L^\gamma \longrightarrow L^{\frac{n\gamma}{n-\beta\gamma}} \qquad \text{for } \gamma > 1 \quad \text{and} \quad \beta\gamma < n, \tag{2.1.3}$$

see [139], one can prove the *a priori estimates*

$$\|Du\|_{L^{\frac{n\gamma}{n-\gamma}}} \lesssim \|\mu\|_{L^\gamma}, \qquad \|u\|_{L^{\frac{n\gamma}{n-2\gamma}}} \lesssim \|\mu\|_{L^\gamma} \tag{2.1.4}$$

for $1 < \gamma < n$ and $1 < \gamma < n/2$, respectively. Here $\lesssim$ means that the inequality holds up to a universal constant that does not depend on the specific values of the norms appearing in both sides and the arrow "$\to$" means that the operator is bounded between the concerned spaces. Note that both the kernels $G(|x-y|)$ and $DG(|x-y|)$ are *locally integrable* in the sense that

$$G(|x-y|) \approx |x-y|^{2-n} \qquad \text{and} \qquad |DG(|x-y|)| \lesssim |x-y|^{1-n}$$

therefore estimates in (2.1.4) invoke, as a matter of fact, essentially *size properties* of the kernel. When looking for *maximal order estimates* for solutions to (2.1.1), i.e. estimates for the full Hessian D^2u, things drastically change. The analysis of the integrability properties of the second derivatives requires a further differentiation of (2.1.2), thereby yielding

$$D^2u(x) \approx \int_{\mathbb{R}^n} K(x-y)\, d\mu(y). \tag{2.1.5}$$

At this point the approach used to deduce (2.1.4) cannot anymore be applied. Here comes into play the work of Calderón and Zygmund [36, 37]. Although the kernel $K(x)$ is singular

$$|K(x)| \lesssim \frac{1}{|x|^n}, \tag{2.1.6}$$

it however enjoys enough *cancelation properties* to ensure that the linear operator $\mu \mapsto CZ(\mu)$ defined as $CZ(\mu)(x) := K(x-\cdot) * \mu$ still has the property

$$CZ : L^\gamma \longrightarrow L^\gamma \qquad \text{for all } \gamma > 1. \tag{2.1.7}$$

Indeed the kernel $K(x)$ is a so-called *Calderón-Zygmund kernel* and has the following properties: it has the singular behavior described in (2.1.6),

- there holds the Hörmander condition

$$\sup_{y \in \mathbb{R}^n, y \neq 0} \int_{|x| \geq 2|y|} \big|K(x-y) - K(x)\big|\, dx < \infty;$$

- $\|\hat{K}\|_{L^\infty}$ is finite, where $\hat{K}$ denotes the Fourier tranform of K.

At this point the *Calderón-Zygmund theory of singular integrals* comes into the play and implies that (2.1.7) holds; indeed the second point in the list above together with the classic Plancherel's Theorem implies

$$\|CZ(\mu)\|_{L^2} \lesssim \|\mu\|_{L^2}; \tag{2.1.8}$$

at this point the first condition, together with usual interpolation and duality arguments, gives plainly the estimate for the second derivatives

$$\|D^2 u\|_{L^\gamma} \lesssim \|\mu\|_{L^\gamma} \qquad \text{for all } \gamma > 1. \tag{2.1.9}$$

Note that (2.1.4) could be inferred by the previous estimate using Sobolev's embedding but, on the other hand, the contrary is not true, i.e. the constant in (2.1.4) depends on β and blows up when $\beta \to 0$. Note moreover that counterexamples show that all the previous estimate fails in the extremal cases.

The extremal case $\gamma = 1$ and extensions. Estimate (2.1.9) fails in the case $\gamma = 1$ together with those in (2.1.4); on the other hand the analysis of the level sets of the fundamental solution

$$\big|\{x \in \mathbb{R}^n : |DG(|x-y|)| > \lambda\}\big| \lesssim \lambda^{\frac{n}{n-1}},$$
$$\big|\{x \in \mathbb{R}^n : |D^2 G(|x-y|)| > \lambda\}\big| \lesssim \lambda$$

for all $y \in \mathbb{R}^n$, suggests the perform the analysis in the so called Marcinkiewcz or weak-Lebesgue spaces $\mathcal{M}^t(\Omega)$, for $\Omega \subset \mathbb{R}^n$ open set and $t \geq 1$, i.e. the space of measurable functions $f : \Omega \to \mathbb{R}^k$, such that

$$\sup_{\lambda>0} \lambda^{-t}\big|\{x \in \Omega : |f(x)| > \lambda\}\big| =: \|f\|^t_{\mathcal{M}^t(\Omega)} < \infty.$$

It is easy to see that $L^t \subsetneqq \mathcal{M}^t \subsetneqq L^{t-\varepsilon}$ for all $\varepsilon > 0$ and then it is also easy to think how these spaces are suitable to treat borderline cases when treating Newtonian potentials: the classical enlightening example is the potential

$$\frac{1}{|x|^{\frac{n}{t}}} \in \mathcal{M}^t(B_1) \smallsetminus L^t(B_1). \tag{2.1.10}$$

A more convincing argument is that inequality (2.1.3) in its extremal case reads indeed as

$$I_\beta : L^1 \longrightarrow \mathcal{M}^{\frac{n}{n-\beta}},$$

see again [139], and therefore we have the borderline estimates

$$\|Du\|_{\mathcal{M}^{\frac{n}{n-1}}} \lesssim \|\mu\|_{L^1} \qquad \text{or} \qquad \|Du\|_{\mathcal{M}^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim |\mu|(\mathbb{R}^n),$$

in the case μ is a measure. For the second derivatives the borderline result is

$$\|CZ(\mu)\|_{\mathcal{M}^1} \lesssim \|\mu\|_{L^1} \qquad \Longrightarrow \qquad \|D^2 u\|_{\mathcal{M}^1} \lesssim \|\mu\|_{L^1}$$

and actually (2.1.9), in the classical approach of Calderón-Zygmund, is obtained interpolating the previous inequality with (2.1.8) and then passing to the super-quadratic range $q > 2$ by duality methods.

It is worth remarking that many of the results on the integrability properties explained above extend to more general linear elliptic equations of the type

$$-\operatorname{div}(A(x)Du) = \mu, \tag{2.1.11}$$

where the matrix with variable, continuous coefficients $A(x) \in C^0(\mathbb{R}^n)$ is such that

$$\nu|\lambda|^2 \leq \langle A(x)\lambda, \lambda\rangle \leq L|\lambda|^2$$

for any $\lambda \in \mathbb{R}^n$, uniformly in Ω and with $0 < \lambda \leq 1 \leq L$. This extension goes through using a perturbation method: by the continuity of the coefficients, the equation can be considered as a local perturbation of the Laplace operator, and estimates follow by mean of fixed point arguments, see for instance [**81**]. The same results do not hold when the coefficient matrix $A(x)$ has just measurable entries, due to well-known counterexamples, but anyway certain types of mild discontinuities for the matrix A can still be allowed, for example so called VMO coefficients work as well [**42**].

A slightly different problem. The techniques outlined in the previous section also extend to equations having a right-hand side in divergence form of the type

$$\Delta u = \operatorname{div} F; \tag{2.1.12}$$

indeed it is enough to (formally) substitute μ with $\operatorname{div} F$, differentiate just once below the integral sign and then integrate by parts to have a singular integral operator like CZ. This scheme yields the first order estimate

$$\|Du\|_{L^\gamma} \lesssim \|F\|_{L^\gamma} \qquad \text{for all } 1 < \gamma < \infty, \tag{2.1.13}$$

where now clearly on the left-hand side we just have the gradient and not anymore the Hessian. The equation in display (2.1.12) offers the opportunity to outline a method to obtain integral estimates that is alternative to original one of Calderón-Zygmund and that was devised in the 60s by Campanato & Stampacchia [**137, 136**]. This goes as follows: defining the linear operator

$$T : F \longrightarrow T(F) = Du$$

where u is the solution of (2.1.12) once boundary data are given. At this point a standard Caccioppoli's inequality gives the continuity of this operator in L^2

$$\|T(F)\|_{L^2} = \|Du\|_{L^2} \lesssim \|F\|_{L^2}, \tag{2.1.14}$$

while a slightly less immediate argument, using Campanato spaces $\mathcal{L}^{2,n} \cong BMO$ [**38**], gives

$$\|T(F)\|_{BMO(B_1)} \lesssim \|F\|_{L^\infty(B_1)}. \tag{2.1.15}$$

BMO is a function space, strictly larger than L^∞ but close enough to it to allow for suitable interpolation between (2.1.14) and (2.1.15). Indeed, by *Stampacchia's interpolation theorem*, see [**137, 136**] and duality arguments one proves (2.1.13) again starting from (2.1.14)-(2.1.15).

2.2. The main nonlinear example: the *p*-Laplace operator

The results in the previous section are concerned with linear equations, and, although explicit representation formulas are not always an unavoidable tool – as for instance outlined in Section 2.1 – all the classical approaches to Calderón-Zygmund theory found till the beginning of the eighties strongly rely on the linearity of the problems considered. In this section we shall describe the first nonlinear results of Calderón-Zygmund type, mainly referring to possibly degenerate quasilinear equations of p-Laplacian type, i.e. involving operators modeled on the following p-Laplacian operator:

$$\Delta_p u := \operatorname{div}\left(|Du|^{p-2}Du\right). \tag{2.2.1}$$

The nonlinearity of the problem poses new issues: even the notion of solution employed must carefully treated and therefore we initially provide a general approach. Consider a problem of the form

$$-\operatorname{div} a(x, Du) = H,$$

where the right-hand side H is at the moment only a distribution and the vector field $a(\cdot)$ is a Carathéodory function satisfying the following strong p-monotonicity and growth assumptions:

$$\begin{cases} \langle a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu\big(s^2 + |\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \\ |a(x,\xi)| \leq L\big(s^2 + |\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.2.2}$$

for all $x \in \Omega$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$and with $0 < \nu \leq 1 \leq L$. $s \in [0,1]$ is a parameter which discriminate between the degenerate ($s = 0$) and the non-degenerate ($s > 0$) case. We say that a weak solution to such problem is a map $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ satisfying the weak formulation

$$\int_\Omega \langle a(x,Du), D\varphi \rangle \, dx = \langle H, \varphi \rangle_{\mathscr{D}'(\Omega)\times C_c^\infty(\Omega)} \quad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{2.2.3}$$

where on the right-hand side we find the duality crochet between $\mathscr{D}'$ and $\mathscr{D}$. This definition turns out to be too general, as it will become clear very soon. Therefore in the following we shall mainly distinguish two situations, and *the first* is when $H \in W^{-1,p'}$, that is the dual of the natural Sobolev space $W_0^{1,p}(\Omega;\mathbb{R}^N)$. In this case standard monotonicity methods apply [**115**], allowing to find - for instance when solving Dirichlet problems - a so called *energy solution*, that is a solution belonging to the natural energy space associated to the problem: $u \in W^{1,p}(\Omega;\mathbb{R}^N)$. This is actually the standard situation and solutions are unique in their Dirichlet class provided strict monotonicity properties, as for instance $(2.2.2)_1$, are assumed.

The second is when $H \notin W^{-1,p'}$ and it is more delicate; indeed in this situation the notion of solution must be specified more carefully since specific phenomena appear. Solutions that do not lie in the natural space $W^{1,p}$, often called *very weak solutions*, have naturally to be considered. They for instance naturally occur when considering measure data problems, as we shall in the following pages. In general, very weak solutions may also exist beside usual energy solutions, even for simple linear homogeneous equations of the type (2.1.11) – here take $\mu \equiv 0$ – as shown by a classical counterexample of Serrin [**134**].

***p*-Laplace equation: the dual case.** We here mean that H belongs to the dual of the energy space related to the operator, $H \in W^{-1,p}(\Omega)$. Here usual monotonicity methods [**115**] apply and therefore there exists a unique solution in the energy space $W_0^{1,p}(\Omega)$. Since by density (2.2.3) can be tested with $W_0^{1,p}(\Omega)$ test functions, one is therefore allowed in this case to test the equation with the solution itself, or some multiples, in order to get energy estimates.

Let us represent the right hand side in divergence form $H = -\operatorname{div} G$, with $G \in L^q$ and $q \geq \frac{p}{p-1}$. A change of variable leads to the more symmetric equation

$$\operatorname{div}\big(|Du|^{p-2}Du\big) = \operatorname{div}\big(|F|^{p-2}F\big), \qquad p > 1, \tag{2.2.4}$$

where $F \in L^q(\Omega)$, $q \geq p$, which is the nonlinear analog of equation (2.1.12). We start by a fundamental result of Iwaniec, which marks the beginning of what may be called *nonlinear Calderón-Zygmund theory*.

THEOREM 2.1 ([**89**]). *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a weak solution to the equation* (2.2.4) *in $\mathbb{R}^n$. Then*

$$F \in L^\gamma(\mathbb{R}^n,\mathbb{R}^n) \Longrightarrow Du \in L^\gamma(\mathbb{R}^n,\mathbb{R}^n) \qquad \textit{for every } \gamma \geq p.$$

Iwaniec's innovative approach consists of replacing singular integrals with another tool from Harmonic Analysis, the *Fefferman-Stein sharp maximal operator*, see [77], while representation formulae are replaced by some kind of local surrogate, i.e. local comparisons with solutions to the homogeneous equation $-\operatorname{div}(|Dv|^{p-2}Dv) = 0$, which enjoys nice $C^{1,\alpha}$ estimates.

Iwaniec result actually extends to bounded domains Ω and more general vector fields $a(Du)$ satisfying the classical Ladyzhenskaya and Ural'tseva assumptions

$$\begin{cases} \langle \partial a(\xi)\lambda, \lambda\rangle \geq \nu\big(s^2+|\xi|^2\big)^{\frac{p-2}{2}}|\lambda|^2, \\ |a(\xi)| + \big(s^2+|\xi|^2\big)^{\frac{1}{2}}|\partial a(\xi)| \leq L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.2.5}$$

for all $\xi, \lambda \in \mathbb{R}^n$. (2.2.5) models the structure of the more general vector field $(s^2 + |Du|^2)^{(p-2)/2}Du$ and clearly reduces to (2.2.1) for $s = 0$. A proof of the following local version can be adapted from [3, 97].

THEOREM 2.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to*

$$\operatorname{div} a(Du) = \operatorname{div}\big(|F|^{p-2}F\big) \qquad \text{in } \Omega$$

under the assumptions (2.2.5). *Then*

$$F \in L^{\gamma}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^{\gamma}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{for every } \gamma \geq p\,. \tag{2.2.6}$$

Moreover for every ball $B_R \subseteq \Omega$ it holds that

$$\left(\fint_{B_{R/2}} |Du|^{\gamma}\,dx\right)^{1/\gamma} \lesssim \left(\fint_{B_R} |Du|^{p}\,dx\right)^{1/p} + \left(\fint_{B_R} |F|^{\gamma}\,dx\right)^{1/\gamma}. \tag{2.2.7}$$

Iwaniec's result has been extended by DiBenedetto and Manfredi in [56] to the vectorial case of the p-Laplacian systems (2.2.4); see also [59]. Such a generalization to general structures as (2.2.5) is not anymore possible. Indeed, already when considering systems of the type

$$\operatorname{div} a(Du) = 0 \qquad u\colon \Omega \to \mathbb{R}^N \qquad N > 1.$$

energy solutions might be unbounded, as recently shown by [140], while a partial result in this direction has been obtained by Kristensen & Mingione [99]. An analog of Theorem 2.2 holds for systems of quasi-diagonal p-Laplacian structure, often called Uhlenbeck structure, i.e.

$$a(Du) = g\big(|Du|\big)Du \qquad \text{where} \qquad g\big(|Du|\big) \approx |Du|^{p-2}. \tag{2.2.8}$$

We have up to now dealt with energy solutions i.e. u to be an *energy solution* to (2.2.4). What is known for *very weak solutions* to (2.2.4), i.e. solvability and a priori estimates in L^{γ} as long as $\gamma > p-1$? Not quite more than just the fact that solving the problem in the full range is an *hard open problem*, since the only result available is due to Iwaniec & Sbordone [90] and Lewis [110], who proved that the estimate (2.2.6) holds true in the range $\gamma \geq p - \varepsilon$, where $\varepsilon > 0$ is a small parameter depending only on the data of the problem.

2.3. *p*-Laplace operator: the non-dual case

In this Section we deal with problems where we assume that the right-hand side *does not* to belong to the dual of space $W^{-1,p'}$. We shall actually deal only with the the case $H = \mu$ is a signed Borel measure with finite total mass ($|\mu|(\Omega) < \infty$). Of particular interest for us will be the case μ is a function which belongs to the space $L^\gamma(\Omega)$; the limitation on γ will be

$$1 < \gamma < (p^*)' = \frac{np}{np-(n-p)}, \qquad p < n$$

and this guarantees that μ is not in the dual of $W_0^{1,p}$, a case that can be reduced to that analyzed in Paragraph 2.2; for this see also (2.3.9) below. For the sake of readability in this case we shall often re-denote by $\mu \equiv g$. Finally, without loss of generality we shall assume in what follows that μ is defined on the whole $\mathbb{R}^n$, by eventually letting $\mu(\mathbb{R}^n \setminus \Omega) = 0$.

Consider now in the most general case a vector field $a(\cdot)$ satisfying measurability, growth and monotonicity assumptions (2.2.2). Although we shall mainly deal with local regularity results, for the sake of exposition we shall restrict to analyze the case of homogeneous Dirichlet problems. The weak formulation reads now

$$\int_\Omega \langle a(x, Du), D\varphi \rangle \, dx = \int_\Omega \varphi \, d\mu \qquad \text{for all } \varphi \in W_0^{1,p} \cap L^\infty(\Omega). \tag{2.3.1}$$

Note that the enlarged class of test functions φ is allowed by density arguments. Note moreover that we are only dealing with *very weak solutions*, since if u were an energy solution, then the left-hand side of (2.3.1) would define an element of $W^{-1,p'}(\Omega)$, differently form of what we are interested in here.

In the case

$$2 - \frac{1}{n} < p \leq n$$

a distributional solution to (2.3.1) can be obtained by regularization methods as showed in [**25, 26**], and this generates a notion of solution called SOLA (Solution Obtained as Limit of Approximations). Let us outline the strategy, which is on the other hand very natural. One considers smooth, C^∞ functions g_k converging to μ in the weak-$*$ topology of measures such that $\|g_k\|_{L^1(\Omega)} \leq |\mu|(\Omega)$ and the regularized problems

$$\begin{cases} -\operatorname{div} a(x, Du_k) = g_k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Solutions of such problems are found by classic monotonicity methods, since $g_k \in W^{-1,p}$. While in [**25, 26**] it is shown a suitable strong convergence $u_k \to u$ in $W_0^{1,p-1}(\Omega)$ towards a function $u \in W_0^{1,p-1}(\Omega)$ which is indeed a SOLA to (2.3.1). This scheme involves a priori estimates and therefore implicitly carries on regularity results. Summarizing, the following, that contains **the lower order Calderón-Zygmund theory for measure data problems**.

THEOREM 2.3 ([**25, 26, 48**]). *Under the assumptions* (2.2.5) *with* $2 - 1/n < p \leq n$, *there exist a SOLA* $u \in W_0^{1,p-1}(\Omega)$ *to* (2.3.1). *Moreover*

$$u \in W_0^{1,\gamma}(\Omega) \quad \text{for every } \gamma < \frac{n(p-1)}{n-1} \quad \text{and} \quad |Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}(\Omega).$$

Finally, there exists a unique SOLA when $\mu \in L^1(\Omega)$ *or* $p = 2$.

Uniqueness in Theorem 2.3 is in the sense that by considering a different approximating sequence $\{\bar{g}_k\}$ converging to μ in $L^1(\Omega)$, we still get the same limiting solution u. See also [**114, 49, 144, 94**] for important, related contributions. Note that we have $n(p-1)/(n-1) > 1$ if and only if $p > 2 - 1/n$. One of the very few cases uniqueness of SOLA is given when μ concentrates at one point, in this case we have a Dirac measure. Indeed the only SOLA to

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = \delta_0 & \text{in } B_1, \\ u \equiv 0 & \text{on } \partial B_1, \end{cases} \tag{2.3.2}$$

is given by the following nonlinear fundamental solution, or nonlinear Green's function:

$$G_p(x) \equiv G_p(|x|) \approx \begin{cases} |x|^{\frac{p-n}{p-1}} - 1, & 1 < p < n, \\ \log |x|, & p = n. \end{cases} \tag{2.3.3}$$

In turn, such uniqueness result allows to test the optimality of the regularity result, such as for instance Theorem 2.3, which is in fact optimal.

When $1 < p \leq 2 - 1/n$, one is lead to consider different notions of solutions, which do not anymore belong to $W^{1,1}$; subsequently different notions of "gradient" must be considered. For instance in the case $\mu \in L^1 + W^{-1,p'}$ one can consider *entropy solutions*, see [**23, 27, 128**]; this concept was then extended to general measures with the notion of *renormalized solutions*, see [**50**]. Despite we are not going to deal here with these notions, note however that for positive measures *renormalized solutions* coincide with SOLA, as recently proved in [**92**]. Finally, just a few words about systems: it is actually an open problem the solvability of the Dirichlet problem of (2.3.1) in the case of vector-valued measures. It is only known the existence in special cases, amongst them the Uhlenbeck structure (2.2.8), see [**62, 63, 64**].

We now want to recall a few recent results aimed at obtaining, on one hand what can be called **the maximal Calderón-Zygmund theory for measure data problems**, and on the other one at outlining a few results aim at going beyond Theorem 2.3 when certain more special measures are considered.

The idea is now very basic: since equations as in (2.3.1) formally involve second order operators, then it is natural to expect for the gradient of solutions a degree of regularity that goes beyond the integrability one considered in Theorem 2.3. More precisely, we consider differentiability rather that integrability properties of the gradient. For this we need to consider assumptions which are stronger than those considered in Theorem 2.3, but that are nevertheless natural towards the forthcoming results. More precisely, we shall consider differentiable and Carathéodory vector fields a such that $\partial_\xi a$ is a Carathéodory function and such that

$$\begin{cases} \langle \partial_\xi a(x,\xi)\lambda, \lambda\rangle \geq \nu\big(s^2 + |\xi|^2\big)^{\frac{p-2}{2}}|\lambda|^2, \\ |a(x,\xi)| + \big(s^2+|\xi|^2\big)^{\frac{1}{2}}|\partial_\xi a(x,\xi)| \leq L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \\ |a(x_1,\xi) - a(x_2,\xi)| \leq L\omega(|x_1-x_2|)\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.3.4}$$

for all $x, x_1, x_2 \in \Omega$ and all $\xi, \xi_1, \xi_2, \lambda \in \mathbb{R}^n$. Again $0 < \nu \leq 1 \leq L$ and $s \in [0,1]$. Here the differentiable dependence on the coefficients x is encoded by the inequality

$$\omega(R) \lesssim R.$$

The first results in this direction are due to Mingione, and can be summarized in the following:

THEOREM 2.4 ([**118, 123**]). *Let $2 - 1/n < p \leq n$, let u be as in previous Theorem and let μ be a signed Borel measure with finite total mass. Then*

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1}_{\text{loc}}(\Omega;\mathbb{R}^n) \qquad \textit{for } \varepsilon > 0, \qquad \textit{if} \quad p \geq 2,$$

$$Du \in W^{\frac{p-np(2-p)-\varepsilon}{2(p-1)},1}_{\text{loc}}(\Omega;\mathbb{R}^n) \qquad \textit{for } \varepsilon > 0, \qquad \textit{if} \quad 2 - 1/n < p < 2.$$

In particular

$$Du \in W^{1-\varepsilon,1}_{\text{loc}}(\Omega;\mathbb{R}^n) \qquad \textit{for } \varepsilon > 0, \qquad \textit{when} \quad p = 2.$$

The previous result is optimal in the sense that we cannot allow for $\varepsilon = 0$, as easily shown by considering the fundamental solution displayed in (2.3.3) together with the following well-known fractional versions of Sobolev embedding:

$$W^{\alpha,\gamma}(\Omega;\mathbb{R}^n) \hookrightarrow L^{\frac{n\gamma}{n-\alpha\gamma}}(\Omega;\mathbb{R}^n), \qquad \text{for} \qquad \gamma \geq 1, \quad \alpha\gamma < n. \tag{2.3.5}$$

Also notice, that in the case $p < 2$ the limiting fractional differentiability exponent $[p - np(2-p))]/(2p-2)$ tends to zero when p approaches $2 - 1/n$.

Theorem 2.3 can be upgraded in a different, in some sense orthogonal direction. As we have seen before, Theorem 2.3 is optimal when considering the fundamental solution. The question is now: What happens if we are considering measures that do not concentrate on sets with small Hausdorff dimension? As observed in [**118**], measure data problems obey the heuristic principle

"the less the measure concentrates, the better the gradient is".

A natural way to quantify this can be, following [**8**], for a given signed finite Borel measure $\mu \in \mathcal{M}_b(\Omega)$, to consider the Morrey type density condition

$$|\mu|(B_R(x)) \lesssim R^{n-\vartheta}, \qquad 0 \leq \vartheta \leq n, \tag{2.3.6}$$

the inequality being valid for all the balls $B_R(x) \Subset \Omega$. We shall denote, with some abuse of notation, the space of such measures $L^{1,\vartheta}(\Omega)$; see also the next paragraph. Assuming (2.3.6) does not allow μ to concentrate on sets with Hausdorff dimension less than $n - \vartheta$, and indeed higher regularity of solutions can be obtained. We shall divide the range $0 \leq \vartheta \leq n$ into two separate sub-ranges: a classic Harmonic Analysis result [**6**] indeed asserts that if $0 \leq \vartheta < p$, then $L^{1,\vartheta}(\Omega) \subset W^{-1,p}(\Omega)$. This will be called *the capacitary case* as in this case the measure in question is absolutely continuous with respect to the p-capacity. Note also that this obviously occurs when $p \leq n$. We shall mainly restrict to this case. The principle in (2.3) finds now the following quantified form, due to Mingione:

THEOREM 2.5 ([**118, 123**]). *Let $2 - 1/n < p \leq n$ and let $u \in W^{1,p-1}(\Omega)$ be a SOLA of problem* (2.3.1)*, where the vector field satisfies assumptions* (2.2.2) *and the measure μ satisfies the density condition* (2.3.6) *for $p \leq \vartheta \leq n$. Then*

$$|Du|^{p-1} \in \mathcal{M}^{\frac{\vartheta}{\vartheta-1}}_{\text{loc}}(\Omega;\mathbb{R}^n). \tag{2.3.7}$$

Moreover, it holds that

$$\sup_{\lambda>0, B_R\subset\Omega} \lambda^{-\frac{\vartheta}{\vartheta-1}} \big|\{x \in B_R : |Du(x)| > \lambda\}\big| \lesssim R^{n-\vartheta}. \tag{2.3.8}$$

Note that the previous result reduces to the one in Theorem 2.3 in the case of general measures $\vartheta = n$, and claims a better integrability of the gradient when $\vartheta < n$, i.e. when the measure diffuses. The inequality in (2.3.8) means that in the information on the density of the measure is inherited by the gradient Marcinkiewicz norm. This result opens the way to those of the following paragraph, indeed concerning Morrey spaces.

Linear and nonlinear Adams theorems. The main viewpoint linking the results of this paragraph is that they show that certain classical potential theory facts apparently linked to the linear setting can be actually reformulated in the context of what is called nonlinear potential theory. We shall in fact present at the same time classical results of Adams [7], Adams & Lewis [9] and Talenti [141], therefore refining gradient estimates of Section 2.1, and their nonlinear extensions given in [121].

Here we shall initially focus on the case where $H \equiv g$ is a function and

$$g \in L^{\gamma}(\Omega), \qquad 1 < \gamma < (p^*)' = \frac{np}{np-(n-p)}, \qquad p < n \tag{2.3.9}$$

and then we will introduce more and more subtle function spaces. We restrict to this range since, as already observed in the previous paragraph, when $\gamma \geq (p^*)'$ then g belongs to the dual of $W^{1,p}_0$ and we are here interested in the sub-dual case. Clearly throughout the section we shall consider only the solution obtained via approximation described in the lines above. Moreover we shall focus from now on on the case

$$2 \leq p \leq n.$$

Note that forcing $p = n$ would give in the limit as $p \nearrow n$, $\gamma \searrow 1$, a case we are not going to consider since it falls in the realm of the previously considered measure data problems.

We begin with the analog of $(2.1.4)_1$ for the nonlinear case we are considering; we consider a quasi-linear equation of the type

$$-\operatorname{div} a(x, Du) = g \qquad \text{in } \Omega, \tag{2.3.10}$$

$u \in W^{1,p-1}_0(\Omega)$, where the vector field satisfies the assumptions (2.2.2). We are therefore allowing for measurable coefficients. In this case we have the following classic result:

THEOREM 2.6 ([141, 26]). *Let $u \in W^{1,p-1}_0(\Omega)$ be a SOLA to equation (2.3.10), where g is as in (2.3.9). Then*

$$|Du|^{p-1} \in L^{\frac{n\gamma}{n-\gamma}}(\Omega).$$

The previous result represents the sharp nonlinear analog of estimate , which is in turn implied by the mapping property of Riesz potentials in (2.1.3). It is therefore natural to see the extent to which other mapping properties of Riesz potentials, and therefore other related a priori estimates for solutions to the Poisson equation, find an optimal nonlinear analog. We start by Morrey spaces: with $0 \leq \vartheta \leq n$ and $0 \leq \beta < \vartheta$, a classical result of Adams [7] asserts that

$$I_\beta : L^{\gamma,\vartheta}(\mathbb{R}^n) \longrightarrow L^{\frac{\vartheta\gamma}{\vartheta-\beta\gamma},\vartheta}(\mathbb{R}^n), \quad \text{for all } \gamma > 1 \text{ such that } \beta\gamma < \vartheta, \tag{2.3.11}$$

and therefore for the Poisson equation (2.1.1) there holds

$$\mu \in L^{\gamma,\vartheta}(\Omega), \quad 1 < \gamma < \vartheta \leq n \quad \Longrightarrow \quad Du \in L^{\frac{\vartheta\gamma}{\vartheta-\gamma},\vartheta}(\Omega;\mathbb{R}^n).$$

We recall that $L^{\gamma,\vartheta}(\Omega)$ is the Morrey space of functions $g \in L^{\gamma}(\Omega)$ such that

$$\int_{B_R(x)} |g|^{\gamma}\, dx \lesssim R^{n-\vartheta}, \tag{2.3.12}$$

with $B_R(x) \Subset \Omega$, cfr. (2.3.6). Mingione in [121] proved that these estimates, valid for the Poisson equation, extend also to nonlinear p-Laplacian elliptic equations of the type (2.3.10), at the same time extending Theorem 2.6 at Morrey level; indeed there holds

THEOREM 2.7 ([121]). *Let u be as in Theorem 2.6 and let $g \in L^{\gamma,\vartheta}(\Omega;\mathbb{R}^n)$, with*

$$1 < \gamma \leq \frac{\vartheta p}{\vartheta p - (\vartheta - p)}, \qquad p < \vartheta \leq n. \tag{2.3.13}$$

Then

$$|Du|^{p-1} \in L^{\frac{\vartheta\gamma}{\vartheta-\gamma},\vartheta}_{\mathrm{loc}}(\Omega). \tag{2.3.14}$$

Note that forcing $\vartheta = p$ would give $\gamma = 1$ and therefore we would fall in the realm of measure data problems. In this case the previous result should be compared with Theorem 2.5. Further relevant instances of spaces that can be considered when using Riesz potentials are the so-called Lorentz spaces. The Lorentz space $L(\gamma, q)(\Omega)$, with $1 \leq \gamma < \infty$ and $0 < q \leq \infty$, is defined prescribing that a measurable map g belongs to $L(\gamma, q)(\Omega)$ iff

$$\|g\|^q_{L(\gamma,q)(\Omega)} := q\int_0^\infty \Big(\lambda^\gamma|\{x \in \Omega : |g(x)| > \lambda\}|\Big)^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda} < \infty \tag{2.3.15}$$

when $q < \infty$; for $q = \infty$ we instead set $L(\gamma,\infty)(\Omega) := \mathcal{M}^\gamma(\Omega)$, and this means finding Marcinkiewicz spaces back. We shall also consider the "Morreyzation" [119] of such spaces, namely the Morrey-Lorenz spaces $L^\vartheta(\gamma, q)(\Omega)$ prescribed by the fact that the set function $A \mapsto \|g\|^q_{L(\gamma,q)(A)}$, defined for measurable subsets of Ω, belongs to $L^{1,\vartheta}(\Omega)$, i.e.

$$\sup_{B_R(x)\Subset\Omega} \|g\|^q_{L(\gamma,q)(B_R(x))} \lesssim R^{n-\vartheta}. \tag{2.3.16}$$

This definition has to be compared with (2.3.8). Note that the quantity defined in (2.3.15) is only a quasi-norm, that is satisfies the triangle inequality only up to a multiplicative factor larger than one. Recalling that here Ω has finite measure, we remark that the spaces $L(\gamma, q)(\Omega)$ "decrease" in the first parameter γ, while increasing in q; moreover, they "interpolate" Lebesgue spaces in the following sense: for $0 < q < \gamma < r \leq \infty$ we have

$$L^r \equiv L(r,r) \subset L(\gamma,q) \subset L(\gamma,\gamma) \subset L(\gamma,r) \subset L(q,q) \equiv L^q,$$

with continuous embeddings. Note by Fubini's theorem that $L(\gamma,\gamma) \equiv L^\gamma$. Lorentz spaces serve to describe finer scales of singularities, not achievable neither via the use of Lebesgue spaces nor of Marcikiewicz ones. We have seen that Marcinkiewicz spaces describe in a sharp way potentials, see (2.1.10). The perturbation of a potential via a logarithmic singularity is then described via Lorentz spaces:

$$\frac{1}{|x|^{\frac{n}{\gamma}}\log^\beta|x|} \in L(\gamma,q)(B_1) \qquad \text{iff } q > \frac{1}{\beta}.$$

At this point, as usual, we come to Adams-Lewis theory [9] which yields

$$I_\beta : L(\gamma,q)(\mathbb{R}^n) \longrightarrow L\left(\frac{n\gamma}{n-\beta\gamma}, q\right)(\mathbb{R}^n), \tag{2.3.17}$$

for $\beta \in (0, n],\ \gamma > 1,\ q > 0,\ \beta\gamma < n$ and

$$I_\beta : L^\vartheta(\gamma,q)(\mathbb{R}^n) \longrightarrow L^\vartheta\left(\frac{\vartheta\gamma}{\vartheta-\beta\gamma}, \frac{\vartheta q}{\vartheta-\beta\gamma}\right)(\mathbb{R}^n), \tag{2.3.18}$$

for $\beta \in (0,\vartheta), \gamma > 1,\ q > 0,\ \beta\gamma < \vartheta$. Note that in the case $\gamma = q$, (2.3.18) reduces to (2.3.11), but when dropping the Morrey condition $\vartheta = n$, (2.3.18) does not reduce to (2.3.17): this is not a gap in the theory, but a genuine discontinuity phenomenon discussed at length, and by mean of counterexamples, in [9]; see also an interesting discussion in [119, Remark 5.7]. The same phenomenon shows in the nonlinear analog of such results:

THEOREM 2.8 ([121]). *Let $u \in W_0^{1,p-1}(\Omega)$ be a SOLA of equation (2.3.10), where $g \in L^{\vartheta}(\gamma, q)(\Omega)$ with γ, ϑ as in (2.3.13) and for $0 < q \le \infty$; then*

$$|Du|^{p-1} \in L^{\vartheta}\left(\frac{\vartheta\gamma}{\vartheta-\gamma}, \frac{\vartheta q}{\vartheta-\gamma}\right) \qquad \textit{locally in } \Omega.$$

The discontinuity phenomenon discussed above obviously appears also in the nonlinear setting: in the case the Morrey condition is dropped, i.e. $\vartheta = n$, the previous Theorem would give, with $0 < q \le \infty$ and γ as in (2.3.13), the following implication:

$$g \in L(\gamma, q)(\Omega) \quad \Longrightarrow \quad |Du|^{p-1} \in L\left(\frac{n\gamma}{n-\gamma}, \frac{nq}{n-\gamma}\right) \qquad \text{locally in } \Omega.$$

The optimal result in this case is

THEOREM 2.9 ([12, 95, 121]). *Let $u \in W_0^{1,p-1}(\Omega)$ be a SOLA of equation (2.3.10), where $g \in L(\gamma, q)(\Omega)$ with $0 < q \le \infty$ and*

$$1 < \gamma \le (p^*)' = \frac{np}{np-(n-p)}, \qquad p < n.$$

Then

$$|Du|^{p-1} \in L\left(\frac{n\gamma}{n-\gamma}, q\right) \qquad \textit{locally in } \Omega.$$

Notice that the previous theorem has been proved in [12, 95] in the case $\gamma < (p^*)$; the delicate borderline case $\gamma = \gamma \le (p^*)$, separating the case of energy solutions form the very weak one since $Du \in L(p, q(p-1))$, has been finally reached in [121].

2.4. Back to the roots: pointwise estimates

Here we describe a more radical approach to Calderón-Zygmund estimates. Recall where we began: for the Poisson equation we have, by the representation formula, the pointwise estimates

$$|u(x)| \le I_2(|\mu|)(x) \qquad \text{and} \qquad |Du(x)| \le I_1(|\mu|)(x); \tag{2.4.1}$$

at this point, since we exactly know the action of Riesz potential over almost every kind of integral spaces, zero and first order estimates become almost trivial; in particular borderline cases, which usually are difficult to handle by itself nature, are treated with no more difficulty than other cases. The same would be for nonlinear functions, if such pointwise estimates held true. Actually the first of the two in (2.4.1), a pointwise estimate for solution to elliptic nonlinear equations of p-Laplacian type, has been proved almost twenty years ago by Kilpeläinen and Malý in a fundamental paper [94]. First we need to define an appropriate nonlinear potential. Indeed an estimate like (2.4.1) would be impossible for solution to p-Laplacian equations, for scaling reasons. Indeed it is clear that multiplying a non-null solution to $-\Delta_p u = \mu$ by a constant $c \neq 0$ we still obtain a solution $\tilde{u} = cu$ to the same equation with on the right-hand side the measure $\tilde{\mu} = c^{p-1}\mu$. Applying therefore $(2.4.1)_1$ to u and coming back to u and μ we would get $|u(x)| \le c^{p-2} I_2(|\mu|)(x)$; letting $c \to 0$ we would get $u \equiv 0$. The same would apply to solution to nonlinear problems of the type

$$-\operatorname{div} a(x, Du) = \mu \tag{2.4.2}$$

where the vector field is modeled upon the p-Laplacian.

Therefore for the nonlinear case we need a potential encoding the information of the rescaling of the problem: this is the nonlinear Wolff potential, introduced in [85], see also

[86], defined in (1.0.9):

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left[\frac{|\mu|(B_\rho(x))}{\rho^{n-\beta p}}\right]^{1/(p-1)} \frac{d\rho}{\rho}, \tag{2.4.3}$$

for $0 < \beta p \leq n$. This one appears as a nonlinear version of the classical truncated Riesz potential

$$\mathbf{I}^{|\mu|}_{\beta}(x,R) := \int_0^R \frac{|\mu|(B_\rho(x))}{\rho^{n-\beta}} \frac{d\rho}{\rho},$$

for for $0 < \beta \leq n$. Then the analog of $(2.4.1)_1$ was given by Kilpeläinen and Malý in the following

THEOREM 2.10 ([94, 70]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to (2.4.2), under the assumptions (2.2.5) and with $p > 2 - 1/n$, where μ is a Borel measure with finite total mass. Then the pointwise estimate*

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \fint_{B_R(x)} \big(|u| + Rs\big)\, d\xi, \tag{2.4.4}$$

holds for almost every $x \in \Omega$, for $B_R(x) \subset \Omega$.

Again noticeable contributions have also been given in [93, 144, 100]. Note that the previous estimate for $p = 2$ becomes

$$|u(x)| \lesssim \int_0^R \frac{|\mu|(B_\rho(x))}{\rho^{n-2}} \frac{d\rho}{\rho} + \fint_{B_R(x)} \big(|u| + Rs\big)\, d\xi,$$

which is already highly non-trivial, due to the nonlinearity of the context. The potential $\mathbf{W}^{\mu}_{1,2}(x,R) =: \mathbf{I}^{|\mu|}_2(x,R)$ appearing in the previous estimate is the truncated Riesz potential. Note that by a simple change of variable $\mathbf{I}^{|\mu|}_2(x,R) \lesssim I_2(|\mu|)(x)$ for all $R > 0$; therefore the result in [94] locally recovers $(2.4.1)_1$ for nonlinear Poisson equation (let $R \to \infty$ and the average on the right-hand side will disappear). Moreover (2.4.4), despite not involving the well-known behavior of the Riesz potential, allows to recover in a local way all the integrability results known for u via the properties of the Wolff potential. Indeed it is possible to estimate Wolff potentials via iteration of Riesz potentials:

$$\mathbf{W}^{\mu}_{\beta,p}(x,\infty) \lesssim I_\beta\Big\{\big[I_\beta(|\mu|)\big]^{1/(p-1)}\Big\}(x) =: \mathbf{V}_{\beta,p}(|\mu|)(x),$$

the potential $\mathbf{V}$ being called Havin-Maz'ya potential of $|\mu|$, see [10, 85].

But what about $(2.4.1)_2$? Its extension to nonlinear equations in the non-degenerate case, i.e. nonlinear vector fields satisfying the following

$$|a(\xi)| + |\partial_\xi a(\xi)|(s + |\xi|) \leq L(s + |\xi|), \qquad \nu|\lambda|^2 \leq \langle \partial_\xi a(\xi)\lambda, \lambda\rangle \tag{2.4.5}$$

for $\xi, \lambda \in \mathbb{R}^n$, has been reached by Mingione, who proved the following

THEOREM 2.11 ([122]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to the equation*

$$-\operatorname{div} a(Du) = \mu,$$

where the vector field $a(\cdot)$ satisfies (2.4.5) and where μ is a Borel measure with finite total mass. Then for every $i \in \{1, \dots, n\}$ the pointwise estimate

$$|D_i u(x)| \lesssim \mathbf{I}^{|\mu|}_1(x,R) + \fint_{B_R(x)} \big(|D_i u| + s\big)\, d\xi, \tag{2.4.6}$$

for almost every point $x \in \Omega$ and whenever $B_R(x) \subset \Omega$.

Here we have $\mathbf{I}_1^{|\mu|}(x,R) := \mathbf{W}^{\mu}_{1/2,2}(x,R)$. (2.4.6) was proved using an interesting technique which involves fractional Caccioppoli's inequality and De Giorgi's iteration performed at a fractional level. Indeed it is not possible, even with a differentiable vector field as in (2.2.5), further differentiate the equation, as done in [94] in order to get bounds on u. However, differentiation can be done at a fractional level, and this is sufficient to get the potential bound (2.4.6). Note that clearly here we have to impose differentiability assumptions on the vector field, when looking for gradient estimates.

At a first sight, one may think that only the nonlinear Wolff potentials play a fundamental role in extending (2.4.6) to p-Laplace like nonlinear vector fields, as estimate (2.4.3) is sharp. As a matter of fact the first estimate for nonlinear elliptic equations (and also for nonlinear heat equation, for the reasons we explained in Section 2.5) was given by Duzaar & Mingione in [70], using techniques different from [122], for the case $p \geq 2$. They proved the following

THEOREM 2.12 ([70]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to (2.4.2), where the vector field satisfies (2.3.4) with $p \geq 2$ and the map $R \mapsto [\omega(R)]^{2/p}$ being Dini continuous. Then the pointwise estimate for the gradient*

$$|Du(x)| \lesssim \mathbf{W}^{\mu}_{1/p,p}(x,R) + \fint_{B_R(x)} \big(|Du| + s\big)\, d\xi \tag{2.4.7}$$

holds true for almost every point $x \in \Omega$ and with $B_R(x) \subset \Omega$.

One might think the previous result as an optimal one but it is indeed not so. In fact, when looking at the equation

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu$$

one may think both as a nonlinear equation in the gradient and as a linear equation in the vector field $|Du|^{p-2}Du$. This heuristic argument leads to think of the possibility of a linear estimate for the quantity $|Du|^{p-1}$. It indeed holds

THEOREM 2.13 ([69, 106]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to (2.4.2), where the vector field satisfies (2.3.4) with $p > 2 - 1/n$ and $\omega(R)$ is Dini continuous. Then the pointwise estimate*

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B_R(x)} \big(|Du| + s\big)\, d\xi\right)^{p-1} \tag{2.4.8}$$

holds true for almost every point $x \in \Omega$ and with $B_R(x) \subset \Omega$.

The main point in the previous result is the case $p > 2$. Indeed, while in the case $p < 2$ the previous estimate does not improve a possible Wolff potential bound of the type in (2.4.7), which is not conjectured to hold in the subquadratic case, when $p > 2$ estimate in display (2.4.8), obtained by Kuusi & Mingione, improves the one in (2.4.7) and requires substantially new technical tools.

Estimate (2.4.8) shows that the first order integrability theory of solutions to equations of p-Laplacian type is completely reduced to the linear one, i.e. there is basically no difference between degenerate quasi-linear equations as (2.2.1) and the classical, linear Poisson equation (2.1.1). The analogy actually extends up to the C^1-regularity of solution. For instance, in [106] Kuusi & Mingione also proved that

$$\mu \in L(n,1)(\Omega) \quad \Longrightarrow \quad Du \text{ is continuous,}$$

exactly as for the classical Poisson equation. This in turn relates to a classical result of Stein [**138**] that claims the continuity of a function f whenever its distributional derivatives belong to $L(n, 1)$.

Interpolation potential estimates. The potential estimates in the previous paragraph allow for *size estimates* of solutions and their gradient in terms of potentials. Clearly estimates regarding the values of functions in *just one point* are not enough to provide estimates in spaces equipped with non-local norms, as for instance fractional Sobolev spaces. More in general, what about oscillation estimates? For instance estimates in Hölder or, indeed, fractional Sobolev spaces? We will here see, for instance, that estimates (2.4.4) and (2.4.7) can be viewed as two special cases of a one parameter family of potential estimates, covering estimates for finite differences of solutions, and carrying sharp information on the regularity of solutions depending on the data. To frame the results, we recall here a proper notion of fractional differentiability introduced by DeVore & Sharpley [**51**]. This notion has the merit to reduce non-locality of norms to a minimal status: two points only are needed. Let $\alpha \in (0, 1]$ and $q \geq 1$. For a measurable function $f : \Omega \to \mathbb{R}^k$ finite almost everywhere is in the Calderón space C^α_q if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|f(x) - f(y)| \leq \big[m(x) + m(y)\big]|x - y|^\alpha \tag{2.4.9}$$

holds for a.e. $x, y \in \Omega$. In a certain sense, (2.4.9) allows to identify $m(\cdot)$ as "a fractional derivative of order α" for f. For this reason, one may wonder if it possible to interpolate estimates (2.4.4) and (2.4.7) with Wolff potential depending on a parameter, so to get for $\alpha = 0$ the potential $\mathbf{W}^\mu_{1,p}$ and for $\alpha = 1$ the other potential $\mathbf{W}^\mu_{1/p,p}$. This can be suggested by the analysis of the Poisson equation: using elementary estimates and the representation formula, one gets

$$|u(x) - u(y)| \lesssim \big[I_{2-\alpha}(|\mu|)(x) + I_{2-\alpha}(|\mu|)(y)\big]|x - y|^\alpha$$

for the solution to the Possion equation (2.1.1), and this may be intuitively thought as

$$|D^\alpha u| \lesssim \mathbf{I}^{|\mu|}_{2-\alpha}$$

for every $\alpha \in [0, 1]$. This possibility has been actually showed true by Kuusi & Mingione [**103**]. Also here we restrict to the case $p \geq 2$, for ease of exposition. One can refer to Section 4.6 for some of the statements in the sub-quadratic case. Depending on the regularity with respect to the coefficient of the vector field in (2.4.2), where a satisfies the first two assumptions of (2.3.4), one has estimates as (2.4.9) up to the maximal regularity (i.e. α varying up to a certain threshold) allowed from the coefficients. Several estimates are in fact achievable and we refer to [**103**] for complete statements. Here we confine ourselves to report a model theorem.

THEOREM 2.14 ([**103**]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to equation*

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu \qquad \text{in } \Omega.$$

Then for $B_{2R} \subset \Omega$ and for every $x, y \in B_{R/8}$ the estimates

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \lesssim \Big[\mathbf{W}^\mu_{1-\alpha\frac{p-1}{p},p}(x, 2R) + \mathbf{W}^\mu_{1-\alpha\frac{p-1}{p},p}(y, 2R)\Big] + R^{-\alpha} \fint_{B_R} (|u| + Rs)\, d\xi$$

and

$$\frac{|Du(x) - Du(y)|}{|x - y|^\alpha} \lesssim \Big[\mathbf{W}^\mu_{1-(1+\alpha)\frac{p-1}{p},p}(x, 2R) + \mathbf{W}^\mu_{1-(1+\alpha)\frac{p-1}{p},p}(y, 2R)\Big]$$

$$+ R^{-\alpha} \fint_{B_R} (|Du| + s)\, d\xi$$

hold uniformly in $\alpha \in [0,1]$ *and locally uniformly in* $\alpha \in [0, \tilde{\alpha})$*, respectively.*

In the previous statement $\tilde{\alpha} < 1$ is the maximal Hölder continuity of solutions to the homogenous equation

$$-\operatorname{div}(|Du|^{p-2}Du) = 0.$$

Indeed, the $C^{1,\tilde{\alpha}}$-regularity for some $\tilde{\alpha} \equiv \tilde{\alpha}(n,p) \in (0,1)$ is the maximal regularity of solutions to the p-LaplacIan equation, as first shown by Uraltseva [145]. Lower bounds are available for $\tilde{\alpha}$ although its precise value is still unknown.

Notice also that in both the estimates of Theorem 2.14 we go back to the potential occurring in Theorem 2.12 when $\alpha = 1$ ("from below") and $\alpha = 0$ ("from above"), respectively. The role of Theorem 2.14 is now clear: for instance, if we want to know whether u is locally in $C^{0,\alpha}$ it is sufficient to ask for $\mathbf{W}^{\mu}_{1-\alpha\frac{p-1}{p},p} \in L^\infty$. By requiring for instance that $\mathbf{W}^{\mu}_{1-\alpha\frac{p-1}{p},p} \in L^q$ for some $q \geq 1$ one gets informations in Besov spaces $B^{\alpha,q}_s$ and so on. In other words, Theorem 2.14 provide a unified approach to the regularity of p-Laplacian operators via potentials.

2.5. Parabolicities

We now want to discuss the extension of the above Calderón-Zygmund theory to the parabolic case. Following the presentation in the elliptic case we shall fist outline a few results in the linear case i.e. we shall talk about the heat equation

$$u_t - \Delta u = \mu$$

and then we shall move to nonlinear equations of the type

$$u_t - \operatorname{div} a(Du) = \mu. \tag{2.5.1}$$

In the case the vector field $a(\cdot)$ satisfies (2.4.5), so that no degeneracy is allowed, the analysis of such equations can be carried out using a family of *standard parabolic cylinders*

$$Q_R(z_0) \equiv Q_R(x_0, t_0) := B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$$

whose shapes is devised to rebalance the lack of one derivative in the time direction, allowing for suitable rescaling arguments. When instead we consider assumptions as for instance (2.3.4) with $p \neq 2$ we cover the very relevant model case of the parabolic evolutionary p-Laplace operator

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0, \quad p \neq 2, \quad \text{in } \Omega_T := \Omega \times (-T, 0). \tag{2.5.2}$$

The lack of scaling of the previous equation makes the standard parabolic cylinders unsuitable to get regularity estimates, and this leads to consider what is called an intrinsic geometry: the size of cylinders is dictated by the behavior of the solution on the same cylinders. This in particular tells that *no universal family of balls is associated to the equation* and, as a consequence, typical harmonic analysis tools like maximal operators are automatically ruled out. This is a major obstruction in the theory, that, as we are going to see in the following, leads to find purely PDE approaches to the Calderón-Zygmund theory.

The intrinsic geometry approach has been introduced by and some details are as follows: consider, for simplicity in the case $p \geq 2$, cylinders of the type

$$Q^\lambda_R(z_0) \equiv Q^\lambda_R(x_0, t_0) := B_R(x_0) \times (t_0 - \lambda^{2-p}R^2, t_0 + \lambda^{2-p}R^2) \tag{2.5.3}$$

with $\lambda \geq 1$ a scaling parameter. The heuristic underneath the choice of the scaling parameter λ is the following. Suppose that on such a certain cylinder the relation

$$\fint_{Q_R^\lambda(z_0)} |Du|^p \, dz \approx \lambda^p \tag{2.5.4}$$

holds. We call such a cylinder *intrinsic*, since the parameter λ appears both in the definition of the cylinder and in the values Du takes over it; therefore everyone of these cylinders *depends explicitly on the solution*. Relation (2.5.4) roughly tells that $|Du| \approx \lambda$ on $Q_R^\lambda(z_0)$ and hence one may think to equation (2.5.2) as actually

$$u_t - \lambda^{p-2} \operatorname{div} Du = 0 \qquad \text{in } Q_R^\lambda(z_0). \tag{2.5.5}$$

Now switching from the intrinsic cylinder $Q_R^\lambda(z_0)$ to Q_1, that is making the change of variables

$$v(x,t) := u(x_0 + Rx, t_0 + \lambda^{2-p} R^2 t), \qquad (x,t) \in B_1 \times (-1,1) \equiv Q_1,$$

we note that (2.5.5) rewrites as $v_t - \Delta v = 0$ in Q_1. This argument tells that on an *intrinsic cylinder* like (2.5.4) the solution u behaves as a solution to the heat equation Note however that the previous argument is clearly only heuristic, and its implementation is far from straightforward. The ultimate outcome is anyway that, when considering solutions on such cylinders a priori estimate for solutions become homogenous and therefore enjoys homogeneous estimates *which is a technical keypoint in the estimates we are dealing with here*. For instance, when considering standard parabolic cylinders, for solutions to (2.5.2) it is only possible to prove bounds of the type

$$\sup_{Q_{r/2}(z_0)} |Du| \leq c(n,p) \fint_{Q_r(z_0)} (|Du| + 1)^{p-1} \, dz, \tag{2.5.6}$$

whose lack of homogeneity precisely reflects that of the equation. In this sense the previous estimate is natural but still unsuitable to be used to develop Calderón-Zygmund estimates. When instead considering intrinsic cylinders with (2.5.4) being in force, *estimates become dimensionally homogeneous*:

$$c(n,p) \left(\fint_{Q_r^\lambda(z_0)} |Du|^{p-1} \, dz \right)^{1/(p-1)} \leq \lambda \quad \Longrightarrow \quad |Du(z_0)| \leq \lambda. \tag{2.5.7}$$

The analysis in the following paragraph is therefore strongly based on the use of intrinsic cylinders, that are the basic objects when dealing with regularity and qualitative properties of solutions. In particular, their use allows to analyze the equation considered on portions of domains where it behaves, roughly speaking, as the heat equation.

Estimates for nonlinear evolutionary *p*-Laplacian. The distinction between duality and non-duality range we made in the previous pages makes sense also in the parabolic setting. The energy space gathered to the Cauchy-Dirichlet problem

$$\begin{cases} u_t - \operatorname{div} a(Du) = H & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}} \Omega_T := \partial \Omega_T \setminus \big(\Omega \times \{0\}\big), \end{cases} \tag{2.5.8}$$

where the vector field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}$ is Carathéodory regular and satisfies at least the monotonicity and p-growth assumptions

$$\begin{cases} \langle a(\xi_1) - a(\xi_2), \xi_1 - \xi_2 \rangle \geq \nu \big(s^2 + |\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \\ |a(\xi)| \leq L \big(s^2 + |\xi|^2\big)^{\frac{p-1}{2}}, \end{cases}$$

for all $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$, ν, L, s as in (2.2.2), is the space

$$\big[(L^p(-T,0;W_0^{1,p}(\Omega))\big])^* \cong L^{p'}(-T,0;W^{-1,p'}(\Omega)); \tag{2.5.9}$$

indeed if $H \in L^{p'}(-T,0;W^{-1,p'}(\Omega))$, then we are in position to apply the classic monotonicity argument of [115, Chapter 2] and conclude that there exists a solution u to the problem (2.5.8) belonging, by Sobolev's embedding, to the space

$$L^p(-T,0;W_0^{1,p}(\Omega)) \cap C^0(-T,0;L^2(\Omega)).$$

We here now consider the duality case

$$H = \operatorname{div}\big(|F|^{p-2}F\big) \tag{2.5.10}$$

with $F \in L^q(\Omega_T)$, $q \geq p$. We again confine ourselves to the case $p \geq 2$ for ease of exposition. In this setting the problem of the parabolic extension of the result of Iwaniec in Theorem 2.2 has remained an open problem for sevreal years. Indeed, due to the lack of scaling of the equation, as described in the previous paragraph, standard parabolic cylinders cannot be any longer used due to lack of scaling of the equation. In turn, this rules out the possibility of using the Harmonic Analysis tools (for instance various maximal operators) who lie at the heart of Iwaniec's approach. The result has been finally achieved by Acerbi & Mingione in [4], who gave a purely PDE approach to the problem, thereby avoiding any use of Harmonic Analysis tools.

THEOREM 2.15 ([4]). *Let u belonging to the energy space defined in* (2.5.9) *be a solution to* (2.5.8)*, where H has the form* (2.5.10) *with $F \in L^\gamma(\Omega_T)$, $\gamma > p \geq 2$. Then*

$$Du \in L^\gamma_{\text{loc}}(\Omega_T;\mathbb{R}^n)$$

and for every parabolic cylinder $Q_{2R} \subset \Omega_T$ the following local estimate holds true:

$$\left(\fint_{Q_R} |Du|^\gamma \, dz\right)^{\frac{1}{\gamma}} \lesssim \left[\left(\fint_{Q_{2R}} (|Du|^p + 1)\, dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{2R}} |F|^\gamma \, dz\right)^{\frac{1}{\gamma}}\right]^{\frac{p}{2}}. \tag{2.5.11}$$

Compare estimate (2.5.11) with (2.2.7); the scaling deficit $d = p/2$ is typical when dealing with energy estimate for the evolutionary p-Laplacian on standard parabolic cylinders, and precisely reflects its lack of scaling. Similar results have been proved for general parabolic systems in [71] with restrictions on γ, following the elliptic case treated in [99].

Non-duality range for evolutionary p*-Laplacian.* The techniques described in the previous lines would allow to treat at least integrability problems for (2.5.8) as those described in Section 2.3, since they all essentially rely on estimates over level sets. A first approach to this program form part of the original of this thesis and can be found in Chapter 3, to which refer for the parabolic version of the elliptic results presented in Section 2.3 and in particular in Paragraph 2.3.

(Intrinsic) potential gradient bounds. We come here to the last Section to describe the very recent approach to potential estimates for the evolutionary p-Laplace operator obtained by Kuusi & Mingione in [104]. The extension of elliptic gradient estimates to the nonlinear heat equation (2.5.1), where a satisfies (2.4.5), follows the one in the elliptic case, and can be found in [70]. The result is

THEOREM 2.16 ([70]). *Let $u \in L^{p-1}(-T,0;W^{1,p-1}(\Omega))$ be a SOLA to equation* (2.5.1)*, with a satisfying* (2.4.5)*. Then for every cylinder $Q_{2R} \equiv Q_{2R}(z) \subset \Omega_T$ the following estimate holds:*

$$|Du(z)| \lesssim \mathbf{I}_1^{|\mu|}(z,2R) + \fint_{Q_R} \big(|Du| + s\big)\, d\xi$$

provided z is a Lebesgue point of Du.

Here clearly the appearing Riesz potential is the parabolic one, i.e. the one obtained replacing balls B_ρ with cylinders Q_ρ and the dimension n with the parabolic one N:

$$\mathbf{I}_1^{|\mu|}(z,R) := \int_0^R \frac{|\mu|(Q_\rho(z))}{\rho^{N-1}}\frac{d\rho}{\rho}.$$

The case of the evolutionary p-Laplacian operator for $p \neq 2$ is a completely different story. It necessitates a new approach, and first of all it imposes to understand how the basic concept of intrinsic geometry – which is something that links the cylinders considered to the solution – links to the one of nonlinear potentials. The two things are apparently inconsistent. Indeed, on one hand one of the ultimate goals of potentials is to "separate", when performing estimates, the solution from the equation and form the assigned datum, using indeed an estimate involving potentials, exactly as when using convolution via fundamental solutions. On the other hand, when using degenerate parabolic problems, one is led to consider intrinsic geometries, that is to consider cylinders linked to the solutions itself, and this, in principle, reflects again on the form of the potential one is going to consider.

The way to match intrinsic geometry and potential estimates has been recently found in [**104, 105**] and consists in defining potentials in an *intrinsic way*, eventually getting non-intrisic estimates, i.e. estimates valid on standard parabolic cylinders. Recall the definition of the cylinders $Q_R^\lambda(z)$ given in (2.5.3), and define the *intrinsic Wolff potential* as

$$\mathbf{W}_\lambda^\mu(z,R) := \int_0^R \left[\frac{|\mu|(Q_\rho^\lambda(z))}{\lambda^{2-p}\rho^{N-1}}\right]^{1/(p-1)}\frac{d\rho}{\rho}.$$

This is the key to the parabolic gradient estimate.

THEOREM 2.17 ([**104**]). *Let $u \in L^p(-T,0;W^{1,p}(\Omega))$ be a solution to (2.5.8)$_1$, where the vector field is differentiable and satisfies assumptions (2.2.5) for $p \geq 2$. Then for every $z \in \Omega_T$ being a Lebesgue's point of Du and every cylinder $Q_{2R}^\lambda(z) \subset \Omega_T$ there holds*

$$\left(\fint_{Q_R^\lambda(z)} (|Du|+s)^{p-1}\,d\xi\right)^{1/(p-1)} + \int_0^{2R}\left[\frac{|\mu|(Q_\rho^\lambda(z))}{\lambda^{2-p}\rho^{N-1}}\right]^{1/(p-1)}\frac{d\rho}{\rho} \lesssim \lambda$$
$$\Longrightarrow \qquad |Du(z)| \leq \lambda. \qquad (2.5.12)$$

Note that (2.5.12) essentially reduces to (2.5.7) when $\mu \equiv 0$. The previous intrinsic estimate in turn implies potential estimates on standard, non-intrinsic parabolic cylinders, where the loss of homogeneity is showed by the appearance of the scaling deficit $p-1$, as in (2.5.6):

COROLLARY 2.18 ([**104**]). *Let u as in Theorem 2.17. Then for every $z \in \Omega_T$ which is a Lebesgue point of Du and every parabolic cylinder $Q_{2R}(z) \subset \Omega_T$ there holds*

$$|Du(z)|^{p-1} \lesssim \int_0^{2R}\left[\frac{|\mu|(Q_\rho(z))}{\rho^{N-1}}\right]^{1/(p-1)}\frac{d\rho}{\rho} + \left(\fint_{Q_R(z)} (|Du|+s+1)^{p-1}\,d\xi\right)^{p-1}.$$

CHAPTER 3

Some technical background

In this Section we first want to fix the notation we shall use throughout all the thesis. Despite having already introduced some of them, we will repeat ourselves for completeness, and for the ease of the reader. Particular notations, used only in certain cases, will be highlighted at beginning of respective Chapters. Moreover here we collect definitions of function spaces.

3.1. Notation

In the elliptic case we shall always consider Ω as a bounded domain of $\mathbb{R}^n$, with $n \geq 2$. We denote by

$$B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

the open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $R > 0$ and by

$$C_R(x_0) := \{x \in \mathbb{R}^n : \max_j |x_i - x_{0,j}| < R\}$$

the open cube with center x_0 and sidelength $2R$. As a general convention B_1 and C_1will be the ball of radius one (resp. cube of sidelength 2) and center 0. Often, when clear from the context, we will drop from notations the centers of families of (concentric) balls. For a given ball $B \subset \mathbb{R}^n$ we denote by $C_{\mathrm{inn}}(B)$ and $C_{\mathrm{out}}(B)$ the largest and the smallest cubes with sides parallel to the coordinate axes concentric to B contained in B or containing B, respectively; i.e. if $B = B_R(x_0)$ we have $C_{\mathrm{inn}}(B) = C_{R/\sqrt{n}}(x_0)$ and $C_{\mathrm{out}}(B) = Q_R(x_0)$. These cubes we shall call inner and outer cubes. Being $D \in \mathbb{R}^m$, a measurable set with positive measure and $f : D \to \mathbb{R}^k$ with $m, k \geq 1$ an integrable map, we denote with $(f)_D$ the average of f over D

$$(f)_D := \fint_D f(\xi)\, d\xi := \frac{1}{|D|} \int_D f(\xi)\, d\xi \tag{3.1.1}$$

and with $E(f, D)$ the (L^1)-excess of f:

$$E(f, D) := \fint_D |f(\xi) - (f)_D|\, d\xi. \tag{3.1.2}$$

Note that a useful property of the excess, which we shall use extensively in the following, is that

$$E(f, D) \leq 2 \fint_D |f(\xi) - \eta|\, d\xi \tag{3.1.3}$$

for all $\eta \in R^k$. The expression $|D|$ will denote the m-dimensional Lebesgue's measure $\mathcal{L}^m$ depending on where D lives: $|D| = \mathcal{L}^m(D)$ if $D \subset \mathbb{R}^m$; here ξ is the variable in $\mathbb{R}^m$. α_m will denote the Lebesgue measure of the unit ball B_1 of $\mathbb{R}^m$. When dealing at the same time with different ambient space, for example with $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ at the same time, we will use Hausordff notation for the measure on the set with the lower dimension, in the case above $\mathcal{H}^n$, to settle possible misunderstandings.

For the parabolic arguments, we shall always consider problems on parabolic cylinders $\Omega_T := \Omega \times (-T, 0)$, with Ω as above and $T > 0$. $\mathbb{R}^{n+1}$ will always be thought as $\mathbb{R}^n \times \mathbb{R}$, so a point $z \in \Omega_T \subset \mathbb{R}^{n+1}$ will be often also denoted as (x, t), z_0 also as (x_0, t_0), and so on, eventually without recalling this convention every time. We shall need several type of parabolic cylinders. The point $z_0 \in \Omega_T$, for each of those defined in the following lines, will be its "vertex". We denote by $Q_{r,s}(z_0)$ the generic cylinder

$$Q_{r,s}(z_0) := B_r(x_0) \times (t_0 - s, t_0 + s)$$

for $r, s > 0$. In the case $r = R$, $s = R^2$ we shall have the standard parabolic cylinders

$$Q_R(z_0) \equiv Q_{R,R^2}(z_0) = B_R(x_0) \times \Lambda_R(t_0) := B_R(x_0) \times (t_0 - R^2, t_0 + R^2),$$

according to the parabolic metric

$$d_{\mathcal{P}}(z_1, z_2) := \max\Big\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\Big\} \tag{3.1.4}$$

for all $z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in \Omega_T$.

$$\mathcal{Q}_R(z_0) = C_R(x_0) \times (t_0 - R^2, t_0 + R^2)$$

will denote instead the open parabolic cylinder having a cube $C_R(x_0)$ of sidelength $2R$ as horizontal slice. Throughout the paper all the cubes considered will have sides parallel to the coordinate axes in $\mathbb{R}^n$ and will have positive sidelength. As above, we shall denote $Q_1 := Q_1(0)$, $\mathcal{Q}_1 := \mathcal{Q}_1(0)$ and $\Lambda_1 := \Lambda_1(0)$. Moreover, for a given parabolic cylinder $Q = B_R(x_0) \times (t_0 - R^2, t_0 + R^2) \subset \mathbb{R}^{n+1}$ we will denote by $\mathcal{Q}_{\text{out}}(\mathcal{C})$ the smallest parabolic cylinder with horizontal cross section a cube with sides parallel to the coordinate axes containing Q, i.e. $C_R(x_0) \times (t_0 - R^2, t_0 + R^2)$. Similarly, the largest parabolic cylinder with cross section a cube contained in Q is denoted by $\mathcal{Q}_{\text{inn}}$ and given by $C_{R/\sqrt{n}}(x_0) \times (t_0 - (R/\sqrt{n})^2, t_0 + (R/\sqrt{n})^2)$. Note that due to the parabolic scaling we have to decrease the time interval in the case of $\mathcal{Q}_{\text{inn}}(Q)$. Without abuse of confusion we will call $\mathcal{Q}_{\text{inn}}(Q)$ and $\mathcal{Q}_{\text{out}}(Q)$ **inner** and **outer parabolic cylinder** (associated to Q). The cylinders $Q_r^\lambda(z_0)$, defined starting from a radius r and a parameter $\lambda \geq 1$, will be of special importance. They are defined in the following way:

$$Q_R^\lambda(z_0) := Q_{R,\lambda^{2-p}R^2}(z_0) = B_R(x_0) \times \Lambda_R^\lambda(t_0), \tag{3.1.5}$$

where

$$\Lambda_R^\lambda(t_0) := (t_0 - \lambda^{2-p}R^2, t_0 + \lambda^{2-p}R^2).$$

Such cylinders will be called "intrinsic" if some intrinsically defined relation between λ and the data (and the solution) of the considered problem will hold, see for example (7.3.6) or (8.3.6). Luckily enough we shall need this kind of cylinder with cross-section a cube only for nonlinear heat equation, and therefore there will be no need of building intrinsic cylinders starting from them. Accordingly with the parabolic metric, for $\alpha > 0$ we shall write $\alpha\Lambda_R^\lambda := \Lambda_{\alpha R}^\lambda(t_0) = (t_0 - \lambda^{2-p}(\alpha R)^2, t_0 + \lambda^{2-p}(\alpha R)^2)$. The same will hold for the cylinders $\alpha Q_R^\lambda := (t_0 - \lambda^{2-p}(\alpha R)^2, t_0 + \lambda^{2-p}(\alpha R)^2)$, in particular when $\lambda = 1$ (and therefore $Q_R^1 \equiv Q_R$). Idem for $\alpha\mathcal{Q}_R := C_{\alpha R} \times (t_0 - (\alpha R)^2, t_0 + (\alpha R)^2)$. When dealing with families of cylinders with the same"vertex" (respectively of balls, of intervals), we will avoid to denote its center, highlighting only when the cylinders considered will not share the vertex.

All the subset of the cylinder Ω_T we are going to consider will be of cylindrical form. Indeed, if $C \subset \Omega_T$, then $C = A \times J$, with $A \subset \Omega$ and $J \subset (-T, 0)$. Subsequently by parabolic boundary of $C := A \times J$, with $A \subset \Omega$, $J \subset (-T, 0)$, we will mean

$$\partial_{\mathcal{P}} C := A \times \{\inf J\} \cup \partial A \times J. \tag{3.1.6}$$

Moreover writing $C \Subset \Omega_T$ we will mean that $A \Subset \Omega$, $J \Subset (-T, 0)$, eventually keeping implied the spacial and temporal sections. Coming back to (3.1.1), for a measurable function over a cylinder as above $g : C = A \times J \subset \mathbb{R}^{n+1} \to \mathbb{R}^k$ we will use the notation

$$(g)_A(t) := \fint_A g(x, t)\, dx$$

for the slice-wise averages, for all $t \in J$. Note that if $g \in L^p(J, W^{1,p}(A))$, $p \geq 1$, then $x \mapsto g(x, t) \in W^{1,p}(A)$ for a.e. $t \in J$. Hence we will be allowed to use Poincaré's inequality slice-wise for almost every instant of time.

Concerning derivatives, we will use different notations throughout of the paper, but all the expressions $\partial_t u$, $\frac{\partial}{\partial t} u$, u_t will mean the derivative of u with respect to t. For the spatial gradient of u we will always use the notation Du and by $D_i u$, $i \in \{1, \dots, n\}$ we shall mean $\frac{\partial}{\partial x_i} u$. With a vector field $a \equiv a(\xi)$, $a(x, \xi)$ or similar, $\partial_\xi a$ will denote its differential $\{D_i a^j\}_{i,j}$.

Finally, we will denote with c a generic constant always greater than one, which will not necessarily be the same at different occurrences throughout the paper. In particular it may also change from line to line. For reasons of readability, dependencies of the constants will often be omitted within the chains of estimates, therefore stated after the estimate. Constants we need to recall will be denoted with special symbols, such as $c_1, c_2, \tilde{c}, c_*$. By $\mathbb{R}^+$ we will mean the half-line $[0, \infty)$, by $\mathbb{N}$ the set $\{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

3.2. Function spaces

Here we collect many of the definitions we gave throughout all the previous pages (and many others we didn't give) regarding function spaces more or less known. Clearly this in just intended to clarify the concepts we are going to use, and will not be nowhere near to as general as possible. *We give the definitions in the elliptic setting, but it is enough to replace balls B_R with parabolic cylinders Q_R, the dimension n with the homogeneous dimension N and Ω with Ω_T to get the parabolic analog.* Moreover we shall give the definitions of some typically parabolic spaces.

First of all we recall that whereas $X = X(C)$ is some space functions over $C \subset \mathbb{R}^m$, $m \in \mathbb{N}$, its local variant X_{loc} is defined in the usual way, that is $f \in X_{\mathrm{loc}}(C)$ if $f \in X(C')$ whenever $C' \Subset C$. This applies also to the parabolic setting, since we shall only deal with symmetric cylinders. Sometimes we will lighten a bit notations writing $X(C)$ for $X(C; \mathbb{R}^k)$, $k \in \mathbb{N}$, $k > 1$, when treating vectorial valued functions where no confusion shall arise. In this spirit, *we restrict our description of the following spaces to the scalar case*: the reader should however keep in mind that they have a trivial generalization for vector valued (and, as we will see, also for Banach-valued) functions.

Spaces measuring size. A measurable map $g\colon \Omega \to \mathbb{R}$ is said to belong to the **Lorentz-space** $L(p, q)(\Omega)$ with $1 \leq p < \infty$ and $0 < q \leq \infty$ iff

$$\|g\|^q_{L(p,q)(\Omega)} := p \int_0^\infty \Big(\lambda^p |\{x \in \Omega : |g(x)| > \lambda\}|\Big)^{\frac{q}{p}} \frac{d\lambda}{\lambda} < \infty \tag{3.2.1}$$

when $q < \infty$, while for $q = \infty$ it is imposed that

$$\sup_{\lambda > 0} \lambda^p \big|\{x \in \Omega : |g(x)| > \lambda\}\big| =: \|g\|^p_{\mathcal{M}^p(\Omega)} < \infty. \tag{3.2.2}$$

The latter is the so called **Marcinkiewicz**, or **weak-L^p space**. Since we always assume Ω to have finite measure the spaces $L(p, q)(\Omega)$ decrease in the first parameter p, which means that for $1 \le \tilde{p} < p \le \infty$ and $0 < q \le \infty$ we have a continuous embedding

$$L(p,q)(\Omega) \hookrightarrow L(\tilde{p},q)(\Omega) \qquad \text{with} \qquad \|g\|_{L(\tilde{p},q)(\Omega)} \le |\Omega_T|^{\frac{1}{\tilde{p}}-\frac{1}{p}} \|g\|_{L(p,q)(\Omega)}.$$

On the other hand the Lorentz-spaces increase in the second parameter q, i.e. we have for $0 < q < \tilde{q} \le \infty$ the continuous embedding

$$L(p,q)(\Omega) \hookrightarrow L(p,\tilde{q})(\Omega) \qquad \text{with} \qquad \|g\|_{L(p,\tilde{q})(\Omega)} \le c(p,q,\tilde{q}) \|g\|_{L(p,q)(\Omega)}.$$

The so-called **parabolic Lorentz-Morrey-spaces** are obtained by coupling definition (3.2.1) with a density condition in the following sense: A measurable map $g\colon \Omega \to \mathbb{R}$ belongs to $L^{\vartheta}(p,q)(\Omega)$ for $1 \le p < \infty$, $0 < q < \infty$ and $0 \le \vartheta \le n$ iff $\|g\|_{L^{\vartheta}(p,q)(\Omega)} < \infty$, where

$$\begin{aligned} \|g\|_{L^{\vartheta}(p,q)(\Omega)} &:= \sup_{B_\varrho \subset \Omega} \varrho^{\frac{\vartheta-n}{p}} \|g\|_{L(p,q)(B_\varrho)} \\ &= \sup_{B_\varrho \subset \Omega} \left[p \int_0^\infty \Big(\lambda^p \varrho^{\vartheta-n} |\{x \in B_\varrho : |g(x)| > \lambda\}| \Big)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \right]^{\frac{1}{q}}, \end{aligned} \tag{3.2.3}$$

while $g \in L^{\vartheta}(p,\infty)(\Omega) = \mathcal{M}^{p,\vartheta}(\Omega)$ iff

$$\begin{aligned} \|g\|_{\mathcal{M}^{p,\vartheta}(\Omega)} &:= \sup_{B_\varrho \subset \Omega} \varrho^{\frac{\vartheta-n}{p}} \|g\|_{\mathcal{M}^p(B_\varrho)} \\ &= \sup_{B_\varrho \subset \Omega} \varrho^{\frac{\vartheta-n}{p}} \sup_{\lambda>0} \Big(\lambda^p \big|\{x \in B_\varrho : |g(x)| > \lambda\}\big| \Big)^{\frac{1}{p}} < \infty. \end{aligned}$$

REMARK 3.1. By Fubini's Theorem we have

$$\|g\|^p_{L^p(\Omega)} = p \int_0^\infty \lambda^p \big|\{x \in \Omega : |g(x)| > \lambda\}\big| \frac{d\lambda}{\lambda} = \|g\|^p_{L(p,p)(\Omega)},$$

so that $L^p(\Omega) = L(p,p)(\Omega)$. As an immediate consequence we also have $L^{p,\vartheta}(\Omega) = L^{\vartheta}(p,p)(\Omega)$ with $\|g\|_{L^{p,\vartheta}(\Omega)} = \|g\|_{L^{\vartheta}(p,p)(\Omega)}$.

A measurable map g defined on Ω belongs to the space $L\log L(\Omega)$ iff

$$\|g\|_{L\log L(\Omega)} := \inf\left\{ \lambda > 0 : \fint_\Omega \left|\frac{g}{\lambda}\right| \log\left(e + \left|\frac{g}{\lambda}\right|\right) dx \le 1 \right\} < \infty. \tag{3.2.4}$$

Note that we have incorporated in the preceding definition a dependence on the measure $|\Omega|$, by considering an averaged integral in (3.2.4). The reason for this will become clear in few lines, when we introduce a Morrey-type variant of the $L\log L$-spaces. Due to a remarkable result by Iwaniec [91] we have

$$\|g\|_{L\log L(\Omega)} \approx |g|_{L\log L(\Omega)} := \fint_\Omega |g| \log\left(e + \frac{g}{(|g|)_\Omega}\right) dx. \tag{3.2.5}$$

The constant connecting the Luxemburg-norm $\|\cdot\|_{L\log L}$ with $|\cdot|_{L\log L}$ is independent of Ω and g. Moreover, and this is the striking fact of Iwaniec's result, $|\cdot|_{L\log L}$ defines a true norm on $L\log L(\Omega)$. Moreover, a fundamental estimate when dealing with non-standard growth conditions, and which is a consequence of this result, is that for any $\gamma > 0$ and $g \in L^\sigma(\Omega)$ for some $\sigma > 1$, there holds

$$\fint_\Omega |g| \log^\beta\left(e + \frac{|g|}{(|g|)_\Omega}\right) dx \le c(\sigma,\beta) \left(\fint_\Omega |g|^\sigma \, dx\right)^{\frac{1}{\sigma}}; \tag{3.2.6}$$

here and later $\log^\gamma(t) := [\log(t)]^\gamma$. Thereby, the constant $c(\sigma,\beta)$ blows up when $\sigma \searrow 1$. See also [3, inequality (28)]. Moreover, $c(\sigma,\beta)$ depends continuously on β and therefore it can be replaced by a constant $c(\sigma,\gamma_1,\gamma_2)$ if $\beta \in [\gamma_2', \gamma_1']$.

In the light of Definition (3.2.3) for $\vartheta \in [0, n]$ the **parabolic Morrey-Orlicz-space** $L \log L^{\vartheta}(\Omega)$ is defined as the space of measurable functions g defined on Ω satisfying

$$\begin{aligned} \|g\|_{L \log L^{\vartheta}(\Omega)} &:= \sup_{B_\varrho \subset \Omega} \varrho^{\vartheta} \|g\|_{L \log L(B_\varrho)} \\ &\approx \sup_{B_\varrho \subset \Omega} \varrho^{\vartheta - n} \int_{B_\varrho} |g| \log \left(e + \frac{g}{(|g|)_{B_\varrho}} \right) dx < \infty. \end{aligned} \tag{3.2.7}$$

Maximal operators. In Chapter 9 we will make use of the **(restricted) centered fractional maximal operator**: for $\beta \in [0, n]$ and being $g \in L^1(\Omega)$ or eventually a measure with finite total mass, we define

$$M_{\beta,R}(g)(x) := \sup_{0<r\leq R} r^{\beta} \fint_{B_r(x)} |g| \, d\xi \qquad \text{or} \qquad \sup_{0<r\leq R} r^{-\beta} \frac{|g|(B_r(x))}{|B_r(x)|}.$$

For $\beta = 0$, the above defined operator $M_R(g) \equiv M_{0,R}(g)$ is the classical (restricted) centered Hardy-Littlewood maximal operator. On the other hand for $\beta \in [0, n]$ and $g \in L^1(\Omega)$ the **(restricted) centered sharp fractional maximal operator** of g is

$$M^{\sharp}_{\beta,R}(g)(x) := \sup_{0<r\leq R} r^{-\beta} \fint_{B_r(x)} |g - (g)_{B_r(x)}| \, d\xi.$$

In the case $\beta = 0$ the definition gives the usual Fefferman-Stein sharp maximal operator $M^{\sharp}_R(g) \equiv M^{\sharp}_{0,R}(g)$. Obviously, by Poincaré's inequality, for any $g \in W^{1,1}(\Omega)$ we have

$$M^{\sharp}_{\alpha,R}(g)(x) \leq c(n) \, M_{1-\alpha,R}(Dg)(x) \qquad \text{for all } \alpha \in [0, 1]. \tag{3.2.8}$$

In the parabolic setting, for fixed $\beta \in [0, N]$ we consider the **(restricted) fractional maximal function operator** relative to a symmetric parabolic cylinder $\widetilde{\mathcal{Q}} = \mathcal{Q}_R(z_0) \subset \mathbb{R}^{n+1}$, which is defined by

$$M^*_{\beta,\widetilde{\mathcal{Q}}}(g)(z) := \sup_{\mathcal{Q} \subset \widetilde{\mathcal{Q}}, z \in \mathcal{Q}} |\mathcal{Q}|^{\frac{\beta}{N}} \fint_{\mathcal{Q}} |g(w)| \, dw, \tag{3.2.9}$$

where the sup is taken with respect to all parabolic cylinders $\mathcal{Q}$ contained in $\widetilde{\mathcal{Q}}$ having sides parallel to those of $\widetilde{\mathcal{Q}}$ and containing the point z. When $\beta = 0$ we write $M^*_{\widetilde{\mathcal{Q}}}$ instead of $M^*_{\beta,\widetilde{\mathcal{Q}}}$. Moreover, in the case $\widetilde{\mathcal{Q}} = \mathbb{R}^{n+1}$ we abbreviate $M_\beta \equiv M^*_{\beta,\mathbb{R}^{n+1}}$ respectively $M \equiv M^*_{\mathbb{R}^{n+1}}$. Completely similar definitions and notations are given when cylinders with a cube as horizontal slice are replaced by those ones with a ball as horizontal slices:

$$M^*_{\beta,\widetilde{Q}}(g)(z) := \sup_{Q \subset \widetilde{Q}, \, z \in Q} |Q|^{\frac{\beta}{N}} \fint_{Q} |g(w)| \, dw,$$

where $\widetilde{Q}$ is a fixed parabolic cylinder and Q is any other parabolic cylinder contained in $\widetilde{Q}$ containing the point z. From [34, 82] we recall the boundedness of the maximal operators in Marcinkiewicz spaces, i.e. if $g \in L^q(\widetilde{\mathcal{Q}})$ then

$$|\{z \in \mathcal{Q}_0 : M^*_{\widetilde{\mathcal{Q}}}(g)(z) \geq \lambda\}| \leq \frac{c_0(n,q)}{\lambda^q} \int_{\widetilde{\mathcal{Q}}} |g|^q \, dz \tag{3.2.10}$$

holds for every $\lambda > 0$ and $q \geq 1$.

Spaces measuring oscillations. The **fractional Sobolev space** $W^{\alpha,q}(\Omega)$ is the subspace of $L^q(\Omega)$ made up of all the functions g whose fractional Sobolev seminorm

$$[g]^q_{W^{\alpha,q}(\Omega)} := \int_A \int_A \frac{|g(x) - g(y)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy$$

is finite. It is endowed with the norm $\|g\|_{W^{\alpha,q}(\Omega)} := \|g\|_{L^q(\Omega)} + [g]_{W^{\alpha,q}(\Omega)}$. For a function $g : \Omega \to \mathbb{R}$, any "small" real number $h \in \mathbb{R}$ and $i \in \{1, \dots, n\}$, we define the spatial finite difference operator $\tau_{i,h}$ as

$$\big[\tau_{i,h} g\big](x) = \tau_{i,h} g(x) := g(x + h\, e_i) - g(x),$$

being e_i the i-th vector of the standard orthonormal basis of $\mathbb{R}^n$. This will make sense, for example, whenever $x \in A \Subset \Omega$, A an open set and $0 < |h| < \text{dist}(A, \partial\Omega)$, an assumption that will be always satisfied whenever we shall use this operator. Analogously, we define also the finite difference operator in time τ_h as

$$\big[\tau_h \tilde{g}\big](t) = \tau_h \tilde{g}(t) := \tilde{g}(t+h) - \tilde{g}(t),$$

again for $|h| > 0$ sufficiently small such that the definition makes sense. For a set $A \Subset \Omega$, we define the **Nikolski space** $\mathcal{N}^{\alpha,q}(A)$ as the space of the $L^q(\Omega)$ functions g such that their $\mathcal{N}^{\alpha,q}$ norm

$$\|g\|_{\mathcal{N}^{\alpha,q}(A)} := \|g\|_{L^q(A)} + [q]_{\mathcal{N}^{\alpha,q}(A)}\,,$$

with

$$[g]_{\mathcal{N}^{\alpha,q}(A)} := \sum_{i=1}^n \sup_{0<h<\text{dist}(A,\partial\Omega)} |h|^{-\alpha} \|\tau_{i,h} g\|_{L^q(A)},$$

is finite. In the following we shall also let $W^{0,q}(A) = \mathcal{N}^{0,q}(A) = L^q(A)$, with an intentional abuse of notation. It is well known that there exists a precise chain of inclusions between fractional Sobolev and Nikolski spaces (see, among the others, [99, Lemma 2.3] or [38]), which reads as

$$W^{\alpha,q}(A) \subset \mathcal{N}^{\alpha,q}(q) \subset W^{\alpha-\varepsilon,q}_{\text{loc}}(A) \qquad \text{for all } \varepsilon \in (0,\alpha), \tag{3.2.11}$$

see Proposition 3.8.

Parabolic and Banach-valued spaces. Here we spend a couple of words with regard to Banach valued spaces of functions, which are quite common in the parabolic setting. In general, take a measurable function $g : A \times B \to \mathbb{R}$, where $A \subset \mathbb{R}^l$ and $B \subset \mathbb{R}^m$ are bounded domains whose points are denoted respectively by y_1 and y_2. Let's moreover take two spaces of integrable functions X and Y, which could be defined over A and B, with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. By writing $g \in X(A; Y(B))$ we will simply mean that the scalar function $\|g(y_1, \cdot)\|_{Y(B)} : A \to \mathbb{R}$ belongs to $X(A)$.

In particular X and Y will always be or a Lebesgue space, or one of the previously defined spaces, and the sets A and B will be, alternatively, a bounded interval of $\mathbb{R}$ and a bounded open subset of $\mathbb{R}^n$. For the particular choice $X \equiv L^r$, $A \equiv (-T, 0)$, $Y(B) \equiv W^{\alpha,q}(\Omega)$ we have $g \in L^r(-T, 0; W^{\alpha,q}(\Omega))$ iff

$$\int_{-T}^0 \left(\int_\Omega \int_\Omega \frac{|g(x,t) - g(y,s)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy \right)^{r/q} dt < \infty;$$

whereas with the choice $E \equiv W^{\alpha,r}$, $A \equiv (-T,0)$, $F(B) \equiv L^q(\Omega)$ we obtain $g \in W^{\alpha,r}(-T,0;L^q(\Omega))$ iff

$$\int_{-T}^{0}\int_{-T}^{0} \frac{\left|\|g(\cdot,t)\|_{L^q(\Omega)} - \|g(\cdot,s)\|_{L^q(\Omega)}\right|^r}{|t-s|^{1+\alpha r}}\,dt\,ds < \infty;$$

similarly interchanging Ω and $(-T,0)$. We shall lighten again notations always denoting $X(-T,0;Y(A)) := X(-T,0;Y(A))$ and similarly, as we already did. Finally a straightforward inclusion, whose proof is simply given by triangle's inequality, in between some of these spaces is the following

REMARK 3.2. For $g \in L^q(-T,0;W^{\vartheta,q}(\Omega))$ we have the inequality

$$\|g\|_{W^{\vartheta,q}(\Omega;L^q(-T,0))} \le \|g\|_{L^q(-T,0;W^{\vartheta,q}(\Omega))}.$$

Obviously the previous Remark can be applied interchanging the sets Ω and $(-T,0)$ so that we also have the continuous immersion

$$L^q(\Omega;W^{\vartheta,q}(-T,0)) \subset W^{\vartheta,q}(-T,0;L^q(\Omega)). \tag{3.2.12}$$

Parabolic factional spaces. We say that a function $g \in L^q(\Omega_T)$ belongs to the **parabolic fractional Sobolev space** $W^{\vartheta,\tilde\vartheta;q}(\Omega_T)$, with $\vartheta,\tilde\vartheta \in (0,1)$ and $1 \le q < \infty$, if it belongs to $L^q(-T,0;W^{\vartheta,q}(\Omega)) \cap L^q(\Omega;W^{\tilde\vartheta,q}(-T,0))$, which is the space consisting of all functions $u \in L^q(-T,0;L^q(\Omega))$ such that

$$\begin{aligned}
[g]^q_{W^{\vartheta,\tilde\vartheta;q}(\Omega_T)} &:= \int_{-T}^{0} [g(\cdot,t)]^q_{W^{\vartheta,q}(\Omega)}\,dt + \int_\Omega [g(x,\cdot)]^q_{W^{\tilde\vartheta,q}(-T,0)}\,dx \\
&= \int_{-T}^{0}\int_\Omega\int_\Omega \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+\vartheta q}}\,dx\,dy\,dt \\
&\quad + \int_\Omega\int_{-T}^{0}\int_{-T}^{0} \frac{|g(x,t)-g(x,s)|^q}{|t-s|^{1+\tilde\vartheta q}}\,ds\,dt\,dx < \infty.
\end{aligned} \tag{3.2.13}$$

It is a Banach space if it is endowed with the norm, see [108],

$$\|g\|^q_{W^{\vartheta,\tilde\vartheta;q}(\Omega_T)} := \|g\|^q_{L^q(\Omega_T)} + [g]^q_{W^{\vartheta,\tilde\vartheta;q}(\Omega_T)}.$$

Also Nikolski spaces have a natural generalization when considered in parabolic shape (see [32]): precisely, we call the **parabolic Nikolski space** $\mathcal{N}^{\vartheta,\tilde\vartheta;q}(A\times J)$, for $A \Subset \Omega$, $J \Subset (-T,0)$ and $\vartheta,\tilde\vartheta \in (0,1]$, as the space of functions $\tilde g \in L^q(\Omega_T)$ such that

$$\begin{aligned}
[\tilde g]_{\mathcal{N}^{\vartheta,\tilde\vartheta,q}(A\times J)} &:= \sup_{0<|h|<\mathrm{dist}(J,\partial(-T,0))} |h|^{-\tilde\vartheta}\|\tau_h g\|_{L^q(A\times J)} \\
&\qquad + \sum_{i=1}^{n} \sup_{0<h<\mathrm{dist}(A,\partial\Omega)} |h|^{-\vartheta}\|\tau_{i,h} g\|_{L^q(A\times J)} < \infty.
\end{aligned}$$

Obviously there is a chain of inclusion similar to (3.2.11) between the $W^{\vartheta,\tilde\vartheta;q}_{\mathrm{loc}}$ and the $\mathcal{N}^{\vartheta,\tilde\vartheta;q}$ spaces, see Proposition 3.8 and Corollary 3.9

Function spaces properties. Here we collect some useful properties of the function spaces defined in the previous lines. We shall use them only in the parabolic framework, so they are directly stated for cylinders instead of balls. First we propose the following standard Hölder type inequality for Marcinkiewicz spaces:

LEMMA 3.3. *Let $g \in \mathcal{M}^p(A)$ with $p > 1$ and $A \subset \mathbb{R}^{n+1}$ a measurable subset with finite measure $|A| < \infty$. Then $g \in L^q(A)$ for any $1 \le q < p$. Moreover, we have the estimate*

$$\|g\|_{L^q(A)} \le \left(\frac{p}{p-q}\right)^{\frac{1}{q}} |A|^{\frac{1}{q}-\frac{1}{p}} \|g\|_{\mathcal{M}^p(A)} .$$

The Lemma below about the scaling properties of $\|\cdot\|_{L^\vartheta(p,q)}$, respectively $\|\cdot\|_{L\log L^\vartheta}$, is an immediate consequence of definitions (3.2.3), (3.2.7).

LEMMA 3.4. *Let $g \in L^\vartheta(p,q)(Q_\varrho(z_0))$ with $1 \le p < \infty$ and $0 < q \le \infty$. Then, the map $\tilde{g}(y,s) := g(x_0 + \varrho y, t_0 + \varrho^2 s)$ for $(y,s) \in Q_1$ belongs to $L^\vartheta(p,q)(Q_1)$ and*

$$\|\tilde{g}\|_{L^\vartheta(p,q)(Q_1)} = \varrho^{-\frac{\vartheta}{p}} \|g\|_{L^\vartheta(p,q)(Q_\varrho(z_0))}.$$

Similarly, if $g \in L\log L^\vartheta(Q_\varrho(z_0))$ then $\tilde{g} \in L\log L^\vartheta(Q_1)$ and

$$\|\tilde{g}\|_{L\log L^\vartheta(Q_1)} = \varrho^{-\vartheta} \|g\|_{L\log L^\vartheta(Q_\varrho(z_0))}.$$

Next theorem is a standard embedding theorem for the maximal function in Lorentz spaces. It can be easily inferred from [121, Theorem 7].

THEOREM 3.5. *Let $\beta \in [0,N)$ and $p > 1$ such that $\beta p < N$; moreover let $0 < q \le \infty$ and Q a parabolic cylinder in $\mathbb{R}^{n+1}$. Then there exists a constant $c \equiv c(n,p,\beta,q)$ such that for every map $g \in L(p,q)(\widetilde{Q})$ there holds*

$$\|M^*_{\beta,Q}(g)\|_{L(\frac{Np}{N-\beta p},q)(Q)} \le c\,\|g\|_{L(p,q)(Q)}.$$

Lower semi-continuity of quasi-norms. As we have pointed out after (2.3.16) the quantity $\|\cdot\|_{L^\vartheta(p,q)(\Omega)}$ is only a quasi-norm. Nevertheless, the mapping $g \mapsto \|g\|_{L^\vartheta(p,q)(\Omega)}$ is lower semi-continuous with respect to a.e. convergence. This can be seen as follows: Take $g_k \in L^\vartheta(p,q)(\Omega)$ with $g_k(x) \to g(x)$ a.e. on Ω as $k \to \infty$. Then by Fatou's Lemma we have

$$\big|\{x \in \Omega : |g(z)| > \lambda\}\big| \le \liminf_{k\to\infty} \big|\{x \in \Omega : |g_k(x)| > \lambda\}\big| \tag{3.2.14}$$

whenever $\lambda \ge 0$. For $q < \infty$ we use (3.2.14) and Fatou's Lemma in (3.2.1) to have that

$$\begin{aligned}
\|g\|_{L(p,q)(\Omega)} &= \left[p\int_0^\infty \Big(\lambda^p\big|\{x \in \Omega : |g(x)| > \lambda\}\big|\Big)^{\frac{q}{p}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{q}} \\
&\le \left[p\int_0^\infty \Big(\lambda^p \liminf_{k\to\infty}\big|\{x \in \Omega : |g_k(x)| > \lambda\}\big|\Big)^{\frac{q}{p}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{q}} \\
&= \liminf_{k\to\infty}\left[p\int_0^\infty \Big(\lambda^p\big|\{x \in \Omega : |g_k(x)| > \lambda\}\big|\Big)^{\frac{q}{p}} \frac{d\lambda}{\lambda}\right]^{\frac{1}{q}} \\
&= \liminf_{k\to\infty} \|g_k\|_{L(p,q)(\Omega)}.
\end{aligned}$$

When $q = \infty$, recalling the definition of the Marcinkiewicz norm, by (3.2.14) for fixed $\lambda > 0$ each of the functionals $g \mapsto \big(\lambda|\{x \in \Omega : |g(x)| > \lambda\}|\big)^{\frac{1}{p}}$ is lower semi-continuous with respect to a.e. convergence. The lower semi-continuity of the $\mathcal{M}^p$-norm now follows from the general fact that the supremum of an arbitrary family of lower semi-continuous functionals is still lower semi-continuous. The same argument also implies the lower semi-continuity of the quantities $\|\cdot\|_{L^\vartheta(p,q)(\Omega)}$ and $\|\cdot\|_{L\log L^\vartheta(\Omega)}$ since they are defined as the supremum over a family of balls of lower semi-continuous functionals.

Additivity of quasi-norms. The following elementary inequality holds

$$\Big(\sum_{i=1}^m a_i\Big)^\beta \le \max\{1, m^{\beta-1}\}\sum_{i=1}^m a_i^\beta \tag{3.2.15}$$

whenever $\beta > 0$ and a_i, $i = 1, \dots, m$ are non-negative numbers. We assume now that $\Omega \subset \bigcup_{i=1}^m \Omega_i$. Then, from (3.2.1) and (3.2.15) we infer that

$$\|g\|_{L(p,q)(\Omega)} \le G(m,p,q)\sum_{i=1}^m \|g\|_{L(p,q)(\Omega_i)}, \tag{3.2.16}$$

holds for every $0 < q \le \infty$, where $G(m,p,q) = 1$ if $1 \le q \le p$ or $q = \infty$, while $G(m,p,q) = m^{1/p-1/q}$ if $q > p$ and $G(m,p,q) = m^{1/p-1}$ if $0 < q < 1$.

For the following fractional Sobolev's embedding result see [**109**, Theorem 14.29] with minor changes, keeping in mind that $B^{s,p,p} \equiv W^{s,p}$, or also [**11**].

PROPOSITION 3.6 (Fractional Sobolev embedding). *Let $\Omega \subset \mathbb{R}^n$ a Lipschitz domain and let $g \in W^{\alpha,q}(\Omega)$ with $1 \le q < \infty$ and $\alpha \in (0,1)$ such that $\alpha q < n$. Then $g \in L^{nq/(n-\alpha q)}(\Omega)$ and there exists a constant $c \equiv c(n,\alpha,q,[\partial\Omega]_{0,1})$ such that*

$$\|g\|_{L^{Nq/(N-\alpha q)}(\Omega)} \le c\,\|g\|_{W^{\alpha,q}(\Omega)}.$$

The next result roughly says that we can increase integrability of a fractional Sobolev function up to lowering their fractional differentiability; its proof can be found in [**109**, Theorem 14.22], see also [**11**, Theorem 7.58].

PROPOSITION 3.7. *Let $g \in W^{\tilde\alpha,\tilde q}(\Omega)$ for $\tilde\alpha \in (0,1)$, $1 \le \tilde q < \infty$ and Ω as in Proposition 3.6. Then for every $\alpha \in (0,\tilde\alpha)$ there exists a constant $c \equiv c(n,p,\tilde\alpha,\alpha,[\partial\Omega]_{0,1})$ such that*

$$[g]_{W^{\alpha,q}(\Omega)} \le c\,[g]_{W^{\tilde\alpha,\tilde q}(\Omega)}$$

if $q \in (\tilde q, \infty)$ satisfies

$$\alpha - \frac{n}{q} = \tilde\alpha - \frac{n}{\tilde q}.$$

In particular $W^{\tilde\alpha,\tilde q}(\Omega) \subset W^{\alpha,q}(\Omega)$ for such q.

The proof of the following inclusion result between parabolic Nikolskii and fractional Sobolev spaces is a straightforward variation on the proof of the elliptic analog, see [**66, 98**]; for this parabolic formulation we refer to [**67**, Proposition 3.4].

PROPOSITION 3.8. *Let $g \in L^q(\Omega_T)$ with $1 \le q < \infty$ and assume that there exists $\bar\alpha \in (0,1]$, two open sets $\widetilde\Omega \Subset \Omega$ and $\tilde J \Subset (-T,0)$ such that*

$$\|\tau_{i,h} g\|_{L^q(\widetilde\Omega\times\tilde J)} \le S\,|h|^{\bar\alpha},$$

for some constant $S > 0$, for every $i \in \{1,\dots,n\}$ and every $h \in \mathbb{R}$ satisfying $0 < |h| < \mathcal{D}$, where $0 < \mathcal{D} \le \min\{1, \operatorname{dist}(\tilde\Omega,\partial\Omega)\}$. Then $g \in L^q(\tilde J; W^{\alpha,q}_{\rm loc}(\widetilde\Omega))$ for every $\alpha \in [0,\bar\alpha)$. In particular for each open set $\mathcal{O} \Subset \widetilde\Omega$ there exists a constant c depending on $q, \bar\alpha-\alpha, \mathcal{D}, \operatorname{dist}(\widetilde\Omega,\partial\Omega), \operatorname{dist}(\mathcal{O},\partial\widetilde\Omega), |\Omega|$ such that

$$\int_{\tilde J}\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy\,dt \le c\Big[S^q + \|g\|^q_{L^q(\widetilde\Omega\times\tilde J)}\Big].$$

Moreover if for some $\bar\beta \in (0,1]$ there holds

$$\|\tau_h g\|_{L^q(\widetilde\Omega\times\tilde J)}\,dt \le \tilde S\,|h|^{\bar\beta},$$

for every $h \in \mathbb{R}$ *satisfying* $0 < |h| < \widetilde{\mathcal{D}}$ *with* $0 < \widetilde{\mathcal{D}} \le \min\{1, \mathrm{dist}(J, \partial(-T, 0))\}$ *and with a constant* $\tilde{S} > 0$*, then* $g \in L^q(\tilde{\Omega}; W^{\beta,q}(\tilde{J}))$ *for every* $\beta \in [0, \bar{\beta})$*; moreover there exists a constant* $\tilde{c}$ *depending only on* $q, \bar{\beta} - \beta, \widetilde{\mathcal{D}}$, $\mathrm{dist}(\tilde{J}, \partial(-T, 0))$ *and* T *such that*

$$\int_{\tilde{\Omega}} \int_{\tilde{J}} \int_{\tilde{J}} \frac{|g(x,t) - g(x,s)|^q}{|t-s|^{1+\beta q}}\, dt\, ds\, dx \le \tilde{c} \left[S^q + \|g\|^q_{L^q(\tilde{\Omega} \times \tilde{J})} \right].$$

We will always use the two results of the previous Proposition coupled together with the choice $\bar{\beta} \equiv \bar{\alpha}/2$; so we state explicitly the following Corollary

COROLLARY 3.9. *Let* $g \in L^q(\Omega_T)$ *satisfy the following estimate*

$$\|\tau_{h^2} g\|_{L^q(\tilde{\Omega} \times \tilde{J})} + \sum_{i=1}^n \|\tau_{i,h} g\|_{L^q(\tilde{\Omega} \times \tilde{J})} \le S\, |h|^{\bar{\vartheta}},$$

for every $0 < |h| < \mathcal{D}$*, with* $\tilde{\Omega}, \tilde{J}$ *as in the Proposition 3.8,* $\bar{\vartheta} \in (0, 1]$*,* $S > 0$ *and* $0 < \mathcal{D} \le \min\{1, \mathrm{dist}(\tilde{\Omega}, \partial\Omega), \mathrm{dist}(\tilde{J}, \partial(-T, 0))\}$*. Then* $g \in W^{\vartheta,\vartheta/2;q}_{\mathrm{loc}}(\tilde{\Omega} \times \tilde{J})$ *for every* $\vartheta \in [0, \bar{\vartheta})$ *with the explicit estimate*

$$[g]_{W^{\vartheta,\vartheta/2;q}(\mathcal{O} \times \mathcal{J})} \le c \left[S + \|g\|_{L^q(\tilde{\Omega} \times \tilde{J})} \right].$$

for $\mathcal{O} \Subset \tilde{\Omega}$ *and* $\mathcal{J} \Subset \tilde{J}$*. The constant* c *depends on* $q, \bar{\vartheta} - \vartheta, \mathcal{D}, \mathrm{dist}(\widetilde{\Omega}, \partial\Omega)$*,* $\mathrm{dist}(\mathcal{O}, \partial\widetilde{\Omega})$*,* $\mathrm{dist}(\tilde{J}, \partial(-T, 0))$*,* $|\Omega|, T$*.*

The final statement of this Section is an appropriate version of the fractional Poincaré inequality. The proof is simple and follows widely the classical ones in the elliptic setting, see [66, 67], so we skip it.

LEMMA 3.10. *Let* $g \in W^{\vartheta,\vartheta/2;q}(Q_\rho)$ *for* $\vartheta \in (0, 1)$ *and* $q \ge 1$*. Then there holds*

$$\fint_{Q_\rho} |g - (g)_{Q_\rho}|\, dz \le c\, \rho^{\vartheta - \frac{n+2}{q}} [g]_{W^{\vartheta,\vartheta/2;q}(Q_\rho)},$$

with a constant $c \equiv c(n, q)$*.*

We stress here that it is also possible to obtain a Poincaré-type inequality involving only the spatial derivatives of solutions to certain parabolic problems, see Proposition 7.11 in Chapter 7.

3.3. Technical tools

Here first we collect standard iteration and algebraic Lemmas which will be used in various point in the proofs in next Chapters. Subsequently we shall give the argument which allows to test the weak formulation of a parabolic problem with the solution itself, i.e. the regularizing Steklov's averaging. Finally we want to give a parabolic version of the Calderón-Zygmund covering, proposing also for the convenience of the reader its proofs, which is a revisitation of the original, elliptic one.

The following Lemmas are standard iteration argument and can for instance be found in [81, Lemma 6.1 & Lemma 7.3].

LEMMA 3.11. *Let* $\phi : [R, 2R] \to [0, \infty)$ *be a function such that*

$$\phi(r_1) \le \frac{1}{2}\phi(r_2) + \mathcal{A} + \frac{\mathcal{B}}{(r_2 - r_1)^\beta} \qquad \text{for every } R \le r_1 < r_2 \le 2R,$$

where $\mathcal{A}, \mathcal{B} \ge 1$ *and* $\beta > 0$*. Then*

$$\phi(R) \le c(\beta) \left[\mathcal{A} + \frac{\mathcal{B}}{R^\beta} \right].$$

LEMMA 3.12. *Let* $\varphi\colon [0,R_0]\to[0,\infty)$ *be a non-decreasing function such that*

$$\varphi(\varrho)\le A\left[\left(\frac{\varrho}{R}\right)^{\delta_0}+\varepsilon\right]\varphi(R)+\mathcal{B}R^{\delta_1} \tag{3.3.1}$$

for every $0<\varrho\le R\le R_0$*, where* $A,\mathcal{B}\ge 0$ *and* $0<\delta_1<\delta_0$*. Then there exist* $\varepsilon_0=\varepsilon_0(\delta_0,\delta_1,A)>0$ *and* $c_1\equiv c_1(\delta_0,\delta_1,A)$ *such that whenever* (3.3.1) *holds for some* $0<\varepsilon\le\varepsilon_0$ *then*

$$\varphi(\varrho)\le c_1\left[\left(\frac{\varrho}{R}\right)^{\delta_1}\varphi(R)+\mathcal{B}\varrho^{\delta_1}\right]\qquad \text{for every } 0<\varrho\le R\le R_0.$$

The following reverse Hölder type inequality encodes part of the self-improving properties of reverse Hölder's inequalities and allows to reduce the integral power on the right-hand side below the natural exponent for the linear growth problems:

LEMMA 3.13. *Let* $g:\Omega_T\to\mathbb{R}^n$ *an integrable map such that*

$$\left[⨍_{Q_\rho}|g|^{\chi_0}\,dz\right]^{1/\chi_0}\le c\left[⨍_{Q_{2\rho}}\big(s+|g|\big)^2\,dz\right]^{1/2}$$

holds whenever $Q_{2\rho}\Subset\Omega_T$*, where* $s\ge 0$*,* $\chi_0>2$ *and* $c>0$*. Then, for every* $\sigma\in(0,2]$*, there exists a constant* $c_0=c_0(n,\sigma,c)$ *such that*

$$\left[⨍_{Q_\rho}|g|^2\,dz\right]^{1/2}\le c_0\left[⨍_{Q_{2\rho}}\big(s+|g|\big)^\sigma\,dz\right]^{1/\sigma}$$

for every $Q_{2\rho}\Subset\Omega_T$*.*

The following one, on the other hand, is a less known version and it is particularly useful when dealing with $p\neq 2$-growth parabolic equations. See [**102**, Lemma 5.1].

LEMMA 3.14. *Let* ν *be a non-negative Borel measure with finite total mass. Let moreover* $1<q<p<\infty$ *and* $\xi\ge 0$*, and let* $\{\theta U\}$ *be a family of open sets with the property*

$$\theta_1 U\subset\theta_2 U\subset 1U=U \tag{3.3.2}$$

whenever $0<\theta_1<\theta_2<1$*. If* $w\in L^q(U)$ *is a non-negative function satisfying*

$$\left(\int_{\theta_1 U}w^p\,d\nu\right)^{1/p}\le\frac{c_0}{(\theta_2-\theta_1)^\xi}\left(\int_{\theta_2 U}w^q\,d\nu\right)^{1/q} \tag{3.3.3}$$

for all $1/2\le\theta_1<\theta_2\le 1$*, then there is a positive constant* $c\equiv c(c_0,\xi,p,q)$ *such that*

$$\left(\int_{\theta U}w^p\,d\nu\right)^{1/p}\le\frac{c}{(1-\theta)^{\xi'}}\int_U w\,d\nu, \tag{3.3.4}$$

for all $0<\theta<1$*, where* $\xi'=\xi q(p-1)/(p-q)$*.*

The following algebric Lemmas are a useful tool when dealing with p-growth problems. The continuous dependence of the constant with respect to p allows to instead consider a constant depending on γ_1,γ_2 when $p\in[\gamma_1,\gamma_2]$, so they fit also the variable growth situation described in Section 4.4.

LEMMA 3.15. *Let* $p\in[\gamma_1,\gamma_2]$*. Then there exists a constant* $c\equiv c(n,N,\gamma_1,\gamma_2)$ *such that for any* A*,* $B\in\mathbb{R}^{Nn}$*, not both zero, there holds*

$$\big(|A|^2+|B|^2\big)^{(p-2)/2}|B-A|^2\le c\big\langle|B|^{p-2}B-|A|^{p-2}A,B-A\big\rangle.$$

LEMMA 3.16 ([47]). *Let $p \in [\gamma_1, \gamma_2]$. Then there exists a constant $c_\ell \equiv c_\ell(\gamma_2)$ such that for any A, $B \in \mathbb{R}^{Nn}$ there holds*

$$|A|^p \leq c_\ell\, |B|^p + c_\ell \left(|A|^2 + |B|^2\right)^{\frac{p-2}{2}} |B - A|^2.$$

Parabolic Dirichlet problems and Steklov averages. Here we briefly explain what has to be intended when dealing with solutions of Cauchy-Dirichlet problems of the type

$$\begin{cases} \partial_t u - \operatorname{div} a(x,t,Du) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases}$$

with for instance a satisfying assumptions (4.3.4). Avoiding any further consideration upon the concepts of solution, here we want to specify the meaning of $u = 0$ on $\partial_{\mathcal{P}}\Omega_T$ and a further question. The fact that u vanishes on the lateral boundary is prescribed by denoting $u(\cdot, t) \in W_0^{1,p-1}(\Omega)$ for a.e. t: the initial boundary value $u(x, -T) = 0$ should be on the other hand understood in the L^1 sense, which means that

$$\lim_{h \searrow 0} \frac{1}{h} \int_{-T}^{-T+h} \int_\Omega |u(x,t)|\, dx\, dt = 0.$$

The other question we want to explain here is the following: as we already said, one usually gets energy estimates testing the equation with the solutions itself; eventually one can truncate the solution when working with *very weak solutions*, but in any case a problem here appears. The solution to parabolic problems in general enjoys a low degree a regularity with respect to time, while for the test function φ we need the existence of the time derivative φ_t in L^2, see the weak formulation (4.1.4).

Therefore here is mandatory, but by now standard, to regularize the solution itself before using it as a test function. This can be done by convolutions or Steklov averaging; in this manuscript we chose the second option and now we explain in which this consists. For $h > 0$ and $t \in (-T, 0)$ we define the so-called **Steklov average** of u by

$$u_h(x,t) := \begin{cases} \displaystyle\frac{1}{h} \int_t^{t+h} u(x, \tilde{t})\, d\tilde{t} & \text{if } t \leq -h, \\ 0 & \text{if } t > -h. \end{cases} \tag{3.3.5}$$

This definition naturally extends to the case when h is negative, averaging backward instead of forward. Being u a weak solution of (4.1.1) with $\mu \in L^1(\Omega_T)$ and u_h the Steklov average of u, the slicewise equality

$$\int_\Omega \Big[\partial_t [u_h]\varphi + \langle [a(\cdot, t, Du)]_h, D\varphi\rangle\Big]\, dx = \int_\Omega \varphi \mu_h\, dx,$$

holds true for any $\varphi \in C_c^\infty(\Omega)$ or $W_0^{1,p}(\Omega)$ and for a.e. $t \in (-T, 0)$ (see [54, Chapter 2]). In Section 5.1-5.2 we shall show, with an example, how to implement this regularizing procedure.

Calderón-Zygmund coverings. Let $\mathcal{Q}_0 = \mathcal{Q}_R(z_0) = C_R(x_0) \times (t_0 - R^2, t_0 + R^2)$ be a parabolic cylinder in $\mathbb{R}^{n+1}$ with horizontal cross section being a cube. By $\mathcal{D}(\mathcal{Q}_0)$ we shall denote the class of all dyadic parabolic cylinders obtained from $\mathcal{Q}_0$ by a finite number of dyadic subdivisions. The construction of a dyadic subdivision is as follows: If $\mathcal{Q}_0$ is as above then we subdivide $C_R(x_0)$ into 2^n congruent sub-cubes C' having sides parallel to $C_R(x_0)$ and $(t_0 - R^2, t_0 + R^2)$ into four disjoint intervals I' of equal length $R^2/2$. Then, the set of all parabolic sub-cylinders obtained by this dyadic subdivision consist of all cylinders of the form $C' \times I'$. The total number of sub-cylinders obtained from a parabolic cylinder $\mathcal{Q}_0$ by one dyadic subdivision is 2^N, $N = n + 2$. We note that $\mathcal{Q}_0 \notin \mathcal{D}(\mathcal{Q}_0)$. For later use we mention a few simple facts of the class $\mathcal{D}(\mathcal{Q}_0)$: First, if $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{D}(\mathcal{Q}_0)$ then either $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$, or one of the parabolic cylinders contains the other one, i.e. $\mathcal{Q}_1 \subset \mathcal{Q}_2$ or $\mathcal{Q}_2 \subset \mathcal{Q}_1$. We shall denote $\widetilde{\mathcal{Q}} \in \mathcal{D}(\mathcal{Q}_0)$ **the predecessor** of $\mathcal{Q}$ if $\mathcal{Q}$ has been obtained by exactly one dyadic subdivision from the parabolic cylinder $\tilde{\mathcal{Q}}$. The following is a Calderón-Zygmund-Krylov-Safanov type covering lemma in the parabolic setting; for the elliptic (classical) version we refer to [35].

PROPOSITION 3.17. *Let $\mathcal{Q}_0 \subset \mathbb{R}^{n+1}$ be a parabolic cylinder. Assume that $X \subset Y \subset \mathcal{Q}_0$ are measurable sets such that the following properties (i) and (ii) hold:*

(i) there exist $\delta > 0$ such that $|X| < \delta|\mathcal{Q}_0|$;
(ii) if $\mathcal{Q} \in \mathcal{D}(\mathcal{Q}_0)$, then $|X \cap \mathcal{Q}| > \delta|\mathcal{Q}|$ implies $\widetilde{\mathcal{Q}} \subset Y$, where $\widetilde{\mathcal{Q}}$ denotes the predecessor of $\mathcal{Q}$.

Then there holds $|X| < \delta|Y|$.

The proof of the preceding proposition can be inferred using arguments from [35]. For convenience of the reader we give the simple adaptation to our parabolic set up. The starting point is the following version of the classical Calderón-Zygmund type covering lemma.

LEMMA 3.18. *Let $\mathcal{Q}_0 \subset \mathbb{R}^{n+1}$ be a parabolic cylinder and X a measurable subset of $\mathcal{Q}_0$ satisfying*

$$0 < |X| < \delta|\mathcal{Q}_0|$$

for some $0 < \delta < 1$. Then there exists a sequence $(\mathcal{Q}_i)_{i\in\mathbb{N}}$ of disjoint dyadic sub-cylinders of $\mathcal{Q}_0$ such that there holds:

(i) $\big|X \setminus \bigcup_{i=1}^\infty \mathcal{Q}_i\big| = 0$,
(ii) $|X \cap \mathcal{Q}_i| \ge \delta|\mathcal{Q}_i|$ and
(iii) $|X \cap \widetilde{\mathcal{Q}}| < \delta|\widetilde{\mathcal{Q}}|$ if $\widetilde{\mathcal{Q}} \in \mathcal{D}(\mathcal{Q}_0)$ and $\mathcal{Q}_i \subsetneqq \widetilde{\mathcal{Q}}$.

PROOF. We divide $\mathcal{Q}_0$ into 2^N dyadic sub-cylinders $\mathcal{Q}_1^{(j)}$ and select those satisfying

$$\big|X \cap \mathcal{Q}_1^{(j)}\big| \ge \delta\big|\mathcal{Q}_1^{(j)}\big|.$$

Now, we take those cylinders that were not chosen, divide each of them again into 2^N dyadic sub-cylinders and repeat the selection argument from above. Proceeding iteratively in this way we obtain a sequence of disjoint dyadic cylinders $\mathcal{Q}_i \in \mathcal{D}(\mathcal{Q}_0)$, $i \in \mathbb{N}$. By construction each of these cylinders satisfies (ii) and (iii). For $z \in \mathcal{Q}_0 \setminus \bigcup_{i=1}^\infty \mathcal{Q}_i$ we have a sequence of dyadic cylinders $\mathcal{P}_k$ with $|\mathcal{P}_k| \to 0$ as $k \to \infty$, each of them containing z, such that

$$|\mathcal{P}_k \cap X| < \delta|\mathcal{P}_k| \quad \text{or equivalently} \quad ⨍_{\mathcal{P}_k} \chi_X(\tilde z)\, d\tilde z = \frac{|\mathcal{P}_k \cap X|}{|\mathcal{P}_k|} < \delta < 1.$$

By Lebesgue's differentiation theorem the left-hand side of the preceding inequality converges to $\chi_X(z)$ for a.e. z as $k \to \infty$, and therefore we have $z \in \mathcal{Q}_0 \setminus X$ for a.e. $z \in \mathcal{Q}_0 \setminus \bigcup_{i=1}^\infty \mathcal{Q}_i$. Hence $\big|X \setminus \bigcup_{i=1}^\infty \mathcal{Q}_i\big| = 0$, proving finally (i). □

PROOF OF PROPOSITION 3.17. We apply Lemma 3.18 to have a sequence of disjoint dyadic cylinders $(\mathcal{Q}_i)_{i\in\mathbb{N}}$ covering almost all of X. By (ii) of Lemma 3.18 we have $|X \cap \mathcal{Q}_i| > \delta|\mathcal{Q}_i|$; therefore by assumption (ii) the predecessor $\widetilde{\mathcal{Q}}_i$ of $\mathcal{Q}_i$ is contained in Y. Now, from the sequence of predecessors $(\widetilde{\mathcal{Q}}_i)_{i\in\mathbb{N}}$ we can extract a sub-covering

$(\widetilde{\mathcal{Q}}_i)_{i\in\mathfrak{K}}$ of X, where the $\widetilde{\mathcal{Q}}_i$ are pairwise disjoint and $\mathfrak{K} \subset \mathbb{N}$. Then, using Lemma 3.18, (iii), the fact that $\widetilde{\mathcal{Q}}_i \subset Y$, as well as the disjointness of the $\widetilde{\mathcal{Q}}_i$ for $i \in \mathfrak{K}$ we obtain

$$|X| \leq \sum_{i\in\mathfrak{K}} |X \cap \widetilde{\mathcal{Q}}_i| < \delta \sum_{i\in\mathfrak{K}} |\widetilde{\mathcal{Q}}_i| \leq \delta |Y|,$$

proving the claim of Proposition 3.17. □

CHAPTER 4

New results

In this section we shall present the original results obtained in this thesis; as a consequence, we shall be necessarily more detailed than in the previous review chapter. The rest of the manuscript will be structured as follows: in this chapter we shall present the statement and everything will be needed in order to put them in the right context, included specific bibliographical informations not presented in the previous chapter. We are going to devote a different section to every argument we are going to present. *All the proofs will make up a subsequent chapter with the same name of the corresponding section.*

4.1. Fractional differentiability for nonlinear heat equation

In this first paragraph we are going to deal with nonlinear heat equations of the type (2.5.1) where the vector field $a(\cdot)$ will satisfy assumptions similar to (2.4.5); we shall also allow for coefficient Lipschitz regular with respect the spatial variables. Our goal will be to provide fractional differentiability results for the spatial gradient of *very weak solutions* similar to those obtained in [**118**] and described in Theorem 2.4 in the elliptic case. With respect to this elliptic result, we here will focus on the case $p = 2$. In particular we are going to consider inhomogeneous Cauchy-Dirichlet problems of the type

$$\begin{cases} \partial_t u - \operatorname{div} a(x,t,Du) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{4.1.1}$$

where μ a signed Borel measure with finite total mass, $|\mu|(\Omega_T) < \infty$. Here, and throughout all the chapter, $\Omega \subset \mathbb{R}^n$ will be a bounded domain with $n \geq 2$ and Ω_T will denote the parabolic cylinder $\Omega \times (-T, 0)$; see (3.1.6) for the definition of its parabolic boundary. $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be a Carathéodory vector field fulfilling the following monotonicity and continuity conditions:

$$\begin{cases} \big\langle a(x,t,\xi_1) - a(x,t,\xi_2),\, \xi_1 - \xi_2 \big\rangle \geq \nu\, |\xi_1 - \xi_2|^2, \\ \big|a(x,t,\xi_1) - a(x,t,\xi_2)\big| \leq L|\xi_1 - \xi_2|, \\ |a(x,t,0)| \leq Ls, \\ \big|a(x_1,t,\xi) - a(x_2,t,\xi)\big| \leq L\,|x_1 - x_2|(s + |\xi|) \end{cases} \tag{4.1.2}$$

for all $x, x_1, x_2 \in \Omega$, $t \in (-T, 0)$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$, with constants

$$0 < \nu \leq 1 \leq L \tag{4.1.3}$$

and degeneracy parameter $s \in [0, 1]$. Note that no regularity of $a(\cdot)$ but measurability is assumed with respect to t. The notion of solution we are going to deal with is the SOLA described in Section 2.3. Similarly as in the elliptic case, we call a *very weak solution* to $(4.1.1)_1$ a function $u \in L^1(-T, 0; W^{1,1}(\Omega))$ solving the distributional formulation

$$\int_{\Omega_T} \Big[-u\, \varphi_t + \langle a(x,t,Du), D\varphi \rangle \Big]\, dz = \int_{\Omega_T} \varphi\, d\mu, \tag{4.1.4}$$

for every $\varphi \in \mathcal{C}_c^\infty(\Omega_T)$. Existence of such distributional solutions has been proved using the already described approximation method, for the general nonlinear parabolic case $p > 2 - 1/(N-1)$, by Boccardo & Gallouët in [25] and Boccardo, Gallouët, Dall'Aglio & Orsina, see [24]. In particular, in the case $p = 2$ these results read as follows: [25] it is proved the existence of a SOLA belonging to $L^q(-T, 0; W_0^{1,q}(\Omega))$ for every exponent q satisfying

$$1 \leq q < \frac{N}{N-1}, \tag{4.1.5}$$

while in [24] the result is refined in an anisotropic sense, in the sense that the solution is shown to belong to $L^r(-T, 0; W_0^{1,q}(\Omega))$, where the couple of exponents (r, q) satisfies the following bounds:

$$1 \leq p < \frac{n}{n-1}, \qquad 1 \leq r < 2, \qquad \frac{2}{r} + \frac{n}{q} > n + 1. \tag{4.1.6}$$

The first Theorem we present here concerns the differentiability and not anymore the integrability of the solution found in [25]; we find that, depending of the exponent satisfying (4.1.5), we have a certain level of (fractional) differentiability that reduces to zero as $q \to N/(N-1)$. The following Theorem quantifies this rough statement:

THEOREM 4.1 ([20]). *Let $u \in L^1(-T, 0; W_0^{1,1}(\Omega))$ be a SOLA to equation (4.1.1) under the assumptions (4.1.2) on the vector field $a(\cdot)$; then*

$$Du \in W_{\text{loc}}^{\delta, \frac{\delta}{2}; q}(\Omega_T; \mathbb{R}^n), \tag{4.1.7}$$

for all q as in (4.1.5) and with

$$0 < \delta < \frac{N}{q} - (N-1) =: \delta(q). \tag{4.1.8}$$

The estimate above in particular implies that

$$Du \in W_{\text{loc}}^{1-\varepsilon, \frac{1-\varepsilon}{2}; 1}(\Omega_T; \mathbb{R}^n)$$

for all $\varepsilon \in (0, 1)$. The most general version of Theorem 2.4, which can be found in [118], asserts that in the elliptic case, under completely similar hypotheses on the vector field, the regularity for the gradient of SOLA is

$$Du \in W_{\text{loc}}^{\delta, q}(\Omega; \mathbb{R}^n), \qquad \text{for all} \qquad 0 < \delta < \frac{n}{q} - (n-1),$$

which is completely analogous to (4.1.7). We moreover stress that using slice-wise fractional Sobolev's embedding (2.3.5) it is possible to recover, at least locally, the anisotropic integrability result proved in [24], also getting a different kind of integrability property. Indeed we have

COROLLARY 4.2 ([20]). *SOLA u to problem (4.1.1) enjoy the following integrability property:*

$$Du \in L_{\text{loc}}^r(-T, 0; L_{\text{loc}}^q(\Omega)) \cap L_{\text{loc}}^q(\Omega; L_{\text{loc}}^r(-T, 0))$$

for all (r, q) satisfying (4.1.6).

Theorem 4.1 comes along with the following local estimates of Calderón-Zygmund type:

THEOREM 4.3 (Local Calderón-Zygmund estimates, [20]). *Under the assumptions of Theorem 4.1, let q be as in (4.1.5), δ as in (4.1.8), let $\sigma := \delta\, q$ and $\sigma \in (0, N-(N-1)/q =: \sigma(q))$. Then for every cylinder $Q_\rho \equiv B_\rho \times \Lambda_\rho \Subset \Omega_T$ it holds*

$$\int_{I_{\rho/2}}\int_{B_{\rho/2}}\int_{B_{\rho/2}} \frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\sigma}}\,dx\,dy\,dt$$
$$+\int_{B_{\rho/2}}\int_{I_{\rho/2}}\int_{I_{\rho/2}} \frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\sigma/2}}\,dt\,ds\,dx$$
$$\leq c\,\rho^{-\sigma}\int_{Q_\rho}(s+|Du|)^q\,dz + c\,\rho^{\sigma(q)-\sigma}\big[|\mu|(\overline{Q_\rho})\big]^q \qquad (4.1.9)$$

for a constant $c \equiv c(n,\nu,L,q,\delta)$. Furthermore, for any open subset ${\Omega_T}' \equiv \Omega' \times J' \Subset \Omega_T$ the estimate

$$\int_{{\Omega_T}'}|Du|^q\,dz + \int_{J'}\int_{\Omega'}\int_{\Omega'} \frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\sigma}}\,dx\,dy\,dt$$
$$+\int_{\Omega'}\int_{J'}\int_{J'} \frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\sigma/2}}\,dt\,ds\,dx \leq c\,\big[s+|\mu|(\Omega_T)\big]^q \qquad (4.1.10)$$

holds true with a constant c depending on n, L/ν, q, $\mathrm{dist}_{\mathrm{par}}({\Omega_T}',\partial\Omega_T)$, $|\Omega|$ and T.

Finally using standard immersion theorems between fractional Sobolev spaces we can deduce the following anisotropic regularity result

THEOREM 4.4 ([20]). *Let $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ be a SOLA to problem (4.1.1). Then we have:*

(i) *for all (r,q) satisfying (4.1.6) and the condition $r < q$ we have*

$$Du \in L^r_{\mathrm{loc}}(-T,0;W^{\delta,q}_{\mathrm{loc}}(\Omega)) \cap W^{\delta,q}_{\mathrm{loc}}(\Omega;L^r_{\mathrm{loc}}(-T,0)) \qquad \text{for all } \delta \in [0,\tilde{\delta}(r,q));$$

(ii) *for all (r,q) satisfying (4.1.6) and the condition $r > q$ on the other hand*

$$Du \in L^q_{\mathrm{loc}}(\Omega;W^{\delta/2,r}_{\mathrm{loc}}(-T,0)) \cap W^{\delta/2,r}_{\mathrm{loc}}(-T,0;L^q_{\mathrm{loc}}(\Omega))$$

for all $\delta \in [0,\tilde{\delta}(r,q))$. In both cases $\tilde{\delta}$ denotes the function

$$\tilde{\delta}(r,q) := \frac{n}{q} + \frac{2}{r} - (n+1) > 0 \qquad \text{for } (r,q) \text{ satisfying (4.1.6).}$$

For the definition and the basic properties of the fractional Sobolev spaces above, we refer the reader to Section 3.2.

4.2. Adams theorems for nonlinear heat equations

In this Section we indeed focus on the sub-dual case for the Cauchy-Dirichlet problem (4.1.1), that is we investigate regularity properties of solutions to (4.1.1) in the case $\mu \equiv g$ is a Lebesgue function with low integrability property in the sense of Paragraph 2.3, adapted to the parabolic case $p = 2$ and the vector-field $a\colon \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be a Carathéodory map which satisfies only the growth and monotonicity conditions:

$$\begin{cases} \langle a(x,t,\xi_1)-a(x,t,\xi_2),\xi_1-\xi_2\rangle \geq \nu|\xi_1-\xi_2|^2, \\ |a(x,t,\xi)| \leq L\big(1+|\xi|\big) \end{cases} \qquad (4.2.1)$$

for every choice of $(x,t) \in \Omega_T$, $\xi_1,\xi_2 \in \mathbb{R}^n$, and ν, L as in (4.1.3). As in the previous Paragraph we shall deal with the (unique) solution obtained by approximation and also in this case the existence and integrability question have been faced by Boccardo, Dall'Aglio, Gallouët and Orsina who in [24, Theorem 1.9] prove the existence of a solution $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ such that

$$Du \in L^q(\Omega_T), \qquad \text{with} \qquad q = \frac{N\gamma}{N-\gamma}, \qquad (4.2.2)$$

provided the datum $\mu \equiv g$ satisfies

$$g \in L^{\gamma}(\Omega_T) \qquad \text{for some} \qquad 1 < \gamma < \frac{2N}{N+2}.$$

Moreover, by Sobolev's parabolic embedding the solution u belongs to $L^{\sigma}(\Omega_T)$ with σ given by $\sigma = \frac{N\gamma}{N-2\gamma}$; this result is optimal in the scale of Lebesgue spaces. Note that the exponent $2N/(N+2)$ is the analog of the duality exponent we met for instance in (2.3.9), the parabolic Sobolev's embedding gives

$$L^2(-T,0;W_0^{1,2}(\Omega)) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega_T)$$

and therefore the space $L^{\frac{2N}{N+2}}(\Omega_T)$ embeds into the dual of the energy space. Coming back to (4.2.2), here we ask for a more accurate scale to describe regularity of Du in dependence on the inhomogeneity g. Before stating the result, we stress here that, since the notion of solution to measure data problems holds uniqueness in the case of L^1 data, our results apply to every class of solutions which provides uniqueness in the case of data in L^1 data. In particular, all our regularity results could therefore also be stated in terms of renormalized solution, [126] or entropy solutions.

The first result we want to propose here is the extension to nonlinear parabolic equations with linear growth of the elliptic result of Theorem 2.7. Here we focus our consideration on the range of coefficients

$$1 < \gamma \leq \frac{2\vartheta}{\vartheta+2} \qquad \text{and} \qquad 2 < \vartheta \leq N. \tag{4.2.3}$$

The following theorem is a special case of Theorem 4.8, which is the main theorem of this paragraph, being concerned with the more general Lorentz-Morrey space regularity. Indeed, Theorem 4.5 will follow from Theorem 4.8 by the special choice $q = \gamma$.

THEOREM 4.5 (Nonlinear parabolic Adams theorem, [21]). *Let $g \in L^{\gamma,\vartheta}(\Omega_T)$ with γ, ϑ as in (4.2.3) and let $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ be a solution to (4.1.1) under the assumptions (4.2.1). Then*

$$Du \in L^{\frac{\vartheta\gamma}{\vartheta-\gamma},\vartheta}_{\text{loc}}(\Omega_T;\mathbb{R}^n).$$

Moreover, there exists a constant $c \equiv c(n,\nu,L,\gamma,\vartheta)$ such that the quantitative local estimate

$$\|Du\|_{L^{\frac{\vartheta\gamma}{\vartheta-\gamma},\vartheta}(Q_R)} \leq c\, R^{\frac{\vartheta-\gamma}{\gamma}-N} \big\||Du|+1\big\|_{L^1(Q_{2R})} + c\,\|g\|_{L^{\gamma,\vartheta}(Q_{2R})}$$

holds for any parabolic cylinder $Q_{2R} \subset \Omega_T$.

We recall here that the parabolic Morrey space $L^{\gamma,\vartheta}(\Omega_T)$ for $0 \leq \vartheta \leq N$ is simply defined substituting in definition (2.3.12) balls B_R with parabolic cylinders Q_R and the resulting Morrey norm is given by

$$\|g\|^{\gamma}_{L^{\gamma,\vartheta}(\Omega_T)} := \sup_{Q_R \subset \Omega_T} R^{\vartheta-N} \int_{Q_R} |g|^{\gamma}\, dz. \tag{4.2.4}$$

Note that the special choice $\vartheta = N$ in the above theorem gives back (4.2.2). On the other hand, Theorem 4.5 fails in the borderline case $\gamma = 1$. Here we recall that also in the elliptic, and even linear case, (2.3.11) and (2.3.14) fail for the borderline choice $\gamma = 1$, and $L \log L$-integrability on the inhomogeneity has to be imposed. Analogously we have to impose some further logarithmic integrability on the datum g and we obtain the following:

THEOREM 4.6 (Borderline parabolic Adams theorem, [21]). *Under the assumptions of Theorem 4.5 and being* $g \in L^{1,\vartheta}(\Omega_T) \cap L\log L(\Omega_T)$ *with* $2 \leq \vartheta \leq N$*, the solution* $u \in L^1(-T, 0; W_0^{1,1}(\Omega))$ *is such that*

$$Du \in L_{\text{loc}}^{\frac{\vartheta}{\vartheta-1}}(\Omega_T; \mathbb{R}^n).$$

Moreover, the quantitative local estimate

$$\left[\fint_{Q_R} |Du|^{\frac{\vartheta}{\vartheta-1}}\, dz \right]^{\frac{\vartheta-1}{\vartheta}} \leq c \fint_{Q_{2R}} \big(|Du| + 1\big)\, dz$$
$$+ c\, \|g\|_{L^{1,\vartheta}(Q_{2R})}^{\frac{1}{\vartheta}} \left[\fint_{Q_{2R}} |g| \log\left(e + \frac{g}{(|g|)_{Q_{2R}}}\right) dz \right]^{\frac{\vartheta-1}{\vartheta}}, \quad (4.2.5)$$

holds for any parabolic cylinder $Q_{2R} \subset \Omega_T$ *with a constant* $c \equiv c(n, \nu, L, \vartheta)$.

Moreover we mention that the particular choice $\vartheta = 2$ is allowed in the above theorem, since we are in the case $\gamma = 1$. With this particular choice we reach the maximal regularity, that is $g \in L^{1,2}(\Omega_T) \cap L\log L(\Omega_T) \Longrightarrow Du \in L^2_{\text{loc}}(\Omega_T; \mathbb{R}^n)$.

REMARK 4.7. In the case we don't impose a $L\log L$ condition on g, still an estimate in Marcinkiewicz spaces holds true:

$$g \in L^{1,\vartheta}(\Omega_T), \quad 2 \leq \vartheta \leq N \qquad \Longrightarrow \qquad Du \in \mathcal{M}_{\text{loc}}^{\frac{\vartheta}{\vartheta-1}}(\Omega_T; \mathbb{R}^n);$$

compare with (2.3.7).

As we have already mentioned above, the result of Theorem 4.5 is a particular case of more general results in Lorentz-Morrey spaces. We recall that the space $L^\vartheta(p,q)(\Omega_T)$ is defined by asking

$$\|g\|_{L^\vartheta(\gamma,q)(\Omega_T)} := \sup_{Q_R \subset \Omega_T} R^{\frac{\vartheta-N}{\gamma}} \|g\|_{L(\gamma,q)(Q_R)} < \infty.$$

The main theorem of this Paragraph is indeed the following:

THEOREM 4.8 ([21]). *Let* $u \in L^1(-T, 0; W_0^{1,1}(\Omega))$ *be the solution to* (4.1.1) *where the structure conditions* (4.2.1) *are in force. Moreover, assume* $g \in L^\vartheta(\gamma, q)(\Omega_T)$ *with* γ, ϑ *as in* (4.2.3) *and* $0 < q \leq \infty$*. Then*

$$|Du| \in L^\vartheta\left(\frac{\vartheta\gamma}{\vartheta - \gamma}, \frac{\vartheta q}{\vartheta - \gamma}\right) \quad \text{locally in } \Omega_T. \quad (4.2.6)$$

Furthermore, we have the local estimate

$$\|Du\|_{L^\vartheta(\frac{\vartheta\gamma}{\vartheta-\gamma}, \frac{\vartheta q}{\vartheta-\gamma})(Q_R)} \leq c\, R^{\frac{\vartheta-\gamma}{\gamma} - N} \big\| |Du| + 1 \big\|_{L^1(Q_{2R})} + c\, \|g\|_{L^\vartheta(\gamma,q)(Q_{2R})}, \quad (4.2.7)$$

for any parabolic cylinder $Q_{2R} \subset \Omega_T$*, where the constant* c *depends on* $n, \nu, L, \gamma, q, \vartheta$.

For the special choice $q = \gamma$, having in mind that $L^\vartheta(p,p)(\Omega_T) \equiv L^{p,\vartheta}(\Omega_T)$, we obtain the statement of Theorem 4.5. As in Theorem 2.9, for the borderline case $\vartheta = N$ this Theorem does not give the sharp regularity in Lorentz spaces for the gradient, which indeed is

THEOREM 4.9 ([21]). *Let* u *be as in Theorem 4.8 and let* $g \in L(\gamma, q)(\Omega_T)$ *with* γ *as in* (4.2.3) *and* $0 < q \leq \infty$*. Then*

$$|Du| \in L\left(\frac{N\gamma}{N - \gamma}, q\right) \qquad \text{locally in } \Omega_T.$$

Moreover, the local quantitative estimate

$$\|Du\|_{L(\frac{N\gamma}{N-\gamma},q)(Q_R)} \le c\, R^{\frac{N-\gamma}{\gamma}-N} \big\| |Du|+1 \big\|_{L^1(Q_{2R})} + \|g\|_{L(\gamma,q)(Q_{2R})} \tag{4.2.8}$$

holds for every parabolic cylinder $Q_{2R} \subset \Omega_T$, with a constant $c = c(n, L, \nu, \gamma, q, \vartheta)$.

Now we treat the borderline case $L\log L^\vartheta$ in full generality: we recall also here that this is the space of measurable functions defined over Ω_T such that

$$\|g\|_{L\log L^\vartheta(\Omega_T)} \approx \sup_{Q_R \subset \Omega_T} R^{\vartheta-N} \int_{Q_R} |g| \log\left(e + \frac{g}{(|g|)_{Q_R}}\right) dz < \infty.$$

Note that this is not the usual norm over $L\log L$, but we used the fact that this norm is equivalent to the expression above by a result of Iwaniec & Verde [91], see also Section 3.2. Note that from this expression it is straightforward that $(L^{1,\vartheta} \cap L\log L) \subset L\log L^\vartheta$, with also continuous embedding, and therefore Theorem 4.6 follows as a consequence of the following

THEOREM 4.10 ([21]). *Assume that (4.2.1) holds and $g \in L\log L^\vartheta(\Omega_T)$ with $2 \le \vartheta \le N$. Then the solution $u \in L^1(-T,0;W_0^{1,1}(\Omega_T))$ to (4.1.1) satisfies*

$$Du \in L_{\mathrm{loc}}^{\frac{\vartheta}{\vartheta-1},\vartheta}(\Omega_T;\mathbb{R}^n).$$

Moreover, the local quantitative estimate

$$\|Du\|_{L^{\frac{\vartheta}{\vartheta-1},\vartheta}(Q_R)} \le c\, R^{\vartheta-1-N} \big\| |Du|+1 \big\|_{L^1(Q_{2R})} + \|g\|_{L\log L^\vartheta(Q_{2R})}$$

holds for every parabolic cylinder $Q_{2R} \subset \Omega_T$, with a constant $c = c(n, \nu, L, \vartheta)$.

Finally we treat the complementary case to (4.2.3):

$$\gamma > \frac{2\vartheta}{\vartheta+2}, \qquad 2 \le \vartheta \le N. \tag{4.2.9}$$

The techniques applied for Theorems 4.5 and 4.6 provide Morrey-Gehring regularity in the following sense:

THEOREM 4.11 (Morrey-Gehring regularity, [21]). *Let $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ be a solution to (4.1.1) under the assumptions (4.2.1) and let $g \in L^{\gamma,\vartheta}(\Omega_T)$, with γ, ϑ as in (4.2.9). Then*

$$Du \in L_{\mathrm{loc}}^{h,\vartheta}(\Omega_T;\mathbb{R}^n) \qquad \textit{for some } h = h(n,L,\nu,\gamma,\vartheta) > 2; \tag{4.2.10}$$

moreover, for a constant $c \equiv c(n,\nu,L,\gamma,\vartheta)$ the quantitative local estimate

$$\|Du\|_{L^{h,\vartheta}(Q_R)} \le c\, R^{\frac{\vartheta}{h}-N} \big\| |Du|+1 \big\|_{L^1(Q_{2R})} + c\, \|g\|_{L^{\gamma,\vartheta}(Q_{2R})} \tag{4.2.11}$$

holds for any parabolic cylinder $Q_{2R} \subset \Omega_T$ with radius $R \le 1$.

Integrability of u. The technique of establishing Calderón-Zygmund type estimates for the maximal function for the spatial gradient Du of the solution leading to the statements of Theorems 4.5 and 4.8 can also be applied on the level of the solution u itself and provides – under certain modifications – also Lorentz-Morrey space estimates for the solution u:

THEOREM 4.12 ([21]). *Under the assumptions (4.2.1), $g \in L^\vartheta(\gamma,q)(\Omega_T)$ with*

$$0 < q \le \infty, \qquad 1 < \gamma < \frac{\vartheta}{2} \qquad \textit{and} \qquad 2 < \vartheta \le N, \tag{4.2.12}$$

the solution $u \in L^1(-T,0,W_0^{1,1}(\Omega))$ to (4.1.1) is such that

$$u \in L^\vartheta\left(\frac{\vartheta\gamma}{\vartheta-2\gamma}, \frac{\vartheta q}{\vartheta-2\gamma}\right) \quad \textit{locally in } \Omega_T.$$

Moreover, the following quantitative estimate

$$\|u\|_{L^{\vartheta}(\frac{\vartheta\gamma}{\vartheta-2\gamma},\frac{\vartheta q}{\vartheta-2\gamma})(Q_R)} \le c\,R^{\frac{\vartheta-2\gamma}{\gamma}-N}\big\||u|+R\big\|_{L^1(Q_{2R})} + c\,\|g\|_{L^{\vartheta}(\gamma,q)(Q_{2R})} \tag{4.2.13}$$

holds for any $Q_{2R} \subset \Omega_T$, where $c = c(n,\nu,L,\gamma,q,\vartheta)$.

Also here, we may establish a "borderline" estimate in Lorentz spaces, coming up in the special case $\vartheta = N$, in the sense that

THEOREM 4.13 ([21]). *The solution $u \in L^1(-T,0,W_0^{1,1}(\Omega))$ to (4.1.1) under the assumptions of Theorem 4.12, where $g \in L(\gamma,q)(\Omega_T)$, with q,γ as in (4.2.12), is such that*

$$u \in L\left(\frac{N\gamma}{N-2\gamma},q\right) \quad \textit{locally in } \Omega_T.$$

Moreover, the following quantitative estimate

$$\|u\|_{L(\frac{N\gamma}{N-2\gamma},q)(Q_R)} \le c\,R^{\frac{N-2\gamma}{\gamma}-N}\big\||u|+R\big\|_{L^1(Q_{2R})} + c\,\|g\|_{L(\gamma,q)(Q_{2R})}$$

holds for any $Q_{2R} \subset \Omega_T$, where $c = c(n,\nu,L,\gamma,q)$.

To conclude, the borderline case $\gamma = \vartheta/2, q = \infty$ provides the following BMO-estimate:

THEOREM 4.14 ([21]). *Under the assumption (4.2.1) and with $g \in \mathcal{M}^{\vartheta/2,\vartheta}(\Omega_T)$, $2 < \vartheta \le N$, the solution $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ to (4.1.1) belongs to $\mathrm{BMO}_{\mathrm{loc}}(\Omega_T)$. Moreover, there exists a constant $c \equiv c(n,L,\nu,\vartheta)$ such that for any parabolic cylinder $\mathcal{C}_R \subset \Omega_T$ holds*

$$[u]_{\mathrm{BMO}(Q_{R/2})} \le c\,R^{1-N}\big\||Du|+1\big\|_{L^1(Q_{2R})} + c\,\|g\|_{\mathcal{M}^{\vartheta/2,\vartheta}(Q_{2R})}.$$

Here $\mathcal{M}^{\vartheta/2,\vartheta}(\Omega_T) \equiv L^{\vartheta}(\vartheta/2,\infty)(\Omega_T)$ denotes the Marcinkiewicz-Morrey, see Section 3.2 and we recall that the BMO semi-norm is given by

$$[u]_{\mathrm{BMO}(Q_{R/2})} := \sup_{Q_\varrho \subset Q_{R/2}} \fint_{Q_\varrho} \big|u-(u)_{Q_\varrho}\big|\,dz.$$

Equations with more regular coefficients. We would now like to focus on the situation where the vector-field $a(x,t,\xi)$ in (4.1.1) satisfies stronger assumptions, especially more regularity with respect to the variable x. We therefore consider weak solutions to the equation (4.1.1) under either one of the following two settings:

- The vector-field $a \equiv a(x,t,\xi)$ is Carathéodory regular and differentiable with respect the ξ variable, $\partial_\xi a(x,t,\xi)$ is a Carathéodory map and moreover a satisfies the structure assumptions

$$\begin{cases} \langle \partial_\xi a(x,t,\xi)\lambda,\lambda\rangle \ge \nu|\lambda|^2, \\ |a(x,t,\xi)| + (1+|\xi|)\big|\partial_\xi a(x,t,\xi)\big| \le L(1+|\xi|), \\ |a(x,t,\xi)-a(x_0,t,\xi)| \le L\omega\big(|x-x_0|\big)(1+|\xi|), \end{cases} \tag{4.2.14}$$

  for any choice of $x, x_0 \in \Omega, t \in (-T,0)$ and $\xi,\lambda \in \mathbb{R}^n$, with the structure constants ν, L satisfying (4.1.3). Finally, we assume that $\omega : [0,\infty) \to [0,1)$ is a bounded, concave modulus of continuity with $\omega(0) = 0$.

- The vector-field has the structure $a(x,t,\xi) := c(x)\bar{a}(t,\xi)$, where $\bar{a}\colon (-T,0)\times\mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory maps, differentiable with respect to ξ and also $\partial_\xi a$ is a Carathéodory maps. Moreover we suppose that a satisfies the growth and ellipticity conditions:

$$\begin{cases} \langle \partial_\xi \bar{a}(t,\xi)\lambda, \lambda\rangle \geq \sqrt{\nu}|\lambda|^2, \\ |\bar{a}(t,\xi)| + (1+|\xi|)\big|\partial_\xi \bar{a}(t,\xi)\big| \leq \sqrt{L}(1+|\xi|), \end{cases} \tag{4.2.15}$$

for any choice of $t \in (-T,0)$ and $\xi, \lambda \in \mathbb{R}^n$, with ν, L satisfying (4.1.3). For the function $c : \Omega \to \mathbb{R}$ we shall assume that

$$0 < \sqrt{\nu} \leq c(x) \leq \sqrt{L} \tag{4.2.16}$$

for all $x \in \Omega$ and VMO-regularity, which means that the function c satisfies

$$\lim_{R\searrow 0} \omega(R) = 0, \qquad \text{where} \quad \omega(R) := \sup_{\substack{B_\rho \Subset \Omega \\ 0<\rho\leq R}} \fint_{B_\rho} \big|c(x) - (c)_{B_\rho}\big|\, dx. \tag{4.2.17}$$

We are going therefore to consider parabolic equations where the vector field $a(\cdot)$ satisfies either the structure assumptions (4.2.15) to (4.2.17) – the VMO-case – or (4.2.14) – the case of a continuous vector-field. In these cases we can weaken the assumption (4.2.3). As we will see below, we can assume

$$1 < \gamma < \vartheta \leq N. \tag{4.2.18}$$

The reason for this comes from the fact that the corresponding solutions to homogeneous Cauchy-Dirichlet problems satisfy reverse Hölder-type inequalities for arbitrarily large integrability exponents; see Theorem 6.16 and equation (6.3.4) of Chapter 6. The last theorem of this section will hence be the following

THEOREM 4.15 ([21]). *Let $u \in L^1(-T,0;W_0^{1,1}(\Omega))$ be the solution to (4.1.1), where either the structure conditions (4.2.14) or (4.2.15) to (4.2.17) are in force. Assume $g \in L^\vartheta(\gamma,q)(\Omega_T)$ with γ,ϑ as in (4.2.18) and $0 < q \leq \infty$. Then*

$$|Du| \in L^\vartheta\left(\frac{\vartheta\gamma}{\vartheta-\gamma}, \frac{\vartheta q}{\vartheta-\gamma}\right) \quad \textit{locally in } \Omega_T.$$

Furthermore, we have the local estimate

$$\|Du\|_{L^\vartheta(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_R)} \leq c\, R^{\frac{\vartheta-\gamma}{\gamma}-N}\big\||Du|+1\big\|_{L^1(Q_{2R})} + c\,\|g\|_{L^\vartheta(\gamma,q)(Q_{2R})}, \tag{4.2.19}$$

for any parabolic cylinder $Q_{2R} \subset \Omega_T$. In the case $\vartheta = N$, still γ, q satisfying (4.2.18), we furthermore have

$$|Du| \in L\left(\frac{N\gamma}{N-\gamma}, q\right) \quad \textit{locally in } \Omega_T$$

and the local estimate

$$\|Du\|_{L(\frac{N\gamma}{N-\gamma},q)(Q_R)} \leq c\, R^{\frac{N-\gamma}{\gamma}-N}\big\||Du|+1\big\|_{L^1(Q_{2R})} + c\,\|g\|_{L(\gamma,q)(Q_{2R})}.$$

The constant c appearing in the previous local estimate depends only on n, ν, L, γ, q, while the constant in (4.2.19) depends also on ϑ.

4.3. Marcinkiewicz regularity for degenerate parabolic equations

In this section we keep on analyzing the arguments developed in the previous one, at the same time moving to p-Laplace equation. Indeed here we consider degenerate equations of the type

$$u_t - \operatorname{div} a(x,t,Du) = \mu \qquad \text{in } \Omega_T, \tag{4.3.1}$$

where the vector field $a(\cdot)$ shall satisfy only minimal measurability and ellipticity assumptions of Ladyzhenskaya & Uralt'seva type, see [**108**] and later for precise statements. Moreover, here μ is a signed Borel measure with finite total mass satisfying a Morrey density condition. The most prominent model we have in mind for (4.3.1) is the parabolic p-Laplace equation with measurable coefficients, i.e.

$$a(x,t,Du) = A(x,t)\big(s^2 + |Du|^2\big)^{\frac{p-2}{2}} Du, \qquad p \geq 2,$$

where $A(\cdot)$ is a measurable, bounded and uniformly elliptic matrix, $p \geq 2$ and $s \in [0,1]$ is the degeneracy parameter.

The phenomenon we want to investigate here is the one which we already showed in Theorem 2.5, that is the improvement of integrability for the gradient of solutions of (4.3.1) in the case the measure on the right-hand side satisfies a Morrey-type density condition. Here we shall naturally consider the density condition

$$\sup_{Q_R(z_0)\Subset\Omega_T} \frac{|\mu|(Q_R(z_0))}{R^{N-\vartheta}} \leq c_d \quad\Longleftrightarrow\quad \mu \in L^{1,\vartheta}(\Omega_T), \quad \vartheta_c < \vartheta \leq N, \tag{4.3.2}$$

compare with (2.3.6) and (4.2.4). The threshold $\vartheta_c \in (1,2)$ we are considering in (4.3.2) is a constant depending on the data of the problem, i.e. upon n, p, ν, L – and it is linked with the higher integrability exponent for homogeneous problems. Indeed it is the solutions to the equation

$$p - 1 + \frac{1}{\vartheta_c - 1} = p\chi \tag{4.3.3}$$

where $\chi \equiv \chi(n,p,\nu,L) > 1$ is the higher integrability exponent for homogeneous problems $v_t - \operatorname{div} a(x,t,Dv) = 0$, see Corollary 7.8.

In the most general case we here consider Carathéodory regular vector fields $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following ellipticity and growth conditions for $p \geq 2$:

$$\begin{cases} \langle a(x,t,\xi_1) - a(x,t,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu\big(s^2 + |\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \\ |a(x,t,\xi)| \leq L\big(s^2 + |\xi|^2\big)^{\frac{p-1}{2}}. \end{cases} \tag{4.3.4}$$

The main Theorem of this Section is directly given for SOLAs, which clearly can be found using the procedure described in [**25, 24**], similarly to the elliptic case. Actually a SOLA $u \in L^{p-1}(-T,0;W^{1,p-1}(\Omega))$ is such that

$$Du \in L^q(\Omega;\mathbb{R}^n) \qquad \text{for all } 1 \leq q < p - 1 + \frac{1}{N-1}, \tag{4.3.5}$$

as proved in [**25**].

THEOREM 4.16 ([18]). *There exists a constant $\vartheta_c \in (1,2)$ depending on n,p,ν,L such that if u is a SOLA to equation (4.3.1) under the structure conditions (4.3.4), with $\mu \in L^{1,\vartheta}(\Omega_T)$ for $\vartheta_c < \vartheta \leq N$, then*

$$Du \in \mathcal{M}^m_{\text{loc}}(\Omega_T;\mathbb{R}^n) \qquad \text{where} \quad m = p-1+\frac{1}{\vartheta-1}. \tag{4.3.6}$$

Moreover for any parabolic cylinder $Q_{2R} \equiv Q_{2R}(z_0) \subset \Omega_T$ there exists a constant depending on n,p,ν,L,ϑ such that the following quantitative estimate holds:

$$\|Du\|^m_{\mathcal{M}^m(Q_R)} \leq c\,R^N\left[\frac{|\mu|(Q_{2R})}{|Q_{2R}|}\right]^{p-1} + c\,R^N\left[\fint_{Q_{2R}}\big(|Du|+s+1\big)^{p-1}\,dz\right]^m. \tag{4.3.7}$$

Regularity (4.3.6) for the non-degenerate evolutionary case $p=2$ has already been stated in Remark 4.7 and subsequently extended in [52] considering also lower order terms. Note that (4.3.6) actually sharpens, at least locally, (4.3.5), since in the borderline case $L^{1,N}(\Omega_T)$ coincides with the full space of Borel measures with finite total mass.

At this point the heuristic behind (4.3.3) should be clear: for what concerns regularity for Du, we obviously cannot overcome the maximal regularity we can get when $\mu \equiv 0$, just $Dv \in L^{p\chi}_{\text{loc}}(\Omega_T;\mathbb{R}^n)$. Things change when considering more regular vector fields, i.e. vector field as those considered in Paragraph 4.2 recast to the polynomial $p \neq 2$-growth. In particular we consider the two following options: the first one is that the vector-field $a(x,t,\xi)$ differentiable with respect the ξ variable, with $\partial_\xi a(x,t,\xi)$ a Carathéodory map and moreover a satisfying the p-growth and monotonicity assumptions

$$\begin{cases} \langle \partial_\xi a(x,t,\xi)\lambda,\lambda\rangle \geq \nu\big(s^2+|\xi|^2\big)^{\frac{p-2}{2}}|\lambda|^2, \\ |a(x,t,\xi)|+|\partial_\xi a(x,t,\xi)|\big(s^2+|\xi|^2\big)^{\frac{1}{2}} \leq L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \\ |a(x,t,\xi)-a(x_0,t,\xi)| \leq L\omega(|x-x_0|)\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{4.3.8}$$

for all $x,x_0 \in \Omega, t \in (-T,0), \xi,\tilde{\xi} \in \mathbb{R}^n$, with $p \geq 2$ and ν, L as in (4.1.3) and $s \in [0,1]$. Moreover we suppose $\tilde{\omega} : [0,\infty) \to [0,1)$ a concave modulus of continuity such that $\lim_{\rho\searrow 0}\omega(\rho)=0$.

The second option is when the vector-field has the structure $a(x,t,\xi) := c(x)\tilde{a}(t,\xi)$, where $\tilde{a}\colon (-T,0)\times\mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory maps, differentiable with respect to ξ and also $\partial_\xi\tilde{a}$ is a Carathéodory maps. Moreover we suppose that $\tilde{a}$ satisfies the growth and ellipticity conditions:

$$\begin{cases} \langle \partial_\xi \tilde{a}(t,\xi)\lambda,\lambda\rangle \geq \sqrt{\nu}\big(s^2+|\xi|^2\big)^{\frac{p-2}{2}}|\lambda|^2, \\ |\tilde{a}(t,\xi)|+|\partial_\xi \tilde{a}(t,\xi)|\big(s^2+|\xi|^2\big)^{\frac{1}{2}} \leq \sqrt{L}\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{4.3.9}$$

for all $t \in (-T,0)$, $\xi,\tilde{\xi} \in \mathbb{R}^n$, $p \geq 2$ and with ν, L, s as above. We moreover suppose $c : \Omega \to \mathbb{R}$ bounded and VMO regular, i.e. $\sqrt{\nu} \leq c(\cdot) \leq \sqrt{L}$ and exactly satisfying (4.2.17). In this cases we have the following

COROLLARY 4.17 ([18]). *Let u be a SOLA of equation (4.3.1), where the vector field $a(\cdot)$ is more regular in the sense that (4.3.8) or (4.3.9) hold. Then if $\mu \in L^{1,\vartheta}(\Omega_T)$ for $1 < \vartheta \leq N$ the conclusions of Theorem 4.16 hold.*

Here we can see that if μ satisfies the condition $|\mu|(Q_R) \leq c_d R^{N-1}$, then $Du \in L^q_{\rm loc}(\Omega_T;\mathbb{R}^n)$ for all $q > 1$. Compare with [**105**, Theorem 1.5] which states that

$$|\mu|(Q_R) \leq c_d R^{N-1+\varepsilon} \quad \text{for some } \varepsilon > 0 \quad \Longrightarrow \quad Du \in C^{0,\beta}_{\rm loc}(\Omega_T)$$

for some $\beta \in (0,1)$ depending on $n, p, \nu, L, \varepsilon$, and moreover the improved borderline case in [**107**]

$$|\mu|(Q_R) \leq c_d R^{N-1} h(R) \qquad \Longrightarrow \qquad Du \in C^0(\Omega_T),$$

where $h(R)$ is Dini continuous (note that actually in [**105**] only the case with no coefficients is considered).

4.4. Variable exponent *p*-Laplacian: an overview

In this section we are going not to present original results; we shall rather give a brief overview on the results available for so called variable exponent operators. In the next section we shall then give new results on such operators.

We start here considering the model equation

$$-\operatorname{div}\left(|Du|^{p(x)-2}Du\right) = 0 \qquad \text{or} \qquad -\operatorname{div}\left(p(x)|Du|^{p(x)-2}Du\right) = 0 \tag{4.4.1}$$

in Ω, the latter being the Euler equations of minimizers of the functional

$$\mathcal{D}_{p(\cdot)}(u) := \int_\Omega |Du|^{p(x)}\, dx. \tag{4.4.2}$$

This kind of functionals has been first considered Zhikov in the context of homogenization of strongly anisotropic material (see [**148**]), and in recent years the subject has gained a relevant importance by providing variational models for many problems from Mathematical Physics: special non-Newtonian fluids, called electro-rheological fluids, as modeled by Rajagopal & Růžička in [**129, 131, 130**], temperature dependent viscosity fluids, as again conceived by Zhikov [**146**], image processing models by by Chen, Levine & Rao [**40**], flows in porous media by Antonsev & Shmarev and Henriques & Urbano [**13, 88**]. More generally, a functional as $\mathcal{D}$ serves when modeling physical situation with strong anisotropicity, the nature of the situation being described by the appearance of the x-variable in the growth exponent.

Spaces where these equations and functionals naturally are posed have however been studied already by Orlicz in [**125**]. These are the so-called generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and generalized Sobolev-Lebesgue spaces $W^{k,p(\cdot)}(\Omega)$, that is the spaces of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f|^{p(x)}\, dx < \infty \qquad \left(\text{respectively} \quad \int_\Omega \sum_{|\alpha|\leq k} |D_\alpha f|^{p(x)}\, dx < \infty\right),$$

with α multindex. In general $p : \Omega \to \mathbb{R}$ is taken to be a continuous function, and these spaces are endowed with a Luxemburg type norm. Many of the properties of classic Lebesgue spaces are inherited by generalized ones: for instance, Sobolev's conjugate exponent is the pointwise one, and it is also possible to establish estimates of Calderón-Zygmund type for Singular integrals in the spaces $L^{p(\cdot)}$, to stay in our setting; see [**60, 61, 72**] and the recent book [**58**]. Regularity for equations and functionals with non-standard growth condition of this kind has been settled down by Acerbi & Mingione in [**1, 2**], while the analog of Calderón-Zygmund type Theorem 2.2 can be found in [**3**]; we shall examine this result later. Several authors in these last years have extended regularity results available for standard p-Laplacian equations, systems and functional to non-standard situation; a good overview on all these results can be [84].

We sketch now some of these results in order to show appropriately assumptions (and problems) naturally appearing when treating equations like (4.4.1) or functionals like (4.4.2). We shall treat only the basic cases of the $p(x)$-Laplacian and the $p(x)$-energy, therefore stressing that results presented here are naturally valid for more general functionals, equations and systems with "$p(x)$-growth", provided suitable distinctions between the various cases are done. An essential regularity assumption for $p(\cdot)$, in addition to its continuity, is the so-called $\log$-continuity assumption, first introduced by Zhikov [147] to treat the Lavrentiev Phenomenon related to $\mathcal{D}_{p(\cdot)}$. This goes as follows: if we denote by $\omega(\cdot)$ the modulus of continuity of the exponent function $p(\cdot)$, then the $\log$-continuity assumption prescribes that

$$\limsup_{\rho \searrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) < \infty. \tag{4.4.3}$$

Such an assumption turns out to be crucial: Zhikov in [147] proved that the failure of (4.4.3) is a possible cause of discontinuities of minima. On the other hand, condition (4.4.3) is sufficient and necessary condition for boundedness of Singular integrals in $L^{p(\cdot)}(\Omega)$; furthermore under the same assumption can be proved higher integrability for minimizer of (4.4.2), even if in [149] Zhikov and Pastukhova proved that a certain form of logarithmic higher integrability does still hold even without assumption (4.4.3). Also Hölder continuity of solutions for some "small" Hölder's exponent $\alpha \in (0,1)$ follows from (4.4.3); this result has been given in full generality by Acerbi & Mingione in the following way:

THEOREM 4.18 ([1]). *Let $u \in W^{1,p(\cdot)}(\Omega)$ a local minimizer of the functional $\mathcal{D}_{p(\cdot)}$. Then for every $\alpha \in (0,1)$, there exists $\varepsilon \equiv \varepsilon(\alpha)$ such that if*

$$\limsup_{\rho \searrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) < \varepsilon,$$

then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$.

In other words, controlling the oscillations of the exponent function $p(\cdot)$ against a logarithmic weight allows to control the degree of regularity of local minimizers. It is at this point straightforward that if

$$\limsup_{\rho \searrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) = 0 \tag{4.4.4}$$

then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for every $\alpha \in (0,1)$. In order to reach first order regularity, Hölder regularity has to be imposed on the coefficient, i.e.

$$\omega(\rho) \lesssim \rho^{\alpha} \tag{4.4.5}$$

for some $\alpha \in (0,1)$. The maximal regularity theorem has been given by Coscia & Mingione and read as follows:

THEOREM 4.19 ([46]). *Let $u \in W^{1,p(\cdot)}(\Omega)$ a local minimizer of the functional $\mathcal{D}_{p(\cdot)}$. If (4.4.5) holds for some $\alpha \in (0,1)$, then*

$$Du \in C^{0,\beta}_{\text{loc}}(\Omega; R^n)$$

for some $\beta < \alpha$.

The previous regularity results extend obviously also to (4.4.1) and equations, systems and functionals whose vector field or integrand, respectively, is in a suitable sense controlled by $|Du|^{p(x)}$; see [**1, 73, 117**] for the general statements.

Concerning parabolic equations with $p(x,t)$-growth, these are simplified versions of the models we already showed in the time-dependent case. For instance, the viscosity of electro-rheological fluids strongly depends on the external electromagnetic field, which can vary both space and time. The mathematical model for electro-rheological fluids developed by Růžička in [**131**] also admits a $p(x,t)$-growth structure in the non linear diffusion term. Regularity for such a model has been studied in [**5**].

However, we note that compared to the stationary case only few regularity results for such problems are in general available, and this is probably due to technical reasons: to treat evolutionary $p(x,t)$-Laplace equation, one has to match the already described intrinsic approach of DiBenedetto with the localization techniques used to handle variable exponent growths. We shall give an example to the reader in Chapter 8 how not immediate this matching can be.

The first regularity result we want to mention here, not only since it is the starting point for almost any other regularity result in this area, is the higher integrability, i.e. the existence of some $\varepsilon > 0$, depending only on the structural constants, such that

$$|Du|^{p(\cdot)} \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega_T).$$

This result was first established in the case of the $p(x,t)$-Laplacian equation by Antontsev & Zhikov [**16**], and later for a quite general class of parabolic systems with $p(x,t)$-growth independently by Zhikov & Pastukhova [**150**] and Bögelein and Duzaar [**29**]. With regard to Hölder regularity, Chen & Xu [**41**] proved that weak solutions of the parabolic $p(x,t)$-Laplacian equation are locally bounded and Hölder continuous, while the local Hölder continuity of Du for parabolic $p(x,t)$ systems has recently been established by Bögelein & Duzaar [**30**].

4.5. Calderón-Zygmund estimates for parabolic $p(x,t)$-Laplacian

In this section we describe our extension of the parabolic result of Calderón-Zygmund type Theorem 2.15, which has been proved for evolutionary p-Laplacian, to parabolic systems with non-standard growth of the following type:

$$\partial_t u - \operatorname{div}\big(a(x,t)|Du|^{p(x,t)-2}Du\big) = \operatorname{div}\big(|F|^{p(x,t)-2}F\big) \tag{4.5.1}$$

Since we consider the case of systems, the solution is a possibly vector valued function $u\colon \Omega_T \to \mathbb{R}^N$ with $N \geq 1$. With respect to the variable exponent $p(x,t)$ we shall assume first of all its boundedness:

$$\frac{2n}{n+2} < \gamma_1 \leq p(z) \leq \gamma_2 < \infty \qquad \text{for all } z \in \Omega_T. \tag{4.5.2}$$

Note that the lower bound $\gamma_1 > 2n/(n+2)$ is unavoidable even in the constant exponent case $p(\cdot) \equiv p$, cf. [**54**, Chapters 5, 8]. With respect to its regularity, we will assume the logarithmic continuity (4.4.4), in the sense that if ω it she modulus of continuity of $p(\cdot)$ with respect to the parabolic metric (3.1.4), i.e.

$$|p(z) - p(\tilde z)| \leq \omega\big(d_{\mathcal{P}}(z,\tilde z)\big) \qquad \text{for any } z, \tilde z \in \Omega_T, \tag{4.5.3}$$

with $\omega : [0,+\infty) \to [0,1]$ nondecreasing, then we assume that ω satisfies (4.4.4). For the coefficient function $a : \Omega_T \to \mathbb{R}$ we are going to assume its measurability and the boundedness

$$\nu \leq a(z) \leq L \qquad \text{for any } z \in \Omega_T \tag{4.5.4}$$

for some constants satisfying (4.1.3). With regard to its regularity, we shall assume not necessarily its continuity but only that it satisfies a VMO condition with respect the spatial variable. More precisely, denoting

$$(a)_{x_0,\rho}(t) := \fint_{B_\rho(x_0)} a(x,t)\,dx \qquad \text{for } B_\rho(x_0)\subset\Omega,$$

we assume that there exists $\tilde\omega\colon [0,\infty)\to[0,1]$ such that

$$\sup_{\substack{B_\rho(x_0)\subset\Omega,\\ 0<\rho\le r}} \fint_{B_\rho(x_0)} |a(x,t)-(a)_{x_0,\rho}(t)|\,dx \le \tilde\omega(r) \tag{4.5.5}$$

for a.e. $t\in(0,T)$ any $r>0$ and

$$\lim_{r\searrow 0}\tilde\omega(r)=0. \tag{4.5.6}$$

Here we stress that with respect to time we assume not more than measurability. Clearly our assumptions on a allow product coefficients of the type $a(x,t)=b(x)c(t)$, with $b\in VMO(\Omega)\cap L^\infty(\Omega)$ and $c\in L^\infty(0,T)$.

As usual we shall consider weak solutions u of (4.5.1), which we define as maps $u\in L^2(\Omega_T;\mathbb{R}^N)\cap L^1(-T,0;W^{1,1}(\Omega;\mathbb{R}^N))$ such that $Du\in L^{p(\cdot)}(\Omega_T;\mathbb{R}^{Nn})$ and satisfying the distributional formulation

$$\int_{\Omega_T}\big[u\cdot\varphi_t-\big\langle a(\cdot)|Du|^{p(\cdot)-2}Du,D\varphi\big\rangle\big]\,dz=\int_{\Omega_T}\big\langle |F|^{p(\cdot)-2}F,D\varphi\big\rangle\,dz \tag{4.5.7}$$

for every test function $\varphi\in C_0^\infty(\Omega_T;\mathbb{R}^N)$. The existence of such weak solutions is ensured by a result of Antontsev and Shmarev [**14, 15**]. For such solutions we proved that the following Calderón-Zygmund result holds:

THEOREM 4.20 ([**19**]). *Let u be a weak solution of the parabolic system (4.5.1), where the exponent p and the coefficient a satisfy the assumptions listed above; moreover, assume that $|F|^{p(\cdot)}\in L^q_{\text{loc}}(\Omega_T)$ for some $q>1$. Then we have*

$$|Du|^{p(\cdot)}\in L^q_{\text{loc}}(\Omega_T).$$

Moreover, for $K\ge 1$ there exist a radius $R_0=R_0(n,N,\nu,L,\gamma_1,\gamma_2,K,\omega(\cdot),\tilde\omega(\cdot),q)>0$ and a constant c depending upon $n,N,\nu,L,\gamma_1,\gamma_2,q$ such that the following holds: If

$$\int_{\Omega_T}|Du|^{p(\cdot)}+(|F|+1)^{p(\cdot)}\,dz\le K, \tag{4.5.8}$$

then for every parabolic cylinder $Q_{2R}\equiv Q_{2R}(\mathfrak{z}_0)\Subset\Omega_T$ with $R\in(0,R_0]$, then the local estimate

$$\begin{aligned}&\fint_{Q_R}|Du|^{p(\cdot)q}\,dz\\ &\le c\left[\fint_{Q_{2R}}\big(|Du|+1\big)^{p(\cdot)}\,dz+\left(\fint_{Q_{2R}}|F|^{p(\cdot)q}\,dz\right)^{\frac1q}\right]^{1+d(p_0)(q-1)},\end{aligned} \tag{4.5.9}$$

where we denoted

$$d(p_0):=\begin{cases}\dfrac{p_0}{2} & \text{if } p_0\ge 2,\\[2mm] \dfrac{2p_0}{p_0(n+2)-2n} & \text{if } p_0<2,\end{cases}\qquad \text{with } p_0:=p(\mathfrak{z}_0). \tag{4.5.10}$$

We here note that the constant c in Theorem 4.20 remains stable when $q\searrow 1$ and it blows up, i.e. $c\to\infty$ when $q\to\infty$.

REMARK 4.21. The same result holds true if we assume, instead of the VMO condition (4.5.6), that the BMO seminorm of a with respect to x is small, i.e. that

$$[a]_{\text{BMO}} := \sup_{r>0} \tilde{\omega}(r) \leq \epsilon_{\text{BMO}}$$

with some constant $\epsilon_{\text{BMO}} > 0$ depending on $n, N, \nu, L, \gamma_1, \gamma_2, q$.

Note that the previous result extends to the parabolic setting the Calderón-Zygmund results due to Acerbi & Mingione for elliptic $p(x)$-growth systems we mentioned above and we report here.

THEOREM 4.22 ([3]). *Let $u \in W^{1,p(\cdot)}(\Omega; \mathbb{R}^n)$ be a weak solution to the system*

$$\operatorname{div}\left(|Du|^{p(x)-2}Du\right) = \operatorname{div}\left(|F|^{p(x)-2}F\right),$$

where the exponent function $p(\cdot)$ satisfies (4.5.2), (4.5.3), (4.4.4) recast to the elliptic case. Then if $|F|^{p(x)} \in L^q_{\text{loc}}(\Omega; R^{nN})$ for some $q > 1$, then

$$|Du|^{p(x)} \in L^q_{\text{loc}}(\Omega; R^{nN}).$$

The previous Theorem is moreover coupled with a local estimate similar to (4.5.9) apart from the fact that the scaling deficit doesn't there appear.

4.6. Interpolation potential estimates for $p(x)$-growth conditions

In this section we keep on considering non-standard, $p(x)$-growth conditions but the topic now is quite different, since we want to face potential estimates, in particular interpolation ones of the type of those considered in Paragraph 2.4. In particular here we are going to consider nonlinear elliptic equations of the form

$$-\operatorname{div}\left[\gamma(x)a(x, Du)\right] = \mu \qquad \text{in } \Omega, \tag{4.6.1}$$

where μ denotes a signed Radon measure with finite total mass. The vector field $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is modeled upon the non-standard $p(\cdot)$–Laplacian so that the most prominent model we want to imitate with (4.6.1) is the following elliptic equation with non-standard growth conditions:

$$-\operatorname{div}\left[\gamma(x)|Du|^{p(x)-2}Du\right] = \mu, \tag{4.6.2}$$

see below for the precise assumptions on the vector field $a(\cdot)$. We tell however in advance that the exponent function $p : \Omega \to (2 - 1/n, +\infty)$ is assumed to be bounded and to satisfy – at least – the classical weak logarithmic continuity condition (4.4.3), and we allow the bounded coefficient function $\gamma : \Omega \to \mathbb{R}$ to be discontinuous, but in a mild way: in particular, we will only consider coefficients with controlled integral oscillation, namely in BMO or VMO classes. The aim of this Section is to show how the interpolation estimates proved in [103] and explained in Section 2.4 can be extended, at various levels, to solution to non-standard elliptic equations of the type (4.6.2), or more generally (4.6.1), in a scale depending on the regularity of both $\gamma(\cdot)$ and $a(\cdot, \xi)$, that is, referring to (4.6.2), the regularity of both coefficients and exponent.

We make the choice here to present all the estimates in this Section in the form of *a priori estimates* for C^1-solutions of problems with L^1 data, but they all *hold true also for energy solutions $u \in W^{1,p(\cdot)}(\Omega)$ provided one only considers x, y Lebesgue's points of u*. Estimates for SOLA $u \in W^{1,q(\cdot)}$ with $q(\cdot) < \min\{n(p(\cdot) - 1)/(n - 1), p(\cdot)\}$ to genuine measure data problems can be obtained therefore using approximation techniques that apply also to the case of non-standard growth conditions, see [33, Chapter 4] and the references therein. Due to the reason just explained, we shall involve not more than the 1-energy bound in our estimates instead of the $p(\cdot)$-energy which is typically used for non standard growth problems – see for instance (4.5.8); hence within the whole section we shall assume that the total 1-energy of the solution u is bounded, i.e.

$$\int_\Omega |Du|\,dx =: M < +\infty. \tag{4.6.3}$$

The estimates described in the following lines involve the following nonlinear Wolff potential for variable exponent functions:

$$\mathbf{W}^{\mu}_{\beta(\cdot),p(\cdot)}(x,R) := \int_0^R \left[\frac{|\mu|(B_\rho(x))}{\rho^{n-\beta(x)p(x)}}\right]^{1/(p(x)-1)} \frac{d\rho}{\rho}\,, \quad \beta(x)\in(0,n/p(x)],$$

which is defined pointwise just as the usual constant exponent Wolff potential (2.4.3). Moreover, in the case $p(x)\equiv 2$, the Wolff potential reduces to the non standard Riesz potential, defined as

$$\mathbf{I}^{\mu}_{\beta(\cdot)}(x,R) := \int_0^R \frac{|\mu|(B_\rho(x))}{\rho^{n-\beta(x)}}\frac{d\rho}{\rho}\,, \qquad \beta(x)\in(0,n].$$

We need to introduce also the following mixed potential, depending explicitly on the value of the function $p(\cdot)$:

$$\mathbf{WI}^{\mu}_{\beta(\cdot),p(\cdot)}(x,R) := \begin{cases} \left[\mathbf{I}^{|\mu|}_{\beta(\cdot)p(\cdot)}(x,R)\right]^{1/(p(x)-1)} & \text{if } p(x)<2,\\ \mathbf{W}^{\mu}_{\beta(\cdot),p(\cdot)}(x,R) & \text{if } p(x)\geq 2. \end{cases} \tag{4.6.4}$$

Notice that both the right-hand side potentials share the same scaling properties with respect to the exponent, and subsequently also $\mathbf{WI}$ does. We introduce this potential since in order to get fractional estimates on u having as a upper borderline case a pointwise bound for the gradient, accordingly with [103], we need to catch different behaviors depending on the value of the exponent in the point considered, see also the discussion before Theorem 4.25.

Before detailing the assumptions on the vector field $a(\cdot)$, we stress that we made explicit the possible presence of coefficients in (4.6.1) since, while we are forced to consider a continuous dependence of the vector field $a(\cdot)$ upon x by the fact that we want to model the $p(\cdot)$-Laplacian (4.6.2), and in this case continuity is essentially an unavoidable condition, we can consider slightly weaker assumptions when considering the regularity of the coefficient of the equation. Now the vector field $a:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is assumed to be C^1-regular in the gradient variable ξ, with $\partial_\xi a(\cdot)$ being Carathéodory regular, and to satisfy the following non standard growth and ellipticity conditions:

$$\begin{cases} \langle \partial_\xi a(x,\xi)\lambda,\lambda\rangle \geq \sqrt{\nu}\big(s^2+|\xi|^2\big)^{\frac{p(x)-2}{2}}|\lambda|^2\\ |a(x,\xi)|+|\partial_\xi a(x,\xi)|\big(s^2+|\xi|^2\big)^{\frac12} \leq \sqrt{L}\big(s^2+|\xi|^2\big)^{\frac{p(x)-1}{2}}, \end{cases} \tag{4.6.5}$$

whenever $x\in\Omega$ and $\xi,\lambda\in\mathbb{R}^n$, where ν, L satisfy (4.1.3) and $s\in[0,1]$ is fixed. The exponent function $p:\Omega\to(2-1/n,+\infty)$ is assumed to be continuous with modulus of continuity $\omega:[0,\infty)\to[0,1]$ and bounded:

$$2-\frac{1}{n} < \gamma_1 \leq p(x) \leq \gamma_2 < \infty \qquad \text{for all } x\in\Omega. \tag{4.6.6}$$

We shall see time by time which kind of regularity we require upon ω. Let us remark that the restriction $\gamma_1 > 2-1/n$ already appears in the constant growth case since this condition guarantees that solutions u to measure data problems belong to the Sobolev space $W^{1,1}$, which in turn allows to speak of the usual gradient of u.

We shall also impose the following continuity assumption on $a(\cdot)$ with respect to x:

$$|a(x,\xi)-a(x_0,\xi)| \le L_1\omega\big(|x-x_0|\big)\times \\ \times\left[\big(|\xi|^2+s^2\big)^{\frac{p(x)-1}{2}}+\big(|\xi|^2+s^2\big)^{\frac{p(x_0)-1}{2}}\right]\Big[1+\big|\log\big(|\xi|^2+s^2\big)\big|\Big], \quad (4.6.7)$$

with $L_1 \ge 1$ and for all $x, x_0 \in \Omega$ and $z \in \mathbb{R}^n$. Now $\gamma : \Omega \to \mathbb{R}$ denotes a possibly discontinuous bounded function with

$$\sqrt{\nu} \le \gamma(x) \le \sqrt{L} \qquad \text{for all } x \in \Omega. \quad (4.6.8)$$

In the course of the Chapter we will impose a variety of conditions on the oscillations of γ. In order to do this we introduce the quantity

$$\mathbf{v}(r) := \frac{1}{2\sqrt{L}} \sup_{\substack{B_\rho(x_0)\subset\Omega,\\ 0<\rho\le r}} \fint_{B_\rho(x_0)} \big|\gamma(x)-(\gamma)_{B_\rho(x_0)}\big|\,dx \in [0,1]$$

and call the coefficient $\gamma(x)$ of *bounded mean oscillation* (or BMO regular) if there exist a constant c and a radius $r_0 > 0$ such that

$$\mathbf{v}(r) \le c \qquad \text{for all radii } r \le r_0. \quad (4.6.9)$$

Moreover, we call $\gamma(x)$ of *vanishing mean oscillation* (or VMO regular), if

$$\mathbf{v}(r) \to 0 \qquad \text{as } r \to 0. \quad (4.6.10)$$

Brief overview on non-standard potential results. In [116] Lukkari, Maeda and Marola generalized the basic results of Kilpeläinen & Malý [94] to the case of non standard growth conditions with variable exponent. They showed a pointwise estimate for the solution to the equation (4.6.1) under the structure conditions (4.6.5) to (4.6.7), where ω satisfies the logarithmic continuity (4.4.3), of the following type:

$$|u(x)| \le c\left[\mathbf{W}^{\mu}_{1,p(\cdot)}(x,2R)+\fint_{B_R(x)}(|u|+s+R)\,d\xi\right], \quad (4.6.11)$$

for all $B_{2R}(x) \Subset \Omega$ and $R \le R_0$, where R_0 is a universal constant, depending only on the structural data of the equation. On the other hand, Bögelein & Habermann in [33] generalized pointwise potential estimates for the gradient of the solution of Theorem 2.12 to the non standard growth situation, i.e. solution to equation (4.6.1) under the conditions (4.6.5)–(4.6.7) and the additional condition $\gamma_1 \ge 2$ the estimate

$$|Du(x)| \le c\left[\mathbf{W}^{\mu}_{\frac{1}{p(\cdot)},p(\cdot)}(x,2R)+\fint_{B_R(x)}(|Du|+s+R)\,d\xi\right], \quad (4.6.12)$$

for all balls $B_{2R}(x) \Subset \Omega$ and radii $R \le R_0$, R_0 as above. For the pointwise potential estimates (4.6.11) and (4.6.12) to hold true, different continuity conditions on the modulus of continuity ω have to be imposed: Whereas for the estimate (4.6.11) it is sufficient to impose the **logarithmic Hölder continuity** condition as (4.4.3), i.e.

$$\omega(\rho)\log\frac{1}{\rho} \le c(\omega(\cdot)) < +\infty, \qquad \text{for all } \rho \le 1, \quad (4.6.13)$$

in order to make estimate (4.6.12) hold true, we need to impose a **logarithmic Dini-condition** of the type

$$\int_0^r \big[\omega(\rho)\log\tfrac{1}{\rho}\big]^{\kappa}\frac{d\rho}{\rho} =: d_\omega(r) < \infty \qquad \text{for some } r > 0, \quad (4.6.14)$$

where

$$\kappa := \min\Big\{\frac{2}{\gamma_2},1\Big\}.$$

Condition (4.6.14) is stronger than (4.6.13). This is in accordance with the standard growth situation in which for the pointwise estimate for u it has merely be imposed measurability of the vector field with respect to x, whereas for the pointwise estimate for the gradient Du one needs to impose a Dini-type condition on $\big[\omega(r)\big]^{\min\{2/p,1\}}$. This comparison is not completely correct, since in our case we uncouple the regularity of the exponent, roughly speaking given by the behavior of ω from the regularity of the coefficient, given by $\mathbf{v}$. We therefore also need to impose a Dini-condition on the coefficient function $\gamma(x)$, involving the function $\mathbf{v}$ which measures its integral oscillation, as follows:

$$\int_0^r [\mathbf{v}(\rho)]^{\sigma_h} \frac{d\rho}{\rho} =: d_{\mathbf{v}}(r) < \infty, \tag{4.6.15}$$

for some $r > 0$ and with $\sigma_h < 1$ depending on data of the problem. Note that it might be difficult to verify condition (4.6.15), since the exponent σ_h depends on the higher integrability exponent for homogeneous equations with $p(x)$-growth (see Lemma 9.9) but for example it is satisfied in the case $\mathbf{v}(\rho) \le c\,\rho^{\gamma}$ for some $\gamma \in (0,1)$.We immediately point out that, despite we uncouple the regularity of the vector field $a(\cdot)$ and the regularity of the coefficient γ, the two conditions (4.6.14) and (4.6.15) will always be coupled, see Theorem 4.25 and Lemma 9.12. This is due to the fact that in order to get estimates for the gradient, we need to perform a comparison argument with the problem where the dependence of the full vector field $\gamma(x)a(x,z)$ on the variable x will be frozen in some fixed point. Hence both the conditions (4.6.14) and (4.6.15), through a dyadic summation process, will attend the result.

Interpolation fractional estimates. Let us first state the results of De Giorgi type, covering fractional differentiability "of order $\alpha < \alpha_0$" – with some $\alpha_0 > 0$ depending on the structural data. We highlight here that for this first result we don't require any further regularity property with respect to x apart from the weak logarithmic continuity (4.4.3) for the vector field $a(\cdot)$, similarly as when the only measurability of the coefficients yields Hölder continuity of solutions to homogeneous problems.

THEOREM 4.23 (Estimates of De Giorgi type, [22]). *Let $u \in C^1(\Omega)$ be a weak solution to the equation (4.6.1) with the assumptions (4.6.5) to (4.6.7) holding for a modulus of continuity fulfilling the weak logarithmic condition (4.6.13). Moreover let the coefficient $\gamma(x)$ be bounded as in (4.6.8). Then there exists $\alpha_m > 0$ depending only on the structural data of the equation and a radius $R_0 \equiv R_0(n,\nu,L,\gamma_1,\gamma_2,\omega(\cdot))$, such that the following holds true: Whenever $B_R \subset \Omega$ with $R \le R_0$ and $x, y \in B_{R/8}$, then*

$$|u(x)-u(y)| \le c\Big[\mathbf{W}^{\mu}_{1-\alpha\frac{p(\cdot)-1}{p(\cdot)},p(\cdot)}(x,2R) + \mathbf{W}^{\mu}_{1-\alpha\frac{p(\cdot)-1}{p(\cdot)},p(\cdot)}(y,2R)\Big]|x-y|^{\alpha} + c\left(\frac{|x-y|}{R}\right)^{\alpha} \fint_{B_R} \big(|u| + Rs + R^{\alpha_m}\big)\, d\xi \tag{4.6.16}$$

holds uniformly in $\alpha \in [0,\tilde{\alpha}]$ for every $\tilde{\alpha} < \alpha_m$, where the constant depends only on $n, \gamma_1, \gamma_2, \nu, L, L_1$ and $\tilde{\alpha}$.

The next results specifies the dependency of the Hölder exponent – and therefore the fractional differentiability – on the continuity property of the exponent function $p(\cdot)$ and the $x-$dependence of the vector field:

THEOREM 4.24 ([22]). *Let $u \in C^1(\Omega)$ be a weak solution to (4.6.1) under the structural assumptions (4.6.5) to (4.6.8). For every $\tilde{\alpha} < 1$ there exists two positive numbers δ_1, δ_2 depending both on $n, \gamma_1, \gamma_2, \nu, L, L_1, \tilde{\alpha}$ such that if*

$$\lim_{\rho\searrow 0} \omega(\rho)\log\left(\frac{1}{\rho}\right) \le \delta_1, \qquad \lim_{\rho\searrow 0} \mathbf{v}(\rho) \le \delta_2, \tag{4.6.17}$$

then the pointwise estimate (4.6.16) *holds uniformly in* $\alpha \in [0, \tilde{\alpha}]$*, for a constant* c *which depends on* n, γ_1, γ_2, ν, L, $\omega(\cdot)$, $\tilde{\alpha}, \mathrm{diam}(\Omega)$*, as soon as* $x, y \in B_{R/8}$ *and* $R \le R_0$*, where* R_0 *is the radius appearing in Theorem 4.23.*

Note that the previous condition on $\mathbf{v}$ can be rephrased as γ is BMO regular and has a small (in the sense specified above) norm; it is otherwise always satisfied in the case γ is VMO regular. The same holds for the condition regarding ω: (4.6.17) would always be satisfied if a strong logarithmic continuity condition (4.4.4) held, or just eventually a smallness condition similar to that in Theorem 4.18. Note moreover that in order to get the borderline case $\tilde{\alpha} = 1$ – which means differentiability – it is not even sufficient to impose condition (4.6.17) for both $\delta_i = 0$. Indeed, we have to impose Dini conditions of the form (4.6.14), (4.6.15) to obtain fractional differentiability in the full range $\alpha \in [0, 1]$. In other words, even not a strong estimate for the integral oscillations of coefficients and the strong logarithmic continuity of the exponent are sufficient to assure differentiability: it is indeed needed a quantitative description of the behavior of the two moduli of continuity *close to zero*.

Another problem here is that we have to match the case $\tilde{\alpha} < 1$ – i.e. no gradient estimate is approached, Theorem 4.23 – involving the non linear Wolff potentials independently of the value of the function $p(\cdot)$, with the case $\tilde{\alpha} = 1$, which is the gradient estimate involving both the Wolff and the Riesz potential, depending on the value of the exponent function $p(\cdot)$. Namely in the case $p(x) \ge 2$ we have the estimate (4.6.12), while in the case $p(x) < 2$, as a byproduct of the following theorem, we will have

$$|Du(x)| \le c \left[\mathbf{I}_1^{|\mu|}(x, 2R)\right]^{\frac{1}{p(x)-1}} + c \fint_{B_R(x)} (|Du| + s + R)\, d\xi,$$

analogously as in the standard case. In order to deal simultaneously with the two different behaviors of the estimates, and for simplicity of notations and readability of the estimates, we will make use of the mixed potential introduced in (4.6.4). In particular, in order to "match" the two borderline estimates we were talking about, we have to replace for every $\alpha \in [0, \tilde{\alpha}]$ the nonlinear Wolff potentials $\mathbf{W}^{\mu}_{1,p(\cdot)}(x, R)$ by the slightly larger Riesz potentials $[\mathbf{I}^{|\mu|}_{p(\cdot)}(x, 2R)]^{1/(p(x)-1)}$ for the points where $p(x) < 2$. See therefore also the comment before [**103**, Theorem 1.5]. After this introduction we can state the following

THEOREM 4.25 ([**22**]). *Let* $u \in C^1(\Omega)$ *be a weak solution to* (4.6.1) *under the structural assumptions* (4.6.5) *to* (4.6.8) *and with* ω *satisfying the log-Dini condition* (4.6.14)*. There exists a constant* σ_h*, depending on* $n, L/\nu, \gamma_1, \gamma_2$*, such that if also* (4.6.15) *holds, then*

$$\begin{aligned}
&|u(x) - u(y)| \\
&\le c \left[\mathbf{WI}^{\mu}_{1-\alpha\frac{p(\cdot)-1}{p(\cdot)}, p(\cdot)}(x, 2R) + \mathbf{WI}^{\mu}_{1-\alpha\frac{p(\cdot)-1}{p(\cdot)}, p(\cdot)}(y, 2R)\right] |x - y|^{\alpha} \\
&\qquad + c \left(\frac{|x - y|}{R}\right)^{\alpha} \fint_{B_R} (|u| + Rs + R^{\varsigma})\, d\xi, \qquad (4.6.18)
\end{aligned}$$

holds uniformly in $\alpha \in [0, 1]$*, whenever* $B_R \Subset \Omega$ *is a ball with radius* $R \le R_0$ *and* $x, y \in B_{R/8}$*, being* $R_0 \equiv R_0(n, \nu, L, L_1, \gamma_1, \gamma_2, \omega(\cdot))$*. Here* ς *has the expression*

$$\varsigma := \begin{cases} \alpha_m & \text{if } \alpha \le \alpha_m/2; \\ 2 & \text{if } \alpha_m < \alpha \le 1. \end{cases}$$

and the constant c *depends on* $n, \gamma_1, \gamma_2, \nu, L, \omega(\cdot), \sigma$.

Note that as a Corollary of some of the estimates used in the proof of the previous Theorem, we have the pointwise gradient estimate for the case $p(x) < 2$, which together with [**33**] completes the theory of non standard potential estimates for the gradient.

COROLLARY 4.26 ([**22**]). *Let $u \in C^1(\Omega)$ be a weak solution to (4.6.1) under the structural assumptions (4.6.5) to (4.6.8); let moreover the logarithmic-Dini conditions (4.6.14) and (4.6.15) hold. Then there exists a contant c and a positive radius R_0, both having the same dependencies listed in Theorem 4.25, such that the pointwise estimate*

$$|Du(x)| \leq c \left[\mathbf{I}_1^{|\mu|}(x, 2R)\right]^{\frac{1}{p(x)-1}} + c \fint_{B_R(x)} (|Du| + s + R)\, d\xi \tag{4.6.19}$$

holds for every $x \in \Omega$ such that $p(x) < 2$ and $B_{2R}(x) \subset \Omega$, with $R \leq R_0$.

REMARK. Notice that the previous Theorem holds with the conditions

$$\int_0^r \left[\omega(\rho) \log \tfrac{1}{\rho} + [\mathbf{v}(\rho)]^{\sigma_h}\right] \frac{d\rho}{\rho} < \infty \qquad \text{for some } r > 0,$$

σ_h as in Theorem 4.25, which is a slightly weaker assumption. This can be seen carefully checking the proof of (the second part of) Theorem 9.1 (or directly proving Corollary 4.26) taking into consideration that we used the quantity κ in order to have an unitary approach, while the correct exponent in the case $p(x) < 2$ is one (see the definition of κ after (4.6.14)).

We finally remark that also estimates for the Maximal and the sharp Maximal functions, which are necessary to prove the various previous theorems, are available, but we shall leave them in Chapter 9 since we prefer to highlight here only the potential interpolation results.

4.7. Linear potential estimates under general growth conditions

In this final Section we consider general growth conditions as those studied by Lieberman in [**112**], and we show that still linear potential estimates as those proved in [**106**] Theorem 2.13. Let us consider the elliptic equation

$$-\operatorname{div}\left(g\big(|Du|\big)\frac{Du}{|Du|}\right) = \mu \qquad \text{in } \Omega, \tag{4.7.1}$$

where μ is a Borel measure with finite total mass and being $g \in \mathcal{C}^1(\mathbb{R}^+)$ a positive function satisfying only

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \qquad t > 0. \tag{4.7.2}$$

Lieberman introduced this kind of conditions in [**112**] since (4.7.1) in turn can be seen as – using his own words – the natural, and, in a sense, the best generalization of the p-Laplace equation (where the function g takes the power-like form $g(t) = t^{p-1}$, with $p > 1$). More in general, we shall consider equations like

$$-\operatorname{div} a(Du) = \mu \qquad \text{in } \Omega, \tag{4.7.3}$$

where the $C^1(\mathbb{R}^n)$ vector field satisfyies the ellipticity and growth conditions

$$\begin{cases} \langle \partial_\xi a(\xi)\lambda, \lambda\rangle \geq \nu \dfrac{g(|\xi|)}{|\xi|}|\lambda|^2 \\ |a(\xi)| + |\partial_\xi a(\xi)||\xi| \leq Lg(|\xi|) \end{cases} \tag{4.7.4}$$

for all $\xi, \lambda \in \mathbb{R}^n$ and with ν, L as in (4.1.3). g is the function considered in the lines above satisfying (4.7.2) and moreover the degeneracy conditions

$$\lim_{t\to 0^+} \frac{g(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{g(t)}{t} = \infty; \tag{4.7.5}$$

this is to say that as the gradient vanishes, the modulus of ellipticity of the equation becomes zero and that the equation is not asymptotically non-degenerate. We make this choice in order to simplify the (already technically heavy) presentation, still considering a case that in many respects can be considered as the most interesting one.

We shall focus here only on the case

$$1 \le \delta < g_0.$$

Notice that without loss of generality we can suppose $\delta \neq g_0$, on the other hand we would have $g(t) \approx t^{\delta-1}$ by integration. Natural examples of functions g satisfying (4.7.2) are the logarithmic perturbations of the monomials, i.e.

$$g(t) = t^{p-1}\big[\log(a+t)\big]^{\alpha}, \qquad p \ge 2, \quad a \ge 1, \quad \alpha \ge 0.$$

The principal aim of this Section is to show that for equations like (4.7.1), or more in general (4.7.3), holds the linear potential result showed in Theorem 2.13; in particular we prove that

THEOREM 4.27 ([**17**]). *Let $u \in W^{1,G}(\Omega)$ be a weak solution to equation (4.7.3), where μ is a Radon measure with finite total mass, or a function in $L^1(\Omega)$, and the vector field satisfies assumption (4.7.4)-(4.7.5). Then there exists a constant c, depending on n, δ, g_0, ν, L, such that the pointwise estimate*

$$g\big(|Du(x)|\big) \le c\,\mathbf{I}_1^{|\mu|}(x, 2R) + c\, g\bigg(\fint_{B_R(x)} |Du|\, d\xi\bigg) \tag{4.7.6}$$

holds for every $x \in \Omega$ Lebesgue's point of Du and for every ball $B_{2R}(x) \subset \Omega$.

Note that also here we give the Theorem in the form of an *a priori estimate*, for the reasons already explained. See Section 10.5 for some more words on measure data problems and SOLA for this kind of growth conditions, and in particular Theorem 10.19.

The reason for Lieberman's words we mentioned above can be found in [**112**]: (4.7.1) is the Euler-Lagrange equation for local minimizers of the functional

$$w \to \int_{\Omega} G\big(|Dw|\big)\, dx, \tag{4.7.7}$$

where $G' = g$. Since again in the case $G(t) = t^p$, $p > 1$ (or, more in general, $G(t) = (s^2 + t^2)^{p/2}$, $p > 1$, $s \in [0, 1]$) (4.7.7) gives back the classical p-Dirichlet energy, whose Euler-Lagrange equation actually inspires the ultra-classic Ladyzhenskaya and Ural'tseva growth conditions, it is therefore natural to investigate more general forms of $G(\cdot)$ than power-like ones. In [**112**] a full basic regularity theory (local boundedness of solution and its gradient, zero and first order Hölder regularity, Harnack's inequalities and properties De Giorgi classes) is proved.

Once having the *a priori* potential bound (4.7.6) at hand, the following Corollary follows in a straightforward way using a covering argument:

COROLLARY 4.28. *Let $u \in W^{1,G}(\Omega)$ be a weak solution to (4.7.3), where the vector field $a(\cdot)$ satisfies the assumption (4.7.4)-(4.7.5). Then*

$$\mathbf{I}_1^{|\mu|}(\cdot, R) \in L^\infty_{\mathrm{loc}}(\Omega) \quad \text{for some } R > 0 \qquad \Longrightarrow \qquad Du \in L^\infty_{\mathrm{loc}}(\Omega; R^n).$$

Moreover the following local estimate holds true:

$$\|Du\|_{L^\infty(B_{R/2})} \le c\, g^{-1}\left(\left\|\mathbf{I}_1^{|\mu|}(\cdot,R)\right\|_{L^\infty(B_R)}\right) + c \fint_{B_R} |Du|\, d\xi,$$

for every ball $B_R \subset \Omega$ and with constant depending upon n, ν, L, δ, g_0.

It is worth to remark that this shows that the classic, sharp Riesz potential criterium implying the Lipschitz continuity of solutions to the Poisson equations still remains valid when considering operators of the type (4.7.1). Only few modifications of the proofs given in [**106**] at this point lead to the following borderline conditions for the continuity of the gradient Du:

PROPOSITION 4.29 ([**17**]). *Let $u \in W^{1,G}(\Omega)$ be as in Corollary 4.28 and suppose that at least one of the following assumptions holds:*

(1) $\lim_{R\to 0} \mathbf{I}_1^{|\mu|}(\cdot,R) = 0$ *locally uniformly in Ω with respect to x;*
(2) $\mu \in L(n,1)$ *locally in Ω;*
(3) $|\mu|(B_R) \le c\, R^{n-1} h(R)$, *for some constant $c \ge 1$ and with* $\int_0 h(\rho)\dfrac{d\rho}{\rho} < \infty$.

Then Du is continuous in Ω.

CHAPTER 5

Fractional differentiability for nonlinear heat equation

We first sketch here (again) the SOLA approach in the parabolic setting, which we shall refer to also in the following: consider the regular problem

$$\begin{cases} \partial_t u - \operatorname{div} a(x,t,Du) = f & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{5.0.8}$$

with $f \in L^2(\Omega_T) \cap C^\infty(\Omega)$ regular function and its unique solution

$$u \in L^2(-T,0;W_0^{1,2}(\Omega)) \cap C^0([-T,0];L^2(\Omega)); \tag{5.0.9}$$

such a solution exists via monotonicity methods, see for instance [**115**]. We shall consider a sequence of regular functions $\{f_k\}$ in the sense above which converges weakly in the sense of the measures to μ, with the properties

$$\|f_k\|_{L^1(\Omega_T)} \le |\mu|(\Omega_T) \qquad \text{and} \qquad \|f_k\|_{L^1(Q_\rho)} \le |\mu|(Q_{\rho+1/k}). \tag{5.0.10}$$

We shall denote by u_k the solution to (5.0.8) with $f \equiv f_k$ and we deduce all regularity theorems of Section 4.1 first for the solutions u_k; finally, we shall obtain the regularity result for the solution u of the original problem with measure data exploiting the fact that the properties are stable when passing to the limit, i.e. they involve only the L^1 norm of f_k, see (5.0.10). Note that the *a priori estimates* we are going to deduce are strong enough to imply pointwise convergence of the gradients, using a classic compactness result by Simon [**135**], without using the *ad hoc* result proved in [**24**].

5.1. A global estimate

The following estimate was also proved in [**25**]. We propose again its proof for two reasons: we compute explicitly the dependence upon the norm of f, which will be useful later, and we show how to use the Steklov averaging when testing the equation with (truncates of) the solution.

LEMMA 5.1 (Global estimate). *Let $u \in L^2(-T,0;W_0^{1,2}(\Omega))$ be a weak solution to the problem (5.0.8) and let q satisfy (4.1.5). Then we have the global estimate*

$$\|Du\|_{L^q(\Omega_T)} \le c\left[s + \|f\|_{L^1(\Omega_T)}\right],$$

with $c \equiv c(n,\nu,L,q,|\Omega|,T)$.

PROOF. We first suppose $\|f\|_{L^1(\Omega_T)} \le 1$ and later show the statement for the general case by a scaling argument. Starting with the Steklov formulation of (5.0.8), for a.e. $t \in (-T,0)$ we have

$$\int_\Omega \left[\partial_t u_h(\cdot,t)\varphi + \langle [a(\cdot,t,Du)]_h, D\varphi\rangle\right] dx = \int_\Omega f_h(\cdot,t)\varphi\, dx, \tag{5.1.1}$$

for any test function $\varphi \in W_0^{1,2}(\Omega)$, where u_h denotes the Steklov average of u defined in (3.3.5). The proof is performed by applying a classical truncation technique (see [**26, 24,** 118]). For $k \in \mathbb{N}$, we define the truncation operators

$$T_k(\varsigma) := \max\big\{-k, \min\{k, \varsigma\}\big\}, \qquad \Phi_k(\varsigma) := T_1\big(\varsigma - T_k(\varsigma)\big), \tag{5.1.2}$$

for each $\varsigma \in \mathbb{R}$. Moreover we define

$$D_k := \{z \in \Omega_T : k < |u(z)| \le k+1\}. \tag{5.1.3}$$

Furthermore let $\Psi_k : \mathbb{R} \to \mathbb{R}$ be defined as $\Psi_k(\varsigma) := \int_0^\varsigma \Phi_k(\zeta)\, d\zeta$. An explicit calculation of Ψ_k shows immediately (see [70]) that

$$\Psi_k(\varsigma) \ge 0 \qquad \text{for any } \varsigma \in \mathbb{R}. \tag{5.1.4}$$

We now test the Steklov formulation (5.1.1) with the function

$$\varphi(x,t) := \zeta(t)\, \Phi_k\big(u_h(x,t)\big), \quad x \in \Omega,$$

for a function $\zeta(t)$ in time. Note that φ is admissible in (5.1.1) for a.e. $t \in (-T, 0)$, i.e. $\varphi(\cdot, t) \in W_0^{1,2}(\Omega)$. Integrating the resulting equation over $(-T, 0)$ with respect to t gives

$$\int_{\Omega_T} \partial_t u_h \Phi_k(u_h)\zeta(t)\, dz + \int_{\Omega_T} \langle [a(\cdot, t, Du)]_h, D\Phi_k(u_h)\rangle \zeta(t)\, dz = \int_{\Omega_T} \Phi_k(u_h) f_h \zeta(t)\, dz.$$

For $\tau \in (-T, 0)$ and $\varepsilon > 0$ let $\zeta \in W^{1,\infty}(\mathbb{R})$ be defined as

$$\zeta(t) := \begin{cases} 1 & \text{if } t \le \tau, \\ 1 - \dfrac{1}{\varepsilon}(t-\tau) & \text{if } \tau < t \le \tau + \varepsilon, \\ 0 & \text{if } t > \tau + \varepsilon. \end{cases} \tag{5.1.5}$$

Using this function in the previous identity and recalling the definition of Ψ_k we obtain

$$\begin{aligned} \int_{\Omega_T} \partial_t u_h \Phi_k(u_h)\zeta(t)\, dz &= \int_{\Omega_T} \partial_t \left[\Psi_k(u_h)\zeta(t)\right] dz - \int_{\Omega_T} \Psi_k(u_h)\zeta'(t)\, dz \\ &= -\int_\Omega \Psi_k(u_h)(x, -T)\, dx - \int_{\Omega_T} \Psi_k(u_h)\zeta'(t)\, dz \end{aligned}$$

for a.e. $\tau \in (-T, 0)$. Now, the second integral on the right–hand side of the preceding equality converges, as $\varepsilon \searrow 0$, to $\int_\Omega \Psi_k(u)(x,\tau)\, dx$ for a.e. $\tau \in (-T, 0)$, whereas the first integral converges to 0 as $h \searrow 0$, since $u_h(\cdot, -T) \to 0$ in the sense of L^2. Therefore, letting first $\varepsilon \searrow 0$ then $h \searrow 0$, we obtain for a.e. $\tau \in (-T, 0)$

$$\int_\Omega \Psi_k(u)(x,\tau)\, dx + \int_{-T}^{\tau} \int_\Omega \big\langle a(x,t,Du), D\Phi_k(u)\big\rangle\, dx\, dt = \int_{-T}^{\tau} \int_\Omega \Phi_k(u) f\, dx\, dt. \tag{5.1.6}$$

Now recalling the definition of D_k and exploiting the explicit calculations of $\Phi_k(u)$, $\Psi_k(u)$ and $D\Phi_k(u)$ (we refer the reader to [70] for a detailed calculation) the terms of the previous identity can be treated as follows:

$$\int_{\Omega_T} \langle a(x,t,Du), D\Phi_k(u)\rangle\, dz = \int_{D_k} \langle a(x,t,Du), Du\rangle\, dz,$$

$$\left| \int_{\Omega_T} \Phi_k(u) f\, dz \right| \le \int_{\Omega_T} |f|\, dz,$$

$$\int_\Omega \Psi_k(u)(x,\tau)\,dx \geq 0 \qquad \text{for all } k \text{ and for every } \tau \in (-T,0),$$

since $u \in C^0([-T,0];L^2(\Omega))$ and by (5.1.4). Now exploiting the structure conditions $(4.1.2)_1$ and $(4.1.2)_3$, then (5.1.6) together with the previous estimates, and finally Young's inequality and the fact that $\|f\|_{L^1(\Omega_T)} \leq 1$, we deduce

$$\begin{aligned}
\nu \int_{D_k} |Du|^2\,dz &\leq \int_{D_k} \langle a(x,t,Du) - a(x,t,0), Du\rangle\,dz \\
&\leq \int_\Omega \Psi_k(u)(x,0)\,dx + \int_{D_k} \langle a(x,t,Du), Du\rangle\,dz \\
&\qquad - \int_{D_k} \langle a(x,t,0), Du\rangle\,dz \\
&\leq \int_{\Omega_T} |f|\,dz + Ls\int_{D_k} |Du|\,dz \\
&\leq 1 + \varepsilon \int_{D_k} |Du|^2\,dz + \frac{L^2 s^2}{4\varepsilon}|D_k|.
\end{aligned}$$

Choosing $\varepsilon = \nu/2$ we therefore conclude

$$\int_{D_k} |Du|^2\,dz \leq c(\nu,L)\Big(1 + s^2|D_k|\Big). \tag{5.1.7}$$

Secondly, (5.1.6) for $k=0$ yields, writing for shortness $D_0(\tau) := D_0 \cap (\Omega \times (-T,\tau))$,

$$\begin{aligned}
\|f\|_{L^1(\Omega_T)} &\geq \int_{D_0(\tau)} \langle a(x,t,Du), Du\rangle\,dz + \int_\Omega \Psi_0(u)(x,\tau)\,dx \\
&= \int_{D_0(\tau)} \langle a(x,t,Du) - a(x,t,0), Du\rangle\,dz \\
&\qquad + \int_{D_0(\tau)} \langle a(x,t,0), Du\rangle\,dz + \int_\Omega \Psi_0(u)(x,\tau)\,dx \\
&\geq \int_\Omega \Psi_0(u)(x,\tau)\,dx - Ls\int_{D_0} |Du|\,dz,
\end{aligned}$$

keeping in mind the structure conditions (4.1.2) and discarding the positive term. Now, calculating Ψ_0 explicitely, we achieve

$$\int_\Omega \Psi_0(u)(x,\tau)\,dx \geq \int_\Omega |u(x,\tau)|\,dx - \frac{1}{2}|\Omega|.$$

Thus, merging this with the last estimate, the fact that $\|f\|_{L^1(\Omega_T)} \leq 1$ and $s \leq 1$, together with Young's inequality and (5.1.7), we finally conclude the L^∞–L^1 estimate

$$\begin{aligned}
\sup_{\tau\in(-T,0)} \int_\Omega |u(x,\tau)|\,dx &\leq 1 + Ls\int_{D_0} |Du|\,dz + \frac{1}{2}|\Omega| \\
&\leq 1 + L^2s^2|\Omega_T| + \int_{D_0} |Du|^2\,dz + \frac{1}{2}|\Omega| \\
&\leq c\Big(1 + s^2|\Omega_T|\Big) + \frac{1}{2}|\Omega| \leq c(\nu,L,|\Omega|,T).
\end{aligned} \tag{5.1.8}$$

Let $\tilde{q} > 1$ be a free parameter, which will be chosen later. Using Hölder's inequality, (5.1.7) and the definition of D_k in (5.1.3) we obtain for $1 \leq q < 2$ and for any $k \in \mathbb{N}$

$$\int_{D_k} |Du|^q\,dz \leq |D_k|^{1-\frac{q}{2}}\Big(\int_{D_k} |Du|^2\,dz\Big)^{\frac{q}{2}}$$

$$\leq c\,|D_k|^{1-\frac{q}{2}} + c\,|D_k|$$
$$\leq c\,k^{-\tilde{q}(1-\frac{q}{2})}\left(\int_{D_k} |u|^{\tilde{q}}\,dz\right)^{1-\frac{q}{2}} + c\,|D_k|,$$

with $c \equiv c(L/\nu, q)$. Now we split in the following way, using also Hölder's inequality and (5.1.7) in order to deduce

$$\begin{aligned}
\int_{\Omega_T} |Du|^q\,dz &= \int_{D_0} |Du|^q\,dz + \sum_{k=1}^{\infty}\int_{D_k} |Du|^q\,dz \\
&\leq c\left[|D_0|^{1-\frac{q}{2}}\left(\int_{D_0} |Du|^2\,dz\right)^{\frac{q}{2}}\right. \\
&\qquad\left. + \sum_{k=1}^{\infty}|D_k| + \sum_{k=1}^{\infty} k^{-\tilde{q}(1-\frac{q}{2})}\left(\int_{D_k} |u|^{\tilde{q}}\,dz\right)^{1-\frac{q}{2}}\right] \\
&\leq c\left[1 + |\Omega_T| + \sum_{k=1}^{\infty} k^{-\tilde{q}(1-\frac{q}{2})}\left(\int_{D_k} |u|^{\tilde{q}}\,dz\right)^{1-\frac{q}{2}}\right] \\
&\leq c\left[1 + \left(\sum_{k=1}^{\infty} k^{-\tilde{q}(\frac{2}{q}-1)}\right)^{\frac{q}{2}}\left(\int_{\Omega_T} |u|^{\tilde{q}}\,dz\right)^{1-\frac{q}{2}}\right],
\end{aligned} \tag{5.1.9}$$

for a constant $c \equiv c(\nu, L, q, |\Omega|, T)$. To treat the integral on the right–hand side we remark that a well–known version of the Gagliardo–Nirenberg embedding (see for example [80, Chapter 7]), applied on time slices $t \in (-T, 0)$, gives us

$$\|u(\cdot,t)\|_{L^{\tilde{q}}(\Omega)} \leq c(n,q)\|Du(\cdot,t)\|^{\theta}_{L^q(\Omega)}\|u(\cdot,t)\|^{1-\theta}_{L^1(\Omega)},$$

for an interpolation parameter $0 \leq \theta \leq 1$ such that $\frac{1}{\tilde{q}} = \theta\left(\frac{1}{q} - \frac{1}{n}\right) + 1 - \theta$. If we choose $\tilde{q} \equiv q(n+1)/n$ an we keep in mind (5.1.8) it is easy to check that

$$\int_{\Omega_T} |u|^{\tilde{q}}\,dz \leq c(n,\nu,L,q,|\Omega|)\int_{\Omega_T} |Du|^q\,dz$$

and that $\tilde{q}\left(\frac{2}{q}-1\right) > 1$, if q satisfies (4.1.5), so the series appearing in (5.1.9) is convergent. Subsequently we can write

$$\int_{\Omega_T} |Du|^q\,dz \leq c\left[1 + \left(\int_{\Omega_T} |Du|^q\,dz\right)^{1-q/2}\right]$$

with $c \equiv c(n,\nu,L,q,|\Omega|,T)$, and finally conclude using Hölder's inequality, since $1-\frac{q}{2} < 1$, to re–absorb the right–hand side norm of Du:

$$u \in L^q(-T,0;W^{1,q}_0(\Omega)), \qquad \text{i.e.} \qquad \int_{\Omega} |Du|^q\,dz \leq c, \tag{5.1.10}$$

for all q satisfying (4.1.5), with a constant c that depends on $n, \nu, L, q, |\Omega|, T$. In a last step, it remains to eliminate the assumptions $\|f\|_{L^1(\Omega_T)} \leq 1$ by a scaling argument: Let $u \in L^2(-T,0;W^{1,2}_0(\Omega))$ be as in the statement of the Lemma. We define $F := \|f\|_{L^1(\Omega_T)} + s > 0$ (otherwise the statement is trivial) and let

$$\bar{u} := \frac{1}{F}\,u, \quad \bar{f} := \frac{1}{F}\,f, \quad \bar{a}(x,t,z) := \frac{1}{F}\,a(x,t,Fz).$$

We therefore easily see that

$$\bar{u}_t - \operatorname{div}\bar{a}(x,t,D\bar{u}) = \bar{f} \quad \text{on } \Omega_T \quad \text{and} \quad \|\bar{f}\|_{L^1(\Omega_T)} \leq 1.$$

Furthermore, $\bar{a}$ fulfills the conditions (4.1.2) with s replaced by $\bar{s} := s/F$ and we have $\bar{s} = s/F \leq 1$. Therefore estimate (5.1.10) holds for $\bar{u}$. Having in mind $\bar{u} = u/F$ we conclude

$$\int_{\Omega_T} |Du|^q \, dz \leq c \left[s + \|f\|_{L^1(\Omega_T)} \right]^q,$$

with $c \equiv c(n, \nu, L, q, |\Omega|, T)$. The proof is now complete. □

5.2. Comparison lemmata

A main tool of the proof of Theorems 4.1 and 4.3 is a series of comparison procedures. Let us first fix $z_0 \in \Omega_T$ and $0 < \rho \leq 1$ such that $Q_\rho(z_0) \subset \Omega_T$, and let $v \in u + L^2(\Lambda_\rho(t_0); W_0^{1,2}(B_\rho(x_0)))$ the unique weak solution to

$$\begin{cases} \partial_t v - \operatorname{div} a(x, t, Dv) = 0 & \text{in } Q_\rho(z_0), \\ v = u & \text{on } \partial_{\mathcal{P}} Q_\rho(z_0). \end{cases} \tag{5.2.1}$$

Existence and uniqueness directly follow from the structure conditions and can be referred from [115]. Since v is the solution of a homogeneous problem, we have the following **higher integrability** property for v (see [79, Theorem 2.1] or [124]):

LEMMA 5.2. *Let $v \in u + L^2(\Lambda_\rho(t_0); W_0^{1,2}(B_\rho(x_0)))$ be the solution of (5.2.1), where the vector field a satisfies the ellipticity and monotonicity assumptions $(4.1.2)_1$ and $(4.1.2)_2$. Then there exists $\chi_0 > 1$, depending on n and L/ν, such that $Dv \in L^{2\chi_0}_{\mathrm{loc}}(Q_\rho(z_0))$. Furthermore there exists a constant $c \equiv c(n, L/\nu)$ such that for any $Q_{2\tilde{\rho}} \Subset Q_\rho(z_0)$ and any $\chi \leq \chi_0$ the following estimate holds true:*

$$\left[\fint_{Q_{\tilde{\rho}}} |Dv|^{2\chi} \, dz \right]^{1/\chi} \leq c \fint_{Q_{2\tilde{\rho}}} (s + |Dv|)^2 \, dz.$$

REMARK 5.3. The higher integrability statement in [79] is done for homogeneous parabolic systems of the special type $v_t - \operatorname{div}(a(z)Dv) = 0$ with bounded, measurable, continuous and elliptic coefficients $a(z)$. However, some minor modifications of the proof in [79], involving the growth and ellipticity conditions $(4.1.2)_1$ and $(4.1.2)_2$, also provide the result for equations (and systems) of the type (5.2.1).

REMARK 5.4. *Once having higher integrability in terms of Lemma 5.2 at hand, Lemma 3.13 allows to reduce the integral power in the sense of*

$$\left[\fint_{Q_{\tilde{\rho}}} |Dv|^2 \, dz \right]^{1/2} \leq c \fint_{Q_{2\tilde{\rho}}} (s + |Dv|) \, dz,$$

with a constant c depending on $n, L/\nu$.

A second step consists in considering the following homogeneous frozen Dirichlet problem on a smaller parabolic cylinder

$$\begin{cases} \partial_t v_0 - \operatorname{div} a(x_0, t, Dv_0) = 0 & \text{in } Q_{\rho/4}(z_0), \\ v_0 = v & \text{on } \partial_{\mathcal{P}} Q_{\rho/4}(z_0), \end{cases} \tag{5.2.2}$$

and its unique solution which belongs to $v + L^2(\Lambda_{\rho/4}(t_0); W_0^{1,2}(B_{\rho/4}(x_0)))$. Again, existence and uniqueness of such a solution can be referred from [115].

We now establish suitable comparison estimates between the solution u of the original problem and the solution v of the homogeneous one, respectively v_0 of the homogeneous frozen one. Note at this point that it is essential to involve nothing more than the L^1 norm of the inhomogeneity f on the right–hand side. Therefore the proofs again involve certain truncation techniques. Define, for the sake of shortness, the following quantity:

$$\delta(q) := \frac{N}{q} - (N-1) \tag{5.2.3}$$

and notice that by the bounds for q in (4.1.5), we have $\delta \leq 1$. Consequently denote by $\sigma(q)$ the quantity

$$\sigma(q) := N - q(N-1) = \delta(q)q.$$

Note again that $\sigma(q) > 0$ for all q satisfying (4.1.5) and also $\sigma(q) \leq q$.

REMARK 5.5. At certain points in the proofs of our results it is useful to scale from an arbitrary parabolic cylinder $Q_\rho(z_0)$ to Q_1 via the following scaling procedure: for $(y,s) \in Q_1$ we define

$$\begin{cases} \tilde{u}(y,s) := \dfrac{1}{\rho}\, u(x_0 + Ry, t_0 + R^2 s), \\ \tilde{v}(y,s) := \dfrac{1}{\rho}\, v(x_0 + Ry, t_0 + R^2 s), \\ \tilde{a}(y,s,\xi) := a(x_0 + Ry, t_0 + R^2 s, \xi), \\ \tilde{g}(y,s) := R\, g(x_0 + Ry, t_0 + R^2 s). \end{cases}$$

Then it is easy to verify that

$$\partial_t \tilde{u} - \operatorname{div}\, \tilde{a}(x,t,D\tilde{u}) = \tilde{f}, \quad \partial_t \tilde{v} - \operatorname{div}\, \tilde{a}(x,t,D\tilde{v}) = 0 \quad \text{in } Q_1.$$

and $\tilde{u} = \tilde{v}$ on $\partial_{\mathrm{par}} Q$. Furthermore it is easy to check that the new vector field $\tilde{a}$ satisfies the growth and and monotonicity properties described in (4.2.1).

We can start now with comparison between u and v:

LEMMA 5.6. *Let u as in* (5.0.9) *be the solution of problem* (5.0.8) *and v the solution of problem* (5.2.1)*. Then the following comparison estimate holds true:*

$$\|Du - Dv\|_{L^q(Q_\rho(z_0))} \leq c\, \rho^{\delta(q)} \|f\|_{L^1(Q_\rho(z_0))},$$

for all q satisfying (4.1.5)*, with $c \equiv c(n,\nu,q)$.*

PROOF. We first consider the case $Q_\rho(z_0) = Q_1(0) \equiv Q \equiv B \times \Lambda$ and suppose $\|f\|_{L^1(Q)} = 1$. The general case will follow again by a scaling argument. We start with the Steklov formulations of the equations which write as

$$\int_B \Big[\partial_t u_h(\cdot,t)\varphi + \langle [a(\cdot,t,Du)]_h, D\varphi\rangle\Big]\, dx = \int_B f_h(\cdot,t)\varphi\, dx, \tag{5.2.4}$$

for all $\varphi \in W_0^{1,2}(B)$ and for a.e. $t \in \Lambda$, respectively

$$\int_B \Big[\partial_t v_h(\cdot,t)\varphi + \langle [a(\cdot,t,Dv)]_h, D\varphi\rangle\Big]\, dx = 0, \tag{5.2.5}$$

for all $\varphi \in W_0^{1,2}(B)$ and for a.e. $t \in \Lambda$.

Defining now the truncation operator $\Phi_k(\varsigma)$ as in (5.1.2), having again $\Psi_k(\varsigma) := \int_0^\varsigma \Phi_k(\zeta)\, d\zeta$ as in the proof of Lemma 5.1 and denoting

$$D_k := \{z \in Q_1 : k < |u(z) - v(z)| \leq k+1\},$$

we test the difference of (5.2.4) and (5.2.5) by $\varphi(x,t) := \Phi_k(u_h - v_h)(x,t)\zeta(t)$, $x \in B$, where $\zeta(\cdot)$ denotes a Lipschitz continuous function in time, and subsequently integrate over Λ with respect to t to achieve

$$\begin{aligned}\int_Q \partial_t(u_h - v_h)\Phi_k(u_h - v_h)\zeta\, dz \\ + \int_Q \langle [a(\cdot,t,Du)]_h - [a(\cdot,t,Dv)]_h, D\Phi_k(u_h - v_h)\rangle\, \zeta\, dz \\ = \int_Q f_h \Phi_k(u_h - v_h)\zeta\, dz.\end{aligned}$$

Now choosing $\zeta(t)$ as in (5.1.5) and arguing exactly as in (5.1.6), letting $\varepsilon \searrow 0$, then $h \searrow 0$ and taking the supremum, we finally arrive at

$$\begin{aligned}&\sup_{-1<\tau<1} \int_B \Psi_k(u-v)(x,\tau)\, dx \\ &+ \int_Q \langle a(x,t,Du) - a(x,t,Dv), D\Phi_k(u-v)\rangle\, \zeta\, dz \le \int_Q |f||\Phi_k(u-v)|\, dz.\end{aligned} \tag{5.2.6}$$

Writing (5.2.6) for $k = 0$ and exploiting $(4.1.2)_2$ we immediately have

$$\sup_{-1<\tau<1} \int_B \Psi_0(u-v)(x,\tau)\, dx \le \int_Q |f|\, dz = 1.$$

On the other hand carefully exploiting Young's inequality and the explicit expression for Ψ_0 we have for a.e. $\tau \in \Lambda$

$$\begin{aligned}\int_B |u(\cdot,\tau) - v(\cdot,\tau)|\, dx &= \int_{B\cap\{|u-v|<1\}} |\ldots|\, dx + \int_{B\cap\{|u-v|\ge 1\}} |\ldots|\, dx \\ &\le \frac{1}{2}\int_{B\cap\{|u-v|<1\}} |u(\cdot,\tau) - v(\cdot,\tau)|^2\, dx + \frac{1}{2}|B\cap\{|u-v|<1\}| \\ &\quad + \int_{B\cap\{|u-v|\ge 1\}} |u(\cdot,\tau) - v(\cdot,\tau)|\, dx \\ &= \frac{1}{2}\int_{B\cap\{|u-v|<1\}} |u(\cdot,\tau) - v(\cdot,\tau)|^2\, dx + \frac{1}{2}|B| - \frac{1}{2}|B\cap\{|u-v|\ge 1\}| \\ &\quad + \int_{B\cap\{|u-v|\ge 1\}} |u(\cdot,\tau) - v(\cdot,\tau)|\, dx \\ &= \int_B \Psi_0(u-v)(\cdot,\tau)\, dx + \frac{1}{2}|B|.\end{aligned}$$

Merging this estimate with the previous one, we arrive at

$$u - v \in L^\infty(-1,1;L^1(B)) \qquad \text{and} \qquad \|u-v\|_{L^\infty(-1,1;L^1(B))} \le c(n).$$

Having again a look at (5.2.6) , keeping in mind that $D\Phi_k(u-v) = Du - Dv$ on the set D_k and $D\Phi_k(u-v) = 0$ otherwise, subsequently exploiting $(4.1.2)_2$, $|\Phi_k| \le 1$ and (5.1.4), we achieve

$$\begin{aligned}\nu \int_{D_k} |Du - Dv|^2\, dz &\le \int_{D_k} \langle a(x,t,Du) - a(x,t,Dv), Du - Dv\rangle\, dz \\ &\le \int_Q |f|\, dz = 1,\end{aligned}$$

and thus

$$\int_{D_k} |Du - Dv|^2\, dx \le \frac{1}{\nu}.$$

Now further proceeding exactly as in the proof of Lemma 5.1, here with the function $u-v$ instead of u, we finally conclude

$$Du - Dv \in L^q(Q_1), \quad \|Du - Dv\|_{L^q(Q_1)} \le c(n,\nu,q), \tag{5.2.7}$$

for all q satisfying (4.1.5) (cfr. (5.1.10)). The case $0 < F := \|f\|_{L^1(Q_1)} \neq 1$ (if $\|f\|_{L^1(Q_1)} = 0$ the thesis is trivial since $u = v$ by the monotonicity of the vector field) is faced exactly as in the proof of Lemma 5.1, considering the functions $\bar{u} := u/F$ and $\bar{v} := v/F$; consequently we get

$$\|Du - Dv\|_{L^q(Q_1)} \le c\,\|f\|_{L^1(Q_1)}.$$

Finally for the general case $Q_\rho(z_0)$ we consider the rescaled functions $\tilde{u}$ and $\tilde{v}$, defined in Q_1, as in Remark 5.5 and by (5.2.7) we arrive at

$$\rho^{-\frac{N}{q}}\,\|Du - Dv\|_{L^q(Q_\rho(x_0))} = \|D\tilde{u} - D\tilde{v}\|_{L^q(Q_1)} \le c\,\big\|\tilde{f}\big\|_{L^1(Q_1)}$$
$$= c\,\rho^{-(N-1)}\big\|f\big\|_{L^1(Q_\rho(z_0))},$$

which is the desired estimate. Let us note that (4.1.5) ensures that the exponent of ρ is positive. The proof is complete. □

Subsequently, we establish a comparison estimate between the solution v of the homogeneous problem and the solution v_0 of the frozen homogeneous one:

LEMMA 5.7. *Let $v \in u + L^2(\Lambda_\rho(t_0); W_0^{1,2}(B_\rho(x_0)))$ be the unique weak solution to (5.2.1) and $v_0 \in v + L^2(\Lambda_{\rho/4}(t_0); W_0^{1,2}(B_{\rho/4}(x_0)))$ the one of (5.2.2). Then the following comparison estimate holds true:*

$$\|Dv - Dv_0\|_{L^q(Q_{\rho/4}(z_0))} \le c\,\rho^{\delta(q)}\left[\int_{Q_\rho(z_0)} \big(s + |Dv|\big)\,dz\right],$$

with $c = c(n, L/\nu, q)$.

PROOF. To focus on the main aspects of the proof, the following argumentation is merely formal, since it would need time derivatives of both v and v_0. On the other hand, the calculations can easily be made rigorous by again involving the Steklov formulation of the equations, thereafter passing to the limit. We test the difference of the equations

$$\int_{Q_{\rho/4}(z_0)} \Big[\partial_t(v - v_0)\varphi + \langle a(x,t,Dv) - a(x_0,t,Dv_0), D\varphi\rangle\Big]\,dz = 0,$$

by the function $\varphi := (v - v_0)\zeta$, with ζ as in (5.1.5), and proceed analogously to the argumentation in the proof of Lemma 5.6 to achieve (5.2.6), arriving at

$$\sup_{\tau \in \Lambda_{\rho/4}(t_0)} \int_{B_{\rho/4}(x_0)} |v - v_0|^2(x,\tau)\,dx$$
$$+ \int_{Q_{\rho/4}(z_0)} \langle a(x,t,Dv) - a(x_0,t,Dv_0), Dv - Dv_0\rangle\,dz \le 0,$$

and therefore by $(4.1.2)_1$ also at

$$\nu \int_{Q_{\rho/4}(z_0)} |Dv - Dv_0|^2\,dz$$
$$\le \int_{Q_{\rho/4}(z_0)} \langle a(x_0,t,Dv) - a(x_0,t,Dv_0), Dv - Dv_0\rangle\,dz$$
$$\le \sup_{\tau \in \Lambda_{\rho/4}(t_0)} \int_{B_{\rho/4}(x_0)} |v - v_0|^2(x,\tau)\,dx$$

$$+\int_{Q_{\rho/4}(z_0)} \langle a(x,t,Dv)-a(x_0,t,Dv_0), Dv-Dv_0\rangle\, dz$$
$$+\int_{Q_{\rho/4}(z_0)} \langle a(x_0,t,Dv)-a(x,t,Dv), Dv-Dv_0\rangle\, dz$$
$$\leq \left|\int_{Q_{\rho/4}(z_0)} \langle a(x_0,t,Dv)-a(x,t,Dv), Dv-Dv_0\rangle\, dz\right|.$$

Exploiting now (4.1.2)$_4$ and using Young's inequality we finally arrive at

$$\nu\int_{Q_{\rho/4}(z_0)} |Dv-Dv_0|^2\, dz$$
$$\leq Lc(\varepsilon)\,\rho^2\int_{Q_{\rho/4}(z_0)} \left(s^2+|Dv|^2\right) dz + L\varepsilon\int_{Q_{\rho/4}(z_0)} |Dv-Dv_0|^2\, dz.$$

Choosing $\varepsilon\equiv\nu/(2L)$ and reabsorbing the last term of the estimate, we get

$$\int_{Q_{\rho/4}(z_0)} |Dv-Dv_0|^2\, dz \leq c(L/\nu)\,\rho^2\int_{Q_{\rho/4}(z_0)} \left(s^2+|Dv|^2\right) dz. \tag{5.2.8}$$

Using now again Hölder's inequality, (5.2.8) and thereafter the reverse Hölder inequality of Remark 5.4, we deduce

$$\int_{Q_{\rho/4}(z_0)} |Dv-Dv_0|^q\, dz \leq c\,\rho^{N(1-\frac{q}{2})}\left[\int_{Q_{\rho/4}(z_0)} |Dv-Dv_0|^2\, dz\right]^{\frac{q}{2}}$$
$$\leq c\,\rho^{N+q}\left[⨍_{Q_{\rho/4}(z_0)} \left(s^2+|Dv|^2\right) dz\right]^{\frac{q}{2}}$$
$$\leq c\,\rho^{N-q(N-1)}\left[\int_{Q_{\rho}(z_0)} \left(s+|Dv|\right) dz\right]^{q}$$

with $c\equiv c(n,L/\nu,q)$, which is the desired comparison estimate. □

Finally we deduce an **energy estimate** for the L^2 norm of Dv_0 in terms of L^q norm of Du, in the following sense:

LEMMA 5.8. *Let u be a weak solution to (5.0.8) with $f\in L^1(\Omega_T)$, v and v_0 respectively as in (5.2.1) and (5.2.2). Then the following estimate holds true:*

$$\left[\int_{Q_{\rho/4}} (s^2+|Dv_0|^2)\, dz\right]^{1/2}$$
$$\leq c\,\rho^{1-\frac{2}{q}+n\frac{q-2}{2q}}\left[\|s+|Du|\|_{L^q(Q_\rho(z_0))}+\|f\|_{L^1(Q_\rho(z_0))}\right],$$

with $c\equiv c(n,L/\nu,q)$.

PROOF. We start, using the intermediate comparison estimate (5.2.8), reverse Hölder's inequality of Remark 5.4 and Hölder's inequality (note that $\rho\leq 1$), to deduce

$$⨍_{Q_{\rho/4}} (s^2+|Dv_0|^2)\, dz \leq 2⨍_{Q_{\rho/4}} (s^2+|Dv|^2)\, dz + 2⨍_{Q_{\rho/4}} |Dv-Dv_0|^2\, dz$$
$$\leq c(L/\nu)⨍_{Q_{\rho/4}} \left(s^2+|Dv|^2\right) dz$$
$$\leq c(n,L/\nu)\left[⨍_{Q_{\rho/2}} \left(s+|Dv|\right) dz\right]^2$$

$$\leq c(n, L/\nu, q)\left[\fint_{Q_{\rho/2}} (s^q + |Dv|^q)\, dz\right]^{\frac{2}{q}}.$$

Now exploiting Lemma 5.6 and recalling $\rho \leq 1$ we get

$$\begin{aligned}
&\left[\int_{Q_{\rho/4}} (s^2 + |Dv_0|^2)\, dz\right]^{\frac{1}{2}} \\
&\qquad \leq c\,\rho^{\frac{N}{2}}\left[\fint_{Q_{\rho/2}} (s^q + |Du|^q)\, dz + \fint_{Q_{\rho/2}} |Du - Dv|^q\, dz\right]^{\frac{1}{q}} \\
&\qquad \leq c\,\rho^{1-\frac{2}{q}+n\frac{q-2}{2q}}\left[\|f\|_{L^1(Q_\rho(z_0))} + \left(\int_{Q_\rho} (s + |Du|)^q\, dz\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $c \equiv c(n, L/\nu, q)$. This finishes the proof.

□

5.3. Fractional estimates for the reference problem

In this chapter we consider the reference problem (5.2.2) which is homogeneous and with no dependence of the vector field on the space variable, while the dependence on the time variable is merely measurable. We will show by approximation that the gradient Dv_0 of its solution v_0 is differentiable with respect to space and at least "almost" half differentiable with respect to time. This is the content of the following

LEMMA 5.9. *Let $Q_\rho(z_0) \subset \Omega_T$ be a parabolic cylinder and let furthermore $v_0 \in v + L^2(\Lambda_{\rho/4}(t_0); W_0^{1,2}(B_{\rho/4}(x_0)))$ be the solution of the frozen Dirichlet problem (5.2.2) on the cylinder $Q_{\rho/4}(z_0)$, where the vector field a satisfies the hypotheses (4.1.2). Then for any $\theta \in (0, 1/2)$ we have*

$$Dv_0 \in L^2_{\mathrm{loc}}(\Lambda_{\rho/4}(t_0); W^{1,2}_{\mathrm{loc}}(B_{\rho/4})) \cap W^{\theta,2}_{\mathrm{loc}}(\Lambda_{\rho/4}(t_0); L^2_{\mathrm{loc}}(B_{\rho/4})).$$

Moreover, there exists a constant $c \equiv c(n, L/\nu)$ such that for arbitrary $\eta \in \mathbb{R}^n$ the following estimates hold true:

$$\left[\fint_{Q_{\rho/16}} |D^2 v_0|^2\, dz\right]^{\frac{1}{2}} \leq c\,\rho^{-1} \fint_{Q_{\rho/4}} |Dv_0 - \eta|\, dz \tag{5.3.1}$$

and

$$\left[\fint_{Q_{\rho/32}} \frac{|\tau_h Dv_0|^2}{|h|}\, dz\right]^{\frac{1}{2}} \leq c\,\rho^{-1} \fint_{Q_{\rho/4}} |Dv_0 - \eta|\, dz, \tag{5.3.2}$$

for any $h \in \mathbb{R}$ with $0 < |h| < (\rho/32)^2$.

PROOF. The proof is done in firmly exploiting Lemma 9.4. of [67], see also [31, 32]. Since the vector field a is not differentiable with respect to the variable z, we proceed analogously to [118, Lemma 3.2], regularizing a in an appropriate way, showing the desired estimates for the solution of the regularized problem and passing to the limit, as in [4].

1st step: Approximation by regularized vector fields. Let us, for the whole proof, use the abbreviation $\tilde{a}(t, p) := a(x_0, t, p)$. We define a standard smooth, radial, nonnegative mollifier $\phi : \mathbb{R}^n \to \mathbb{R}$, such that $\phi \in C_c^\infty(B_1)$, $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$ and impose the additional condition

$$\int_{B_1 \setminus B_{1/2}} \phi(\xi)\, d\xi \geq \frac{1}{1000},$$

which is a technical condition needed for this kind of approximation procedures (see also [118, 75]). For $k \in \mathbb{N}$ we set $\phi_k(\xi) := k^n \phi(k\xi)$ and define the smooth vector fields $\tilde{a}_k$ by convolution

$$\tilde{a}_k(t,p) := \big(\tilde{a}(t,\cdot) * \phi_k\big)(p) := \int_{B_1(0)} \tilde{a}(t, p + k^{-1}y)\phi_k(y)\, dy.$$

Proceeding analogously to [75, Lemma 3.1] and having in mind (4.1.2), defining $s_k := s + 1/k$, we find that the smoothened vector fields satisfy the following structure conditions

$$\begin{cases} \langle \partial_\xi \tilde{a}_k(t,\xi)\lambda, \lambda\rangle \geq \tilde{c}^{-1}|\lambda|^2, \\ |\tilde{a}_k(t,\xi)| + |\partial_\xi \tilde{a}_k(t,p)| \left(s_k^2 + |\xi|^2\right)^{1/2} \leq \tilde{c}\left(s_k^2 + |\xi|^2\right)^{1/2}, \\ |\tilde{a}(t,\xi) - \tilde{a}_k(t,\xi)| \leq \dfrac{\tilde{c}}{k}, \end{cases} \tag{5.3.3}$$

for all $\lambda, \xi \in \mathbb{R}^n$, $t \in (-T, 0)$, with a constant $\tilde{c} \equiv \tilde{c}(n, L/\nu)$. Moreover each vector field $\tilde{a}_k$ satisfies the assumptions (4.1.2) with s replaced by s_k, for different growth and ellipticity constants $\tilde{\nu}$, $\tilde{L}$ but still depending on the original ones and independent of k. Therefore the Dirichlet problem

$$\begin{cases} \partial_t v_k - \operatorname{div} \tilde{a}_k(t, Dv_k) = 0 & \text{in } Q_{\rho/4}(z_0), \\ v_k = v_0 & \text{on } \partial_{\mathcal{P}} Q_{\rho/4}(z_0). \end{cases} \tag{5.3.4}$$

has a unique solution $v_k \in v_0 + L^2(\Lambda_{\rho/4}; W_0^{1,2}(B_{\rho/4}))$.

2nd step: Estimates for the regularized problems. We start with the estimate corresponding to (5.3.1) for the second spatial derivatives. By Nash-Moser's theory (see [71]) we conclude that $v_k \in L^2_{\text{loc}}(\Lambda_{\rho/4}; W^{2,2}_{\text{loc}}(B_{\rho/4}))$; moreover $w_k := D_i v_k$ for $i \in \{1, \ldots, n\}$ belongs to $C^0_{\text{loc}}(\Lambda_{\rho/4}; W^{1,2}_{\text{loc}}(B_{\rho/4}))$ and is a weak solution of the differentiated equation

$$\partial_t w_k - \operatorname{div}\left(\bar{a}_k(x,t) D w_k\right) = 0, \tag{5.3.5}$$

with $\bar{a}_k(x,t) := \partial_\xi \tilde{a}_k(t, Dv_k(x,t))$. Furthermore $\bar{a}_k(x,t)$ has measurable entries and by (5.3.3) is elliptic and bounded by a constant which does not depend on k, i.e.

$$\tilde{c}^{-1}|\lambda|^2 \leq \langle \bar{a}_k(x,t)\lambda, \lambda\rangle, \qquad \big|\bar{a}_k(x,t)\big| \leq \tilde{c},$$

for every $(x,t) \in Q_{\rho/4}(z_0)$ and all $\lambda \in \mathbb{R}^n$, where $\tilde{c} \equiv \tilde{c}(n, L/\nu)$ is the constant from (5.3.3). Thus, [39, Lemma 2.10] provides for any $\eta_i \in \mathbb{R}$ the estimate

$$\int_{Q_{\rho/16}} |DD_i v_k|^2\, dz \leq \frac{c}{\rho^2} \int_{Q_{\rho/8}} |D_i v_k - \eta_i|^2\, dz,$$

with $c = c(n, L/\nu)$. Since $D_i v_k - \eta_i$ is a solution to (5.3.5), we can apply the higher integrability Lemma 5.2 and Remark 5.4, which hold – with $s = 0$ – also for equations like (5.3.5) (see Remark 5.3), getting

$$\left[\fint_{Q_{\rho/16}} |DD_i v_k|^2\, dz \right]^{1/2} \leq c\,\rho^{-1} \fint_{Q_{\rho/4}} |D_i v_k - \eta_i|\, dz. \tag{5.3.6}$$

To prove the existence of the fractional time derivative of Dv_0 we argue as follows: Taking the approximated problem (5.3.4) and having in mind that $w_k = D_i v_k$ solves the linear equation (5.3.5) in $Q_{\rho/8}$, write the Steklov formulation of (5.3.5) at "level" h (we consider only the case $h > 0$, the $h < 0$ one is very similar), noting that $\tau_h w_k = h\,\partial_t [w_k]_h$:

$$\int_{B_{\rho/4}} \frac{\tau_h w_k}{h}\, \varphi + \langle [\bar{a}_k(x,t) D w_k]_h, D\varphi\rangle\, dx = 0 \qquad \varphi \in W_0^{1,2}(B_{\rho/8}).$$

Choosing as testing function $\varphi(x,t) := \xi^2(x)\tau_h w_k$, where $\xi \in C_c^\infty(B_{\rho/8})$ denotes a cut-off function, $0 \le \xi \le 1$, $\xi \equiv 1$ on $B_{\rho/32}$ and $\xi \equiv 0$ outside $B_{\rho/16}$, with $|D\xi| \le c/\rho$ and integrating with respect to time over $\Lambda_{\rho/32}$, we deduce

$$\int_{\Lambda_{\rho/32}}\int_{B_{\rho/8}} \frac{|\tau_h w_k|^2}{h}\xi^2\,dx\,dt = -\int_{\Lambda_{\rho/32}}\int_{B_{\rho/8}} \big\langle [\bar{a}_k Dw_k]_h, D(\xi^2\tau_h w_k)\big\rangle\,dx\,dt.$$

Now we take into account $(5.3.3)_1$, apply Young's inequality and use $|D\xi| \le c/\rho$ to arrive at

$$\begin{aligned}\int_{\Lambda_{\rho/32}}\int_{B_{\rho/8}} \frac{|\tau_h w_k|^2}{h}\xi^2\,dx\,dt &\le \varepsilon \int_{\Lambda_{\rho/32}}\int_{B_{\rho/16}} \frac{|\tau_h w_k|^2}{\rho^2}\xi^2\,dx\,dt \\ &\quad + \frac{\tilde{c}}{4\varepsilon}\int_{\Lambda_{\rho/32}}\int_{B_{\rho/16}} \Big[|Dw_k|^2 + \eta^2|\tau_h Dw_k|^2\Big]\,dx\,dt.\end{aligned}$$

Finally, estimating $|\tau_h Dw_k|^2 \le 2(|Dw_k(x,t)|^2 + |Dw_k(x,t+h)|^2)$ and exploiting that $h \le (\rho/32)^2$ we may choose $\varepsilon = \frac{1}{2\cdot 32^2}$ to absorb the first term of the right–hand side on the left and conclude

$$\int_{Q_{\rho/32}} \frac{|\tau_h D_i v_k|^2}{h}\,dz \le c\int_{Q_{\rho/16}} |DD_i v_k|^2\,dz.$$

At this point we may exploit estimate (5.3.6) which we already derived before to achieve

$$\fint_{Q_{\rho/32}} \frac{|\tau_h D_i v_k|^2}{h}\,dz \le c\,\rho^{-1}\fint_{Q_{\rho/4}} |D_i v_k - \eta_i|\,dz. \tag{5.3.7}$$

3rd step: Passing to the limit. We now prove the strong L^2-convergence of $\{Dv_k\}_k$. Since both v_k and v_0 are solutions and coincide on the parabolic boundary, arguing analogously to Lemma 5.7, taking $(4.1.2)_1$ adapted for $\tilde{a}_k$, subsequently Young's inequality we achieve

$$\begin{aligned}\tilde{\nu}\int_{Q_{\rho/4}} |Dv_k - Dv_0|^2\,dz &\le \int_{Q_{\rho/4}} \big\langle \tilde{a}_k(t,Dv_k) - \tilde{a}_k(t,Dv_0), Dv_k - Dv_0\big\rangle\,dz \\ &\le \frac{\tilde{\nu}}{2}\int_{Q_{\rho/4}} |Dv_k - Dv_0|^2\,dz \\ &\qquad + c\int_{Q_{\rho/4}} \big|\tilde{a}(t,Dv_0) - \tilde{a}_k(t,Dv_0)\big|^2\,dz\end{aligned}$$

where $Q_{\rho/4} \equiv Q_{\rho/4}(z_0)$; hence absorbing the first term of the right–hand side on the left one, and noting that by $(5.3.3)_3$ the second integral on the right–hand side goes to zero as $k \to \infty$, we immediately deduce that $Dv_k \longrightarrow Dv_0$ strongly in $L^2(Q_{\rho/4};\mathbb{R}^n)$ and also in $L^1(Q_{\rho/4};\mathbb{R}^n)$. In consequence, using the strong convergence for the right–hand side of the inequalities (5.3.6) and (5.3.7) and lower semicontinuity for the left–hand sides, we may pass to the limit $k \to \infty$ and obtain both estimates for the limit function v_0. Summing over $i = 1,\dots,n$ finally provides the desired inequalities (5.3.1) and (5.3.2). □

5.4. Uniform fractional estimates

In this section we will take use of the previous Lemmata to construct the proof of Theorem 4.1. First, we recall the definition of δ in (5.2.3) and we define, once fixed q

$$\gamma(\kappa) := \frac{\delta}{\delta + 1 - \kappa} \qquad \text{for every } \kappa \in [0, \delta+1). \tag{5.4.1}$$

The strategy of the proof is now the following: In a first step, by comparison techniques, we show initial fractional differentiability of Du, i.e.

$$Du \in W^{\tilde{\kappa},\tilde{\kappa}/2;q}_{\mathrm{loc}}(\Omega_T) \qquad \text{for some } \tilde{\kappa} > 0,$$

(see (5.4.7) for $\gamma(0) = \delta/(\delta+1)$). This is the starting point of an iteration procedure: Once having fractional estimates to some quantified exponent (coupled with an explicit local estimate), one may exploit this information in order to increase the amount of differentiability in space and time. Thus, this procedure can be iterated to finally prove the desired result. Let us mention that for the whole proof, we argue on the finite differences of step h in space and step h^2 in time, whereas the estimates are established on cylinders Q of "radius" $|h|^\beta$. Thus, the step size of the finite differences is linked to the size of the radii of appearing parabolic cylinders.

Let us first fix a notation: for subsets $A \subset \Omega$ and $J \subset (-T, 0)$, with $C := A \times J$, we denote with $\lambda_0[C]$ the quantity

$$\lambda_0[C] := \|s + |Du|\|_{L^q(C)} + \|f\|_{L^1(C)}. \tag{5.4.2}$$

Moreover, for a cylinder $Q \equiv Q_\rho(z_0)$ with $32Q \Subset \Omega_T$, let v be the solution of the homogeneous problem (5.2.1) on the cylinder $32Q$ and v_0 the solution of the frozen homogeneous problem (5.2.2) on the cylinder $8Q$. Later in this chapter, Q will be a cylinder of radius $\rho \equiv |h|^\beta$ (see the definition in (5.4.12)), where $h \in \mathbb{R}$ denotes the step size of the finite differences in space and time. However, for the first Lemma, we leave step size and radius uncoupled.

Let us first recall the definitions of the finite difference operator of step $\xi \in \mathbb{R}$ in space

$$[\tau_{i,\xi} f](x,t) := f(x + \xi e_i, t) - f(x,t),$$

for $i \in \{1, \ldots, n\}$ with e_i denoting the unit vector in direction i, as well as the finite difference operator of step ξ^2 in time

$$[\tau_{\xi^2} f](x,t) := f(x, t + \text{sign}(\xi)\xi^2) - f(x),$$

both for $|\xi|$ small enough to assure that the expressions are well defined.

LEMMA 5.10. *There exists a constant $c \equiv c(n, L/\nu, q)$ such that for any $\xi \in \mathbb{R}$ with $|\xi| \le \rho$ and for any $\eta \in \mathbb{R}^n$ the following estimate holds true:*

$$\|\tau_{\xi^2} Du\|_{L^q(Q)} + \sum_{i=1}^n \|\tau_{i,\xi} Du\|_{L^q(Q)} \le c\,\rho^{\delta(q)} \lambda_0[32Q] + c\,\rho^{\frac{n+2}{q}-1} |\xi| \fint_{8Q} |Du - \eta|\, dz.$$

PROOF. For the finite difference operator in space we argue as follows: For $i = 1, \ldots, n$, keeping in mind that $|\xi| \le \rho$, we obtain

$$\begin{aligned}\|\tau_{i,\xi} Du\|_{L^q(Q)} &\le \|\tau_{i,\xi} Dv_0\|_{L^q(Q)} + \|Du - Dv\|_{L^q(Q)} + \|Dv - Dv_0\|_{L^q(Q)} \\ &\quad + \Big(\int_Q |Du(x+\xi e_i, t) - Dv(x + \xi e_i, t)|^q \,dx\,dt\Big)^{1/q} \\ &\quad + \Big(\int_Q |Dv(x+\xi e_i, t) - Dv_0(x + \xi e_i, t)|^q \,dx\,dt\Big)^{1/q} \\ &\le I + II + III,\end{aligned}$$

where we define

$$I := \|\tau_{i,\xi} Dv_0\|_{L^q(Q)}, \quad II := \|Du - Dv\|_{L^q(2Q)}, \quad III := \|Dv - Dv_0\|_{L^q(2Q)}.$$

Using Lemma 5.6 we estimate II:

$$II \le \|Du - Dv\|_{L^q(8Q)} \le c(n,\nu,q)\, \rho^{\delta(q)} \|f\|_{L_1(8Q)}.$$

Secondly, we estimate III in the following way: using Lemma 5.7 and the estimate for II we established before, always having in mind $|\xi| \le \rho \le 1$, we deduce

$$\begin{aligned} III &\le c\,\rho^{\delta(q)} \int_{8Q} \big(s + |Dv|\big)\, dz \\ &\le c\,\rho\Big[\|Du - Dv\|_{L^q(8Q)} + \|s + |Du|\|_{L^q(8Q)}\Big] \\ &\le c\,\rho\Big[\|s + |Du|\|_{L^q(8Q)} + \|f\|_{L^1(8Q)}\Big] \end{aligned}$$

where $c = c(n, \nu, L, q)$. Hence, summarizing the estimates for II and III, taking into account $\delta \le 1$, we get

$$\begin{aligned} II + III &\le c\,\big(\rho + \rho^{\delta(q)}\big)\Big[\|s + |Du|\|_{L^q(8Q)} + \|f\|_{L^1(8Q)}\Big] \\ &\le c\,\rho^{\delta}\Big[\|s + |Du|\|_{L^q(8Q)} + \|f\|_{L^1(8Q)}\Big] \\ &= c\,\rho^{\delta}\lambda_0[8Q], \end{aligned}$$

with a constant depending on n, ν, L, q. To estimate I, we take use of Lemma 5.9. First, noting that $Dv_0(\cdot, t) \in W^{1,2}(B)$ for a.e. t, elementary properties of Sobolev functions together with $|\xi| \le \rho$ provide that

$$\int_B |\tau_{i,\xi} Dv_0(\cdot, t)|^2\, dx \le c(n)\, |\xi|^2 \int_{2B} \big|D^2 v_0(\cdot, t)\big|^2\, dx.$$

Secondly, applying Lemma 5.9, equation (5.3.1) with $Q_{\rho/16} \equiv 2Q$ we obtain

$$\left[\int_{2Q} |D^2 v_0|^2\, dz\right]^{1/2} \le c(n, L/\nu)\rho^{\frac{n}{2}} \fint_{8Q} |Dv_0 - \eta|\, dz.$$

Merging the second last estimate (integrated with respect to time) and the last one, using twice Hölder's inequality, we therefore conclude

$$\begin{aligned} I = \Big(\int_Q |\tau_{i,\xi} Dv_0|^q\, dz\Big)^{1/q} &\le c\,\rho^{\frac{N}{q}(1-\frac{q}{2})}\left[\int_Q |\tau_{i,h} Dv_0|^2\, dz\right]^{1/2} \\ &\le c\,\rho^{\frac{N}{q}(1-\frac{q}{2})}|\xi|\left[\int_{2Q} \big|D^2 v_0\big|^2\, dz\right]^{1/2} \\ &\le c\,\rho^{\frac{N}{q}-1}|\xi| \fint_{8Q} |Dv_0 - \eta|\, dz \end{aligned}$$

for any $\eta \in \mathbb{R}^n$, with a constant $c \equiv c(n, L/\nu, q)$. For the last term in the preceding inequality, we write, using again Hölder's inequality:

$$\begin{aligned} \fint_{8Q} |Dv_0 - \eta|\, dz &\le \fint_{8Q} |Dv_0 - Du|\, dz + \fint_{8Q} |Du - \eta|\, dz \\ &\le c\,\rho^{-\frac{N}{q}}\|Dv_0 - Du\|_{L^q(8Q)} + \fint_{8Q} |Du - \eta|\, dz \\ &\le c\,\rho^{-\frac{N}{q}}\big[\widetilde{II} + \widetilde{III}\big] + \fint_{8Q} |Du - \eta|\, dz \qquad (5.4.3) \end{aligned}$$

with the definitions

$$\widetilde{II} := \|Du - Dv\|_{L^q(8Q)}, \qquad \text{and } \widetilde{III} := \|Dv - Dv_0\|_{L^q(8Q)}.$$

Note that the quatities $\widetilde{II}$ and $\widetilde{III}$ similar to the expressions II and III which we defined before, just being integrated over the cylinder $8Q$ instead of $2Q$. However, the same argumentation which lead to the estimate of $II + III$ also applies here and gives

$$\widetilde{II} + \widetilde{III} \leq c\,\rho^{\delta(q)}\lambda_0[32Q].$$

Merging this estimate with the one before, which gives an estimate for I, combining this with the estimate we established for $II + III$, and having in mind that $|\xi| \leq \rho$, we finally conclude

$$\|\tau_{i,\xi} Du\|_{L^q(Q)} \leq c\,\rho^{\frac{N}{q}-1}|\xi| \fint_{8Q} |Du - \eta|\,dz + c\,\rho^{\delta}\lambda_0[32Q]. \tag{5.4.4}$$

Let us now have a look at the finite difference operator in time. We argue analogously, first writing

$$\left\|\tau_{\xi^2} Du\right\|_{L^q(Q)} \leq \tilde{I} + II + III,$$

where we define

$$\tilde{I} := \left\|\tau_{\xi^2} Dv_0\right\|_{L^q(Q)},$$

and II, III are exactly as before. Consequently, it remains here to estimate the quantity $\tilde{I}$. We use Hölder's inequality, subsequently Lemma 5.9, estimate (5.3.2) with h replaced by sign$(\xi)\xi^2$, and Lemma 5.8 to conclude

$$\tilde{I} \leq c\,\rho^{\frac{n+2}{q}} \left[\fint_{Q} |\tau_{\xi^2} Dv_0|^2\,dz\right]^{1/2} \leq c\,\rho^{\frac{n+2}{q}-1}|\xi| \fint_{8Q} |Dv_0 - \eta|\,dz,$$

with $c \equiv c(n, L/\nu, r, q)$. To replace Dv_0 in the last integral of the preceding estimate, we proceed again as in (5.4.3). We conclude the proof of the Lemma by merging together the estimates for $\tilde{I}$, II and III with (5.4.4). □

The following Proposition is the key to the proof of Theorem 4.1. For the seek of brevity, we define for sets $C := A \times J$ with subsets $A \subset \Omega$, $J \subset (-T, 0)$ the mapping

$$\lambda_\kappa[C] := \lambda_0[C] + \chi(\kappa)[Du]_{W^{\kappa,\kappa/2;q}(C)}, \tag{5.4.5}$$

where $\chi(\kappa) = 0$, if $\kappa = 0$, and $\chi(\kappa) = 1$, whenever $\kappa > 0$; λ_0 is the function defined in (5.4.2). Note that λ_κ is a true extension of λ_0. Let's also use the following notation, regarding the sets mentioned in the statement of the Proposition: for $i = 1, 2$ we denote

$$\Omega_{T,i} := \Omega_i \times J_i, \qquad {\Omega_T}' := \Omega' \times J' \qquad \text{and naturally} \qquad {\Omega_T}'' := \Omega'' \times J'',$$

and we recall the meaning of the compact inclusion for a product set.

Our aim is to prove the following estimates for the finite differences of step h, h^2 respectively, in space and time:

PROPOSITION 5.11. *Let $u \in L^2(-T, 0; W_0^{1,2}(\Omega))$ be the unique weak solution to (5.0.8), under the assumptions (4.1.2) and let q be as in (4.1.5). Assume that for some $\kappa \in [0, \delta)$, where δ is defined in (5.2.3), and that for any couple of subsets ${\Omega_T}' \Subset {\Omega_T}'' \Subset \Omega_T$, there exists a constant c_1 such that the estimate*

$$[Du]_{W^{\kappa,\kappa/2;q}({\Omega_T}')} \leq c_1\lambda_0[{\Omega_T}''] \tag{5.4.6}$$

holds true. Then

$$Du \in W^{\tilde{\kappa},\tilde{\kappa}/2;q}_{\text{loc}}(\Omega_T) \qquad \text{for all } \tilde{\kappa} \in [0, \gamma(\kappa)), \tag{5.4.7}$$

where $\gamma(\cdot)$ is the function defined in (5.4.1). Moreover, for every couple of subsets $\Omega_{T,1} \Subset \Omega_{T,2} \Subset \Omega_T$ the following statements hold:

(i) *There exists a constant* $\mathcal{D} \in (0,1)$*, depending on* $\delta, \kappa, \operatorname{dist}(\Omega_1, \partial\Omega_2), \operatorname{dist}(J_1, \partial J_2)$ *and a constant* c_2 *depending on* $\mathcal{D}, c_1, n, L/\nu, q$ *such that for any* $0 < |h| < \mathcal{D}$ *there holds*

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,1})} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(\Omega_{T,1})} \le c_2\, |h|^{\gamma(\kappa)} \lambda_0[\Omega_{T,2}]; \tag{5.4.8}$$

(ii) *There exists a constant* $\tilde{c}_1$ *depending on* c_1, n, q*,* $\delta - \gamma(\kappa)$*,* $\gamma(\kappa) - \tilde{\kappa}$*,* $\operatorname{dist}(\Omega_2, \partial\Omega)$*,* $\operatorname{dist}(\Omega_1, \partial\Omega_2)$*,* $\operatorname{dist}(J_1, \partial J_2)$*,* $\operatorname{dist}(J_1, \partial J_2)$ *such that*

$$[Du]_{W^{\tilde{\kappa}, \tilde{\kappa}/2, q}(\Omega_{T,1})} \le \tilde{c}_1 \lambda_0[\Omega_{T,2}]. \tag{5.4.9}$$

PROOF. *Step 1: Choice of suitable parabolic cylinders.* Let us take a parabolic cylinder $Q \equiv Q_R(z_0) \Subset \Omega_T$ of radius R and center $z_0 = (x_0, t_0)$. We denote by $\mathcal{Q}_R$ the cuboid of the form

$$\mathcal{Q}_R(z_0) := \Big\{ (x,t) \in \mathbb{R}^{n+1} : \max\Big\{ \max_j \frac{|x_j - (x_0)_j|}{\sqrt{n}}, \sqrt{t - t_0} \Big\} < R \Big\},$$

which is the largest cuboid centered in $z_0 = (x_0, t_0)$ and contained in Q_R. Therefore we denote this cuboid also by $\mathcal{Q}_{\text{inn}} \equiv \mathcal{Q}_{\text{inn}}(Q)$. Analogously we denote by $\mathcal{Q}_{\text{out}} \equiv \mathcal{Q}_{\text{out}}(Q)$ the smallest cuboid containing Q. Denoting by $\hat{Q} \equiv 32Q$ the enlarged cylinder $\hat{Q}$, we denote $\mathcal{Q}_{\text{inn}} \equiv \mathcal{Q}_{\text{inn}}(Q)$ and $\hat{\mathcal{Q}}_{\text{out}} \equiv \mathcal{Q}_{\text{out}}(\hat{Q})$ and finally have the following inclusions:

$$\mathcal{Q}_{\text{inn}} \subset Q \Subset 2Q \Subset 32Q = \hat{Q} \subset \hat{\mathcal{Q}}_{\text{out}}. \tag{5.4.10}$$

Now we fix arbitrary open sets $\Omega_{T,1} \Subset \Omega_{T,2} \Subset \Omega_T$, and find an intermediate subset $\Omega_{T,3} = \Omega_3 \times J_3$ such that $\Omega_{T,1} \Subset \Omega_{T,3} \Subset \Omega_{T,2}$. It is easy to see that

$$\begin{aligned} \Omega_{T,3} := \{ z = (x,t) \in \Omega_{T,2} :& \operatorname{dist}(x, \partial\Omega_2) > \operatorname{dist}(\partial\Omega_2, \Omega_1)/2, \\ & \operatorname{dist}(t, \partial J_2) > \operatorname{dist}(\partial J_2, J_1)/2 \} \end{aligned}$$

is an appropriate choice. Take $\beta \in (0,1)$ to be chosen later, and let $h \in \mathbb{R}$ be a real number satisfying

$$0 < |h| < \min\Big\{ \Big(\frac{\operatorname{dist}(\Omega_1, \partial\Omega_3)}{100\sqrt{n}} \Big)^{\frac{1}{\beta}}, \Big(\frac{\sqrt{\operatorname{dist}(J_1, \partial J_3)}}{100} \Big)^{\frac{1}{\beta}}, 1 \Big\} =: \mathcal{D}. \tag{5.4.11}$$

We take $z_0 \in \Omega_{T,1}$ and fix a cylinder of radius $|h|^\beta$, i.e.

$$Q := Q(h) := Q_{|h|^\beta}(z_0) = B_{|h|^\beta}(x_0) \times \big(t_0 - |h|^{2\beta}, t_0 + |h|^{2\beta} \big). \tag{5.4.12}$$

Let us recall that for $\alpha > 0$ we write

$$\alpha Q := B_{\alpha|h|^\beta}(x_0) \times \big(t_0 - \alpha^2 |h|^{2\beta}, t_0 + \alpha^2 |h|^{2\beta} \big).$$

Note that by condition (5.4.11) we have that $\hat{\mathcal{Q}}_{\text{out}} \Subset \Omega_{T,3}$ and since $\beta \in (0,1)$ we moreover have $|h| \le |h|^\beta$. Finally, let v and v_0 respectively be the solutions of (5.2.1) and (5.2.2) with $\rho = 32|h|^\beta$, which means that v solves (5.2.1) on the cylinder $32Q \equiv Q_{32|h|^\beta}(z_0)$, whereas v_0 solves (5.2.2) on $8Q \equiv Q_{8|h|^\beta}(z_0)$.

Step 2: Estimates on certain parabolic cylinders: We start by Lemma 5.10, which we apply with $\rho = |h|^\beta$ and $\xi = h$, to deduce

$$\begin{aligned} &\|\tau_{h^2} Du\|_{L^q(Q)} + \sum_{i=1}^n \|\tau_{i,h} Du\|_{L^q(Q)} \\ &\qquad\qquad \le c\, |h|^{\beta\delta} \lambda_0[32Q] + c\, |h|^{\beta[\frac{N}{q} - 1] + 1} \fint_{8Q} |Du - \eta|\, dz, \end{aligned} \tag{5.4.13}$$

with $c \equiv c(n, L/\nu, q)$ and where we recall the definition of $\lambda_0[C]$ in (5.4.2). Let us now distinguish two cases: In **case of $\kappa = 0$** we choose $\eta \equiv 0$ and obtain by Hölder's inequality

$$\fint_{8Q} |Du|\,dz \le c(n,q)|h|^{-\beta\frac{n+2}{q}}\|Du\|_{L^q(8Q)},$$

and therefore

$$\|\tau_{h^2}Du\|_{L^q(Q)} + \sum_{i=1}^{n}\|\tau_{i,h}Du\|_{L^q(Q)} \le c\left[|h|^{\beta\delta} + |h|^{(1-\beta)}\right]\lambda_0[32Q],$$

with a constant $c \equiv c(n, L/\nu, q)$. In **case of $\kappa > 0$** we choose $\eta \equiv (Du)_{8Q}$ and apply the fractional Poincaré inequality in terms of Lemma 3.10 to deduce

$$\fint_{8Q} |Du - (Du)_{8Q}|\,dz \le c\,|h|^{\beta(\kappa - \frac{N}{q})}[Du]_{W^{\kappa,\kappa/2;q}(8Q)},$$

with $c \equiv c(n,q)$ and thus, merging this with (5.4.13), and having in mind the definition of λ_κ in (5.4.5), we arrive at

$$\|\tau_{h^2}Du\|_{L^q(Q)} + \sum_{i=1}^{n}\|\tau_{i,h}Du\|_{L^q(Q)} \le c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right]\lambda_\kappa[32Q],$$

for a constant depending on n, L/ν and q.

Step 3: Covering argument: Recalling the choice of the involved cylinders in (5.4.10), i.e. $\mathcal{Q}_{\text{inn}} \equiv \mathcal{Q}_{\text{inn}}(Q) \subset Q$ and $32Q \equiv \hat{Q} \subset \mathcal{Q}_{\text{out}}(\hat{Q}) \equiv \hat{\mathcal{Q}}_{\text{out}}$, we immediately have

$$\|\tau_{h^2}Du\|_{L^q(\mathcal{Q}_{\text{inn}})} + \sum_{i=1}^{n}\|\tau_{i,h}Du\|_{L^q(\mathcal{Q}_{\text{inn}})} \le c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right]\lambda_\kappa[\hat{\mathcal{Q}}_{\text{out}}], \quad (5.4.14)$$

with a constant $c \equiv c(n, L/\nu, q)$.

Let's now observe that, even if the set function defined in (5.4.5) is not a measure – due to the presence of the term $[Du]_{W^{\kappa,\kappa/2;q}}$ – it is nevertheless countably super-additive, that is

$$\sum_j \lambda_\kappa[C_j] \le \lambda_\kappa\Big[\bigcup_j C_j\Big],$$

whenever $\{C_j\}$ is a countable family of mutually disjoint subsets. The covering argument is now the following: First, we recall that the sets $\mathcal{Q}$ involved here are cuboids with sides parallel to the coordinate axis. Then, for each $h \in \mathbb{R}$, satisfying the smallness condition (5.4.11) we can find cylinders $Q_1 \equiv Q_{|h|^\beta}(z_1), \ldots, Q_m \equiv Q_{|h|^\beta}(z_m)$ of the type considered in (5.4.12) such that the corresponding inner cuboids $\mathcal{Q}_{\text{inn}}(Q_1), \ldots, \mathcal{Q}_{\text{inn}}(Q_m)$ are disjoint and cover $\Omega_{T,1}$ up to a negligible set, i.e.

$$\mathcal{L}^{n+1}\left(\Omega_{T,1} \setminus \cup\mathcal{Q}_{\text{inn}}(Q_j)\right) = 0, \quad \mathcal{Q}_{\text{inn}}(Q_k) \cap \mathcal{Q}_{\text{inn}}(Q_j) = \emptyset \text{ for } k \ne j. \quad (5.4.15)$$

Precisely we proceed as follows: for the two sets $\Omega_{T,1}$ and $\Omega_{T,3}$, we first take cuboids $\{\mathcal{Q}_j\}$, all centered in $\Omega_{T,1}$, with sides parallel to the coordinate axes and side length comparable to $|h|^\beta$ in order to obtain (5.4.15). Then we see them as inner cuboids of the cylinders $Q_{|h|^\beta}(z_j)$, according to (5.4.10). Now, we sum up the inequalities (5.4.14) for $j \le m$ and obtain

$$\sum_{j=1}^{m}\left[\|\tau_{h^2}Du\|_{L^q(\mathcal{Q}_{\text{inn}}(Q_j))} + \sum_{i=1}^{n}\|\tau_{i,h}Du\|_{L^q(\mathcal{Q}_{\text{inn}}(Q_j))}\right]$$

$$\leq c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right] \sum_{j=1}^{m} \lambda_\kappa[\mathcal{Q}_{\text{out}}(\hat{Q}_j)]. \quad (5.4.16)$$

By construction, and in particular by (5.4.11) we have that $\mathcal{Q}_{\text{out}}(\hat{Q}_j) \subset \Omega_{T,3}$ for any $j \leq m$. Moreover, each of the dilated cuboids $\mathcal{Q}_{\text{out}}(\hat{Q}_k)$ intersects the similar ones $\mathcal{Q}_{\text{out}}(\hat{Q}_j)$ less than $c(n)$ times. Therefore, using these facts, i.e. (5.4.15), (5.4.16) together with the countably super-additivity of the set-function λ_κ we end up with

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,1})} + \sum_{i=1}^{n} \|\tau_{i,h} Du\|_{L^q(\Omega_{T,1})} \leq c\left[|h|^{\beta\delta} + |h|^{1-\beta+\beta\kappa}\right] \lambda_\kappa[\Omega_{T,3}].$$

In a next step, we determine β in order to minimize the right–hand side of the preceding inequalities with respect to $|h|$. I.e. we choose β in such a way that $1 - \beta + \beta\kappa = \beta\delta$, that is $\beta = \gamma(\kappa)/\delta$, where we recall the definition of $\gamma(\kappa)$ in (5.4.1). Note at this point, that since $\kappa < \delta$ implies $\gamma(\kappa)/\delta < 1$, this choice of $\beta \in (0, 1)$ is admissible. Therefore, for h satisfying (5.4.11), the preceding estimate becomes

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,1})} + \sum_{i=1}^{n} \|\tau_{i,h} Du\|_{L^q(\Omega_{T,1})} \leq c_0 \, |h|^{\gamma(\kappa)} \lambda_\kappa[\Omega_{T,3}], \quad (5.4.17)$$

with a constant $c_0 \equiv c_0(n, L/\nu, q)$.

Now we are at the point to conclude the assertions of Proposition 5.11. First, we prove (5.4.8): In the case $\kappa = 0$, we have directly

$$\lambda_\kappa[\Omega_{T,3}] = \lambda_0[\Omega_{T,3}] \leq \lambda_0[\Omega_{T,2}],$$

whereas in the case $\kappa > 0$, we take (5.4.6) with $\Omega_{T,3}$ as inner subset, $\Omega_{T,2}$ as outer one, and achieve

$$\begin{aligned}\lambda_\kappa[\Omega_{T,3}] &= \lambda_0[\Omega_{T,3}] + [Du]_{W^{\kappa,\kappa/2;q}(\Omega_{T,3})} \\ &\leq \lambda_0[\Omega_{T,3}] + c_1[\Omega_{T,2}] \leq (1 + c_1)\, \lambda_0[\Omega_{T,2}].\end{aligned}$$

Merging these two estimates with (5.4.17), we conclude (5.4.8) for $0 < |h| < \mathcal{D}$ with $c_2 := c_0(1 + c_1)$.

Having (5.4.17) at hand, the proof of (5.4.9) and (5.4.7) is performed via the Corollary 3.9: We retrace the proof in the previous lines in order to get the finite differences on the set $\Omega_{T,3}$ estimated by $\lambda_0[\Omega_{T,2}]$, using a further intermediate set. We hence have

$$\|\tau_{h^2} Du\|_{L^q(\Omega_{T,3})} + \sum_{i=1}^{n} \|\tau_{i,h} Du\|_{L^q(\Omega_{T,3})} \leq (1 + c_1)\, |h|^{\gamma(\kappa)} \lambda_0[\Omega_{T,2}],$$

for every $0 < |h| < \mathcal{D}$. This estimate enables us to apply Corollary 3.9 with $\tilde{J} \equiv J_3$, $\widetilde{\Omega} \equiv \Omega_3$, $\mathcal{O} \equiv \Omega_1$, $\mathcal{J} \equiv J_1$, $\bar{\theta}$ replaced by $\gamma(\kappa)$ and $S \equiv (1 + c_1)\lambda_0[\Omega_{T,2}]$ in order to obtain

$$\begin{aligned}[Du]_{W^{\tilde{\kappa},\tilde{\kappa}/2;q}(\Omega_{T,1})} &\leq \tilde{c}_1 \lambda_0[\Omega_{T,2}] \\ &= \tilde{c}_1 \left[\| s + |Du| \|_{L^q(\Omega_{T,2})} + \|f\|_{L^1(\Omega_{T,2})}\right],\end{aligned}$$

for all $\tilde{\kappa} \in [0, \gamma(\kappa))$, with the constant $\tilde{c}_1$ depending on $c_1, n, q, \mathcal{D}, \gamma(\kappa) - \tilde{\kappa}$, $\text{dist}(\Omega_2, \partial\Omega)$, $\text{dist}(\Omega_1, \partial\Omega_2)$, $\text{dist}(J_1, \partial J_2)$, $\text{dist}(J_1, \partial J_2)$ so that, since all our subsets are arbitrary,

$$Du \in W^{\tilde{\kappa},\tilde{\kappa}/2;q}_{\text{loc}}(\Omega_T) \qquad \text{for all } \tilde{\kappa} \in [0, \gamma(\kappa)).$$

□

The main Theorem 4.1 is now proved for the approximate sequence by an iteration argument:

PROPOSITION 5.12 (Iteration). *Let $u \in L^2(-T,0;W_0^{1,2}(\Omega))$ the (unique) solution to* (5.0.8) *under the assumptions* (4.1.2) *and let q satisfies* (4.1.5). *Then*

$$Du \in W_{\text{loc}}^{\kappa,\kappa/2;q}(\Omega_T) \qquad \text{for every } \kappa \in [0,\delta) \tag{5.4.18}$$

where δ is as in (5.2.3). *Furthermore, for every couple of subsets $\Omega_{T,1} \Subset \Omega_{T,2} \Subset \Omega_T$ there exists a constant c depending only on n, ν, L, q, $\delta - \kappa$,* $\operatorname{dist}(\Omega_1, \partial\Omega)$*,* $\operatorname{dist}(\Omega_2, \partial\Omega_1)$*,* $\operatorname{dist}(J_1, \partial J_2)$*,* $\operatorname{dist}(J_1, \partial J_2)$ *such that*

$$[Du]_{W^{\kappa,\kappa/2;q}(\Omega_{T,1})} \le c\left[\|s + |Du|\|_{L^q(\Omega_{T,2})} + \|f\|_{L^1(\Omega_{T,2})}\right]. \tag{5.4.19}$$

PROOF. The proof of this lemma follows essentially the lines of the one in [118, Lemma 6.3]. However, for the convenience of the reader we sketch at least the argumentation: The function $\gamma(\cdot)$ in (5.4.1) is easily seen to be non-decreasing and to satisfy

$$\kappa \in (0,\delta) \Rightarrow \gamma(\kappa) \in (\kappa,\delta) \qquad \text{and} \qquad \gamma(\delta) = \delta. \tag{5.4.20}$$

Let's define by induction the two sequences $\{\ell_k\}$ and $\{\kappa_k\}$ as follows:

$$\ell_1 := \frac{\delta}{4(\delta+1)},\ \kappa_1 := \frac{\delta}{2(\delta+1)},\ \ell_{k+1} := \gamma(\ell_k),\ \kappa_{k+1} := \frac{\gamma(\kappa_k)+\gamma(\ell_k)}{2}.$$

From (5.4.20) it follows that $\ell_k \nearrow \delta$; since $\gamma(\cdot)$ is increasing we have $\ell_k < \kappa_k < \delta$, hence also $\kappa_k \nearrow \delta$. Applying in a first step Proposition 5.11 with $\kappa = 0$, we get that $Du \in W_{\text{loc}}^{\tilde\kappa,\tilde\kappa/2;q}(\Omega_T)$, with corresponding estimates of the type (5.4.9) for $\tilde\kappa$, for any $\tilde\kappa \in [0,\gamma(0))$, where $\gamma(0) = \delta/(\delta+1)$. Since $\gamma(\cdot)$ is increasing, we have in particular that $Du \in W_{\text{loc}}^{\kappa_1,\kappa_1/2;q}(\Omega_T)$, with corresponding estimate of the type (5.4.8) and (5.4.9) for $\tilde\kappa \equiv \kappa_1$. Having once at hand the estimates on level κ_k, we once again apply Proposition 5.11 with $\kappa = \kappa_k$ and we get that $Du \in W_{\text{loc}}^{\tilde\kappa,\tilde\kappa/2;q}(\Omega_T)$ for all $\tilde\kappa < \gamma(\kappa_k)$ and in particular, since $\gamma(\cdot)$ is increasing and thus $\ell_k < \kappa_k$, we have $\kappa_{k+1} < \gamma(\kappa_k)$ and therefore also $Du \in W_{\text{loc}}^{\kappa_{k+1},\kappa_{k+1}/2;q}(\Omega_T)$. Moreover, (5.4.19) holds for $\kappa = \kappa_{k+1}$. Then by induction we get both (5.4.18) and (5.4.19). □

REMARK 5.13. *It can be proved in particular, exploiting the iterative process of the previous Proposition together with estimate* (5.4.8)*, that*

$$\|\tau_h Du\|_{L^q(\Omega_{T,1})} \le c\,|h|^{\kappa/2}\left[\|s + |Du|\|_{L^q(\Omega_{T,2})} + \|f\|_{L^1(\Omega_{T,2})}\right] \tag{5.4.21}$$

for every $\kappa \in [0,\delta/2)$ and $|h|$ small, with a constant depending essentially on δ and on the distance between $\Omega_{T,1}$ and the boundary of $\Omega_{T,2}$.

PROOF OF THEOREM 4.1 AND ESTIMATE (4.1.10). We consider the approximation sequence $\{u_k\}$ built as solutions of (5.0.8) with data $f \equiv f_k$ as stated in Section 2.3, adapted to our case. The strong convergence in $L^1(\Omega_T)$ of the sequence u_k to u can be deduced exactly as in [24], using the fact that from the equation $\partial_t u_k$ is uniformly bounded in $L^1(-T,0;W^{-1,1}(\Omega))$, and deducing the convergence by compactness arguments, see [135]. For the convergence of the gradients, our stronger estimates allow a simpler, independent proof. The global estimate in Lemma 5.1 applied to any u_k, together with (5.0.10), leads to

$$\|Du_k\|_{L^q(\Omega_T)} \le c\left[s + \|f_k\|_{L^1(\Omega_T)}\right] \le c\big[s + |\mu|(\Omega_T)\big], \tag{5.4.22}$$

which coupled with (5.4.19) and (5.4.21) gives the following two facts: for $J \Subset (-T,0)$, $\Omega_1 \Subset \Omega$, for every $\kappa < \delta$

$$\left(\int_J [Du_k(\cdot,t)]^q_{W^{\kappa,q}(\Omega_1)}\,dt\right)^{1/q} \le c\big[s + |\mu|(\Omega_T)\big]$$

and

$$\|\tau_h Du\|_{L^q(\Omega_1\times J)} \leq c\,|h|^{\kappa/2}\big[s+|\mu|(\Omega_T)\big].$$

In particular, $\{Du_k\}$ is uniformly bounded in $L^1(J;W^{\kappa,q}(\Omega_1))$ and $\|\tau_h Du\|_{L^q(\Omega_1\times J)} \to 0$ as $h\to 0$ uniformly with respect to k. Hence we can apply the compactness result [**135**, Theorem 3] to deduce, after extracting a non relabeled subsequence, the convergence of Du_k to Du strongly in $L^1_{\text{loc}}(\Omega_T)$ and almost everywhere. Note that we made the choice $X\equiv W^{\kappa,q}(\Omega_1)$ which is compactly (see [**11**]) embedded into $B\equiv L^q(\Omega_1)$.

Hence finally we can prove our theorem for the function u which is, a SOLA. Indeed it is now easy to see, using Lipschitz continuity (4.1.2)$_2$ and the convergences just proved, that u solves (4.1.4). Writing estimate (5.4.19) for u_k in particular we find, for $\Omega_{T,1}\Subset\Omega_T$

$$[Du_k]_{W^{\kappa,\kappa/2;q}(\Omega'\times J')} \leq c\,\big[s+|\mu|(\Omega_T)\big],$$

where c depends on n, L/ν, q, $\operatorname{dist}(\Omega'_{T'},\partial\Omega_T)$, $|\Omega|$ and T. Here we have used (5.0.10) and the previous global estimate (5.4.22). Now estimate (4.1.10) follows by treating the left-hand sides of the previous inequality with Fatou's Lemma. □

PROOF OF LOCAL ESTIMATES ON CYLINDERS. Finally in order to prove (4.1.9) we make use of a scaling argument. Fix $Q_\rho(z_0)\Subset\Omega_T$ and take $u\in L^2(-T,0;W^{1,2}_0(\Omega))$ the unique solution to (5.0.8) for a fixed regular f; then restrict u to $Q_\rho(z_0)$ and then rescale it to Q_1, as in Lemma 5.6, in order to get $\tilde u\in L^2(-1,1;W^{1,2}(B_1))$. Now observe that we may apply Lemma 5.12 to $\tilde u$ since the whole argument is just local and no boundary information is needed. Hence we can deduce by estimate (5.4.19) applied to $\tilde u$ with $\Omega_{T,1}\equiv Q_{1/2}$ and $\Omega_{T,2}\equiv Q_1$, up to a little change in notation, for $\sigma<\delta q$:

$$[D\tilde u]^q_{W^{\sigma/q,\sigma/(2q);q}(Q_{1/2})} \leq c\Big[\|s+|D\tilde u|\|^q_{L^q(Q_1)}+\|\tilde f\|^q_{L^1(Q_1)}\Big].$$

Scaling back to $Q_\rho(z_0)$ yields to

$$\begin{aligned}\rho^{\sigma-N}&\int_{\Lambda_{\rho/2}}\int_{B_{\rho/2}}\int_{B_{\rho/2}}\frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\sigma}}\,dx\,dy\,dt\\ &+\rho^{\sigma-N}\int_{B_{\rho/2}}\int_{\Lambda_{\rho/2}}\int_{\Lambda_{\rho/2}}\frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\sigma/2}}\,dt\,ds\,dx\\ &\qquad\leq c\Big[\rho^{-N}\|s+|Du|\|^q_{L^q(Q_1)}+\rho^{-q(N-1)}\|f\|^q_{L^1(Q_1)}\Big],\end{aligned}$$

that is

$$[Du]^q_{W^{\sigma/q,\sigma/(2q);q}(Q_{1/2})} \leq c\,\rho^{-\sigma}\Big[\|s+|Du|\|^q_{L^q(Q_1)}+\rho^{\sigma(q)}\|f\|^q_{L^1(Q_1)}\Big].$$

Now it's enough to write the latter estimate for $u\equiv u_k$, u_k being the approximated solution described in the beginning of this proof, and follow again the scheme described just above, using also (5.0.10). □

PROOF OF COROLLARY 4.2. Recall that by the definition of fractional Sobolev space (3.2.13) we have (the reason of the changing in the notation from q to r will become clear in a moment)

$$Du\in L^r_{\text{loc}}(-T,0;W^{\bar\delta,r}_{\text{loc}}(\Omega))\qquad\text{for all }\bar\delta\in(0,\delta),\tag{5.4.23}$$

with

$$r\in\Big[1,\frac{N}{N-1}\Big)\qquad\text{and}\qquad\delta\equiv\delta(r)=\frac{N}{r}-(N-1).$$

Using fractional Sobolev embedding of Proposition 3.6 slicewise in space, after a simple computation we have

$$Du \in L^r_{\text{loc}}(-T,0;L^q_{\text{loc}}(\Omega)) \qquad \text{for all } q \in [1,q^*),$$
$$q^* \equiv q^*(r) := \frac{nr}{r(n+1)-2}. \qquad (5.4.24)$$

Moreover by immersion (3.2.12) we have that

$$Du \in W^{\bar{\delta}/2,q}_{\text{loc}}(-T,0;L^q_{\text{loc}}(\Omega)) \qquad \text{for all } \bar{\delta} \in (0,\delta), \qquad (5.4.25)$$

with this time

$$q \in \Big[1, \frac{N}{N-1}\Big) \qquad \text{and} \qquad \delta \equiv \delta(q) = \frac{N}{q} - (N-1).$$

Applying Proposition 3.6 this time slicewise in time (which in this case means applied to the function $\|Du(\cdot,t)\|_{L^q}$), with $N \equiv 1$, gives

$$Du \in L^r_{\text{loc}}(-T,0;L^q_{\text{loc}}(\Omega)) \qquad \text{for all } r \in [1,r^*),$$
$$r^* \equiv r^*(q) := \frac{2q}{q(n+1)-n}. \qquad (5.4.26)$$

It is easy to check that the bounds for r and q appearing in (5.4.24) and (5.4.26) fully recover the bounds in (4.1.6), since the images of $q = q^*(r)$ and $r = r^*(q)$ are the same arc of hyperbola in the (r,q) plane.

Reasoning exactly as above, using the facts that

$$Du \in L^q_{\text{loc}}(\Omega; W^{\bar{\delta}/2,q}_{\text{loc}}(-T,0)), \qquad q \in \Big[1, \frac{N}{N-1}\Big) \qquad (5.4.27)$$

and

$$Du \in W^{\bar{\delta},r}_{\text{loc}}(\Omega; L^r_{\text{loc}}(-T,0)) \qquad r \in \Big[1, \frac{N}{N-1}\Big) \qquad (5.4.28)$$

we obtain $Du \in L^q_{\text{loc}}(\Omega; L^r_{\text{loc}}(-T,0))$ for all (r,q) satisfying (4.1.6). □

PROOF OF ANISOTROPIC REGULARITY THEOREM 4.4. We argue similarly as above. Since by Proposition 3.7

$$W^{\delta,r}_{\text{loc}}(\Omega) \subset W^{\delta-n/r+n/q,q}_{\text{loc}}(\Omega)$$

plugging last result slicewise into (5.4.23) gives

$$Du \in L^r_{\text{loc}}(-T,0;W^{\delta,q}_{\text{loc}}(\Omega)) \qquad \text{for all } q > r \text{ and } \delta < \tilde{\delta}(r,q)$$

with

$$\tilde{\delta}(r,q) := \frac{n}{q} + \frac{2}{r} - (n+1).$$

Applying Proposition 3.7 to the function $\|Du(x,\cdot)\|_{L^r(J)}$ with $J \Subset (-T,0)$ generic we get by (5.4.28)

$$\|Du(x,\cdot)\|_{L^r(J)} \in W^{\delta,q}_{\text{loc}}(\Omega) \qquad \text{that is} \qquad Du \in W^{\delta,q}_{\text{loc}}(\Omega; L^r_{\text{loc}}(-T,0))$$

for $\delta \in (0, \tilde{\delta}(r,q))$ and $r < q$. Finally we get results involving time regularity ($r > q$) exactly in the same way, using Proposition 3.7 in dimension 1 together with inclusions (5.4.27) and (5.4.25). This finally finishes the proof. □

CHAPTER 6

Adams theorems for nonlinear heat equations

In this Chapter we give the proof of the results of Section 4.2. We shall here consider the Cauchy-Dirichlet problem (4.1.1) where $H \equiv g$ is a Lebesgue function and the vector-field $a\colon \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies assumptions (4.2.1). Theorem 4.14.

Some notes about the techniques of the proofs

The proof of our theorems is based on the method developed in [**121**] for elliptic equations, which we carry over to the framework of parabolic equations. The key point to the proof of Theorem 4.8 is an estimate which allows to control the level set of the Hardy-Littlewood maximal function of the spatial gradient $|Du|$ locally by the level sets of a suitable parabolic Riesz potential operator. More precisely (see (6.2.16) together with Lemma 6.1 for the exact estimate) we establish for some exponent $\chi > 1$ an estimate of the type

$$\big|\{M(|Du|) \geq T\lambda\}\big| \lesssim T^{-2\chi}\big|\{M(|Du|) \geq \lambda\}\big| + c(T)\big|\{I_1(|g|) \geq \lambda\}\big|, \tag{6.0.29}$$

where $M(|Du|)$ denotes the maximal operator of $|Du|$, λ is a number large enough and $T \gg 1$ is a constant. Here, the parabolic Riesz potential operator for $\beta \in (0, N]$ is defined as

$$I_\beta(|g|)(z) := \int_{\mathbb{R}^{n+1}} \frac{|g(w)|}{d_{\mathcal{P}}(z,w)^{N-\beta}}\, dw, \quad z \in \mathbb{R}^{n+1}.$$

and $d_{\mathcal{P}}$ denotes the parabolic metric (see (6.0.30) and (3.1.4)). The precise definitions of the other involved quantities can be found in Section 3.2 and in a few lines. All the Lorentz- and Lorentz-Morrey estimates and also the borderline cases can then be derived with the help of (6.0.29). In order to prove the decay estimate (6.0.29), we apply the classical Calderón-Zygmund covering Lemma in the parabolic setting to suitable level sets of the maximal function. Indeed, to verify the conditions necessary to apply the Calderón-Zygmund Lemma in our setting, we take use of a comparison strategy to the solution v to an associated homogeneous problem.The proof of such comparison estimates is similar to that given in Chapter 5 already going back to the contributions of Boccardo & Gallouët [**25**]. For the solution to homogeneous problems, well known Hölder continuity and higher integrability results coming up from the De Giorgi-Nash-Moser theory provide suitable reference estimates.

In the case of general equations, as (4.1.1), fulfilling the structure conditions (4.2.1), the decay estimate of the type (6.0.29) can be established for some fixed exponent $\chi > 1$, depending on the data of the equation. This, in turn leads to restrictions on the range of exponents which are allowed in the Lorentz and Lorentz-Morrey estimates, and we come up with the desired estimates for exponents fulfilling (4.2.3). On the other hand, assuming more regularity on the data as those considered in Paragraph 4.2, integrability and Hölder continuity estimates for solutions to homogeneous equations have been established by Duzaar, Mingione & Steffen in [**71**] in a stronger form (in particular Hölder continuity holds to any exponent $\alpha \in (0,1)$) and therefore the decay estimate (6.0.29) can be found to hold true for any exponent $\chi > 1$. As a consequence, we derive Lorentz and Lorentz-Morrey estimates in this case for the full range of exponents, as in (4.2.18).

Concerning the level of the solution u itself, the decay estimate (6.0.29) can be substituted by an estimate of the form

$$\big|\{M(u) \geq T\lambda\}\big| \lesssim T^{-2\chi}\big|\{M(u) \geq \lambda\}\big| + c(T)\big|\{I_2(|g|) \geq \lambda\}\big|,$$

which holds for any λ large enough and $T \gg 1$, and for any exponent $\chi > 1$. Here, instead of the Riesz potential $I_1(|g|)$, we have the potential $I_2(|g|)$ involved on the right hand side, clearly according to (2.4.1). This finally allows to establish the desired Lorentz-Morrey estimate for u for the range of exponents declared in (4.2.12).

For $\beta \in (0, N]$ the fractional integral operator $I_\beta(\cdot)$, also called parabolic Riesz potential, is the linear operator defined by

$$I_\beta(g)(z) := \int_{\mathbb{R}^{n+1}} \frac{g(\tilde z)}{d_{\mathcal{P}}(z,\tilde z)^{N-\beta}}\, d\tilde z \qquad z \in \mathbb{R}^{n+1}, \tag{6.0.30}$$

for all measurable functions $g\colon \mathbb{R}^{n+1} \to \mathbb{R}$. This specializes the definition given in [83, p. 24, (31)] for a doubling metric space (X, d, μ), i.e.

$$I_\beta(g)(z) := \int_X \frac{g(w) d^\beta(z,w)}{\mu(B_{d(z,w)}(z))}\, dw, \qquad z \in X,$$

to the case $(\mathbb{R}^{n+1}, d_{\mathcal{P}}, \mathcal{P}^N)$. We recall that the Parabolic metric $d_{\mathcal{P}}$ has been defined in (3.1.4). Note that the parabolic Hausdorff-measure $\mathcal{P}^N$ is equivalent to the Lebesgue measure in $\mathbb{R}^{n+1}$. The following Lemma is an immediate consequence of the definitions of the fractional Riesz potential and the fractional maximal operators.

LEMMA 6.1. *For every non-negative measurable function g defined on $\mathbb{R}^{n+1}$ there holds*

$$I_\beta(g)(z) \geq 2^{\beta-N}|\mathcal{Q}_1|^{1-\frac{\beta}{N}} M_\beta(g)(z) \qquad \text{for every } z \in \mathbb{R}^{n+1}.$$

PROOF. Let $\mathcal{Q}_\varrho(z_0) \subset \mathbb{R}^{n+1}$ be an arbitrary but fixed symmetric parabolic cylinder containing the point z. Then $\mathcal{Q}_\varrho(z_0) \subset \mathcal{Q}_{2\varrho}(z)$ and therefore

$$\begin{aligned} I_\beta(g)(z) &\geq \int_{\mathcal{Q}_{2\varrho}(z)} \frac{g(w)}{d_{\mathcal{P}}(z,w)^{N-\beta}}\, dw \geq (2\varrho)^{\beta-N} \int_{\mathcal{Q}_\varrho(z_0)} g(w)\, dw \\ &= 2^{\beta-N}|\mathcal{Q}_1|^{1-\frac{\beta}{N}}\big|\mathcal{Q}_\varrho(z_0)\big|^{\frac{\beta}{N}} \fint_{\mathcal{Q}(z_0,\varrho)} g(w)\, dw. \end{aligned}$$

Taking the sup with respect to all parabolic cylinders $\mathcal{Q}_\varrho(z_0)$ containing z then yields the result. □

LEMMA 6.2. *Let $0 < \beta < N$, $p \geq 1$, $\vartheta > 0$ be such that $\beta < \vartheta/p \leq N$, and let g be a non-negative measurable function on $\mathbb{R}^{n+1}$. Then the pointwise estimate*

$$I_\beta(g)(z) \leq c\left[M_{\vartheta/p}(g)(z)\right]^{\frac{\beta p}{\vartheta}}\left[M(g)(z)\right]^{1-\frac{\beta p}{\vartheta}}$$

holds for every $z \in \mathbb{R}^{n+1}$ with a constant $c \equiv c(n, p, \vartheta, \beta)$.

PROOF. Without loss of generality we may assume that $g \not\equiv 0$. Let $z \in \mathbb{R}^{n+1}$. For $\delta > 0$ to be chosen later we decompose $\mathbb{R}^{n+1}$ into $\mathcal{Q}_\delta(z)$ and $\mathbb{R}^{n+1} \setminus Q_\delta(z)$ and write

$$I_\beta(g)(z) = \int_{\mathcal{Q}_\delta(z)} \ldots\, dw + \int_{\mathbb{R}^{n+1}\setminus\mathcal{Q}_\delta(z)} \ldots\, dw =: I_1 + I_2,$$

with the obvious meaning of I_1 and I_2. Moreover, for $k \in \mathbb{Z}$ we let

$$A_k(z) := \{w \in \mathbb{R}^{n+1} : 2^k \delta \leq d_{\mathcal{P}}(z,w) < 2^{k+1}\delta\}.$$

We first treat the integral I_1:

$$\begin{aligned}
I_1 &\leq \sum_{k=1}^{\infty} \int_{A_{-k}(z)} \frac{g(w)}{d_{\mathcal{P}}(z,w)^{N-\beta}}\,dw \leq \sum_{k=1}^{\infty} (2^{-k}\delta)^{\beta-N} \int_{\mathcal{Q}_{2^{-k+1}\delta}(z)} g(w)\,dw \\
&\leq \sum_{k=1}^{\infty} (2^{-k}\delta)^{\beta-N} \left|\mathcal{Q}_{2^{-k+1}\delta}(z)\right| \fint_{\mathcal{Q}_{2^{-k+1}\delta}(z)} g(w)\,dw \\
&\leq \omega_n \delta^{\beta} \sum_{k=1}^{\infty} 2^{-\beta k} M(g)(z) = \frac{\alpha(n)}{2^{\beta}-1}\,\delta^{\beta} M(g)(z).
\end{aligned}$$

On the other hand we have for the integral I_2:

$$\begin{aligned}
I_2 &\leq \sum_{k=0}^{\infty} \int_{A_k(z)} \frac{g(w)}{d_{\mathcal{P}}(z,w)^{N-\beta}}\,dw \\
&\leq \sum_{k=0}^{\infty} (2^{k}\delta)^{\beta-N} \left|\mathcal{Q}_{2^{k+1}\delta}(z)\right| \fint_{\mathcal{Q}_{2^{k+1}\delta}(z)} g(w)\,dw \\
&\leq \sum_{k=0}^{\infty} (2^{k}\delta)^{\beta-N} \left|\mathcal{Q}_{2^{k+1}\delta}(z)\right|^{1-\frac{\vartheta}{Np}} M_{\vartheta/p}(g)(z) \\
&= [2^{N+1}\omega_n]^{1-\frac{\vartheta}{Np}} \delta^{\beta-\frac{\vartheta}{p}} \sum_{k=0}^{\infty} 2^{k(\beta-\frac{\vartheta}{p})} M_{\vartheta/p}(g)(z) \\
&= \frac{[2^{N+1}\omega_n]^{1-\frac{\vartheta}{Np}}}{1-2^{\beta-\frac{\vartheta}{p}}} \delta^{\beta-\frac{\vartheta}{p}} M_{\vartheta/p}(g)(z).
\end{aligned}$$

Having arrived at this stage we choose

$$\delta = \delta(z) := \left[\frac{M_{\vartheta/p}(g)(z)}{M(g)(z)}\right]^{\frac{p}{\vartheta}},$$

and finally obtain

$$\begin{aligned}
I_\beta(g)(z) \leq c \Bigg[&\left(\frac{M_{\vartheta/p}(g)(z)}{M(g)(z)}\right)^{\frac{\beta p}{\vartheta}} M(g)(z) \\
&+ \left(\frac{M_{\vartheta/p}(g)(z)}{M(g)(z)}\right)^{\frac{p}{\vartheta}(\beta-\frac{\vartheta}{p})} M_{\vartheta/p}(g)(z)\Bigg] \\
\leq 2c &\left[M_{\vartheta/p}(g)(z)\right]^{\frac{\beta p}{\vartheta}} \left[M(g)(z)\right]^{1-\frac{\beta p}{\vartheta}},
\end{aligned}$$

where $c \equiv c(n,p,\vartheta,\beta)$. □

REMARK 6.3. We note that the constant in Lemma 6.2 blows up either when $\beta \searrow 0$ - the case of singular integrals - or when $\beta p \nearrow \vartheta$ - the limiting case in the Sobolev-embedding. This settles in a certain sense the dependencies of the constants in all following results. Moreover, as an immediate consequence of Lemma 6.2 we obtain that for any measurable function g defined on $\mathbb{R}^{n+1}$ the pointwise Hedberg-type estimate

$$I_\beta(|g|)(z) \leq c\left[M_{\vartheta/p}(g)(z)\right]^{\frac{\beta p}{\vartheta}} \left[M(g)(z)\right]^{1-\frac{\beta p}{\vartheta}}$$

holds true for every $z \in \mathbb{R}^{n+1}$ with a constant $c \equiv c(n,p,\vartheta,\beta)$.

COROLLARY 6.4. *Let* $0 < \beta < N$, $0 < \vartheta \le N$, $1 < p < \frac{\vartheta}{\beta}$, $1 \le q \le \infty$, $g \in L^p(\mathbb{R}^{n+1})$, $E \subset \mathbb{R}^{n+1}$ *and* $M_{\vartheta/p}(g) \in L^q(E)$. *Then*

$$\|I_\beta(g)\|_{L^r(E)} \le c\,\big\|M_{\vartheta/p}(g)\big\|_{L^q(E)}^{\frac{\beta p}{\vartheta}} \|g\|_{L^p(\mathbb{R}^{n+1})}^{1-\frac{\beta p}{\vartheta}},$$

where

$$\frac{1}{r} = \frac{1}{p} - \frac{\beta}{\vartheta} + \frac{\beta p}{\vartheta q}. \tag{6.0.31}$$

PROOF. The case $q < \infty$: Integrating the Hedberg-Type inequality from Lemma 6.2 over E, using Hölder's inequality and (6.0.31) we infer

$$\begin{aligned}
\int_E \big[I_\beta(|g|)\big]^r\,dz &\le c \int_E \big[M_{\vartheta/p}(g)\big]^{r\frac{\beta p}{\vartheta}} \big[M(g)\big]^{r(1-\frac{\beta p}{\vartheta})}\,dz \\
&\le c\left(\int_E \big[M_{\vartheta/p}(g)\big]^q\,dz\right)^{\frac{r\beta p}{\vartheta q}} \left(\int_E \big[M(g)\big]^{r\frac{1-\beta p/\vartheta}{1-r\beta p/\vartheta q}}\,dz\right)^{1-\frac{r\beta p}{\vartheta q}} \\
&\le c\,\|M_{\vartheta/p}(g)\|_{L^q(E)}^{\frac{r\beta p}{\vartheta}} \left(\int_{\mathbb{R}^{n+1}} \big[M(g)\big]^p\,dz\right)^{1-\frac{r\beta p}{\vartheta q}}.
\end{aligned}$$

This leads to the estimate

$$\begin{aligned}
\left(\int_E \big[I_\beta(|g|)\big]^r\,dz\right)^{\frac{1}{r}} &\le c\,\big\|M_{\vartheta/p}(g)\big\|_{L^q(E)}^{\frac{\beta p}{\vartheta}} \left(\int_{\mathbb{R}^{n+1}} \big[M(g)\big]^p\,dz\right)^{\frac{1}{r}-\frac{\beta p}{\vartheta q}} \\
&\le c\,\big\|M_{\vartheta/p}(g)\big\|_{L^q(E)}^{\frac{\beta p}{\vartheta}} \|g\|_{L^p(\mathbb{R}^{n+1})}^{1-\frac{\beta p}{\vartheta}},
\end{aligned}$$

where we have used the boundedness of the maximal operator between L^p-spaces and the identity (6.0.31). This proves $I_\beta(|g|) \in L^r(E)$ and the desired estimate follows easily.

In the case $q = \infty$, instead of using Hölder's inequality in the first step, we have the trivial estimate

$$\int_E \big[I_\beta(|g|)\big]^r\,dz \le c\,\big\|M_{\vartheta/p}(g)\big\|_{L^\infty(E)}^{\frac{r\beta p}{\vartheta}} \int_{\mathbb{R}^{n+1}} \big[M(g)\big]^p\,dz$$

and we immediately obtain the desired estimate, taking into account (6.0.31) and $\frac{\beta p}{\vartheta q} = 0$. □

LEMMA 6.5. *Let* $0 < \beta < N$, $p \ge 1$ *such that* $\beta p < N$, *and assume* $g \in L^p(\mathbb{R}^{n+1})$. *Then*

$$\left(\int_{\mathbb{R}^{n+1}} \big|I_\beta(g)\big|^{\frac{Np}{N-\beta p}}\,dz\right)^{\frac{N-\beta p}{Np}} \le c(n,p,\beta)\left(\int_{\mathbb{R}^{n+1}} |g|^p\,dz\right)^{\frac{1}{p}},$$

i.e. $I_\beta \colon L^p(\mathbb{R}^{n+1}) \hookrightarrow L^{\frac{Np}{N-\beta p}}(\mathbb{R}^{n+1})$ *is a continuous embedding.*

PROOF. We apply Corollary 6.4 with $E \equiv \mathbb{R}^{n+1}$, $\vartheta = N$ and $q = \infty$. Note that

$$\begin{aligned}
M_{\vartheta/p}(g)(z) = M_{N/p}(g)(z) &= \sup_{\mathcal{Q}\subset\mathbb{R}^{n+1},\, z\in\mathcal{Q}} |\mathcal{Q}|^{\frac{1}{p}} \fint_{\mathcal{Q}} |g(w)|\,dw \\
&\le \sup_{\mathcal{Q}\subset\mathbb{R}^{n+1},\, z\in\mathcal{Q}} \left(\int_{\mathcal{Q}} |g(w)|^p\,dw\right)^{\frac{1}{p}} \le \|g\|_{L^p(\mathbb{R}^{n+1})},
\end{aligned}$$

so that $\|M_{N/p}(g)\|_{L^\infty(\mathbb{R}^{n+1})} \le \|g\|_{L^p(\mathbb{R}^{n+1})}$. Moreover, $1/r = 1/p - \beta/N = \frac{N-\beta p}{Np}$. Inserting this in the estimate of Corollary 6.4 then yields the result. □

REMARK 6.6. The preceding lemma yields in combination with the pointwise estimate from Lemma 6.1 that also the fractional maximal operator M_β is a continuous embedding from $L^p(\mathbb{R}^{n+1})$ into $L^{\frac{Np}{N-\beta p}}(\mathbb{R}^{n+1})$. Moreover, we have the estimate

$$\bigg(\int_{\mathbb{R}^{n+1}} \big|M_\beta(g)\big|^{\frac{Np}{N-\beta p}}\,dz\bigg)^{\frac{N-\beta p}{Np}} \le c(n,p,\beta)\bigg(\int_{\mathbb{R}^{n+1}} |g|^p\,dz\bigg)^{\frac{1}{p}},$$

whenever $0<\beta<N$ and $p\ge 1$ such that $\beta p<N$.

LEMMA 6.7. *Let $g\in L^\vartheta(p,q)(\mathbb{R}^{n+1})$ and $Q_R\subset\mathbb{R}^{n+1}$ be a parabolic cylinder with radius $R>0$. Then, for any $s>1$ there holds*

$$\|g\chi_{Q_R}\|_{L^\vartheta(p,q)(\mathbb{R}^{n+1})} \le \max\big\{1,\big[(s-1)/2\big]^{\frac{\vartheta-N}{p}}\big\}\|g\|_{L^\vartheta(p,q)(Q_{sR})}.$$

PROOF. We consider Q_ϱ such that $Q_\varrho\cap Q_R\neq\emptyset$ and remark that

$$|\{z\in Q_\varrho : |(g\chi_{Q_R})(z)|>\lambda\}| \le |\{z\in Q_\varrho : |g(z)|>\lambda\}|.$$

In the case $Q_\varrho\subset Q_{sR}$ we have

$$\begin{aligned}
&\varrho^{\vartheta-N}\|g\chi_{Q_R}\|^p_{L(p,q)(Q_\varrho)}\\
&\quad= \bigg[p\int_0^\infty \Big(\lambda^p\varrho^{\vartheta-N}|\{z\in Q_\varrho : |(g\chi_{Q_R}(z)|>\lambda\}|\Big)^{\frac{q}{p}}\frac{d\lambda}{\lambda}\bigg]^{\frac{p}{q}}\\
&\quad\le \bigg[p\int_0^\infty \Big(\lambda^p\varrho^{\vartheta-N}|\{z\in Q_\varrho : |g(z)|>\lambda\}|\Big)^{\frac{q}{p}}\frac{d\lambda}{\lambda}\bigg]^{\frac{p}{q}}\\
&\quad\le \bigg[p\int_0^\infty \Big(\lambda^p\varrho^{\vartheta-N}|\{z\in Q_{sR} : |g(z)|>\lambda\}|\Big)^{\frac{q}{p}}\frac{d\lambda}{\lambda}\bigg]^{\frac{p}{q}}\\
&\quad\le \|g\|^p_{L^\vartheta(p,q)(Q_{sR})}.
\end{aligned}$$

In the remaining case $Q_\varrho\not\subset Q_{sR}$ (taking also into account $Q_\varrho\cap Q_{sR}\neq\emptyset$) we have $2\varrho>(s-1)R$. This implies

$$\begin{aligned}
&\varrho^{\vartheta-N}|\{z\in Q_\varrho : |(g\chi_{Q_R})(z)|>\lambda\}|\\
&\qquad\qquad\le [(s-1)/2]^{\vartheta-N}R^{\vartheta-N}|\{z\in Q_R : |g(z)|>\lambda\}|,
\end{aligned}$$

and similarly to the first case this leads us now to the estimate

$$\varrho^{\vartheta-N}\|g\chi_{Q_R}\|^p_{L(p,q)(Q_\varrho)} \le [(s-1)/2]^{\vartheta-N}\|g\|^p_{L^\vartheta(p,q)(Q_R)}.$$

Combining the two cases yields the desired estimate. □

REMARK 6.8. Let Q_ϱ be a parabolic cylinder with radius $\varrho>0$. Then, from the definition of the Lorentz-Morrey-norm (see (3.2.3)) we infer the bound

$$\begin{aligned}
\|g\|_{L(p,q)(Q_\varrho)} &= \varrho^{\frac{N-\vartheta}{p}}\bigg[p\int_0^\infty \Big(\lambda^p\varrho^{\vartheta-N}|\{z\in Q_\varrho : |g(z)|>\lambda\}|\Big)^{\frac{q}{p}}\frac{d\lambda}{\lambda}\bigg]^{\frac{1}{q}}\\
&\le \max\big\{1,\varrho^{\frac{N-\vartheta}{p}}\big\}\|g\|_{L^\vartheta(p,q)(Q_\varrho)}.
\end{aligned}$$

LEMMA 6.9. *Let Q be a parabolic cylinder in $\mathbb{R}^{n+1}$. Then for every $g\in L\log L(Q)$ with support in Q we have*

$$⨍_Q M(g)\,dz \le c(n)\|g\|_{L\log L(Q)} \approx c(n)|g|_{L\log L(Q)}.$$

PROOF. We define

$$h := \frac{g}{\|g\|_{L \log L(Q)}}.$$

Then, we have $\|h\|_{L \log L(Q)} = 1$ and therefore

$$\fint_Q |h| \log\big(e + |h|\big)\, dz \le 1.$$

Applying (the parabolic analog of) [65, Theorem 2.15] we conclude

$$\int_Q M(h)\, dz \le c(n)|Q| + c(n) \int_Q |h| \log\big(e + |h|\big)\, dz \le c(n)|Q|,$$

so that $\fint_Q M(h)\, dz \le c(n)$. Re-scaling back from h to g then yields the desired estimate. □

THEOREM 6.10. *Let $0 < \beta < \vartheta \le N$, Q a parabolic cylinder in $\mathbb{R}^{n+1}$ and $s > 1$. Then there exists a constant $c \equiv c(n, \beta, \vartheta, s)$ such that the estimate*

$$\|M^*_{\beta,Q}(g)\|_{L^{\frac{\vartheta}{\vartheta-\beta}}(Q)} \le c\,|Q|^{1-\frac{\beta}{\vartheta}} \|g\|^{\frac{\beta}{\vartheta}}_{L^{1,\vartheta}(sQ)} \|g\|^{1-\frac{\beta}{\vartheta}}_{L \log L(Q)}$$

holds, whenever $g\colon sQ \to \mathbb{R}$ is measurable.

PROOF. Without loss of generality we may assume that $g \ge 0$. Then, $I_\beta(g) \ge 0$. Let $\tilde g = g\chi_Q$. Applying Lemma 6.2 with $p = 1$ we obtain that

$$\big[I_\beta(\tilde g)(z)\big]^{\frac{\vartheta}{\vartheta-\beta}} \le c\,\big[M_\vartheta(\tilde g)(z)\big]^{\frac{\beta}{\vartheta-\beta}} M(\tilde g)(z) \le c\,\|\tilde g\|^{\frac{\beta}{\vartheta-\beta}}_{L^{1,\vartheta}(\mathbb{R}^{n+1})} M(\tilde g)(z)$$

holds for every $z \in \mathbb{R}^{n+1}$. Here we have also used the obvious estimate $M_\vartheta(\tilde g)(z) \le c\,\|\tilde g\|_{L^{1,\vartheta}(\mathbb{R}^{n+1})}$ which follows directly from the definitions of the fractional maximal operators given in (3.2.9) and the usual one of a Morrey-space. Using the pointwise bound from below for $I_\beta(\tilde g)$ from Lemma 6.1 we infer that

$$\big[M_\beta(\tilde g)(z)\big]^{\frac{\vartheta}{\vartheta-\beta}} \le c\,\|\tilde g\|^{\frac{\beta}{\vartheta-\beta}}_{L^{1,\vartheta}(\mathbb{R}^{n+1})} M(\tilde g)(z)$$

holds for every $z \in \mathbb{R}^{n+1}$. Integrating the preceding inequality on Q and using Lemma 6.9 yields

$$\begin{aligned} \|M_\beta(\tilde g)\|^{\frac{\vartheta}{\vartheta-\beta}}_{L^{\frac{\vartheta}{\vartheta-\beta}}(Q)} &\le c\,|Q|\|\tilde g\|^{\frac{\beta}{\vartheta-\beta}}_{L^{1,\vartheta}(\mathbb{R}^{n+1})} \fint_Q M(\tilde g)\, dz \\ &\le c\,|Q|\,\|\tilde g\|^{\frac{\beta}{\vartheta-\beta}}_{L^{1,\vartheta}(\mathbb{R}^{n+1})} \|\tilde g\|_{L \log L(Q)}. \end{aligned}$$

Recalling the obvious inequality $M^*_{\beta,Q}(g) \le M_\beta(\tilde g)$ in order to estimate the left-hand side from below and Lemma 6.7 to estimate the right-hand side from above, i.e. the fact that $\|\tilde g\|_{L^{1,\vartheta}(\mathbb{R}^{n+1})} \le c\|g\|_{L^{1,\vartheta}(sQ)}$, we conclude the assertion of the lemma. □

THEOREM 6.11. *Let $\beta, \vartheta \in (0, N]$, $p > 1$, such that $\beta p < \vartheta$, and let $q \in (0, \infty]$. Furthermore, let Q be a parabolic cylinder in $\mathbb{R}^{n+1}$ and $s > 1$. Then there exists a constant $c \equiv c(n, p, q, \beta, \vartheta, s)$ such that*

$$\big\|M^*_{\beta,Q}(g)\big\|_{L\left(\frac{\vartheta p}{\vartheta-\beta p}, \frac{\vartheta q}{\vartheta-\beta p}\right)(Q)} \le c\,\|g\|^{\frac{\beta p}{\vartheta}}_{L^\vartheta(p,q)(sQ)} \|g\|^{1-\frac{\beta p}{\vartheta}}_{L(p,q)(Q)}$$

holds whenever g is a measurable map defined on sQ. Moreover, if $|sQ| \le 100^N$ we have

$$\big\|M^*_{\beta,Q}(g)\big\|_{L\left(\frac{\vartheta p}{\vartheta-\beta p}, \frac{\vartheta q}{\vartheta-\beta p}\right)(Q)} \le c\,\|g\|_{L^\vartheta(p,q)(sQ)}.$$

The constant c blows up, i.e. $c \to \infty$, when $q \searrow 0$ or $p \searrow 1$.

PROOF. In the case $q = \infty$ we let $\frac{\vartheta q}{\vartheta - \beta p} := \infty$. Once again we may assume without loss of generality that $g \geq 0$. We define $\tilde{g} := g\chi_Q$. Then for $Q_R \subset \mathbb{R}^{n+1}$ we have

$$\int_{Q_R} |\tilde{g}|\, dz \leq \frac{p}{p-1}|Q_R|^{1-\frac{1}{p}}\|\tilde{g}\|_{\mathcal{M}^p(Q_R)} = \frac{cp}{p-1}\, R^{N(1-\frac{1}{p})}\|\tilde{g}\|_{\mathcal{M}^p(Q_R)}$$
$$= \frac{cp}{p-1}\, R^{N-\frac{\vartheta}{p}} R^{\frac{\vartheta - N}{p}}\|\tilde{g}\|_{\mathcal{M}^p(Q_R)} \leq \frac{cp}{p-1}\, R^{N-\frac{\vartheta}{p}}\|\tilde{g}\|_{\mathcal{M}^{p,\vartheta}(\mathbb{R}^{n+1})},$$

where $c \equiv c(n,p)$. This implies in particular that $M_{\vartheta/p}(\tilde{g})(z) \leq \frac{cp}{p-1}\|\tilde{g}\|_{\mathcal{M}^{p,\vartheta}(\mathbb{R}^{n+1})}$ holds for every $z \in \mathbb{R}^{n+1}$. Using this in the Hedberg-type inequality from Lemma 6.2 yields

$$\big[I_\beta(\tilde{g})(z)\big]^{\frac{\vartheta}{\vartheta-\beta p}} \leq c\,\big[M_{\vartheta/p}(\tilde{g})(z)\big]^{\frac{\beta p}{\vartheta - \beta p}} M(\tilde{g})(z) \leq c\,\|\tilde{g}\|^{\frac{\beta p}{\vartheta-\beta p}}_{\mathcal{M}^{p,\vartheta}(\mathbb{R}^{n+1})} M(\tilde{g})(z),$$

for every $z \in \mathbb{R}^{n+1}$. In the preceding inequality we want to replace the Marcinkiewicz norm of $\tilde{g}$ by an appropriate Lorentz-Morrey norm. For this we recall that for $q > 0$ we have $\|\tilde{g}\|_{\mathcal{M}^p(Q_R)} \leq (q/p)^{\frac{1}{q}}\|\tilde{g}\|_{L(p,q)(Q_R)}$, so that $\|\tilde{g}\|_{\mathcal{M}^{p,\vartheta}(\mathbb{R}^{n+1})} \leq c\,\|\tilde{g}\|_{L^\vartheta(p,q)(\mathbb{R}^{n+1})}$. Inserting this above we immediately find

$$[I_\beta(\tilde{g})(z)]^{\frac{\vartheta}{\vartheta-\beta p}} \leq c\,\|\tilde{g}\|^{\frac{\beta p}{\vartheta - \beta p}}_{L^\vartheta(p,q)(\mathbb{R}^{n+1})} M(\tilde{g})(z),$$

which leads after integrating in an appropriate way over $\mathbb{R}^{n+1}$ to

$$\|[I_\beta(\tilde{g})]^{\frac{\vartheta}{\vartheta-\beta p}}\|^{\frac{\vartheta-\beta p}{\vartheta}}_{L(p,q)(\mathbb{R}^{n+1})} \leq c\,\|\tilde{g}\|^{\frac{\beta p}{\vartheta}}_{L^\vartheta(p,q)(\mathbb{R}^{n+1})}\|M(\tilde{g})\|^{1-\frac{\beta p}{\vartheta}}_{L(p,q)(\mathbb{R}^{n+1})}. \tag{6.0.32}$$

Using definition (3.2.1) and a simple change-of-variable argument we find for the left-hand side of (6.0.32) the identity

$$\begin{aligned}
&\big\|[I_\beta(\tilde{g})]^{\frac{\vartheta}{\vartheta-\beta p}}\big\|_{L(p,q)(\mathbb{R}^{n+1})}\\
&= \left[p\int_0^\infty \Big(\lambda^p|\{z \in \mathbb{R}^{n+1} : [I_\beta(\tilde{g})(z)]^{\frac{\vartheta}{\vartheta-\beta p}} > \lambda\}|\Big)^{\frac{q}{p}}\frac{d\lambda}{\lambda}\right]^{\frac{1}{q}}\\
&= \left[p\int_0^\infty \Big(\lambda^p|\{z \in \mathbb{R}^{n+1} : I_\beta(\tilde{g})(z) > \lambda^{\frac{\vartheta-\beta p}{\vartheta}}\}|\Big)^{\frac{q}{p}}\frac{d\lambda}{\lambda}\right]^{\frac{1}{q}}\\
&= \left[\frac{\vartheta p}{\vartheta-\beta p}\, ds\int_0^\infty \Big(\mu^{\frac{\vartheta p}{\vartheta-\beta p}}|\{z \in \mathbb{R}^{n+1} : I_\beta(\tilde{g})(z) > \mu\}|\Big)^{\frac{\vartheta q/(\vartheta-\beta p)}{\vartheta p/(\vartheta-\beta p)}}\frac{d\mu}{\mu}\right]^{\frac{1}{q}}\\
&= \big\|I_\beta(\tilde{g})\big\|^{\frac{\vartheta}{\vartheta-\beta p}}_{L(\frac{\vartheta p}{\vartheta-\beta p},\frac{\vartheta q}{\vartheta-\beta p})(\mathbb{R}^{n+1})}.
\end{aligned} \tag{6.0.33}$$

On the other hand the boundedness of the maximal operator in Lorentz-spaces allows us to estimate the second term on the right-hand side of (6.0.32) from above; to be precise we have

$$\|M(\tilde{g})\|_{L(p,q)(\mathbb{R}^{n+1})} \leq c(n,p,q)\|\tilde{g}\|_{L(p,q)(\mathbb{R}^{n+1})}.$$

Using this and (6.0.33) in (6.0.32) we arrive at

$$\|I_\beta(\tilde{g})\|_{L(\frac{\vartheta p}{\vartheta-\beta p},\frac{\vartheta q}{\vartheta-\beta p})(\mathbb{R}^{n+1})} \leq c\,\|\tilde{g}\|^{\frac{\beta p}{\vartheta}}_{L^\vartheta(p,q)(\mathbb{R}^{n+1})}\|\tilde{g}\|^{1-\frac{\beta p}{\vartheta}}_{L(p,q)(\mathbb{R}^{n+1})}.$$

The first term in the right-hand side of the preceding inequality is estimated by Lemma 6.7, i.e $\|\tilde{g}\|_{L^\vartheta(p,q)(\mathbb{R}^{n+1})} \leq c(s)\,\|g\|_{L^\vartheta(p,q)(sQ)}$, while the second term is equal to $\|g\|_{L(p,q)(Q)}$. On the other hand from Lemma 6.1 and the definition of the restricted maximal operator in (3.2.9) we infer the pointwise estimate $c^{-1}I_\beta(\tilde{g})(z) \geq M_\beta(\tilde{g})(z)$. Inserting this above yields

$$\|M_\beta(\tilde{g})\|_{L(\frac{\vartheta p}{\vartheta-\beta p},\frac{\vartheta q}{\vartheta-\beta p})(Q)} \leq c\,\|g\|^{\frac{\beta p}{\vartheta}}_{L^\vartheta(p,q)(sQ)}\|g\|^{1-\frac{\beta p}{\vartheta}}_{L(p,q)(Q)}.$$

Combining this with $M_\beta(\tilde{g})(z) \geq M_{\beta,Q}(g)(z)$, $z \in Q$, leads to the first asserted estimate of the theorem. In order to obtain the second assertion we use Remark 6.8 and the assumption $|sQ| \leq 100^N$ to estimate the right-hand side from above. □

6.1. Basic Regularity

In this section we will consider

$$u \in C^0([-T,0]; L^2(\Omega)) \cap L^2(-T,0; W^{1,2}(\Omega)),$$

defined as the unique solution to the regularized Cauchy-Dirichlet problems (4.1.1) with $H \equiv g \in L^\infty(\Omega_T)$ under the assumptions (4.2.1). For the ease of the reader we re-write it here:

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) = g \in L^\infty(\Omega_T) & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T. \end{cases} \tag{6.1.1}$$

For a fixed parabolic cylinder $Q_R = Q_R(z_0) \subset \Omega_T$ we consider the unique solution

$$v \in C^0([t_0-R^2, t_0+R^2]; L^2(B_R(x_0))) \cap L^2(t_0-R^2, t_0+R^2; W^{1,2}(B_R(x_0)))$$

to the following comparison homogeneous Cauchy-Dirichlet problem:

$$\begin{cases} v_t - \operatorname{div} a(z,Du) = 0 & \text{in } Q_R, \\ v = u & \text{on } \partial_{\mathcal{P}}Q_R. \end{cases} \tag{6.1.2}$$

The following comparison lemma can be inferred by Lemma 5.6 using also Poincaré's inequality.

LEMMA 6.12. *Let u be a solution to (6.1.1) under the assumption (4.2.1) and $Q_R(z_0)$ a parabolic cylinder in Ω_T. Moreover, let v be a solution to the Cauchy-Dirichlet problem(6.1.2). Then there exists a constant $c \equiv c(n)$ such that*

$$\int_{Q_R(z_0)} \frac{|u-v|}{R}\, dz + \int_{Q_R(z_0)} |Du - Dv|\, dz \leq c\, \nu^{-1} R \int_{Q_R(z_0)} |g|\, dz. \tag{6.1.3}$$

LEMMA 6.13. *Let u, v be as in the previous Lemma; moreover let $g \in L^\vartheta(\gamma,q)(Q_R)$ for some $\gamma > 1$. Then there exists a constant $c \equiv c(n,\nu,\gamma)$ such that*

$$\int_{Q_R(z_0)} \frac{|u-v|}{R}\, dz + \int_{Q_R(z_0)} |Du - Dv|\, dz \\ \leq c\, R^{N - \frac{\vartheta-\gamma}{\gamma}} \|g\|_{L^\vartheta(\gamma,q)(Q_R(z_0))}. \tag{6.1.4}$$

PROOF. Using Lemma 3.3 and the embedding $\|g\|_{\mathcal{M}^\gamma(Q_R)} \leq (q/\gamma)^{\frac{1}{q}} \|g\|_{L(\gamma,q)(Q_R)}$ we can conclude for the right-hand side in (6.1.3) that

$$\begin{aligned} R\int_{Q_R} |g|\, dz &\leq c\left(\frac{\gamma}{\gamma-1}\, ds\right) R^{1+N(1-\frac{1}{\gamma})} \|g\|_{\mathcal{M}^\gamma(Q_R)} \\ &\leq c\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{q}{\gamma}\right)^{\frac{1}{q}} R^{1+N(1-\frac{1}{\gamma})} \|g\|_{L(\gamma,q)(Q_R)} \\ &\leq c\, R^{N-\frac{\vartheta-\gamma}{\gamma}} R^{\frac{\vartheta-N}{\gamma}} \|g\|_{L(\gamma,q)(Q_R)} \leq c\, R^{N-\frac{\vartheta-\gamma}{\gamma}} \|g\|_{L^\vartheta(\gamma,q)(Q_R)}, \end{aligned}$$

where $c \equiv c(n,\nu,\gamma)$. Using the preceding inequality in 6.1.3 yields the result. □

LEMMA 6.14. *Let u, v as above and let $g \in L^{1,\vartheta}(Q_R)$. Then there exists a constant $c \equiv c(n, \nu, \gamma)$ such that*

$$\int_{Q_R(z_0)} R^{-1}|u - v| + |Du - Dv|\, dz \le c\, R^{N-(\vartheta-1)} \|g\|_{L^{1,\vartheta}(Q_R(z_0))}. \tag{6.1.5}$$

PROOF. For the proof it is sufficient to note that $\|g\|_{L^1(Q_R)} \le R^{N-\vartheta}\|g\|_{L^{1,\vartheta}(Q_R)}$. □

Homogeneous problems. The results of this chapter summarize the basic Hölder regularity results from the De Giorgi-Nash-Moser theory of solutions to nonlinear, homogeneous parabolic equations as well as the higher integrability theory.

THEOREM 6.15. *Let $v \in C^0(I; L^2(A)) \cap L^2(I; W^{1,2}(A))$ be a weak solution to the parabolic equation*

$$v_t - \operatorname{div} a(x, t, Dv) = 0 \qquad \text{in } A \times I \subset \Omega_T, \tag{6.1.6}$$

under the assumptions

$$|a(x, t, \xi)| \le L(1 + |\xi|), \qquad \nu|\xi|^2 - L^2/\nu \le \langle a(x, t, \xi), \xi\rangle, \tag{6.1.7}$$

for every choice of $(x, t) \in A \times I$ and $\xi \in \mathbb{R}^n$ where $0 < \nu \le 1 \le L < \infty$ and $a\colon A \times I \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector field. Then, there exists $\alpha \in (0, 1/2]$ depending only on n and L/ν, such that for every $q \in (0, 2]$ there exists a constant $c \equiv c(n, L, \nu, q)$ such that the following holds: Whenever $Q_R \subset A \times I$ and $0 < \varrho \le R$ there holds

$$\int_{Q_\varrho} (|Dv|^q + 1)\, dz \le c \left(\frac{\varrho}{R}\right)^{N-q+\alpha q} \int_{Q_R} (|Dv|^q + 1)\, dz \tag{6.1.8}$$

and

$$\int_{Q_\varrho} (|v|^q + \varrho^q)\, dz \le c \left(\frac{\varrho}{R}\right)^{N} \int_{Q_R} (|v|^q + R^q) dz. \tag{6.1.9}$$

Furthermore, there exists $\chi \equiv \chi(n, L, \nu) > 1$ such that $Dv \in L^{2\chi}_{\mathrm{loc}}(A \times I; \mathbb{R}^n)$ and

$$\left(\fint_{Q_{R/2}} (|Dv| + 1)^{2\chi}\, dz\right)^{\frac{1}{2\chi}} \le c \left(\fint_{Q_R} (|Dv| + 1)^q\, dz\right)^{\frac{1}{q}} \tag{6.1.10}$$

holds for every $q \in (0, 2]$, while for every $\chi_0 > 1$ there holds

$$\left(\fint_{Q_{R/2}} (|v| + R)^{2\chi_0}\, dz\right)^{\frac{1}{2\chi_0}} \le c \left(\fint_{Q_R} (|v| + R)^q\, dz\right)^{\frac{1}{q}}. \tag{6.1.11}$$

In both (6.1.10) and (6.1.11) we have the following dependence of the constant c from the structural constants: $c \equiv c(n, L, \nu, q)$

PROOF. The statement is a direct consequence of De Giorgi-Nash-Moser's theory. We give a very brief hint how to retrieve the estimates (6.1.8) to (6.1.11). (6.1.10) for $q = 2$ can be inferred for instance from [**124**], and we refer the reader also to [**32**, Lemma 3.1], where the statement is directly proved at the boundary. From this, the reduction of the exponent 2 on the right hand side to any exponent $q \in (0, 2]$ follows by a standard result on reverse Hölder inequalities. For details, we refer the reader for example to [**70**, Chapter 4] and Lemma 3.1. The estimates (6.1.8), (6.1.9) and also (6.1.11) with $q = 2$ follow for instance from [**113**], Chapter 6, where De Giorgi's proof is performed in the parabolic setting. Then, again the arguments of [**70**] allow to reduce the exponent 2 on the right hand side to any exponent $q \in (0, 2]$. Note here, that the arguments in [**113**] are worked out for linear parabolic equations, but as mentioned in the notes in Chapter 6, the linearity of the equation is actually irrelevant for the estimates and they hold also for quasi-linear equations fulfilling the structure conditions (6.1.7). □

The next theorem is the homogeneous case of a much more general result concerning Calderón-Zygmund estimates for weak solutions to nonlinear parabolic equations (systems); see [**71**, Theorem 1.8, 1.9] for the specific form of the statement. We consider weak solutions v to (6.1.6) where the vector-field $a(x,t,\xi)$ satisfies either the structure conditions (4.2.14) or the vector field has the special form $a(x,t,\xi) = c(x)\bar{a}(t,\xi)$, where $c(x)$ and $\bar{a}(t,\xi)$ satisfy the conditions (4.2.15), (4.2.16) and (4.2.17). Then the following theorem holds:

THEOREM 6.16. *Let $v \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W^{1,2}(\Omega))$ be a weak solution to the homogeneous nonlinear parabolic equation* (6.1.6) *where either the structure assumptions* (4.2.14) *or the conditions* (4.2.15) *to* (4.2.17) *are in force. Then for any $\alpha \in (0,1)$ and $q \in (0,2]$ there exists a constant $c \equiv c(n,L,\nu,\alpha,q)$ such that the following holds: Whenever $Q_R \subset \Omega_T$ and $0 < \varrho \leq R$ there holds*

$$\int_{Q_\varrho} (|Dv|^q + 1)\,dz \leq c\left(\frac{\varrho}{R}\right)^{N-q+\alpha q} \int_{Q_R} (|Dv|^q + 1)\,dz. \tag{6.1.12}$$

Furthermore, $Dv \in L^{2\chi_0}_{\mathrm{loc}}(\Omega_T;\mathbb{R}^n)$ for any $\chi_0 > 1$. Moreover, for any given $\chi_0 > 1$ and $q \in (0,2]$ there exits a constant $c \equiv c(n,\nu,L,\chi_0,\omega(\cdot),q)$ such that for any $Q_R \Subset \Omega_T$ there holds

$$\left(\fint_{Q_{R/2}} (|Dv|+1)^{2\chi_0}\,dz\right)^{\frac{1}{2\chi_0}} \leq c\left(\fint_{Q_R} (|Dv|+1)^q\,dz\right)^{\frac{1}{q}}. \tag{6.1.13}$$

PROOF. Estimate (6.1.13) is the statement of [**71**, Theorems 1.8, 1.9]. Then, once having (6.1.12) for the case $q = 2$, the general estimate for $q \in (0,2]$ can be retrieved by a simple application of Hölder's inequality to pass from exponent $q < 2$ to exponent 2, then exploiting (6.1.12) for $q = 2$, and subsequently using (6.1.13) to reduce the exponent 2 again to exponent $q < 2$. However, (6.1.12) for the special case $q = 2$ is a consequence of the Hölder continuity to any Hölder exponent $\alpha \in (0,1)$ for solutions to parabolic equations with linear growth. On the other hand, Hölder continuity to every exponent $\alpha \in (0,1)$ is a standard consequence of the fact that the vector field a is sufficiently regular with respect to x. In this case, Hölder continuity can be shown via suitable comparison procedures to differentiable or constant coefficient equations (see [**71**, Chapter 8] for comparison estimates in the case of VMO-regular vector fields as well as continuous ones). We note here, that actually the estimates in [**71**] are shown for much more general possibly degenerate p growth equations and systems. Standard references for Hölder regularity in the constant coefficient case are for example [**108, 113**]. □

REMARK 6.17. In the estimates (6.1.10) and (6.1.13) we can replace $R/2, R$ by $\sigma R, R$ for any $\sigma \in [\frac{1}{2},1)$ as long as we enlarge the constant by factor $\approx (1-\sigma)^{N(\frac{1}{2\chi}-\frac{1}{q})}$. This can be inferred along the arguments form [**81**, Remark 6.12]. On the other hand inequalities (6.1.8)–(6.1.13) continue to hold when replacing the parabolic cylinders Q with a ball as horizontal slice by the cylinders $\mathcal{Q}$ having a cube as horizontal cross section.

6.2. Integrability of Du

Parabolic Lorentz spaces estimates. Here we give the proof of Theorem 4.8. Note that Theorem 4.5 follows directly from the more general Theorem 4.8 by the choice $q = \gamma$. The proof is divided into several steps.

Step 1: Level sets decay. On a fixed parabolic cylinder $\mathcal{Q}_0$ satisfying $n^2\mathcal{Q}_0 \Subset \Omega_T$ and $|\mathcal{Q}_0| \le 1$ we consider the following maximal operators

$$M^* := M^*_{0,n^2\mathcal{Q}_0} = M^*_{n^2\mathcal{Q}_0} \qquad \text{and} \qquad M^*_1 := M^*_{1,n^2\mathcal{Q}_0}.$$

For the definitions of these restricted maximal function operators we refer the reader to Section 3.2.

LEMMA 6.18. *Let u be a weak solution to (6.1.1) where the assumptions (4.2.1) are in force and $g \in L^\infty(\Omega_T)$. Then, for every $S > 1$ there exists a constant $\varepsilon = \varepsilon(n, L, \nu, S) \in (0,1)$ such that if $\lambda > 1$ and $\mathcal{Q}$ is a dyadic sub-cylinder of $\mathcal{Q}_0$ such that*

$$\big|\mathcal{Q}\cap\big\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > AS\lambda \ \text{ and } \ M^*_1(g) \le \varepsilon\lambda\big\}\big| > \frac{|\mathcal{Q}|}{S^{2\chi}}, \quad (6.2.1)$$

then the predecessor $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ satisfies

$$\tilde{\mathcal{Q}} \subset \big\{z \in \mathcal{Q}_0 : M^*(1 + |Du|)(z) > \lambda\big\}. \quad (6.2.2)$$

Here $\chi \equiv \chi(n, L/\nu) > 1$ is the higher integrability exponent introduced in Theorem 6.15, while $A \equiv A(n, L/\nu) > 1$ is an absolute constant.

PROOF. We shall prove the assertion of the lemma by a contradiction argument. We therefore assume that (6.2.1) holds but (6.2.2) fails. Hence we can find $\tilde{z}$ such that there holds

$$M^*(1 + |Du|)(\tilde{z}) \le \lambda \qquad \text{and} \qquad \tilde{z} \in \tilde{\mathcal{Q}}.$$

Since $\tilde{\mathcal{Q}} \subset 3\mathcal{Q} \subset n^2\mathcal{Q}_0$, and trivially $\tilde{z} \in 3\mathcal{Q}$ we have

$$\fint_{3\mathcal{Q}} (1 + |Du|)\, dz \le M^*(1 + |Du|)(\tilde{z}) \le \lambda. \quad (6.2.3)$$

Moreover, from (6.2.1) we infer the existence of $\bar{z}$ satisfying

$$M^*_1(g)(\bar{z}) \le \varepsilon\lambda \qquad \text{and} \qquad \bar{z} \in \mathcal{Q}. \quad (6.2.4)$$

Now, let Q denote – in the sense of Section 3.1 – the unique parabolic cylinder having $3\mathcal{Q}$ as inner cylinder, i.e. $\mathcal{Q}_{\text{inn}}(Q) = 3\mathcal{Q}$. If $\mathcal{Q} = C_\varrho(x_1) \times (t_1 - \varrho^2, t_1 + \varrho^2)$ then Q is given by $B(x_1, 3\sqrt{n}\varrho) \times (t_1 - (3\sqrt{n}\varrho)^2, t_1 + (3\sqrt{n}\varrho)^2)$. It is easy to check that $Q \subset n^2\mathcal{Q}_0$. Next we denote by

$$v \in C^0\big([t_1 - (3\sqrt{n}\varrho)^2, t_1 + (3\sqrt{n}\varrho)^2]; L^2(B_3\sqrt{n}\varrho(x_1)\big) \\ \cap L^2\big(t_1 - (3\sqrt{n}\varrho)^2, t_1 + (3\sqrt{n}\varrho)^2; W^{1,2}(B_{3\sqrt{n}\varrho}(x_1)\big)$$

the unique solution to the homogeneous Cauchy-Dirichlet problem (6.1.2)

$$\begin{cases} v_t - \operatorname{div} a(z, Dv) = 0 & \text{in } Q, \\ v = u & \text{on } \partial_{\mathcal{P}} Q. \end{cases} \quad (6.2.5)$$

We consider the outer parabolic cylinder to Q, i.e. $\mathcal{Q}_{\text{out}}(Q) = Q_{3\sqrt{n}\varrho}(x_1) \times (t_1 - (3\sqrt{n}\varrho)^2, t_1 + (3\sqrt{n}\varrho)^2)$, which satisfies also $\mathcal{Q}_{\text{out}}(Q) \subset n^2\mathcal{Q}_0$. Then the definition of the fractional maximal operator M^*_1 and (6.2.4), i.e. $\bar{z} \in \mathcal{Q} \subset \mathcal{Q}_{\text{out}}(Q)$ and $M^*_1(g)(\bar{z}) \le \varepsilon\lambda$, yield that

$$|Q|^{\frac{1}{N}} \fint_Q |g|\, dz \le \left(\frac{|\mathcal{Q}_{\text{out}}(Q)|}{|Q|}\right)^{1-\frac{1}{N}} |\mathcal{Q}_{\text{out}}(Q)|^{\frac{1}{N}} \fint_{\mathcal{Q}_{\text{out}}(Q)} |g|\, dz \le c(n)\varepsilon\lambda. \quad (6.2.6)$$

Combining (6.2.6) with the universal comparison estimate from (6.1.3) we obtain

$$\int_{3\mathcal{Q}} |Du - Dv|\, dz \le \int_{Q} |Du - Dv|\, dz \le c\,\nu^{-1}|Q|^{\frac{1}{N}} \int_{Q} |g|\, dz \le c\,\varepsilon\lambda|Q|,$$

for a constant $c \equiv c(n)/\nu$. Using $|Q| = c(n)|3\mathcal{Q}|$ in the preceding inequality we arrive at

$$\fint_{3\mathcal{Q}} |Du - Dv|\, dz \le c(n,\nu)\varepsilon\lambda. \tag{6.2.7}$$

Next, we observe that the hypothesis of Theorem 6.15 are fulfilled for the solution v to the homogeneous Cauchy-Dirichlet problem(6.1.2) on Q. Therefore, we have the local higher integrability of Dv on $2\mathcal{Q} \subset 3\mathcal{Q} \subset Q$ with the estimate

$$\left(\fint_{2\mathcal{Q}} (1+|Dv|)^{2\chi}\, dz\right)^{\frac{1}{2\chi}} \le c(n,\nu,L) \fint_{3\mathcal{Q}} (1+|Dv|)\, dz, \tag{6.2.8}$$

where $\chi \equiv \chi(n,\nu,L)$ is the higher integrability exponent introduced in Theorem 6.15. Using the comparison estimate (6.2.7), (6.2.3) and $0 < \varepsilon \le 1$ the right-hand side of the preceding inequality is estimated as follows:

$$\fint_{3\mathcal{Q}} (1+|Dv|)\, dz \le \fint_{3\mathcal{Q}} (1+|Du|)\, dz + \fint_{3\mathcal{Q}} |Du - Dv|\, dz \le \lambda + c\,\varepsilon\lambda \le c\,\lambda,$$

with a constant $c \equiv c(n,\nu)$. Combining the preceding estimate with (6.2.8) yields

$$\fint_{2\mathcal{Q}} (1+|Dv|)^{2\chi}\, dz \le c(n,\nu,L)\lambda^{2\chi}. \tag{6.2.9}$$

In order to proceed further we use the restricted maximal operator on $2\mathcal{Q}$ and here we abbreviate $M^{**} := M^{*}_{0,2\mathcal{Q}}$. Using (3.2.10) twice, (6.2.9) and (6.2.7) we obtain

$$\begin{aligned}
&\big|\{z \in \mathcal{Q} : M^{**}(1+|Du|)(z) > AS\lambda\}\big| \\
&\quad\le \big|\{z \in \mathcal{Q} : M^{**}(1+|Dv|)(z) > \tfrac{1}{2}AS\lambda\}\big| \\
&\qquad\quad + \big|\{z \in \mathcal{Q} : M^{**}(|Du - Dv|)(z) > \tfrac{1}{2}AS\lambda\}\big| \\
&\quad\le \frac{c(n,\chi)}{(AS\lambda)^{2\chi}} \int_{2\mathcal{Q}} (1+|Dv|)^{2\chi}\, dz + \frac{c(n)}{AS\lambda} \int_{2\mathcal{Q}} |Du - Dv|\, dz \\
&\quad\le \frac{c(n,\nu,L)}{(AS)^{2\chi}}|2\mathcal{Q}| + \frac{c(n,\nu)\varepsilon}{AS}|3\mathcal{Q}| \\
&\quad= \left[\frac{c_1(n,\nu,L)}{(AS)^{2\chi}} + \frac{c_2(n,\nu)\varepsilon}{AS}\right]|\mathcal{Q}|.
\end{aligned} \tag{6.2.10}$$

Having arrived at this stage we perform the following choices of A and ε: We first choose $A \equiv A(n,\nu,L) > 1$ such that

$$A = 4 \cdot 10^{N}\left[1 + c_1(n,\nu,L)\right] \quad\Longrightarrow\quad \frac{c_1}{(AS)^{2\chi}} \le \frac{1}{4S^{2\chi}}. \tag{6.2.11}$$

Then we the choose $\varepsilon = \varepsilon(n,L,\nu,S) \in (0,1)$ such that

$$\varepsilon = \frac{1}{4S^{2\chi-1}[1+c_2]} \quad\Longrightarrow\quad \frac{c_2\varepsilon}{AS} \le \frac{1}{4S^{2\chi}}. \tag{6.2.12}$$

Using these choices in (6.2.10) we find that

$$\big|\{z \in \mathcal{Q} : M^{**}(1+|Du|)(z) > AS\lambda\}\big| < S^{-2\chi}|\mathcal{Q}|. \tag{6.2.13}$$

At this stage it remains to replace in (6.2.13) the restricted maximal operator $M^{**} = M^{*}_{0,2\mathcal{Q}}$ by the restricted maximal operator $M^{*} = M^{*}_{0,n^2\mathcal{Q}_0}$. Let $\ell = 2\varrho$ be the side-length of $\mathcal{Q}$ and $z \in \mathcal{Q}$ arbitrary. Moreover, let $\widehat{\mathcal{Q}}$ denote an arbitrary parabolic cylinder with side-length $\hat{\ell} = 2\hat{\varrho}$ contained in $n^2\mathcal{Q}_0$ and containing the point z. We distinguish two cases: **In the case $\hat{\ell} \le \frac{1}{2}\ell$** we have $\widehat{\mathcal{Q}} \subset 2\mathcal{Q} \subset n^2\mathcal{Q}_0$ and therefore

$$\fint_{\widehat{\mathcal{Q}}} (1+|Du|)\,dz \le M^{**}(1+|Du|)(z).$$

In **the other case $2\hat{\ell} > \ell$** or equivalently $2\hat{\varrho} > \varrho$, it is possible to enlarge the cylinder $\widehat{\mathcal{Q}}$ to another cylinder $\mathcal{Q}'$ in such that $\widehat{\mathcal{Q}} \subset \mathcal{Q}' \subset n^2\mathcal{Q}_0$, $|\mathcal{Q}'| \le 5^N|\widehat{\mathcal{Q}}|$ and finally $\widetilde{\mathcal{Q}} \subset \mathcal{Q}'$, where $\widetilde{\mathcal{Q}}$ is the predecessor of $\mathcal{Q}$. In particular we have $\tilde{z} \in \mathcal{Q}'$. Therefore, we find

$$\fint_{\widehat{\mathcal{Q}}} (1+|Du|)\,dz \le 5^N \fint_{\mathcal{Q}'} (1+|Du|)\,dz \le 5^N\lambda,$$

where we have also used (6.2.3). Since $\widetilde{\mathcal{Q}}$ is an arbitrary cylinder in $n^2\mathcal{Q}_0$ we have shown

$$M^*(1+|Du|)(z) \le \max\left\{M^{**}(1+|Du|)(z),\ 5^N\lambda\right\} \qquad \forall\, z \in \mathcal{Q}.$$

Combining the preceding inequality with (6.2.13) and the particular choice of A in (6.2.11) leads us to the estimate

$$\left|\{z \in \mathcal{Q} : M^*(1+|Du|)(z) > AS\lambda\}\right| < S^{-2\chi}|\mathcal{Q}|, \tag{6.2.14}$$

which contradicts (6.2.1) and therefore proves the assertion of the Lemma. □

Step 2: Application of Proposition 3.17. Let $\mathcal{Q}_0$ as in Step 1. Then, we define

$$\lambda_0 := 2c_0(n)n^{2N}S^{2\chi} \fint_{n^2\mathcal{Q}_0} (1+|Du|)\,dz, \tag{6.2.15}$$

where c_0 is taken from (3.2.10). Obviously we have that $\lambda_0 > 0$. The strategy of proof is now to apply Lemma 6.18 for the choice $\lambda := (AS)^k\lambda_0$ for $k \in \mathbb{N}_0$. We first show that the hypotheses of Lemma 6.18 are fulfilled for every $k \in \mathbb{N}_0$. Using (3.2.10) and (6.2.15) we infer that

$$\begin{aligned}
&\left|\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > (AS)^k\lambda_0\}\right| \\
&\qquad \le \left|\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > \lambda_0\}\right| \\
&\qquad \le \frac{c_0(n)}{\lambda_0}\int_{\mathcal{Q}_0} (1+|Du|)\,dz < S^{-2\chi}|\mathcal{Q}_0|.
\end{aligned}$$

In the light of Lemma 6.18 we can therefore apply Proposition 3.17 with $\delta := S^{-2\chi}$,

$$\begin{aligned}
X := \{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) &> (AS)^{k+1}\lambda_0 \\
&\text{and} \quad M_1^*(g)(z) \le (AS)^k\varepsilon\lambda_0\}
\end{aligned}$$

and

$$Y := \left\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > (AS)^k\lambda_0\right\}.$$

The application of Proposition 3.17 and the definition of X and Y yield that

$$\begin{aligned}
&\left|\left\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > (AS)^{k+1}\lambda_0\right\}\right| \\
&\qquad \le S^{-2\chi}\left|\left\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > (AS)^k\lambda_0\right\}\right| \\
&\qquad\qquad + \left|\left\{z \in \mathcal{Q}_0 : M_1^*(g)(z) > (AS)^k\varepsilon\lambda_0\right\}\right|
\end{aligned} \tag{6.2.16}$$

holds for every $k \in \mathbb{N}_0$. With

$$\begin{aligned}
\mu_1(H) &:= \left|\left\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > H\right\}\right|, \\
\mu_2(H) &:= \left|\left\{z \in \mathcal{Q}_0 : M_1^*(g)(z) > H\right\}\right|,
\end{aligned}$$

the preceding inequality turns into

$$\mu_1\left((AS)^{k+1}\lambda_0\right) \le S^{-2\chi}\mu_1\left((AS)^k\lambda_0\right) + \mu_2\left((AS)^k\varepsilon\lambda_0\right). \tag{6.2.17}$$

Multiplying (6.2.17) by $(AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}}$ we find

$$\begin{aligned}(AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1\left((AS)^{k+1}\lambda_0\right) &\le A^{\frac{\vartheta\gamma}{\vartheta-\gamma}}S^{\frac{\vartheta\gamma}{\vartheta-\gamma}-2\chi}(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1\left((AS)^k\lambda_0\right)\\ &\quad + \left(AS/\varepsilon\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}}(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}}\left(\varepsilon\lambda_0\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_2\left((AS)^k\varepsilon\lambda_0\right).\end{aligned} \tag{6.2.18}$$

Note that $\gamma < \vartheta$ by (4.2.3) and therefore $\frac{\vartheta\gamma}{\vartheta-\gamma} > 0$. On the other hand, condition $(4.2.3)_1$, i.e. $\gamma \le \frac{2\vartheta}{\vartheta+2}$, is equivalent to require $\frac{\vartheta\gamma}{\vartheta-\gamma} \le 2$, and therefore, since $\chi > 1$, we have

$$d := 2\chi - \frac{\vartheta\gamma}{\vartheta-\gamma} \ge 2(\chi-1) > 0. \tag{6.2.19}$$

We now choose

$$S := \left[4A^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\right]^{\frac{1}{d}}, \tag{6.2.20}$$

where A has been determined in (6.2.11). Note that $y^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \le y^2$, whenever $y \ge 1$, and therefore $S \le \left[4A^2\right]^{\frac{1}{d}} \le \left[4A^2\right]^{\frac{1}{2(\chi-1)}} = [2A]^{\frac{1}{\chi-1}}$. Recalling the dependencies of A and χ, i.e. $A \equiv A(n,L,\nu)$ and $\chi \equiv \chi(n,L,\nu)$ we easily infer that S from (6.2.20) is bounded by a universal constant depending on n, L, ν. On the other hand, we have the estimate

$$AS/\varepsilon = 4\,[1+c_2]AS^{2\chi} \le 2\,[1+c_2](2A)^{1+\frac{2\chi}{\chi-1}} = \frac{1}{2}c_*(n,L,\nu),$$

so that $\left(AS/\varepsilon\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \le (c_*/2)^{\frac{\vartheta\gamma}{\vartheta-\gamma}}$. Using this and (6.2.20) in (6.2.18) we conclude that

$$\begin{aligned}(AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1\left((AS)^{k+1}\lambda_0\right) &\le \frac{1}{4}(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1\left((AS)^k\lambda_0\right)\\ &\quad + \left(\frac{c_*}{2}\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}}(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}}\left(\varepsilon\lambda_0\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_2\left((AS)^k\varepsilon\lambda_0\right)\end{aligned} \tag{6.2.21}$$

holds for every $k \in \mathbb{N}_0$.

Step 3: Parabolic Lorentz spaces estimates on level sets. We take $\tau \in (0,\infty)$ and raise the terms appearing in (6.2.21) to the power $\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}$. This leads us to

$$\begin{aligned}&\left[(AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1\left((AS)^{k+1}\lambda_0\right)\right]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}\\ &\le \max\left\{(1/4)^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}},(1/2)\right\}\left[(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1\left((AS)^k\lambda_0\right)\right]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}\\ &\qquad + c_*^\tau\left[(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}}\left(\varepsilon\lambda_0\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_2\left((AS)^k\varepsilon\lambda_0\right)\right]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}.\end{aligned}$$

We note that here we have also used that $\frac{1}{2} \le \frac{\vartheta-\gamma}{\vartheta\gamma} < 1$ and therefore $2^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}-1} \le 2^\tau$ in order to obtain the constant c_*^τ in the second term. Now, we sum up the preceding inequality for $k = 0,\dots,H$ and finally add the quantity $\lambda_0^\tau|\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}$ to both sides. In this way we obtain:

$$I_1(H) \le \lambda_0^\tau|\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + \max\left\{(1/4)^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}},(1/2)\right\}I_1(H) + c_*^\tau I_2(\infty),$$

where we have defined

$$I_1(H) := \sum_{k=0}^{H+1} \left[(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\left((AS)^k\lambda_0\right)\right]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \quad \text{and}$$

$$I_2(\infty) := \sum_{k=0}^{\infty} \left[(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} (\varepsilon\lambda_0)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2\left((AS)^k\varepsilon\lambda_0\right)\right]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}.$$

We note that $I_2 = I_2(\infty)$ is finite since $g \in L^\infty(\Omega_T)$, but this fact is not needed here. Reabsorbing the second therm appearing in the right-hand side of the preceding inequality on the left and then letting $H \to \infty$ yields

$$I_1(\infty) \le c_1 c_*^\tau \Big[\lambda_0^\tau |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + I_2(\infty)\Big], \tag{6.2.22}$$

where we have abbreviated

$$c_1 := \Big[1 - \max\left\{(1/4)^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}, (1/2)\right\}\Big]^{-1}.$$

Using the fact $\gamma \le \frac{2\vartheta}{\vartheta+2}$ we see that $(1/4)^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \le (1/2)^\tau$ so that the constant c_1 in (6.2.22) can be replaced by the larger quantity $[1-\max\{(1/2)^\tau,(1/2)\}]^{-1}$. This implies in particular that $c_1 c_*^\tau$ can be replaced by a constant of the form c^τ where $c(n,L,\nu,\tau) := c_*[1-\max\{(1/2)^\tau,(1/2)\}]^{-1/\tau}$. We note that c is a decreasing function of τ with $c \to \infty$ when $\tau \searrow 0$ and $c \to c_*$ when $\tau \to \infty$. Therefore, $c \equiv c(\tau)$ stays bounded on any interval $[\tau_0,\infty)$ with $\tau_0 > 0$. We will keep this kind of dependence for the rest of the proof. Now, if $k \ge 0$ and $\lambda \in [(AS)^k\lambda_0, (AS)^{k+1}\lambda_0)$ then

$$\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1(\lambda) \le (AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\big((AS)^k\lambda_0\big), \tag{6.2.23}$$

and similarly when $k \ge 1$ and $\lambda \in [(AS)^{k-1}\varepsilon\lambda_0, (AS)^k\varepsilon\lambda_0)$ we have

$$(AS)^{\frac{(k-1)\vartheta\gamma}{\vartheta-\gamma}} (\varepsilon\lambda_0)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2\big((AS)^k\varepsilon\lambda_0\big) \le \lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2(\lambda). \tag{6.2.24}$$

Using (6.2.23) we find

$$\begin{aligned}
\int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \frac{d\lambda}{\lambda} &= \int_0^{\lambda_0} (\dots)\,\frac{d\lambda}{\lambda} + \sum_{k=0}^\infty \int_{(AS)^k\lambda_0}^{(AS)^{k+1}\lambda_0} (\dots) \frac{d\lambda}{\lambda} \\
&\le \frac{\lambda_0^\tau}{\tau} |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \\
&\qquad + (AS)^\tau \log(AS) \sum_{k=0}^\infty \left[(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\left((AS)^k\lambda_0\right)\right]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \\
&= \frac{\lambda_0^\tau}{\tau} |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + (AS)^\tau \log(AS) I_1(\infty),
\end{aligned}$$

and similarly using (6.2.24) and the fact that we have chosen $\varepsilon \le 1$ we infer that

$$\begin{aligned}
I_2(\infty) &= (\lambda_0\varepsilon)^\tau |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + \sum_{k=1}^\infty \left[(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} (\varepsilon\lambda_0)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2\left((AS)^k\varepsilon\lambda_0\right)\right]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \\
&\le \lambda_0^\tau |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + \frac{(AS)^\tau}{\log(AS)} \sum_{k=1}^\infty \int_{(AS)^{k-1}\varepsilon\lambda_0}^{(AS)^k\varepsilon\lambda_0} \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \frac{d\lambda}{\lambda} \\
&= \lambda_0^\tau |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + \frac{(AS)^\tau}{\log(AS)} \int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \frac{d\lambda}{\lambda}.
\end{aligned}$$

Combining the preceding estimates with (6.2.22) we conclude

$$\int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \frac{d\lambda}{\lambda}$$

$$
\begin{aligned}
&\le \frac{\lambda_0^\tau}{\tau}|\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + c^\tau (AS)^\tau \log(AS)\Big[\lambda_0^\tau|\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + I_2(\infty)\Big]\\
&= \Big[\frac{1}{\tau} + c^\tau (AS)^\tau \log(AS)\Big]\lambda_0^\tau|\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + c^\tau (AS)^\tau \log(AS) I_2(\infty)\\
&\le c_2^\tau \lambda_0^\tau |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + c_2^\tau \int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_2(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}\frac{d\lambda}{\lambda},
\end{aligned}
\tag{6.2.25}
$$

where we have abbreviated $c_2 = \max\big\{\big[\frac{1}{\tau}\,ds + 2c^\tau(AS)^\tau \log(AS)\big]^{\frac{1}{\tau}}, c\,(AS)^2\big\}$. As for the constant c the constant $c_2 = c_2(n, L, \nu, \tau)$ blows up when $\tau \searrow 0$, while c_2 remains bounded when τ is bounded away from zero. Taking into account the definition (3.2.1) and (3.2.15) the preceding inequality turns into

$$
\begin{aligned}
&\|M^*(1+|Du|)\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\tau)(\mathcal{Q}_0)}\\
&\quad = \left(\frac{\vartheta\gamma}{\vartheta-\gamma}\int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}\frac{d\lambda}{\lambda}\right)^{\frac{1}{\tau}}\\
&\quad \le \left[\frac{\vartheta\gamma}{\vartheta-\gamma}c_2^\tau\lambda_0^\tau|\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + \frac{\vartheta\gamma}{\vartheta-\gamma}c_2^\tau\int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_2(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}\frac{d\lambda}{\lambda}\right]^{\frac{1}{\tau}}\\
&\quad \le c(\tau)c_2\Bigg[\left(\frac{\vartheta\gamma}{\vartheta-\gamma}\right)^{\frac{1}{\tau}}\lambda_0|\mathcal{Q}_0|^{\frac{\vartheta-\gamma}{\vartheta\gamma}}\\
&\qquad\qquad + \left(\frac{\vartheta\gamma}{\vartheta-\gamma}\int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_2(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}}\frac{d\lambda}{\lambda}\right)^{\frac{1}{\tau}}\Bigg]\\
&\quad \le c(\tau)c_2\Big[\lambda_0|\mathcal{Q}_0|^{\frac{\vartheta-\gamma}{\vartheta\gamma}} + \big\|M_1^*(g)\big\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\tau)(\mathcal{Q}_0)}\Big],
\end{aligned}
$$

where $c(\tau) = 4^{1/\tau}$. Here we have used in the last line that $\big(\frac{\vartheta\gamma}{\vartheta-\gamma}\big)^{\frac{1}{\tau}} \le 2^{1/\tau}$. With the obvious inequality $|Du(z)| \le M^*(1+|Du|)(z)$ for almost every $z \in \mathcal{Q}_0$ we conclude from the preceding inequality that

$$
\|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\tau)(\mathcal{Q}_0)} \le c\lambda_0|\mathcal{Q}_0|^{\frac{\vartheta-\gamma}{\vartheta\gamma}} + c\,\big\|M_1^*(g)\big\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\tau)(\mathcal{Q}_0)},
\tag{6.2.26}
$$

where $c \equiv c(n, L, \nu, \tau)$ stays bounded as long τ is bounded away from zero, and $c \to \infty$ when $\tau \searrow 0$. For $0 < q < \infty$ the choice $\tau = \frac{\vartheta q}{\vartheta-\gamma}$ in (6.2.26) yields

$$
\|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(\mathcal{Q}_0)} \le c\lambda_0|\mathcal{Q}_0|^{\frac{\vartheta-\gamma}{\vartheta\gamma}} + c\,\big\|M_1^*(g)\big\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(\mathcal{Q}_0)}.
\tag{6.2.27}
$$

Having arrived at this stage we can apply Theorem 6.11 with $\beta = 1$ and $p = \gamma$ (note that $\beta p = \gamma < \vartheta$) passing to the outer parabolic cylinder Q, i.e. $\mathcal{Q}_{\rm inn}(Q) = \mathcal{Q}_0$ and choosing $s = 2$. Note that, if $\mathcal{Q}_0 = C_R \times (-R^2, R^2)$ then $Q = B_{\sqrt{n}R} \times (-nR^2, nR^2)$ and $2Q = B_{2\sqrt{n}R} \times (-4nR^2, 4nR^2) \subset C_{n^2R} \times (-n^4R^2, n^4R^2) = n^2\mathcal{Q}_0$. Furthermore, $|\mathcal{Q}_0| \le 1$ implies $R < 1$, so that $\sqrt{n}R < \sqrt{n}$. Hence, the application of Remark 6.8 at the end of the proof of Theorem 6.11 yields a constant $\max\{1, \sqrt{n}\}^{(N-\vartheta)\frac{\vartheta-\gamma}{\vartheta\gamma}} \le \sqrt{n}^{\frac{n(\vartheta-\gamma)}{\vartheta\gamma}} \le c(n,\gamma)$. Applying Theorem 6.11 with the choice of the parameters described before leads to

$$
\big\|M_1^*(g)\big\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(\mathcal{Q}_0)} \le c(n,\gamma,q)\|g\|_{L^\vartheta(\gamma,q)(n^2\mathcal{Q}_0)}.
\tag{6.2.28}
$$

Combining (6.2.28) and (6.2.15), i.e. the choice of λ_0, with (6.2.27) and noting that $S^{2\chi} \le c(n, L, \nu)$ by the choice in (6.2.20), we finally arrive at

$$
\begin{aligned}
\|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(\mathcal{Q}_0)} &\le c\left(\fint_{n^2\mathcal{Q}_0}(1+|Du|)\,dz\right)|\mathcal{Q}_0|^{\frac{\vartheta-\gamma}{\vartheta\gamma}}\\
&\quad + c\,\|g\|_{L^\vartheta(\gamma,q)(n^2\mathcal{Q}_0)},
\end{aligned}
\tag{6.2.29}
$$

where $c \equiv c(n, L, \nu, \gamma, q)$.

We now show how the previous inequality, i.e. (6.2.29), can be extended to the case $q = \infty$. We proceed as follows: We first go back to (6.2.21) and obtain for $H \in \mathbb{N}$ that

$$\begin{aligned} I_3(H) &:= \sup_{0\le k\le H+1} (AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\left((AS)^k\lambda_0\right) \\ &\le \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} |\mathcal{Q}_0| + \frac{1}{4} I_3(H) + c_*^2 \sup_{k\ge 0} (AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} (\varepsilon\lambda_0)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2\left((AS)^k\varepsilon\lambda_0\right). \end{aligned}$$

Here we have used that $c_*^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \le c_*^2$, since $\frac{\vartheta\gamma}{\vartheta-\gamma} \le 2$ and $c_* \ge 1$. Re-absorbing as usual $\frac{1}{4}I_3(H)$ in the left-hand side, and then letting $H \to \infty$ we deduce

$$\begin{aligned} I_3(\infty) &\le (4/3)\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} |\mathcal{Q}_0| + c \sup_{k\ge 0} (AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} (\varepsilon\lambda_0)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2\left((AS)^k\varepsilon\lambda_0\right) \\ &\le (4/3)\lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} |\mathcal{Q}_0| + c \sup_{\lambda>0} \lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2(\lambda). \end{aligned}$$

Here we have used (6.2.24) in the last line. On the other hand using (6.2.23) we can bound the left-hand side of the preceding inequality from below and obtain

$$\sup_{\lambda>0} \lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1(\lambda) \le \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} |\mathcal{Q}_0| + (AS)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} I_3(\infty).$$

We note that $(AS)^{\frac{\vartheta\gamma}{\vartheta-\gamma}}$ can be bounded by a constant $c \equiv c(n, L, \nu)$. Combining the last two inequalities we have

$$\sup_{\lambda>0} \lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1(\lambda) \le c\, \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} |\mathcal{Q}_0| + c \sup_{\lambda>0} \lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2(\lambda),$$

where $c \equiv c(n, L, \nu)$. Taking into account the definition of the Marcinkiewicz space from (3.2.2) and again the obvious a.e. estimate $|Du(z)| \le M^*(1+|Du|)(z)$ we conclude that

$$\|Du\|_{\mathcal{M}^{\frac{\vartheta\gamma}{\vartheta-\gamma}}(\mathcal{Q}_0)} \le c\, \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} |\mathcal{Q}_0| + c \left\|M_1^*(g)\right\|_{\mathcal{M}^{\frac{\vartheta\gamma}{\vartheta-\gamma}}(\mathcal{Q}_0)}. \tag{6.2.30}$$

Similarly to (6.2.28) we use Theorem 6.11, now with the choice $q = \infty$, in order to get

$$\left\|M_1^*(g)\right\|_{\mathcal{M}^{\frac{\vartheta\gamma}{\vartheta-\gamma}}(\mathcal{Q}_0)} \le c(n, \gamma)\, \|g\|_{\mathcal{M}^{\gamma,\vartheta}(n^2\mathcal{Q}_0)}.$$

Connecting the last two inequalities and recalling the choice of λ_0 from (6.2.15) we infer that (6.2.29) extends to the case $q = \infty$.

PROOF OF REMARK 4.7. Note that (6.2.30) holds also for $\gamma = 1$, when $\vartheta \ge 2$. In this case it is enough to estimate the Marcinkiewicz norm on the right-hand side with the same norm of $I_1(g)$ and then recall the classical result of Adams [11]: $I_1 : L^{1,\vartheta} \to \mathcal{M}^{\frac{\vartheta}{\vartheta-1}}$. See also Step 6. □

Step 4: Intermediate parabolic Morrey-space regularity of Du.

PROPOSITION 6.19. *Let u be a weak solution to (6.1.1) where the assumptions (4.2.1) are in force and let $g \in L^\vartheta(\gamma, q)(\Omega_T)$ with $2 < 2\gamma \le \vartheta \le N$. Then, for every pair of concentric parabolic cylinders $Q_\sigma \Subset Q_\varrho \subset \Omega_T$ there holds*

$$\begin{aligned} \big\|1 + |Du|\big\|_{L^{1,\frac{\vartheta-\gamma}{\gamma}}(Q_\sigma)} &\le c\,(\varrho-\sigma)^{\frac{\vartheta-\gamma}{\gamma}-N} \big\|1 + |Du|\big\|_{L^1(Q_\varrho)} \\ &\quad + c\, \|g\|_{L^\vartheta(\gamma,q)(Q_\varrho)}, \end{aligned} \tag{6.2.31}$$

where $c \equiv c(n, L, \nu, \gamma, q)$.

PROOF. Let $Q_\sigma \Subset Q_\varrho \subset \Omega_T$ be two fixed concentric cylinders and let z_0 be a point in Q_σ. Moreover let $Q_R(z_0)$ be a parabolic cylinder with $0 < R \le d_{\mathcal{P}}(z_0, \partial Q_\varrho)$, i.e. $Q_R(z_0) \subset Q_\varrho$. Moreover, let $v \in C^0([t_0 - R^2, t_0 + R^2]; L^2(B_R(x_0)) \cap L^2(t_0 - R^2, t_0+R^2; W^{1,2}(B_R(x_0))$ be the unique solution to the Cauchy-Dirichlet problem(6.1.2) in $Q_R(z_0)$. Then, using (6.1.8) for the choice $q = 1$ we infer that for any $0 < r \le R$ we have

$$\int_{Q_r(z_0)} (1+|Du|)\,dz \le \int_{Q_r(z_0)} (1+|Dv|)\,dz + \int_{Q_R(z_0)} |Du-Dv|\,dz$$
$$\le c\left(\frac{r}{R}\right)^{N-1+\alpha} \int_{Q_R(z_0)} (1+|Dv|)\,dz + \int_{Q_R(z_0)} |Du-Dv|\,dz,$$

where $c \equiv c(n, L, \nu)$ and $\alpha \equiv \alpha(n, L, \nu) \in (0, 1/2]$. Now, the second integral appearing on the right-hand side of the preceding inequality is estimated by (6.1.4) from Lemma 6.13, i.e. we have

$$\int_{Q_R(z_0)} |Du-Dv|\,dz \le c\,R^{N-\frac{\vartheta-\gamma}{\gamma}} \|g\|_{L^\vartheta(\gamma,q)(Q_R(z_0))},$$

where $c \equiv c(n, \nu, \gamma)$. Inserting this above yields

$$\int_{Q_r(z_0)} (1+|Du|)\,dz$$
$$\le c\left(\frac{r}{R}\right)^{N-1+\alpha} \int_{Q_R(z_0)} (1+|Du|)\,dz + c\,\|g\|_{L^\vartheta(\gamma,q)(Q_\varrho)} R^{N-\frac{\vartheta-\gamma}{\gamma}},$$

for any choice of $0 < r \le R$. We remark that $2\gamma \le \vartheta$ implies $\frac{\vartheta-\gamma}{\gamma} \ge 1$ and therefore $N - \frac{\vartheta-\gamma}{\gamma} \le N-1 < N-1+\alpha$. This allows us to apply the iteration Lemma 3.12 with

$$\varphi(r) := \int_{Q_r(z_0)} (1+|Du|)\,dz, \qquad A := c, \qquad \mathcal{B} := c\,\|g\|_{L^\vartheta(\gamma,q)(Q_\varrho)},$$

$R_0 := d_{\mathcal{P}}(z_0, \partial Q_\varrho)$ and

$$\delta_0 := N-1+\alpha > N - \frac{\vartheta-\gamma}{\gamma} =: \delta_1.$$

Applying (3.3.1) and taking the choice $R = R_0 > \varrho - \sigma$ we obtain in particular

$$\int_{Q_r(z_0)} (1+|Du|)\,dz \tag{6.2.32}$$
$$\le c\left[(\varrho-\sigma)^{\frac{\vartheta-\gamma}{\gamma}-N} \int_{Q_\varrho} (1+|Du|)\,dz + \|g\|_{L^\vartheta(\gamma,q)(Q_\varrho)}\right] r^{N-\frac{\vartheta-\gamma}{\gamma}},$$

whenever $Q_r(z_0) \subset Q_\sigma$, and with a constant $c \equiv c(n, L, \nu, \gamma)$. Here we have used the dependence $\alpha \equiv \alpha(n, L, \nu)$. Inequality (6.2.32) immediately implies (6.2.31), and this completes the proof of the Lemma. □

REMARK 6.20 (Extensions). Proposition 6.19 holds under the assumption $2\gamma \le \vartheta$ and this is implied by the assumptions of Theorem 4.8. In fact, $\gamma \le \frac{2\vartheta}{\vartheta+2}$ and $2 < \vartheta \le N$ imply that $2\gamma < \vartheta$. Therefore, the assumption $2\gamma \le \vartheta$ can be replaced by the weaker assumption $\gamma(2 - \frac{\alpha}{2}) < \vartheta$, where $\alpha > 0$ is the exponent from (6.1.8). This condition serves for $\frac{\vartheta-\gamma}{\gamma} > 1 - \frac{\alpha}{2}$ and therefore also for $\delta_1 = N - \frac{\vartheta-\gamma}{\gamma} < N - 1 + \frac{\alpha}{2} < N-1+\alpha = \delta_0$, which was needed in the proof of Proposition 6.19. When $\gamma = 1$ the proof of Proposition 6.19 still works, provided $2 \le \vartheta \le N$ and that we use the comparison estimate (6.1.5) instead of (6.1.4). Note that in this case $N-1+\alpha > N-(\vartheta-1)$. The final outcome is instead of (6.2.32) the following estimate:

$$\int_{Q_r} (1+|Du|)\,dz \tag{6.2.33}$$
$$\leq c\Big[(\varrho-\sigma)^{\vartheta-1-N}\int_{Q_\varrho}(1+|Du|)\,dz + \|g\|_{L^{1,\vartheta}(Q_\varrho)}\Big] r^{N-(\vartheta-1)},$$

where now $c \equiv c(n, L, \nu)$.

With respect to u (instead of Du) we have a statement similar to (6.2.31), assuming

$$1 < \gamma < \frac{\vartheta}{2} \qquad \text{and} \qquad 2 < \vartheta \leq N$$

instead of (4.2.3). This can be seen as follows: keeping in mind the notation introduced at the beginning of the proof of Proposition 6.19 we obtain using (6.1.9) instead of (6.1.8) and again (6.1.4) the following decay estimate:

$$\int_{Q_r(z_0)} (r+|u|)\,dz$$
$$\leq c\left(\frac{r}{R}\right)^N \int_{Q_R(z_0)} (R+|u|)\,dz + c\,\|g\|_{L^{\vartheta}(\gamma,q)(Q_\varrho)} R^{N-\frac{\vartheta-2\gamma}{\gamma}},$$

for any $0 < r \leq R$. We note that $2\gamma < \vartheta$ implies that $\frac{\vartheta-2\gamma}{\gamma} > 0$. Therefore we can apply Lemma 3.12 to the quantity $\varphi(r) := \int_{Q_r(z_0)}(r+|u|)\,dz$. The final outcome, that follows along the lines of the proof of (6.2.32), is

$$\|u\|_{L^{1,\frac{\vartheta-2\gamma}{\gamma}}(Q_\sigma)} \leq c\,(\varrho-\sigma)^{\frac{\vartheta-2\gamma}{\gamma}-N} \|\varrho+|u|\|_{L^1(Q_\varrho)} + c\,\|g\|_{L^{\vartheta}(\gamma,q)(Q_\varrho)}. \tag{6.2.34}$$

We note that $c \to \infty$ when $\gamma \nearrow \vartheta/2$.

Step 5: Full Morrey space regularity of Du. In this section we prove (4.2.7) for the approximating solutions $u = u_k$. We consider a parabolic cylinder $Q_\varrho \subset \Omega_T$ and scale the problem as in Remark 5.5 to $Q_1 = Q(0,1)$, switching from u, g, a to $\tilde{u}, \tilde{g}, \tilde{a}$. Applying (6.2.29) with $\tilde{u}, \tilde{g}$ with $\mathcal{Q}_0 := \mathcal{Q}_{1/n^4}$ (note, with this choice of $\mathcal{Q}_0$ we have $n^2\mathcal{Q}_0 = \mathcal{Q}_{1/n^2} \subset Q_{1/n}$) we conclude that

$$\begin{aligned}
\|D\tilde{u}\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_{1/n^4})} &\leq c\,\big\|1+|D\tilde{u}|\big\|_{L^1(\mathcal{Q}_{1/n^2})} + c\,\|\tilde{g}\|_{L^{\vartheta}(\gamma,q)(\mathcal{Q}_{1/n^2})}\\
&\leq c\,\big\|1+|D\tilde{u}|\big\|_{L^1(Q_{1/n})} + c\,\|\tilde{g}\|_{L^{\vartheta}(\gamma,q)(Q_{1/n})}\\
&\leq c\,\big\|1+|D\tilde{u}|\big\|_{L^{1,\frac{\vartheta-\gamma}{\gamma}}(Q_{1/n})} + c\,\|\tilde{g}\|_{L^{\vartheta}(\gamma,q)(Q_{1/n})}\\
&\leq c\,\big\|1+|D\tilde{u}|\big\|_{L^{1,\frac{\vartheta-\gamma}{\gamma}}(Q_{9/10})} + c\,\|\tilde{g}\|_{L^{\vartheta}(\gamma,q)(Q_1)}.
\end{aligned}$$

At this stage we scale back to Q_ϱ and find

$$\|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_{\varrho/n^4})} \leq c(n,L,\nu,\gamma,q)\,\Psi(Q_\varrho)\varrho^{(N-\vartheta)\frac{\vartheta-\gamma}{\vartheta\gamma}}, \tag{6.2.35}$$

where we have defined

$$\Psi(Q_\varrho) := \big\|1+|Du|\big\|_{L^{1,\frac{\vartheta-\gamma}{\gamma}}(Q_{9\varrho/10})} + \|g\|_{L^{\vartheta}(\gamma,q)(Q_\varrho)}$$

for every choice of $Q_\varrho \subset \Omega_T$. For a general parabolic cylinder $Q_{2R} \subset \Omega_T$ we conclude the proof by means of a covering argument. Let $Q_\varrho \subset Q_R$ be a parabolic cylinder not necessary concentric to Q_{2R}. If $Q_{n^4\varrho} \subset Q_R$ then applying (6.2.35) we have

$$\varrho^{(\vartheta-N)\frac{\vartheta-\gamma}{\vartheta\gamma}} \|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_\varrho)} \le c\,\Psi(Q_{n^4\varrho}) \le c\,\Psi(Q_R).$$

On the other hand, if $Q_{n^4\varrho} \not\subset Q_R$ we cover Q_ϱ with a finite number of parabolic cylinders Q_i of radius $\varrho/(8n^4)$ and center in Q_ϱ. Note, that the total number of these cylinders is bounded by a constant $m(n)$ independently on the radius ϱ. Moreover, for each i we have $n^4 Q_i \subset Q_{3R/2}$. Therefore, (3.2.16) and (6.2.35) imply

$$\begin{aligned}
&\varrho^{(\vartheta-N)\frac{\vartheta-\gamma}{\vartheta\gamma}} \|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_\varrho)} \\
&\qquad \le c(n,q) \sum_i \varrho^{(\vartheta-N)\frac{\vartheta-\gamma}{\vartheta\gamma}} \|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_i)} \\
&\qquad \le c(n,L,\nu,\gamma,q) \sum_i \Psi(n^4 Q_i) \le c\, m \Psi(Q_{3R/2}).
\end{aligned}$$

Together the last two inequalities (recall definition (3.2.3)) imply that

$$\begin{aligned}
\|Du\|_{L^\vartheta(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_R)} &\le c(n,L,\nu,\gamma,q)\,\Psi(Q_{3R/2}) \\
&= c\Big[\|1+|Du|\|_{L^{1,\frac{\vartheta-\gamma}{\gamma}}(Q_{27R/20})} + \|g\|_{L^\vartheta(\gamma,q)(Q_{3R/2})}\Big].
\end{aligned} \tag{6.2.36}$$

It remains to estimate the first term on the right-hand side of (6.2.36). For this we proceed by applying (6.2.31), i.e. Proposition 6.19 (note that the conditions imposed in (4.2.3), i.e. $1 < \gamma \le \frac{2\vartheta}{\vartheta+2}$ and $2 < \vartheta \le N$, yield $1 < \gamma < \frac{\vartheta}{2}$ so that the hypothesis $2\gamma \le \vartheta$ from Proposition 6.19 is fulfilled) with $\sigma := 27R/20$ and $\varrho := R$ and conclude that

$$\big\|1+|Du|\big\|_{L^{1,\frac{\vartheta-\gamma}{\gamma}}(Q_{27R/20})} \le c\,R^{\frac{\vartheta-\gamma}{\gamma}-N}\big\|1+|Du|\big\|_{L^1(Q_{2R})} + c\|g\|_{L^\vartheta(\gamma,q)(Q_{2R})},$$

and inserting this into (6.2.36) yields

$$\|Du\|_{L^\vartheta(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_R)} \le c\,R^{\frac{\vartheta-\gamma}{\gamma}-N}\big\|1+|Du|\big\|_{L^1(Q_{2R})} + c\|g\|_{L^\vartheta(\gamma,q)(Q_{2R})},$$

where $c \equiv c(n,L,\nu,\gamma,q)$. Recall that we are applying the preceding estimate to the approximating solutions and data $u \equiv u_k$, $g \equiv g_k \in L^\infty(\Omega_T)$, see (6.1.1) and beginning of Chapter 5, we see that we have established

$$\begin{aligned}
&\|Du_k\|_{L^\vartheta(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_R)} \\
&\qquad \le c\,R^{\frac{\vartheta-\gamma}{\gamma}-N}\big\|1+|Du_k|\big\|_{L^1(Q_{2R})} + c\,\|g_k\|_{L^\vartheta(\gamma,q)(Q_{2R})},
\end{aligned}$$

with a constant $c \equiv c(n,L,\nu,\gamma,q)$ independent of k. Since the right-hand side $g_k \in L^\infty(\Omega_T)$ of the approximating problems is constructed in such a way that it satisfies $|g_k| \le |g|$ we have $\|g_k\|_{L^\vartheta(\gamma,q)(Q_{2R})} \le \|g\|_{L^\vartheta(\gamma,q)(Q_{2R})}$ – we can indeed truncate

$$g_k(z) := \max\{-k, \min\{g(z),k\}\} \qquad k \in \mathbb{N}.$$

Therefore the preceding estimate can be replaced by

$$\begin{aligned}
&\|Du_k\|_{L^\vartheta(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(Q_R)} \\
&\qquad \le c\,R^{\frac{\vartheta-\gamma}{\gamma}-N}\big\|1+|Du_k|\big\|_{L^1(Q_{2R})} + c\,\|g\|_{L^\vartheta(\gamma,q)(Q_{2R})},
\end{aligned} \tag{6.2.37}$$

and this is exactly the estimate for the approximating solutions we were looking for.

Step 6: Approximation and conclusion. The proof of (4.2.7) and therefore the one of Theorem 4.8 follows by the use of the lower semi-continuity of the Lorentz-Morrey-norm with respect to a.e. convergence. In fact, the approximating solutions u_k converge as $k \to \infty$ to the solution u in $L^1(-T,0;W_0^{1,1}(\Omega))$ and a.e. on Ω_T (see Section 2.3). Therefore, we can pass to the limit $k \to \infty$ in (6.2.37) using the lower semi-continuity of the Lorentz-Morrey-norms from Section 3.2. This finishes the proof of Theorem 4.8 and therefore also of Theorem 4.5.

PROOF OF THEOREM 4.11. We first recall that $\gamma = \frac{2\vartheta}{\vartheta+2}$ is equivalent to $\frac{\vartheta\gamma}{\vartheta-\gamma} = 2$. Since $\vartheta \geq 2$ we have $\frac{2\vartheta}{\vartheta+2} \leq \frac{\vartheta}{2} < \frac{\vartheta}{2-\alpha/2}$. Therefore the assumption $\frac{2\vartheta}{\vartheta+2} < \gamma$ yields the existence of γ_0 such that

$$\frac{2\vartheta}{\vartheta+2} < \gamma_0 \leq \min\Big\{\gamma, \frac{\vartheta}{2-\alpha/2}\Big\} \quad \text{and} \quad d := 2\chi - \frac{\vartheta\gamma_0}{\vartheta-\gamma_0} \geq \chi - 1, \tag{6.2.38}$$

where $\chi \equiv \chi(n,L,\nu) > 1$ is the higher integrability exponent from Theorem 6.15 and $\alpha \equiv \alpha(n,L,\nu) \in (0,\frac{1}{2}]$ is the Hölder exponent from the same Theorem. Using Hölder's inequality we easily obtain the following embedding for parabolic Morrey spaces:

$$\|g\|_{L^{\gamma_0,\vartheta}(Q_{2R})} \leq c(n)(2R)^{\frac{\vartheta(\gamma-\gamma_0)}{\gamma\gamma_0}} \|g\|_{L^{\gamma,\vartheta}(Q_{2R})} \tag{6.2.39}$$

for any parabolic cylinder $Q_R \subset \Omega_T$. With these preliminaries we proceed along the lines of the proof of Theorem 4.8 taking into account the following changes: We replace γ by γ_0 and choose $q = \gamma_0$. Then everything can be carried out, since (6.2.19) holds with d defined in (6.2.38). Moreover, from the definition of γ_0 we see that $\gamma_0(2-\frac{\alpha}{2}) \leq \vartheta$, so that (6.2.33) is applicable with γ_0 instead of γ (see Remark 6.20). On the other hand we can also apply Theorem 6.11 in this setting with $p = \gamma_0$ and $\beta = 1$ as in (6.2.28), since $\beta p = \gamma_0 \leq \frac{\vartheta}{2-\alpha/2} < \vartheta$ by (6.2.38). Having arrived at this stage we let

$$h := \frac{\vartheta\gamma_0}{\vartheta-\gamma_0},$$

and note that by (6.2.38) we have $h > 2$. Then (4.2.10) follows from (4.2.6) specialized to $\gamma = q = \gamma_0$, and the quantitative estimate (4.2.11) follows from (4.2.7) and (6.2.39) for radii $R \leq 1$ as follows:

$$\begin{aligned}
\|Du\|_{L^{h,\vartheta}(Q_R)} &\leq c\, R^{\frac{\vartheta-\gamma_0}{\gamma_0}-N} \big\|1+|Du|\big\|_{L^1(Q_{2R})} + c\|g\|_{L^{\gamma_0,\vartheta}(Q_{2R})} \\
&\leq c\, R^{\frac{\vartheta-\gamma_0}{\gamma_0}-N} \big\|1+|Du|\big\|_{L^1(Q_{2R})} + c\, R^{\frac{\vartheta(\gamma-\gamma_0)}{\gamma\gamma_0}} \|g\|_{L^{\gamma,\vartheta}(Q_{2R})} \\
&\leq c\, R^{\frac{\vartheta}{h}-N} \big\|1+|Du|\big\|_{L^1(Q_{2R})} + c\,\|g\|_{L^{\gamma,\vartheta}(Q_{2R})}.
\end{aligned}$$

This completes the proof of Theorem 4.11. □

Borderline estimates. Here we consider the cases that $g \in L\log L(\Omega_T)$, resp. $g \in L\log L^\vartheta(\Omega_T)$. We start with the case $g \in L^{1,\vartheta}(\Omega_T) \cap L\log L(\Omega_T)$ and the following

PROOF OF THEOREM 4.6. We proceed as in the proof of Theorem 4.8 taking $\gamma = 1$; we note that the proof works with this choice up to (6.2.25), i.e.

$$\int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_1(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \frac{d\lambda}{\lambda} \leq c_2^\tau \lambda_0^\tau |\mathcal{Q}_0|^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} + c_2^\tau \int_0^\infty \Big[\lambda^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\mu_2(\lambda)\Big]^{\frac{\tau(\vartheta-\gamma)}{\vartheta\gamma}} \frac{d\lambda}{\lambda}.$$

Taking $\tau = \frac{\vartheta}{\vartheta-1}$ and recalling that $\gamma = 1$ the preceding inequality turns into the following analog of (6.2.26):

$$\int_{\mathcal{Q}_0} |Du|^{\frac{\vartheta}{\vartheta-1}}\,dz \le \big(c_2\lambda_0\big)^{\frac{\vartheta}{\vartheta-1}}|\mathcal{Q}_0| + c_2^{\frac{\vartheta}{\vartheta-1}} \int_{\mathcal{Q}_0} |M_1^*(g)|^{\frac{\vartheta}{\vartheta-1}}\,dz.$$

In order to bound the integral appearing on the right-hand side of the preceding inequality we argue as in the proof of Theorem 4.8 (the paragraph before (6.2.28)) passing to the outer parabolic cylinder Q and then applying Theorem 6.10 for the choice $\beta = 1$ and $s = 2$ instead of Theorem 6.11. Proceeding in this way we have

$$\begin{aligned}
\|M_1^*(g)\|^{\frac{\vartheta}{\vartheta-1}}_{L^{\frac{\vartheta}{\vartheta-1}}(\mathcal{Q}_0)} &\le c^{\frac{\vartheta}{\vartheta-1}}\,|Q|\,\|g\|^{\frac{1}{\vartheta-1}}_{L^{1,\vartheta}(2Q)}\|g\|_{L\log L(Q)}\\
&\le c^{\frac{\vartheta}{\vartheta-1}}\,|n^2\mathcal{Q}_0|\,\|g\|^{\frac{1}{\vartheta-1}}_{L^{1,\vartheta}(n^2\mathcal{Q}_0)}\|g\|_{L\log L(n^2\mathcal{Q}_0)}\\
&\le c^{\frac{\vartheta}{\vartheta-1}}\,|\mathcal{Q}_0|\,\|g\|^{\frac{1}{\vartheta-1}}_{L^{1,\vartheta}(n^2\mathcal{Q}_0)}\|g\|_{L\log L(n^2\mathcal{Q}_0)},
\end{aligned}$$

where $c \equiv c(n, L, \nu) \ge 1$. Inserting this in the second last inequality and recalling the definition of λ_0 from (6.2.15) we find

$$\begin{aligned}
&\bigg(\fint_{\mathcal{Q}_0} |Du|^{\frac{\vartheta}{\vartheta-1}}\,dz\bigg)^{\frac{\vartheta-1}{\vartheta}}\\
&\qquad\le \Big[\big(c_2\lambda_0\big)^{\frac{\vartheta}{\vartheta-1}} + (c_2c)^{\frac{\vartheta}{\vartheta-1}}\,\|g\|^{\frac{1}{\vartheta-1}}_{L^{1,\vartheta}(n^2\mathcal{Q}_0)}\|g\|_{L\log L(n^2\mathcal{Q}_0)}\Big]^{\frac{\vartheta-1}{\vartheta}}\\
&\qquad\le c\Big[\lambda_0 + \|g\|^{\frac{1}{\vartheta}}_{L^{1,\vartheta}(n^2\mathcal{Q}_0)}\|g\|^{\frac{\vartheta-1}{\vartheta}}_{L\log L(n^2\mathcal{Q}_0)}\Big]\\
&\qquad\le c\fint_{n^2\mathcal{Q}_0}(1+|Du|)\,dz + c\,\|g\|^{\frac{1}{\vartheta}}_{L^{1,\vartheta}(n^2\mathcal{Q}_0)}\|g\|^{\frac{\vartheta-1}{\vartheta}}_{L\log L(n^2\mathcal{Q}_0)},
\end{aligned}\tag{6.2.40}$$

with a constant $c \equiv c(n, L, \nu)$. Apart from the fact that the preceding estimate (6.2.40) holds for the approximating solutions $u = u_k$ on the concentric parabolic cylinders $\mathcal{Q}_0$, $n^2\mathcal{Q}_0$, having in mind (3.2.5), it has exactly the structure of (4.2.5) from Theorem 4.6. Therefore, all assertions of Theorem 4.6 follow by a standard covering argument combined with the usual approximation argument. □

PROOF OF THEOREM 4.10. Since $g \in L\log L^{\vartheta}(\Omega_T)$ we have $g \in L^{1,\vartheta}(\Omega_T) \cap L\log L(\Omega_T)$. Therefore, the arguments from the proof of Theorem 4.6 apply and we initially end up with (6.2.40) from above. At this stage we go on using the strategy from the proof of Theorem 4.8, Step 5, and scale everything back to Q_1. Using the thereby introduced notation, in particular passing to inner and outer parabolic cylinders, we obtain for the re-scaled function $\tilde{u}$ the following estimate:

$$\begin{aligned}
&\|D\tilde{u}\|_{L^{\frac{\vartheta}{\vartheta-1}}(Q_{1/n^4})}\\
&\qquad\le c\,\big\|1+|D\tilde{u}|\big\|_{L^1(\mathcal{Q}_{1/n^2})} + c\,\|\tilde{g}\|^{\frac{1}{\vartheta}}_{L^{1,\vartheta}(\mathcal{Q}_{1/n^2})}\|\tilde{g}\|^{\frac{\vartheta-1}{\vartheta}}_{L\log L(\mathcal{Q}_{1/n^2})}\\
&\qquad\le c\,\big\|1+|D\tilde{u}|\big\|_{L^1(Q_{9/10})} + c\,\|\tilde{g}\|^{\frac{1}{\vartheta}}_{L^{1,\vartheta}(Q_1)}\|\tilde{g}\|^{\frac{\vartheta-1}{\vartheta}}_{L\log L(Q_1)}\\
&\qquad\le c\,\big\|1+|D\tilde{u}|\big\|_{L^{1,\vartheta-1}(Q_{9/10})} + c\,\|\tilde{g}\|^{\frac{1}{\vartheta}}_{L^{1,\vartheta}(Q_1)}\|\tilde{g}\|^{\frac{\vartheta-1}{\vartheta}}_{L\log L(Q_1)}\\
&\qquad\le c\,\big\|1+|D\tilde{u}|\big\|_{L^{1,\vartheta-1}(Q_{9/10})} + c\,\|\tilde{g}\|_{L\log L^{\vartheta}(Q_1)},
\end{aligned}$$

where $c \equiv c(n, L, \nu)$. Here we have used the fact that $\|\tilde{g}\|_{L^{1,\vartheta}(Q_1)} \lesssim \|\tilde{g}\|_{L\log L^{\vartheta}(Q_1)}$ in the last line. Scaling back to Q_ϱ we have

$$\|Du\|_{L^{\frac{\vartheta}{\vartheta-1}}(Q_{\varrho/n^4})} \le c(n, L, \nu)\,\Psi(Q_\varrho)\varrho^{(N-\vartheta)\frac{\vartheta-1}{\vartheta}},$$

where this time we have defined

$$\Psi(Q_\varrho) := \big\|1+|Du|\big\|_{L^{1,\vartheta-1}(Q_{9\varrho/10})} + \|g\|_{L\log L^\vartheta(Q_\varrho)}.$$

Having arrived at this stage we can use the covering argument from the proof of Theorem 4.8, Step 5; more precisely, the argument leading us to (6.2.36) now yields

$$\begin{aligned}\|Du\|_{L^{\frac{\vartheta}{\vartheta-1},\vartheta}(Q_R)} &\le c(n,L,\nu)\,\Psi(Q_{3R/2})\\ &= c\Big[\big\|1+|Du|\big\|_{L^{1,\vartheta-1}(Q_{27R/20})} + \|g\|_{L\log L^\vartheta(Q_R)}\Big],\end{aligned} \tag{6.2.41}$$

whenever $Q_R \subset \Omega_T$ is a parabolic cylinder. Now, as observed in Remark 6.20 Proposition 6.19 works also when $\gamma = 1$ and $g \in L^{1,\vartheta}(\Omega_T)$. Therefore, we apply (6.2.33) with $\sigma := 27R/40$ and $\varrho := R$ in order to bound $\|1+|Du|\|_{L^{1,\vartheta-1}(Q_{27R/20})}$. This leads us to the estimate

$$\begin{aligned}\|1+|Du|\|_{L^{1,\vartheta-1}(Q_{27R/20})} &\le c\Big[R^{\vartheta-1-N}\big\|1+|Du|\big\|_{L^1(Q_{2R})} + \|g\|_{L^{1,\vartheta}(Q_{2R})}\Big]\\ &\le c\Big[R^{\vartheta-1-N}\big\|1+|Du|\big\|_{L^1(Q_{2R})} + \|g\|_{L\log L^\vartheta(Q_{2R})}\Big],\end{aligned}$$

where we have used once again the trivial bound $\|\tilde g\|_{L^{1,\vartheta}(Q_1)} \lesssim \|\tilde g\|_{L\log L^\vartheta(Q_1)}$ in the second line. Using the preceding inequality in (6.2.41) we finally arrive at

$$\|Du\|_{L^{\frac{\vartheta}{\vartheta-1},\vartheta}(Q_R)} \le c\Big[R^{\vartheta-1-N}\big\|1+|Du|\big\|_{L^1(Q_{2R})} + \|g\|_{L\log L^\vartheta(Q_{2R})}\Big], \tag{6.2.42}$$

with a constant $c \equiv c(n,L,\nu)$. Note that in the preceding inequality we have $u \equiv u_k$, where u_k are the approximating solutions with right-hand side $g_k \in L^\infty(\Omega_T)$ satisfying $|g_k| \le |g|$. From the definition of the $L\log L^\vartheta$-norm we easily have that $\|g_k\|_{L\log L^\vartheta(\Omega_T)} \le \|g\|_{L\log L^\vartheta(\Omega_T)}$, so that (6.2.42) turns into

$$\|Du_k\|_{L^{\frac{\vartheta}{\vartheta-1},\vartheta}(Q_R)} \le c\Big[R^{\vartheta-1-N}\big\|1+|Du_k|\big\|_{L^1(Q_{2R})} + \|g\|_{L\log L^\vartheta(Q_{2R})}\Big],$$

where again $c \equiv c(n,L,\nu)$. This is the desired estimate for the approximating solutions we were looking for and the final result follows again by passing to the limit $k \to \infty$ in the right-hand side and the lower-semicontinuity on the left-hand side. □

PROOF OF THEOREM 4.9. Once again we refer to the proof of Theorem 4.8. We start with (6.2.26) with the choices $\tau = q$ and $\vartheta = N$ and obtain

$$\|Du\|_{L(\frac{N\gamma}{N-\gamma},q)(\mathcal{Q}_0)} \le c\lambda_0|\mathcal{Q}_0|^{\frac{N-\gamma}{N\gamma}} + c\big\|M_1^*(g)\big\|_{L(\frac{N\gamma}{N-\gamma},q)(\mathcal{Q}_0)}, \tag{6.2.43}$$

with λ_0 from (6.2.15). The second term appearing on the right-hand side of (6.2.43) is treated via Theorem 3.5 (again switching to outer and inner cylinders)

$$\big\|M_1^*(g)\big\|_{L(\frac{N\gamma}{N-\beta\gamma},q)(\mathcal{Q}_0)} \le c\,\|g\|_{L(\gamma,q)(n^2\mathcal{Q}_0)}.$$

Combining this with (6.2.43) and recalling the definition (6.2.15) of λ_0 yields the following analog of (6.2.29):

$$\|Du\|_{L(\frac{N\gamma}{N-\gamma},q)(\mathcal{Q}_0)} \le c\bigg(\fint_{n^2\mathcal{Q}_0}(1+|Du|)\,dz\bigg)|\mathcal{Q}_0|^{\frac{N-\gamma}{N\gamma}} + c\|g\|_{L(\gamma,q)(n^2\mathcal{Q}_0)}.$$

Modulo a standard covering argument and the additivity of quasi-norms from Remark 3.2 the preceding inequality is essentially equivalent to (4.2.8). The conclusion of the Theorem then follows by approximation. □

6.3. Integrability of u and more regular coefficients

Parabolic equations with more regular coefficients. In this section we consider parabolic equations where the vector field a satisfies either the structure assumptions (4.2.15) to (4.2.17) – the VMO-case – or (4.2.14) – the case of a continuous vector-field. In these cases we can weaken the assumption (4.2.3). As we saw in Theorem 4.15 see below, we can assume that

$$1 < \gamma < \vartheta \le N$$

holds. The reason for this comes from the fact that the corresponding solutions to homogeneous Cauchy-Dirichlet problems satisfy reverse Hölder-type inequalities for arbitrarily large integrability exponents; see Theorem 6.16.

Since the ideas in the proof of Theorem 4.15 are very close to the ones of Theorem 4.8 we confine ourselves to sketch the necessary modifications. We deal with the approximating solutions $u \equiv u_k$ where $g \equiv g_k \in L^\infty(\Omega_T)$. Keeping in mind the notation introduced in the proof of Theorem 4.8 we must replace Lemma 6.18 by

LEMMA 6.21. *Let $u(= u_k) \in C^0([-T,0]; L^2(\Omega)) \cap L^2(-T,0; W_0^{1,2}(\Omega))$ be a weak solution to (6.1.1) where the assumptions (4.2.1) are in force and $g \in L^\infty(\Omega_T)$. Then, for every choice of $\chi_0, S > 1$ there exist constants $\varepsilon \in (0,1)$ and $A > 1$ depending on $n, L, \nu, \omega(\cdot), S$ and χ_0 such that if $\lambda > 1$ and $\mathcal{Q}$ a dyadic sub-cylinder of $\mathcal{Q}_0$ such that if*

$$\big|\mathcal{Q} \cap \big\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > AS\lambda \ \text{ and } \ M_1^*(g) \le \varepsilon\lambda\big\}\big| > \frac{|\mathcal{Q}|}{S^{2\chi_0}}, \tag{6.3.1}$$

then the predecessor $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ satisfies

$$\tilde{\mathcal{Q}} \subset \big\{z \in \mathcal{Q}_0 : M^*(1+|Du|)(z) > \lambda\big\}.$$

PROOF. Due to the fact that we are dealing with parabolic equations with more regular vector-fields we can use the better higher integrability result from Theorem 6.16 instead of the ones from Theorem 6.15. Again we shall prove the assertion by a contradiction argument. We proceed as in the proof of Lemma 6.18 until (6.2.7). Having arrived at this stage we observe that the hypothesis of Theorem 6.16 are fulfilled for the solution v of the homogeneous Cauchy-Dirichlet problem (6.2.5) on Q. Therefore, we have the local higher integrability (6.1.13) of Dv on $2\mathcal{Q} \subset 3\mathcal{Q} \subset Q$. This means that for any given $\chi_0 > 1$ there exist a constant $c \equiv c(n, L, \nu, \omega(\cdot), \chi_0)$ such that the estimate

$$\left(\fint_{2\mathcal{Q}} (1+|Dv|)^{2\chi_0}\, dz\right)^{\frac{1}{2\chi_0}} \le c \fint_{3\mathcal{Q}} (1+|Dv|)\, dz,$$

holds. Exactly as in the proof of Lemma 6.18 this leads us to

$$\fint_{2\mathcal{Q}} (1+|Dv|)^{2\chi_0}\, dz \le c(n, L, \nu, \omega(\cdot), \chi_0)\, \lambda^{2\chi_0}.$$

We proceed further using again the restricted maximal operator $M^{**} := M^*_{0,2\mathcal{Q}}$ on $2\mathcal{Q}$ and obtain the following analog of (6.2.10):

$$\big|\{z \in \mathcal{Q} : M^{**}(1+|Du|)(z) > AS\lambda\}\big| \le \Big[\frac{c_1(n, L, \nu, \omega(\cdot), \chi_0)}{(AS)^{2\chi_0}} + \frac{c_2(n,\nu)\varepsilon}{AS}\Big]|\mathcal{Q}|. \tag{6.3.2}$$

Now we perform the following choices of A and ε: first we choose $A \equiv A(n, L, \nu, \omega(\cdot), \chi_0) > 1$ such that

$$A = 4 \cdot 10^N [1 + c_1] \quad \Longrightarrow \quad \frac{c_1}{(AS)^{2\chi_0}} \leq \frac{1}{4S^{2\chi_0}}, \tag{6.3.3}$$

and then choose $\varepsilon = \varepsilon(n, \nu, S, \chi_0) \in (0, 1)$ accordingly to

$$\varepsilon = \frac{1}{4S^{2\chi_0 - 1}[1 + c_2]} \quad \Longrightarrow \quad \frac{c_2 \varepsilon}{AS} \leq \frac{1}{4S^{2\chi_0}}.$$

These choices in (6.3.2) yield the following analog of (6.2.13):

$$\left|\{z \in \mathcal{Q} : M^{**}(1 + |Du|)(z) > AS\lambda\}\right| < S^{-2\chi_0}|\mathcal{Q}|.$$

Having arrived at this stage we can argue exactly as in the proof of Proposition 6.18 after (6.2.13) to derive the analog of (6.2.14), i.e.

$$\left|\{z \in \mathcal{Q} : M^{*}(1 + |Du|)(z) > AS\lambda\}\right| < S^{-2\chi_0}|\mathcal{Q}|,$$

which contradicts (6.3.1). This proves the assertion of the proposition. □

To proceed with the proof of Theorem 4.15 we choose λ_0 accordingly to

$$\lambda_0 := 2c_0(n) n^{2N} S^{2\chi_0} \fint_{n^2 \mathcal{Q}_0} (1 + |Du|)\, dz.$$

With the arguments form the proof of Theorem 4.8, Step 3, replacing χ by χ_0 everywhere we arrive at the following proper version of (6.2.18) (the only change here is the replacement of χ by χ_0):

$$\begin{aligned}
&(AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\left((AS)^{k+1}\lambda_0\right) \\
&\qquad \leq A^{\frac{\vartheta\gamma}{\vartheta-\gamma}} S^{\frac{\vartheta\gamma}{\vartheta-\gamma} - 2\chi_0} (AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\left((AS)^k \lambda_0\right) \\
&\qquad\quad + \left(AS/\varepsilon\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} (AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} (\varepsilon\lambda_0)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2\left((AS)^k \varepsilon\lambda_0\right).
\end{aligned}$$

Since $\gamma < \vartheta$ by assumption, the quantity $\frac{\vartheta\gamma}{\vartheta-\gamma}$ can be arbitrarily large. Nevertheless, since $\chi_0 > 0$ is at our disposal, we can choose χ_0 large enough to have

$$d := 2\chi_0 - \frac{\vartheta\gamma}{\vartheta - \gamma} > 0, \tag{6.3.4}$$

a relation playing the same role as (6.2.19) before. Note, that here we really need the possibility of taking χ_0 large. This fixes $\chi_0 = \chi_0(\vartheta, \gamma)$ (for example we could choose $\chi_0 = \frac{\vartheta\gamma}{\vartheta-\gamma}$), $d = d(\vartheta, \gamma)$ and also $A \equiv A(n, L, \nu, \omega(\cdot), \vartheta, \gamma)$ by (6.3.3). Having fixed χ_0 we choose

$$S := \left[4A^{\frac{\vartheta\gamma}{\vartheta-\gamma}}\right]^{\frac{1}{d}}, \tag{6.3.5}$$

where A has been determined in (6.3.3). Then S admits the same dependencies as A, i.e. $S = S(n, L, \nu, \omega(\cdot), \vartheta, \gamma)$, and therefore we can write $AS/\varepsilon =: c_*(n, L, \nu, \omega(\cdot), \vartheta, \gamma)$. In view of (6.3.4) and (6.3.5) we find that the analog of (6.2.21), i.e.

$$\begin{aligned}
&(AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\left((AS)^{k+1}\lambda_0\right) \\
&\qquad \leq \frac{1}{4}(AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} \lambda_0^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_1\left((AS)^k \lambda_0\right) \\
&\qquad\quad + \left(\frac{c_*}{2}\right)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} (AS)^{\frac{k\vartheta\gamma}{\vartheta-\gamma}} (\varepsilon\lambda_0)^{\frac{\vartheta\gamma}{\vartheta-\gamma}} \mu_2\left((AS)^k \varepsilon\lambda_0\right),
\end{aligned} \tag{6.3.6}$$

holds for every $k \in \mathbb{N}_0$ with a constant c_*. The preceding estimate for the level sets allows us to proceed as in the proof of Theorem 4.8 after (6.2.21); i.e. we first sum up (6.3.6) upon $k \in \mathbb{N}$ and then re-absorb the intermediate sum in the left-hand side. Arguing exactly as in (6.2.21)–(6.2.29) we arrive at the following analog of (6.2.29):

$$\|Du\|_{L(\frac{\vartheta\gamma}{\vartheta-\gamma},\frac{\vartheta q}{\vartheta-\gamma})(\mathcal{Q}_0)} \le c\left(\fint_{n^2\mathcal{Q}_0}(1+|Du|)\,dz\right)|\mathcal{Q}_0|^{\frac{\vartheta-\gamma}{\vartheta\gamma}} + c\|g\|_{L^\vartheta(\gamma,q)(n^2\mathcal{Q}_0)}, \tag{6.3.7}$$

where now $c \equiv c(n,L,\nu,\vartheta,\gamma,q)$. In the next step we have to replace Proposition 6.19 by an appropriate version valid under the weaker assumption $1<\gamma<\vartheta\le N$. This is achieved in the following

PROPOSITION 6.22. *Let $u(=u_k) \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ be a weak solution to (6.1.1) where either the structure conditions (4.2.15) to (4.2.17) or the condition (4.2.14) are in force and $g \in L^\vartheta(\gamma,q)(\Omega_T)$ with $1<\gamma\le\vartheta\le N$. Then, for every pair of concentric parabolic cylinders $Q_\sigma \subset Q_\varrho \subset \Omega_T$ there holds*

$$\|1+|Du|\|_{L^{1,\frac{\vartheta-\gamma}{\gamma}}(Q_\sigma)} \le c\,(\varrho-\sigma)^{\frac{\vartheta-\gamma}{\gamma}-N}\|1+|Du|\|_{L^1(Q_\varrho)} + c\|g\|_{L^\vartheta(\gamma,q)(Q_\varrho)},$$

where $c \equiv c(n,L,\nu,\gamma,\vartheta,q)$.

PROOF. We first note that by Proposition 6.19 we only have to treat the case $\gamma \in (\vartheta/2,\vartheta)$. Let z_0 be a point in Q_σ and $Q_R(z_0)$ a parabolic cylinder with $0<R\le d_{\mathcal{P}}(z_0,\partial Q_\varrho)$, i.e. $Q_R(z_0)\subset Q_\varrho$. Moreover, let $v \in C^0([t_0-R^2,t_0+R^2];L^2(B_R(x_0)))\cap L^2(t_0-R^2,t_0+R^2;W^{1,2}(B_R(x_0))$ be the unique solution to the Cauchy-Dirichlet problem (6.2.5) in $C(z_0,R)$. Then, using (6.1.12) for the choice $q=1$ and with $\alpha\in(0,1)$ to be fixed later we infer by the argument from the beginning of the proof of Proposition 6.19 that

$$\int_{Q_r(z_0)}(1+|Du|)\,dz \le c\left(\frac{r}{R}\right)^{N-1+\alpha}\int_{Q_R(z_0)}(1+|Du|)\,dz + c\int_{Q_R(z_0)}|Du-Dv|\,dz,$$

holds for any $0<r\le R$ where $c\equiv c(n,L,\nu,\alpha)$. Using (6.1.4) from Lemma 6.13 the previous inequality leads us to

$$\int_{Q_r(z_0)}(1+|Du|)\,dz \le c\left(\frac{r}{R}\right)^{N-1+\alpha}\int_{Q_R(z_0)}(1+|Du|)\,dz + c\,\|g\|_{L^\vartheta(\gamma,q)(Q_\varrho)}R^{N-1+(1-\frac{\vartheta-\gamma}{\gamma})},$$

where $c\equiv c(n,L,\nu,q,\alpha)$. At this stage we remark that $\gamma\in(\vartheta/2,\vartheta)$ yields $1-\frac{\vartheta-\gamma}{\gamma}\in(0,1)$. Therefore we choose $\alpha\in(0,1)$ such that $1>\alpha>1-\frac{\vartheta-\gamma}{\gamma}>0$. Note, that here we really need the possibility of taking α close to 1 at our disposal. For example we could choose $\alpha := 1-\frac{\vartheta-\gamma}{2\gamma}$, fixing $\alpha\equiv\alpha(\vartheta,\gamma)$. Now, we can finish the proof exactly as in the proof of Proposition 6.19 by the application of Lemma 3.3.1. □

Having arrived at this stage the local Lorentz integrability of Du from (6.3.7) can be turned into the desired Lorentz-Morrey space estimate via the scaling argument along the lines of the proof of Theorem 4.8, Step 5, combined with the intermediate Morrey space information for Du from Proposition 6.22. See also the Proof of Theorem 4.9.

Integrability of u. Since the proofs of Theorem 4.12 is very close to the one of Theorem 4.8 we confine ourselves to outline the necessary modifications only. Again, we deal with the approximating solutions $u \equiv u_k$, abbreviating again $g \equiv g_k$. Now, we go back to the proof of Theorem 4.8 and keep in mind the notation introduced thereby. Then, Lemma 6.18 must be replaced by

LEMMA 6.23. *Let $u(= u_k) \in C^0([-T,0]; L^2(\Omega)) \cap L^2(-T,0; W_0^{1,2}(\Omega))$ be a weak solution to (6.1.1) where the assumptions (4.2.1) are in force and $g \in L^\infty(\Omega_T)$. Then, there exists an absolute constant $A \equiv A(n,L,\nu) > 1$ such that: For every $S > 1$ and $\chi_0 > 1$ there exists a constant $\varepsilon = \varepsilon(n,L,\nu,S,\chi_0) \in (0,1)$ such that if $\lambda > 1$ and $\mathcal{Q}$ a dyadic sub-cylinder of $\mathcal{Q}_0$ such that*

$$\big|\mathcal{Q} \cap \big\{z \in \mathcal{Q}_0 : M^*(1+|u|)(z) > AS\lambda \ \text{ and } \ M_2^*(g) \le \varepsilon\lambda\big\}\big| > \frac{|\mathcal{Q}|}{S^{2\chi_0}}, \quad (6.3.8)$$

then the predecessor $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ satisfies

$$\tilde{\mathcal{Q}} \subset \big\{z \in \mathcal{Q}_0 : M^*(1+|u|)(z) > \lambda\big\}.$$

The main changes in the statement of Lemma 6.23 are essentially the replacement of $M_1^*(g) = M_{1,n^2\mathcal{Q}_0}^*(g)$ by $M_2^*(g) = M_{2,n^2\mathcal{Q}_0}^*(g)$ and the introduction of the parameter $\chi_0 > 1$ which is at our disposal, i.e. χ_0 can be picked large at will, while the quantity χ in Lemma 6.18 was fixed. In principle, the proof of Lemma 6.23 follows the one of Lemma 6.18 replacing $M^*(1+|Du|)$, $M_1^*(g)$ by $M^*(1+|u|)$, $M_2^*(g)$. But for convenience of the reader we describe the main differences. Since we are dealing with $M_2^*(g)$, (6.2.4) has to be replaced by $M_2^*(g)(\bar{z}) \le \varepsilon\lambda$ for some $\bar{z} \in \mathcal{Q}$, yielding in turn the following analog of (6.2.6):

$$|Q|^{\frac{2}{N}} \fint_Q |g|\,dz \le c(n)\varepsilon\lambda.$$

Therefore, by (7.2.2) we have

$$\int_{3\mathcal{Q}} |u-v|\,dz \le c(n,\nu)|Q|^{\frac{2}{N}} \int_Q |g|\,dz \le c(n,\nu)\varepsilon\lambda.$$

At this stage the proof proceeds exactly along the lines of Lemma 6.18, starting with the preceding inequality instead of (6.2.6). This leads us to the following analog of (6.2.9):

$$\int_{2\mathcal{Q}} (1+|v|)^{2\chi_0}\,dz \le c(n,L,\nu)\,\lambda^{2\chi_0}.$$

Here we have taken into account that the side-length of the cylinders is bounded by 1. Now, as in the proof of Lemma 6.18 we compare the level set of $M^{**}(1+|u|)$ in the cylinder $\mathcal{Q}$ with the one of $M^{**}(1+|v|)$, where $M^{**} := M_{0,2\mathcal{Q}}^*$. This procedure implies that

$$\big|\{z \in \mathcal{Q} : M^{**}(1+|u|)(z) > AS\lambda\}\big| \le \Big[\frac{c_1(n,L,\nu)}{(AS)^{2\chi_0}} + \frac{c_2(n,\nu)\varepsilon}{AS}\Big]|\mathcal{Q}|.$$

The choices of A and ε are performed exactly as in (6.2.11), (6.2.12), but everywhere replacing χ by χ_0. This fixes $A \equiv A(n,L,\nu) > 1$ and $\varepsilon = \varepsilon(n,L,\nu,S,\chi_0) \in (0,1)$ and leads first to the analog of (6.2.13), and secondly with the arguments from the proof of Lemma 6.18 to the following analog of (6.2.14):

$$\big|\{z \in \mathcal{Q} : M^*(1+|u|)(z) > AS\lambda\}\big| < S^{-2\chi_0}|\mathcal{Q}|,$$

contradicting (6.3.8). This completes the proof of Lemma 6.23.

We now proceed with the proof of Theorem 4.12 along the lines of Theorem 4.8, starting at Step 2. We initially choose

$$\lambda_0 := 2c_0(n)n^{2N}S^{2\chi_0}\fint_{n^2\mathcal{Q}_0}(1+|u|)\,dz, \tag{6.3.9}$$

and define $\mu_1(\cdot)$, $\mu_2(\cdot)$ by

$$\mu_1(H) := \left|\left\{z\in\mathcal{Q}_0 : M^*(1+|u|)(z)>H\right\}\right|,$$

and

$$\mu_2(H) := \left|\left\{z\in\mathcal{Q}_0 : M_2^*(g)(z)>H\right\}\right|,$$

respectively, for $H\geq 0$. At this stage we start replacing $\frac{\vartheta}{\vartheta-\gamma}$ by $\frac{\vartheta}{\vartheta-2\gamma}$ everywhere. Applying Lemma 6.23 and Proposition 3.17 at levels $H=(AS)^{k+1}\lambda_0$, $(AS)^k\lambda_0$ for $k=0,1,2,\dots$ we arrive at the following analog of (6.2.18):

$$\begin{aligned}(AS)^{\frac{(k+1)\vartheta\gamma}{\vartheta-2\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-2\gamma}}\mu_1\left((AS)^{k+1}\lambda_0\right)&\\ \leq A^{\frac{\vartheta\gamma}{\vartheta-2\gamma}}S^{\frac{\vartheta\gamma}{\vartheta-2\gamma}-2\chi_0}(AS)^{\frac{k\vartheta\gamma}{\vartheta-2\gamma}}\lambda_0^{\frac{\vartheta\gamma}{\vartheta-2\gamma}}\mu_1\left((AS)^k\lambda_0\right)&\\ +\left(AS/\varepsilon\right)^{\frac{\vartheta\gamma}{\vartheta-2\gamma}}(AS)^{\frac{k\vartheta\gamma}{\vartheta-2\gamma}}\left(\varepsilon\lambda_0\right)^{\frac{\vartheta\gamma}{\vartheta-2\gamma}}\mu_2\left((AS)^k\varepsilon\lambda_0\right).&\end{aligned} \tag{6.3.10}$$

We observe that this time we can choose χ_0 large enough to have $2\chi_0-\frac{\vartheta\gamma}{\vartheta-2\gamma}>0$; see (6.2.19) for the corresponding relation in the proof of Theorem 4.8. Note also that at this stage we need the possibility for choosing χ_0 large at will, since 2γ can be arbitrarily close ϑ making $\frac{\vartheta\gamma}{\vartheta-2\gamma}$ large. This motivates the following definitions

$$d := 2\chi_0-\frac{\vartheta\gamma}{\vartheta-2\gamma}>0, \qquad S := \left[4A^{\frac{\vartheta\gamma}{\vartheta-2\gamma}}\right]^{\frac{1}{d}},$$

so that $A^{\frac{\vartheta\gamma}{\vartheta-2\gamma}}S^{\frac{\vartheta\gamma}{\vartheta-2\gamma}-2\chi_0}\leq\frac14$. Using this in (6.3.10) allows us to proceed as in the proof of Theorem 4.8 after (6.2.21); i.e. we first sum up (6.3.10) upon $k\in\mathbb{N}$ and then re-absorb the intermediate sum in the left-hand side. Arguing exactly as in (6.2.21)–(6.2.26) we arrive at the following analog of (6.2.26):

$$\|u\|_{L(\frac{\vartheta\gamma}{\vartheta-2\gamma},\tau)(\mathcal{Q}_0)}\leq c\lambda_0|\mathcal{Q}_0|^{\frac{\vartheta-2\gamma}{\vartheta\gamma}}+c\left\|M_2^*(g)\right\|_{L(\frac{\vartheta\gamma}{\vartheta-2\gamma},\tau)(\mathcal{Q}_0)}, \tag{6.3.11}$$

which holds for every $\tau>0$, and where $c\equiv c(n,L,\nu,\gamma,\vartheta,\tau)$. At this stage we take $\tau=\frac{\vartheta q}{\vartheta-2\gamma}$ in (6.3.11) and apply Theorem 6.11 with $\beta=2$ and $p=\gamma$ (note that $\beta p=2\gamma<\vartheta$ by assumption (4.2.12); this leads us to

$$\left\|M_2^*(g)\right\|_{L(\frac{\vartheta\gamma}{\vartheta-2\gamma},\frac{\vartheta q}{\vartheta-2\gamma})(\mathcal{Q}_0)}\leq c\,\|g\|_{L^\vartheta(\gamma,q)(n^2\mathcal{Q}_0)}.$$

It is worth to remark that this is exactly the point where we use the fact that M_2^* admits a higher regularizing effect then M_1^*. Combining the preceding inequality with (6.3.11) and recalling the definition of λ_0 from (6.3.9) we obtain

$$\|u\|_{L(\frac{\vartheta\gamma}{\vartheta-2\gamma},\frac{\vartheta q}{\vartheta-2\gamma})(\mathcal{Q}_0)}\leq c\left(\fint_{n^2\mathcal{Q}_0}(1+|u|)\,dz\right)|\mathcal{Q}_0|^{\frac{\vartheta-2\gamma}{\vartheta\gamma}}+c\,\|g\|_{L^\vartheta(\gamma,q)(n^2\mathcal{Q}_0)}.$$

Having arrived at this stage the local Lorentz integrability of u can be turned into the desired Lorentz-Morrey space estimate via a scaling argument along the lines of the proof of Theorem 4.8, Step 5, combined with the intermediate Morrey space information for u from Remark 6.20, (6.2.34). Consider $Q_\varrho\subset\Omega_T$. Scaling back to Q_1 as in Remark 5.5 and arguing along the lines of Step 5 we find

$$\|\tilde u\|_{L(\frac{\vartheta\gamma}{\vartheta-2\gamma},\frac{\vartheta q}{\vartheta-2\gamma})(Q_{1/n^4})}\leq c\left\|1+|\tilde u|\right\|_{L^{1,\frac{\vartheta-2\gamma}{\gamma}}(Q_{9/10})}+c\,\|\tilde g\|_{L^\vartheta(\gamma,q)(Q_1)}.$$

Scaling back to Q_ϱ via Lemma 3.4, we find for every parabolic cylinder $Q_\varrho \subset \Omega_T$ that

$$\|u\|_{L(\frac{\vartheta\gamma}{\vartheta-2\gamma},\frac{\vartheta q}{\vartheta-2\gamma})(Q_{\varrho/n^4})} \le c\,\Psi(Q_\varrho)\,\varrho^{(N-\vartheta)\frac{\vartheta-2\gamma}{\vartheta\gamma}},$$

where this time we have set

$$\Psi(Q_\varrho) := \big\|\varrho + |u|\big\|_{L^{1,\frac{\vartheta-2\gamma}{\gamma}}(Q_{9\varrho/10})} + \|g\|_{L^\vartheta(\gamma,q)(Q_\varrho)}.$$

Having arrived at this stage we follow exactly the proof of Theorem 4.8, Step 5, after (6.2.35). The only difference occurs when using the intermediate Morrey-space estimate (6.2.34) instead of (6.2.31). The desired estimate (4.2.13) then follows by the approximation argument from the proof of Theorem 4.8, Step 6.

The proof of the second part follows similarly to the one of Theorem 4.9, taking into account Theorem 3.5 for the choice $\beta = 2$.

PROOF OF THEOREM 4.14. We once again consider the approximating solutions $u \equiv u_k \in C^0([-T,0];L^2(\Omega)) \cap L^2(-T,0;W_0^{1,2}(\Omega))$ to (6.1.1). From [28, Lemma 4.3] we recall that the following Poincaré-type inequality holds

$$\int_{Q_{\varrho/2}} |u-(u)_{Q_{\varrho/2}}|\,dz \le c\,\varrho\int_{Q_\varrho}|Du|\,dz + c\,\varrho^2\int_{Q_\varrho}|g|\,dz,$$

for any parabolic cylinder $Q_\varrho \subset \Omega_T$, with a constant $c \equiv c(n,L,\nu)$. Therefore, we have

$$\begin{aligned}[u]_{\mathrm{BMO}(Q_{R/2})} &= \sup_{Q_\varrho\subset Q_{R/2}} \fint_{Q_\varrho}|u-(u)_{Q_\varrho}|\,dz \\ &\le c\Big[\|Du\|_{L^{1,1}(Q_R)} + \sup_{Q_\varrho\subset Q_R}\varrho^2\fint_{Q_\varrho}|g|\,dz\Big].\end{aligned}$$

The first term appearing on the right-hand side of the preceding inequality can be estimated with Proposition 6.19 for the choice $\gamma = \vartheta/2$ and $q = \infty$ (note that $\frac{\vartheta-\vartheta/2}{\vartheta/2} = 1$); we infer that

$$\|Du\|_{L^{1,1}(Q_R)} \le c\,R^{1-N}\big\|1+|Du|\big\|_{L^1(Q_{2R})} + c\,\|g\|_{\mathcal{M}^{\vartheta/2,\vartheta}(Q_{2R})},$$

where $c \equiv c(n,L,\nu,\vartheta)$. On the other hand, the second term can be treated by use of Lemma 3.3 as follows:

$$\begin{aligned}\varrho^2\fint_{Q_\varrho}|g|\,dz &\le \frac{\vartheta}{\vartheta-2}\,ds|Q_\varrho|^{-2/\vartheta}\varrho^2\|g\|_{\mathcal{M}^{\vartheta/2}(Q_\varrho)} \\ &\le \frac{\vartheta}{\vartheta-2}[2\alpha(n)]^{-2/\vartheta}\varrho^{\frac{\vartheta-N}{\vartheta/2}}\|g\|_{\mathcal{M}^{\vartheta/2}(Q_\varrho)} \\ &= c(n,\vartheta)\varrho^{\frac{\vartheta-N}{\vartheta/2}}\|g\|_{\mathcal{M}^{\vartheta/2}(Q_\varrho)}.\end{aligned}$$

Therefore, we have

$$\sup_{Q_\varrho\subset Q_R}\varrho^2\fint_{Q_\varrho}|g|\,dz \le c(n,\vartheta)\|g\|_{\mathcal{M}^{\vartheta/2,\vartheta}(Q_R)}.$$

Combining the preceding estimates leads us to

$$[u]_{\mathrm{BMO}(Q_{R/2})} \le c\,R^{1-N}\|1+|Du|\|_{L^1(Q_{2R})} + c\,\|g\|_{\mathcal{M}^{\vartheta/2,\vartheta}(Q_{2R})},$$

where $c \equiv c(n,L,\nu,\vartheta)$ we note that the constant c blows up, i.e. $c\to\infty$, when $\vartheta \searrow 2$. Again the desired result follows by approximation. □

CHAPTER 7

Marcinkiewicz regularity for degenerate parabolic equations

To begin here we spend a few words about the main problem, but at the same time the interesting point, of the proof of the results of Section 4.3. As we saw in Section 2.5 when dealing with evolutionary p-Laplacian operator it is customary to consider estimates over intrinsic cylinders, i.e. cylinders of the form (3.1.5) in which $|Du| \approx \lambda$, in some integral sense. The problem when dealing with the techniques we have seen in the previous two Chapters is that there are two possibilities which are to be accommodated. The correct geometry to be used when approaching questions regarding homogeneous problems are energy ones, i.e.

$$\fint_{Q_R^\lambda} \big(s + |Dv|\big)^p \, dz \approx \lambda^p,$$

see (7.1.18) and also the Caccioppoli estimate Lemma 7.3; this could be seen as the *energy geometry*. On the other hand, since we are dealing with *very weak solutions* to equation (4.3.1), these functions might even have infinite energy. Consequently, also comparison estimate Lemma 7.9 does not hold for the natural integrability exponent p. Therefore, the only meaningful definition for intrinsic cylinders for u is

$$\fint_{Q_R^\lambda} \big(s + |Du|\big)^{p-1} \, dz \approx \lambda^{p-1} \quad \Longrightarrow \quad \fint_{Q_R^\lambda} \big(s + |Dv|\big)^{p-1} \, dz \approx \lambda^{p-1}$$

via comparison, see (7.3.6) and (7.3.12); we could call this the *weak geometry*. It will turn out in Section 7.2 that the right geometry for the problem is the *weak* one, as one could expect, since the regularity for solutions to homogenous will allow to show that the two geometries for v are actually equivalent, as proved in Proposition 7.13.

7.1. Estimates for homogeneous problems

Let us consider $v \in C^0(I; L^2(A)) \cap L^p(I; W^{1,p}(A))$ solution to the problem

$$v_t - \operatorname{div} a(x,t,Dv) = 0 \qquad \text{in } A \times I \subset \Omega_T, \tag{7.1.1}$$

being A, I open sets and with $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}$ the vector field appearing in (4.3.1), therefore satisfying (4.3.4) and

$$\langle a(x,t,\xi), \xi\rangle \geq c\,\big(s^2 + |\xi|^2\big)^{\frac{p-2}{2}} |\xi|^2 - c^{-1}\, L s^p, \tag{7.1.2}$$

by a simple use of Young's inequality, with $c \equiv c(p,\nu,L)$. In this section we collect some regularity and comparison results for weak solutions to (7.1.1).

The following is the $\sup$ bound for solutions to degenerate parabolic equations. It can be found in [**127**], see obviously also [**54**, Chapter V, Theorems 3.1 & 4.1]. Some modifications of the proofs are needed; in particular we followed the proof of the homogeneous case and the last term in (7.1.3) is due to the fact that in (7.1.2) the term containing s^p appears. Moreover we introduced the parameter ε: following [**54**, §12, Proof of Theorem 4.1], once we fix $\varepsilon \in (0,1)$, we just need in (12.1) to take k so big that $k \geq \varepsilon(\rho^p/\theta)^{1/(p-2)}$. Note that DiBenedetto's book notation differs from ours (in particular $\theta \leftrightarrow \sigma$).

PROPOSITION 7.1. *Any positive local sub-solution w to (7.1.1) in $A \times I$ is locally bounded with the following quantitative estimate: for $Q_{\rho,\sigma}(z_0) \equiv Q_{\rho,\sigma} \subset A \times I$, $\theta \in (0,1)$ and for every $\varepsilon \in (0,1]$ there holds*

$$\sup_{Q_{\theta\rho,\theta\sigma}} w \leq c\left(\frac{1}{(1-\theta)\varepsilon}\right)^{\frac{n+p}{2}}\left(\frac{\sigma}{\rho^p}\right)^{\frac{1}{2}}\left(\fint_{Q_{\rho,\sigma}} w^p\,dz\right)^{1/2} + \varepsilon\left(\frac{\rho^p}{\sigma}\right)^{1/(p-2)} + s\,\rho \quad (7.1.3)$$

for a constant depending only on n, p, ν, L.

Starting on the other hand from (7.1.3) and following exactly [54, §12, Proof of Theorem 4.1] we can lower the exponent appearing on the right-hand side, therefore getting the following corollary:

COROLLARY 7.2. *Let w be as in Proposition 7.1; then for $Q_{\rho,\sigma}, \theta, \varepsilon$ as above there holds*

$$\sup_{Q_{\theta\rho,\theta\sigma}} w \leq c\left(\frac{1}{(1-\theta)\varepsilon}\right)^{n+p}\frac{\sigma}{\rho^p}\fint_{Q_{\rho,\sigma}} w^{p-1}\,dz + \varepsilon\left(\frac{\rho^p}{\sigma}\right)^{1/(p-2)} + s\,\rho \quad (7.1.4)$$

with $c \equiv c(n,p,\nu,L)$.

The next Lemma is a standard Caccioppoli's inequality on generic cylinders. For its proof see [96, Lemma 3.2].

LEMMA 7.3 (Caccioppoli's inequality). *Let $v \in L^p(I; W^{1,p}(A))$ be a weak solution to (7.1.1) and let $Q_{\rho_2,\sigma_2} \equiv Q_{\rho_2,\sigma_2}(z_0) \subset A \times I$ be a cylinder. Moreover let $k \in \mathbb{R}$. Then the following estimate holds:*

$$\begin{aligned}\int_{Q_{\rho_1,\sigma_1}} &\left(s^2+|Dv|^2\right)^{\frac{p-2}{2}}|Dv|^2\,dz + \sup_{t\in(t_o-\sigma_1,t_o+\sigma_1)}\int_{B_{\rho_1}(x_o)}|v-k|^2\,dx \\ &\leq \frac{c}{\sigma_2-\sigma_1}\int_{Q_{\rho_2,\sigma_2}}|v-k|^2\,dz \\ &\quad + \frac{c}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2,\sigma_2}}|v-k|^p\,dz + c\,s^p|Q_{r_2,s_2}|, \quad (7.1.5)\end{aligned}$$

for all concentric cylinders $Q_{\rho_1,\sigma_1} \equiv Q_{\rho_1,\sigma_1}(z_0) \Subset Q_{\rho_2,\sigma_2}(z_0)$ and with a constant depending on n, p, ν, L.

The next is [96, Lemma 3.1]:

LEMMA 7.4. *Let $v \in L^p(I; W^{1,p}(A))$ be a weak solution to (7.1.1). If $Q_{\rho_2,\sigma}(z_0) \Subset A \times I$ and $\rho_1 < \rho_2$, then there exists a radius $\hat\rho \in (\rho_1,\rho_2)$ and a constant c depending on n, p, L such that*

$$|(v)_{B_{\hat\rho}(x_o)}(t_1) - (v)_{B_{\hat\rho}(x_o)}(t_2)| \leq \frac{c\,\sigma}{\rho_2-\rho_1}\fint_{Q_{\rho_2,\sigma}}\left(|Dv|+s\right)^{p-1}dz \quad (7.1.6)$$

for a.e. $t_1, t_2 \in (t_0-\sigma, t_0+\sigma)$.

The following is instead a Sobolev-type inequality. See again [96, Lemma 3.3].

LEMMA 7.5. *Let* $1 \le q < \infty$ *and suppose* $v \in L^q(t_0 - \sigma_2, t_0 + \sigma_2; W^{1,q}(B_{\rho_2}(x_0)))$. *Then, for a constant depending only on* n, q, *there holds*

$$\int_{Q_{\rho_1,\sigma_1}} \left|v - (v)_{B_{\rho_1}}(t)\right|^{q(1+2/n)} dz \le c \left(\frac{R}{\rho_2 - \rho_1}\right)^q \int_{Q_{\rho_2,\sigma_2}} |Dv|^q \, dz \times$$
$$\times \left(\sup_{t \in (t_o - \sigma_2, t_o + \sigma_2)} \int_{B_{\rho_2}} \left|v(\cdot,t) - (v)_{B_{\rho_2}}(t)\right|^2 dx \right)^{q/n} \tag{7.1.7}$$

for every couple of radii $R/2 \le \rho_1 < \rho_2 \le R$. Q_{ρ_1,σ_1} *and* Q_{ρ_2,σ_2} *share the vertex* $z_0 = (x_0, t_0)$.

PROPOSITION 7.6. *Let* $v \in L^p(I; W^{1,p}(A))$ *be a weak solution to* (7.1.1) *and let* $Q_R^\lambda(z_0) \subset A \times I$ *be a cylinder such that*

$$\left(\frac{\lambda}{\kappa}\right)^p \le \fint_{Q^\lambda_{R/2}} (s + |Dv|)^p \, dz \quad \text{and} \quad \fint_{Q^\lambda_R} (s + |Dv|)^p \, dz \le \kappa^p \lambda^p \tag{7.1.8}$$

hold for a constant $\kappa \ge 1$. *Then there exist constants* c_1 *depending on* n, p, ν, L, κ *and an exponent* $\xi \equiv \xi(n,p)$ *such that*

$$\left(\fint_{Q^\lambda_{r_1}} (s+|Dv|)^p \, dz\right)^{1/p} \le c_1 \left(\frac{R}{r_2 - r_1}\right)^\xi \left(\fint_{Q^\lambda_{r_2}} (s+|Dv|)^q \, dz\right)^{1/q} \tag{7.1.9}$$

for all $R/2 \le r_1 < r_2 \le R$ *and with* $q := \max\left\{p-1, np/(n+2)\right\} < p$. *Here* $Q^\lambda_{r_1}$ *and* $Q^\lambda_{r_2}$ *are concentric cylinders having the same vertex of* Q^λ_R.

PROOF. Since this proof is very similar to that of [96, Lemma 3.4], in some points the arguments are only sketched. We refer to the aforementioned paper for the missing details. For shortness of notation from now on we suppose $z_0 = 0$. If this were not the case, a simple translation would be sufficient to recover this situation.

We begin by defining $r_3 := (r_2 - r_1)/5$. Caccioppoli's inequality (7.1.5) applied with $Q_{\rho_1,\sigma_1} \equiv Q^\lambda_{r_1}$, $Q_{\rho_2,\sigma_2} \equiv B_{r_1+r_3} \times \Lambda^\lambda_{r_1+2r_3} = B_{r_1+r_3} \times (-\lambda^{2-p}(r_1+2r_3)^2, \lambda^{2-p}(r_1 + 2r_3)^2)$ gives, for $\varepsilon \in (0,1)$ to be chosen

$$\begin{aligned}\fint_{Q^\lambda_{r_1}} (s^2 + |Dv|^2)^{\frac{p-2}{2}} |Dv|^2 \, dz &\le \frac{c\,\lambda^{p-2}}{r_3^2} \fint_{B_{r_1+r_3} \times \Lambda^\lambda_{r_1+2r_3}} |v-k|^2 \, dz \\ &\quad + \frac{c}{r_3^p} \fint_{B_{r_1+r_3} \times \Lambda^\lambda_{r_1+2r_3}} |v-k|^p \, dz + c\, s^p \\ &\le \frac{\varepsilon}{2}\lambda^p + \frac{c_\varepsilon}{r_3^p} \fint_{B_{r_1+r_3} \times \Lambda^\lambda_{r_1+2r_3}} |v-k|^p \, dz + c\, s^p,\end{aligned} \tag{7.1.10}$$

with the constant c_ε depending on n, p, ν, L and on ε. In the last line we used Young's and Hölder's inequalities. Note moreover that we could take averages in (7.1.5) since $|B_{r_1+r_3} \times \Lambda^\lambda_{r_1+2r_3}|/|Q^\lambda_{r_1}| \le c(n)$. Now we choose

$$k \equiv (v)_{B_{r_1+\hat r} \times \Lambda^\lambda_{r_1+2r_3}} = \fint_{B_{r_1+\hat r} \times \Lambda^\lambda_{r_1+2r_3}} v \, dz,$$

recalling that $B_{r_1+\hat r} \times \Lambda^\lambda_{r_1+2r_3} = B_{r_1+\hat r} \times (-\lambda^{2-p}(r_1+2r_3)^2, \lambda^{2-p}(r_1+2r_3)^2)$ and $\hat r \in (r_3, 2r_3)$ is such that $r_1 + \hat r$ is the radius $\hat\rho$ of Lemma 7.4 with $\rho_1 \equiv r_1 + r_3$, $\rho_2 \equiv r_1 + 2r_3$ and $\sigma \equiv \lambda^{2-p}(r_1 + 2r_3)^2$. Note that

$$B_{r_1+r_3} \times \Lambda^\lambda_{r_1+2r_3} \subset B_{r_1+\hat r} \times \Lambda^\lambda_{r_1+2r_3} \subset Q^\lambda_{r_1+2r_3}$$

and their measures differ only by a constant $c(n)$, since $R/2 \leq r_1 < r_2 \leq R$. With this choice of k we can estimate the integral term in (7.1.10) in the following way, writing for shortness $\hat{Q} := B_{r_1+\hat{r}} \times \Lambda^{\lambda}_{r_1+2r_3}$ and $\hat{B} := B_{r_1+\hat{r}}$:

$$\fint_{B_{r_1+r_3}\times\Lambda^{\lambda}_{r_1+2r_3}} |v-(v)_{\hat{Q}}|^p\,dz \leq c(p)\fint_{B_{r_1+r_3}\times\Lambda^{\lambda}_{r_1+2r_3}} |v-(v)_{\hat{B}}(t)|^p\,dz + c(p)\sup_{t\in\Lambda^{\lambda}_{r_1+2r_3}} |(v)_{\hat{B}}(t)-(v)_{\hat{Q}}|^p, \quad (7.1.11)$$

where sup is to be understood in the sense of essential supremum. Applying (7.1.6) from Lemma 7.4 we have

$$|(v)_{\hat{B}}(t_1)-(v)_{\hat{B}}(t_2)| \leq c\,\frac{\lambda^{2-p}(r_1+2r_3)^2}{r_3}\fint_{Q^{\lambda}_{r_1+2r_3}} \big(|Du|+s\big)^{p-1}\,dz \quad (7.1.12)$$

for a.e. $t_1, t_2 \in \Lambda^{\lambda}_{r_1+2r_3}$. This will be useful to estimate the *second term* in (7.1.11): indeed for a.e. $t \in \Lambda^{\lambda}_{r_1+2r_3}$ by (7.1.12), using the intrinsic estimate (7.1.8), we have

$$\begin{aligned}|(v)_{\hat{B}}(t)-(v)_{\hat{Q}}| &\leq c\,\frac{\lambda^{2-p}(r_1+2r_3)^2}{r_3}\fint_{Q^{\lambda}_{r_1+2r_3}} \big(|Dv|+s\big)^{p-1}\,dz\\ &\leq c\,\frac{R^2}{r_3}\left(\fint_{Q^{\lambda}_{r_2}} \big(|Dv|+s\big)^{p-1}\,dz\right)^{1/(p-1)} \quad (7.1.13)\end{aligned}$$

for a constant depending on n, p, L, κ. Note again that $|Q^{\lambda}_{r_1+2r_3}| \approx |Q^{\lambda}_{r_2}| \approx |Q^{\lambda}_{R}|$, up to a constant depending on n. Now we estimate the *first term* of (7.1.11). Using the Sobolev's estimate of Lemma 7.5 with $q = np/(n+2)$, $Q_{\rho_1,\sigma_1} \equiv \hat{Q} = B_{r_1+\hat{r}} \times \Lambda^{\lambda}_{r_1+2r_3}$, $Q_{\rho_2,\sigma_2} \equiv Q^{\lambda}_{r_1+3r_3}$ we gain

$$\begin{aligned}&\fint_{B_{r_1+r_3}\times\Lambda^{\lambda}_{r_1+2r_3}} |v-(v)_{\hat{B}}(t)|^p\,dz\\ &\leq c\left(\frac{R}{r_3}\right)^{np/(n+2)}\fint_{Q^{\lambda}_{r_1+3r_3}} |Dv|^{np/(n+2)}\,dz\,\times\\ &\quad\times\left(\sup_{t\in\Lambda_{r_1+3r_3}}\int_{B_{r_1+3r_3}} \big|v(\cdot,t)-(v)_{B_{r_1+3r_3}}(t)\big|^2\,dx\right)^{p/(n+2)}. \quad (7.1.14)\end{aligned}$$

We now need to further estimate the supremum on the right-hand side of the previous inequality. By a simple argument, using triangle's inequality, we get

$$\int_{B_{r_1+3r_3}} \big|v(\cdot,t)-(v)_{B_{r_1+3r_3}}(t)\big|^2\,dx \leq 2\int_{B_{r_1+3r_3}} \big|v(\cdot,t)-(v)_{Q^{\lambda}_{r_1+4r_3}}\big|^2\,dx.$$

Using the previous estimate together with the Caccioppoli's Lemma 7.3, this time with $Q_{r_1,s_1} \equiv Q^{\lambda}_{r_1+3r_3}$, $Q_{r_2,s_2} \equiv Q^{\lambda}_{r_1+4r_3}$ and $k \equiv (v)_{Q^{\lambda}_{r_1+4r_3}}$ gives

$$\begin{aligned}&\frac{1}{|Q^{\lambda}_{r_1+4r_3}|}\sup_{t\in\Lambda^{\lambda}_{r_1+3r_3}}\int_{B_{r_1+3r_3}} \big|v(\cdot,t)-(v)_{B_{r_1+3r_3}}(t)\big|^2\,dx\\ &\leq \frac{c\lambda^{p-2}}{r_3^2}\fint_{Q^{\lambda}_{r_1+4r_3}} \big|v-(v)_{Q^{\lambda}_{r_1+4r_3}}\big|^2\,dz\\ &\qquad + \frac{c}{r_3^p}\fint_{Q^{\lambda}_{r_1+4r_3}} \big|v-(v)_{Q^{\lambda}_{r_1+4r_3}}\big|^p\,dz + c\,s^p, \quad (7.1.15)\end{aligned}$$

the constant depending upon n, p, ν, L. We estimate, again by triangle's inequality

$$\fint_{Q^\lambda_{r_1+4r_3}} |v-(v)_{Q^\lambda_{r_1+4r_3}}|^2\,dz \le 2\fint_{Q^\lambda_{r_1+4r_3}} \left|v-(v)_{B_{r_1+4r_3}}(t)\right|^2 dz$$
$$+\,2\sup_{t\in\Lambda^\lambda_{r_1+4r_3}} \left|(v)_{B_{r_1+4r_3}}(t)-(v)_{Q^\lambda_{r_1+4r_3}}\right|^2. \quad (7.1.16)$$

At this point Poincaré's inequality applied slicewise gives

$$\fint_{Q^\lambda_{r_1+4r_3}} \left|v-(v)_{B_{r_1+4r_3}}(t)\right|^2 dz \le c(n)\,(r_1+4r_3)^2 \fint_{Q^\lambda_{r_1+4r_3}} |Dv|^2\,dz$$
$$\le c(n)\,(r_1+4r_3)^2\lambda^2,$$

using Hölder's inequality and (7.1.8). As for the second term of the right-hand side of (7.1.16) we deduce, similarly as done in (7.1.11)-(7.1.12), using Lemma 7.4 with $\rho_1 \equiv r_1+4r_3$, $\rho_2 \equiv r_1+5r_3 = r_2$ and $\sigma = \lambda^{2-p}(r_1+4r_3)^2$

$$\frac{1}{|Q^\lambda_{r_1+4r_3}|}\sup_{t\in\Lambda_{r_1+4r_3}} \left|(v)_{B_{r_1+4r_3}}(t)-(v)_{Q^\lambda_{r_1+4r_3}}\right|^2$$
$$\le c\,\frac{\lambda^{2(2-p)}(r_1+4r_3)^4}{r_3^2}\left(\fint_{Q^\lambda_{r_2}} \left(|Dv|^p+s^p\right)dz\right)^{2(p-1)/p}$$
$$\le c\left(\frac{(r_1+4r_3)^2}{r_3}\right)^2\lambda^2$$

with $c\equiv c(n,p,L)$, so merging the last two inequalities into (7.1.16) gives

$$\int_{Q^\lambda_{r_1+4r_3}} |v-(v)_{Q^\lambda_{r_1+4r_3}}|^2\,dz \le c\,(r_1+4r_3)^2\left(1+\frac{r_1+4r_3}{r_3}\right)^2 |Q^\lambda_{r_1+4r_3}|\lambda^2$$
$$\le c\,R^2\left(\frac{R}{r_3}\right)^2|Q^\lambda_{r_1+4r_3}|\lambda^2$$

c having the same dependencies as the constant above. Similar estimates give

$$\fint_{Q^\lambda_{r_1+4r_3}} |v-(v)_{Q^\lambda_{r_1+4r_3}}|^p\,dz \le c\,R^p\left(\frac{R}{r_3}\right)^p\lambda^p;$$

putting these two estimates into (7.1.15) gives (notice that $s\le\kappa\,\lambda$)

$$\frac{1}{|Q^\lambda_{r_1+4r_3}|}\sup_{t\in\Lambda_{r_1+3r_3}} \int_{B_{r_1+3r_3}} \left|v(\cdot,t)-(v)_{B_{r_1+3r_3}}(t)\right|^2 dx$$
$$\le c\left(\frac{R}{r_3}\right)^{2p}\lambda^p + c\,s^p \le c\left(\frac{R}{r_3}\right)^{2p}\lambda^p,$$

that is

$$\sup_{t\in\Lambda_{r_1+3r_3}} \int_{B_{r_1+3r_3}} \left|v(\cdot,t)-(v)_{B_{r_1+3r_3}}(t)\right|^2 dx \le c\,R^{n+2}\left(\frac{R}{r_3}\right)^{2p}\lambda^p;$$

using in turn this estimate into (7.1.14) and using again Young's inequality, ε being the same quantity already chosen in (7.1.10), finally gives an estimate for the *first term* of (7.1.11)

$$\frac{1}{r_3^p}\fint_{B_{r_1+r_3}\times\Lambda^\lambda_{r_1+2r_3}} |v-(v)_{\hat B}(t)|^p\,dz$$
$$\le c\,\lambda^{2p/(n+2)}\left(\frac{R}{r_3}\right)^{2p(n+p+1)/(n+2)}\fint_{Q^\lambda_{r_2}} |Dv|^{np/(n+2)}\,dz$$
$$\le \frac{\varepsilon}{2}\lambda^p + c_\varepsilon(\kappa)\left(\frac{R}{r_3}\right)^{2p(n+p+1)/n}\left(\fint_{Q^\lambda_{r_2}} |Dv|^{np/(n+2)}\,dz\right)^{(n+2)/n}$$

with $c_\varepsilon \equiv c_\varepsilon(n,p,\nu,L,\kappa,\varepsilon)$; putting this estimate inside (7.1.11) together with (7.1.13) and in turn the result into (7.1.10)

$$
\begin{aligned}
&\fint_{Q^\lambda_{r_1}} \left(s^2+|Dv|^2\right)^{\frac{p-2}{2}} |Dv|^2\,dz \\
&\leq \varepsilon\lambda^p + c_\varepsilon \left(\frac{R}{r_3}\right)^{2p} \left(\fint_{Q^\lambda_{r_2}} \left(|Dv|+s\right)^{p-1}\,dz\right)^{p/(p-1)} \\
&\qquad\qquad + c_\varepsilon \left(\frac{R}{r_3}\right)^{2p(n+p+1)/n} \left(\fint_{Q^\lambda_{r_2}} |Dv|^{np/(n+2)}\,dz\right)^{(n+2)/n},
\end{aligned}
$$

c_ε depending upon n,p,ν,L,κ and obviously ε. Now we only need some algebraic manipulations to get (7.1.9). In particular we first recall the definitions of r_3 and $q<p$, we estimate from below the left-hand side and we sum to both sides s. Then we estimate

$$
\lambda^p \leq \kappa^p \fint_{Q^\lambda_{R/2}} |Dv|^p\,dz \leq \kappa^p\, 2^{n+2} \fint_{Q^\lambda_{r_1}} |Dv|^p\,dz
$$

and choose ε, depending on n,p,κ, small enough to make reabsorption possible. This finishes the proof. □

Matching the previous Proposition with Lemma 3.14 immediately implies the following homogeneous reverse-Hölder estimate:

COROLLARY 7.7. *Let $v \in L^p(I;W^{1,p}(A))$ be a weak solution to (7.1.1) and let $Q^\lambda_R(z_0) \subset A\times I$ be a cylinder such that*

$$
\frac{\lambda^p}{\kappa^p} \leq \fint_{Q^\lambda_{R/2}} \left(s+|Dv|\right)^p\,dz \qquad \text{and} \qquad \fint_{Q^\lambda_R} \left(s+|Dv|\right)^p\,dz \leq \kappa^p\,\lambda^p
$$

hold for a constant $\kappa \geq 1$. Then there exists constants c depending on n,p,ν,L,q,κ such that

$$
\left(\fint_{Q^\lambda_{\theta R}} \left(s+|Dv|\right)^p\,dz\right)^{1/p} \leq \frac{c}{(1-\theta)^{\xi'}} \left(\fint_{Q^\lambda_R} \left(s+|Dv|\right)^q\,dz\right)^{1/q}, \tag{7.1.17}
$$

for any $\theta\in(0,1)$, $q\in[1,p]$ and with $\xi' \equiv \xi'(n,p,q)$.

PROOF. We apply Lemma 3.14 with the choices

$$
\nu = \frac{1}{|Q^\lambda_R|}\mathcal{L}^{n+1}, \qquad U = Q^\lambda_R(z_0), \qquad \sigma_i = \frac{r_i}{R}, \quad i=1,2,
$$

so that $\sigma_i U = (r_i/R)Q^\lambda_R(z_0) = Q^\lambda_{r_i}(z_0)$, $i=1,2$. (3.3.2) obviously holds. With these agreements (7.1.9) looks exactly like (3.3.3), apart from a constant depending on n and p (the reader might recall that $|Q^\lambda_R| \approx |Q^\lambda_{r_1}| \approx |Q^\lambda_{r_2}|$). Then (7.1.17) follows straight from (3.3.4). □

Finally we can state the higher integrability for the parabolic p-Laplacian in the homogeneous form we needed. With all the preceding results at hand, its derivation is straightforward.

COROLLARY 7.8. *Let v as in Proposition 7.6 and in particular let*

$$
\left(\frac{\lambda}{\kappa}\right)^p \leq \fint_{Q^\lambda_{R/2}} \left(s+|Dv|\right)^p\,dz \qquad \text{and} \qquad \fint_{Q^\lambda_R} \left(s+|Dv|\right)^p\,dz \leq (\kappa\,\lambda)^p \tag{7.1.18}
$$

hold in some cylinder Q_R^λ*, with* $\lambda \geq 1$ *and for a constant* $\kappa \geq 1$*. Then there exists* $\chi \equiv \chi(n,p,\nu,L) > 1$ *such that*

$$\left(\fint_{Q^\lambda_{R/2}} (s+|Dv|)^{p\chi}\,dz\right)^{1/(p\chi)} \leq c\left(\fint_{Q^\lambda_R} (s+|Dv|)^q\,dz\right)^{1/q} \tag{7.1.19}$$

for any $q \in [1,p]$*, for a constants* c *depending on* n,p,ν,L,κ*.*

PROOF. The estimate

$$\left(\fint_{Q^\lambda_{R/2}} (s+|Dv|)^{p\chi}\,dz\right)^{1/(p\chi)} \leq c\left(\fint_{Q^\lambda_{3R/4}} (s+|Dv|)^p\,dz\right)^{1/p}$$

is deduced starting from [**96**, Proposition 4.1] and (7.1.18) very similarly to [**4**, Lemma 3]. At this point using (7.1.17) in the previous estimate gives (7.1.19). □

7.2. Comparison lemmata and merging the geometries

In this section we approach to the proof of Theorem 4.16, first collecting some comparison result and then showing how to accommodate the two geometries of the problem, as explained in the beginning of the Chapter.

First of all, from now on we will choose for the set $A \times I$ a cylinder $Q_R^\lambda(z_0) \Subset \Omega_T$ and we introduce therein the comparison function solution to the Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x,t,Dv) = 0 & \text{in } Q_R^\lambda;\\ v = u & \text{on } \partial_{\mathcal{P}} Q_R^\lambda, \end{cases} \tag{7.2.1}$$

where u is a solution to (4.3.1). Recall we are dealing with approximating, regular solutions $u \equiv u_k$; therefore existence and uniqueness of v are well known arguments (see [**54**]) and so it is the fact that $v \in u + C^0\big(\Lambda; L^2(B)\big) \cap L^p\big(\Lambda; W_0^{1,p}(B)\big)$ if $Q_R^\lambda = \Lambda \times B$. The following comparison result is [**104**, Lemma 4.1].

LEMMA 7.9. *Let* u *be a weak solution to* (4.3.1) *and let* v *be the comparison function defined in* (7.2.1)*. Then*

$$\left(\fint_{Q^\lambda_R} |Du-Dv|^q\,dz\right)^{1/q} \leq c\left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}} \tag{7.2.2}$$

for every $q \in \left[1, p-1+\dfrac{1}{N-1}\right)$ *and for a constant* $c \equiv c(n,p,\nu,q)$*.*

We note that this comparison estimate has a non-homogenous character as the elliptic corresponding ones, see [**123**, Lemma 9.5], since as we already said equation (4.3.1) does not show homogeneity. However this estimates perfectly fit our situation, once having intrinsic relations at hand, as we will see several times in the sequel. The first one is the following

LEMMA 7.10. *Let* u *be a weak solution to* (4.3.1) *and let the density condition* (4.3.2) *for some* $1 < \vartheta \leq N$ *hold. Moreover suppose that the intrinsic relation*

$$\left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|}\right]^{\frac{1}{m}} \leq \kappa\,\lambda \tag{7.2.3}$$

holds true for some constant $\kappa \in (0,1)$*, where* m *is defined in* (4.3.6)*. Then*

$$\left(\fint_{Q^\lambda_R} |Du-Dv|^q\,dz\right)^{1/q} \leq c_*\,\kappa^{\frac{N}{(N-1)(p-1)+1}}\,\lambda$$

for all $q \in \Big[1, p-1+\dfrac{1}{N-1}\Big)$, *for a constant* c_* *depending on* n, p, ν, c_d, q.

PROOF. From (7.2.3) and (4.3.2) we obviously have

$$\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|} \le (\kappa\,\lambda)^m \qquad \text{and} \qquad \frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|} \le c_d \lambda^{p-2} R^{-\vartheta}.$$

Hence by Lemma 7.9 for q as in the statement ,we deduce, using the previous relations

$$\begin{aligned}\left(\fint_{Q_R^\lambda} |Du - Dv|^q \, dz\right)^{1/q} &\le c \left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|^{\frac{N-1}{N}}}\right]^{\frac{N}{(N-1)(p-1)+1}} \\ &= c \left[\frac{|\mu|(Q_R^\lambda)}{|Q_R^\lambda|}\right]^{\alpha+\beta} |Q_R^\lambda|^{\frac{\alpha+\beta}{N}} \\ &\le c\,\kappa^{\alpha m}\,\lambda^{\alpha m + \beta(p-2) + \frac{2-p}{(N-1)(p-1)+1}} R^{-\beta\vartheta + \frac{N}{(N-1)(p-1)+1}},\end{aligned}$$

where α, β are two positive constants such that $\alpha + \beta = N/[(N-1)(p-1)+1]$. We can choose α and β in such a way that the exponent of R is zero and the exponent of λ is s, i.e.

$$\alpha := \left(1 - \frac{1}{\vartheta}\right)\frac{N}{(N-1)(p-1)+1}, \qquad \beta := \frac{1}{\vartheta}\frac{N}{(N-1)(p-1)+1}.$$

At this point a direct calculation shows that $\alpha m \ge N/[(N-1)(p-1)+1]$. □

At this point, in order to prove that the two geometries of the problem, we first a Poincaré-type estimate for the function v.

PROPOSITION 7.11. *Let* v *as in* (7.2.1) *and let* $Q_R^\lambda \equiv Q_R^\lambda(z_0) \subset A \times I$ *be a parabolic cylinder, not necessarily intrinsic. Then*

$$\fint_{Q_R^\lambda} \left|\frac{v - (v)_{Q_R^\lambda}}{R}\right|^q dz \le c \fint_{Q_R^\lambda} (s + |Dv|)^q \, dz + c\,\lambda^{q(2-p)} \left(\fint_{Q_R^\lambda} (s + |Dv|)^{p-1} \, dz\right)^q \tag{7.2.4}$$

for all $1 \le q \le p$, *for a constant* c *depending on* n, p, L, q.

PROOF. For the sake of readability we denote $Q_R^\lambda \equiv Q_R^\lambda(z_0) = B_R \times \Lambda_R^\lambda$. Take a positive weight function $\eta \in C_c^\infty(B_R)$ satisfying

$$\fint_{B_R} \eta \, dx = 1, \qquad \eta(x) + R|D\eta(x)| \le c(n) \quad \text{for all } x \in B_R$$

and define the weigthted mean of $v(\cdot, t)$ on B_R by

$$(v)^\eta_{B_R} := \fint_{B_R} u(\cdot, t)\eta \, dx.$$

Now we split the integral on the left-hand side of (7.2.4) in the following way:

$$\begin{aligned}\fint_{Q_R^\lambda} \left|\frac{v - (v)_{Q_R^\lambda}}{R}\right|^q dz &\le c(q) \fint_{Q_R^\lambda} \left|\frac{v - (v)^\eta_{B_R}(t)}{R}\right|^q dz \\ &+ \frac{c(q)}{R^q} \fint_{\Lambda_R^\lambda} \left|(v)^\eta_{B_R}(t) - \fint_{\Lambda_R^\lambda} (v)_{B_R}(\tau)\, d\tau\right|^q dt \\ &+ \frac{c(q)}{R^q} \left|\fint_{\Lambda_R^\lambda} (v)_{B_R}(\tau)\, d\tau - (v)_{Q_R^\lambda}\right|^q dt = I + II + III.\end{aligned}$$

We have, by a standard variation of Poincaré's inequality applied slice-wise

$$III \le I \le c(n,p) \fint_{Q_R^\lambda} |Dv|^q \, dz.$$

To estimate II we rather use the equation. Test indeed directly (7.2.1)$_1$ with the test function η independent of time: for a.e. $t_1, t_2 \in \Lambda_R^\lambda$

$$\begin{aligned}|(v)^\eta_{B_R}(t_1) - (v)^\eta_{B_R}(t_2)| &= \left|\int_{t_1}^{t_2} \partial_t\big[(v)^\eta_{B_R}(t)\big]\, dt\right| = \left|\int_{t_1}^{t_2} \fint_{B_R} \partial_t v \, \eta \, dx\, dt\right| \\ &= \left|\int_{t_1}^{t_2} \fint_{B_R} \langle a(\cdot, Dv), D\eta\rangle \, dx\, dt\right| \\ &\le \frac{c(n)L}{R} \int_{\Lambda_R^\lambda} \fint_{B_R} \left(s^2 + |Dv|^2\right)^{(p-1)/2} dz \\ &\le c(n,L)\, \lambda^{2-p} R \fint_{Q_R^\lambda} \left(s + |Dv|\right)^{p-1} dz,\end{aligned}$$

so

$$II \le c(n,L,q) \lambda^{q(2-p)} \left(\fint_{Q_R^\lambda} \left(s + |Dv|\right)^{p-1} dz \right)^q.$$

Note that the previous estimate is just formal: a precise proof can be done using a regularizing procedure in time, for example Steklov's averaging. See for example the analogous [**29**, Lemma 5.1] or [**71**, Lemma 4.11]. Merging together the estimates for I, II, III gives (7.2.4). □

The previous proposition immediately translates in the following corollary, once we know that the cylinder Q_R^λ is intrinsic:

COROLLARY 7.12. *Let v as in Proposition (7.11) and moreover let us suppose the intrinsic relation*

$$\fint_{Q_R^\lambda} \left(s + |Dv|\right)^{p-1} dz \le (\kappa\lambda)^{p-1}$$

is satisfied. Then

$$\fint_{Q_R^\lambda} \left| \frac{v - (v)_{Q_R^\lambda}}{R} \right|^{p-1} dz \le c \fint_{Q_R^\lambda} \left(s + |Dv|\right)^{p-1} dz$$

for a constant c depending on n, p, L, κ.

The following Proposition finally shows that the *weak* geometry for Dv is equivalent to the standard one.

PROPOSITION 7.13. *Let v be the weak solution to (7.1.1). Then if*

$$\left(\frac{\lambda}{\kappa_1}\right)^{p-1} \le \fint_{Q_{R/4}^\lambda} \left(s + |Dv|\right)^{p-1} dz, \quad \fint_{Q_R^\lambda} \left(s + |Dv|\right)^{p-1} dz \le (\kappa_2\lambda)^{p-1} \tag{7.2.5}$$

hold for constants $\kappa_1, \kappa_2 \ge 1$, then

$$\left(\frac{\lambda}{\kappa_1}\right)^{q} \le \fint_{Q_{R/4}^\lambda} \left(s + |Dv|\right)^{q} dz, \quad \fint_{Q_{R/2}^\lambda} \left(s + |Dv|\right)^{q} dz \le c\,(\kappa_2\lambda)^{q} \tag{7.2.6}$$

holds for every $q \in [p-1, p]$, with c depending on n, p, ν, L, κ_1 and κ_2.

PROOF. The first inequality of (7.2.6) follows immediately from (7.2.5). For the second one, we use Caccioppoli's inequality together with Hölder's and Young's inequalities to infer

$$\begin{aligned}
\fint_{Q^\lambda_{R/2}} (s+|Dv|)^q \, dx &\le c \left[\lambda^{p-2} \left(\fint_{Q^\lambda_{3R/4}} \left| \frac{v-(v)_{Q^\lambda_{3R/4}}}{R} \right|^p dz \right)^{2/p} \right. \\
&\qquad \left. + \fint_{Q^\lambda_{3R/4}} \left(\left| \frac{v-(v)_{Q^\lambda_{3R/4}}}{R} \right|^p + s^p \right) dz \right]^{q/p} \\
&\le c \left[\lambda^p + \fint_{Q^\lambda_{3R/4}} \left(\left| \frac{v-(v)_{Q^\lambda_{3R/4}}}{R} \right|^p + s^p \right) dz \right]^{q/p} \\
&\le \frac{c}{R^q} \left[\operatorname*{osc}_{Q^\lambda_{3R/4}} v \right]^q + c\, s^q + c\,(\kappa_2 \lambda)^q, \qquad (7.2.7)
\end{aligned}$$

where $c \equiv c(n,p,\nu,L,q)$. Now we estimate the oscillation using Corollary 7.2: we indeed apply (7.1.4) with $\varepsilon = 1$, $Q_{\rho,\sigma} \equiv Q^\lambda_R$, $\theta \equiv 3/4$, to the positive sub-solutions $w = (v-(v)_{Q^\lambda_{3R/4}})_\pm$; then we sum up the resulting inequalities and, noting that $Q^\lambda_{3R/4} \subset Q_{3R/4, 3\lambda^{2-p}R^2/4}$, we infer

$$\operatorname*{osc}_{Q^\lambda_{3R/4}} v \le c\,\lambda^{2-p} R \fint_{Q^\lambda_R} \left| \frac{v-(v)_{Q^\lambda_R}}{R} \right|^{p-1} dz + 2\lambda R + 2sR.$$

with $c \equiv c(n,p,\nu,L)$; note that we replaced the average in the right-hand side. Therefore, using Poincaré's inequality Corollary 7.12, and this is allowed since $(7.2.5)_2$ holds, we infer

$$\frac{1}{R} \operatorname*{osc}_{Q^\lambda_{3R/4}} v \le c\,\lambda^{2-p} \fint_{Q^\lambda_R} (s+|Dv|)^{p-1} \, dz + 2\lambda + 2s \le c\,\kappa_2 \lambda, \qquad (7.2.8)$$

using again $(7.2.5)_2$, which also yields $s \le \kappa_2\, \lambda$. Using again this fact together with (7.2.8) into (7.2.7) concludes the proof. □

7.3. Proof of Theorem 4.16

Finally we come to the proof of the Theorem. We take a parabolic cylinders $Q_{2R} \equiv Q_{2R}(\mathfrak{z}_0) \subset \Omega_T$, $R > 0$, and following [123] we let $M \ge 1$ be a free parameter to be chosen and we define the Calderón-Zygmund functional

$$CZ(\mathfrak{Q}) := \left(\fint_{\mathfrak{Q}/20} (|Du|+s)^{p-1} \, dz \right)^{\frac{1}{p-1}} + \left[M \frac{|\mu|(\mathfrak{Q}/20)}{|\mathfrak{Q}/20|} \right]^{\frac{1}{m}}$$

for cylinders $\mathfrak{Q} \equiv \mathfrak{Q}(z_0) \subset \Omega_T$, where

$$m := p-1+\frac{1}{\vartheta-1} > p-1.$$

Now, after fixing two radii $R \le r_1 < r_2 \le 2R$, we define

$$\lambda_0^{\frac{1}{p-1}} := \left(\fint_{Q_{r_2}} (|Du|+s)^{p-1} \, dz \right)^{\frac{1}{p-1}} + \left[M \frac{|\mu|(Q_{r_2})}{|Q_{r_2}|} \right]^{\frac{1}{m}} + 1, \qquad (7.3.1)$$

we take λ such that

$$\lambda > B\lambda_0 \qquad \text{where} \qquad B := \left(\frac{800 r_2}{r_2 - r_1} \right)^N \ge 1 \qquad (7.3.2)$$

and subsequently consider radii satisfying

$$\frac{r_2 - r_1}{40} \le r \le \frac{r_2 - r_1}{2}. \qquad (7.3.3)$$

Note that due to such a bound $Q_r^\lambda(z) \Subset Q_{r_2}$ for any $z \in Q_{r_1}$ and for all r satisfying (7.3.3). Hence by (7.3.1) and (7.3.2), enlarging the domain of integration, we have

$$CZ(Q_r^\lambda(z)) \le \left[\frac{|Q_{r_2}|}{|Q^\lambda_{r/20}|}\right]^{\frac{1}{p-1}} \lambda_0^{\frac{1}{p-1}} < \lambda^{\frac{p-2}{p-1}} \left(\frac{20 r_2}{r}\right)^{\frac{N}{p-1}} \lambda^{\frac{1}{p-1}} B^{-\frac{1}{p-1}} \le \lambda < 4\lambda. \tag{7.3.4}$$

Now we prove by Lebesgue's theorem that the converse inequality holds for cylinders centered in points where the gradient takes big values. More precisely, for $\lambda > 0$ and radii $\gamma \in [R, 2R]$, define the level sets

$$E(\lambda, \gamma) := \Big\{ z \in Q_\gamma(z_0) : |Du(z)| + s > \lambda \Big\}. \tag{7.3.5}$$

Note that the cylinder $Q_\gamma(z_0)$ have the same "vertex" as Q_R and Q_{2R}. Take then a point $z \in E(4\lambda, r_1)$ with $\lambda > B\lambda_0$. By Lebesgue's differentiation theorem, for almost every such points it holds

$$\lim_{r\searrow 0} CZ(Q_r^\lambda(z)) \ge \lim_{r \searrow 0} \left(\fint_{Q^\lambda_{r/20}(z)} (|Du| + s)^{p-1}\, dz \right)^{\frac{1}{p-1}} > 4\lambda.$$

Hence for small radii $0 < r \ll 1$ we have by continuity $CZ(Q_r^\lambda(z)) > 4\lambda$. From this consideration and the fact that (7.3.4) holds, together with the absolute continuity of the integral, we infer the existence of a maximal radius r_z such that

$$CZ(Q^\lambda_{r_z}) = \left(\fint_{Q^\lambda_{r_z/20}} (|Du| + s)^{p-1}\, dz \right)^{\frac{1}{p-1}} + \left[M \frac{|\mu|(Q^\lambda_{r_z/20})}{|Q^\lambda_{r_z/20}|} \right]^{\frac{1}{m}} = 4\lambda. \tag{7.3.6}$$

The word "maximal" refers to the fact that for all radii $\tilde r \in (r_z, (r_2 - r_1)/2]$ the inequality $CZ(Q^\lambda_{\tilde r})(z) < 4\lambda$ holds. In particular for $\tilde r = 20 r_z$ we have

$$\left(\fint_{Q^\lambda_{r_z}} (|Du| + s)^{p-1}\, dz \right)^{\frac{1}{p-1}} + \left[M \frac{|\mu|(Q^\lambda_{r_z})}{|Q^\lambda_{r_z}|} \right]^{\frac{1}{m}} = CZ\big(Q^\lambda_{20 r_z}\big) < 4\lambda. \tag{7.3.7}$$

Note also that obviously $r_z < (r_2 - r_1)/40$ and hence $Q^\lambda_{20 r_z}(z) \subset Q_{r_2}$.

A favorable case. Now we single out an intrinsic cylinder $Q^\lambda_{r_z}(z)$, $z \in E(4\lambda, r_1)$, $\lambda > B\lambda_0$ where (7.3.6) holds. For ease of notation, from now on let's denote $Q := Q^\lambda_{r_z}(z)$. Assume now that, in addition to condition (7.3.6), also

$$(2\lambda)^{p-1} \le \fint_{Q/20} (|Du| + s)^{p-1}\, dx \tag{7.3.8}$$

holds true. The reason for this additional assumption will become clear in the remainder of the proof.

We introduce now the comparison function solution to the Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x, t, Dv) = 0 & \text{in } Q; \\ v = u & \text{on } \partial_{\mathcal{P}} Q. \end{cases}$$

We apply Lemma 7.10 with $\kappa = 4^m/M$ and we get

$$\left(\fint_Q |Du - Dv|^q\, dz \right)^{1/q} \le \frac{c_*}{M^{\frac{N}{(N-1)(p-1)+1}}} \lambda \le c\,\lambda, \tag{7.3.9}$$

with c depending on $n, p, \nu, L, \vartheta, c_d$, for any $q \in \left[1, p - 1 + \dfrac{1}{N-1}\right)$.

The previous estimate (7.3.9) in particular holds for the choice $q = p-1$. Hence first we have

$$\fint_Q \big(|Dv|+s\big)^{p-1}\,dz \le 2^{p-2}\fint_Q \big(|Du|+s\big)^{p-1}\,dz + 2^{p-2}\fint_Q |Du-Dv|^{p-1}\,dz \le c\,\lambda^{p-1} \tag{7.3.10}$$

by (7.3.7) and (7.3.9); moreover

$$\fint_{Q/4} \big(|Dv|+s\big)^{p-1}\,dz \ge \frac{1}{2^{p-1}5^N}\fint_{Q/20} \big(|Du|+s\big)^{p-1}\,dz - 4^N\fint_Q |Du-Dv|^{p-1}\,dz \ge \frac{2}{5^N}\lambda^{p-1} - \frac{4^N c_*^{p-1}}{M}\lambda^{p-1},$$

since we are assuming (7.3.8), we can use (7.3.9) and $\frac{N(p-1)}{(N-1)(p-1)+1} \ge 1$. Now we impose that $M \ge 1$ is so big that

$$\frac{4^N c_*^{p-1}}{M} \le \frac{1}{5^N} \tag{7.3.11}$$

and this, making M depend on $n, p, \nu, c_d, \vartheta$, yields, together with (7.3.10)

$$\left(\frac{\lambda}{c}\right)^{p-1} \le \fint_{Q/4} \big(|Dv|+s\big)^{p-1}\,dz, \qquad \fint_Q \big(|Dv|+s\big)^{p-1}\,dz \le c\,\lambda^{p-1}; \tag{7.3.12}$$

therefore we can apply Proposition 7.13 which gives

$$\left(\frac{\lambda}{c}\right)^{q} \le \fint_{Q/4} \big(|Dv|+s\big)^{q}\,dx, \qquad \fint_{Q/2} \big(|Dv|+s\big)^{q}\,dx \le c\lambda^q \tag{7.3.13}$$

for all $q \in [p-1, p]$ and with a constant c depending on n, p, ν, L, c_d. Since in particular (7.3.13) holds for $q = p$, we are finally in position to apply Corollary 7.8:

$$\left(\fint_{Q_{R/4}} \big(s+|Dv|\big)^{p\chi}\,dz\right)^{1/(p\chi)} \le c\left(\fint_{Q_{R/2}} \big(s+|Dv|\big)^{p-1}\,dz\right)^{1/(p-1)} \le c\,\lambda, \tag{7.3.14}$$

for $\chi \equiv \chi(n, p, \nu, L) > 1$ and the constants c at this point depending only on n, p, ν, L, c_d. We moreover have, for any $q \in \left[p-1, p-1+\frac{1}{N-1}\right)$, using the previous (7.3.14) and (7.3.9)

$$\left(\fint_{Q_{R/4}} \big(s+|Du|\big)^{q}\,dz\right)^{1/q} \le \left(\fint_{Q_{R/4}} \big(s+|Dv|\big)^{q}\,dz\right)^{1/q} + \left(\fint_{Q_{R/4}} |Du-Dv|^{q}\,dz\right)^{1/q} \le c\,\lambda. \tag{7.3.15}$$

Splitting the intrinsic cylinder – a density estimate. First we show why the additional assumption (7.3.8) can be assumed and, at the same time, we make use of the results of the preceding section. Clearly, by the definition of the CZ operator and by (7.3.6), one of the following inequalities must hold true:

$$(2\lambda)^{p-1} \le \fint_{Q/20} \big(|Du|+s\big)^{p-1}\,dz \qquad \text{or} \qquad (2\lambda)^m \le M\frac{|\mu|(Q/20)}{|Q/20|}. \tag{7.3.16}$$

Suppose **we are in the first case**, so we can use the results of the previous section: we split the integral, observing that $Q/20 \equiv Q^{\lambda}_{r_z/20}(z) \subset Q_{r_2}$ and we use Hölder's inequality to infer

$$\begin{aligned}\fint_{Q/20} \big(|Du|+s\big)^{p-1}\,dx &\le \frac{|Q/20 \setminus E(\lambda,r_2)|}{|Q/20|}\lambda^{p-1}\\ &\quad+\frac{1}{|Q/20|}\int_{Q/20\cap E(\lambda,r_2)}\big(|Du|+s\big)^{p-1}\,dx\\ &\le \lambda^{p-1}+c\left(\frac{|Q/20\cap E(\lambda,r_2)|}{|Q|}\right)^{1-\frac{p-1}{\tilde q}}\left(\fint_{Q/20}\big(|Du|+s\big)^{\tilde q}\,dx\right)^{\frac{p-1}{\tilde q}},\end{aligned} \tag{7.3.17}$$

for $c \equiv c(n,p)$ and for the exponent

$$\tilde q := p-1+\frac{1}{2(N-1)} \in \Big(p-1, p-1+\frac{1}{N-1}\Big). \tag{7.3.18}$$

Plugging the density estimate (7.3.15) into (7.3.17) (the reader might recall now (7.3.18)) and taking into account the fact that $(7.3.16)_1$ holds, we infer

$$(2\lambda)^{p-1} \le \fint_{Q/20}\big(|Du|+s\big)^{p-1}\,dx \le \lambda^{p-1}\left[1+c\left(\frac{|Q/20\cap E(\lambda,r_2)|}{|Q|}\right)^{1-\frac{p-1}{\tilde q}}\right];$$

in turn, dividing by λ^{p-1} and reabsorbing the first term, we get

$$\frac{1}{c} \le \frac{|Q/20\cap E(\lambda,r_2)|}{|Q|},$$

with $c \equiv c(n,p,\nu,L)$. Merging this estimate with **the second alternative** $(7.3.16)_2$, we get finally the estimate for $|Q|$ we were looking for:

$$|Q| \le c\,|Q/20\cap E(\lambda,r_2)| + c\,\frac{M}{\lambda^m}|\mu|(Q/20). \tag{7.3.19}$$

We recall that $Q \equiv Q^{\lambda}_{r_z}(z)$.

A covering argument. The following covering argument has been developed in [96]. The technique we use here is very similar to the variant which can be found in [4]. We saw in the preceding step of the proof that, once we fix $\lambda > B\lambda_0$, then for every $z \in E(4\lambda, r_1)$ we can find a cylinder $Q^{\lambda}_{r_z}(z)$ such that (7.3.6) and subsequently (7.3.19) hold.

Then we consider the collection of all the cylinders $\mathcal{E}_\lambda := \{Q^{\lambda}_{r_z/20}(z)\}_{z\in E(4\lambda,r_1)}$ and, by a Vitali type argument, we can extract a countable sub-collection $\mathcal{F}_\lambda \subset \mathcal{E}_\lambda$ such that the 5-times enlarged cylinders cover almost all $E(4\lambda, r_1)$ and the cylinders are pairwise disjoints. I.e. if we denote the cylinders of $\mathcal{F}_\lambda$ by $Q^0_i := Q^{\lambda}_{r_{z_i}/20}(z_i)$, for $i \in \mathcal{I}_\lambda$, being eventually $\mathcal{I}_\lambda = \mathbb{N}$, and with $z_i \in E(4\lambda, r_1)$, we have

$$Q^0_i \cap Q^0_j = \emptyset \quad \text{whenever } i \neq j \quad \text{and} \quad E(4\lambda,r_1) \subset \bigcup_{i\in\mathcal{I}_\lambda} Q^1_i \cup \mathcal{N}, \tag{7.3.20}$$

with $|\mathcal{N}| = 0$. We denoted $Q^1_i := 5Q^0_i = Q^{\lambda}_{r_{z_i}/4}(z_i)$; note that by (7.3.3) we have the inclusion $Q^1_i \subset Q_{r_2}$ for all $i \in \mathcal{I}_\lambda$. We now fix $H \ge 4$ to be chosen later and we estimate

$$|E(H\lambda,r_1)| \le \sum_{i\in\mathcal{I}_\lambda} |Q^1_i \cap E(H\lambda,r_2)|,$$

where $\{Q^1_i\} = \mathcal{F}_\lambda$ is the family related to $E(4\lambda, r_1)$ just defined. We split every term in the following way:

$$|Q^1_i \cap E(H\lambda,r_2)| = \big|\{z \in Q^1_i : s+|Du(x)| > H\lambda\}\big| \tag{7.3.21}$$

$$\leq \left|\{z \in Q_i^1 : |Du(x) - Dv_i(x)| > H\lambda/2\}\right| \\ + \left|\{z \in Q_i^1 : s + |Dv_i(x)| > H\lambda/2\}\right| =: I_i + II_i.$$

Here v_i is the comparison function, solution to (7.2.1) with $Q \equiv Q_i^2 \equiv Q_{r_{z_i}}^{\lambda} = 4Q_i^1$. We estimate separately the two pieces: for the first one we use (7.3.9) and subsequently (7.3.19) to infer

$$I_i \leq \left(\frac{2}{H\lambda}\right)^{p-1} \int_{Q_i^2} |Du - Dv_i|^{p-1}\, dz \leq \frac{c}{(H\lambda)^{p-1} M} |Q_i^2| \lambda^{p-1} \\ \leq \frac{c}{H^{p-1}} \left[\frac{|Q_i^0 \cap E(\lambda, r_2)|}{M} + \frac{|\mu|(Q_i^0)}{\lambda^m} \right]. \tag{7.3.22}$$

On the other hand we use higher integrability (7.3.14) to get

$$II_i \leq \left(\frac{2}{H\lambda}\right)^{p\chi} \int_{Q_i^1} \big(|Dv_i| + s\big)^{p\chi}\, dz \leq \frac{c}{(H\lambda)^{p\chi}} |Q_i^2| \lambda^{p\chi} \\ \leq \frac{c}{H^{p\chi}} \left[|Q_i^0 \cap E(\lambda, r_2)| + M \frac{|\mu|(Q_i^0)}{\lambda^m} \right]. \tag{7.3.23}$$

Connecting the two estimates (7.3.22) and (7.3.23) and plugging into (7.3.21), taking into account that $H \geq 1$, gives

$$|E(H\lambda, r_2) \cap Q_i^1| \leq \left[\frac{c_2}{H^{p-1} M} + \frac{c_2}{H^{p\chi}} \right] |Q_i^0 \cap E(\lambda, r_2)| + c\, M \frac{|\mu|(Q_i^0)}{\lambda^m}.$$

At this point, since the $\{Q_i^0\}$ are disjoint, see (7.3.20), summing up and multiplying both sides of the previous inequality by $(H\lambda)^m$ gives

$$(H\lambda)^m |E(H\lambda, r_1)| \leq \left[\frac{c_*}{H^{p-1-m} M} + \frac{c_*}{H^{p\chi - m}} \right] \lambda^m |E(\lambda, r_2)| \\ + c\, M H^m |\mu|(Q_{2R}). \tag{7.3.24}$$

Finally we perform the choice of M and H: recall that $p - 1 < m < p\chi$ by (4.3.3). First choose H so big that

$$\frac{c_*}{H^{p\chi - m}} \leq \frac{1}{4} \qquad \text{and} \qquad H \geq 4.$$

Then at this point, having fixed $H \equiv H(n, p, \nu, L, c_d, \vartheta)$, choose $M \geq 1$ satisfying (7.3.11) and such that

$$\frac{c_*}{M} \leq \frac{1}{4} H^{p-1-m}.$$

This choice makes also M depend on $n, p, \nu, L, c_d, \vartheta$. Having such choices at hand, after taking the supremum with respect to $\lambda > B\lambda_0$, (7.3.24) rewrites as

$$\sup_{\lambda > HB\lambda_0} \lambda^m |E(\lambda, r_1)| \leq \frac{1}{2} \sup_{\lambda > B\lambda_0} \lambda^m |E(\lambda, r_2)| + c\, |\mu|(Q_{2R}) \\ \leq \frac{1}{2} \| |Du| + s \|_{\mathcal{M}^m(Q_{r_2})}^m + c\, |\mu|(Q_{2R}) \tag{7.3.25}$$

and therefore, recalling that $E(\lambda, \gamma)$ denotes the super-level set (7.3.5)

$$\| |Du| + s \|_{\mathcal{M}^m(Q_{r_1})}^m \leq \frac{1}{2} \| |Du| + s \|_{\mathcal{M}^m(Q_{r_2})}^m + c \left[B\lambda_0 \right]^m R^N + c\, |\mu|(Q_{2R})$$

for all $R \leq r_1 < r_2 \leq 2R$, since $B\lambda_0 \geq 1$. We now, recalling the definitions of λ_0 and B, apply Lemma 3.11 with $\phi(r) := \| |Du| + s \|_{\mathcal{M}^m(Q_r)}^m$, $\mathcal{A} := c\, |\mu|(Q_{2R})$,

$$\mathcal{B} = c\, R^{N(m+1)} \left(\left[\fint_{Q_{2R}} (s + |Du|)^{p-1}\, dz \right]^m + \left[\frac{|\mu|(Q_{2R})}{|Q_{2R}|} \right]^{p-1} + 1 \right)$$

and $\beta = Nm$. Note that this is possible since we are dealing with approximate energy solutions and therefore, since $Du \in L^{p\chi}_{\text{loc}}(\Omega_T)$, we have $\||Du| + s\|_{\mathcal{M}^m(Q_{2R})} < \infty$ for $m < p\chi$. This yields, using also Young's inequality with conjugate exponents $p-1$ and $(p-1)/(p-2)$

$$\||Du| + s\|^m_{\mathcal{M}^m(Q_R)} \le c \left[\frac{|\mu|(Q_{2R})}{|Q_{2R}|^{\frac{p-2}{p-1}}}\right]^{p-1} + c\,R^N + c\,R^N \left[\fint_{Q_{2R}} (s + |Du|)^{p-1}\,dz\right]^m$$

which finally gives (4.3.7).

PROOF OF COROLLARY 7.8. In the case we consider more regular vector fields as ones satisfying (4.3.9) or (4.3.8), higher integrability Corollary 7.8 holds for every $\chi > 1$, see [**71**, Theorems 1.8 & 1.9] and also [**21**, Theorem 5.7], with the constant appearing in the right-hand side depending critically upon χ. Therefore an argument similar to the one carried in Section 7.1 can be performed, in order to get that Corollary 7.8 holds for every $\chi > 1$ and with the constant depending also on χ. Now the only different points in Section 7.3 are two: in (7.3.24) now we can choose, given $\vartheta > 1$, $\chi(p, \vartheta)$ so big that $p\chi = m + 1$; this reflects in the critical dependence of the constant upon ϑ, as $\vartheta \to 1$. The same reason and the same [**71**, Theorems 1.8 & 1.9] justify the reabsorption after (7.3.25): since the data for the approximating problems are regular, at least L^∞, then the energy solution u_k, under assumptions (4.3.8) or (4.3.9), are as integrable as needed. □

CHAPTER 8

Calderón-Zygmund estimates for parabolic $p(x,t)$-Laplacian

Here we give the proof of the result of Section 4.5. We have already seen the problems that appear when dealing with parabolic p-Laplace equations or systems.

The case of non-standard growth is even more involved, the construction of a uniform system of intrinsic cylinders, as done in the previous Chapter, which would allow to certain covering argument as done in [4] is not anymore possible: the exponent p appearing in the scaling parameter λ^{2-p} and therefore the scaling would depend on the particular point z_0. This means that the scaling of the intrinsic cylinder will in fact depend on space and time, so that we have to deal with a *non uniform intrinsic geometry*. Additionally, the structure of the problem and of the proof requires to handle with cylinders different from those defined in (3.1.5), but essentially equivalent in the standard constant exponent case. We shall consider cylinders of the type

$$Q_\rho^{(\lambda)}(z_0) := B_\rho(x_0) \times \big(t_0 - \lambda^{\frac{2-p_0}{p_0}}\rho^2, t_0 + \lambda^{\frac{2-p_0}{p_0}}\rho^2\big), \tag{8.0.26}$$

where $p_0 := p(z_0)$, with an intrinsic coupling of the form (where for simplicity we omit the role of the right-hand side F):

$$\fint_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(z)}\, dz \approx \lambda.$$

Note that compared to (2.5.4) we performed a change of parameter $\lambda^{p_0} \leftrightarrow \lambda$, so that the right-hand side is independent of p_0 and hence independent of z_0. The main difficulty now comes from the fact that the heuristics we described above for the standard growth case do not apply for the case of non-standard growth, i.e. on $Q_\rho^{(\lambda)}(z_0)$ the parabolic $p(x,t)$-Laplacian system behaves like $\partial_t u = \lambda^{\frac{p(z)-2}{p(z)}}\Delta u$ such that the multiplicative factor $\lambda^{\frac{p(z)-2}{p(z)}}$ does not cancel out with the scaling factor $\lambda^{\frac{2-p_0}{p_0}}$. This problem will be solved by a parabolic localization argument which has its origin in [29], see Section 8.

Now, we briefly describe the strategy of proof of our main result. The technique we use goes back to [4]: since we have to work on a system of non uniform intrinsic cylinders of the type (8.0.26) there is no uniform maximal function available. Instead of maximal operators we construct a covering of the super level sets

$$\big\{|Du(z)|^{p(z)} > \lambda\big\}, \qquad \lambda \gg 1$$

by exit cylinders $Q_{\rho_i}^{(\lambda)}(z_i)$, $i = 1, \ldots, \infty$ defined according to (8.0.26) on which we have

$$\fint_{Q_{\rho_i}^{(\lambda)}(z_i)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz \approx \lambda.$$

Thereby, $M \gg 1$ is a suitably chosen parameter depending on the structural constants of the problem. Then, we know that

$$\fint_{Q_{\rho_i}^{(\lambda)}(z_i)} |Du|^{p(\cdot)}\, dz \lesssim \lambda \quad \text{and} \quad \fint_{Q_{\rho_i}^{(\lambda)}(z_i)} (|F|+1)^{p(\cdot)}\, dz \lesssim \frac{\lambda}{M}.$$

Therefore, if M is large u solves approximately (here we suppose $a \equiv 1$ for simplicity)

$$\partial_t u - \operatorname{div}\left(|Du|^{p(z)-2}Du\right) \approx 0 \qquad \text{on } Q^{(\lambda)}_{\rho_i}(z_i).$$

This heuristic suggests to compare u to the solution w of

$$\begin{cases} \partial_t w - \operatorname{div}\left(|Dw|^{p(z_i)-2}Dw\right) = 0 & \text{in } Q^{(\lambda)}_{\rho_i}(z_i), \\ w = u & \text{on } \partial_{\mathcal{P}} Q^{(\lambda)}_{\rho_i}(z_i). \end{cases}$$

To be precise, this will be done in a two step comparison argument. Here, we stress that the comparison argument strongly relies on the parabolic localization technique, since we replaced the variable exponent $p(z)-2$ by the constant exponent $p(z_i)-2$. The advantage now is that the theory of DiBenedetto and Friedman [55] ensures that Dw satisfies an a priori L^∞-estimate. Via the comparison argument this L^∞-estimate can be transferred into estimates for Du on the super level sets. At this stage the final result follows by a standard argument using Fubini's theorem.

Particular notations and tools

In this proof we shall employ some different, even slightly, objects respect to the ones used in the other pages of the manuscript and fixed in the notation Section 3.1. We collect them here, together with some additional observations regard our assumptions.

In particular note that by virtue of (4.4.4) we may assume that there exists $R_1 \in (0,1]$ depending on $\omega(\cdot)$ such that

$$\omega(\rho)\log\left(\frac{1}{\rho}\right) \leq 1 \quad \text{for all } \rho \in (0, R_1]. \tag{8.0.27}$$

The cylinders we are going to use in this proof as we already said slightly differ from the ones defined in (3.1.5). In particular we shall deal with scaled cylinders of the form

$$Q^{(\lambda)}_\rho(z_0) := B_\rho(x_0) \times \Lambda^{(\lambda)}_\rho(z_0), \tag{8.0.28}$$

where $\lambda > 0$ and

$$\Lambda^{(\lambda)}_\rho(z_0) := \left(t_0 - \lambda^{\frac{2-p_0}{p_0}}\rho^2, t_0 + \lambda^{\frac{2-p_0}{p_0}}\rho^2\right).$$

In any case, when considering a certain cylinder $Q^{(\lambda)}_\rho(z_0)$ with center z_0, by p_0 we denote the value of $p(\cdot)$ at the center of the cylinder, i.e. $p_0 \equiv p(z_0)$. Note that such a system of scaled cylinders is non-uniform in the sense that the scaling $\lambda^{\frac{2-p_0}{p_0}}$ depends on the particular point z_0 via $p_0 \equiv p(z_0)$. Also for the cylinders $Q^{(\lambda)}_R$, in the particular case $\lambda = 1$ the cylinders $Q^{(1)}_\rho(z_0)$ reduce to the standard parabolic ones, i.e. $Q^{(1)}_\rho(z_0) \equiv Q_\rho(z_0)$. By $\chi Q^{(\lambda)}_\rho(z_0)$, for a constant $\chi > 1$, we denote the χ-times enlarged cylinder, i.e. $\chi Q^{(\lambda)}_\rho(z_0) := Q^{(\lambda)}_{\chi\rho}(z_0)$.

For shortness of notation we will denote by the word $data$ exactly the set of parameters $n, N, \nu, L, \gamma_1, \gamma_2$, so that writing $c(data, M)$ we will mean that the constant c depends on $n, N, \nu, L, \gamma_1, \gamma_2$ and moreover upon M.

Non uniform intrinsic geometry. In the following lemma we provide a parabolic localization technique. Obviously the difficulty stems from the necessity to couple the technique of intrinsic geometry with the localization needed to treat the variable exponent growth conditions. As we already pointed out in the introduction, this will be achieved by a *non uniform intrinsic geometry*, i.e. a system of cylinders as defined in (8.0.28) whose scaling depends on the particular point considered. Most of this technique goes back to [29].

LEMMA 8.1. *Let $\kappa, K, M \geq 1$ and $p\colon \Omega_T \to [\gamma_1,\gamma_2]$ satisfy (4.6.6) and (8.0.27). Then there exists a radius $\rho_0 \equiv \rho_0(n,\gamma_1,\kappa,K,M,\omega(\cdot)) \in (0,R_1]$ such that the following holds: whenever $Du, F \in L^{p(\cdot)}(\Omega_T;\mathbb{R}^{Nn})$ satisfy (4.5.8) and $Q_\rho^{(\lambda)}(z_0) \subset \Omega_T$ is a parabolic cylinder with $\rho \in (0,\rho_0]$ and $\lambda \geq 1$ such that*

$$\lambda \leq \kappa \fint_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\,dz, \tag{8.0.29}$$

then we have

$$\lambda \leq \left(\frac{\Gamma}{4\rho^{n+2}}\right)^{\frac{p_0}{2}}, \quad p_2 - p_1 \leq \omega(\Gamma\rho^\alpha) \quad \text{and} \quad \lambda^{\omega(\Gamma\rho^\alpha)} \leq e^{\frac{3np_0}{\alpha}}, \tag{8.0.30}$$

where

$$p_0 := p(z_0), \quad p_1 := \inf_{Q_\rho^{(\lambda)}(z_0)} p(\cdot), \quad p_2 := \sup_{Q_\rho^{(\lambda)}(z_0)} p(\cdot)$$

and

$$\Gamma := 4\beta_n\kappa KM, \quad \beta_n := \max\{1,(2\omega_n)^{-1}\}, \quad \alpha := \min\Big\{1, \gamma_1\frac{n+2}{4} - \frac{n}{2}\Big\}. \tag{8.0.31}$$

PROOF. We first deduce from (8.0.29), (4.5.8) (recall that $Q_\rho^{(\lambda)}(z_0) \subset \Omega_T$) and the definitions of Γ and β_n in (8.0.31) the following bound for λ:

$$\lambda \leq \frac{\kappa KM}{|Q_\rho^{(\lambda)}(z_0)|} = \frac{\kappa KM}{2\omega_n\rho^{n+2}}\lambda^{\frac{p_0-2}{p_0}} \leq \frac{\beta_n\kappa KM}{\rho^{n+2}}\lambda^{\frac{p_0-2}{p_0}} = \frac{\Gamma}{4\rho^{n+2}}\lambda^{\frac{p_0-2}{p_0}}.$$

Rewriting this inequality we obtain $(8.0.30)_1$. Now, we come to the proof of $(8.0.30)_2$. We define

$$\rho_0 := R_1^{\frac{1}{\alpha}}\Gamma^{-\frac{2}{\alpha}} \leq R_1 \leq 1 \tag{8.0.32}$$

and assume that $\rho \leq \rho_0$. Keeping in mind the definition of α and Γ this determines ρ_0 as a constant depending on $n,\gamma_1,K,M,\kappa,\omega(\cdot)$. From (4.6.6) and the fact that $\lambda \geq 1$ we obtain the following preliminary bound for the oscillation of $p(\cdot)$ on $Q_\rho^{(\lambda)}(z_0)$:

$$p_2 - p_1 \leq \omega\big(2\rho + \sqrt{2}\lambda^{\frac{2-p_0}{2p_0}}\rho\big) \leq \omega\big(2\rho + \sqrt{2}\lambda^{\frac{2-\gamma_1}{2p_0}}\rho\big).$$

In the case $\gamma_1 \geq 2$ this leads us to

$$p_2 - p_1 \leq \omega(4\rho),$$

while in the case $\frac{2n}{n+2} < \gamma_1 < 2$ we infer from $(8.0.30)_1$ that

$$p_2 - p_1 \leq \omega\big(4\lambda^{\frac{2-\gamma_1}{2p_0}}\rho\big) \leq \omega\Big(4\big(\tfrac{\Gamma}{4}\big)^{\frac{2-\gamma_1}{4}}\rho^{1-\frac{(2-\gamma_1)(n+2)}{4}}\Big) \leq \omega\big(\Gamma\rho^{\gamma_1\frac{n+2}{4}-\frac{n}{2}}\big).$$

Note that the restriction $\gamma_1 > \frac{2n}{n+2}$ ensures that the exponent of ρ is positive, i.e. $\gamma_1\frac{n+2}{4} - \frac{n}{2} > 0$. Combining the estimates from the cases $\gamma_1 \geq 2$ and $\gamma_1 < 2$ and recalling that $\rho \leq 1$ we arrive at:

$$p_2 - p_1 \leq \omega(\Gamma\rho^\alpha),$$

which proves $(8.0.30)_2$. Finally, we come to the proof of $(8.0.30)_3$. Using the definition of ρ_0 in (8.0.32) and the logarithmic bound (8.0.27) (which is applicable since $R_1/\Gamma \leq R_1$) we obtain

$$\Gamma^{\omega(\Gamma\rho^\alpha)} \leq \Gamma^{\omega(\Gamma\rho_0^\alpha)} \leq \Gamma^{\omega(R_1/\Gamma)} \leq \big(\tfrac{\Gamma}{R_1}\big)^{\omega(R_1/\Gamma)} = \exp\big[\omega\big(\tfrac{R_1}{\Gamma}\big)\log\big(\tfrac{\Gamma}{R_1}\big)\big] \leq e.$$

Moreover, by a similar reasoning and using the last inequality we get

$$\rho^{-\omega(\Gamma\rho^\alpha)} = \Gamma^{\frac{\omega(\Gamma\rho^\alpha)}{\alpha}}(\Gamma\rho^\alpha)^{-\frac{\omega(\Gamma\rho^\alpha)}{\alpha}} \leq e^{\frac{1}{\alpha}}(\Gamma\rho^\alpha)^{-\frac{\omega(\Gamma\rho^\alpha)}{\alpha}}$$

$$= e^{\frac{1}{\alpha}} \exp\left[\tfrac{\omega(\Gamma\rho^\alpha)}{\alpha} \log \tfrac{1}{\Gamma\rho^\alpha}\right] \le e^{\frac{2}{\alpha}}.$$

At this stage (8.0.30)$_3$ follows from (8.0.30)$_1$ and the previous two inequalities since

$$\lambda^{\omega(\Gamma\rho^\alpha)} \le \left(\Gamma\rho^{-(n+2)}\right)^{\frac{p_0\omega(\Gamma\rho^\alpha)}{2}} \le e^{\frac{p_0}{2}+\frac{p_0(n+2)}{\alpha}} \le e^{\frac{3np_0}{\alpha}}.$$

This completes the proof of the lemma. □

Since the family of intrinsic cylinders is non uniform – in the sense that the scaling depends on the center of the cylinder – we need the following non uniform version of Vitali's covering theorem, which can be found in [29, Lemma 7.1]. Note that we may choose $L_1 = 1$ due to assumption (8.0.27) and that we replaced M by KM which is more suitable in our setting.

LEMMA 8.2. *Let $K, M, \lambda \ge 1$ and let $p : \Omega_T \to [\gamma_1, \gamma_2]$ fulfill assumptions (4.6.6) and (8.0.27). Then there exists $\chi \equiv \chi(n, \gamma_1) \ge 5$ and $\rho_1 = \rho_1(n, \gamma_1, K, M) \in (0, 1]$ such that the following is true: Let $\mathcal{F} = \{Q_i\}_{i\in\mathcal{I}}$ be a family of axially parallel parabolic cylinders of the form*

$$Q_i \equiv Q^{(\lambda)}_{\rho_i}(z_i) \equiv B_{\rho_i}(x_i) \times \left(t_i - \lambda^{\frac{2-p(z_i)}{p(z_i)}}\rho_i^2, t_i + \lambda^{\frac{2-p(z_i)}{p(z_i)}}\rho_i^2\right)$$

with uniformly bounded radii, in the sense that there holds

$$\rho_i \le \min\left\{\rho_1, \left[\beta_n KM\lambda^{-\frac{2}{p(z_i)}}\right]^{\frac{1}{n+2}}\right\} \qquad \forall\, i \in \mathcal{I} \tag{8.0.33}$$

with β_n defined in (8.0.31). Then there exists a countable subcollection $\mathcal{G} \subset \mathcal{F}$ of disjoint parabolic cylinders, such that

$$\bigcup_{Q\in\mathcal{F}} Q \subset \bigcup_{Q\in\mathcal{G}} \chi Q.$$

8.1. Higher integrability and a priori estimates

In this Section we provide a higher integrability result for solutions to homogeneous parabolic $p(x,t)$-Laplacian systems that will be crucial later in the proof of certain comparison estimates. We consider the parabolic system

$$\partial_t v - \operatorname{div}\left(a(z)|Dv|^{p(z)-2}Dv\right) = 0 \qquad \text{on } A \times (t_1, t_2) =: \mathfrak{A}, \tag{8.1.1}$$

where $A \subset \mathbb{R}^n$ is an open set and $t_1 < t_2$. Then, we have the following higher integrability result from [29, Theorem 2.2].

THEOREM 8.3. *Suppose that $p\colon \mathfrak{A} \to [\gamma_1, \gamma_2]$ satisfies (4.6.6) and (8.0.27) and that $a\colon \mathfrak{A} \to \mathbb{R}$ satisfies (4.5.4). Then there exists $\epsilon_0 \equiv \epsilon_0(data) > 0$ such that the following holds: whenever a function $v \in L^2(\mathfrak{A}, \mathbb{R}^N) \cap L^1(t_1, t_2; W^{1,1}(A, \mathbb{R}^N))$ with $Dv \in L^{p(\cdot)}(\mathfrak{A}, \mathbb{R}^{Nn})$ is a weak solution to the parabolic system (8.1.1) on $\mathfrak{A}$, we have that*

$$Dv \in L^{p(\cdot)(1+\epsilon_0)}_{\text{loc}}\left(\mathfrak{A}, \mathbb{R}^{Nn}\right). \tag{8.1.2}$$

Moreover, for any $K \ge 1$ there exists a radius $\rho_2 \equiv \rho_2(n, \gamma_1, \gamma_2, K, \omega(\cdot)) \in (0, R_1]$ such that there holds: If

$$\int_{\mathfrak{A}} (|Dv| + 1)^{p(\cdot)}\, dz \le K \tag{8.1.3}$$

and $\epsilon \in (0, \epsilon_0]$, then for any parabolic cylinder $Q_{2\rho}(z_0) \subset \mathfrak{A}$ with $\rho \in (0, \rho_2]$ we have

$$\fint_{Q_\rho(z_0)} |Dv|^{p(\cdot)(1+\epsilon)}\, dz \le c\left(\fint_{Q_{2\rho}(z_0)} |Dv|^{p(\cdot)}\, dz\right)^{1+\epsilon d(p(z_0))} + c \tag{8.1.4}$$

for a constant $c \equiv c(data)$ and with $d(\cdot)$ defined in (4.5.10).

Note that the quantitative higher integrability estimate (8.1.4) is non homogeneous, in the sense that the exponents of $|Dv|$ on both sides of the inequality are different. In the following Corollary we deduce a homogeneous version of this estimate valid on intrinsic cylinders of the type (8.1.5). In order to understand that inequality (8.1.7) is homogeneous one has to interpret $\lambda \approx \fint |Dv|^{p(\cdot)}\,dz$ in a heuristic sense which will become clear later on.

COROLLARY 8.4. *Let $K, c_*, \hat{c} \geq 1$ and suppose that $p\colon \mathfrak{A} \to [\gamma_1, \gamma_2]$ satisfies (4.6.6) and (8.0.27) and that $a\colon \mathfrak{A} \to \mathbb{R}$ fulfills (4.5.4). Then, there exist $\epsilon_0 \equiv \epsilon_0(data) > 0$, $c \equiv c(data, c_*, \hat{c}) \geq 1$ and $\rho_2 \equiv \rho_2(n, \gamma_1, \gamma_2, K, \omega(\cdot)) \in (0, R_1]$ such that the following holds: whenever $v \in L^2(\mathfrak{A}, \mathbb{R}^N) \cap L^1(t_1, t_2; W^{1,1}(A, \mathbb{R}^N))$ with $Dv \in L^{p(\cdot)}(\mathfrak{A}, \mathbb{R}^{Nn})$ is a weak solution to the parabolic system (8.1.1) satisfying (8.1.3) and*

$$\fint_{Q_{2\rho}^{(\lambda)}(z_0)} |Dv|^{p(\cdot)}\,dz \leq c_* \lambda \tag{8.1.5}$$

for some cylinder $Q_{2\rho}^{(\lambda)}(z_0) \subset \mathfrak{A}$ with $\rho \in (0, \rho_2]$ and $\lambda \geq 1$ satisfying

$$\lambda^{p_2 - p_1} \leq \hat{c}, \qquad \text{where } p_1 := \inf_{Q_{2\rho}^{(\lambda)}(z_0)} p(\cdot), \quad p_2 := \sup_{Q_{2\rho}^{(\lambda)}(z_0)} p(\cdot), \tag{8.1.6}$$

then we have (8.1.2) and

$$\fint_{Q_{\rho}^{(\lambda)}(z_0)} |Dv|^{p(\cdot)(1+\epsilon_0)}\,dz \leq c\,\lambda^{1+\epsilon_0}. \tag{8.1.7}$$

PROOF. Without loss of generality we assume that $z_0 = 0$. We let ϵ_0 and ρ_2 be the constants appearing in Theorem 8.3. The strategy now is to rescale the problem from $Q_\rho^{(\lambda)}, Q_{2\rho}^{(\lambda)}$ to the standard parabolic cylinders $Q_\rho, Q_{2\rho}$ via a transformation in time and then apply Theorem 8.3. We start with the case $p_0 := p(0) \geq 2$ and define for $(x,t) \in Q_{2\rho}$ the rescaled exponent

$$\tilde{p}(x,t) := p\big(x, \lambda^{\frac{2-p_0}{p_0}} t\big),$$

the rescaled function

$$\tilde{v}(x,t) := \lambda^{-\frac{1}{p_0}}\, v\big(x, \lambda^{\frac{2-p_0}{p_0}} t\big)$$

and the rescaled coefficient

$$\tilde{a}(x,t) := \lambda^{\frac{\tilde{p}(x,t)-p_0}{p_0}}\, a\big(x, \lambda^{\frac{2-p_0}{p_0}} t\big).$$

Then, $\tilde{v}$ is a weak solution of the parabolic system

$$\partial_t \tilde{v} - \operatorname{div}\big(\tilde{a}(\cdot)|D\tilde{v}|^{\tilde{p}(\cdot)-2} D\tilde{v}\big) = 0 \quad \text{in } Q_{2\rho}. \tag{8.1.8}$$

In order to apply the higher integrability Theorem 8.3 to $\tilde{v}$ we have to ensure that the hypotheses on $\tilde{p}$ and $\tilde{a}$ are satisfied. Since $p_0 \geq 2$ and $\lambda \geq 1$ we have

$$\begin{aligned}
|\tilde{p}(x_1,t_1) - \tilde{p}(x_2,t_2)| &= \big|p\big(x_1, \lambda^{\frac{2-p_0}{p_0}} t_1\big) - p\big(x_2, \lambda^{\frac{2-p_0}{p_0}} t_2\big)\big| \\
&\leq \omega\Big(\max\big\{|x_1 - x_2|, \lambda^{\frac{2-p_0}{2p_0}} \sqrt{|t_1 - t_2|}\big\}\Big) \\
&\leq \omega\Big(\max\big\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\big\}\Big) \\
&= \omega\Big(d_{\mathcal{P}}\big((x_1,t_1),(x_2,t_2)\big)\Big).
\end{aligned} \tag{8.1.9}$$

Moreover by (4.5.4) and (8.1.6) it holds that

$$\frac{\nu}{\hat{c}} \leq \nu\,\lambda^{-\frac{p_2-p_1}{p_0}} \leq \tilde{a}(x,t) \leq L\,\lambda^{\frac{p_2-p_1}{p_0}} \leq \hat{c}\,L. \tag{8.1.10}$$

Therefore, we are allowed to apply Theorem 8.3 with $(\nu/\hat{c}, \hat{c}\, L)$ instead of (ν, L) to the function $\tilde{v}$ on $Q_\rho, Q_{2\rho}$ to infer that $D\tilde{v} \in L^{p(\cdot)(1+\epsilon_0)}_{\text{loc}}(Q_{2\rho}, \mathbb{R}^{Nn})$ and moreover the following quantitative estimate holds:

$$\fint_{Q_\rho} |D\tilde{v}|^{\tilde{p}(\cdot)(1+\epsilon_0)}\, dz \le c\Big(\fint_{Q_{2\rho}} |D\tilde{v}|^{\tilde{p}(\cdot)}\, dz\Big)^{1+\epsilon_0 d(p_0)} + c$$

for a constant $c \equiv c(data)$. Note that $p_0 = p(0) = \tilde{p}(0)$. Scaling back from v to $\tilde{v}$ and back and using the preceding estimate, (8.1.5) and (8.1.6) several times we find that

$$\begin{aligned}
\fint_{Q_\rho^{(\lambda)}} |Dv|^{p(\cdot)(1+\epsilon_0)}\, dz &= \fint_{Q_\rho} \lambda^{\frac{\tilde{p}(\cdot)}{p_0}(1+\epsilon_0)} |D\tilde{v}|^{\tilde{p}(\cdot)(1+\epsilon_0)}\, dz \\
&\le c\,\lambda^{1+\epsilon_0} \fint_{Q_\rho} |D\tilde{v}|^{\tilde{p}(\cdot)(1+\epsilon_0)}\, dz \\
&\le c\,\lambda^{1+\epsilon_0}\Big[\Big(\fint_{Q_{2\rho}} |D\tilde{v}|^{\tilde{p}(\cdot)}\, dz\Big)^{1+\epsilon_0 d(p_0)} + 1\Big] \\
&= c\,\lambda^{1+\epsilon_0}\Big[\Big(\fint_{Q_{2\rho}^{(\lambda)}} \lambda^{-\frac{p(\cdot)}{p_0}} |Dv|^{p(\cdot)}\, dz\Big)^{1+\epsilon_0 d(p_0)} + 1\Big] \\
&\le c\,\lambda^{\epsilon_0(1-d(p_0))}\Big(\fint_{Q_{2\rho}^{(\lambda)}} |Dv|^{p(\cdot)}\, dz\Big)^{1+\epsilon_0 d(p_0)} + c\,\lambda^{1+\epsilon_0} \\
&\le c(data, c_*, \hat{c})\,\lambda^{1+\epsilon_0}.
\end{aligned} \tag{8.1.11}$$

This proves the lemma in the case $p_0 \ge 2$. In the case $p_0 < 2$ we define similarly as above

$$\tilde{p}(x,t) := p\big(\lambda^{\frac{p_0-2}{2p_0}} x, t\big), \quad \tilde{v}(x,t) := \lambda^{-\frac{1}{2}}\, v\big(\lambda^{\frac{p_0-2}{2p_0}} x, t\big)$$

and

$$\tilde{a}(x,t) := \lambda^{\frac{\tilde{p}(x,t)-p_0}{p_0}}\, a\big(\lambda^{\frac{p_0-2}{2p_0}} x, t\big) \tag{8.1.12}$$

for $(x,t) \in Q_{2\tilde{\rho}}$, where $\tilde{\rho} := \lambda^{\frac{2-p_0}{2p_0}}\rho$. A straightforward computation shows that

$$D\tilde{v}(x,t) = \lambda^{-\frac{1}{p_0}} Dv\big(\lambda^{\frac{p_0-2}{2p_0}} x, t\big) \qquad \text{in } Q_{2\tilde{\rho}}$$

and that $\tilde{v}$ is a weak solution of the system (8.1.8) in $Q_{2\tilde{\rho}}$, where $\tilde{a}$, $\tilde{v}$, $\tilde{p}$ are this time the quantities defined just above. Notice that estimate (8.1.10) holds also for the vector field defined in (8.1.12), while the verification of (8.1.9) in this case is analoguous to the previous one. Applying again Theorem 8.3 and repeating the computations in (8.1.11) we obtain the assertion of the lemma also in the case $p_0 < 2$. □

In the next Theorem we state the gradient bound of DiBenedetto and Friedman [54, 55] for parabolic standard growth problems. Later on, we will transfer these a priori estimates via a comparison argument to our non-standard growth problem. Therefore, in this Section we consider parabolic systems with constant p-growth of the type

$$w_t - \operatorname{div}\big(\tilde{a}(t)|Dw|^{p-2}Dw\big) = 0 \qquad \text{on } A \times (t_1, t_2) =: \mathfrak{A}, \tag{8.1.13}$$

with $p > 2n/(n+2)$ and $\tilde{a}\colon (t_1, t_2) \to \mathbb{R}$. Thereby, A is an open set in $\mathbb{R}^n$ and $t_1 < t_2$. Moreover, we denote

$$\mathfrak{Q}_\rho^{(\lambda)}(z_0) := B_\rho(x_0) \times \big(t_0 - \lambda^{\frac{2-p}{p}}\rho^2, t_0 + \lambda^{\frac{2-p}{p}}\rho^2\big).$$

Note that the scaling of this system of cylinders does not depend on the center z_0. Later on, we will apply the subsequent Theorem with the choice $p = p_0 \equiv p(z_0)$, and hence the cylinders $\mathfrak{Q}_\rho^{(\lambda)}(z_0)$ will coincide with the ones defined in (8.0.28). As mentioned above, the next Theorem is a consequence of the gradient bounds proved in [55, 54]. The precise statement for the case $p \geq 2$ can be found in [4, Lemma 1] (replacing λ by $\lambda^{\frac{1}{p}}$), and for the case $2n/(n+2) < p < 2$ in [4, Lemma 2] (replacing ρ by $\lambda^{\frac{p-2}{2}}\rho$ and subsequently λ by $\lambda^{\frac{1}{p}}$).

THEOREM 8.5. *Let $w \in C^0(t_1,t_2;L^2(A,\mathbb{R}^N)) \cap L^p(t_1,t_2;W^{1,p}(A,\mathbb{R}^N))$ be a weak solution to (8.1.13) in $\mathfrak{A}$ with $\tilde{a}\colon (t_1,t_2) \to \mathbb{R}$ satisfying $\nu \leq \tilde{a} \leq L$ for some constants $0 < \nu \leq 1 \leq L$. Moreover suppose that*

$$\fint_{\mathfrak{Q}_{2\rho}^{(\lambda)}(z_0)} |Dw|^p \, dz \leq c_* \lambda$$

holds for some cylinder $\mathfrak{Q}_{2\rho}^{(\lambda)}(z_0) \Subset \mathfrak{A}$, where c_* is a given positive constant. Then there exists a constant $c_{DiB} \geq 1$, depending on n, N, p, ν, L and c_* such that

$$\sup_{\mathfrak{Q}_{\rho}^{(\lambda)}(z_0)} |Dw| \leq c_{DiB}\, \lambda^{\frac{1}{p}}.$$

8.2. Comparison estimates

In this Section we prove two different comparison estimates. The first one compares the weak solution u of the original inhomogeneous parabolic system (4.5.1) to the solution v of the associated homogeneous parabolic system (8.2.2) below. The second one compares v to the solution w of the frozen parabolic system (8.2.14). Both, the parabolic localization Lemma 8.1 and the homogeneous version of the higher integrability estimate from Corollary 8.4 will be crucial in order to achieve homogeneous comparison estimates.

Now, we let $K \geq 1$ and suppose that (4.5.8) is satisfied. Next, we fix $\kappa, M \geq 1$ to be specified later. In the following we consider a cylinder $Q := Q_\rho^{(\lambda)}(z_0)$ with $z_0 = (x_0,t_0) \in \Omega_T$, $\rho \in (0,1]$ and $\lambda \geq 1$ defined according to (8.0.28) and which satisfies $2Q := Q_{2\rho}^{(\lambda)}(z_0) \Subset \Omega_T$ and

$$\frac{\lambda}{\kappa} \leq \fint_{2Q} |Du|^{p(\cdot)} \, dz + \fint_{2Q} M(|F|+1)^{p(\cdot)} \, dz \leq \lambda. \tag{8.2.1}$$

Moreover, we abbreviate $B := B_\rho(x_0)$ and $\Lambda := \Lambda_\rho^{(\lambda)}(t_0)$ so that $Q \equiv B \times \Lambda$ and define

$$p_0 := p(z_0), \quad p_1 := \inf_{2Q} p(\cdot) \quad \text{and} \quad p_2 := \sup_{2Q} p(\cdot).$$

By $v \in L^2(2Q,\mathbb{R}^N) \cap L^1(2\Lambda;W^{1,1}(2B,\mathbb{R}^N))$ with $Dv \in L^{p(\cdot)}(2Q,\mathbb{R}^{Nn})$ we denote the unique solution of the homogeneous initial-boundary value problem

$$\begin{cases} \partial_t v - \operatorname{div}\big(a(z)|Dv|^{p(z)-2}Dv\big) = 0 & \text{in } 2Q, \\ v = u & \text{on } \partial_{\mathcal{P}} 2Q. \end{cases} \tag{8.2.2}$$

Thereby, the parabolic boundary $\partial_{\mathcal{P}}2Q$ is given by $\partial_{\mathcal{P}}2Q := (\partial 2B \times 2\Lambda) \cup (\overline{2B} \times \{t_0 - \lambda^{2-p_0}(2\rho)^2\})$. Note that the existence of v can be inferred from [15] by small modifications. Our first aim is to prove suitable energy and comparison estimates for the comparison function v. We hence subtract the weak formulation of the parabolic system (8.2.2) from the one of (4.5.1) given in (4.5.7). This yields

$$\int_{2Q} (u-v)\cdot \partial_t \varphi \, dz - \int_{2Q} a(\cdot)\big\langle |Du|^{p(\cdot)-2}Du - |Dv|^{p(\cdot)-2}Dv, D\varphi\big\rangle \, dz = \int_{2Q} \big\langle |F|^{p(\cdot)-2}F, D\varphi\big\rangle \, dz$$

for any $\varphi \in C_0^\infty(2Q,\mathbb{R}^N)$. For $\theta > 0$ and $\tau := t_0 + \lambda^{\frac{2-p_0}{p_0}}(2\rho)^2$ we define

$$\chi_\theta(t) := \begin{cases} 1 & \text{on } (-\infty, \tau-\theta], \\ -\frac{1}{\theta}(t-\tau) & \text{on } (\tau-\theta,\tau), \\ 0 & \text{on } [\tau,\infty). \end{cases} \tag{8.2.3}$$

Since $Du - Dv \in L^{p(\cdot)}(2Q,\mathbb{R}^{Nn})$ and $u = v$ on $\partial_{\mathcal{P}}2Q$ in the sense of traces, we are (formally) allowed to choose $\varphi = (u-v)\chi_\theta$ in the preceding identity. We note that the argument can be made rigorous via the use of Steklov averages and an approximation argument; since this is standard we omit the details. This choice of φ together with the observation that

$$\begin{aligned} \int_{2Q} (u-v)\cdot\partial_t[(u-v)\chi_\theta]\,dz &= -\int_{2Q} \partial_t(u-v)\cdot(u-v)\chi_\theta\,dz \\ &= -\frac{1}{2}\int_{2Q} \partial_t|u-v|^2\chi_\theta\,dz \\ &= \frac{1}{2}\int_{2Q} |u-v|^2\partial_t\chi_\theta\,dz \\ &= -\frac{1}{2\theta}\int_{\tau-\theta}^{\tau}\int_{2B} |u-v|^2\,dz \\ &\overset{\theta\downarrow 0}{\to} -\frac{1}{2}\int_{2B} |u-v|^2(\cdot,\tau)\,dx \le 0 \end{aligned} \tag{8.2.4}$$

leads us after letting $\theta \downarrow 0$ to

$$\begin{aligned} \int_{2Q} a(\cdot)\big\langle |Du|^{p(\cdot)-2}Du - |Dv|^{p(\cdot)-2}Dv, D(u-v)\big\rangle\,dz \\ \le -\int_{2Q} \big\langle |F|^{p(\cdot)-2}F, D(u-v)\big\rangle\,dz. \end{aligned} \tag{8.2.5}$$

This inequality will be used in the following in two different directions. The first one will lead to an energy inequality for Dv. Rearranging terms and taking into account that $\nu \le a(\cdot) \le L$ we find

$$\begin{aligned} \nu\int_{2Q} |Dv|^{p(\cdot)}\,dz &\le L\int_{2Q} \big(|Du|^{p(\cdot)-1}|Dv| + |Dv|^{p(\cdot)-1}|Du|\big)\,dz \\ &\quad + \int_{2Q} |F|^{p(\cdot)-1}\big(|Du| + |Dv|\big)\,dz \\ &\le \frac{\nu}{2}\int_{2Q} |Dv|^{p(\cdot)}\,dz + c\int_{2Q} \big(|Du|^{p(\cdot)} + |F|^{p(\cdot)}\big)\,dz, \end{aligned}$$

where in the last line we applied Young's inequality and $c = c(\nu, L, \gamma_1, \gamma_2)$. Reabsorbing the first integral of the right-hand side into the left and subsequently using (8.2.1) we get the following *energy estimate for Dv*:

$$\int_{2Q} |Dv|^{p(\cdot)}\,dz \le c\int_{2Q} \big(|Du|^{p(\cdot)} + |F|^{p(\cdot)}\big)\,dz \le c(data)\,\lambda|Q|. \tag{8.2.6}$$

We now come to the proof of the comparison estimate. Starting again from (8.2.5) we use Lemma 3.15 and Young's inequality to infer

$$\begin{aligned} \nu\int_{2Q} \big(|Du|^2 + |Dv|^2\big)^{\frac{p(\cdot)-2}{2}}|Du-Dv|^2\,dz \\ \le c\int_{2Q} |F|^{p(\cdot)-1}\big(|Du| + |Dv|\big)\,dz \end{aligned}$$

$$\leq c\, M^{-\frac{\gamma_1-1}{\gamma_1}} \int_{2Q} \big(|Du|^{p(\cdot)} + |Dv|^{p(\cdot)} + M|F|^{p(\cdot)}\big)\, dz,$$

where the constant c depends on $n, N, L, \gamma_1, \gamma_2$. Finally, using (8.2.1) and the energy estimate (8.2.6) this leads us to the *first comparison estimate* we were looking for:

$$\int_{2Q} \big(|Du|^2 + |Dv|^2\big)^{\frac{p(\cdot)-2}{2}} |Du - Dv|^2\, dz \leq c(data)\, M^{-\frac{\gamma_1-1}{\gamma_1}} \lambda |Q|. \tag{8.2.7}$$

We now let $\epsilon_0 = \epsilon_0(data) > 0$ be the higher integrability exponent from Corollary 8.4 and set

$$\rho_3 := \min\{\rho_0/2, \rho_2\} \in (0,1],$$

where ρ_0 is the radius appearing in the localization Lemma 8.1 and ρ_2 the one for the higher integrability from Corollary 8.4. Note that ρ_3 depends on $data, \kappa, K, M, \omega(\cdot)$. In the course of the proof we shall further reduce the value of ρ_3 when necessary, but without changing its dependencies. In the following we assume that

$$\rho \leq \rho_3.$$

Thanks to assumption (8.2.1) we are allowed to apply Lemma 8.1 on $2Q$ which yields that

$$p_2 - p_1 \leq \omega\big(\Gamma(2\rho)^\alpha\big) \quad \text{and} \quad \lambda^{p_2-p_1} \leq \lambda^{\omega(\Gamma(2\rho)^\alpha)} \leq e^{\frac{3np_0}{\alpha}} \leq e^{\frac{3n\gamma_2}{\alpha}}, \tag{8.2.8}$$

where Γ and α are defined according to (8.0.31). Note that for the second estimate we also used that $\lambda \geq 1$. Therefore, assumption (8.1.6) of Corollary 8.4 is satisfied with $\hat{c} \equiv \hat{c}(n, \gamma_1, \gamma_2) := e^{\frac{3n\gamma_2}{\alpha}}$. Due to the energy estimate (8.2.6) we know that also assumption (8.1.5) is satisfied with c_* replaced by the constant $c(data)$ from (8.2.6). The application of the Corollary then ensures that $Dv \in L^{p(\cdot)(1+\epsilon_0)}(Q, \mathbb{R}^{Nn})$ and moreover

$$\fint_Q |Dv|^{p(\cdot)(1+\epsilon_0)}\, dz \leq c(data)\, \lambda^{1+\epsilon_0}. \tag{8.2.9}$$

Next, we reduce the value of ρ_3 in such a way that

$$\omega\big(\Gamma(2\rho_3)^\alpha\big) \leq \frac{\epsilon_1}{\gamma_1'}, \quad \text{where } \epsilon_1 := \sqrt{1+\epsilon_0} - 1 \leq \epsilon_0 \tag{8.2.10}$$

is satisfied. Then, by (8.2.8), for any $z \in 2Q$ there holds

$$\begin{aligned} p_0(1+\epsilon_1) &\leq p(z)\big(1 + \omega\big(\Gamma(2\rho)^\alpha\big)\big)(1+\epsilon_1) \\ &\leq p(z)\big(1 + \omega\big(\Gamma(2\rho_3)^\alpha\big)\big)(1+\epsilon_1) \\ &< p(z)(1+\epsilon_1)^2 = p(z)(1+\epsilon_0) \end{aligned}$$

and therefore we have $Dv \in L^{p_0(1+\epsilon_1)}(Q, \mathbb{R}^{Nn})$ together with the estimate

$$\begin{aligned} \fint_Q |Dv|^{p_0(1+\epsilon_1)}\, dz &\leq \fint_Q |Dv|^{p(\cdot)(1+\omega(\Gamma(2\rho)^\alpha))(1+\epsilon_1)}\, dz + 1 \\ &\leq \bigg(\fint_Q |Dv|^{p(\cdot)(1+\epsilon_0)}\, dz\bigg)^{\frac{(1+\omega(\Gamma(2\rho)^\alpha))(1+\epsilon_1)}{1+\epsilon_0}} + 1 \\ &\leq c\, \lambda^{(1+\omega(\Gamma(2\rho)^\alpha))(1+\epsilon_1)} + 1 \\ &= c\, \lambda^{1+\epsilon_1} \lambda^{\omega(\Gamma(2\rho)^\alpha)(1+\epsilon_1)} + 1 \\ &\leq c(data)\, \lambda^{1+\epsilon_1}, \end{aligned} \tag{8.2.11}$$

where we used Hölder's inequality, (8.2.9), (8.2.8) and the fact that $\lambda \geq 1$. For later reference we also provide the following estimate using (8.2.8) and (8.2.10):

$$p_0'(p_2-1) = p_0\Big(1+\frac{p_2-p_0}{p_0-1}\Big) \leq p_0\Big(1+\frac{\omega(\Gamma(2\rho_3)^\alpha)}{\gamma_1-1}\Big)$$
$$\leq p_0\big(1+\tfrac{\epsilon_1}{\gamma_1}\big) < p_0(1+\epsilon_1). \tag{8.2.12}$$

Together with (8.2.11), Hölder's inequality and (8.2.8) this implies

$$\fint_Q |Dv|^{p_0'(p_2-1)}\,dz \leq \Big(\fint_Q |Dv|^{p_0(1+\epsilon_1)}\,dz\Big)^{\frac{p_2-1}{(p_0-1)(1+\epsilon_1)}}$$
$$\leq c\,\lambda^{\frac{p_2-1}{p_0-1}} = c\,\lambda^{1+\frac{p_2-p_0}{p_0-1}} \leq c(data)\,\lambda. \tag{8.2.13}$$

We now define

$$\tilde a(t) := (a)_{x_0,\rho}(t) := \fint_{B_\rho(x_0)} a(\cdot,t)\,dx \qquad \text{for any } t\in(0,T).$$

Note that $\nu \leq \tilde a(t) \leq L$ for any $t \in (0,T)$ as a consequence of (4.5.4). By

$$w \in C^0\big(\Lambda, L^2(B;\mathbb{R}^N)\big) \cap L^{p_0}\big(\Lambda, W^{1,p_0}(B;\mathbb{R}^N)\big)$$

we denote the unique solution to the initial-boundary value problem

$$\begin{cases} \partial_t w - \operatorname{div}\big(\tilde a(t)|Dw|^{p_0-2}Dw\big) = 0 & \text{in } Q,\\ w = v & \text{on } \partial_{\mathcal P}Q. \end{cases} \tag{8.2.14}$$

We now start deriving energy and comparison estimates for w. As before, we subtract the weak formulations of (8.2.2) and (8.2.14) and test the result with $\varphi := (v-w)\chi_\theta$, where χ_θ is defined in (8.2.3). Here, we recall that $Dv \in L^{p_0}(Q,\mathbb{R}^{Nn})$ by (8.2.11) and therefore φ is (formally) admissible as a test function. Proceeding as before, i.e. treating the terms involving the time derivatives with the argument performed in (8.2.4) and passing to the limit $\theta \downarrow 0$ we obtain

$$\int_Q \big\langle a(\cdot)|Dv|^{p(\cdot)-2}Dv - \tilde a(t)|Dw|^{p_0-2}Dw, D(v-w)\big\rangle\,dz \leq 0. \tag{8.2.15}$$

Firstly, we shall use this inequality to get an energy estimate for Dw. Rearranging terms, taking into account that $\nu \leq a(\cdot), \tilde a(\cdot) \leq L$ and applying Young's inequality we find

$$\nu \fint_Q |Dw|^{p_0}\,dz \leq L \fint_Q \big(|Dw|^{p_0-1}|Dv| + |Dv|^{p(\cdot)-1}|Dw|\big)\,dz$$
$$\leq \frac{\nu}{2}\fint_Q |Dw|^{p_0}\,dz + c\fint_Q \big(|Dv|^{p_0} + |Dv|^{p_0'(p(\cdot)-1)}\big)\,dz$$

with a constant $c = c(\nu, L, \gamma_1, \gamma_2)$. Reabsorbing the first integral of the right-hand side into the left and using Hölder's inequality, (8.2.11), (8.2.13) and the fact that $\lambda \geq 1$ we get the following *energy estimate for Dw*:

$$\fint_Q |Dw|^{p_0}\,dz \leq c\Big[\fint_Q |Dv|^{p_0}\,dz + \fint_Q |Dv|^{p_0'(p_2-1)}\,dz + 1\Big] \leq c(data)\,\lambda. \tag{8.2.16}$$

In order to obtain a comparison estimate we once again start from (8.2.15) which can be rewritten as follows:

$$\fint_Q \tilde a(t)\big\langle |Dv|^{p_0-2}Dv - |Dw|^{p_0-2}Dw, D(v-w)\big\rangle\,dz$$
$$\leq \fint_Q \big(\tilde a(t) - a(\cdot)\big)\big\langle |Dv|^{p_0-2}Dv, D(v-w)\big\rangle\,dz$$

$$+ \fint_Q a(\cdot)\big\langle |Dv|^{p_0-2}Dv - |Dv|^{p(\cdot)-2}Dv, D(v-w)\big\rangle\,dz.$$

Using Lemma 3.15 and the fact that $\nu \le \tilde a(\cdot) \le L$ we obtain

$$\begin{aligned}
&\fint_Q \big(|Dv|^2+|Dw|^2\big)^{\frac{p_0-2}{2}}|Dv-Dw|^2\,dz\\
&\quad\le c\bigg[\fint_Q |\tilde a(t)-a(\cdot)||Dv|^{p_0-1}|Dv-Dw|\,dz\\
&\qquad + \fint_Q \big||Dv|^{p_0-1}-|Dv|^{p(\cdot)-1}\big||Dv-Dw|\,dz\bigg] =: c\,\big[I+II\big],
\end{aligned} \tag{8.2.17}$$

where $c = c(\nu, L, \gamma_1, \gamma_2)$. Now we estimate separately the two terms. For the first one we use Hölder's inequality several times, (8.2.11), (8.2.16), the fact that $a(\cdot), \tilde a(t) \le L$, (4.5.5) and $\tilde\omega \le 1$ to infer that

$$\begin{aligned}
I &\le c\bigg(\fint_Q |a(t)-a(\cdot)|^{p_0'}|Dv|^{p_0}\,dz\bigg)^{\frac{1}{p_0'}}\bigg(\fint_Q \big(|Dv|^{p_0}+|Dw|^{p_0}\big)\,dz\bigg)^{\frac{1}{p_0}}\\
&\le c\bigg(\fint_Q |a(t)-a(\cdot)|^{\frac{p_0'(1+\epsilon_1)}{\epsilon_1}}\,dz\bigg)^{\frac{\epsilon_1}{p_0'(1+\epsilon_1)}}\bigg(\fint_Q |Dv|^{p_0(1+\epsilon_1)}\,dz\bigg)^{\frac{1}{p_0'(1+\epsilon_1)}}\lambda^{\frac{1}{p_0}}\\
&\le c\,\lambda^{\frac{1}{p_0'}+\frac{1}{p_0}}\big[\tilde\omega(\rho)\big]^{\frac{\epsilon_1}{\gamma_1'(1+\epsilon_1)}} \le c(data)\,\big[\tilde\omega(\rho)\big]^{\frac{\epsilon_1}{2\gamma_1'}}\lambda.
\end{aligned}$$

In order to estimate II we first use (8.2.8) to find that for any $z\in Q$ and $b\ge 0$ there holds

$$\begin{aligned}
|b^{p_0-1}-b^{p(z)-1}| &\le |p_0-p(z)| \sup_{\sigma\in[p_1-1,p_2-1]} b^\sigma|\log b|\\
&\le \omega\big(\Gamma(2\rho)^\alpha\big)\,\Big[b^{p_2-1}\log\big(e+b^{p_0'(p_2-1)}\big)+\frac{1}{e(\gamma_1-1)}\Big],
\end{aligned}$$

where in the last line we used $b^\sigma|\log b| \le \frac{1}{\mathrm{e}(\gamma_1-1)}$ for $b\in[0,1]$ and $\sigma\in[p_1-1,p_2-1]$ and $b^\sigma|\log b| \le b^{p_2-1}\log(e+b^{p_0'(p_2-1)})$ for $b>1$ and $\sigma\in[p_1-1,p_2-1]$. This together with Hölder's inequality, (8.2.11) and (8.2.16) yields

$$\begin{aligned}
II &\le c\,\omega\big(\Gamma(2\rho)^\alpha\big)\fint_Q \Big[|Dv|^{p_2-1}\log\big(e+|Dv|^{p_0'(p_2-1)}\big)+1\Big]|Dv-Dw|\,dz\\
&\le c\,\omega\big(\Gamma(2\rho)^\alpha\big)\bigg(\fint_Q \Big[|Dv|^{p_2-1}\log\big(e+|Dv|^{p_0'(p_2-1)}\big)+1\Big]^{p_0'}dz\bigg)^{\frac{1}{p_0'}}\\
&\qquad\cdot\bigg(\fint_Q |Dv-Dw|^{p_0}\,dz\bigg)^{\frac{1}{p_0}}\\
&\le c\,\omega\big(\Gamma(2\rho)^\alpha\big)\bigg(\fint_Q \Big[|Dv|^{p_2-1}\log\big(e+|Dv|^{p_0'(p_2-1)}\big)+1\Big]^{p_0'}dz\bigg)^{\frac{1}{p_0'}}\lambda^{\frac{1}{p_0}},
\end{aligned}$$

where $c = c(data)$. Next, we note that the monotonicity of the logarithm implies

$$\log(e+ab) \le \log(e+a)+\log(e+b) \qquad \forall\, a,b\ge 0,$$

which together with Young's inequality allows to further estimate II as follows:

$$\begin{aligned}
II &\le c\,\omega\big(\Gamma(2\rho)^\alpha\big)\,\lambda^{\frac{1}{p_0}}\bigg[\fint_Q |Dv|^{p_0'(p_2-1)}\log^{p_0'}\bigg(e+\frac{|Dv|^{p_0'(p_2-1)}}{\big(|Dv|^{p_0'(p_2-1)}\big)_Q}\bigg)dz\\
&\qquad + \log^{p_0'}\Big(e+\big(|Dv|^{p_0'(p_2-1)}\big)_Q\Big)\fint_Q |Dv|^{p_0'(p_2-1)}\,dz+1\bigg]^{\frac{1}{p_0'}}
\end{aligned}$$

$$= c(data)\, \omega\big(\Gamma(2\rho)^{\alpha}\big)\, \lambda^{\frac{1}{p_0}} \big[II_1 + II_2 + 1\big]^{\frac{1}{p_0'}}, \tag{8.2.18}$$

with the obvious meaning of II_1 and II_2. In order to estimate II_1 we apply inequality (3.2.6) with the choices $g = |Dv|^{p_0'(p_2-1)}$ and

$$\sigma := \frac{1+\epsilon_1}{1+\frac{\epsilon_1}{\gamma_1}} = c(data) > 1$$

to infer that

$$II_1 \le c(data)\bigg(\fint_Q |Dv|^{p_0'(p_2-1)\sigma}\, dz\bigg)^{\frac{1}{\sigma}}.$$

To the integral on the right-hand side we apply Hölder's inequality (which is justified by (8.2.12)). Subsequently using (8.2.11) and (8.2.8) we obtain

$$\begin{aligned} II_1 &\le c\bigg(\fint_Q |Dv|^{p_0(1+\epsilon_1/\gamma_1)\sigma}\, dz\bigg)^{\frac{1}{\sigma}\cdot\frac{p_2-1}{(p_0-1)(1+\epsilon_1/\gamma_1)}} \\ &= c\bigg(\fint_Q |Dv|^{p_0(1+\epsilon_1)}\, dz\bigg)^{\frac{p_2-1}{(p_0-1)(1+\epsilon_1)}} \\ &\le c\,\lambda^{\frac{p_2-1}{p_0-1}} = c\,\lambda^{1+\frac{p_2-p_0}{p_0-1}} \le c(data)\,\lambda. \end{aligned}$$

Now, we come to the estimate for II_2 in (8.2.18). From (8.2.13) and $(8.0.30)_1$ we get

$$\big(|Dv|^{p_0'(p_2-1)}\big)_Q = \fint_Q |Dv|^{p_0'(p_2-1)}\, dz \le c\,\lambda \le c(data,\kappa)\left(\frac{KM}{\rho^{n+2}}\right)^{\frac{p_0}{2}}.$$

Using this estimate, again (8.2.13), the fact that $\log(cx) \le c\log(x)$ for $c \ge 1$ and that we can always assume $c(KM/\rho^{n+2})^{p_0/2} \ge \mathrm{e}$ by possibly reducing the value of ρ_3 we find

$$II_2 \le \log^{p_0'}\Bigg(e + c\left(\frac{KM}{\rho^{n+2}}\right)^{\frac{p_0}{2}}\Bigg)\fint_Q |Dv|^{p_0'(p_2-1)}\, dz \le c\, M^{p_0'}\log^{p_0'}\left(\frac{K}{\rho}\right)\lambda$$

with $c = c(data,\kappa)$. Joining the estimates for II_1 and II_2 with (8.2.18) we end up with

$$II \le c(data,\kappa)\,\omega\big(\Gamma(2\rho)^{\alpha}\big) M\log\left(\frac{K}{\rho}\right)\lambda.$$

Merging the preceding estimates for I and II into (8.2.17) we get

$$\begin{aligned} &\fint_Q \big(|Dv|^2 + |Dw|^2\big)^{\frac{p_0-2}{2}} |Dv - Dw|^2 dz \\ &\qquad \le c(data,\kappa)\Big[\omega\big(\Gamma(2\rho)^{\alpha}\big) M\log\left(\frac{K}{\rho}\right) + \big[\tilde\omega(\rho)\big]^{\frac{\epsilon_1}{2\gamma_1'}}\Big]\lambda. \end{aligned}$$

Here, we still want to replace the exponent p_0 in the integral on the left-hand side by $p(\cdot)$. This is achieved with the help of Hölder's inequality as follows:

$$\begin{aligned} &\fint_{\frac12 Q} \big(|Dv|^2 + |Dw|^2\big)^{\frac{p(\cdot)-2}{2}} |Dv - Dw|^2\, dz \\ &\quad \le \bigg(\fint_{\frac12 Q} \big(|Dv|^2 + |Dw|^2\big)^{\frac{p_0-2}{2}} |Dv - Dw|^2\, dz\bigg)^{\frac12} \\ &\qquad \cdot \bigg(\fint_{\frac12 Q} \big(|Dv|^2 + |Dw|^2\big)^{\frac{2p(\cdot)-p_0-2}{2}} |Dv - Dw|^2\, dz\bigg)^{\frac12} \\ &\quad \le c\Big[\omega\big(\Gamma(2\rho)^{\alpha}\big) M\log\left(\frac{K}{\rho}\right) + \big[\tilde\omega(\rho)\big]^{\frac{\epsilon_1}{2\gamma_1'}}\Big]^{\frac12}\lambda^{\frac12} \end{aligned}$$

$$\cdot \left(\fint_{\frac{1}{2}Q} |Dv|^{2p(\cdot)-p_0} + |Dw|^{2p(\cdot)-p_0}\, dz \right)^{\frac{1}{2}}.$$

In order to further estimate the integral on the right-hand side we use that fact that $2p(\cdot) - p_0 \le p(\cdot)(1+\omega(\Gamma(2\rho)^\alpha)) \le p(\cdot)(1+\omega(\Gamma\rho_3^\alpha)) \le p(\cdot)(1+\epsilon_0)$ which is a consequence of (8.2.10), Hölder's inequality, (8.2.9), (8.2.8) and $\lambda \ge 1$ to infer that

$$\begin{aligned} \fint_{\frac{1}{2}Q} |Dv|^{2p(\cdot)-p_0}\, dz &\le 2^{n+2} \fint_Q |Dv|^{p(\cdot)(1+\omega(\Gamma(2\rho)^\alpha))}\, dz + 1 \\ &\le 2^{n+2} \left(\fint_Q |Dv|^{p(\cdot)(1+\epsilon_0)}\, dz \right)^{\frac{1+\omega(\Gamma(2\rho)^\alpha)}{1+\epsilon_0}} + 1 \\ &\le c\, \lambda^{1+\omega(\Gamma(2\rho)^\alpha)} + 1 \le c(data,\kappa)\, \lambda. \end{aligned}$$

Moreover, since the parabolic system $(8.2.14)_1$ is of the same type as (8.1.13) we are allowed by (8.2.16) to apply Theorem 8.5 which yields that

$$\sup_{\frac{1}{2}Q} |Dw| \le c_{DiB}\, \lambda^{\frac{1}{p_0}}. \tag{8.2.19}$$

Note that c_{DiB} initially depends on n, N, ν, L, p_0. Since the dependence upon p_0 is continuous it can be replaced by a larger constant depending on γ_1 and γ_2 instead of p_0, i.e. $c_{DiB} = c_{DiB}(data)$. Therefore, using (8.2.19) and (8.2.8) we can bound also the integral involving Dw in terms of λ. Inserting this above we deduce the *second comparison estimate* we were looking for:

$$\begin{aligned} &\int_{\frac{1}{2}Q} \left(|Dv|^2 + |Dw|^2\right)^{\frac{p(\cdot)-2}{2}} |Dv - Dw|^2\, dz \\ &\quad \le c(data,\kappa) \left[\omega\big(\Gamma(2\rho)^\alpha\big) M \log\left(\frac{K}{\rho}\right) + \left[\tilde{\omega}(\rho)\right]^{\frac{\epsilon_1}{2\gamma_1'}} \right]^{\frac{1}{2}} \lambda |Q|. \end{aligned} \tag{8.2.20}$$

Note that this estimate holds for any cylinder $\frac{1}{2}Q \equiv Q^{(\lambda)}_{\rho/2}(z_0)$ with $\lambda \ge 1$ and $\rho \in (0, \rho_3]$ such that $2Q$ satisfies the intrinsic relation (8.2.1) and $2Q \Subset \Omega_T$. We recall that $\rho_3 \in (0,1]$ depends on $data, \kappa, K, M, \omega(\cdot)$.

8.3. Proof of the Calderón-Zygmund estimate

This Section is devoted to the proof of Theorem 4.20. We shall proceed in several steps.

A stopping-time argument and estimates on intrinsic cylinders. Here, we shall construct a covering of the upper level set of $|Du|^{p(\cdot)}$ with respect to some parameter λ by intrinsic cylinders. The argument uses a certain *stopping time argument* which takes its origin in [96] together with the non uniform version of Vitali's covering argument from Lemma 8.2.

We let $K \ge 1$ and suppose that (4.5.8) is satisfied and consider a standard parabolic cylinder $Q_R \equiv Q_R(\mathfrak{z}_0)$ such that $Q_{2R} \Subset \Omega_T$. Then, we fix $M \ge 1$ to be specified later and define

$$\lambda_0 := \left[\fint_{Q_{2R}} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz \right]^d \ge 1, \quad \text{where } d := \sup_{Q_{2R}} d(p(\cdot)) \tag{8.3.1}$$

and $d(\cdot)$ is defined according to (4.5.10). Next, as in [4, Section 4], we fix two numbers $R \le r_1 < r_2 \le 2R$ such that $Q_R \subset Q_{r_1} \subset Q_{r_2} \subset Q_{2R}$, all the cylinders sharing the same center $\mathfrak{z}_0$. In the following we shall consider λ such that

$$\lambda > B\lambda_0, \qquad \text{where } B := \Big(\frac{8\chi R}{r_2 - r_1}\Big)^{(n+2)d} \tag{8.3.2}$$

and for $z_0 \in Q_{r_1}$ we consider radii ρ satisfying

$$\min\big\{1, \lambda^{\frac{p_0-2}{2p_0}}\big\}\frac{r_2 - r_1}{4\chi} \le \rho \le \min\big\{1, \lambda^{\frac{p_0-2}{2p_0}}\big\}\frac{r_2 - r_1}{2}, \tag{8.3.3}$$

where $p_0 := p(z_0)$ and $\chi \equiv \chi(n, \gamma_1) \ge 5$ denotes the constant appearing in Lemma 8.2. Note that these choices of λ and ρ ensure that $Q_\rho^{(\lambda)}(z_0) \subset Q_{r_2}$ for any $z_0 \in Q_{r_1}$. Next, we want to prove that for any $z_0 \in Q_{r_1}$ there holds

$$\fint_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz < \lambda. \tag{8.3.4}$$

Indeed, enlarging the domain of integration from $Q_\rho^{(\lambda)}(z_0)$ to Q_{2R} and recalling the definition of λ_0 from (8.3.1) we infer that

$$\begin{aligned}
&\fint_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz \\
&\qquad \le \frac{|Q_{2R}|}{|Q_\rho^{(\lambda)}(z_0)|} \fint_{Q_{2R}} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz \\
&\qquad = \Big(\frac{2R}{\rho}\Big)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d}}.
\end{aligned}$$

Now we distinguish the cases $p_0 \ge 2$ and $p_0 < 2$. If $p_0 \ge 2$, then $1/d \le 1/d(p_0) = 2/p_0$ and $\min\{1, \lambda^{\frac{p_0-2}{2p_0}}\} = 1$, so that, using also the choice of ρ from (8.3.3) we obtain

$$\begin{aligned}
\fint_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz &\le \Big(\frac{8\chi R}{r_2 - r_1}\Big)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d}} \\
&< B^{\frac{1}{d}} \lambda^{\frac{p_0-2}{p_0}} B^{-\frac{1}{d}} \lambda^{\frac{1}{d}} = \lambda^{\frac{p_0-2}{p_0}} \lambda^{\frac{1}{d}} \le \lambda.
\end{aligned}$$

If $\gamma_1 \le p_0 < 2$, we have $1/d \le 1/d(p_0) = 1 - n(2-p_0)/(2p_0)$ and $\min\{1, \lambda^{\frac{p_0-2}{2p_0}}\} = \lambda^{\frac{p_0-2}{2p_0}}$ and therefore we get

$$\begin{aligned}
&\fint_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz \le \Big(\frac{8\chi R \lambda^{\frac{2-p_0}{2p_0}}}{r_2 - r_1}\Big)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d}} \\
&\qquad = B^{\frac{1}{d}} \lambda^{\frac{n(2-p_0)}{2p_0}} \lambda_0^{\frac{1}{d}} < B^{\frac{1}{d}} \lambda^{\frac{n(2-p_0)}{2p_0}} B^{-\frac{1}{d}} \lambda^{\frac{1}{d}} = \lambda^{\frac{n(2-p_0)}{2p_0}} \lambda^{\frac{1}{d}} \le \lambda.
\end{aligned}$$

Hence, in any case we proved that (8.3.4) holds.

For λ as in (8.3.2) we consider the upper level set

$$E(\lambda, r_1) := \big\{z \in Q_{r_1} : z \text{ is a Lebesgue point of } |Du| \text{ and } |Du(z)|^{p(z)} > \lambda\big\}.$$

In the following we show that also a converse inequality holds true for small radii and for points $z_0 \in E(\lambda, r_1)$. Indeed, by Lebesgue's differentiation theorem (see [29, (7.9)]) we infer for any $z_0 \in E(\lambda, r_1)$ that

$$\lim_{\rho \downarrow 0} \fint_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz \ge |Du(z_0)|^{p(z_0)} > \lambda.$$

From the preceding reasoning we conclude that the last inequality yields a radius for which the considered integral takes a value larger than λ, while (8.3.4) states that the integral is smaller than λ for any radius satisfying (8.3.3). Therefore, the continuity of the integral yields the existence of a maximal radius ρ_{z_0} in between, i.e.

$$0 < \rho_{z_0} < \min\left\{1, \lambda^{\frac{p_0-2}{2p_0}}\right\} \frac{r_2 - r_1}{4\chi} \tag{8.3.5}$$

such that

$$\fint_{Q^{(\lambda)}_{\rho_{z_0}}(z_0)} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz = \lambda. \tag{8.3.6}$$

By saying that ρ_{z_0} is maximal we mean that for every $\rho \in (\rho_{z_0}, \min\{1, \lambda^{\frac{p_0-2}{2p_0}}\}(r_2 - r_1)/2]$ inequality (8.3.4) holds. With this choice of ρ_{z_0} we define concentric parabolic cylinders centered at $z_0 \in E(\lambda, r_1)$ as follows:

$$\begin{aligned} Q^0_{z_0} &:= Q^{(\lambda)}_{\rho_{z_0}}(z_0), \quad Q^1_{z_0} := Q^{(\lambda)}_{\chi\rho_{z_0}}(z_0), \\ & \qquad Q^2_{z_0} := Q^{(\lambda)}_{2\chi\rho_{z_0}}(z_0), \quad Q^3_{z_0} := Q^{(\lambda)}_{4\chi\rho_{z_0}}(z_0). \end{aligned} \tag{8.3.7}$$

Then, we have $Q^0_{z_0} \subset Q^1_{z_0} \subset Q^2_{z_0} \subset Q^3_{z_0} \subset Q_{r_2}$ and for $j \in \{0, \ldots, 3\}$ there holds

$$\frac{\lambda}{(4\chi)^{n+2}} \le \fint_{Q^j_{z_0}} |Du|^{p(\cdot)} + M(|F|+1)^{p(\cdot)}\, dz \le \lambda. \tag{8.3.8}$$

Note that the upper bound follows from (8.3.6) and the maximal choice of the stopping radius ρ_{z_0}, while the lower bound follows from (8.3.6) by enlarging the domain of integration from $Q^0_{z_0}$ to $Q^j_{z_0}$ and taking into account that $|Q^j_{z_0}|/|Q^0_{z_0}| \le (4\chi)^{n+2}$.

We now fix one particular cylinder $Q^0_{z_0}$ and define the comparison functions v and w as the unique solutions to the initial-boundary value problems (8.2.2) and (8.2.14) with $Q^3_{z_0}$ and $Q^2_{z_0}$ instead of $2Q$ and Q. Thanks to (8.3.8) we know that (8.2.1) is satisfied with $\kappa = \kappa(n, \gamma_1) = (4\chi)^{n+2}$. Moreover, we assume that

$$R \le R_0 \le \rho_3,$$

where $\rho_3 = \rho_3(data, K, M, \omega(\cdot)) \in (0,1]$ denotes the radius introduced after (8.2.20) for the choice $\kappa = (4\chi)^{n+2}$. This ensures that we may apply (8.2.7), (8.2.19) and (8.2.20) with $\kappa = (4\chi)^{n+2}$ for any radius smaller than ρ_3. Therefore, from (8.2.19) applied with $\kappa = (4\chi)^{n+2}$ we infer that

$$\sup_{Q^1_{z_0}} |Dw| \le c_{DiB}\, \lambda^{\frac{1}{p_0}}, \tag{8.3.9}$$

where $c_{DiB} = c_{DiB}(data) \ge 1$. In the following by $c_\ell = c_\ell(\gamma_2) \ge 1$ we denote the constant from Lemma 3.16. For A chosen in dependence on $data$ according to

$$A \ge 2c_\ell^2\, c_{DiB}^{\gamma_2}\, e^{\frac{3n}{\alpha}} \ge 1$$

we now consider $z \in Q^1_{z_0} \cap E(A\lambda, r_1)$. Our aim now is to deduce a suitable estimate for $|Du(z)|^{p(z)}$. Applying Lemma 3.16 twice yields

$$\begin{aligned} |Du(z)|^{p(z)} &\le c_\ell^2\, |Dw(z)|^{p(z)} \\ &\quad + c_\ell^2 \left(|Dv(z)|^2 + |Dw(z)|^2\right)^{\frac{p(z)-2}{2}} |Dv(z) - Dw(z)|^2 \\ &\quad + c_\ell \left(|Du(z)|^2 + |Dv(z)|^2\right)^{\frac{p(z)-2}{2}} |Du(z) - Dv(z)|^2. \end{aligned} \tag{8.3.10}$$

Next, we prove that

$$\begin{aligned} |Dw(z)|^{p(z)} &\le \left(|Du(z)|^2 + |Dv(z)|^2\right)^{\frac{p(z)-2}{2}} |Du(z) - Dv(z)|^2 \\ &\quad + \left(|Dv(z)|^2 + |Dw(z)|^2\right)^{\frac{p(z)-2}{2}} |Dv(z) - Dw(z)|^2 \end{aligned} \tag{8.3.11}$$

holds. Indeed, if (8.3.11) fails to hold we obtain from (8.3.9), $(8.0.30)_{2,3}$ from Lemma 8.1 (which is applicable due to (8.3.8)), the fact that $z \in E(A\lambda, r_1)$ and (8.3.10) that

$$|Dw(z)|^{p(z)} \le c_{DiB}^{p(z)} \lambda^{\frac{p(z)}{p_0}} \le c_{DiB}^{\gamma_2} e^{\frac{3n}{\alpha}} \lambda < \frac{c_{DiB}^{\gamma_2} e^{\frac{3n}{\alpha}}}{A} |Du(z)|^{p(z)} \le \frac{2c_\ell^2 c_{DiB}^{\gamma_2} e^{\frac{3n}{\alpha}}}{A} |Dw(z)|^{p(z)}.$$

But this contradicts the choice of A and hence (8.3.11) is proved. Therefore, combining (8.3.10) and (8.3.11) we get

$$|Du(z)|^{p(z)} \le 2c_\ell^2 \left(|Du(z)|^2 + |Dv(z)|^2\right)^{\frac{p(z)-2}{2}} |Du(z) - Dv(z)|^2 + 2c_\ell^2 \left(|Dv(z)|^2 + |Dw(z)|^2\right)^{\frac{p(z)-2}{2}} |Dv(z) - Dw(z)|^2.$$

Integrating over $Q_{z_0}^1 \cap E(A\lambda, r_1)$ and using the comparison estimates (8.2.7) and (8.2.20) applied with $\kappa = (4\chi)^{n+2}$ we obtain

$$\begin{aligned} \int_{Q_{z_0}^1 \cap E(A\lambda, r_1)} |Du|^{p(\cdot)} \, dz &\le 2c_\ell^2 \int_{Q_{z_0}^1} \left(|Du|^2 + |Dv|^2\right)^{\frac{p(\cdot)-2}{2}} |Du - Dv|^2 \, dz \\ &\quad + 2c_\ell^2 \int_{Q_{z_0}^1} \left(|Dv|^2 + |Dw|^2\right)^{\frac{p(\cdot)-2}{2}} |Dv - Dw|^2 \, dz \\ &\le c(data)\, G(M,R)\, \lambda |Q_{z_0}^0|, \end{aligned} \tag{8.3.12}$$

where

$$G(M,R) := \sup_{\rho \in (0,R]} \left[\frac{1}{M^{2-\frac{2}{\gamma_1}}} + \omega\big(\Gamma(2\rho)^\alpha\big) M \log\left(\frac{K}{\rho}\right) + [\tilde{\omega}(\rho)]^{\frac{\epsilon_1}{2\gamma_2'}} \right]^{\frac{1}{2}}. \tag{8.3.13}$$

Note that $M \ge 1$ is yet to be chosen and α and Γ are defined according to (8.0.31). Moreover, we recall that this estimate holds for any $\lambda > B\lambda_0$ and $z_0 \in E(\lambda, r_1)$.

Next, we will infer a bound for the measure of the cylinder $Q_{z_0}^0$. From (8.3.6) we have

$$|Q_{z_0}^0| = \frac{1}{\lambda} \int_{Q_{z_0}^0} |Du|^{p(\cdot)} \, dz + \frac{1}{\lambda} \int_{Q_{z_0}^0} M(|F|+1)^{p(\cdot)} \, dz. \tag{8.3.14}$$

We split the first integral of (8.3.14) as follows:

$$\begin{aligned} \int_{Q_{z_0}^0} |Du|^{p(\cdot)} \, dz &= \int_{Q_{z_0}^0 \cap \{|Du|^{p(\cdot)} \le \lambda/4\}} |Du|^{p(\cdot)} \, dz \\ &\quad + \int_{Q_{z_0}^0 \cap E(\lambda/4, r_2)} |Du|^{p(\cdot)} \, dz \\ &\le \frac{\lambda}{4} |Q_{z_0}^0| + \int_{Q_{z_0}^0 \cap E(\lambda/4, r_2)} |Du|^{p(\cdot)} \, dz, \end{aligned}$$

and similarly the second one

$$\int_{Q_{z_0}^0} M(|F|+1)^{p(\cdot)} \, dz \le \frac{\lambda}{4} |Q_{z_0}^0| + \int_{Q_{z_0}^0 \cap \{M(|F|+1)^{p(\cdot)} > \lambda/4\}} M(|F|+1)^{p(\cdot)} \, dz.$$

Inserting the last two estimates into (8.3.14) we can reabsorb the term $|Q^0_{z_0}|/2$ from the right-hand side into the left, yielding the following estimate:

$$|Q^0_{z_0}| \le \frac{2}{\lambda}\int_{Q^0_{z_0}\cap E(\lambda/4,r_2)} |Du|^{p(\cdot)}\,dz + \frac{2}{\lambda}\int_{Q^0_{z_0}\cap\{M(|F|+1)^{p(\cdot)}>\lambda/4\}} M(|F|+1)^{p(\cdot)}\,dz.$$

Using this estimate in (8.3.12) we obtain for a constant $c \equiv c(data)$ that

$$\int_{Q^1_{z_0}\cap E(A\lambda,r_1)} |Du|^{p(\cdot)}\,dz \le c\,G(M,R)\int_{Q^0_{z_0}\cap E(\lambda/4,r_2)} |Du|^{p(\cdot)}\,dz + c\,G(M,R)\int_{Q^0_{z_0}\cap\{M(|F|+1)^{p(\cdot)}>\lambda/4\}} M(|F|+1)^{p(\cdot)}\,dz. \tag{8.3.15}$$

Estimates on level sets. Here, we will extend estimate (8.3.15) to the super level set $E(A\lambda, r_1)$. To this aim we first construct a suitable covering of $E(\lambda, r_1)$ by intrinsic cylinders of the type as considered in the preceding steps. Here, we recall from the preceding two steps that for every $z_0 \in E(\lambda, r_1)$ there exists a radius ρ_{z_0} satisfying (8.3.5) such that on the cylinders $Q^j_{z_0}$, $j \in \{0,\dots,3\}$ the estimates (8.3.8) and (8.3.15) hold. Next, we want to apply the Vitali-type covering argument from Lemma 8.2. For this aim we note that (8.3.6) and $(8.0.30)_1$ (with $\kappa = 1$) imply that

$$\lambda \le \left(\frac{\beta_n MK}{\rho_{z_0}^{n+2}}\right)^{\frac{p(z_0)}{2}}.$$

This ensures that assumption (8.0.33) of Lemma 8.2 is satisfied for the family $\mathcal{F} := \{Q^0_{z_0}\}$ of parabolic cylinders with center $z_0 \in E(\lambda, r_1)$ (note that by possibly reducing the value of R_0 we can ensure that $\rho_{z_0} \le R \le R_0 \le \rho_1$, where ρ_1 is the radius from Lemma 8.2). The application of the Lemma then yields the existence of a countable subfamily $\{Q^0_{z_i}\}_{i=1}^\infty \subset \mathcal{F}$ of pairwise disjoint parabolic cylinders, such that the χ-times enlarged cylinders $Q^1_{z_i}$ cover the set $E(A\lambda, r_1)$, i.e.

$$E(A\lambda, r_1) \subset E(\lambda, r_1) \subset \bigcup_{i\in\mathbb{N}} Q^1_{z_i}.$$

Moreover, for the 4χ-times enlarged cylinders $Q^3_{z_i}$ we know that $Q^3_{z_i} \subset Q_{r_2}$. Here, we have used the notation from (8.3.7) with z_0 replaced by z_i. Since we know that on any of the cylinders $Q^1_{z_i}$, $i \in \mathbb{N}$ estimate (8.3.15) holds, we obtain after summing over $i \in \mathbb{N}$ that

$$\int_{E(A\lambda,r_1)} |Du|^{p(\cdot)}\,dz \le c\,G(M,R)\int_{E(\lambda/4,r_2)} |Du|^{p(\cdot)}\,dz + c\,G(M,R)\int_{Q_{r_2}\cap\{M(|F|+1)^{p(\cdot)}>\lambda/4\}} M(|F|+1)^{p(\cdot)}\,dz, \tag{8.3.16}$$

where $c = c(data)$. We recall that this estimate holds for every $\lambda > B\lambda_0$.

Raising the integrability exponent. Having arrived at this stage we would like to multiply both sides of (8.3.16) by λ^{q-2} and subsequently integrate with respect to λ over $(B\lambda_0, \infty)$. This, formally would lead to an $L^{p(\cdot)q}$ estimate of Du after reabsorbing $\int |Du|^{p(\cdot)q}\,dz$ on the left-hand side. However, this step is not allowed since the integral might be infinite. This problem will be overcome in the following by a truncation argument. For $k \ge B\lambda_0$ we define the truncation operator

$$T_k : [0,+\infty) \to [0,k], \qquad T_k(\sigma) := \min\{\sigma, k\}$$

and

$$E_k(A\lambda, r_1) := \big\{z \in Q_{r_1} : T_k\big(|Du(z)|^{p(z)}\big) > A\lambda\big\}.$$

Then, from inequality (8.3.16) we deduce that

$$\int_{E_k(A\lambda,r_1)} |Du|^{p(\cdot)}\,dz \le c\,G(M,R)\int_{E_k(\lambda/4,r_2)} |Du|^{p(\cdot)}\,dz + c\,G(M,R)\int_{Q_{r_2}\cap\{M(|F|+1)^{p(\cdot)}>\lambda/4\}} M(|F|+1)^{p(\cdot)}\,dz. \quad (8.3.17)$$

This can be seen as follows: In the case $k \le A\lambda$ we have $E_k(A\lambda, r_1) = \emptyset$ and therefore (8.3.17) holds trivially. In the case $k > A\lambda$ inequality (8.3.17) follows since $E_k(A\lambda, r_1) = E(A\lambda, r_1)$ and $E_k(\lambda/4, r_2) = E(\lambda/4, r_2)$. Therefore, multiplying both sides of (8.3.17) by λ^{q-2} and integrating with respect to λ over $(B\lambda_0, +\infty)$, we obtain

$$\begin{aligned}
&\int_{B\lambda_0}^{\infty} \lambda^{q-2}\int_{E_k(A\lambda,r_1)} |Du|^{p(\cdot)}\,dz\,d\lambda \qquad (8.3.18)\\
&\quad\le c\,G(M,R)\int_{B\lambda_0}^{\infty} \lambda^{q-2}\int_{E_k(\lambda/4,r_2)} |Du|^{p(\cdot)}\,dz\,d\lambda\\
&\quad+ c\,G(M,R)\int_{B\lambda_0}^{\infty} \lambda^{q-2}\int_{Q_{r_2}\cap\{M(|F|+1)^{p(\cdot)}>\lambda/4\}} M(|F|+1)^{p(\cdot)}\,dz\,d\lambda.
\end{aligned}$$

Using Fubini's theorem we get for the integral on the left-hand side of (8.3.18) that

$$\begin{aligned}
&\int_{B\lambda_0}^{\infty} \lambda^{q-2}\int_{E_k(A\lambda,r_1)} |Du|^{p(\cdot)}\,dz\,d\lambda\\
&\quad= \int_{E_k(AB\lambda_0,r_1)} |Du|^{p(\cdot)}\int_{B\lambda_0}^{T_k(|Du(z)|^{p(z)})/A} \lambda^{q-2}\,d\lambda\,dz\\
&\quad= \frac{1}{q-1}\bigg[\frac{1}{A^{q-1}}\int_{E_k(AB\lambda_0,r_1)} |Du|^{p(\cdot)}T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz\\
&\qquad\qquad - (B\lambda_0)^{q-1}\int_{E_k(AB\lambda_0,r_1)} |Du|^{p(\cdot)}\,dz\bigg]\\
&\quad\ge \frac{1}{q-1}\bigg[\frac{1}{A^{q-1}}\int_{Q_{r_1}} |Du|^{p(\cdot)}T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz\\
&\qquad\qquad - (B\lambda_0)^{q-1}\int_{Q_{r_1}} |Du|^{p(\cdot)}\,dz\bigg],
\end{aligned}$$

where in the last line we used the decomposition

$$Q_{r_1} = E_k(AB\lambda_0, r_1) \cup (Q_{r_1} \setminus E_k(AB\lambda_0, r_1))$$

and the fact that $T_k(|Du|^{p(\cdot)}) \le AB\lambda_0$ on $Q_{r_1} \setminus E_k(AB\lambda_0, r_1)$. Again by Fubini's theorem we obtain for the first integral on the right-hand side of (8.3.18)

$$\begin{aligned}
&\int_{B\lambda_0}^{\infty} \lambda^{q-2}\int_{E_k(\lambda/4,r_2)} |Du|^{p(\cdot)}\,dz\,d\lambda\\
&\quad= \int_{E_k(B\lambda_0/4,r_2)} |Du|^{p(\cdot)}\int_{B\lambda_0}^{4T_k(|Du|^{p(\cdot)})} \lambda^{q-2}\,d\lambda\,dz\\
&\quad\le \frac{4^{q-1}}{q-1}\int_{Q_{r_2}} |Du|^{p(\cdot)}T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz
\end{aligned}$$

and analogously for the integral involving the right-hand side F:

$$\begin{aligned}
&\int_{B\lambda_0}^{\infty} \lambda^{q-2} \int_{Q_{r_2}\cap\{M(|F|+1)^{p(\cdot)}>\lambda/4\}} M(|F|+1)^{p(\cdot)}\,dz\,d\lambda \\
&\quad= \int_{Q_{r_2}\cap\{M(|F|+1)^{p(\cdot)}>B\lambda_0/4\}} M(|F|+1)^{p(\cdot)} \int_{B\lambda_0}^{4M(|F|+1)^{p(\cdot)}} \lambda^{q-2}\,d\lambda\,dz \\
&\quad\le \frac{4^{q-1}M^q}{q-1}\int_{Q_{r_2}} (|F|+1)^{p(\cdot)q}\,dz.
\end{aligned}$$

Hence, joining the preceding estimates with (8.3.18) we get

$$\begin{aligned}
&\int_{Q_{r_1}} |Du|^{p(\cdot)} T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz \\
&\quad\le (AB\lambda_0)^{q-1}\int_{Q_{r_1}} |Du|^{p(\cdot)}\,dz \\
&\qquad+ \bar{c}\,A^{q-1}G(M,R)\int_{Q_{r_2}} |Du|^{p(\cdot)} T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz \\
&\qquad+ \bar{c}\,A^{q-1}M^q G(M,R)\int_{Q_{r_2}} (|F|+1)^{p(\cdot)q}\,dz,
\end{aligned} \tag{8.3.19}$$

where $\bar{c}=\bar{c}(data)$. Note that the estimate stays stable as $q\downarrow 1$.

Choice of the parameters. We now perform the choices of the parameters M and R_0 in such a way that $\bar{c}A^{q-1}G(M,R)\le\frac{1}{2}$ whenever $R\le R_0$. First, we choose $M=M(data,q)\ge 1$ large enough to have

$$\frac{\bar{c}\,A^{q-1}}{M^{1-\frac{1}{\gamma_1}}}\le\frac{1}{4}.$$

Next, we reduce the value of R_0, now depending on $data, K, \omega(\cdot), \tilde{\omega}(\cdot), q$, in such a way that for any $\rho\le R_0$ we have

$$\bar{c}\,A^{q-1}\Big[\omega\big(\Gamma(2\rho)^{\alpha}\big)M\log\Big(\frac{K}{\rho}\Big)+\big[\tilde{\omega}(\rho)\big]^{\frac{\epsilon_1}{2\gamma_1'}}\Big]^{\frac{1}{2}}\le\frac{1}{4}. \tag{8.3.20}$$

Note that this is possible due to the assumptions (4.4.4) and (4.5.6). Recalling the definition of G in (8.3.13) we therefore have $\bar{c}A^{q-1}G(M,R)\le\frac{1}{2}$ for any $R\le R_0$. Using this in (8.3.19) we get

$$\begin{aligned}
&\int_{Q_{r_1}} |Du|^{p(\cdot)} T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz \\
&\quad\le \frac{1}{2}\int_{Q_{r_2}} |Du|^{p(\cdot)} T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz \\
&\qquad+ c\Big(\frac{R}{r_1-r_1}\Big)^{\beta}\lambda_0^{q-1}\int_{Q_{2R}} |Du|^{p(\cdot)}\,dz + c\int_{Q_{2R}} (|F|+1)^{p(\cdot)q}\,dz,
\end{aligned}$$

where $\beta\equiv(n+2)(q-1)d$ and with a constant $c=c(data,q)$. At this point we apply Lemma 3.11 with

$$\phi(r)\equiv\int_{Q_r} |Du|^{p(\cdot)} T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\,dz,$$

and

$$\mathcal{A}\equiv c\int_{Q_{2R}} (|F|+1)^{p(\cdot)q}\,dz \quad\text{and}\quad \mathcal{B}\equiv c\,R^{\beta}\lambda_0^{q-1}\int_{Q_{2R}} |Du|^{p(\cdot)}\,dz,$$

yielding that

$$\int_{Q_R} |Du|^{p(\cdot)} T_k\big(|Du|^{p(\cdot)}\big)^{q-1}\, dz \le c(\beta)\Big[\mathcal{A} + \frac{\mathcal{B}}{R^\beta}\Big].$$

Passing to the limit $k \to \infty$ which is possible by Fatou's lemma and taking averages we find that

$$⨍_{Q_R} |Du|^{p(\cdot)q}\, dz \le c\left[\lambda_0^{q-1} ⨍_{Q_{2R}} |Du|^{p(\cdot)}\, dz + ⨍_{Q_{2R}} (|F|+1)^{p(\cdot)q}\, dz\right]. \tag{8.3.21}$$

Note that $c = c(data, q)$, since β depends continuously on $p(\cdot)$, i.e. the dependence upon $p(\cdot)$ via the parameter d can be replaced by a dependence on γ_1 and γ_2. Since $Q_{2R} \Subset \Omega_T$ was arbitrary, we have thus proved the first assertion in Theorem 4.20, i.e. that $|Du|^{p(\cdot)} \in L^q_{\mathrm{loc}}(\Omega_T)$. It now remains to show the estimate (4.5.9).

Adjusting the exponent. Here, we first observe that (8.3.21) together with the definition of λ_0 in (8.3.1) lead to estimate (4.5.9) in Theorem 4.20, but with d instead of $d(p_0)$, with $p_0 := p(\mathfrak{z}_0)$ and $\mathfrak{z}_0$ is the center of the cylinder $Q_{2R} \equiv Q_{2R}(\mathfrak{z}_0)$. We recall that d was defined in (4.5.10) and $d \ge d(p_0)$. In order to reduce the exponent from d to $d(p_0)$ we need to show a bound of the following form

$$\mathfrak{E}^{d-d(p_0)} \le c(n,\gamma_1), \quad \text{where } \mathfrak{E} := ⨍_{Q_{2R}} |Du|^{p(\cdot)} + (|F|+1)^{p(\cdot)q}\, dz. \tag{8.3.22}$$

To this aim we first shall deduce an upper bound for $d - d(p_0)$ in terms of $\omega(R)$. Since $d(p(\cdot))$ is continuous there exists $\hat{z} \in \overline{Q}_R$ such that $d = d(p(\hat{z}))$. From the definition of $d(\cdot)$ in (4.5.10) we observe that

$$d(p_0) \ge \max\left\{\frac{p_0}{2}, \frac{2p_0}{p_0(n+2)-2n}\right\}.$$

In the following we distinguish the cases where $p(\hat{z})$ is larger, respectively smaller than 2. In the case $p(\hat{z}) \ge 2$ we get from (8.0.27) that

$$d - d(p_0) = \frac{p(\hat{z})}{2} - d(p_0) \le \frac{p(\hat{z})}{2} - \frac{p_0}{2} \le \frac{1}{2}\,\omega(R),$$

while in the case $p(\hat{z}) < 2$ we have $p(\hat{z}) \le p_0$ and therefore we find in a similar way that

$$\begin{aligned} d - d(p_0) &\le \frac{2p(\hat{z})}{p(\hat{z})(n+2)-2n} - \frac{2p_0}{p_0(n+2)-2n} \\ &= \frac{4n(p_0 - p(\hat{z}))}{[p(\hat{z})(n+2)-2n][p_0(n+2)-2n]} \le \frac{4n}{[\gamma_1(n+2)-2n]^2}\,\omega(R). \end{aligned}$$

Hence, in any case we have proved that $d - d(p_0) \le c(n,\gamma_1)\omega(R)$. Recalling the definition of $\mathfrak{E}$ from (8.3.22) and using (4.5.8) we thus obtain

$$\mathfrak{E}^{d-d(p_0)} \le c(n,\gamma_1)\big[R^{-(n+2)}K\big]^{c(n,\gamma_1)\omega(R)} \le c(n,\gamma_1).$$

We note that the last inequality is a consequence of the logarithmic continuity of ω from (8.0.27), since $R^{-\omega(R)} \le e$ and

$$K^{\omega(R)} = \exp\big[\omega(R)\log K\big] \le \exp\big[\omega(R)\log\big(\tfrac{1}{R}\big)\big] \le e$$

provided $R \le R_0 \le \min\{R_1, 1/K\}$, where R_1 is the radius from (8.0.27). This finishes the proof of (8.3.22) and by the reasoning from above we therefore obtain the asserted estimate (4.5.9). We have thus completed the proof of Theorem 4.20. □

Proof of Remark 4.21. Here, it is enough to ensure that we can choose $R_0 > 0$ and ϵ_{BMO} in such a way that (8.3.20) is satisfied. Assuming for instance

$$[a]_{\text{BMO}} \le \epsilon_{\text{BMO}} := \left(\frac{1}{8\bar{c}A^{q-1}}\right)^{\frac{4\gamma_1'}{\epsilon_1}} \text{ and } \omega\big(\Gamma(2\rho)^\alpha\big) M \log\left(\frac{K}{\rho}\right) \le \left(\frac{1}{8\bar{c}A^{q-1}}\right)^2$$

for any $\rho \le R_0$ we conclude that (8.3.20) holds, since $\tilde{\omega}(\rho) \le [a]_{\text{BMO}}$. The rest of the proof is completely the same as the one of Theorem 4.20.

Finally, note that the constant c in Theorem 4.20 remains stable when $q \downarrow 1$ and it blows up, i.e. $c \to \infty$ when $q \to \infty$.

CHAPTER 9

Interpolation potential estimates for $p(x)$-growth conditions

We begin this Chapter by stating an intermediate result which will allow, starting from Theorem 4.23, to infer Theorem 4.24 and 4.25 and also Corollary 4.26.

THEOREM 9.1. *Let $u \in C^1(\Omega)$ be a weak solution to (4.6.1) under the assumptions (4.6.5), (4.6.6) and (4.6.8), with $\omega(\cdot)$ satisfying (4.4.3). Let $B_R \subset \Omega$ be a ball, centered at x. Then for every $\tilde{\alpha} < 1$ there exist positive numbers δ_1 and δ_2 depending on $n, \gamma_1, \gamma_2, \nu, L, L_1, \tilde{\alpha}$ such that if (4.6.17) is satisfied, then the pointwise estimate*

$$M^{\sharp}_{\alpha,R}(u)(x) + M_{1-\alpha,R}(Du)(x) \le c\big[M_{p(\cdot)-\alpha(p(\cdot)-1),R}(\mu)(x)\big]^{\frac{1}{p(x)-1}} + c\,R^{1-\alpha} \fint_{B_R} (|Du| + s + R)\,dy \tag{9.0.23}$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, for a constant $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, \omega(\cdot), \tilde{\alpha}, \mathrm{diam}(\Omega))$.

If in addition the continuity assumption (4.6.7) on the vector field together with the conditions (4.6.14) and (4.6.15) – for σ_h as in Theorem 4.25 – are in force, and moreover $B_{2R} \subset \Omega$, then the estimate

$$M^{\sharp}_{\alpha,R}(u)(x) + M_{1-\alpha,R}(Du)(x) \le c\mathbf{WI}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}(x, 2R) + c\,R^{1-\alpha} \fint_{B_R} (|Du| + s + R)\,dy \tag{9.0.24}$$

holds true uniformly in $\alpha \in [0, 1]$, with a constant $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, \omega(\cdot), \mathrm{diam}(\Omega))$.

9.1. Preliminaries

Since the prototype for equations we handle in this manuscript is the $p(x)$-Laplacian operator, it is convenient that we work with an operator for the gradient Du, involving the growth behavior of this equation. For $s \in [0, 1]$ and $p \in [\gamma_1, \gamma_2]$, we introduce the function

$$V_p(z) := \big(s^2 + |z|^2\big)^{\frac{p-2}{4}} z, \qquad z \in \mathbb{R}^n.$$

A basic property of the map $V_p(\cdot)$ reads as follows: for any $z_1, z_2 \in \mathbb{R}^n$, any $s \in [0, 1]$ and any $p \in [\gamma_1, \gamma_2]$ it holds

$$c^{-1}(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} \le \frac{|V_p(z_2) - V_p(z_1)|^2}{|z_2 - z_1|^2} \le c(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}. \tag{9.1.1}$$

Here the constant c depends on n and p and we notice that for $p \in [\gamma_1, \gamma_2]$ it can be replaced by one depending only on γ_1 and γ_2 instead of p. On the other hand, in case of a function $p : \Omega \to [\gamma_1, \gamma_2]$, the estimate (9.1.1) can be written pointwise for every x and again the constant c depends only on n and the global bounds γ_1 and γ_2 of the function $p(\cdot)$.

At this point we recall that assumption (4.6.5)$_2$ implies the following monotonicity property of the vector field $z \mapsto a(\cdot, z)$: There exists a constant $c \equiv c(\gamma_2) \geq 1$ such that

$$c^{-1}\sqrt{\nu}\left(|z_1|^2+|z_2|^2+s^2\right)^{\frac{p(x)-2}{2}}|z_2-z_1|^2 \leq \langle a(x,z_2)-a(x,z_1), z_2-z_1\rangle\,, \tag{9.1.2}$$

whenever $x \in \Omega$ and $z_1, z_2 \in \mathbb{R}^n$. Taking into account (9.1.1) we have for all $p(x) > 1$ the estimate

$$c^{-1}\sqrt{\nu}\big|V_{p(x)}(z_2)-V_{p(x)}(z_1)\big|^2 \leq \langle a(x,z_2)-a(x,z_1), z_2-z_1\rangle\,, \tag{9.1.3}$$

for a constant $c \equiv c(n, \gamma_1, \gamma_2) \geq 1$. In particular, in the case $p(x) \geq 2$, the previous inequality directly implies

$$c^{-1}\sqrt{\nu}|z_2-z_1|^{p(x)} \leq \langle a(x,z_2)-a(x,z_1), z_2-z_1\rangle\,.$$

For a fixed ball $B_{2R}(x_0) \subset \Omega$ we define

$$p_1 := \inf_{x\in B_{2R}(x_0)} p(x) \qquad \text{and} \qquad p_2 := \sup_{x\in B_{2R}(x_0)} p(x)\,. \tag{9.1.4}$$

Then, assumption (4.6.6) directly gives

$$p_2 - p_1 \leq \omega(4R) \quad \text{and} \quad \frac{p_2}{p_1} \leq 1+\omega(4R). \tag{9.1.5}$$

Furthermore, an elementary computation shows that (4.4.3) and (9.1.5) imply

$$R^{-(p_2-p_1)} \leq R^{-\omega(4R)} \leq c(\omega(\cdot)), \tag{9.1.6}$$

for all radii $0 < R \leq 1$. Finally, from [33, estimate (2.8)] we take the following elementary estimate, which we shall use several times in the course of the proof. For any $\alpha, \sigma > 0$, $R \in (0,1]$ and $\tilde{\omega} \in [0, \omega(R)]$ we have

$$A^{\sigma} \leq c(\alpha, \omega(\cdot))\,(A+R^{\alpha})^{\sigma+\tilde{\omega}}\,, \qquad \text{for all } A \geq 0\,. \tag{9.1.7}$$

Elementary facts on Wolff potentials. The statement in the following remark is a consequence of the fact that the non standard potentials are defined pointwise, see for the standard case [133, Lemma 2.3].

REMARK. *For $1 < p(x) \leq 2$ and $\beta >\in (0, n/p(x)]$, the estimate*

$$\mathbf{W}^{\mu}_{\beta,p(\cdot)}(x,R) \leq c(\gamma_1,\gamma_2,\beta)\left[\mathbf{I}^{\mu}_{\beta p(\cdot)}(x,2R)\right]^{\frac{1}{p(x)-1}}$$

holds true.

The following simple Proposition shows how to estimate the series of the density of the Wolff/Riesz potential, on some dyadic sequence, with the whole Wolff/Riesz potential.

PROPOSITION 9.2. *Let $R > 0$ and let $R_i := R/K^i$, $i = 0, 1, \ldots$ be a sequence of geometrically shrinking radii with $K > 1$. Then if $p(x) \geq 2$, for every $m \in \mathbb{N}$ and $\theta(x) \in (0, n/p(x)]$ we have*

$$\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-\theta(x)p(x)}}\right]^{\frac{1}{p(x)-1}} \leq c(n,\gamma_1,K)\,\mathbf{W}^{\mu}_{\theta(\cdot),p(\cdot)}(x,2R), \tag{9.1.8}$$

and for every $q(x) \in (0, n]$ there holds

$$\sum_{i=0}^{m}\frac{|\mu|(B_i)}{R_i^{n-q(x)}} \leq c(n,\gamma_1,K)\,\mathbf{I}^{|\mu|}_{q(x)}(x,2R), \tag{9.1.9}$$

where $B_i := B(R_i, x)$.

PROOF. Since the nonstandard potentials are defined pointwise, the proof is exactly the one for the standard potentials, which can be found for instance in [70, 103]. Indeed from [70, p. 20] we deduce that the constant in estimate (9.1.8) is

$$c \equiv \frac{2^{\frac{n-\theta(x)p(x)}{p(x)-1}}}{\log 2} + \frac{K^{\frac{n-\theta(x)p(x)}{p(x)-1}}}{\log K} \leq \frac{2^{\frac{n}{\gamma_1 - 1}}}{\log 2} + \frac{K^{\frac{n}{\gamma_1-1}}}{\log K},$$

Estimate (9.1.9) is just estimate (9.1.8) with the choices $p(x) \equiv 2$ and $\theta(x) = q(x)/2$. □

The following Lemma, whose proof is just a pointwise revisitation of the proof of [103, Lemma 4.1], will be useful in order to estimate maximal operators associated to the measure μ with related Wolff/Riesz potentials.

LEMMA 9.3. *Let μ be a Borel measure with finite total mass on Ω and let $\varsigma \in (0,1)$, $\beta(x) \in [0,n]$, $p(x) \in [\gamma_1, \gamma_2]$ and $B_R \subset \Omega$. Then there holds*

$$\big[M_{\beta(x),\varsigma R}(\mu)(x)\big]^{\frac{1}{p(x)-1}} \leq c(n,\gamma_1,\gamma_2,\varsigma)\mathbf{W}^{\mu}_{\beta(\cdot)/p(\cdot),p(\cdot)}(x,R)$$

and

$$M_{\beta(x),\varsigma R}(\mu)(x) \leq c(n,\gamma,\gamma_1,\varsigma)\mathbf{I}^{|\mu|}_{\beta(\cdot)}(x,R).$$

Note that the constant appearing in [103, Lemma 4.1] is continuous and increasing with respect to $\beta(x)$, so we replaced the dependence on $\beta(x)$ with a dependence on n, and the dependence on $p(x)$ with a dependence upon γ_1, γ_2. The proof of the following Lemma can be found in [51]:

LEMMA 9.4. *Let $f \in L^1(\Omega;\mathbb{R}^k)$ and $B_R \subset \Omega$; then for every $\alpha \in (0,1]$ there exists a constant depending only on n such that the inequality*

$$|f(x) - f(y)| \leq \frac{c}{\alpha}\Big[M^{\sharp}_{\alpha,R}(f)(x) + M^{\sharp}_{\alpha,R}(f)(y)\Big]|x-y|^{\alpha}$$

holds for every $x, y \in B_{R/4}$.

9.2. Regularity for the reference problems

Since we will prove the main theorem by suitable comparison procedures to homogeneous and "frozen" problems, in this section we collect several regularity results for problems with non standard growth.

Decay estimates for the reference problem. For a sub-domain $A \subset \Omega$ we consider the homogeneous equation

$$-\operatorname{div}\big[\gamma(x)a(x,Dv)\big] = 0 \qquad \text{in } A. \tag{9.2.1}$$

Then De Giorgi's theory is available for solutions v to this equation, since the vector field a satisfies the ellipticity and $p(x)$-growth conditions (4.6.5) and $p(\cdot)$ is logarithmic Hölder continuous. More precisely we have estimates of Morrey-type for the gradient Dv, as the following Theorem shows:

THEOREM 9.5. *Let $v \in W^{1,p(\cdot)}(A)$ be a weak solution to (9.2.1) under the structure conditions (4.6.5) with a growth exponent $p(\cdot)$ satisfying (4.6.6) and (4.4.3) and with coefficient $\gamma(\cdot)$ bounded in the sense of (4.6.8). Then there exist constants $\alpha_m \in (0,1)$ and $c \geq 1$, both depending at most on n, ν, L, γ_1 and γ_2, such that the estimate*

$$\fint_{B_\rho} (|Dv|+s)^{p(x)}\,dx \leq c\left(\frac{\rho}{R}\right)^{p_2(\alpha_m-1)} \fint_{B_R} (|Dv|+s)^{p(x)}\,dx + c\,\rho^{p_2(\alpha_m-1)},$$

holds, whenever $B_\rho \subset B_R \subset A$ are concentric balls.

PROOF. The proof works by showing that v lies in an appropriate generalized De Giorgi class and subsequently via the embedding of De Giorgi classes into the space of Hölder continuous functions. In fact, this is shown for local minimizers of functionals with $p(x)$-growth structure in [**76**] and in [**74**] (in the latter paper more general problems involving obstacle conditions are treated). In the context of solutions to $p(x)$-growth equations the argument can be established completely analogously. □

Next, we consider the homogeneous frozen equation

$$-\operatorname{div} a(x_0, Dw) = 0 \qquad \text{in } A, \tag{9.2.2}$$

for a sub-domain $A \subset \Omega$. Since the vector-field $z \mapsto a(x_0, z)$ is frozen in the point $x_0 \in \Omega$, it fulfills the structure conditions (4.6.5) with a constant exponent $p(x_0)$. Therefore, [**69**, Theorem 3.3] and [**70**, Theorem 3.1] provide the following reference estimate:

THEOREM 9.6. *Let $w \in W^{1,p(x_o)}(A)$ be a weak solution to* (9.2.2) *under the structure conditions* (4.6.5) *with constant growth exponent $p(x_0) > 2 - 1/n$. Then there exists $\beta_m \in (0, 1]$ and $c \geq 1$, both depending only on $n, \nu, L, p(x_0)$ such that the estimate*

$$\fint_{B_\rho} |Dw - (Dw)_{B_\rho}|\, dx \leq c \left(\frac{\rho}{R}\right)^{\beta_m} \fint_{B_R} |Dw - (Dw)_{B_R}|\, dx, \tag{9.2.3}$$

holds whenever $B_\rho \subset B_R \subset A$ are concentric balls. Moreover it holds that

$$\fint_{B_\rho} (|Dw| + s)\, dx \leq c \fint_{B_R} (|Dw| + s)\, dx, \tag{9.2.4}$$

again for a constant $c \equiv c(n, p(x_0), \nu, L)$.

REMARK. *(Dependence of the constants)* As also mentioned in [**69**, Remark 3.2], the constants β and c in the estimate above depend continuously on the data. This means that for $p(x_0) \in [\gamma_1, \gamma_2]$ we may replace the dependence upon $p(x_0)$ by a dependence on the bounds γ_1 and γ_2. Let us in particular point out that the constants remain stable when $p(x_0) \to 2$, since they rely on estimates for a linearized elliptic equation as considered in [**70**, Lemma 3.2].

We state a result concerning boundary regularity and nonlinear Calderón-Zygmund theory for solutions to the frozen homogeneous equation. We refer the reader for instance to [**103**, Theorem 2.3] and [**99**, Theorem 7.7] for more details and a comment on the proof.

THEOREM 9.7. *Let $w \in W^{1,p_0}(\Omega)$ be a weak solution to the Dirichlet problem*

$$\begin{cases} -\operatorname{div} a(x_0, Dw) = 0 & \text{on } B_R \\ w = v & \text{on } \partial B_R, \end{cases}$$

where the vector field $z \mapsto a(x_0, z)$ satisfies (4.6.5) *with constant exponent $p_0 = p(x_0)$, $B_R \subset \Omega$ denotes a ball and $v \in W^{1,q}(B_R)$ denotes an assigned boundary datum with $p_0 \leq q < \infty$. Then $w \in W^{1,q}(B_R)$ and the estimate*

$$\|Dw\|_{L^q(B_R)} \leq c \left(\|Dv\|_{L^q(B_R)} + s\right) \tag{9.2.5}$$

holds true for a constant $c \equiv c(n, p_0, \nu, L, q)$.

REMARK. *(Dependence of the constant)* Again, a careful look at the proofs of Theorem 7.7. in [**99**] shows that the appearing constant can be replaced by one which depends only on the global bounds γ_1 and γ_2 instead of p_0. Later on we will apply Theorem 9.7 for a choice $q \equiv p_2(1 + \delta_1/2)$, where p_2 denotes the supremum of $p(\cdot)$ on a small ball and δ_1 a higher integrability exponent depending on the data. However the constant in Theorem 9.7 depends in a monotone way on the parameter q and blows up when $q \to \infty$. Thus, we can replace the dependency of the constant on q by the upper bound of $p_2(1+\delta_1/2)$ and therefore make it independent of $p(\cdot)$ itself.

9.3. Comparison estimates for reference problems

The proof of the main theorems will be performed by a series of comparison procedures to suitable "simpler" problems. Let us denote by $u \in C^1(\Omega)$ the solution to the equation (4.6.1) with bounded 1-energy. We consider on a fixed ball $B_{2R}(x_0) \subset \Omega$ with suitable small radius which will be specified later, the solution $v \in W^{1,p(\cdot)}(\Omega)$ to the Dirichlet problem

$$\begin{cases} \operatorname{div}\big[\gamma(x)\, a(x,Dv)\big] = 0 & \text{on } B_{2R}, \\ v = u & \text{on } \partial B_{2R} \end{cases} \tag{9.3.1}$$

and the solution $w \in W^{1,p_0}(\Omega)$, where $p_0 \equiv p(x_0)$, to the Dirichlet problem

$$\begin{cases} \operatorname{div} a(x_0,Dw) = 0 & \text{on } B_R, \\ w = v & \text{on } \partial B_R. \end{cases} \tag{9.3.2}$$

Existence and uniqueness of v and w are guaranteed by standard monotonicity methods. In order to handle the cases $p_0 \geq 2$ and $p_0 < 2$ widely simultaneously, we introduce the notation

$$\chi_{\{p_0<2\}} := \begin{cases} 0 & \text{if } p_0 \geq 2 \\ 1 & \text{if } p_0 < 2. \end{cases}$$

Comparison to the homogeneous problem. We first establish a comparison estimate between the solution $u \in C^1(\Omega)$ to the original measure data problem (4.6.1) and the unique solution $v \in W^{1,p(\cdot)}(\Omega)$ to the homogeneous Dirichlet problem (9.3.1). Our result is the following

LEMMA 9.8. *Under the structure conditions* (4.6.5)*,* (4.6.6)*,* (4.4.3)*, and being* $\gamma(\cdot)$ *bounded in the sense of* (4.6.8)*, let* $u \in C^1(\Omega)$ *be the solution to the equation* (4.6.1) *and let* $v \in u + W_0^{1,p(\cdot)}(B_{2R})$ *be the unique solution to the Dirichlet problem* (9.3.1)*, where* $0 < R \leq 1$*. Then there exists a constant c depending upon* $n, \nu, \gamma_1, \gamma_2, |\mu|(\Omega), |\Omega|, M, \omega(\cdot)$ *such that the following estimate holds true:*

$$\fint_{B_{2R}} |Du - Dv|\,dx \leq c\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p_0-1}} + c\,\chi_{\{p_0<2\}}\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} (|Du|+s)\,dx\right)^{2-p_0} + c\,R. \tag{9.3.3}$$

PROOF. Estimate (9.3.3), in the case $p_1 \geq 2$, can be almost directly inferred from [33, Lemma 3.1]. The only difference consists in the presence of the coefficient function $\gamma(x)$ here. However, it can be easily seen that only slight modifications are sufficient, involving in particular the bound (4.6.8) for $\gamma(x)$, to get the estimate also in our case. So in this proof we only concentrate on the case $p_1 < 2$.

The proof consists in three steps. First, we reduce the situation to the one where $x_0 = 0$, $2R = 1$, thus $B_{2R} \equiv B_1$, and $|\mu|(B_1) + \big[|\mu|(B_1)\big]\big(\int_{B_1}(|Du|+s)\,dx\big)^{2-p_1} \leq \bar{c}$ for a constant $\bar{c} \equiv \bar{c}(n,\gamma_1,\gamma_2,M,|\Omega|,\omega(\cdot))$. Here we have set $p_2 \equiv \sup_{x\in B_1} p(\cdot)$. Then, in a second step we justify this simplification by a scaling argument. Finally, we have to adjust the outcoming estimates by replacing the "wrong" exponent p_2 by the exponent p_0.

Step 1: Dimensionless estimate. We here show that in the case $B_{2R}(x_0) \equiv B_1$, setting

$$p_2 := \sup_{x \in B_1} p(\cdot), \qquad p_1 := \inf_{x \in B_1} p(\cdot)$$

and supposing the estimate

$$|\mu|(B_1) + |\mu|(B_1)\left(\int_{B_1} (|Du| + s)\, dx\right)^{2-p_1} \leq \bar{c}, \tag{9.3.4}$$

holding for a constant $\bar{c} < \infty$, we have

$$\int_{B_1} |Du - Dv|\, dx \leq c(\bar{c}, \nu, \gamma_1, \gamma_2, n). \tag{9.3.5}$$

First we introduce for $k \in \mathbb{N}_0$ the truncation operators

$$T_k(t) := \max\{-k, \min\{k, t\}\}, \qquad \Phi_k(t) := T_1(t - T_k(t)), \quad t \in \mathbb{R},$$

and the sets

$$C_k := \{x \in B_1 :\ k < |u(x) - v(x)| \leq k + 1\}.$$

Substracting the weak formulations of (4.6.1) and (9.3.1), testing the resulting equation

$$\int_{B_1} \gamma(x)\, \langle a(x, Du) - a(x, Dv), D\varphi \rangle\, dx = \int_{B_1} \varphi\, d\mu$$

with $\varphi := \Phi_k(u - v)$ and using that $D\varphi = Du - Dv$ on C_k, $D\varphi = 0$ on $B_1 \setminus C_k$ and $|\varphi| \leq 1$, we obtain by (9.1.3) and the bound (4.6.8) for every $k \in \mathbb{N}$

$$\int_{C_k} \left|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\right|^2 dx \leq c\, |\mu|(B_1), \tag{9.3.6}$$

with $c \equiv c(\nu, \gamma_2)$. Observing that the lower bound γ_1 for the exponent function $p(\cdot)$ satisfies $\gamma_1 > 2 - \frac{1}{n}$, we find $\eta \equiv \eta(\gamma_1, n) \in (0,1)$ such that

$$p_1 \geq \gamma_1 > 2 - \frac{\eta}{n}$$

and therefore also

$$\frac{n(p_1 - 1)}{n - \eta} > 1. \tag{9.3.7}$$

For every integer $k \in \mathbb{N}$ we then obtain

$$\begin{aligned}
&\int_{C_k} \left|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\right|^{\frac{2}{p_1}} dx \\
&\qquad \leq |C_k|^{\frac{p_1 - 1}{p_1}} \left(\int_{C_k} \left|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\right|^2 dx\right)^{\frac{1}{p_1}} \\
&\qquad \leq c\, |C_k|^{\frac{p_1 - 1}{p_1}} \left[|\mu|(B_1)\right]^{\frac{1}{p_1}} \\
&\qquad \leq \frac{c}{k^{\frac{n(p_1 - 1)}{p_1 (n - \eta)}}} \left(\int_{C_k} |u - v|^{\frac{n}{n - \eta}}\, dx\right)^{\frac{p_1 - 1}{p_1}} \left[|\mu|(B_1)\right]^{\frac{1}{p_1}},
\end{aligned}$$

for a constant $c \equiv c(\nu, \gamma_1, \gamma_2)$. Moreover, by Hölder's inequality we obtain for $k = 0$:

$$\int_{C_0} \left|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\right|^{\frac{2}{p_1}} dx \leq c(n, \nu, \gamma_1, \gamma_2) \left[|\mu|(B_1)\right]^{\frac{1}{p_1}}.$$

Now, having in mind (9.3.7), we proceed exactly as in [69, p. 2981] with p replaced by p_1 and obtain

$$\int_{B_1} \left|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\right|^{\frac{2}{p_1}} dx$$

$$\leq c\big[|\mu|(B_1)\big]^{\frac{1}{p_1}} + c\bigg(\int_{B_1} |Du - Dv|\, dx\bigg)^{\frac{n(p_1-1)}{p_1(n-\eta)}} \big[|\mu|(B_1)\big]^{\frac{1}{p_1}}. \quad (9.3.8)$$

In the preceding estimate, the constant c depends on n, ν, γ_1, γ_2 and η, where – in view of (9.3.7) – the dependence upon η can be replaced by a dependence on γ_1 and n. In a next step we use (9.1.1) to write

$$\begin{aligned}
|Du - Dv| &= \big[(|Du|^2 + |Dv|^2 + s^2)^{\frac{p(x)-2}{2}} |Du - Dv|^2\big]^{\frac{1}{2}} \\
&\qquad\qquad \cdot \big(|Du|^2 + |Dv|^2 + s^2\big)^{\frac{2-p(x)}{4}} \\
&\leq c\,\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|\big(|Du|^2 + |Dv|^2 + s^2\big)^{\frac{2-p(x)}{4}} \\
&\leq c\,\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|\big(|Du|^2 + |Dv|^2 + s^2 + 1\big)^{\frac{2-p_1}{4}} \\
&\leq c\,\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big||Du - Dv|^{\frac{2-p_1}{2}} \\
&\qquad + c\,\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|\big(|Du|^2 + s^2 + 1\big)^{\frac{2-p_1}{4}} \\
&\leq \frac{1}{2}|Du - Dv| + c\,\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|^{\frac{2}{p_1}} \\
&\qquad + c\,\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|\big(|Du|^2 + s^2 + 1\big)^{\frac{2-p_1}{4}}.
\end{aligned}$$

Here the constant c depends on n, γ_1 and γ_2. Thus, by absorbing the first term on the right-hand side of the preceding inequality into the left-hand side and subsequently applying Hölder's inequality, we get

$$\begin{aligned}
\int_{B_1} |Du - Dv|\, dx &\leq c \int_{B_1} \big|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\big|^{\frac{2}{p_1}}\, dx \\
&+ c\bigg[\int_{B_1} \big|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\big|^{\frac{2}{p_1}}\, dx\bigg]^{\frac{p_1}{2}} \bigg[1 + \int_{B_1} \big(|Du| + s\big)\, dx\bigg]^{\frac{2-p_1}{2}}.
\end{aligned}$$

Combining the last estimate with (9.3.8) and (9.3.4) we arrive at

$$\begin{aligned}
&\int_{B_1} |Du - Dv|\, dx \\
&\quad\leq c\bigg[\big[|\mu|(B_1)\big]^{\frac{1}{p_1}} + \big[|\mu|(B_1)\big]^{\frac{1}{p_2}}\bigg(\int_{B_1} |Du - Dv|\, dx\bigg)^{\frac{n(p_1-1)}{p_1(n-\eta)}}\bigg] \\
&\qquad + c\bigg[|\mu|(B_1)\bigg(\int_{B_1} \big(|Du| + s\big)\, dx + 1\bigg)^{2-p_1}\bigg]^{\frac{1}{2}} \\
&\qquad + c\bigg[|\mu|(B_1)\bigg(\int_{B_1} \big(|Du| + s\big)\, dx + 1\bigg)^{2-p_1}\bigg]^{\frac{1}{2}} \times \\
&\qquad\qquad\qquad \times \bigg(\int_{B_1} |Du - Dv|\, dx\bigg)^{\frac{n(p_1-1)}{2(n-\eta)}} \\
&\quad\leq c + c\bigg[\int_{B_1} |Du - Dv|\, dx\bigg]^{\frac{n(p_1-1)}{p_1(n-\eta)}},
\end{aligned}$$

for a constant c that depends on $n, \nu, \gamma_1, \gamma_2, \bar{c}$. Here we have used in the last step also that $p_1 \leq 2$ and therefore $\frac{n(p_1-1)}{p_1(n-\eta)} \geq \frac{n(p_1-1)}{2(n-\eta)}$. Having moreover $p_1 \leq 2 \leq n$, we observe that

$$\frac{n(p_1 - 1)}{p_1(n - \gamma)} < \frac{p_1(n - 1)}{p_1(n - \gamma)} \leq 1,$$

and as a consequence Young's inequality leads to the desired estimate (9.3.5).

Step 2: Scaling procedures and adjusting of exponents. We show the comparison estimate (9.3.3) by re-scaling. Here we have to distinguish carefully several cases of the appearing exponents p_1, p_2 and p_0. We recall that we are supposing here we are in **the case $p_1 < 2$**. In **the case $p_2 \leq 2$**, which implies that $p_0 \leq 2$, we define on the ball $B_{2R} \equiv B(x_0, 2R) \subset \Omega$ the quantity

$$A := \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p_2-1}} + \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} (|Du| + s)\, dx\right)^{2-p_2} + R > 0 \quad (9.3.9)$$

and consider the new functions

$$\tilde{u}(y) := \frac{u(x_0 + 2Ry)}{2AR}, \quad \tilde{v}(y) := \frac{v(x_0 + 2Ry)}{2AR}, \quad \tilde{\mu}(y) := \frac{2R\mu(x_0 + 2Ry)}{A^{p_2-1}}, \quad (9.3.10)$$

together with the new vector field

$$\tilde{a}(y, z) := \frac{a(x_0 + 2Ry, Az)}{A^{p_2-1}}, \qquad \tilde{\gamma}(y) := \gamma(x_o + 2Ry), \quad (9.3.11)$$

for $y \in B_1, z \in \mathbb{R}^n$. It is now easy to see that $\tilde{u}$ and $\tilde{v}$ solve the equations

$$-\operatorname{div}\left[\tilde{\gamma}(y)\tilde{a}(y, D\tilde{u})\right] = \tilde{\mu} \quad \text{and} \quad -\operatorname{div}\left[\tilde{\gamma}(y)\tilde{a}(y, D\tilde{v})\right] = 0 \quad (9.3.12)$$

on B_1. On the other hand, we want to ensure that $\tilde{a}(\cdot,\cdot)$ satisfies the assumption $(4.6.5)_2$. To see this, we estimate by $(4.6.5)_2$, which holds for the vector field $a(\cdot,\cdot)$, and $x := x_0 + 2Ry$:

$$\begin{aligned}
\langle \tilde{a}_z(y, z)\lambda, \lambda\rangle &= A^{2-p_2}\langle a_z(x_0 + 2Ry, Az)\lambda, \lambda\rangle \\
&\geq \sqrt{\nu} A^{2-p_2}\left(|Az|^2 + s^2\right)^{\frac{p(x_o+2Ry)-2}{2}} |\lambda|^2 \\
&= \sqrt{\nu} A^{p(x)-p_2}\left(|z|^2 + (s/A)^2\right)^{\frac{\tilde{p}(y)-2}{2}} |\lambda|^2, \quad (9.3.13)
\end{aligned}$$

where $\tilde{p}(y) := p(x_0 + 2Ry)$. Note that $\inf_{B_{2R}} p(\cdot) \leq \tilde{p}(\cdot) \leq \sup_{B_{2R}} p(\cdot)$. Our aim is now to estimate the expression $A^{p(x)-p_2}$. In a first step we write

$$\begin{aligned}
A &\leq \left[1 + |\mu|(\Omega)\right]^{\frac{1}{\gamma_1-1}} R^{-\frac{n-1}{\gamma_1}} + |\mu|(\Omega)\left[M + c(n)s + 1\right]^{2-\gamma_1} R^{-(n-1)-n(2-\gamma_1)} + 1 \\
&\leq c(n, \gamma_1, |\mu|(\Omega), M) R^{-c(n,\gamma_1)},
\end{aligned}$$

since $R \leq 1$. Having in mind (9.1.6), we therefore get

$$A^{p(x)-p_2} \geq c(n, \gamma_1, \gamma_2, |\mu|(\Omega), M) R^{c(n,\gamma_1)\omega(2R)} \geq c_\star(n, \gamma_1, \gamma_2, |\mu|(\Omega), M, \omega(\cdot)),$$

which gives with (9.3.13) in turn

$$\langle \tilde{a}_z(y, z)\lambda, \lambda\rangle \geq \frac{\sqrt{\nu}}{c_\star}\left(|z|^2 + (s/A)^2\right)^{\frac{\tilde{p}(y)-2}{2}} |\lambda|^2, \quad (9.3.14)$$

and means that $\tilde{a}(\cdot,\cdot)$ satisfies $(2.2.5)_2$ with $(\sqrt{\nu}, s)$ replaced by $(\sqrt{\nu}/c_\star, s/A)$, where $c_\star$ denotes the constant in the above estimate. Note that in order to prove (9.3.5) we only used $p_1 \leq p(\cdot) \leq p_2$. On the other hand we have to check that the assumption (9.3.4) is satisfied for the measure $\tilde{\mu}$ and the function $\tilde{u}$. Here we have to be careful since in the definitions of $\tilde{\mu}$ and $\tilde{u}$ we used the quantity A of (9.3.9) which involves the exponent p_2, whereas the assumption (9.3.4) is formulated with the exponent p_1. First, by the definition of A we directly see that $|\tilde{\mu}|(B_1) \leq A^{1-p_2}\left[|\mu|(B_{2R})/R^{n-1}\right] \leq 1$. Moreover we have

$$
\begin{aligned}
&|\tilde{\mu}|(B_1)\Big[\int_{B_1}\big(|D\tilde{u}|+s/A\big)\,dx\Big]^{2-p_1}\\
&\qquad\qquad\le c(p_1)\,A^{p_1-p_2-1}\,\frac{|\mu|(B_{2R})}{R^{n-1}}\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{2-p_1}. \qquad (9.3.15)
\end{aligned}
$$

Having in mind that $A \ge R$ we deduce by (9.1.6) that $A^{p_1-p_2} \le c(\omega(\cdot))$ and moreover that

$$
\begin{aligned}
&\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{2-p_1}\\
&\quad=\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{2-p_2}\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{p_2-p_1}\\
&\quad\le c\,R^{-n(p_2-p_1)}\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{2-p_2}\Big[\int_{\Omega}\big(|Du|+s\big)\,dx\Big]^{p_2-p_1}\\
&\quad\le c\,(M+1)^{p_2-p_1}\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{2-p_2},
\end{aligned}
$$

with $c \equiv c(n,\gamma_1,\gamma_2,\omega(\cdot))$, and therefore

$$
\begin{aligned}
&|\tilde{\mu}|(B_1)\Big[\int_{B_1}\big(|D\tilde{u}|+s/A\big)\,dy\Big]^{2-p_1}\\
&\qquad\qquad\le c\,A^{-1}\frac{|\mu|(B_{2R})}{R^{n-1}}\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{2-p_2}\le \tilde{c},
\end{aligned}
$$

for a constant $\tilde{c} \equiv \tilde{c}(n,\gamma_1,\gamma_2,|\Omega|,M,\omega(\cdot))$. Thus, the assumption (9.3.4) is satisfied and we can apply (9.3.5) to $\tilde{u}$ and $\tilde{v}$ and obtain

$$
\fint_{B_1}|D\tilde{u}-D\tilde{v}|\,dy \le c(n,\nu,\gamma_1,\gamma_2,|\mu|(\Omega),|\Omega|,M,\omega(\cdot)).
$$

The dependence of the constant on $|\mu|(\Omega)$ and $|\Omega|$ comes from the replacement of $\sqrt{\nu}$ by $\sqrt{\nu}/c_\star$, and we note in particular that the constant does not depend on A, since the constant in (9.3.5) is independent of s. Recalling the definitions of $\tilde{u}$, $\tilde{v}$ and A we arrive at

$$
\begin{aligned}
&\fint_{B_{2R}}|Du-Dv|\,dx\\
&\qquad\le c\Big[\frac{|\mu|(B_{2R})}{R^{n-1}}\Big]^{\frac{1}{p_2-1}}+c\Big[\frac{|\mu|(B_{2R})}{R^{n-1}}\Big]\Big[\fint_{B_{2R}}\big(|Du|+s\big)\,dx\Big]^{2-p_2}+c\,R. \qquad (9.3.16)
\end{aligned}
$$

It remains now to replace the exponent p_2 in the preceding estimate by p_0. For this aim we take use of (9.1.6) and (9.1.7) as follows: Replacing p_2 by p_0 in the first term of the right hand side makes difficulties only in the case that $|\mu|(B_{2R})/R^{n-1} \le 1$, since $\frac{1}{p_2-1} \le \frac{1}{p_0-1}$. Using (9.1.7) with the choices $A = \big[|\mu|(B_{2R})/R^{n-1}\big]^{\frac{1}{p_0-1}}$, $\alpha = 1$, $\sigma = \frac{p_0-1}{p_2-1}$ and $\tilde{\omega} = \frac{p_2-p_0}{p_2-1}$ we infer the following estimate

$$
\Big[\frac{|\mu|(B_{2R})}{R^{n-1}}\Big]^{\frac{1}{p_2-1}} \le c\Big[\frac{|\mu|(B_{2R})}{R^{n-1}}\Big]^{\frac{1}{p_0-1}}+c\,R, \qquad (9.3.17)
$$

with a constant $c \equiv c(\omega(\cdot))$. On the other hand, to replace p_2 by p_0 in the second term of the right-hand side in estimate (9.3.16) we use again (9.1.7), this time with the choices

$A = \fint_{B_{2R}}(|Du|+s)\,dx$, $\sigma = 2-p_2 > 0$, $\tilde\omega := p_2 - p_0$ and $\alpha = (2-p_0)^{-1} > 0$ to obtain

$$\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx\Big]^{2-p_2} \le c\,\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx\Big]^{2-p_0} + c\,R.$$

Here we used in the last estimate (9.1.6). Combining this with (9.3.16) we conclude the desired comparison estimate (9.3.3) in **the case $p_2 \le 2$**. It remains to consider **the case $p_1 < 2 \le p_2$**. Here we define the quantity

$$A := \Big[\frac{|\mu|(B_{2R})}{R^{n-1}}\Big]^{\frac{1}{p_2-1}} + \chi_{\{p_0<2\}}\frac{|\mu|(B_{2R})}{R^{n-1}}\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx\Big]^{2-p_0} + R,$$

and perform the same scaling as in (9.3.10) and (9.3.11). Then with the new quantity A, (9.3.12) and (9.3.13) hold true. With the same argumentation as before we observe that the ellipticity condition (9.3.14) holds and it remains to check condition (9.3.4). The condition $|\tilde\mu|(B_1) \le 1$ is again easy to see. Moreover, we observe that again (9.3.15) holds. It remains to consider the right-hand side of the preceding estimate. In **the case $p_0 \ge 2$** we have

$$\begin{aligned}
&\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx\Big]^{2-p_1}\\
&\quad\le \Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx+1\Big]^{2-p_0}\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx+1\Big]^{p_0-p_1}\\
&\quad\le c\,\big(M+1\big)^{p_1-p_0}R^{-n(p_o-p_1)}\Big[\int_{B_{2R}} \big(|Du|+s\big)\,dx+1\Big]^{2-p_0}\\
&\quad\le c(n,\gamma_1,\gamma_2,|\Omega|,M,\omega(\cdot)),
\end{aligned}$$

where we have used (9.1.6) and $2-p_0 \le 0$ in the last step. On the other hand we observe that $|\mu|(B_{2R})/R^{n-1} \le A^{p_2-1}$ and obtain in this way

$$|\tilde\mu|(B_1)\Big[\int_{B_1} \big(|D\tilde u|+s/A\big)\,dy\Big]^{2-p_1} \le c\,A^{p_1-p_2-1}A^{p_2-1}.$$

In the case that $A \ge 1$, we have $A^{p_1-2} \le 1$, whereas in the case $A < 1$ we can exploit that $A^{p_2-1} \le A$ and thus $A^{p_1-p_2-1}A^{p_2-1} \le A^{p_1-p_2} \le c(\omega(\cdot))$. Again we have used (9.1.6) for the last inequality. Altogether we have shown that (9.3.4) is fulfilled in **the case $p_0 \ge 2$**. In the remaining **case $p_0 < 2$** observe that by an analog argumentation to above we again see that

$$\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx\Big]^{2-p_1} \le c\,\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx\Big]^{2-p_0}$$

and thus

$$\begin{aligned}
&|\tilde\mu|(B_1)\Big[\int_{B_1} \big(|D\tilde u|+s/A\big)\,dy\Big]^{2-p_1}\\
&\qquad\le c\,A^{p_1-p_2-1}\frac{|\mu|(B_{2R})}{R^{n-1}}\Big[\fint_{B_{2R}} \big(|Du|+s\big)\,dx\Big]^{2-p_0}\\
&\qquad\le c\,A^{p_1-p_2} \le c,
\end{aligned}$$

hence also in this case (9.3.4) is satisfied. Applying (9.3.5) to to the functions $\tilde u$ and $\tilde v$ and subsequently using again (9.3.17) to exchange p_2 against p_0 on the right-hand side finally gives the estimate

$$\fint_{B_{2R}} |Du-Dv|\,dx \le c\,\Big[\frac{|\mu|(B_{2R})}{R^{n-1}}\Big]^{\frac{1}{p_0-1}}$$

$$+ c\,\chi_{\{p_0<2\}} \frac{|\mu|(B_{2R})}{R^{n-1}} \left[\fint_{B_{2R}} \big(|Du| + s\big)\, dx \right]^{2-p_0} + c\,R.$$

The constant c depends on n, γ_1, γ_2, ν, $|\mu|(\Omega)$, M and $\omega(\cdot)$ and the lemma is proved. □

Higher integrability and energy bounds. In this chapter we collect integrability properties for the solution v of the homogeneous problem. We note at this point that we have no higher integrability for the original solution u at hand, since the right hand side μ of the equation (4.6.1) is merely a L^1-function. However, the solution v of the homogenous problem shows at least local higher integrability properties and so does the solution w of the frozen homogeneous problem. We start with a higher integrability Lemma for v, which can be found in [147]; we refer the reader in particular to the discussion in [33, Remark 3.3] concerning the dependence of the constant.

LEMMA 9.9. *Let $u \in W^{1,p(\cdot)}(\Omega)$ be a solution to (4.6.1) under the assumptions (4.6.5), (4.6.6), (4.6.8) and (4.4.3). There exists a radius $R_1 \equiv R_1(n, L, \gamma_1, \gamma_2, \omega(\cdot)) \le 1$ such that the following holds: Let $v \in u + W_0^{1,p(\cdot)}(B_{2R})$ be the function defined in (9.3.1), with $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$, $0 < R \le R_1$. Then there exists $\delta_0 \equiv \delta_0(n, L/\nu, \gamma_1, \gamma_2) \in (0,1]$ such that $|Dv|^{p(\cdot)} \in L^{1+\delta_0}_{\mathrm{loc}}(B_{2R})$ and for every $\theta \in (0,1)$ and $\delta \in [0, \delta_0]$ the estimate*

$$\left[\fint_{B_{\theta\rho}} \big(|Dv| + s\big)^{p(x)(1+\delta)}\, dx \right]^{\frac{1}{1+\delta}} \le c \fint_{B_\rho} \big(|Dv| + s + \rho\big)^{p(x)}\, dx,$$

holds true whenever $B_\rho \subset B_{3R/2}$. Note that $c \equiv c(n, L/\nu, \gamma_1, \gamma_2, M, |\mu|(\Omega), |\Omega|, \theta)$, where M was defined in (4.6.3) and $c \to \infty$ as $\theta \nearrow 1$.

We point out that the higher integrability Lemma in [33] is formulated for the special situation of equations with non standard growth exponent $p : \Omega \to [2, \gamma_2]$. However, a closer look at [33, Remark 3.3] shows that only slight modifications have to be done to adapt the Lemma to the case $p(\cdot) \in [\gamma_1, \gamma_2]$. We sketch the proof since several intermediate results will be useful. We start from the following reverse Hölder's inequality which, in a slight different form, can be found before equation (3.15) in [33].

$$\int_{B_{\rho_1}} \big(|Dv|+s\big)^{p(x)}\, dx \le \frac{c}{(\rho_2 - \rho_1)^{\frac{\alpha}{1-\beta}}} \left[\int_{B_{\rho_2}} \big(|Dv|+s+R\big)\, dx \right]^{\frac{\gamma}{1-\beta}}, \tag{9.3.18}$$

for every $0 < \rho_1 < \rho_2 < 2R$, with $c \equiv c(n, \nu, \gamma_2, \omega(\cdot))$. Here we have

$$\alpha := n\Big(\frac{\vartheta p_2}{p_1} - 1\Big), \quad \beta := \frac{p_2}{p_1}\frac{p_1 - \vartheta}{p_1 - 1}, \qquad \gamma := \frac{p_2}{p_1}\frac{p_1(\vartheta - 1)}{p_1 - 1},$$

where $\vartheta := \sqrt{1 + 1/n}$ and we eventually reduced the value of R_1, in a fashion depending on $n, \gamma_2, \omega(\cdot)$, so that $\beta < 1$, see [33, (3.14)]. Now we fix $\rho_1 = 3R/2$ and the first result we derive from this inequality is the following:

REMARK. *Let v be the solution of (9.3.1). Then the following estimate holds:*

$$\left[\fint_{B_{3R/2}} \big(|Dv| + s\big)^{p(x)}\, dx \right]^{\frac{1}{p_0}} \le c \fint_{B_{7R/4}} \big(|Dv| + s + R\big)\, dx, \tag{9.3.19}$$

with c depending on $n, \nu, L, L_1, \gamma_1, \gamma_2, |\mu|(\Omega), |\Omega|, M$ and $\omega(\cdot)$. Recall that $R \le 1$.

PROOF. The proof of this Remark follows the lines of the one of [33, Lemma 3.5], here we just consider the case $p_0 < 2$ which is more involved due to the presence of an additional term. The case $p_0 \ge 2$ can be deduced adapting this proof or following [33].

In order to reduce the exponent $p(x)$ to the level 1, we take use of the reverse Hölder inequality (9.3.18). Note that by their definitions

$$\frac{n\gamma}{1-\beta} - n - \frac{\alpha}{1-\beta} = 0;$$

eventually reducing again R_1 we also have

$$0 \leq \frac{\gamma}{1-\beta} - p_0 \leq c(n,\gamma_2)\omega(4R),$$

as proved in [33, Lemma 3.5]. Hence, choosing $\rho_2 \equiv 7R/4$ and taking averages in (9.3.18), the goal now is estimating the quantity

$$\left[\fint_{B_{7R/4}} (|Dv| + s + R)\,dx\right]^{\frac{\gamma}{1-\beta}-p_0} \leq c\,\tilde{J}^{\frac{\gamma}{1-\beta}-p_0}$$

by Lemma 9.8, where we defined

$$\tilde{J} := \fint_{B_{2R}} (|Du| + s + R)\,dx + \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p_0-1}} + \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} (|Du| + s)\,dx\right)^{2-p_0},$$

The estimate of $\tilde{J}^{\frac{\gamma}{1-\beta}-p_0}$ is performed similarly as the estimate of J in [33, Lemma 3.5]; note that since $p_0 \geq \gamma_1 > 2 - 1/n$ and $R \leq 1$, we have $R^{(p_0-1)\left(1-\frac{n-1}{p_0-1}\right)} \leq 1$ and also the fact that both the exponents $(1-n)/(p_0-1)$ and $-n(2-p_0)-(n-1)$ are greater or equal to $-n$. Therefore, for instance, $R^{\frac{1-n}{p_o-1}(\frac{\gamma}{1-\beta}-p_o)} \leq R^{-c(n,\gamma_2)\omega(4R)} \leq c(n,\gamma_2,\omega(\cdot))$. This yields

$$\tilde{J}^{\frac{\gamma}{1-\beta}-p_0} \leq c(n, L, \gamma_2, M, |\mu|(\Omega), \omega(\cdot)),$$

which finishes the proof. □

Using once more Lemma 9.8 we get the following corollary:

COROLLARY 9.10. *Under the assumptions of Lemma 9.8, the following estimate holds true:*

$$\left[\fint_{B_{3R/2}} (s + |Dv|)^{p(x)}\,dx\right]^{\frac{1}{p_0}} \leq c \fint_{B_{2R}} (|Du| + s + R)\,dx + c\,\chi_{\{p_0<2\}}\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} (|Du| + s)\,dx\right)^{2-p_0} + c\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p_0-1}}, \tag{9.3.20}$$

for a constant and radii as in Remark 9.3.

REMARK. Corollary 9.10 shows that the $p(x)$-energy of Dv can be bounded in terms of a constant essentially depending on M, i.e. on the 1-energy of Du. Is is indeed easy to show this, using also the facts stated in the proof of Remark 9.3. To be more precise, our goal is to show that the constant appearing in higher integrability Lemma 9.9 depend on M and not upon $\int_\Omega |Dv|^{p(x)}\,dx$ as in the original paper of Zhikov [147]. Indeed in that paper the constant depends, roughly speaking, on $[\int_{B_\rho} |Dv|^{p(x)}\,dx]^{\omega(\rho)}$. It is now easy, using (9.1.6) and following [33], to show that the constant in Lemma 9.9 has the dependencies therein stated. See also the following

REMARK. From the estimate (9.3.20) we also directly conclude the following estimate, which will be useful later:

$$\fint_{B_R} \big(|Dv| + s\big)^{p(x)}\, dx \le c\, R^{-c(n,\gamma_2)},$$

for a constant c depending on $n, \nu, \gamma_1, \gamma_2, |\mu|(\Omega), |\Omega|, M$ and $\omega(\cdot)$.

Let $\delta_0 \equiv \delta_0(n, L/\nu, \gamma_1, \gamma_2) \in (0, 1]$ be the higher integrability exponent from Lemma 9.9; choose $R_2 \in (0, R_1]$ such that $\omega(4R_2) \le \delta_0/2$; thus $R_2 \equiv R_2(n, L/\nu, \gamma_1, \gamma_2, \omega(\cdot))$. For a radius $R \le R_2$ we define the exponents p_1 and p_2 as in (9.1.4) on the ball B_{2R}. By the choice of R_2, with $\delta_0 < 1$ and having in mind that $p_1 > 2 - 1/n \ge 1 + \delta_0/2$, we then find that

$$p_2\big(1 + \tfrac{\delta_0}{2}\big) \le \big(p_1 + \omega(4R)\big)\big(1 + \tfrac{\delta_0}{2}\big) \le \big(p_1 + \tfrac{\delta_0}{2}\big)\big(1 + \tfrac{\delta_0}{2}\big) \le p_1(1 + \delta_0),$$

and hence $p_2\big(1 + \frac{\delta_0}{2}\big) \le p(x)(1 + \delta_0)$ for all $x \in B_{2R}$. This implies that

$$Dv \in L^{p_2(1+\delta_0/2)}(B_R).$$

Moreover, following the argumentation in [33, Chapter 3.2] we infer the following energy bound for the function v: For any $\sigma \in [p_1, p_2(1 + \delta_0/2)]$ and $\tilde p \in [p_1, p_2 + \omega(4R)]$ there holds

$$\left[\fint_{B_R} \big(|Dv| + s\big)^{\sigma}\, dx\right]^{\frac{\tilde p}{\sigma}} \le c \fint_{B_{3R/2}} \big(|Dv| + s\big)^{p(x)}\, dx + c\, R^{2\tilde p}, \tag{9.3.21}$$

for a constant $c \equiv c(n, \nu, L, \gamma_1, \gamma_2, |\mu|(\Omega), |\Omega|, M, \omega(\cdot))$ and for any radius $0 < R \le R_2 \equiv R_2(n, L/\nu, \gamma_1, \gamma_2, \omega(\cdot))$.

Decay estimate for the reference problem II. We go once again back to Theorem 9.5 where we stated a decay estimate for the solution v to the homogeneous problem (9.2.1). However, for our aim, we have to replace the exponents $p(x)$ in this decay estimates by the exponent 1 on both sides of the inequality. This can now be done with the help of Lemma 9.8, following basically the argumentation of Remark 9.3. The outcome is the following

LEMMA 9.11. *Let $v \in u + W_0^{1,p(\cdot)}(B_{2R})$ be the weak solution to the Dirichlet problem (9.3.1) on B_{2R}, where the structure conditions (4.6.5), (4.6.6), (4.4.3) and (4.6.8) are in charge. Then there exists an exponent $\alpha_m \in (0, 1)$, depending on n, ν, L, γ_1 and γ_2, a constant $c \ge 1$, depending on n, ν, L, L_1, γ_1, γ_2, M, $|\mu|(\Omega)$, $|\Omega|$ and $\omega(\cdot)$ and a radius $R_1 \le 1$ which depends on n, L, γ_1, γ_2 and $\omega(\cdot)$, such that the estimate*

$$\fint_{B_\rho} (|Dv| + s)\, dx \le c \left(\frac{\rho}{r}\right)^{\alpha_m - 1} \fint_{B_r} (|Dv| + s)\, dx + c\, \rho^{\alpha_m - 1}$$

holds true for all concentric balls $B_\rho \subset B_r$ with radius $R \le R_1$ lying in the ball B_{2R}.

PROOF. We do not perform the proof in detail but we give the main arguments. In a first step we use Hölder's inequality and subsequently (9.1.7) with $\widetilde\omega := p(x) - p_1$ to obtain

$$\fint_{B_\rho} (|Dv| + s)\, dx \le c(\bar\alpha, \gamma_1, \gamma_2, \omega(\cdot)) \left[\fint_{B_\rho} (|Dv| + s + \rho^{\bar\alpha})^{p(x)}\, dx\right]^{\frac{1}{p_1}},$$

where $\bar\alpha > 0$ is arbitrary but fixed. The right-hand side of the previous inequality involves the exponent $p(x)$ and can therefore now be estimated from above with the help of Theorem 9.5 by the quantity

$$c\left[\left(\frac{\rho}{r}\right)^{p_2(\alpha_m - 1)} \fint_{B_{6r/7}} (|Dv| + s)^{p(x)}\, dx + c\, \rho^{p_2(\alpha_m - 1)} + \rho^{\bar\alpha p_1}\right]^{\frac{1}{p_1}}.$$

In a final step we use the argumentation analogous to (9.3.19) to reduce the exponent $p(x)$ inside the integral on the righ-hand side to exponent 1. Here, the comparison estimate of Lemma 9.8 is also needed and the restriction on the maximal radius R_1 comes into play. We finally adjust the appearing exponents by the localization technique also used in Remark 9.3 and since $\alpha_m - 1 < 0 < \bar{\alpha}$ to conclude the desired decay estimate with the claimed dependencies of the constants. □

Comparison to the homogeneous frozen problem. In this chapter we establish comparison estimates between Dv and Dw. We consider here a fixed ball $B_R(x_o)$ with $B_{2R}(x_o) \subset \Omega$ and denote $p_0 := p(x_o)$. In order to simultaneously deal with the cases $p_0 \geq 2$ and $p_0 < 2$, we introduce the following quantity:

$$p_* := \min\{2, p_0\}. \tag{9.3.22}$$

Moreover we define

$$\kappa := \min\Big\{1, \frac{2}{\gamma_2}\Big\} \leq \frac{p_*}{p_0} = \begin{cases} \dfrac{2}{p_0} & \text{if } p_0 \geq 2 \\ 1 & \text{if } p_0 < 2, \end{cases} \tag{9.3.23}$$

and we will take use of this quantity at various stages in the course of the proof.

LEMMA 9.12. *Let v be as in Lemma 9.9; moreover let the continuity assumption (4.6.7) hold and let $w \in v + W_0^{1,p_0}(B_R(x_0))$ be the solutions of the Dirichlet problem (9.3.2). Then there exists a constant c depending only on $n, \nu, L, L_1, \gamma_1, \gamma_2, M, |\mu|(\Omega)$ and $\omega(\cdot)$ and a radius $R_2 \equiv R_2(n, L/\nu, \gamma_1, \gamma_2, \omega(\cdot)) \leq 1$ such that whenever $0 < R \leq R_2$ the following estimate holds:*

$$\begin{aligned} &\fint_{B_R} |Dv - Dw|^{p_0}\, dx \\ &\quad \leq c\Big[L_1\omega(R)\log\big(\tfrac{1}{R}\big) + \big[\mathbf{v}(R)\big]^{\sigma_h}\Big]^{p_*}\Big[\fint_{B_{3R/2}} \big(|Dv| + s\big)^{p(x)}\, dx + R^{p_0}\Big], \end{aligned} \tag{9.3.24}$$

where p_* is defined in (9.3.22). Here

$$\sigma_h := \frac{\delta_0}{2(4+\delta_0)} \in (0,1)$$

and δ_0 denotes the higher integrability exponent coming from Lemma 9.9.

PROOF. The proof models on the one in [33], given for the case $\gamma_1 \geq 2$. Therefore we will only give the necessary modifications for the other case, namely when $p_0 < 2$. We note that the proof for $p_0 \geq 2$ is given in [33] without the presence of the coefficient $\gamma(x)$. However, the necessary modifications are similar to the ones we also need to carry out in the case $p_0 < 2$ and therefore again we only consider the case $p_0 < 2$ here. Let R_2 be the radius appearing in the discussion just above this Lemma, possibly again reduced in order to have $\omega(4R) \leq \delta_0/4$ for all $R \leq R_2$, δ_0 being the higher integrability exponent from Lemma 9.9. We have in a first step by Hölder's inequality:

$$\begin{aligned} \fint_{B_R} |Dv - Dw|^{p_0}\, dx \leq \Big[\fint_{B_R} \big(s^2 + |Dv|^2 + |Dw|^2\big)^{\frac{p_0-2}{2}} |Dv - Dw|^2\, dx\Big]^{\frac{p_0}{2}} \\ \times \Big[\fint_{B_R} \big(s^2 + |Dv|^2 + |Dw|^2\big)^{\frac{p_0}{2}}\, dx\Big]^{\frac{2-p_0}{2}}. \end{aligned} \tag{9.3.25}$$

Taking into account that

$$\fint_{B_R} \langle \gamma(x)a(x, Dv) - (\gamma)_{x_0,R}a(x_0, Dw), Dv - Dw\rangle\, dx = 0,$$

we estimate the first term of the right-hand side, using (9.1.2) and (4.6.8) in the following way:

$$\begin{aligned} \frac{\nu}{c(\gamma_2)} & \fint_{B_R} \big(|Dv|^2+|Dw|^2+s^2\big)^{\frac{p_0-2}{2}}|Dv-Dw|^2\,dx \\ & \le \fint_{B_R} (\gamma)_{x_o,R}\,\langle a(x_0,Dv)-a(x_0,Dw),Dv-Dw\rangle\,dx \\ & = \fint_{B_R} (\gamma)_{x_o,R}\,\langle a(x_0,Dv)-a(x,Dv),Dv-Dw\rangle\,dx \\ & \quad + \fint_{B_R} \Big((\gamma)_{x_o,R}-\gamma(x)\Big)\,\langle a(x,Dv),Dv-Dw\rangle\,dx \;=: I+II. \end{aligned} \tag{9.3.26}$$

The **first term** $\boldsymbol{I}$ is now treated with the continuity condition (4.6.7) and the bound (4.6.8) as follows:

$$\begin{aligned} I \le c\,L_1\omega(R) \fint_{B_R} & \Big[(s^2+|Dv|^2)^{\frac{p(x)-1}{2}}+(s^2+|Dv|^2)^{\frac{p_0-1}{2}}\Big] \\ & \times\Big[1+\big|\log(s^2+|Dv|^2)\big|\Big]|Dw-Dv|\,dx \end{aligned}$$

with $c\equiv c(n,L,\gamma_2)$. Using on the right-hand side above the elementary pointwise estimate

$$\begin{aligned} (s^2+|Dv|^2)^{\frac{p(x)-1}{2}}+(s^2+|Dv|^2)^{\frac{p_0-1}{2}} & \le (s^2+|Dv|^2+|Dw|^2)^{\frac{p_0-2}{4}} \\ & \times\big[(s^2+|Dv|^2+|Dw|^2)^{\frac{2p(x)-p_0}{4}}+(s^2+|Dv|^2+|Dw|^2)^{\frac{p_0}{4}}\big], \end{aligned}$$

and subsequently applying Young's inequality and re-absorbing one of the resulting terms into the left hand side, we infer

$$\begin{aligned} & \fint_{B_R} \big(s^2+|Dv|^2+|Dw|^2\big)^{\frac{p_0-2}{2}}|Dv-Dw|^2\,dx \\ & \quad \le c\,L_1^2\omega^2(R) \fint_{B_R} \Big[(s^2+|Dv|^2+|Dw|^2)^{\frac{2p(x)-p_0}{2}} \\ & \qquad\qquad + (s^2+|Dv|^2+|Dw|^2)^{\frac{p_0}{2}}\Big]\Big[1+\big|\log(s^2+|Dv|^2)\big|\Big]^2\,dx \\ & \quad \le c\,L_1^2\omega^2(R)\Big[I_1+I_2+I_3\Big], \end{aligned}$$

where we have abbreviated

$$\begin{aligned} I_1 &:= \fint_{B_R} \Big[(s^2+|Dv|^2)^{\frac{2p(x)-p_0}{2}}+(s^2+|Dv|^2)^{\frac{p_0}{2}}\Big]\times \\ & \qquad\qquad \times\Big[1+\big|\log(s+|Dv|)\big|\Big]^2\,dx, \\ I_2 &:= \fint_{B_R} |Dw|^{p_0}\Big[1+\big|\log(s+|Dv|)\big|\Big]^2\,dx, \\ I_3 &:= \fint_{B_R} |Dw|^{2p(x)-p_0}\Big[1+\big|\log(s+|Dv|)\big|\Big]^2\,dx. \end{aligned}$$

I_1 is treated exactly as in [33]; hence we can write

$$I_1 \le c\,\log^2\left(\tfrac{1}{R}\right)\fint_{B_{3R/2}} \big(s+|Dv|\big)^{p(x)}\,dx+R^{p_0}. \tag{9.3.27}$$

To estimate I_2, we split B_R into the sets $B_R\cap\{|Dv|\ge|Dw|\}$ and $B_R\cap\{|Dv|<|Dw|\}$. In this way we obtain

$$I_2 \le \fint_{B_R} \mathcal{V}_{p_0}(Dv)\,dx+\fint_{B_R} \mathcal{V}_{p_0}(Dw)\,dx \tag{9.3.28}$$

where we denote for the moment

$$\mathcal{V}_q(\xi) := |\xi|^q \Big[1 + \big|\log(s + |\xi|)\big|\Big]^2$$

and analogously, since $2p(x) - p_0 \geq 0$

$$I_3 \leq \fint_{B_R} \mathcal{V}_{2p(x)-p_0}(Dv)\,dx + \fint_{B_R} \mathcal{V}_{2p(x)-p_0}(Dw)\,dx. \tag{9.3.29}$$

We want to deal with both estimates simultaneously in the following way: we establish a pointwise estimate of $\mathcal{V}_q(Dv)$ and $\mathcal{V}_q(Dw)$ for $q \in [p_1 - \omega(4R), p_2 + \omega(4R)]$. Note that both p_0, $2p(x) - p_0$ lie in this interval. We proceed widely analogously to [33], but we nevertheless present the argumentation here since our assumption $p_0 \in (2 - \frac{1}{n}, 2)$ requires some additional comments. First we estimate $\mathcal{V}_q(Dw)$; to do so we split again B_R into three sets: $S_1 := \{x \in B_R : s + |Dw| \geq 1\}$, $S_2 := \{x \in B_R : s + |Dw| \in [R^{2\tilde{n}}, 1)\}$ and $S_3 := \{x \in B_R : s + |Dw| \in (0, R^{2\tilde{n}})\}$, where $\tilde{n} := 1 + \frac{n}{n-1}$.

Estimate on S_1: Using the fact that $s+|Dw| \geq 1$ on S_1 and the elementary inequality $\log(e + AB) \leq \log(e + a) + \log(e + b)$ for all $a, b > 0$, we deduce the pointwise estimate

$$\begin{aligned}\mathcal{V}_q(Dw) &\leq 4\big(s + |Dw|\big)^{p_2+\omega(4R)} \log^2\Big[e + (s + |Dw|)^{p_2+\omega(4R)}\Big] \qquad (9.3.30)\\ &\leq 8\big(s + |Dw|\big)^{p_2+\omega(4R)} \log^2\left[e + \frac{(s + |Dw|)^{p_2+\omega(4R)}}{\big((s + |Dw|)^{p_2+\omega(4R)}\big)_{B_R}}\right]\\ &\quad + 8\big(s + |Dw|\big)^{p_2+\omega(4R)} \log^2\Big[e + \big((s + |Dw|)^{p_2+\omega(4R)}\big)_{B_R}\Big].\end{aligned}$$

To calculate the integral over S_1, we first need to recall from [91] the inequality

$$\fint_{\mathcal{A}} |f| \log^2\Big(e + \frac{|f|}{(f)_{\mathcal{A}}}\Big)\,dx \leq c(q)\left(\fint_{\mathcal{A}} |f|^q\,dx\right)^{\frac{1}{q}} \tag{9.3.31}$$

which holds for all $f \in L^q(\mathcal{A})$, $\mathcal{A} \subset \mathbb{R}^n$, $|\mathcal{A}| > 0$ and for all $q > 1$. Integrating (9.3.30) over S_1 and dividing by $|B_R|$ we get

$$\frac{1}{|B_R|}\int_{S_1} \mathcal{V}_q(Dw)\,dx \leq 8(A + B),$$

where we define

$$A := \fint_{B_R} \big(s + |Dw|\big)^{p_2+\omega(4R)} \log^2\left[e + \frac{(s + |Dw|)^{p_2+\omega(4R)}}{\big((s + |Dw|)^{p_2+\omega(4R)}\big)_{B_R}}\right] dx$$

and

$$B := \fint_{B_R} \big(s + |Dw|\big)^{p_2+\omega(4R)} \log^2\Big[e + \big((s + |Dw|)^{p_2+\omega(4R)}\big)_{B_R}\Big]\,dx.$$

The **first integral A** is estimated using the logarithmic bound (9.3.31) with exponent $q(n, L/\nu, \gamma_1, \gamma_2) \equiv (4 + 2\delta_0)/(4 + \delta_0)$ and $f \equiv (s + |Dw|)^{p_2+\omega(4R)}$, Theorem 9.7 with q replaced by $q(p_2 + \omega(4R))$ to replace $|Dw|$ on the right hand side by $|Dv|$ and subsequently (9.3.21) with the choices $\sigma := q(p_2 + \omega(4R))$ and $\tilde{p} := p_2 + \omega(4R)$ (note that $q(p_2 + \omega(4R)) \leq p_2 q(1 + \delta_0/4) = p_2(1 + \delta_0/2)$, since $\omega(4R) \leq \delta_0/4$). We therefore achieve

$$\begin{aligned}A &\leq c\left[\fint_{B_R} \big(s + |Dw|\big)^{q(p_2+\omega(4R))}\,dx\right]^{\frac{1}{q}}\\ &\leq c\left[\fint_{B_R} \big(s + |Dv|\big)^{q(p_2+\omega(4R))}\,dx\right]^{\frac{1}{q}}\end{aligned}$$

$$\leq c \fint_{B_{3R/2}} \big(s + |Dv|\big)^{p(x)}\, dx + c\, R^{p_2+\omega(4R)}, \tag{9.3.32}$$

for a constant $c \equiv c(n, \nu, L, \gamma_1, \gamma_2, M, |\mu|(\Omega))$. We note that the constant in the first line depends also on $p_2(1 + \delta_0/2)$, but by Remark 9.2 and with $\delta_0 \equiv \delta_0(n, L/\nu, \gamma_1, \gamma_2)$ it can be replaced by $c \equiv c(n, L/\nu, \gamma_1, \gamma_2)$. To estimate the **integral B** we use Theorem 9.7 in order to pass over from the energy of Dw to the one of Dv, thereafter the higher integrability (9.3.21) for Dv together with the energy estimate of Remark 9.3. In particular we estimate as follows:

$$\begin{aligned}\fint_{B_R} \big(s + |Dw|\big)^{p_2+\omega(4R)}\, dx &\leq \fint_{B_R} \big(s + |Dv|\big)^{p_2+\omega(4R)}\, dx \\ &\leq c \fint_{B_{3R/2}} \big(s + |Dv|\big)^{p(x)}\, dx + c\, R^{2(p_2+\omega(4R))} \\ &\leq c \fint_{B_{3R/2}} \big(s + |Dv|\big)^{p(x)}\, dx + c\, R^{2p_0} \leq c\, R^{-c(n,\gamma_2)},\end{aligned} \tag{9.3.33}$$

with $c = c(n, L/\nu, \gamma_1, \gamma_2, M, |\mu|(\Omega), \omega(\cdot))$. Here we used moreover that $R^{2(p_2+\omega(4R))} \leq R^{2p_0} \leq R^{-c(n,\gamma_2)}$. Now, using twice the last estimate we easily deduce

$$\begin{aligned}B &\leq c\, \log^2\big(e + R^{-c(n,\gamma_2)}\big) \left[\fint_{B_{3R/2}} \big(s + |Dv|\big)^{p(x)}\, dx + c\, R^{2p_0} \right] \\ &\leq c\, \log^2\left(\tfrac{1}{R}\right) \left[\fint_{B_{3R/2}} \big(s + |Dv|\big)^{p(x)}\, dx + c\, R^{2p_0} \right],\end{aligned} \tag{9.3.34}$$

for a constant depending on $n, L/\nu, \gamma_1, \gamma_2, M, |\mu|(\Omega), \omega(\cdot)$. Recall that we again assumed that $R_0 \leq 1/e$ in the last step.

Estimate on S_2: We first observe the pointwise estimate

$$\big|\log(s + |Dw|)\big| \leq 2\tilde{n} \log\left(\tfrac{1}{R}\right) \quad \text{since} \quad R^{2\tilde{n}} \leq |Dw| + s < 1 \quad \text{on} \quad S_2.$$

Moreover, by possibly reducing R_0 to a value less than e^{-1} we obtain

$$1 + \big|\log(s + |Dw|)\big| \leq (1 + 2\tilde{n}) \log\left(\tfrac{1}{R}\right),$$

for $R \leq R_0$. On the other hand, noting that $(|Dw| + s)^{q-p(x)} \leq 1$ if $q \geq p(x)$ and $(|Dw| + s)^{q-p(x)} \leq R^{-2\tilde{n}(p(x)-q)} \leq R^{-2\tilde{n}(p(x)+\omega(4R)-p_1)} \leq c(\omega(\cdot))$ if $q < p(x)$ we conclude

$$\mathcal{V}_q(Dw) \leq c(n, \omega(\cdot))\, \log^2\left(\tfrac{1}{R}\right) \big(s + |Dw|\big)^{p(x)}.$$

Integrating the previous inequality over S_2 directly gives, using (9.3.21) and (9.1.6)

$$\frac{1}{|B_R|} \int_{S_2} \mathcal{V}_q(Dw)\, dx \leq c\, \log^2\left(\tfrac{1}{R}\right) \left[\fint_{B_{3R/2}} \big(s + |Dv|\big)^{p(x)}\, dx + R^{2p_0} \right], \tag{9.3.35}$$

with the constant depending on $n, \nu, L, \gamma_1, \gamma_2, M, |\mu|(\Omega), \omega(\cdot)$.

Estimate on S_3: Since $s + |Dw| < R^{2\tilde{n}}$, we have by elementary calculus $(s + |Dw|)^{1/2} \log^2(s + |Dw|) \leq 16e^{-2}$, so we can estimate pointwise

$$\mathcal{V}_q(Dw) \leq c\, |Dw|^q \log^2(s + |Dw|) \leq c\, \big(s + |Dw|\big)^{q-\frac{1}{2}};$$

note that $1 + |\log(s + |Dw|)| \leq 5|\log(s + |Dw|)|$ since $s + |Dw| \leq R_0^{2\tilde{n}} \leq e^{-4}$ and moreover

$$q - \frac{1}{2} \geq p_1 - \omega(4R) - \frac{1}{2} \geq p_1 - 1 \geq \frac{p_1}{\tilde{n}} \geq \frac{p_2}{2\tilde{n}} \geq \frac{p_0}{2\tilde{n}}.$$

In the previous chain of inequalities we used the lower bound of the exponent function $p(\cdot) > 2 - \frac{1}{n}$ together with the definition of $\tilde{n}$ and (9.1.5). Combining the previous two estimates we arrive at

$$\mathcal{V}_q(Dw) \le c\, R^{2\tilde{n}\frac{p_0}{2\tilde{n}}} = c\, R^{p_0},$$

which holds pointwise on S_3. Integrating this over S_3 therefore gives

$$\frac{1}{|B_R|}\int_{S_3} \mathcal{V}_q(Dw) \le c\, R^{p_0}. \tag{9.3.36}$$

Finally we combine the estimates (9.3.32), (9.3.34), (9.3.35) and (9.3.36) to arrive at

$$\begin{aligned}
&\fint_{B_R} \Big(\mathcal{V}_{p_0}(Dw) + \mathcal{V}_{2p(x)-p_0}(Dw)\Big)\,dx \\
&\qquad\qquad \le c\,\log^2\big(\tfrac{1}{R}\big)\fint_{B_{3R/2}} \big(s+|Dv|\big)^{p(x)}\,dx + c\,R^{p_0},
\end{aligned} \tag{9.3.37}$$

where the constant c depends on $n, L/\nu, \gamma_1, \gamma_2, M, |\mu|(\Omega), \omega(\cdot)$. It finally remains to **estimate the integrals** of $\mathcal{V}_{p_0}(Dv)$ and $\mathcal{V}_{2p(x)-p_0}(Dv)$, where – in contrary to (9.3.37) – the function w is replaced by v. However, this case is even easier to see, since we can argue directly on the energy of Dv and omit the pass-over from Dw to Dv on the right hand side, as for example done in (9.3.33). Indeed, we can repeat the pointwise argumentation above, replacing w by v (also in the definition of the sets S_1 to S_3). Then we integrate over B_R and obtain finally

$$\begin{aligned}
&\fint_{B_R} \Big(\mathcal{V}_{p_0}(Dv) + \mathcal{V}_{2p(x)-p_0}(Dv)\Big)\,dx \\
&\qquad\qquad \le c\,\log^2\big(\tfrac{1}{R}\big)\fint_{B_{3R/2}} \big(s+|Dv|\big)^{p(x)}\,dx + c\,R^{p_0},
\end{aligned} \tag{9.3.38}$$

for a constant with the same dependencies as the one in (9.3.37). Combining the estimates (9.3.37) and (9.3.38) with (9.3.28) and (9.3.29) and merging this with (9.3.27) we conclude

$$I \le c\, L_1^2 \omega^2(R) \log^2\big(\tfrac{1}{R}\big)\bigg[\fint_{B_{3R/2}} \big(s+|Dv|\big)^{p(x)}\,dx + R^{p_0}\bigg].$$

The constant here depends on n, ν, γ_1, γ_2, M, $|\mu|(\Omega)$ and $\omega(\cdot)$. In a second step we **consider the expression II** in (9.3.26). By the growth condition $(4.6.5)_2$ and Young's inequality we have in a first step

$$\begin{aligned}
|II| &\le \sqrt{L}\fint_{B_R} \big|\gamma(x) - (\gamma)_{x_o,R}\big|(s^2+|Dv|^2)^{\frac{p_0-1}{2}}|Dw-Dv|\,dx \\
&\le \varepsilon \fint_{B_R} \big(s^2+|Dv|^2+|Dw|^2\big)^{\frac{p_0-2}{2}}|Dv-Dw|^2\,dx \\
&\qquad + c(\varepsilon) L \fint_{B_R} \big|\gamma(x) - (\gamma)_{x_o,R}\big|^2\big(s^2+|Dv|^2+|Dw|^2\big)^{\frac{p_0}{2}}\,dx \\
&=: II_1 + II_2.
\end{aligned}$$

II_1 can be absorbed into the left hand side of (9.3.26) by choosing $\varepsilon := \frac{1}{2}\frac{\nu}{c(\gamma_2)}$. The term II_2 can be handled by (4.6.8), (4.6.10), the higher integrability result for Dv in terms of the Lemma 9.9 and the Calderón-Zygmund type estimate for Dw in terms of Theorem 9.7 as follows:

$$II_2 \le c\bigg[\fint_{B_R} \big|\gamma(x) - (\gamma)_{x_o,R}\big|^{2(1+\frac{4}{\delta_0})}\,dx\bigg]^{\frac{\delta_0}{4+\delta_0}}$$

$$
\begin{aligned}
&\times \left[\fint_{B_R} \left(s^2+|Dv|^2+|Dw|^2\right)^{\frac{p_0}{2}(1+\frac{\delta_0}{4})}\,dx \right]^{\frac{4}{4+\delta_0}} \\
&\le cL^{\frac{8+\delta_0}{4+\delta_0}} \left[\fint_{B_R} \left|\gamma(x)-(\gamma)_{x_o,R}\right|\,dx \right]^{\frac{\delta_0}{4+\delta_0}} \times \\
&\times \left[\fint_{B_R} \left(s^2+|Dv|^2\right)^{\frac{p_0}{2}(1+\frac{\delta_0}{4})}\,dx \right]^{\frac{4}{4+\delta_0}} \\
&\le c\,\mathbf{v}(R)^{\frac{\delta_0}{4+\delta_0}} \left[\fint_{B_{3R/2}} \left(s^2+|Dv|^2\right)^{\frac{p(x)}{2}}\,dx + cR^{2p_0} \right].
\end{aligned}
$$

Here we have used in the last step the estimate (9.3.21) with the choices $\sigma = p_0(1+\delta_0/4)$ and $\tilde{p} = p_0$. Thus, combining the estimates for I and II we arrive at

$$
\begin{aligned}
&\fint_{B_R} \left(|Dv|^2+|Dw|^2+s^2\right)^{\frac{p_0-2}{2}}|Dv-Dw|^2\,dx \\
&\le c\Big[L_1^2\big(\omega(R)\log\tfrac{1}{R}\big)^2 + \mathbf{v}(R)^{\frac{\delta_0}{4+\delta_0}}\Big]\Big[\fint_{B_{3R/2}} (s+|Dv|)^{p(x)}\,dx + cR^{p_0}\Big].
\end{aligned}
\tag{9.3.39}
$$

Notice that the previous lines apply also to the case $p_0 \ge 2$ with minor changes and give the missing estimate of [33] we need. In a very last step, we consider the second term on the right hand side of (9.3.25). Here we proceed analogously to (9.3.33): We first use the Calderón-Zygmund type estimate for Dw in terms of Theorem 9.7 and thereafter the higher integrability estimate (9.3.21) (with $\sigma = \tilde{p} \equiv p_0$) to conclude

$$
\begin{aligned}
III := \fint_{B_R} \left(s^2+|Dv|^2+|Dw|^2\right)^{\frac{p_0}{2}}\,dx &\le c\fint_{B_R} \left(s^2+|Dv|^2\right)^{\frac{p_0}{2}}\,dx \\
&\le c\Big[\fint_{B_{3R/2}} (s+|Dv|)^{p(x)}\,dx + R^{2p_0}\Big],
\end{aligned}
$$

for a constant c which depends on n, ν, L, γ_1, γ_2, M and $|\mu|(\Omega)$. Combining this estimate with (9.3.39) and (9.3.25) proves the comparison estimate (9.3.24) in the case $p_0 < 2$; taking into account that the case $p_0 \ge 2$ is Lemma 3.4 in [33], the proof is complete. □

Combining the two comparison results Lemma 9.8 and Lemma 9.12 in terms of Remark 9.10, using Hölder's inequality together with the fact that (4.4.3) and $\mathbf{v}(\rho) \le c(L)$ hold, leads to the following Lemma, which will show to be useful when assuming conditions (4.6.14) and (4.6.15) holding true.

LEMMA 9.13. *Under the assumptions of Lemma 9.12, let $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$, $p_0 \equiv p(x_o)$, u the solution to (4.6.1) and w the solution to (9.3.2) on B_R. Then there exists a constant $c \equiv c(n,\nu,L,\gamma_1,\gamma_2,M,|\mu|(\Omega),|\Omega|,\omega(\cdot))$ and a radius $R_2 \le 1$ depending upon $n, L/\nu, L_1, \gamma_1, \gamma_2, \omega(\cdot)$ such that whenever $0 < R \le R_2$ the following estimate holds:*

$$
\begin{aligned}
\fint_{B_R} |Du-Dw|\,dx &\le c\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p_0-1}} \\
&+ c\,\chi_{\{p_0<2\}}\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} (|Du|+s)\,dx\right)^{2-p_0} \\
&+ c\Big[L_1\omega(R)\log\big(\tfrac{1}{R}\big) + \big[\mathbf{v}(R)\big]^{\sigma_h}\Big]^{\frac{p_*}{p_0}}\Big[\fint_{B_{2R}} (|Du|+s)\,dx + R\Big].
\end{aligned}
$$

□

The further immediate consequences of the comparison estimates we gained for solutions to homogeneous equations with $p(x)$-growth and measurable coefficients and for homogeneous equations with constant p_0 growth are the following reference estimates for the initial solution u. For the convenience of the reader we recall that the exponent $\alpha_m \in (0,1)$ denotes the maximal Hölder exponent available by Theorem 9.5 or Lemma 9.11, for solutions to homogeneous equations (9.2.1) with $p(x)$-growth structure.

LEMMA 9.14. *Let $u \in C^1(\Omega)$ be a weak solution to equation (4.6.1) under the conditions (4.6.5) to (4.6.8) and the logarithmic continuity condition (4.4.3) on $\omega(\cdot)$. Then there exists $c \equiv c(n, \nu, L, L_1, \gamma_1, \gamma_2, M, |\mu|(\Omega), |\Omega|, \omega(\cdot))$ and a radius $R_1 \le 1$, depending on $n, \nu, L, \gamma_1, \gamma_2, \omega(\cdot)$, such that for all concentric balls $B_\rho \subset B_R \subset \Omega$ with radius $R \le R_1$ there holds*

$$
\begin{aligned}
\fint_{B_\rho} (|Du| + s)\, dx &\le c\left(\frac{\rho}{R}\right)^{-1+\alpha_m} \fint_{B_R} (|Du| + s)\, dx \\
&+ c\left(\frac{R}{\rho}\right)^n \left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{p_0-1}} + c\, R\left(\frac{R}{\rho}\right)^n + c\, \rho^{\alpha_m - 1} \\
&\qquad + c\, \chi_{\{p_0<2\}} \left(\frac{R}{\rho}\right)^n \left[\frac{|\mu|(B_R)}{R^{n-1}}\right]\left[\fint_{B_R} (|Du| + s)\, dx\right]^{2-p_0}.
\end{aligned}
$$

PROOF. The proof is done via comparison. We take the estimate of Lemma 9.11 for the solution v to the homogeneous equation and apply twice the comparison Lemma 9.8 to transfer this estimate to the solution u. Indeed we have the chain of estimates

$$
\begin{aligned}
\fint_{B_\rho} (|Du| + s)\, dx &\le \fint_{B_\rho} (|Dv| + s)\, dx + c\left(\frac{R}{\rho}\right)^n \fint_{B_R} |Du - Dv|\, dx \\
&\le c\left(\frac{\rho}{R}\right)^{-1+\alpha_m} \fint_{B_R} (|Dv| + s)\, dx + c\, \rho^{\alpha_m - 1} \\
&\qquad + c\left(\frac{R}{\rho}\right)^n \fint_{B_R} |Du - Dv|\, dx \\
&\le c\left(\frac{\rho}{R}\right)^{-1+\alpha_m} \fint_{B_R} (|Du| + s)\, dx + c\, \rho^{\alpha_m - 1} \\
&\quad + c\left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^{-1+\alpha_m}\right] \fint_{B_R} |Du - Dv|\, dx.
\end{aligned}
$$

The statement now follows by applying Lemma 9.8 to estimate the last integral in the preceding estimate. □

LEMMA 9.15. *Let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution to (4.6.1) under the structure conditions (4.6.5) to (4.6.8) and (4.4.3). Then there exists a constant $c \ge 1$ depending at most on n, ν, L, γ_1 and γ_2, such that for all concentric balls $B_\rho(x_0) \subset B_R(x_0) \subset \Omega$ with radius $R \le R_2$ – denoting by $R_2 \equiv R_2(n, L/\nu, L_1, \gamma_1, \gamma_2, \omega(\cdot))$ the maximal radius appearing in Lemma 9.13 – the following estimate holds:*

$$
\begin{aligned}
\fint_{B_\rho} (|Du| + s)\, dx &\le c \fint_{B_R} (|Du| + s)\, dx + c\left(\frac{R}{\rho}\right)^n \left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{p_0-1}} \\
&+ c\, \chi_{\{p_0<2\}} \left(\frac{R}{\rho}\right)^n \left[\frac{|\mu|(B_R)}{R^{n-1}}\right]\left[\fint_{B_R} (|Du| + s)\, dx\right]^{2-p_0} + c\, R\left(\frac{R}{\rho}\right)^n \\
&\qquad + c\Big[L_1 \omega(R) \log \tfrac{1}{R} + \big[\mathbf{v}(R)\big]^{\sigma_h}\Big]^{\frac{p_*}{p_0}} \left(\frac{R}{\rho}\right)^n \fint_{B_R} (|Du| + s)\, dx.
\end{aligned}
$$

Here we have $p_0 := p(x_0)$ and $p_* := \min\{2, p_0\}$.

PROOF. The proofs work exactly as the one of Lemma 9.14 via comparison: for $\rho \leq R/2$ this time we involve as "reference estimates" (9.2.4) and comparison Lemma 9.13, while the case $\rho \in (R/2, R]$ is trivial. □

Again for the convenience of the reader we recall that the exponent $\beta_m \in (0,1)$ denotes the maximal Hölder exponent due to Theorem 9.6 for the gradient of solutions to homogeneous frozen equations (9.2.2) with constant growth p_0. At this point also the following Lemma follows plainly:

LEMMA 9.16. *Let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution to* (4.6.1) *under the structure conditions* (4.6.5) *to* (4.6.8) *and* (4.4.3). *Then there exists a constant $c \geq 1$ depending at most on $n, \nu, L, L_1, \gamma_1, \gamma_2$ such that for all concentric balls $B_\rho(x_0) \subset B_R(x_0) \subset \Omega$ with radius $R \leq R_2$, R_2 being the radius appearing in Lemma 9.13, the following estimate holds:*

$$\begin{aligned}\fint_{B_\rho} |Du - (Du)_{B_\rho}|\,dx &\leq c\Big(\frac{\rho}{R}\Big)^{\beta_m} \fint_{B_R} |Du - (Du)_{B_R}|\,dx \\ &\quad + c\Big[L_1\omega(R)\log\tfrac{1}{R} + \big[\mathbf{v}(R)\big]^{\sigma_h}\Big]^{\frac{p_*}{p_0}} \Big(\frac{R}{\rho}\Big)^n \fint_{B_R} (|Du| + s)\,dx \\ &\quad + c\,\chi_{\{p_0<2\}}\Big(\frac{R}{\rho}\Big)^n \Big[\frac{|\mu|(B_R)}{R^{n-1}}\Big]\Big[\fint_{B_R}(|Du|+s)\,dx\Big]^{2-p_0} \\ &\qquad + c\Big(\frac{R}{\rho}\Big)^n\Big[\frac{|\mu|(B_R)}{R^{n-1}}\Big]^{\frac{1}{p_0-1}} + c\,R\Big(\frac{R}{\rho}\Big)^n.\end{aligned}$$

PROOF. The proof is completely similar to that of Lemma 9.15, once using comparison estimate (9.13) into (9.2.3). □

9.4. Proofs of the Theorems

PROOF OF THEOREM 9.1. In the following let $R_0 > 0$ be a "maximal radius" which will at several stages be restricted to smaller values, in dependence of the structure conditions, in particular we demand R_0 to be, from now on, smaller than the occurring maximal radii appearing in Lemma 9.9 and Lemma 9.12. Therefore we have

$$R_0 \equiv R_0(n, L, \nu, \gamma_1, \gamma_2, L_1, \omega(\cdot)).$$

Further restrictions may possibly come up in the course of the proof. We prove Theorem 9.1, basically following widely the ideas of [103].

Proof of estimate (9.0.23)*:* Our aim is to show in a first step the following estimate:

$$\begin{aligned}M_{1-\alpha,R}(Du)(x) \leq c\Big[M_{p(\cdot)-\alpha(p(\cdot)-1),R}(\mu)(x)\Big]^{\frac{1}{p(x)-1}} \\ + c\,R^{1-\alpha}\fint_{B_R}(|Du| + s + R)\,d\xi;\end{aligned} \tag{9.4.1}$$

then (9.0.23) follows from this estimate via (3.2.8). We shall first show the estimate **for a sufficiently small radius**. Take concentric balls $B_\rho \subset B_{r/2} \subset B_r \subset B_R$ with center x and $R \leq R_0$. Having at hand the identities

$$r^{1-\alpha}\Big[\frac{|\mu|(B_r)}{r^{n-1}}\Big]^{\frac{1}{p(x)-1}} = \Big[\frac{|\mu|(B_r)}{r^{n-p(x)+\alpha(p(x)-1)}}\Big]^{\frac{1}{p(x)-1}}, \tag{9.4.2}$$

and

$$r^{1-\alpha}\Big[\frac{|\mu|(B_r)}{r^{n-1}}\Big]\Big[\fint_{B_r}(|Du| + s)\,d\xi\Big]^{2-p(x)}$$

$$= \frac{|\mu|(B_r)}{r^{n-p(x)+\alpha(p(x)-1)}} \left[r^{1-\alpha} \fint_{B_r} \big(|Du| + s\big)\, d\xi \right]^{2-p(x)}, \quad (9.4.3)$$

the estimate of Lemma 9.15 with $R = r$ multiplied by $\rho^{1-\alpha}$ reads as follows

$$\begin{aligned} \rho^{1-\alpha} \fint_{B_\rho} (|Du| + s)\, d\xi &\le c \left(\frac{\rho}{r}\right)^{1-\alpha} r^{1-\alpha} \fint_{B_r} (|Du| + s)\, d\xi \\ &+ c\, \chi_{\{p(x)<2\}} \left(\frac{r}{\rho}\right)^{n-1+\alpha} \frac{|\mu|(B_r)}{r^{n-p(x)+\alpha(p(x)-1)}} \times \\ &\times \left[r^{1-\alpha} \fint_{B_r} \big(|Du| + s\big)\, d\xi \right]^{2-p(x)} \\ &+ c \left(\frac{r}{\rho}\right)^{n-1+\alpha} \left[\frac{|\mu|(B_r)}{r^{n-p(x)+\alpha(p(x)-1)}} \right]^{\frac{1}{p(x)-1}} + c\, \frac{r^{n+1}}{\rho^{n+\alpha-1}} \\ &+ c \left(\frac{r}{\rho}\right)^{n-1+\alpha} \Big[L_1 \omega(r) \log \tfrac{1}{r} + \big[\mathbf{v}(r)\big]^{\sigma_h} \Big]^{\frac{p_*(x)}{p(x)}} r^{1-\alpha} \fint_{B_r} (|Du| + s)\, d\xi, \end{aligned} \quad (9.4.4)$$

with a constant $c \equiv c(n, \nu, L, \gamma_1, \gamma_2)$ and this estimates holds for all $\rho \le r \le R$. Recall that $p_*(x) = \min\{2, p(x)\}$. Now, we choose $H \equiv H(n, \nu, L, \gamma_1, \gamma_2, \tilde{\alpha}) > 2$ large enough to have

$$c\left(\frac{1}{H}\right)^{1-\alpha} \le c\left(\frac{1}{H}\right)^{1-\tilde{\alpha}} \le \frac{1}{8}, \quad (9.4.5)$$

and moreover we choose $\delta_i \equiv \delta_i(n, \gamma_1, \gamma_2, \nu, L, L_1, \tilde{\alpha}) < 1/(2L_1)$, $i = 1, 2$, so small that

$$H^n \big[L_1 \delta_1 + [\delta_2]^{\sigma_h} \big]^{\frac{2}{\gamma_2}} \le \frac{1}{8}, \quad (9.4.6)$$

and finally we decrease $R_0 \equiv R_0(n, \gamma_1, \gamma_2, \nu, L, \tilde{\alpha}, \omega(\cdot)) > 0$ taking use of condition (4.6.17) in order to have

$$\begin{aligned} H^n \Big[L_1 \omega(r) \log \tfrac{1}{r} + \big[\mathbf{v}(r)\big]^{\sigma_h} \Big]^{\frac{p_*(x)}{p(x)}} & \\ \le H^n \Big[L_1 \sup_{r \in (0, R_0]} \omega(r) \log \tfrac{1}{r} + \big[\mathbf{v}(R_0)\big]^{\sigma_h} \Big]^{\frac{p_*(x)}{p(x)}} & \\ \le H^n \big[L_1 \delta_1 + [\delta_2]^{\sigma_h} \big]^{\kappa} \le \frac{1}{8}. & \end{aligned} \quad (9.4.7)$$

Choosing in (9.4.4) $\rho = r/H$ and exploiting step by step the smallness conditions above, we finally end up with the estimate

$$\begin{aligned} &\left(\frac{r}{H}\right)^{1-\alpha} \fint_{B_{r/H}} (|Du| + s)\, d\xi \\ &\le \frac{r^{1-\alpha}}{4} \fint_{B_r} (|Du| + s)\, d\xi + c \left[\frac{|\mu|(B_r)}{r^{n-p(x)+\alpha(p(x)-1)}} \right]^{\frac{1}{p(x)-1}} \\ &+ c\, \chi_{\{p(x)<2\}} \frac{|\mu|(B_r)}{r^{n-p(x)+\alpha(p(x)-1)}} \left[r^{1-\alpha} \fint_{B_r} \big(|Du| + s\big)\, d\xi \right]^{2-p(x)} + c\, r^{2-\alpha}. \end{aligned}$$

At this point we proceed exactly as the authors in [103]. We take the supremum over all radii, noting that $r \le R$ is still arbitrary, and therefore introduce the maximal functions, use Young's inequality and reabsorb into the left-hand side to arrive at

$$\begin{aligned} M_{1-\alpha,R}(|Du| + s)(x) &\le c\, R^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi \\ &+ c \Big[M_{p(\cdot)+\alpha(p(\cdot)-1),R}(\mu)(x) \Big]^{\frac{1}{p(x)-1}} + c\, R^{2-\alpha}, \end{aligned} \quad (9.4.8)$$

for a constant $c \equiv c(n, \nu, L, \gamma_1, \gamma_2, L_1, \tilde{\alpha})$. All in all, we conclude that this estimate holds true for all $R \le R_0$, smaller than R_1, R_2 and satisfying (9.4.7). Now we **remove the smallness condition on R** by a standard argumentation (see for example the proof of (1.35), Step 2 in [**103**]), which we will sketch for the convenience of the reader and for the fact that we will use this argumentation at some points also later in the proofs. Having (9.4.1) at hand for radii $R \le R_0$, we find in the case $R > R_0$ that

$$M_{1-\alpha,R}(Du)(x) \le M_{1-\alpha,R_0}(Du)(x) + \left(\frac{R}{R_0}\right)^n R^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi.$$

On the other hand we trivially have

$$M_{p(\cdot)-\alpha(p(\cdot)-1),R_0}(\mu)(x) \le M_{p(\cdot)-\alpha(p(\cdot)-1),R}(\mu)(x).$$

Therefore, starting with the second-last inequality, then exploiting (9.4.1) with the radius $R = R_0$ and thereafter using the last inequality, we eventually obtain

$$\begin{aligned} M_{1-\alpha,R}(Du)(x) \le c \left[M_{p(\cdot)-\alpha(p(\cdot)-1),R}(\mu)(x)\right]^{\frac{1}{p(x)-1}} \\ + c\left(\frac{R}{R_0}\right)^n \fint_{B_R} (|Du| + s + R)\, d\xi. \end{aligned}$$

Since $R \le \operatorname{diam}(\Omega)$ and the constant c here depends on n, L, ν, γ_1, γ_2, $\tilde{\alpha}$ and $\omega(\cdot)$, we conclude the estimate (9.4.1) for all radii R such that $B_R \subset \Omega$ and with a constant enlarged by the factor $(\operatorname{diam}(\Omega)/R_0)^n$. Since R_0 depends on $n, \nu, L, L_1, \gamma_1, \gamma_2$, and $\omega(\cdot)$, we obtain the final dependence of the constant c as stated in estimate (9.0.23).

Proof of estimate (9.0.24) *for small radii:* Since the estimate is a pointwise one valid in the fixed point x, we can follow exactly the argumentation in [**103**]. For the convenience of the reader we mention the main steps of the argumentation, but refer to [**103**] for a detailed discussion.

Dyadic sequence: We let $H > 1$ and define the dyadic sequence of balls

$$B_i := B_{R/H^i}(x) := B_{R_i}(x), \quad i = 0, 1, 2, \ldots$$

Moreover we define

$$A_i := \fint_{B_i} |Du - (Du)_{B_i}|\, d\xi, \qquad k_i := |(Du)_{B_i}|.$$

Choosing now $H \equiv H(n, \nu, L, \gamma_1, \gamma_2)$ large enough to have

$$c\left(\frac{1}{H}\right)^{\beta_m} \le \frac{1}{16},$$

where β_m denotes the maximal exponent appearing in (9.2.3) and c the constant therein appearing, and applying Lemma 9.16 on the balls $B_\rho \equiv B_{i+1} \subset B_i \equiv B_R$, we achieve

$$\begin{aligned} A_{i+1} \le \frac{1}{16} A_i + \tilde{c}\, H^n \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{p(x)-1}} \\ + \tilde{c}\, H^n \left[L_1 \omega(R_i) \log \tfrac{1}{R_i} + [\mathbf{v}(R_i)]^{\sigma_h}\right]^{\frac{p_*(x)}{p(x)}} \fint_{B_i} (|Du| + s)\, d\xi \\ + \tilde{c}\, \chi_{\{p(x)<2\}} H^n \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right] \left[\fint_{B_i} (|Du| + s)\, dx\right]^{2-p(x)} + \tilde{c}\, H^n R_i, \end{aligned}$$

for a constant $\tilde{c} \equiv \tilde{c}(n, \nu, L, \gamma_1, \gamma_2)$. Notice that, in contrary to (3.16) of [**103**] this estimate holds true, provided that $R \le R_2$ where $R_2 \equiv R_2(n, L, \nu, \gamma_1, \gamma_2, \omega(\cdot))$ denotes the maximal radius determined in Lemma 9.16. Now, we further restrict the maximal radius by imposing the smallness condition $R \le R_3$, where R_3 is chosen in dependence on n, ν, L, L_1, γ_1, γ_2 and $\omega(\cdot)$, such that

$$\tilde{c}\, H^n \Big[L_1\omega(R_i)\log \tfrac{1}{R_i} + \big[\mathbf{v}(R_i)\big]^{\sigma_h}\Big]^{\frac{p_*(x)}{p(x)}} \le \tilde{c}\, H^n \Big[L_1\omega(R_3)\log \tfrac{1}{R_3} + \big[\mathbf{v}(R_3)\big]^{\sigma_h}\Big]^{\kappa} \le \frac{1}{16}.$$

This is possible, since the logarithmic Dini conditions (4.6.14) and (4.6.15) surely imply $\omega(\rho)\log\frac{1}{\rho} \to 0$ and $\mathbf{v}(\rho) \to 0$ as $\rho \to 0$. By this fact, the above estimate can be written as

$$\begin{aligned} A_{i+1} &\le \frac{1}{8}A_i + c\Big[L_1\omega(R_i)\log \tfrac{1}{R_i} + \big[\mathbf{v}(R_i)\big]^{\sigma_h}\Big]^{\frac{p_*(x)}{p(x)}}(k_i+s) \\ &\quad + c\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{p(x)-1}} + c\,\chi_{\{p(x)<2\}}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left[\fint_{B_i}(|Du|+s)\,dx\right]^{2-p(x)} \\ &\quad + c\,R_i, \end{aligned} \tag{9.4.9}$$

for all $i \in \mathbb{N}_0$, since

$$\fint_{B_i}|Du|\,d\xi \le A_i + k_i.$$

Now follow line by line the argumentation in (3.20) to (3.24) of [103]: applying iteratively the preceding estimate on the dyadic sequence, we get

$$\begin{aligned} k_{m+1} &\le c\fint_{B_R}\big|Du-(Du)_{B_R}\big|\,d\xi + c\fint_{B_R}\big|Du\big|\,d\xi + c\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{p(x)-1}} \\ &\quad + c\sum_{i=0}^{m}\Big[L_1\omega(R_i)\log \tfrac{1}{R_i} + \big[\mathbf{v}(R_i)\big]^{\sigma_h}\Big]^{\frac{p_*(x)}{p(x)}}(k_i+s) \\ &\quad + c\,\chi_{\{p(x)<2\}}\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left[\fint_{B_i}\big(|Du|+s\big)\,d\xi\right]^{2-p(x)} + c\sum_{i=0}^{m}R_i. \end{aligned}$$

Now multiplying the inequality with $R_m^{1-\alpha}$ and rearranging terms, taking also into account that $R_{m+1} \le R_m \le R_i$, we find that

$$\begin{aligned} R_{m+1}^{1-\alpha}(k_{m+1}+s) &\le c\,R^{1-\alpha}\fint_{B_R}(|Du|+s)\,d\xi \\ &\quad + c\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-p(x)+\alpha(p(x)-1)}}\right]^{\frac{1}{p(x)-1}} + c\sum_{i=0}^{m}R_i^{2-\alpha} \\ &\quad + c\sum_{i=0}^{m}\Big[L_1\omega(R_i)\log \tfrac{1}{R_i} + \big[\mathbf{v}(R_i)\big]^{\frac{\sigma}{2}}\Big]^{\frac{p_*(x)}{p(x)}}R_i^{1-\alpha}(k_i+s) \\ &\quad + c\,\chi_{\{p(x)<2\}}\big[M_{1-\alpha,R}(|Du|+s)(x)\big]^{2-p(x)}\sum_{i=0}^{m}\frac{|\mu|(B_i)}{R_i^{n-p(\cdot)+\alpha(p(\cdot)-1)}}. \end{aligned} \tag{9.4.10}$$

A uniform upper bound: In a next step we prove

LEMMA 9.17. *There exists a constant* $c \equiv c(n,\nu,L,\gamma_1,\gamma_2)$ *and a positive radius* $R_4 \equiv R_4(n,\nu,L,L_1,\gamma_1,\gamma_2,\omega(\cdot))$*, both independent of* α*, such that*

$$R_m^{1-\alpha}(k_{m+1}+s) \le c\,\mathcal{M},$$

where the quantity $\mathcal{M}$ *is defined as*

$$\mathcal{M} := R^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi + \mathbf{WI}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}(x, 2R)$$
$$+ \chi_{\{p(x)<2\}} \big[M_{1-\alpha,R}(|Du| + s)(x)\big]^{2-p(x)} \mathbf{I}^{|\mu|}_{p(\cdot)-\alpha(p(\cdot)-1)}(x, 2R) + R^{2-\alpha}.$$

PROOF. Since the mixed potential appears in the estimate we want to prove, we have to distinguish the two cases $p(x) \geq 2$ and $p(x) < 2$. In the first case, estimate (9.1.8), with $\theta \equiv 1 - \alpha\frac{p(x)-1}{p(x)}$, reads as

$$\sum_{i=0}^{\infty} \left[\frac{|\mu|(B_i)}{R_i^{n-p(x)+\alpha(p(x)-1)}}\right]^{\frac{1}{p(x)-1}} \leq c\, \mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}(x, 2R);$$

for the second one we use (9.1.9) together with the following elementary estimate:

$$\sum_{i=0}^{\infty} \left[\frac{|\mu|(B_i)}{R_i^{n-p(x)+\alpha(p(x)-1)}}\right]^{\frac{1}{p(x)-1}} \leq \left[\sum_{i=0}^{\infty} \frac{|\mu|(B_i)}{R_i^{n-p(x)+\alpha(p(x)-1)}}\right]^{\frac{1}{p(x)-1}}$$
$$\leq c\Big[\mathbf{I}^{|\mu|}_{p(\cdot)-\alpha(p(\cdot)-1)}(x, 2R)\Big]^{\frac{1}{p(x)-1}};$$

see (9.4.16) for the reason of this estimate. Matching this estimate with (9.4.10) implies that

$$R_{m+1}^{1-\alpha}(k_{m+1} + s) \leq c_4\, \mathcal{M}$$
$$+ c_3 \sum_{i=0}^{m} \Big[L_1\omega(R_i) \log \tfrac{1}{R_i} + \big[\mathbf{v}(R_i)\big]^{\sigma_h}\Big]^{\frac{p_*(x)}{p(x)}} R_i^{1-\alpha}(k_i + s). \quad (9.4.11)$$

The proof of the lemma follows now by induction. In a first step, also for later use, we mention that an argumentation analog to the one in [33], estimate (3.32), provides the estimate

$$\sum_{i=0}^{\infty} \Big[L_1\omega(R_i) \log \tfrac{1}{R_i} + \big[\mathbf{v}(R_i)\big]^{\frac{\vartheta}{2}}\Big]^{\frac{p_*(x)}{p(x)}}$$
$$\leq c \int_0^{2R} \Big[L_1\omega(\rho) \log \tfrac{1}{\rho} + \big[\mathbf{v}(\rho)\big]^{\sigma_h}\Big]^{\kappa} \frac{d\rho}{\rho} \leq \tilde{c}\, d_\omega(2R) + \tilde{c}\, d_{\mathbf{v}}(2R). \quad (9.4.12)$$

With this estimate, exploiting (4.6.14) and (4.6.15), we further restrict the maximal radius R_4 to achieve

$$d_\omega(2R)+d_{\mathbf{v}}(2R) \leq d_\omega(2R_4)+d_{\mathbf{v}}(2R_4) \leq \frac{1}{2c_3\tilde{c}}, \qquad \text{for all } R \leq R_4. \quad (9.4.13)$$

Thus we have the dependence $R_4 \equiv R_4(n, \nu, L, L_1, \gamma_1, \gamma_2, \omega(\cdot))$. This smallness condition, together with (9.4.11), (9.4.12) and (9.4.13) allows to conclude inductively that for every positive integer $m \in \mathbb{N}$ we have

$$R_{m+1}^{1-\alpha}(k_{m+1} + s) \leq [2c_4 + H^n]\mathcal{M},$$

from which the statement of the lemma follows immediately by noting that $R_m^{1-\alpha} \leq H^{1-\alpha}R_{m+1}^{1-\alpha} \leq HR_{m+1}^{1-\alpha}$. □

Maximal inequality and inclusion: We define the quantities

$$C_m := R_m^{1-\alpha}A_m = R_m^{1-\alpha} \fint_{B_m} |Du - (Du)_{B_m}|\, d\xi, \qquad h_m := \fint_{B_m} |Du|\, d\xi$$

and we want to show that

$$R_m^{1-\alpha}h_m \leq c\,\mathcal{M}. \quad (9.4.14)$$

To prove this, we note in a fist step that by Lemma 9.17 we deduce

$$R_m^{1-\alpha} h_m \le R_m^{1-\alpha} k_m + C_m \le c\,\mathcal{M} + C_m, \tag{9.4.15}$$

with $c \equiv c(n,\nu,L,L_1,\gamma_1,\gamma_2)$ and we therefore search for an appropriate bound for C_m. To find this, we first see that by (9.1.8) or (9.1.9) and (9.4.2) we have

$$\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{p(x)-1}} \le c\,R_i^{\alpha-1}\mathbf{WI}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}(x,2R) \le c\,R_i^{\alpha-1}\mathcal{M}$$

and similarly by (9.4.3)

$$\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left[\fint_{B_i}(|Du|+s)\,dx\right]^{2-p(x)}$$
$$\le R_i^{\alpha-1}\big[M_{1-\alpha,R}(|Du|+s)(x)\big]^{2-p(x)}\mathbf{I}^{|\mu|}_{p(\cdot)-\alpha(p(\cdot)-1)}(x,2R) \le c\,R_i^{\alpha-1}\mathcal{M}. \tag{9.4.16}$$

On the other hand, again Lemma 9.17 gives

$$k_m + s \le c\,R_m^{\alpha-1}\mathcal{M},$$

and combining these two facts, (9.4.12) and (9.4.13) with (9.4.9) we deduce easily

$$C_{m+1} \le \frac{1}{8}C_m + c_5\mathcal{M},$$

from which in turn follows by induction that

$$C_m \le 2c_5\mathcal{M},$$

with c_5 depending on $n,\nu,L,\gamma_1,\gamma_2$. Combining this with (9.4.15), the asserted estimate (9.4.14) follows. Having (9.4.14) at hand, we see that for $r \le R$, determining the integer $i \in \mathbb{N}_0$ in such a way that $R_{i+1} < r \le R_i$, we deduce

$$r^{1-\alpha}\fint_{B_r}|Du|\,d\xi \le \left(\frac{R_i}{R_{i+1}}\right)^n R_i^{1-\alpha}\fint_{B_i}|Du|\,d\xi \le c\,H^n R_i^{1-\alpha}h_i \le c\,\mathcal{M},$$

which means that

$$M_{1-\alpha,R}(Du)(x) \le c\,\mathcal{M};$$

at this point using Young's inequality, in the case $p(x) < 2$, with conjugate exponents

$$\frac{1}{2-p(x)}, \quad \frac{1}{p(x)-1}$$

as in (9.4.8) gives (9.0.24) for radii $R \le R_0(n,\nu,L,L_1,\gamma_1,\gamma_2,\omega(\cdot))$.

Finally, in order to remove the conditions $R \le R_0$ in the estimate (9.0.24), we argue basically as in the proof of estimate (9.0.23).

□

The coefficient case: We remark at this stage that estimate (9.0.23) takes a slightly different form in the case analog to the sole measurability of the coefficients in the standard growth case. In the case we only suppose the weak logarithmic continuity (4.4.3) to hold and that $\gamma(x)$ is merely supposed to be bounded (4.6.8),we have that estimate (9.0.23) holds not for every $\tilde{\alpha} < 1$, but only for $\tilde{\alpha} < \alpha_m$. In particular we have

PROPOSITION 9.18. *Let $u \in C^1(\Omega)$ be a weak solution to (4.6.1) under the assumptions (4.6.5), (4.6.6) ,(4.6.8) and (4.4.3). Let $B_R \subset \Omega$; then for every $\tilde{\alpha} < \alpha_m$ the pointwise estimate*

$$M^{\sharp}_{\alpha,R}(u)(x) + M_{1-\alpha,R}(Du)(x) \le c\big[M_{p(\cdot)-\alpha(p(\cdot)-1),R}(\mu)(x)\big]^{\frac{1}{p(x)-1}}$$

$$+ c\, R^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi + c\, R^{\alpha_m - \alpha} \quad (9.4.17)$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, *for a constant* $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, \tilde{\alpha}, \operatorname{diam}(\Omega))$.

PROOF. The proof of (9.4.17) is completely similar to those of (9.0.23) – and even simpler. We only sketch the steps which differ. The main difference is that now we use Lemma 9.14 instead of Lemma 9.15. Hence, going along again the previous Section, we can forget about the term involving $\omega(r) \log \frac{1}{r} + [\mathbf{v}(r)]^{\sigma_h}$. In estimate (9.4.4) we also need to substitute $\left(\frac{\varrho}{r}\right)^{1-\alpha}$ in the first therm of the right-hand side with $\left(\frac{\varrho}{r}\right)^{\alpha_m - \alpha}$. The last change we need to do is substituting (9.4.5) with the analog

$$c \left(\frac{1}{H}\right)^{\alpha_m - \alpha} \le c \left(\frac{1}{H}\right)^{\alpha_m - \tilde{\alpha}} = \frac{1}{8},$$

where we use the fact that $\tilde{\alpha} < \alpha_m$, while (9.4.6) and (9.4.7) are no more necessary, but a term $r^{\alpha_m - \alpha}$ appears. Now the proof goes ahead exactly as sketched in the previous lines. □

PROOF OF THEOREM 4.23 – DE GIORGI TYPE INTERPOLATION ESTIMATES. Goal of this section is proving the interpolation estimate of Theorem 4.23. Take a ball $B_R \subset \Omega$ with $R \le 1$ and consider a point $x \in B_{R/8}$ and a radius $r \le R/2$. In the course of the following argumentation will occur several limitations of the size of the maximal radius. We now want to consider a geometric sequence of radii whose spread $4H > 1$ will be later chosen as a function of the parameters of our problem $n, \nu, L, \gamma_1, \gamma_2$. Consider the families of shrinking balls

$$B_i := B_{r_i}(x) := B_{r/(4H)^i}(x) \qquad \text{and} \qquad \tilde{B}_i := B_{r_i/2}(x)$$

for $i = 0, 1, \ldots$ and $H \ge 1$, so that $B_{i+1} \subset \tilde{B}_i \subset B_i$. Moreover set

$$A_i := \fint_{B_i} |u - (u)_{B_i}|\, d\xi, \qquad k_i := |(u)_{B_i}|.$$

Applying Lemma 9.14 with $B_\rho \equiv B_{i+1}$ and $B_R \equiv \tilde{B}_i$ and Poincaré inequality, after some easy manipulations, recalling the definition of A_i just given, we obtain

$$A_{i+1} \le c \left(\frac{1}{H}\right)^{\alpha_m} r_i \fint_{\tilde{B}_i} \big(|Du| + s\big)\, d\xi + c\, H^n \left[\frac{|\mu|(B_i)}{r_i^{n-p(x)}}\right]^{\frac{1}{p(x)-1}}$$
$$+ c\chi_{\{p(x)<2\}} H^n \left[\frac{|\mu|(B_i)}{r_i^{n-p(x)}}\right] \left[r_i \fint_{\tilde{B}_i} (|Du| + s)\, d\xi\right]^{2-p(x)}$$
$$+ c\, r_i^2 H^{n-1} + c\, H r_i^{\alpha_m}, \quad (9.4.18)$$

for a constant $c \equiv c(n, \nu, L, \gamma_1, \gamma_2)$. This estimate holds if we impose the smallness condition $R \le R_1 \equiv R_1(n, \nu, L, \gamma_1, \gamma_2, \omega(\cdot))$. To estimate the right-hand averaged integrals in terms of A_i we need the following Caccioppoli estimate:

PROPOSITION 9.19. *(Caccioppoli's inequality) Let* $u \in W^{1,p(\cdot)}(\Omega)$ *a weak solution of equation* (4.6.1) *under the only growth and ellipticity assumptions* (4.6.5)*, with* $p(\cdot) > 2 - \frac{1}{n}$*, eventually dropping the hypothesis – and subsequently the associate growth requirement – of existence of the derivative* a_z *with respect to the gradient variable. Then there exists a radius* $R_C \equiv R_C(n, L, \omega(\cdot)) \le 1$ *such that the following holds true: For every* $\varsigma \in (0, 1)$ *there exists a constant, depending only on* $n, \gamma_1, \gamma_2, \nu, L, \varsigma, M, |\mu|(\Omega)$*, such that*

$$\fint_{B_{\varsigma R}} |Du|\, d\xi \le \frac{c}{R} \fint_{B_R} |u - k|\, d\xi + \left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{p(x)-1}} + c\, s,$$

for any $k \in \mathbb{R}$, where $B_{\varsigma R} \subset B_R \subset \Omega$ are concentric balls with center x and radius $R \leq R_C$.

We postpone the proof of this version of Caccioppoli's inequality to the end of this section. Combining this estimate, which we apply with the choices $k \equiv (u)_{B_i}$, $B_R \equiv B_i$ and $\varsigma \equiv \frac{1}{2}$ with (9.4.18) we arrive at

$$A_{i+1} \leq c_2 \left[\left(\frac{1}{H} \right)^{\alpha_m} + \varepsilon \right] A_i$$

$$+ c_3 \left(H^n + H^{n/(p(x)-1)} + 1 \right) \left[\frac{|\mu|(B_i)}{r_i^{n-p(x)}} \right]^{\frac{1}{p(x)-1}} + c\, H^n r_i^{\alpha_m} + c\, r_i s,$$

for all $\varepsilon \in (0,1)$. This ε appears when we estimate $(A_i + r_i s)^{2-p(x)}$, in the case $p(x) < 2$, with Young's inequality. In the last estimate the constants c_2, c depend on $n, \nu, L, \gamma_1, \gamma_2$ and c_3 on the same quantities and also on ε. Now, choosing ε small and H big enough to make the coefficient of A_i smaller than $\frac{1}{2}$ (and this gives a dependence of H, ε and subsequently of c_3 on $n, \gamma_1, \gamma_2, \nu, L$ - this depends also on the dependence of the exponent α_m on the same quantities, see Theorem 9.5), we can write

$$A_{i+1} \leq \frac{1}{2} A_i + c \left[\frac{|\mu|(B_i)}{r_i^{n-p(x)}} \right]^{\frac{1}{p(x)-1}} + c\, r_i^{\alpha_m} + c\, r_i s.$$

Now we can iterate the previous relation in a standard way – see for example the detailed calculation after (3.18) in [103], for an analog case – getting

$$k_{m+1} \leq c\, A_0 + c\, k_0 + c\, r^{\alpha} \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{r_i^{n-1+\alpha(p(x)-1)}} \right]^{\frac{1}{p(x)-1}} + c\, r s + c\, r^{\alpha_m}$$

$$\leq c \fint_{B_r(x)} (|u| + r s)\, d\xi + c\, r^{\alpha} \mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot), p(\cdot)}(x, R) + c\, r^{\alpha_m},$$

where we used again (9.1.8) and $r \leq R/2$. Letting $m \to \infty$ now gives

$$|u(x)| = \lim_{m \to \infty} k_{m+1} \leq c \fint_{B_r(x)} \big(|u| + r s \big)\, d\xi$$

$$+ c\, r^{\alpha} \mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot), p(\cdot)}(x, R) + c\, r^{\alpha_m}.$$

Now we observe that also $u - g$, whenever $g \in \mathbb{R}$, is a solution to (4.6.1); therefore

$$|u(x) - g| \leq c \fint_{B_r(x)} \big(|u - g| + r s \big)\, d\xi$$

$$+ c\, r^{\alpha} \mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot), p(\cdot)}(x, R) + c\, r^{\alpha_m},$$

with the constant depending on $n, \gamma_1, \gamma_2, \nu, L$. Writing the same estimate for $y \in B_{R/8}$ and using the triangle inequality gives

$$|u(x) - u(y)| \leq c \fint_{B_r(x)} |u - g|\, d\xi + c \fint_{B_r(y)} |u - g|\, d\xi$$

$$+ c\, r^{\alpha} \Big[\mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot), p(\cdot)}(x, R) + \mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot), p(\cdot)}(y, R) \Big]$$

$$+ c\, r s + c\, r^{\alpha_m}. \quad (9.4.19)$$

Now take $g = (u)_{B_{3r}(x)}$ and $r = \frac{|x-y|}{2}$; notice this choice is allowed since $|x - y| < \frac{R}{4}$. Moreover we have $B_r(y) \subset B_{3r}(x)$ and we can estimate, using also (9.4.17),

$$\fint_{B_r(x)} |u - g|\, d\xi + \fint_{B_r(y)} |u - g|\, d\xi \leq 6^n \fint_{B_{3r}(x)} \big| u - (u)_{B_{3r}(x)} \big|\, d\xi \quad (9.4.20)$$

$$
\begin{aligned}
&\leq c\,r^{\alpha} M^{\sharp}_{\alpha,R/2}(u)(x)\\
&\leq c\,r^{\alpha}\big[M_{p(\cdot)-\alpha(p(\cdot)-1),R/2}(\mu)(x)\big]^{\frac{1}{p(x)-1}}\\
&\qquad + c\Big(\frac{r}{R}\Big)^{\alpha} R \fint_{B_{R/2}(x)} (|Du|+s)\,d\xi + c\,R^{\alpha_m}\Big(\frac{r}{R}\Big)^{\alpha},
\end{aligned}
$$

where the constant c depends on $n, \gamma_1, \gamma_2, \nu, L, \tilde{\alpha}, \operatorname{diam}(\Omega)$ and $\tilde{\alpha} < \alpha_m$. To estimate the last integral we use Proposition 9.19, with an appropriate choice of the radii and of k, and Lemma 9.3:

$$
\begin{aligned}
R &\fint_{B_{R/2}(x)} \big(|Du|+s\big)\,d\xi\\
&\leq c \fint_{B_{2R/3}(x)} \big(|u|+Rs\big)\,d\xi + R^{\alpha}\Big[\frac{|\mu|(B_{2R/3}(x))}{R^{n-p(x)+\alpha(p(x)-1)}}\Big]^{\frac{1}{p(x)-1}}\\
&\leq c \fint_{B_R} \big(|u|+Rs\big)\,d\xi + c\,R^{\alpha}\big[M_{p(\cdot)-\alpha(p(\cdot)-1),2R/3}(\mu)(x)\big]^{\frac{1}{p(x)-1}}\\
&\leq c \fint_{B_R} \big(|u|+Rs\big)\,d\xi + c\,R^{\alpha}\mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}(x,R).
\end{aligned}
\tag{9.4.21}
$$

We used (9.4.2) and the fact that $B_{2R/3}(x) \subset B_R$ since $x \in B_{R/8}$. Finally, using the facts that $rs \leq (r/R)^{\alpha} Rs$ and $r^{\alpha_m} \leq (r/R)^{\alpha} R^{\alpha_m}$, merging (9.4.19), (9.4.20), and (9.4.21), we complete the proof of Theorem 4.23, for a radius $R \leq R_o := \min\{R_1, R_C\} \equiv R_o(n, \nu, L, \gamma_1, \gamma_2, \omega(\cdot))$.

□

It remains here to deliver the proof of Caccioppoli's inequality in the version of Proposition 9.19.

PROOF OF PROPOSITION 9.19. The proof consists in a combination of the chain of argumentations in [103, Proposition 4.1] and the localization arguments and we will only sketch the main arguments here. We will frequently have to exchange exponents and therefore use (9.1.7) at many stages. Without loss of generality we assume that $(u)_{B_R} = 0$. Moreover we denote $p_1 := \inf_{x\in B_R} p(x)$ and $p_2 := \sup_{x\in B_R} p(x)$.

For $\varsigma R < r \leq R$ we denote by $w_r \in u + W^{1,p(\cdot)}_0(B_r)$ the unique solution to the Dirichlet problem

$$
\begin{cases}
\operatorname{div}\big[\mu(y)\,a(y, Dw_r)\big] = 0, & \text{in } B_r(x),\\
w_r = u, & \text{on } \partial B_r(x).
\end{cases}
\tag{9.4.22}
$$

For a function $\phi \in C^{\infty}_c(B_r)$ with $0 \leq \psi \leq 1$ we test the weak formulation of (9.4.22) with the testing function $\phi^{p_2} w_r$. Exploiting the structure assumptions (4.6.5), we obtain in a standard way

$$
\begin{aligned}
\int_{B_r} &\phi^{p_2}|Dw_r|^{p(\cdot)}\,d\xi\\
&\leq c\int_{B_r} \phi^{p_2-1}\big(s+|Dw_r|\big)^{p(\cdot)-1}|D\phi||w_r|\,d\xi + c\int_{B_r} s^{p(\cdot)}\phi^{p_2}\,d\xi\\
&\leq \tfrac{1}{2}\int_{B_r} \phi^{p_2}|Dw_r|^{p(\cdot)}\,d\xi + c\int_{B_r} |D\phi|^{p(\cdot)}|w_r|^{p(\cdot)}\,d\xi + c\int_{B_r} s^{p(\cdot)}\phi^{p_2}\,d\xi.
\end{aligned}
$$

Here we have used Young's inequality and we exploited that $p(\xi)\frac{p_2-1}{p(y)-1} \geq p_2$ and $\phi \leq 1$ in the last step. Now absorbing the first term on the right-hand side into the left, we come up with

$$\fint_{B_r} \phi^{p_2}|Dw_r|^{p(\cdot)}\,d\xi \le c\fint_{B_r} |D\phi|^{p(\cdot)}|w_r|^{p(\cdot)}\,d\xi + c\int_{B_r} s^{p(\cdot)}\phi^{p_2}\,d\xi,$$

for a constant $c \equiv c(n, L, \nu, \gamma_1, \gamma_2)$. Now, by the estimate (9.1.7), applied with $A = |Dw_r|$, $\sigma = p_1$, $\tilde{\omega} = p(\xi) - p_1$ and $\alpha = p_2/p(\xi)$ we have $|Dw_r|^{p_1} \le c(R^{p_2} + |Dw_r|^{p(\xi)})$ and therefore arrive at

$$\fint_{B_r} \phi^{p_2}|Dw_r|^{p_1}\,d\xi \le c\fint_{B_r} |D\phi|^{p(\cdot)}|w_r|^{p(\cdot)}\,d\xi + c\fint_{B_r} \big(s^{p(\cdot)} + R^{p_2}\big)\,d\xi. \tag{9.4.23}$$

For ρ and σ with $\varsigma R \le \rho < \sigma < r$ let now $\phi \in C_c^\infty(B_r)$ be a cut-off function with $0 \le \phi \le 1$, $\phi \equiv 1$ on B_σ and $|D\phi| \le \frac{4}{r-\sigma}$. For such a special function ϕ, we obtain with the help of (9.4.23) and using again (9.1.7), $|w_r|^{p(\xi)}|D\phi|^{p(\xi)} \le c(R^{p_2} + |w_r|^{p_2}|D\phi|^{p_2})$ (and the same for s):

$$\fint_{B_\sigma} \big|D(\phi w_r)\big|^{p_1}\,d\xi \le c\Big(\frac{r}{\sigma}\Big)^n\bigg[\fint_{B_r} |D\phi|^{p_2}|w_r|^{p_2}\,d\xi + s^{p_2} + R^{p_2}\bigg]. \tag{9.4.24}$$

Exploiting this estimate in combination with the Sobolev-Poincaré inequality which we apply to the function w_r, we eventually arrive at

$$\bigg[\fint_{B_\sigma} |w_r|^\ell\,d\xi\bigg]^{\frac{1}{\ell}} \le c\Big(\frac{r}{\sigma}\Big)^{\frac{n}{p_1}}\bigg[\frac{1}{(r-\sigma)^{p_2}}\fint_{B_r} |w_r|^{p_2}\,d\xi + s^{p_2} + R^{p_2}\bigg]^{\frac{1}{p_1}},$$

for a constant $c \equiv c(n, L, \nu, \gamma_1, \gamma_2)$ and for all $\ell \le \frac{np_1}{n-p_1}$. Since we have $r \le R$ and $\sigma > \varsigma R$, we can estimate the expression $(r/\sigma)^{n/p_1}$ by a constant which depends only on n and p_1 and ς. Imposing in a next step the condition

$$p_2 - p_1 \le \omega(2R) < \frac{1}{n},$$

which gives a smallness condition on the radius R in the sense of $R \le R_C \equiv R_C(n, \omega(\cdot))$, we have $p_2 - p_1 < \frac{p_2 p_1}{n}$ and therefore $\frac{np_1}{n-p_1} > p_2$. This, in turn means that we have the following reverse Hölder inequality

$$\bigg[\fint_{B_\sigma} |w_r|^\ell\,d\xi\bigg]^{\frac{p_1}{\ell}} \le \frac{c}{(r-\sigma)^{p_2}}\fint_{B_r} |w_r|^{p_2}\,d\xi + c\big(s^{p_2} + R^{p_2}\big). \tag{9.4.25}$$

In a next step we would like to replace the power p_1 on the left hand side of the preceding inequality by the power p_2. However, this can be done by an argument which is analog to the one in [33, p. 654-655]. Indeed we have by the localization (9.1.6)

$$\begin{aligned}\bigg[\fint_{B_\sigma} |w_r|^\ell\,d\xi\bigg]^{\frac{p_2-p_1}{\ell}} &= \bigg[\Big(\frac{R}{\sigma}\Big)^n R^{-n}\int_{B_\sigma} |w_r|^\ell\,d\xi\bigg]^{\frac{p_2-p_1}{\ell}} \\ &\le c(n, \gamma_1, \gamma_2, \varsigma, \ell)R^{-(p_2-p_1)\frac{n}{\ell}}\bigg[\int_{B_\sigma} |w_r|^\ell\,d\xi\bigg]^{\frac{p_2-p_1}{\ell}} \\ &\le c(n, \gamma_1, \gamma_2, \varsigma, \ell, L)\bigg[\int_{B_\sigma} |w_r|^\ell\,d\xi\bigg]^{\frac{p_2-p_1}{\ell}}.\end{aligned}$$

The Sobolev-Poincaré inequality and subsequently an argumentation analog to the one in [33, p. 654] allows now to estimate the last integral appearing in the previous estimate by an integral involving only the L^1-norm of $|Dw_k|$ which is in turn again estimated by a comparison estimate analog to (9.3.3). Having the energy bound (4.6.3) at hand we therefore finally arrive at

$$\left[\fint_{B_\sigma} |w_r|^\ell \, d\xi\right]^{\frac{p_2-p_1}{\ell}} \le c(n,\gamma_1,\gamma_2,\sigma,\ell,L,M,|\mu|(\Omega)). \tag{9.4.26}$$

This in turn means that we may replace the exponent p_1 on the left hand side of (9.4.25) by an exponent p_2 and therefore catch the additional dependencies of the constant c on the quantities M and $|\mu|(\Omega)$. Having arrived at this stage, the self-improving property of reverse Hölder inequalities (see the argumentation in [**103**, Proof of Proposition 4.1] and [**87**, Lemma 3.38]) then provides the estimate

$$\left[\fint_{B_\sigma} |w_r|^\ell \, d\xi\right]^{\frac{1}{\ell}} \le \frac{c}{(r-\sigma)^q} \fint_{B_r} |w_r| \, d\xi + c(s+R). \tag{9.4.27}$$

for some $q = q(n,\gamma_1,\gamma_2) > 1$. Now we write (9.4.23) again, this time with a cut-off function on the pair of balls (B_ρ, B_σ), i.e. $\phi \equiv 1$ on B_ρ, $\phi \in C_c^\infty(B_\sigma) \subset C_c^\infty(B_r)$ and $|D\phi| \le \frac{4}{\sigma-\rho}$. This gives (9.4.24) with r replaced by σ and σ replaced by ρ. Using this together with Hölder's inequality and finally combining it with (9.4.27) we therefore arrive at

$$\fint_{B_\rho} |Dw_r| \, d\xi \le c\frac{1}{(\sigma-\rho)^{p_2/p_1}} \left[\frac{1}{(r-\sigma)^q} \fint_{B_r} |w_r| \, d\xi\right]^{\frac{p_2}{p_1}} + c(s+R)^{\frac{p_2}{p_1}}.$$

Choosing here $\sigma := \frac{\rho+r}{2}$ we eventually obtain

$$\fint_{B_\rho} |Dw_r| \, d\xi \le \frac{c}{(r-\rho)^{\frac{p_2}{p_1}(1+q)}} \left[\fint_{B_r} |w_r| \, d\xi\right]^{\frac{p_2}{p_1}} + c(s+R)^{\frac{p_2}{p_1}},$$

and also here we can replace the power p_2/p_1 appearing on the right hand side by a power 1, using again the argumentation as in (9.4.26). In turn we finally have

$$\fint_{B_\rho} |Dw_r| \, d\xi \le \frac{c}{(r-\rho)^{\frac{p_2}{p_1}(1+q)}} \fint_{B_r} |w_r| \, d\xi + c(s+R).$$

At this point we now argue completely analogous to [**103**], exploiting the comparison estimate (9.3.3), Poincaré's and Young's inequality and finally a standard iteration Lemma to conclude the final form of the desired Caccioppoli inequality. □

PROOF OF THEOREMS 4.24 AND 4.25. Once having at hand Theorem 4.23 and the maximal operator bounds (9.0.23) and (9.0.24), the proof is quite simple.

Notice that to prove Theorem 4.24 is sufficient to prove that there exists positive numbers δ and σ such estimate (4.6.16) holds uniformly when α runs in $(\alpha_m/2, \tilde{\alpha}]$ if (4.6.17) is satisfied. Notice that Theorem 4.23 does not even require the fulfillment of (4.6.17) to ensure that (4.6.16) holds uniformly when α belongs to $[0, \alpha_m/2]$. We recall that α_m is the maximal Hölder exponent, appearing in Theorem 9.5, for the operator associated to the vector field $\gamma(\cdot)a(\cdot)$, and it depends on $n, \gamma_1, \gamma_2, \nu, L$. With $x, y \in B_{R/8}$ Lemma 9.4 and inequality (9.0.23) yields

$$\begin{aligned} |u(x)-u(y)| &\le \frac{2c}{\alpha_m}\left[M^\sharp_{\alpha,R/2}(u)(x) + M^\sharp_{\alpha,R/2}(u)(y)\right]|x-y|^\alpha \\ &\le \frac{c}{\alpha_m}\left[M_{p(\cdot)-\alpha(p(\cdot)-1),R/2}(\mu)(x) + M_{p(\cdot)-\alpha(p(\cdot)-1),R/2}(\mu)(y)\right]^{\frac{1}{p(x)-1}}|x-y|^\alpha \\ &\quad + \frac{c}{\alpha_m}\left[R\fint_{B_{R/2}(x)}(|Du|+s)\,d\xi + R\fint_{B_{R/2}(y)}(|Du|+s)\,d\xi\right]\left(\frac{x-y}{R}\right)^\alpha. \end{aligned} \tag{9.4.28}$$

Now we estimate the maximal functions appearing on the right-hand side with the Wolff potentials via Lemma 9.3

$$\big[M_{p(\cdot)-\alpha(p(\cdot)-1),R/2}(\mu)(x)\big]^{\frac{1}{p(x)-1}} \leq c\, \mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}(x,R),$$

where $c \equiv c(n,\gamma_1,\gamma_2,\alpha)$, while the remaining integrals are estimated exactly as in (9.4.21). This concludes the proof of Theorem 4.24.

The proof of Theorem 4.25 is similar: this time we can cover uniformly the whole $(\alpha_m/2, 1]$ taking advantage of the improved spatial regularity assumed. Instead of (9.0.23) we can exploit (9.0.24) in estimating the maximal operators appearing in (9.4.28), and this estimate is uniform up to 1. Moreover in order to have a "coherent" estimate, as we already pointed out in the second Chapter, we have to replace the Wolff potential $\mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}$ with the mixed one $\mathbf{WI}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}$. That is, in the points $\bar{x}$ where $p(\bar{x}) < 2$, replace

$$\mathbf{W}^{\mu}_{1-\alpha(p(\cdot)-1)/p(\cdot),p(\cdot)}(\bar{x},R) \qquad \text{with} \qquad [\mathbf{I}^{|\mu|}_{p(\cdot)-\alpha(p(\cdot)-1)}(\bar{x},R)]^{1/(p(\bar{x})-1)},$$

and this is just Remark 9.1. □

We end this Chapter with the simple proof of Corollary 4.26:

PROOF OF COROLLARY 4.26. Theorem 4.26 follows plainly from (9.0.24) with $\alpha = 1$, considering the expression of the mixed potential in the case $p(x_o) < 2$. Notice moreover that carefully checking the proof of (9.0.24), one can see that we used the quantity κ in order to have a unitary approach, while the exponent of $\omega(\rho)\log\frac{1}{\rho}$, in the case $p(x_o) < 2$, can be taken as one (see (9.3.23)). □

CHAPTER 10

Linear potential estimates under general growth conditions

We recall we are considering a $C^1(\mathbb{R}^n)$ vector field satisfying the ellipticity and growth conditions

$$\begin{cases} \langle \partial_\xi a(\xi)\lambda, \lambda\rangle \geq \nu \, \dfrac{g(|\xi|)}{|\xi|}|\lambda|^2 \\ |a(\xi)| + |\partial_\xi a(\xi)||\xi| \leq L g(|\xi|) \end{cases}$$

for all $\xi, \lambda \in \mathbb{R}^n$. $g \in \mathcal{C}^1(\mathbb{R}^+)$ is the function satisfying

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \qquad t > 0, \qquad 1 \leq \delta < g_0.$$

By defining

$$V_g(\xi) := \left[\frac{g(|\xi|)}{|\xi|}\right]^{1/2} \xi. \tag{10.0.29}$$

we have the analog of a well-known quantity in the study of the p-Laplace operator, and also in our case the following relation holds:

$$|V_g(\xi_1) - V_g(\xi_2)|^2 \approx \frac{g\big(|\xi_1| + |\xi_2|\big)}{|\xi_1| + |\xi_2|}|\xi_1 - \xi_2|^2 \approx g'\big(|\xi_1| + |\xi_2|\big)|\xi_1 - \xi_2|^2. \tag{10.0.30}$$

We introduce here a notation we shall use many times through the whole Chapter. By writing $A \lesssim B$ we will mean that there exists a positive constant $\tilde c$, depending only upon δ and/or g_0, such that $A \leq \tilde c\, B$. With the expression $A \approx B$ we will mean that both $A \lesssim B$ and $B \lesssim A$ hold. Moreover in the case the constant $\tilde c$ will depend also on other quantities, we will write them below these signs. For example, if $A \leq \tilde c(n, \delta, g_0)\, B$, we shall write $A \lesssim_n B$. This notation will show to be very useful, besides lightening notations, since it will also highlight how (4.7.2) plays a fundamental role in our proofs and will be used mainly for equivalences of functions. For example, using (4.7.2), we have

$$g(t) \approx \int_0^t \frac{g(s)}{s}\, ds,$$

being the second function convex (see (10.1.1)) while $g(\cdot)$ is not. Using $(4.7.4)_1$ it is easy to prove, or see [57, Lemma 20], the following monotonicity inequality

$$\begin{aligned} \langle a(\xi_1) - a(\xi_2), \xi_1 - \xi_2\rangle &\gtrsim c(\nu) g'(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2|^2 \\ &\gtrsim c\, |V_g(\xi_1) - V_g(\xi_2)|^2 \end{aligned} \tag{10.0.31}$$

and the Lipschitz continuity

$$|a(\xi_1) - a(\xi_2)| \lesssim_L g(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2|.$$

Note that we shall use the notation

$$\int_0 f(s)\,ds < \infty \qquad \text{for } f:\mathbb{R}^+\to\mathbb{R}^+ \text{ continuous function}$$

to mean $\int_0^t f(s)\,ds < \infty$ for some (and then for all) $t > 0$. Similarly for $\int_0 f(s)\,ds = \infty$; the same for improper integrals at infinity $\int^\infty$. By $\mathbb{R}^+$ we will mean the half-line $[0,\infty)$, by $\mathbb{N}$ the set $\{1,2,\dots\}$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

10.1. Basic properties of the g function and Orlicz-Sobolev spaces.

First of all note that we can suppose without loss of generality $\delta < g_0$ and that the lower bound $\delta \geq 1$ implies that

$$t \to \frac{g(t)}{t} \qquad \text{is increasing;} \tag{10.1.1}$$

the proof is a simple explicit computation of its derivative. We can also assume in full generality that

$$\int_0^1 g(s)\,ds = 1. \tag{10.1.2}$$

At this point elementary calculus shows that $g(1) \approx 1$. Define now the function $G \in \mathcal{C}^2(0,+\infty)$ as the (positive) primitive of g:

$$G(t) := \int_0^t g(s)\,ds. \tag{10.1.3}$$

It is straightforward to see that $G(1) = 1$ from (10.1.2), G is convex and there holds

$$\frac{tg(t)}{1+g_0} \leq G(t) \leq \frac{tg(t)}{1+\delta} \qquad \text{if } t \geq 0. \tag{10.1.4}$$

Note now that under the only assumption (4.7.2) a simple computation of derivatives gives that $t \mapsto g(t)t^{-\delta}$ is increasing and $t \mapsto g(t)t^{-g_0}$ is decreasing, so we have

$$\min\{\alpha^\delta, \alpha^{g_0}\}\, g(t) \leq g(\alpha t) \leq \max\{\alpha^\delta, \alpha^{g_0}\}\, g(t) \qquad \text{for all } t \geq 0, \alpha \geq 0. \tag{10.1.5}$$

Inequality (10.1.4) can be rewritten in a more expressive way as

$$1+\delta \leq \frac{tG'(t)}{G(t)} \leq 1+g_0 : \tag{10.1.6}$$

as above, this implies that the function $t^{-(1+\delta)}G(t)$ is increasing and $t^{-(1+g_0)}G(t)$ is decreasing, so we have

$$\min\{\alpha^{1+\delta}, \alpha^{1+g_0}\}\, G(t) \leq G(\alpha t) \leq \max\{\alpha^{1+\delta}, \alpha^{1+g_0}\}\, G(t) \tag{10.1.7}$$

for all $t \geq 0, \alpha \geq 0$. In the customary terminology, the right-hand side inequalities of (10.1.5) and (10.1.7) mean that g and G satisfy a global Δ_2-condition. We recall that a function $A : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy a *global* Δ_2*- (or doubling) condition* if there exists a constant $k \geq 1$ such that

$$A(2t) \leq k\,A(t) \quad \text{for all } t \geq 0.$$

Note the peculiar form of (10.1.7) when $t = 1$ (and then $G(1) = 1$). Being G strictly increasing and with infinite limit, then G^{-1} exists, is defined for all $t \in \mathbb{R}$ and it is strictly increasing. Writing $t = G\big(G^{-1}(t)\big)$, (10.1.7) implies

$$\min\{\alpha^{\frac{1}{1+\delta}}, \alpha^{\frac{1}{1+g_0}}\}\, G^{-1}(t) \leq G^{-1}(\alpha t) \leq \max\{\alpha^{\frac{1}{1+\delta}}, \alpha^{\frac{1}{1+g_0}}\}\, G^{-1}(t) \tag{10.1.8}$$

for $\alpha, t \geq 0$; something similar holds for g. We shall use this estimate, and also (10.1.7) and (10.1.5), mainly with the purpose of confining constants outside the Young functions we are going to consider.

REMARK 10.1. A useful argument to keep in mind when dealing with N-functions is the following. Usually, and in particular in several occasions in this proof, it could be difficult to check whether a (regular) function D is convex or not, while it will be easier to verify its monotonicity, after computing its derivative. In the case we know that $1 \leq d_0 \leq \mathcal{O}_D(t) \leq d_1$, then, if we define

$$\bar{D}(t) := \int_0^t \frac{D(s)}{s}\, ds,$$

we have that $d_0 D(t) \leq \bar{D}(t) \leq d_1 D(t)$ and moreover, since $d_0 \geq 1$, then $t \mapsto D(t)/t$ has positive derivative and hence it is increasing. Hence, $\bar{D}(t)$ turns out to be convex. On the other hand, if $\tilde{d}_0 \leq \mathcal{O}_D(t) \leq \tilde{d}_1 \leq 1$, then $t \mapsto D(t)/t$ is decreasing and hence $\bar{D}$ is concave. Often, in order to not overburden notation, we shall simply and directly suppose D convex, leaving to the reader the simple task of properly adapting the proof, in the spirit of the proofs of Proposition 10.5 or Lemma 10.9; see for instance (10.3.2).

REMARK 10.2. Note that for an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying a doubling Δ_2 condition $f(2t) \lesssim f(t)$ for $t \geq 0$, it is easy to prove that $f(t+s) \lesssim f(t) + f(s)$ holds for every $t, s \geq 0$. Indeed $f(t+s) \leq f(2t) + f(2s)$. Analogue things happen e.g. if $f(2t) \lesssim_n f(t)$ or similar conditions.

The Remark above shows that for both g and G a sort of "triangle inequality" holds:

$$G(t+s) \lesssim G(t) + G(s), \qquad g(t+s) \lesssim g(t) + g(s). \tag{10.1.9}$$

Finally note that, since we have at hand the monotonicity (10.1.1), then

$$\begin{aligned} G\big(|Du - Dv|\big) &\lesssim \frac{g\big(|Du - Dv|\big)}{|Du - Dv|}|Du - Dv|^2 \lesssim \frac{g\big(|Du| + |Dv|\big)}{|Du| + |Dv|}|Du - Dv|^2 \\ &\lesssim c\big|V_g(Du) - V_g(Dv)\big|^2. \end{aligned} \tag{10.1.10}$$

Young functions and Young's inequality. We call a Young function a left-continuous convex function $A : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ such that $A(0) = 0$. A N-function is a finite valued (therefore continuous) Young function B such that

$$\lim_{t \to 0} \frac{B(t)}{t} = 0, \qquad \lim_{t \to \infty} \frac{B(t)}{t} = \infty.$$

Many of the functions we are going to consider will be N-functions, but not all. E.g. g is a Young's function but not necessarily a N-function. G defined in (10.1.3) is instead a N-function. The *Young's conjugate* of a Young function is defined by

$$\widetilde{A}(t) := \sup_{s > 0} \big\{st - A(s)\big\}$$

and throughout the whole Chapter when using the *tilde* notation over a function we shall always mean its Young's conjugate. Needless to say the Young's conjugate is such that Young's inequality holds; moreover, in the case a condition of the type $A(\alpha t) \leq \alpha^q A(t)$ for $\alpha \in (0,1)$ and with some positive exponent q, the choice of an appropriate power of $\varepsilon \in (0,1)$ leads to the following improved form:

$$ts \leq \varepsilon\, A(t) + c(\varepsilon, q)\widetilde{A}(s) \tag{10.1.11}$$

for all $t, s \geq 0$. Another important feature of Young's conjugate function is the following inequality, which can be found in [11, Chapter 8, (6)]:

$$\widetilde{A}\left(\frac{A(t)}{t}\right) \leq A(t) \qquad (10.1.12)$$

for $t > 0$. The previous inequality can be inferred by the similar one:

$$t \leq A^{-1}(t)\widetilde{A}^{-1}(t) \leq 2t \qquad \text{for all } t \geq 0. \qquad (10.1.13)$$

Orlicz and Orlicz-Sobolev spaces. Given a Young function A satisfying a global Δ_2-condition, the Orlicz space $L^A(\Omega)$ is the Banach space of all measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega A(|f|)\, dx < \infty$, endowed with the Luxemburg norm

$$\|f\|_{L^A(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|f|}{\lambda}\right) dx \leq 1\right\}.$$

Note that for the above norm there holds the inequality

$$\|f\|_{L^A(\Omega)} \leq \int_\Omega A(|f|)\, dx + 1, \qquad (10.1.14)$$

see for example [58, Corollary 2.1.15] in the most general setting. The Orlicz-Sobolev space $W^{1,A}(\Omega)$ is just made up of the functions $f \in W^{1,1}(\Omega)$ such that $Df \in L^A(\Omega)$. Finally, by $W_0^{1,A}(\Omega)$ we mean the subspace of $W^{1,A}(\Omega)$ made up of the functions whose continuation by zero outside Ω belongs to $W^{1,A}(\mathbb{R}^n)$. Note that for $\partial\Omega$ smooth enough, say Lipschitz regular, this space coincides with the closure of $C_c^\infty(\Omega)$ in $W^{1,A}(\Omega)$, at least when A satisfies a Δ_2 condition, that is our case. We shall need this observation in particular for Ω a ball.

Sobolev's embedding. For the Sobolev-Orlicz spaces usual embeddings theorems hold true. In particular, we can still find what can be roughly distinguished as the two different behaviors of function belonging to $W^{1,A}(\Omega)$ depending on the growth of the Young function A at infinity. If the function A grows "slowly" at infinity, then we get an embedding into a bigger Orlicz space, as in the standard case where we have $p \leq n$. Note that the borderline case $p = n$ can be "embedded" in this case, due to the general structure of Orlicz spaces (Trudinger's Theorem [143] is nothing else than the embedding of $W^{1,n}(\Omega)$ into the Orlicz space $L^B(\Omega)$, where $B = e^{s^{n'}} - 1$). In order to be more precise, let's suppose that the Young function A satisfies the following bounds:

$$\int_0 \left(\frac{s}{A(s)}\right)^{\frac{1}{n-1}} ds < \infty \qquad \text{and} \qquad \int^\infty \left(\frac{s}{A(s)}\right)^{\frac{1}{n-1}} ds = \infty. \qquad (10.1.15)$$

In this case we have the space $W^{1,A}(\Omega)$ embeds into $L^{A_n}(\Omega)$, where we define the Young function A_n in the following line:

$$H_n(t) := \left(\int_0^t \left[\frac{s}{A(s)}\right]^{\frac{1}{n-1}} ds\right)^{\frac{n-1}{n}}, \qquad A_n(t) := (A \circ H_n^{-1})(t). \quad (10.1.16)$$

Note that the function $H_n(\cdot)$ depends on the starting function A, but we don't explicit this dependence for ease of notation. We will however recall this fact often in order to avoid misunderstandings. Moreover observe that the first condition in (10.1.15), call it $(10.1.15)_1$, is not really restrictive: given a Young function satisfying $(10.1.15)_2$, we can appropriately modify it near zero in order to satisfy $(10.1.15)_1$. This does not invalidate the function as belonging to the Orlicz-Sobolev space, and also in our context will lead to minor changes, see Section 10.3. The following (sharp) integral form of Sobolev's embedding can be found in this form in [44, Theorem 3] by Cianchi.

PROPOSITION 10.3 (Sobolev's embedding). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded open set and let A be a Young function satisfying* (10.1.15)*. Then there exists a constant c_S depending only on n such that for every weakly differentiable function $u \in W_0^{1,A}(\Omega)$ there holds*

$$\int_\Omega A_n\left(\frac{u}{c_S(n)\big(\int_\Omega A\big(|Du|\big)\,dx\big)^{1/n}}\right)dx \leq \int_\Omega A\big(|Du|\big)\,dx, \tag{10.1.17}$$

where $A_n(t) := A\big(H_n^{-1}(t)\big)$ is the function defined in (10.1.16)*.*

If indeed A grows "quickly" at infinity, i.e. if

$$\int^\infty \left(\frac{s}{A(s)}\right)^{\frac{1}{n-1}} ds < \infty, \tag{10.1.18}$$

we have the embedding into L^∞ by Talenti [**142**]. The more transparent version we propose here can be found in the paper [**43**] by Cianchi.

PROPOSITION 10.4 (Sobolev's embedding - II). *Let Ω as in the previous proposition and let A be a Young function satisfying* (10.1.18)*. Then there exists a constant depending on $n, \delta, g_0, |\Omega|$ such that for every function $u \in W_0^{1,A}(\Omega)$*

$$\sup_\Omega |u| \leq c\,\|Du\|_{L^A(\Omega)}. \tag{10.1.19}$$

Finally an easy Sobolev-type embedding for the function g. We state it explicitly here since in the following we shall often need to refer to it.

PROPOSITION 10.5. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive increasing $C^1(0,\infty)$ function such that $g(0) = 0$ and*

$$\frac{tg'(t)}{g(t)} \leq g_0 \qquad \text{for all } t > 0 \text{ and with } g_0 > 0 \tag{10.1.20}$$

and let B_R be a ball. Then there exists a constant $c \equiv c(n, g_0)$ such that

$$\fint_{B_R} \left[g\left(\frac{|u|}{R}\right)\right]^{\frac{n}{n-1}} dx \leq c\left(\fint_{B_R} g\big(|Du|\big)\,dx\right)^{\frac{n}{n-1}}$$

for every weakly differentiable function $u \in W_0^{1,g}(B_R)$.

PROOF. Define f as

$$f(t) := \int_0^t \frac{g(s)}{s}\,ds$$

and note that $f \approx g$, f is convex and also satisfies (4.7.2). By Sobolev's embedding in $W^{1,1}$, using (4.7.2) for f we know

$$\begin{aligned}\left(\fint_{B_R}\left[f\left(\frac{|u|}{R}\right)\right]^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} &\leq c(n)\,R\fint_{B_R} f'\left(\frac{|u|}{R}\right)\frac{|Du|}{R}\,dx\\ &\leq c\,g_0\,R\fint_{B_R} f\left(\frac{|u|}{R}\right)\frac{R}{|u|}\frac{|Du|}{R}\,dx.\end{aligned}$$

Now we use Young's inequality (10.1.11) with conjugate functions f and $\tilde{f}$ and $\varepsilon \in (0,1)$ appropriate; (10.1.12) and Hölder's inequality then yield

$$\begin{aligned}&\fint_{B_R} f\left(\frac{|u|}{R}\right)\frac{R}{|u|}|Du|\,dx\,dx\\ &\qquad\leq c(\varepsilon)\fint_{B_R} f\big(|Du|\big)\,dx + \varepsilon\fint_{B_R}\tilde{f}\left(f\left(\frac{|u|}{R}\right)\frac{R}{|u|}\right)dx\end{aligned}$$

$$\leq c(\varepsilon) \fint_{B_R} f(|Du|)\, dx + \varepsilon \left(\fint_{B_R} \left[f\left(\frac{|u|}{R} \right) \right]^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}}.$$

To conclude we choose ε small enough to reabsorb the latter term in the left-hand side and we recall that $f \approx g$. □

10.2. Homogeneous equations

In this section we collect some results for homogeneous equations of the form

$$-\operatorname{div} a(Dv) = 0 \qquad \text{on } A \subset \mathbb{R}^n \text{ bounded open set.} \tag{10.2.1}$$

We will assume that the vector field $a : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the ellipticity and growth conditions (4.7.4) and in the following we will propose some variations on the classical themes of Lieberman [**111, 112**]. The following Lemma is indeed essentially a little variation of [**112**, Lemma 5.1]:

LEMMA 10.6. *Let $v \in W^{1,G}(A)$ be a solution to (10.2.1) under the assumptions (4.7.4)–(4.7.2). Then for every ball $B_R \equiv B_R(x_0) \subset A$ the following De Giorgi type estimate holds:*

$$\sup_{B_{R/4}} |Dv| \leq c \fint_{B_R} |Dv|\, dx. \tag{10.2.2}$$

Moreover $Dv \in C^{0,\alpha}(A)$ for some $\alpha \in (0,1)$ and the following estimate for the excess decay holds:

$$\fint_{B_r} \left| Dv - (Dv)_{B_r} \right| dx \leq c_h \left(\frac{r}{R} \right)^{\alpha} \fint_{B_R} \left| Dv - (Dv)_{B_R} \right| dx, \tag{10.2.3}$$

for $0 < r < R$ and B_r having the same center of B_R. Finally we have

$$|Dv(x_1) - Dv(x_2)| \leq c_o \left(\frac{r}{R} \right)^{\alpha} \fint_{B_R} |Dv|\, dx \tag{10.2.4}$$

for every $x_1, x_2 \in B_{r/2}$. The three constants and the exponent α share the same dependence on n, ν, L, δ, g_0.

PROOF. For (10.2.2) we merge the De Giorgi estimate present in [**112**, Lemmata 5.1 & 5.2] with the following Lemma 10.7, which allows to reduce the integrability of Dv on the right-hand side. We have

$$\sup_{B_{R/4}} G(|Dv|) \leq c \fint_{B_{R/2}} G(|Dv|)\, dx \leq c\, G\left(\fint_{B_R} |Dv|\, dx \right).$$

For (10.2.3) and (10.2.4) we take inspiration from [**70**, Theorem 3.1], which is in turn a revisitation of [**112**, Lemma 5.1] for the case of Hölder estimates below the natural growth exponent in the standard super-quadratic case. Therefore at some points we shall only sketch the proof leaving the reader the task of completing the missing details with the help of [**70**, Theorem 3.1]. Take a ball $B_{\tilde r} \subset B_R$, recall the definition of the excess in (3.1.2) and set

$$M(r) := \max_{k \in \{1,\ldots,n\}} \sup_{B_r} |D_k v|.$$

It is a well known regularity fact, see [**53, 111**], that there exists constant μ_0, η depending only on n, ν, L, δ, g_0 such that if one of the following two alternatives holds

$$|\{D_k v < M(\tilde r)/2\} \cap B_{\tilde r}| \leq \mu_0 |B_{\tilde r}| \qquad \text{for some } k \in \{1,\ldots,n\}, \tag{10.2.5}$$

$$|\{D_k v > -M(\tilde r)/2\} \cap B_v| \leq \mu_0 |B_{\tilde r}| \qquad \text{for some } k \in \{1,\ldots,n\}, \tag{10.2.6}$$

then

$$|Dv| \geq \frac{1}{4}M(\tilde{r}) \qquad \text{in } B_{\tilde{r}/2}, \tag{10.2.7}$$

while if neither (10.2.5) nor (10.2.6) holds for any k, then

$$M(\tilde{r}/2) \leq \eta M(\tilde{r}). \tag{10.2.8}$$

Now we fix $\rho \leq R$ and define first the constant $H_1 \in \mathbb{N}$, then $K_1 \in \mathbb{N}$, both depending on n, ν, L, δ, g_0, such that they satisfy

$$8c_* \sqrt{n}\, \eta^{H_1} \leq 1, \qquad\qquad 2^{n(H_1+3)}\eta^{K_1} \leq 1, \tag{10.2.9}$$

where $\eta \in (0,1)$ is the quantity appearing in (10.2.8) and c_* the constant appearing in (10.2.2). Moreover we denote $j_0 := H_1 + K_1$. We distinguish now two alternatives:

The first alternative. Consider the case where one of (10.2.5), (10.2.6) holds in $B_{2^{-j}\rho}$ for some $j \in \mathbb{N}$, and therefore (10.2.7) holds in $B_{2^{-j-1}\rho}$. For $i = 1, \ldots, n$, $\tilde{v} := D_i v$ is a weak solution to a *uniformly elliptic* linear equation:

$$\operatorname{div}\left(\tilde{a}(x)D\tilde{v}\right) = 0, \qquad \text{where} \qquad \tilde{a}(x) = \partial_\xi a(Dv(x)). \tag{10.2.10}$$

Note that this differentiation is possible only in the non-degenerate case $g(t) \geq \varepsilon t$, see [**111**, Lemma 1]: however is possible to consider approximating solutions v_ε solutions of vector fields satisfying this condition and then passing to the limit, as shown in [**112**, Lemma 5.2]. Therefore we shall simply consider (10.2.10) valid in our possibly degenerate case without no loss of generality. We can now use (10.2.7) in (4.7.4) getting

$$\begin{aligned}
\langle \tilde{a}(x)\lambda, \lambda\rangle &\geq \nu \frac{g(|Dv|)}{|Dv|}|\lambda|^2 \geq \nu \frac{2^{-2g_0}}{\sqrt{n}} \frac{g(M(2^{-j}\rho))}{M(2^{-j}\rho)}|\lambda|^2, \\
|\tilde{a}(x)| &\leq L\frac{g(|Dv|)}{|Dv|} \leq 4Ln^{\frac{g_0}{2}} \frac{g(M(2^{-j}\rho))}{M(2^{-j}\rho)},
\end{aligned}$$

both the inequalities being valid in $B_{2^{-(j+1)}\rho}$. Hence here $D_i v$ satisfies a uniformly elliptic linear equation in $B_{2^{-(j+1)}\rho}$ and hence classic theory, see e.g. [**70**, Lemma 3.2], gives

$$\fint_{B_r} \left|D_i v - (D_i v)_{B_r}\right| dx \leq c\left(\frac{r}{\rho}\right)^{\alpha_1} \fint_{B_{2^{-(j+1)}\rho}} \left|D_i v - (D_i v)_{B_\rho}\right| dx. \tag{10.2.11}$$

for all $r \leq 2^{-(j+1)}\rho$ and for every $i = 1, \ldots, n$, with α_1 and c depending on $n, g_0, L/\nu$ and since $j \leq j_0(n, \nu, L, \delta, g_0)$. Note that the important point here is that the dependence of the Hölder exponent and the constant is upon the *ellipticity ratio*, and therefore they do not depend on $M(2^{-j}\rho)$.

The second alternative. For $\rho \leq R$ fixed as above, suppose that both (10.2.5) and (10.2.6) fail in $B_{2^{-j}\rho}$ for $j = 1, \ldots, j_0$, for every k. Then, iterating (10.2.8) we get

$$M(2^{-(H_1+2)}\rho) \leq \eta^{H_1} M(\rho/4), \quad M(2^{-(j_0+1)}\rho) \leq \eta^{K_1} M(2^{-(H_1+1)}\rho). \tag{10.2.12}$$

Note moreover that there holds $E(Dv, B_{\tilde{r}}) \leq 2\sqrt{n}M(\tilde{r})$. Now we consider two different cases. In *the case* $|(Dv)_{B_\rho}| \leq 2\sqrt{n}M(2^{-(H_1+2)}\rho)$ we have, using estimate (10.2.2)

$$M(\rho/4) \leq c_* \fint_{B_\rho} |Dv|\, dx \leq c_*\, E(Dv, B_\rho) + 2c_*\sqrt{n}\, M(2^{-(H_1+2)}\rho).$$

Combining the last two estimates and taking into account the definition of H_1 in (10.2.9) we can reabsorb the second term on the right-hand side obtaining

$$M(2^{-(H_1+2)}\rho) \leq 2c_* \eta^{H_1} E(Dv, B_\rho)$$

and consequently

$$\begin{aligned} E(Dv, B_{2^{-(j_o+1)}\rho}) &\leq 2\sqrt{n} M(2^{-(H_1+K_1+1)}\rho) \leq 2\sqrt{n} M(2^{-(H_1+2)}\rho) \\ &\leq 4c_* \sqrt{n}\, \eta^{H_1} E(Dv, B_\rho) \leq \frac{1}{2} E(Dv, B_\rho) \end{aligned}$$

by (10.2.9) again. In *the case* $|(Dv)_{B_\rho}| > 2\sqrt{n}\, M(2^{-(H_1+2)}\rho)$ we have

$$|Dv - (Dv)_{B_\rho}| > \sqrt{n}\, M(2^{-(H_1+2)}\rho) \qquad \text{in } B_{2^{-(H_1+2)}\rho}.$$

Taking averages of the previous relation and using $(10.2.12)_2$ yields

$$\begin{aligned} 2\sqrt{n} M(2^{-(H_1+K_1+1)}\rho) &\leq 2\sqrt{n}\, \eta^{K_1} M(2^{-(H_1+2)}\rho) \\ &\leq 2\eta^{K_1} \fint_{B_{2^{-(H_1+2)}\rho}} |Dv - (Dv)_{B_\rho}|\, dx \\ &\leq 2^{n(H_1+3)} \eta^{K_1} \fint_{B_\rho} |Dv - (Dv)_{B_\rho}|\, dx \\ &\leq \frac{1}{2} E(Dv, B_\rho) \end{aligned}$$

by the choice of K_1 in (10.2.9).

Hence we proved that in the case neither (10.2.5) nor (10.2.6) holds in $B_{2^{-j}\rho}$ for $j = 1, \ldots, j_0$, then there exists $\tau = 2^{-(j_0+1)} \in (0, 1/2]$ depending only on n, ν, L, δ, g_0 such that $E(Dv, B_{\tau\rho}) \leq \frac{1}{2} E(Dv, B_\rho)$ for $\rho \leq R$. The way this previous inequality together with (10.2.11) leads to (10.2.3) is quite standard and we refer to [**70**] for its proof.

(10.2.4) follows now by a Campanato type argument, see [**81**, Theorem 2.9]. □

Now we give the proof of the Reverse Hölder's inequality we used to deduce (10.2.2):

LEMMA 10.7 (Reverse Hölder's inequality). *Let* $v \in W^{1,G}(A)$ *be a solution to* (10.2.1) *under the conditions* (4.7.4)–(4.7.2)*. Then for every ball* $B_R(x_0) \equiv B_R \subset A$ *there holds*

$$\fint_{B_{R/2}} G(|Dv|)\, dx \leq c\, G\Big(\fint_{B_R} |Dv|\, dx\Big). \tag{10.2.13}$$

for a constant depending on n, ν, L, δ, g_0.

PROOF. In order to lower the integrability on the right-hand side we first consider a preliminary reverse Hölder inequality and then we exploit the self-improving character of such kind of inequalities, see [**132**], with an approach which wants to mimic [**81**, Remark 6.12]. In particular we have the following inequality, which can be found in [**45**, Equation (1.11)]:

$$\fint_{B_{\rho/2}(y)} G(|Dv|)\, dx \leq c\, (G \circ B^{-1}) \Big(\fint_{B_\rho(y)} B(|Dv|)\, dx\Big) \tag{10.2.14}$$

for a constant depending upon n, ν, L, δ, g_0, valid for balls $B_\rho(y) \subset B_R$, and where the function B, which grows essentially slower than G at infinity, is given by

$$B(t) := G(t) \left[\frac{G(t)}{t}\right]^{-\frac{1}{n}}.$$

Note that (10.2.14) is proved for minimizers of functionals like (4.7.7) combining a Caccioppoli's inequality with an appropriate Sobolev's type inequality involving the function B; its proof for our case of equations requires however only slight modification. See also [**132**] for a Caccioppoli's inequality for minimizers satisfying hypotheses similar than ours.

Suppose now $R = 1$; we will prove the general case with the help of a scaling argument. Moreover take $r \leq 1$, $\alpha \in (0,1)$ and a point $y \in B_{\alpha r}(x_0)$. Apply inequality (10.2.14) for $\rho = (1-\alpha)r$, i.e. over $B_{(1-\alpha)r}(y)$. Note that we have $B_\rho(y) \subset B_r$. We have

$$\fint_{B_{(1-\alpha)r/2}(x)} G\big(|Dv|\big)\,dx \leq c\,(G\circ B^{-1})\bigg(\fint_{B_{r(1-\alpha)}(y)} B\big(|Dv|\big)\,dx\bigg).$$

Now we come to a bit of algebra. By its definition, with $B_\rho \equiv B_{(1-\alpha)r}(y)$, we have

$$\begin{aligned}\fint_{B_\rho} B\big(|Dv|\big)\,dx &= \fint_{B_\rho} \big[G(|Dv|)\big]^{\frac{n-1}{n}}|Dv|^{\frac{1}{n}}\,dx \\ &\leq \bigg(\fint_{B_\rho} G\big(|Dv|\big)\,dx\bigg)^{\frac{n-1}{n}}\bigg(\fint_{B_\rho}|Dv|\,dx\bigg)^{\frac{1}{n}},\end{aligned} \tag{10.2.15}$$

using Hölder's inequality. Now we want to use Young's inequality with conjugate functions $C(t) := B(t^n)$ and $\widetilde{C}(t)$. Using an argument we will use also in the rest of the Chapter, changing variable ($s = \sigma^{\frac{1}{n}}$) in the definition of the Young's conjugate function, for $\alpha > 0$,

$$\begin{aligned}\widetilde{C}(\alpha^{\frac{n-1}{n}}) &:= \sup_{s>0} \alpha^{\frac{n-1}{n}} s - B(s^n) \\ &\leq \Big[\sup_{\sigma>0} \alpha^{n-1}\sigma - \big[B(\sigma)\big]^n\Big]^{\frac{1}{n}} =: \Big[\widetilde{T}\big(\alpha^{n-1}\big)\Big]^{\frac{1}{n}}\end{aligned} \tag{10.2.16}$$

where the function T is obviously defined by $T(t) := [B(t)]^n$. Note that when taking the supremum in (10.2.16), we can just consider the values $s > 0$ such that $I(s) := \alpha^{\frac{n-1}{n}} s - B(s^n) \geq 0$; this set is not empty and this can be seen, for instance, by computating $I'(0) = \alpha^{\frac{n-1}{n}} > 0$. Analogously when taking the supremum with respect to σ. We also used $A^\alpha - B^\alpha \leq (A-B)^\alpha$ for $\alpha \in (0,1)$ and $A \geq B \geq 0$. At this point the reader might recall that

$$\widetilde{T}\big([G(\tau)]^{n-1}\big) = \widetilde{T}\bigg(\frac{[B(\tau)]^n}{\tau}\bigg) = \widetilde{T}\bigg(\frac{T(\tau)}{\tau}\bigg) \leq T(\tau) = \big[B(\tau)\big]^n$$

from the definition of B; the choice $\tau = G^{-1}(\alpha)$ leads now to

$$\widetilde{T}\big(\alpha^{n-1}\big) \leq \big[B\big(G^{-1}(\alpha)\big)\big]^n,$$

and plugging the latter estimate into (10.2.16) and choosing $\alpha = \fint_{B_r} G(|Dv|)\,dx$ gives the bound

$$\widetilde{C}\Bigg(\bigg(\fint_{B_\rho} G\big(|Dv|\big)\,dx\bigg)^{\frac{n-1}{n}}\Bigg) \lesssim_n (B\circ G^{-1})\bigg(\fint_{B_\rho} G\big(|Dv|\big)\,dx\bigg). \tag{10.2.17}$$

Note now that $t \mapsto (G\circ B^{-1})(t)$ is increasing and therefore a sort of triangle's inequality (10.1.11) holds, similarly to (10.1.9). Using this fact and Young's inequality with appropriate $\varepsilon \in (0,1)$, together with (10.2.17) into (10.2.15), and recalling that $\rho = (1-\alpha)r$ gives

$$\fint_{B_{(1-\alpha)r/2}} G\big(|Dv|\big)\,dx \leq \frac{1}{2}\fint_{B_{(1-\alpha)r}} G\big(|Dv|\big)\,dx + c\,G\bigg(\fint_{B_{(1-\alpha)r}}|Dv|\,dx\bigg)$$

with $c \equiv c(n,\delta,g_0)$; in turn

$$\int_{B_{(1-\alpha)r/2}} G\big(|Dv|\big)\,dx \le \frac{1}{2}\int_{B_{(1-\alpha)r}} G\big(|Dv|\big)\,dx + c\big[(1-\alpha)r\big]^{-ng_0} G\bigg(\int_{B_{(1-\alpha)r}} |Dv|\,dx\bigg).$$

Note now that the ball $B_{\alpha r}$ can be covered by balls of this kind in such a way that only a finite and independent of α number of balls of double radius intersect, all included in B_r. We then have calling $\alpha r =: s < r$

$$\int_{B_s} G\big(|Dv|\big)\,dx \le \frac{1}{2}\int_{B_r} G\big(|Dv|\big)\,dx + \frac{c}{(r-s)^{-ng_0}} G\bigg(\int_{B_r} |Dv|\,dx\bigg);$$

at this point iteration Lemma 3.11 gives (10.2.13) for the case $R = 1$. For the general case rescale in the following way: define $\tilde v(x) := v(x_0 + Rx)/R$. $\tilde v$ solves $-\operatorname{div} a(D\tilde v) = 0$ on $B_1(0)$ and therefore we can apply (10.2.13) to $\tilde v$. Rescaling back gives the reverse Hölder's inequality in the general case. □

Finally, a so-called "density improvement Lemma":

LEMMA 10.8. *Suppose that the two conditions*

$$\frac{\lambda}{C} \le \fint_{\sigma^m B} |Dv|\,dx \qquad \textit{and} \qquad \sup_{B/4} |Dv| \le C\lambda, \tag{10.2.18}$$

hold for some $m \in \mathbb{N}$, some numbers $C \ge 1$ and $\lambda \ge 0$ and with

$$0 < \sigma^\alpha \le \frac{1}{2^{3\alpha+2} c_o C^2} < \frac{1}{8^\alpha}, \tag{10.2.19}$$

where $\alpha \in (0,1)$ and c_o appear in Lemma 10.6. Then

$$\frac{\lambda}{4C} \le |Dv| \qquad \textit{in } \sigma B.$$

PROOF. From $(10.2.18)_1$ we deduce that there exists a point $x_0 \in \sigma^m B$ such that $|Dv(x_0)| > \lambda/2C$. On the other hand, (10.2.4) and $(10.2.18)_2$ give $|Dv(x) - Dv(x_0)| \le c_o(2\sigma)^\alpha C\lambda$ whenever $x \in B_{r/2} \equiv \sigma B \subset B/8$. The choice above for σ together with the last two inequalities gives

$$|Dv(x)| \ge |Dv(x_0)| - |Dv(x) - Dv(x_0)| \ge \frac{\lambda}{2C} - \frac{\lambda}{4C} = \frac{\lambda}{4C}$$

for all $x \in \sigma B$. □

10.3. Various comparison estimates

In this Section we want to derive comparison estimates between the solution to equation (4.7.3) and to a suitable homogeneous Cauchy problem. In particular, given a ball $B_R \equiv B_R(x_0) \subset \Omega$, we consider the solution $v \in u + W_0^{1,G}(B_R)$ to the Cauchy problem

$$\begin{cases} -\operatorname{div} a(Dv) = 0 & \text{in } B_R,\\ v = u & \text{on } \partial B_R. \end{cases} \tag{10.3.1}$$

Existence and uniqueness of such functions are given with approximation and monotonicity arguments, see [**112**, Lemma 5.2].

In his first part of this Section we are going to introduce and study separately some auxiliary functions we shall use in the proofs. First we introduce two functions directly depending on g:

$$f_\chi(t) := \int_0^t \left[\frac{g(s)}{s}\right]^{1+\chi} ds, \qquad g_\chi(t) := \left[\frac{g(t)}{t}\right]^{1+\chi} t, \tag{10.3.2}$$

for $\chi \geq -1$. Note that functions similar to g_χ have been already used for example in [78]. We immediately stress that, by a simple computation of derivatives, the use of (4.7.2) and integration over $(0,t)$ we have

$$[\delta(1+\chi)-\chi]f_\chi(t) \leq g_\chi(t) \leq \big[g_0(1+\chi)-\chi\big]f_\chi(t) \tag{10.3.3}$$

and therefore $f_\chi(t) \approx_\chi g_\chi(t)$. Note also that

$$f_\chi(\alpha t) \lesssim \max\big\{\alpha^{(g_\circ-1)(1+\chi)+1}, \alpha^{(\delta-1)(1+\chi)+1}\big\} f_\chi(t)$$

for $\alpha \geq 0$. Moreover we need to introduce the function $H_\chi(t)$ defined *through* the following formula

$$H_\chi^{-1}(t) := t^{-\chi}\left[G^{-1}\left(\frac{1}{t}\right)\right]^{-(2\chi+1)};$$

here (and in the sequel) we eventually use the conventions that $1/0=\infty$, $1/\infty=0$ and $G^{-1}(\infty)=\infty$, so that H_χ^{-1} (and other functions) are defined in zero in a direct way. Note that a computation shows that

$$\begin{aligned}\frac{d}{dt}H_\chi^{-1}(t) &= \big[G(\tau)\big]^{\chi+1}\tau^{-(2\chi+2)}\left[(2\chi+1)\frac{G(\tau)}{g(\tau)}-\chi\tau\right]\\ &\geq \left[\frac{2\chi+1}{1+g_0}-\chi\right]\big[G(\tau)\big]^{\chi+1}\tau^{-(2\chi+1)} \geq 0\end{aligned}$$

with $\tau := G^{-1}(\frac{1}{t})$ if $\chi \leq \frac{1}{g_\circ-1}$. $t \mapsto H_\chi^{-1}(t)$ is hence increasing and it makes sense to define its inverse, namely $H_\chi(t)$. Note moreover that by (10.3.3) we have

$$H_\chi^{-1}(t) \approx_\chi t\, g_\chi\left(G^{-1}\left(\frac{1}{t}\right)\right) \approx_\chi t\, f_\chi\left(G^{-1}\left(\frac{1}{t}\right)\right) \tag{10.3.4}$$

and by (10.1.13) we deduce

$$[\widetilde{H_\chi}]^{-1}\left(\frac{1}{G(t)}\right) \approx \frac{1}{G(t)}\frac{1}{H_\chi^{-1}(\frac{1}{G(t)})} \approx \frac{1}{G(t)}\frac{t^{2\chi+1}}{\big[G(t)\big]^\chi} \approx \frac{1}{f_\chi(t)}.$$

Matching this estimate together with the one inferred from the left-hand side inequality of (10.1.13) and (10.3.4) we deduce

$$H_\chi^{-1}\left(\widetilde{H_\chi}\left(\frac{1}{f_\chi(t)}\right)\right) \approx H_\chi^{-1}\left(\frac{1}{G(t)}\right) = \frac{\big[G(t)\big]^\chi}{t^{2\chi+1}} \approx \frac{\big[g(t)\big]^\chi}{t^{\chi+1}} = \frac{g_\chi(t)}{g(t)t}. \tag{10.3.5}$$

Finally, since the function $H_\chi^{-1}(\cdot)$ is increasing and there holds the doubling property $H_\chi^{-1}(2t) \lesssim_\chi H_\chi^{-1}(t)$, by Remark 10.2 we have that the following inequality

$$H_\chi^{-1}(t+s) \lesssim_\chi H_\chi^{-1}(t) + H_\chi^{-1}(s) \tag{10.3.6}$$

for $t,s \geq 0$. Finally we come to the proof of the comparison estimate:

Lemma 10.9. *Let $u \in W^{1,G}(\Omega)$ be the solutions to the equation (4.7.3) and $v \in u + W_0^{1,G}(B_R)$ the solution to the problem (10.3.1) on B_R. Then the following estimate holds true:*

$$\fint_{B_R} g_\chi\big(|Du-Dv|\big)\,dx \leq c_1 g_\chi(A) \quad \text{where} \quad A := g^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right), \tag{10.3.7}$$

where g_χ is the functions defined in (10.3.2), for

$$\chi \in \left[-1, \min\left\{\frac{1}{g_0-1}, \frac{g_0}{(g_0-1)(n-1)}\right\}\right) \tag{10.3.8}$$

and with a constant c_1 depending on $n, \nu, \delta, g_0, \chi$.

PROOF. *Step 1: rescaling.*

Define A as in (10.3.7); we can suppose without loss of generality that $A > 0$, since in the case $|\mu|(B_R) = 0$ the monotonicity of the vector field ensures $u = v$ on B_R and then (10.3.7) is trivially true. Consequently we define

$$\bar{u}(x) := \frac{u(x_0 + Rx)}{AR}, \qquad \bar{v}(x) := \frac{v(x_0 + Rx)}{AR},$$
$$\bar{a}(z) := \frac{a(Az)}{g(A)}, \qquad \bar{\mu}(x) := R\,\frac{\mu(x_0 + Rx)}{g(A)}, \tag{10.3.9}$$

then, subtracting the weak formulations of (4.7.3) and (10.3.1) and rescaling we have

$$-\operatorname{div}\big[\bar{a}(D\bar{u}) - \bar{a}(D\bar{v})\big] = \bar{\mu} \qquad \text{in } B_1; \tag{10.3.10}$$

note that the growth function $\bar{g}$ of the vector field $\bar{a}$ is given by

$$\bar{g}(t) := \frac{g(At)}{g(A)} :$$

indeed

$$\langle \partial\tilde{a}(z)\lambda, \lambda\rangle = \frac{A}{g(A)}\langle \partial a(Az)\lambda, \lambda\rangle \geq \nu\,\frac{A}{g(A)}\,\frac{g(A|z|)}{A|z|}|\lambda|^2 = \nu\,\frac{\bar{g}(|z|)}{|z|}|\lambda|^2$$

for all $z, \lambda \in \mathbb{R}^n$. Since we are treating measure data problems, this is enough since we will use only the ellipticity of the vector field. However a similar estimate holds true for the growth of the vector field. Moreover note that

$$\frac{t\bar{g}'(t)}{\bar{g}(t)} = \frac{At\,g'(At)}{g(At)} \in [\delta, g_0]$$

for all $t > 0$. The aim of this substitution is twofold: we can restrict ourselves to prove the Lemma in the case $B_R(x_0) = B_1$; moreover we can exploit the following estimate

$$|\tilde{\mu}|(B_1) = \frac{1}{g(A)}\,\frac{|\mu|(B_R)}{R^{n-1}} = 1. \tag{10.3.11}$$

In this case what we want to prove is simply

$$\int_{B_1} \bar{g}_\chi\big(|D\bar{u} - D\bar{v}|\big)\,dx \leq c(n, \nu, \delta, g_0), \tag{10.3.12}$$

where $\bar{g}_\chi$ is obtained starting from $\bar{g}$ instead of g in the expression appearing in (10.3.2). At the very end of the proof we will show how to recover the full result from (10.3.12).

Step 2: measure data estimates.

From now on we will drop the tilde notation, recovering it only in Step 3, equation (10.3.32). We recall we are working under the assumptions $B_R = B_1$ and $|\mu|(B_1) = 1$. Since we want estimates involving only the mass of the measure μ, we shall at least initially follow the standard truncation method for which the unavoidable references are the works of Boccardo and Gallouët [**26, 25**]. Some changes are however needed in order to handle the growth condition we are considering. Moreover, two different approach are needed to treat the two different kind of growth G could have at infinity. In the standard case, this correspond to consider the two cases $p \leq n$ and $p > n$. In both cases we will need to consider the weak formulation of (10.3.10)

$$\int_{B_1} \langle a(Du) - a(Dv), D\varphi\rangle\,dx = \int_{B_1} \varphi\,d\mu \tag{10.3.13}$$

holding true for bounded functions $\varphi \in W^{1,G}_0 \cap L^\infty(B_1)$.

Step 2.1: The slow growth case. With this expression we want to suggest the case where

$$\int^{\infty}\left(\frac{s}{G(s)}\right)^{\frac{1}{n-1}}ds=\infty. \tag{10.3.14}$$

In order to use Sobolev's embedding, we need to introduce a slightly modified function in order to have the integrability property (10.1.15)$_1$. We therefore define the continuous function

$$\mathfrak{f}_\chi(t):=\begin{cases}0 & t=0,\\ f_\chi(1)t & \text{for } t\in(0,1),\\ f_\chi(t) & \text{for } t\in[1,\infty).\end{cases} \tag{10.3.15}$$

Let's begin putting into (10.3.13) the test function

$$\varphi:=T_k\left(\frac{u-v}{c_S(n)\big(\int_{B_1}\mathfrak{f}_\chi(|Du-Dv|)\,dx\big)^{\frac{1}{n}}}\right)=:T_k\left(\frac{u-v}{c_S(n)\,\mathcal{F}}\right),$$

for any $k\in\mathbb{N}_0$, being $c_S(n)$ the constant appearing in (10.1.17) and $\mathfrak{f}_\chi(\cdot)$ the function defined in (10.3.15). The classical truncation operators are defined as

$$T_k(\sigma):=\max\big\{-k,\min\{k,\sigma\}\big\},\qquad \Phi_k(\sigma):=T_1\big(\sigma-T_k(\sigma)\big) \tag{10.3.16}$$

for $k\in\mathbb{N}_0$ and $\sigma\in\mathbb{R}$. Note that we can clearly suppose $\mathcal{F}\geq 1$ and that $\varphi\in W^{1,G}_0(B_1)\cap L^\infty(B_1)$, since $\sigma\to T_k(\sigma)$ is Lipschitz; then φ is allowed as test function. We moreover have $D\varphi=\frac{D(u-v)}{c_S(n)\mathcal{F}}\chi_{C_k}$, being χ_{C_k} the characteristic function of the set C_k, where

$$C_k:=\left\{x\in B_1:\frac{|u(x)-v(x)|}{c_S(n)\big(\int_{B_1}\mathfrak{f}_\chi(|Du-Dv|)\,dx\big)^{\frac{1}{n}}}\leq k\right\}.$$

Using (10.1.10) we have

$$\begin{aligned}\int_{B_1}\langle a(Du)-a(Dv),D\varphi\rangle\,dx&=\frac{1}{c_S\,\mathcal{F}}\int_{C_k}\langle a(Du)-a(Dv),Du-Dv\rangle\,dx\\&\geq\frac{c}{c(n)\,\mathcal{F}}\int_{C_k}G\big(|Du-Dv|\big)\,dx,\end{aligned}$$

Estimating the right-hand side in the trivial way

$$\left|\int_{B_1}T_k\Big(\frac{u-v}{c_S(n)\,\mathcal{F}}\Big)\,d\mu\right|\leq\int_{B_1}k\,d|\mu|=k|\mu|(B_1)=k$$

by (10.3.11), we deduce the estimate

$$\int_{C_k}G\big(|Du-Dv|\big)\,dx\leq c\,k\Big(\int_{B_1}\mathfrak{f}_\chi\big(|Du-Dv|\big)\,dx\Big)^{\frac{1}{n}}=c\,k\mathcal{F}, \tag{10.3.17}$$

for all $k\in\mathbb{N}_0$, where $c\equiv c(n,\nu,\delta,g_0)$. Reasoning in an analogous way, using as test function $\Phi_k\big((u-v)/(c_S(n)\mathcal{F})\big)\in W^{1,G}_0(B_1)\cap L^\infty(B_1)$, we infer

$$\int_{D_k}G\big(|Du-Dv|\big)\,dx\leq c(n,\nu)\Big(\int_{B_1}\mathfrak{f}_\chi\big(|Du-Dv|\big)\,dx\Big)^{\frac{1}{n}}=c\,\mathcal{F}$$

since $\Phi_k\leq 1$, where we have denoted

$$D_k:=\left\{x\in B_1:k<\frac{|u(x)-v(x)|}{c_S(n)\big(\int_{B_1}\mathfrak{f}_\chi(|Du-Dv|)\,dx\big)^{\frac{1}{n}}}\leq k+1\right\}.$$

Now we come back to f_χ defined in (10.3.2) and we note that $t \mapsto f_\chi\big(G^{-1}(t)\big)$ is increasing and concave: indeed a computation of derivatives, denoting $\tau := G^{-1}(t)$, gives

$$\begin{aligned}
\frac{d}{dt} f_\chi\big(G^{-1}(t)\big) &= \frac{f'_\chi(\tau)}{g(\tau)} = \frac{[g(\tau)]^\chi}{\tau^{1+\chi}}, \\
\frac{d^2}{dt^2} f_\chi\big(G^{-1}(t)\big) &= \frac{\chi\tau^{1+\chi}[g(\tau)]^{\chi-1}g'(\tau) - [g(\tau)]^\chi(1+\chi)\tau^\chi}{g(\tau)\tau^{2(1+\chi)}} \\
&= \frac{\big[g(\tau)\big]^{\chi-2}}{\tau^{\chi+2}}\big[\chi\tau g'(\tau) - (1+\chi)g(\tau)\big] \\
&\leq \frac{\big[g(\tau)\big]^{\chi-1}}{\tau^{\chi+2}}\big[\chi g_0 - (1+\chi)\big] < 0
\end{aligned} \tag{10.3.18}$$

by (4.7.2) and the fact that $\chi < \frac{1}{g_0-1}$. Therefore using Jensen's inequality and (10.3.17) we get

$$\begin{aligned}
\fint_{C_k} f_\chi\big(|Du - Dv|\big)\,dx &\leq \big(f_\chi \circ G^{-1}\big)\bigg(\fint_{C_k} G\big(|Du-Dv|\big)\,dx\bigg) \\
&\lesssim c\,\big(f_\chi \circ G^{-1}\big)\bigg(\frac{k\mathcal{F}}{|C_k|}\bigg).
\end{aligned} \tag{10.3.19}$$

So using (10.3.4) and doing an easy algebraic manipulation we infer

$$\int_{C_k} f_\chi\big(|Du-Dv|\big)\,dx \lesssim c\,k\mathcal{F}H_\chi^{-1}\bigg(\frac{|C_k|}{k\mathcal{F}}\bigg). \tag{10.3.20}$$

with $c \equiv c(n,\nu,\delta,g_0,\chi)$. By a similar argument we have for the integrals over D_k

$$\int_{D_k} f_\chi\big(|Du-Dv|\big)\,dx \lesssim c\,\mathcal{F}H_\chi^{-1}\bigg(\frac{|D_k|}{\mathcal{F}}\bigg). \tag{10.3.21}$$

We hence have, using (10.3.20) and (10.3.21)

$$\begin{aligned}
\int_{B_1} f_\chi\big(|Du-Dv|\big)\,dx &= \int_{C_1} f_\chi\big(|Du-Dv|\big)\,dx \\
&\qquad + \sum_{k=1}^{\infty}\int_{D_k} f_\chi\big(|Du-Dv|\big)\,dx \\
&\leq \tilde{c}\,\mathcal{F}\bigg[H_\chi^{-1}\bigg(\frac{|B_1|}{\mathcal{F}}\bigg) + \sum_{k=1}^{\infty} H_\chi^{-1}\bigg(\frac{|D_k|}{\mathcal{F}}\bigg)\bigg]
\end{aligned} \tag{10.3.22}$$

with $\tilde{c} \equiv \tilde{c}(n,\nu,\delta,g_0,\chi)$. Here to estimate the summation appearing on the right-hand side, we have to work on the modified function $\mathfrak{f}_\chi$ defined in (10.3.15). Note that $\mathfrak{f}_\chi(\cdot)$ is a Young function and

$$\int_0 \bigg(\frac{s}{\mathfrak{f}_\chi(s)}\bigg)^{\frac{1}{n-1}} ds < \infty \quad \text{and} \quad \int^\infty \bigg(\frac{s}{\mathfrak{f}_\chi(s)}\bigg)^{\frac{1}{n-1}} ds = +\infty; \tag{10.3.23}$$

the first by construction and the second by (10.3.14), since for $s \geq 1$

$$\mathfrak{f}_\chi(s) = f_\chi(s) \approx \bigg[\frac{g(s)}{s}\bigg]^{1+\chi} s \lesssim G(s)\frac{[g(s)]^\chi}{s^{1+\chi}} \lesssim_\chi G(s)\,s^{\chi g_0-(1+\chi)} \leq G(s)$$

being $\chi \leq \frac{1}{g_0-1}$. We can therefore define the Sobolev's conjugate function $(\mathfrak{f}_\chi)_n := \mathfrak{f}_\chi \circ H_n^{-1}$, where in this case H_n is given by $(10.1.16)_1$ with the choice $A \equiv \mathfrak{f}_\chi$. Moreover, since

$$f_\chi(1) = \int_0^1 \bigg[\frac{g(s)}{s}\bigg]^{1+\chi} ds \leq \int_0^1 s^{(\delta-1)(1+\chi)}\,ds = \frac{1}{(\delta-1)(1+\chi)+1} \leq 1,$$

then we have, for $t \geq 1$

$$H_n^{-1}(t) \geq \left[\int_0^1 \left(\frac{s}{\mathfrak{f}_\chi(s)}\right)^{\frac{1}{n-1}} ds\right]^{\frac{n-1}{n}} = \left(\frac{1}{f_\chi(1)}\right)^{\frac{1}{n}} \geq 1. \tag{10.3.24}$$

At this point we have

$$|D_k| \leq \frac{1}{(\mathfrak{f}_\chi)_n(k)} \int_{D_k} (\mathfrak{f}_\chi)_n \left(\frac{|u-v|}{c_S(n)\mathcal{F}}\right) dx \tag{10.3.25}$$

for every $k \in \mathbb{N}$ by the definition of the set D_k. Having now both assumptions (10.1.15) at hand, we can deduce, using Sobolev's embedding (10.1.17), the following estimate for the summation: taking into account Young's inequality with conjugate functions H_χ, $\widetilde{H_\chi}$ with $\varepsilon \in (0,1)$ to be chosen, (10.1.17) and triangle's inequality (10.3.6)

$$\begin{aligned}
\sum_{k=1}^{\infty} H_\chi^{-1}\left(\frac{|D_k|}{\mathcal{F}}\right) &\leq \sum_{k=1}^{\infty} H_\chi^{-1}\left(\frac{1}{\mathcal{F}(f_\chi)_n(k)} \int_{D_k} (\mathfrak{f}_\chi)_n \left(\frac{|u-v|}{c_S(n)\mathcal{F}}\right) dx\right) \\
&\leq \frac{\varepsilon}{\mathcal{F}} \int_{B_1} (\mathfrak{f}_\chi)_n \left(\frac{|u-v|}{c_S(n)\mathcal{F}}\right) dx + c(\delta, g_0, \chi, \varepsilon) \sum_{k=1}^{\infty} H_\chi^{-1}\left(\widetilde{H_\chi}\left(\frac{1}{(f_\chi)_n(k)}\right)\right) \\
&\leq \frac{\varepsilon}{\mathcal{F}} \int_{B_1} \mathfrak{f}_\chi\big(|Du - Dv|\big)\, dx + c_\varepsilon \sum_{k=1}^{\infty} H_\chi^{-1}\left(\widetilde{H_\chi}\left(\frac{1}{(f_\chi)_n(k)}\right)\right) \\
&\leq \frac{\varepsilon}{\mathcal{F}} \int_{B_1} f_\chi\big(|Du - Dv|\big)\, dx + c(n, \delta, g_0, \chi) + c_\varepsilon \sum_{k=1}^{\infty} \frac{\big[g\big(H_n^{-1}(k)\big)\big]^\chi}{\big[H_n^{-1}(k)\big]^{1+\chi}}
\end{aligned} \tag{10.3.26}$$

by (10.3.5), since

$$(\mathfrak{f}_\chi)_n(k) = \mathfrak{f}_\chi\big(H_n^{-1}(k)\big) = f_\chi\big(H_n^{-1}(k)\big) \qquad \text{since } H_n^{-1}(k) \geq 1 \text{ by (10.3.24);}$$

here $c_\varepsilon \equiv c_\varepsilon(\delta, g_0, \chi, \varepsilon)$. Moreover in the last line we also replaced $\mathfrak{f}_\chi$ with f_χ in the first term, since $\mathfrak{f}_\chi(t) \leq f_\chi(1) + f_\chi(t) \lesssim_\chi 1 + f_\chi(t)$ and $\varepsilon/\mathcal{F} \leq 1$. Now we inquire the convergence of the series on the right-hand side. A quite long but elementary calculation of its derivative, similar to (10.3.18), shows that $\sigma \mapsto [g(H_n^{-1}(\sigma))]^\chi/[H_n^{-1}(\sigma)]^{1+\chi}$ is decreasing, since $\chi < \frac{1}{g_0-1}$, and hence it is easily seen that the series is dominated by the quantity

$$\frac{[g(H_n^{-1}(1))]^\chi}{[H_n^{-1}(1)]^{1+\chi}} + \int_1^\infty \left[\frac{g\big(H_n^{-1}(\sigma)\big)}{H_n^{-1}(\sigma)}\right]^\chi \frac{d\sigma}{H_n^{-1}(\sigma)}.$$

Therefore now we want to show that

$$\int_1^\infty \left[\frac{g\big(H_n^{-1}(\sigma)\big)}{H_n^{-1}(\sigma)}\right]^\chi \frac{d\sigma}{H_n^{-1}(\sigma)} = c(n, \delta, g_0, \chi) < \infty. \tag{10.3.27}$$

We use the change of variable $s = H_n^{-1}(\sigma)$; this is allowed by (10.3.23) and the fact that $\sigma \mapsto H_n^{-1}(\sigma)$ is increasing. We note that

$$d\sigma = H_n'(s)\, ds = c(n)\big[H_n(s)\big]^{\frac{1}{1-n}} \left[\frac{s}{\mathfrak{f}_\chi(s)}\right]^{\frac{1}{n-1}} ds \tag{10.3.28}$$

and by monotonicity

$$H_n^{-1}(s) = \left(\int_0^s \left[\frac{\tau}{\mathfrak{f}_\chi(\tau)}\right]^{\frac{1}{n-1}} d\tau\right)^{\frac{n-1}{n}} \geq s^{\frac{n-1}{n}} \left[\frac{s}{\mathfrak{f}_\chi(s)}\right]^{\frac{1}{n}}.$$

Hence for $s \geq H_n^{-1}(1) \geq 1$ we have

$$d\sigma \leq c(n)\left[\frac{s}{\mathfrak{f}_\chi(s)}\right]^{\frac{1}{n}} s^{-\frac{1}{n}} \leq c(n)\left[\frac{s}{f_\chi(s)}\right]^{\frac{1}{n}} s^{-\frac{1}{n}} \lesssim_n \left[\frac{s}{g_\chi(s)}\right]^{\frac{1}{n}} s^{-\frac{1}{n}},$$

since we have $f_\chi(s) \leq \mathfrak{f}_\chi(s)$. We therefore have by (10.3.3) and $H_n^{-1}(1) \geq 1$

$$\int_1^\infty \left[\frac{g\big(H_n^{-1}(\sigma)\big)}{H_n^{-1}(\sigma)}\right]^\chi \frac{d\sigma}{H_n^{-1}(\sigma)} \lesssim_{n,\chi} \int_1^\infty \big[g_\chi(s)\big]^{1-\frac{1}{n}} \frac{ds}{sg(s)} < \infty.$$

The latter integral is finite since in the case $\chi \geq 1/(n-1)$ (i.e. in the case we can use the estimate from above (10.1.5) in the second inequality of the next line)

$$\frac{\big[g_\chi(s)\big]^{1-\frac{1}{n}}}{sg(s)} = \big[g(s)\big]^{(1+\chi)(1-\frac{1}{n})-1} s^{-1-\chi(1-\frac{1}{n})} \lesssim_{n,\chi} s^{\mathfrak{e}(g_0)},$$

where $\mathfrak{e}(\alpha) := \alpha(1+\chi)\big(1-\dfrac{1}{n}\big) - \alpha - 1 - \chi\big(1-\dfrac{1}{n}\big) < -1$ and $\mathfrak{e}(g_0) < -1$ by the fact that $\chi < \frac{g_0}{(g_0-1)(n-1)}$. In the case $\chi \in [-1, 1/(n-1))$ we instead have

$$\frac{\big[g_\chi(s)\big]^{1-\frac{1}{n}}}{sg(s)} \lesssim_\chi \big[g(t)\big]^{(1+\chi)(1-\frac{1}{n})-1} s^{-1-\chi(1-\frac{1}{n})} \lesssim_{n,\chi} s^{\mathfrak{e}(\delta)}$$

and $\mathfrak{e}(\delta) < -1$ since $\chi < \frac{g_0}{(g_0-1)(n-1)} < \frac{\delta}{(\delta-1)(n-1)}$. Therefore in both cases (10.3.27) holds. Coming then back to (10.3.26) and (10.3.22)

$$\int_{B_1} f_\chi\big(|Du-Dv|\big)\,dx \leq \tilde{c}\,\mathcal{F}H_\chi^{-1}\left(\frac{|B_1|}{\mathcal{F}}\right) + \varepsilon\tilde{c}\int_{B_1} f_\chi\big(|Du-Dv|\big)\,dx + \tilde{c}_\varepsilon\mathcal{F}.$$

First we choose ε, depending on $n, \nu, \delta, g_0, \chi$, so small that we can reabsorb the second term of the right-hand side into the left-hand side, i.e. $\varepsilon = 1/(4\tilde{c})$. This fixes the value of $\tilde{c}_\varepsilon$ as a constant depending on $n, \nu, \delta, g_0, \chi$. Then we recall the definition of $\mathcal{F}$ and we estimate

$$\mathcal{F} = \left(\int_{B_1} \mathfrak{f}_\chi\big(|Du-Dv|\big)\,dx\right)^{\frac{1}{n}} \leq \tilde{\varepsilon}\int_{B_1} f_\chi\big(|Du-Dv|\big)\,dx + c(n,\delta,g_0,\tilde{\varepsilon})$$

with $\tilde{\varepsilon}$ small in order to reabsorb also this term, i.e. $\tilde{\varepsilon} := 1/(4\tilde{c}_\varepsilon)$. To conclude note that (10.3.4) gives

$$\mathcal{F}H_\chi^{-1}\left(\frac{|B_1|}{\mathcal{F}}\right) \lesssim_\chi |B_1|\, g_\chi\left(G^{-1}\left(\frac{\mathcal{F}}{|B_1|}\right)\right) \lesssim_{n,\chi} \mathcal{F}^{1+\chi}\Big[G^{-1}\big(\mathcal{F}\big)\Big]^{-(2\chi+1)}.$$

from the definition of g_χ and the fact that $g(t) \approx G(t)/t$. Recall again we are supposing $\mathcal{F} \geq 1$, and therefore using (10.1.8) we infer

$$\mathcal{F}^{1+\chi}\Big[G^{-1}\big(\mathcal{F}\big)\Big]^{-(2\chi+1)} \lesssim_\chi \mathcal{F}^{1+\chi-\frac{1+2\chi}{1+g_0}}.$$

Since the exponent of $\mathcal{F}$ reveals to be strictly smaller than one by $\chi < \frac{1}{g_0-1}$, we use for the third time Young's inequality together with $f_\chi \approx g_\chi$ to finally get (10.3.12).

Step 2.2: The fast growth case. We here approach the simpler case where

$$\int^\infty \left(\frac{s}{G(s)}\right)^{\frac{1}{n-1}} ds < \infty.$$

In this case, since both u and v belong to $W^{1,G}(B_1)$, their difference is bounded and we can directly choose $\varphi = u - v \in W_0^{1,G}(B_1) \cap L^\infty$ as a test function in (10.3.13). Therefore using (10.0.31) we infer

$$\int_{B_1} G(|Du - Dv|)\, dx \le \int_{B_1} (u - v)\, d\mu \le \sup_{B_1} |u - v| |\mu|(B_1) \le c\, \|Du - Dv\|_{L^G(B_1)} \qquad (10.3.29)$$

by (10.1.19) and the fact that $|\mu|(B_1) = 1$, with $c \equiv c(n, \delta, g_0)$. Let's apply inequality (10.1.14) to the function εf with $\varepsilon \in (0,1)$. We have

$$\varepsilon \|f\|_{L^G(B_1)} = \|\varepsilon f\|_{L^G(B_1)} \le \int_{B_1} G(\varepsilon|f|)\, dx + 1 \le \varepsilon^{1+\delta} \int_{B_1} G(|f|)\, dx + 1$$

by (10.1.7) and therefore we can use the following version of Young's inequality:

$$\|f\|_{L^G(B_1)} \le \varepsilon^{\delta} \int_{B_1} G(|f|)\, dx + \varepsilon^{-1}. \qquad (10.3.30)$$

Let's make use of it with $f = Du - Dv$ into (10.3.29): choosing ε small enough and reabsorbing the right-hand side term gives

$$\fint_{B_1} G(|Du - Dv|)\, dx \le c(n, \delta, g_0). \qquad (10.3.31)$$

Arguing as in (10.3.19) we infer

$$\int_{B_1} g_\chi(|Du - Dv|)\, dx \lesssim c(n) \fint_{B_1} f_\chi(|Du - Dv|)\, dx \le c\,(f_\chi \circ G^{-1}) \left(\fint_{B_1} G(|Du - Dv|)\, dx \right) \le c$$

with $c \equiv c(n, \delta, g_0, \chi)$, from the fact that $t \mapsto (f_\chi \circ G^{-1})(t)$ is concave.

Step 3: Recovering the situation.

Now we recover the tilde notation: recalling the definitions given in (10.3.9), denoting for shortness $y = x_0 + Rx$, (10.3.12) can be rephrased as

$$\left[\frac{A}{g(A)}\right]^{1+\chi} \frac{1}{A} \fint_{B_1} \left[\frac{g(|Du(y) - Dv(y)|)}{|Du(y) - Dv(y)|}\right]^{1+\chi} |Du(y) - Dv(y)|\, dx = \fint_{B_1} \left[\frac{\bar g(|D\bar u - D\bar v|)}{|D\bar u - D\bar v|}\right]^{1+\chi} |D\bar u - D\bar v|\, dx \le c, \qquad (10.3.32)$$

that is (10.3.7), once performing a simple change of variable on the left-hand side and recalling the definition of g_χ in (10.3.2). □

Once having the previous Lemma at hand, a minor modification of the proof allows to get the following similar result which, despite being surely not optimal, it is therefore sufficient for our purposes. We introduce the further following function for the sake of shortness:

$$h_\chi(t) := \left[\frac{g(t)}{t}\right]^{1+\chi} = \frac{g_\chi(t)}{t}.$$

COROLLARY 10.10. *Let $u \in W^{1,G}(\Omega)$ and $v \in u + W_0^{1,G}(B_R)$ as in Lemma 10.9. Then the following comparison estimate holds true:*

$$\fint_{B_R} h_\chi(|Du - Dv|)\, dx \le c\, h_\chi(A) \quad \textit{with} \quad A := g^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right),$$

for χ *as in* (10.3.8) *and with a constant* c *depending on* $n, \nu, \delta, g_0, \chi$.

PROOF. We rescale both the function as in *Step 1* of the proof of Lemma 10.9. Having after *Step 2* estimate (10.3.12) at hand, we can estimate

$$
\begin{aligned}
\fint_{B_1} h_\chi\big(|Du-Dv|\big)\,dx &= c(n)\int_{B_1}\left[\frac{g\big(|Du-Dv|\big)}{|Du-Dv|}\right]^{1+\chi} dx \\
&= c\int_{B_1\cap\{|Du-Dv|\le 1\}} \dots\, dx + c\int_{B_1\cap\{|Du-Dv|>1\}} \dots\, dx \\
&\le c(n,\delta,g_0,\chi) + c\int_{B_1}\left[\frac{g\big(|Du-Dv|\big)}{|Du-Dv|}\right]^{1+\chi} |Du-Dv|\,dx \le c.
\end{aligned}
$$

At this point performing a rescaling similar to that in (10.3.32) gives (10.3.33). □

Now another, albeit similar, proof of this kind:

LEMMA 10.11. *Let* $u \in W^{1,G}(\Omega)$ *be the solutions to the equation* (4.7.3) *and* $v \in u + W_0^{1,G}(B_R)$ *the solution to the problem* (10.3.1) *on* B_R. *Then the following estimate holds true:*

$$
\fint_{B_R} \big[g\big(|Du-Dv|\big)\big]^{\xi}\,dx \le c\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\xi} \tag{10.3.33}
$$

for

$$
\xi \in \Big[1, \min\Big\{\frac{g_0+1}{g_0}, \frac{n}{n-1}\Big\}\Big) \tag{10.3.34}
$$

and with a constant c *depending on* n, ν, δ, g_0, ξ.

PROOF. Since the proof is very similar to that of Lemma 10.9, we will only highlight the main points. First of all we perform a scaling as in (10.3.9) and subsequent lines; therefore from now on we can suppose $B_R = B_1$ and moreover $|\mu|(B_1) = 1$. Introducing the auxiliary function

$$
f_\xi(t) := \xi\int_0^t \frac{[g(s)]^{\xi}}{s}\,ds;
$$

for ξ as in (10.3.34) and noting that we have $f_\xi(t) \approx [g(t)]^{\xi}$ (and $f_\xi(1) \le 1$, for later use), all that we want to prove now is

$$
\int_{B_1} f_\xi\big(|Du-Dv|\big)\,dx \le c(n,\nu,\delta,g_o,g(\cdot)). \tag{10.3.35}
$$

Exactly as in *Step 3* of the proof of Lemma 10.9, (10.3.33) will follow simply by coming back to $[g(Du-Dv)]^{\xi}$ and using the scaling of the equation.

The slow growth case. We first consider the case where

$$
\int^{\infty}\left(\frac{s}{G(s)}\right)^{\frac{1}{n-1}} ds = \infty. \tag{10.3.36}
$$

We moreover define, for $\mathfrak{f}_\xi$ defined in (10.3.40)

$$
\mathcal{F} := \left(\int_{B_1} \mathfrak{f}_\xi\big(|Du-Dv|\big)\,dx\right)^{\frac{1}{n}}.
$$

We choose in (10.3.13) the test function

$$
\varphi \equiv T_k\left(\frac{u-v}{c_S(n)\,\mathcal{F}}\right) \in W_0^{1,G}(B_1)
$$

for $k \in \mathbb{N}_0$, being $c_S(n)$ being the constant appearing in (10.1.17) and recalling the definition of the truncation operator T_k in (10.3.16). Also here we can suppose $\mathcal{F} \geq 1$ and we have $D\varphi = \frac{D(u-v)}{c_S(n)\mathcal{F}}\chi_{C_k}$; this time $C_k = B_1 \cap \{|u-v|/(c_S(n)\mathcal{F}) \leq k\}$. Therefore we deduce

$$\int_{C_k} G(|Du - Dv|)\, dx \lesssim c(n)\frac{1}{\mathcal{F}}\int_{B_1} \langle a(Du) - a(Dv), D\varphi\rangle\, dx \tag{10.3.37}$$

$$\leq c\,\mathcal{F}\left|\int_{B_1} \varphi\, dx\right| \leq c\, k\mathcal{F} \tag{10.3.38}$$

for $k \in \mathbb{N}_0$, with $c \equiv c(n, \nu, \delta, g_0)$. Similarly

$$\int_{D_k} G(|Du - Dv|)\, dx \leq c\,\mathcal{F},$$

where $D_k := B_1 \cap \{k < |u-v|/(c_S(n)\mathcal{F}) \leq k+1\}$. Now we compute the first derivatives of the function $G(f_\xi^{-1}(\cdot))$: note that

$$f_\xi'(t) = \xi\frac{[g(t)]^\xi}{t} \quad \text{and then} \quad \frac{d}{dt} f_\xi\big(G^{-1}(t)\big) = \xi\frac{\big[g(\tau)\big]^{\xi-1}}{\tau}$$

with $\tau := G^{-1}(t)$; moreover

$$\begin{aligned}\frac{d^2}{dt^2} f_\xi\big(G^{-1}(t)\big) &= \xi\frac{\tau(\xi-1)[g(\tau)]^{\xi-2}g'(\tau) - [g(\tau)]^{\xi-1}}{g(\tau)\tau^2} \\ &= \xi\frac{\big[g(\tau)\big]^{\xi-1}}{\tau}\Big[(\xi-1)\frac{\tau g'(\tau)}{g(\tau)} - 1\Big] < 0\end{aligned}$$

by (4.7.2) and since $\xi < \frac{1+g_0}{g_0}$. Hence here $t \mapsto f_\xi\big(G^{-1}(t)\big)$ is increasing and concave. Jensen's inequality and (10.3.37) yield

$$\begin{aligned}\int_{C_k} f_\xi\big(|Du - Dv|\big)\, dx &\leq |C_k|\big(f_\xi \circ G^{-1}\big)\left(⨍_{C_k} G\big(|Du - Dv|\big)\, dx\right) \\ &\lesssim c(n,\nu,\xi)|C_k|\big(f_\xi \circ G^{-1}\big)\left(\frac{k\mathcal{F}}{|C_k|}\right) = c\, k\mathcal{F}H_\xi^{-1}\left(\frac{|C_k|}{k\mathcal{F}}\right),\end{aligned} \tag{10.3.39}$$

where

$$H_\xi^{-1}(t) := t^{1-\xi}\left[G^{-1}\Big(\frac{1}{t}\Big)\right]^{-\xi} \approx_\xi t f_\xi\left(G^{-1}\Big(\frac{1}{t}\Big)\right).$$

Also here computation of derivatives yields $t \mapsto H_\xi^{-1}(t)$ increasing: indeed using (10.1.4)

$$\frac{d}{dt} H_\xi^{-1}(t) \geq \Big(1 - \xi + \frac{\xi}{1+g_0}\Big)t^{-\xi}\left[G^{-1}\Big(\frac{1}{t}\Big)\right]^\xi > 0$$

since $\xi < \frac{1+g_0}{g_0}$. Hence there holds a sort of triangle's inequality as in (10.3.6). By arguments we have for the integrals over D_k

$$\int_{D_k} f_\xi\big(|Du - Dv|\big)\, dx \lesssim \mathcal{F}H_\xi^{-1}\left(\frac{|D_k|}{\mathcal{F}}\right).$$

We have as in (10.3.22):

$$\int_{B_1} f_\xi\big(|Du - Dv|\big)\, dx \leq c\,\mathcal{F}\left[H_\xi^{-1}\left(\frac{|B_1|}{\mathcal{F}}\right) + \sum_{k=1}^{\infty} H_\xi^{-1}\left(\frac{|D_k|}{\mathcal{F}}\right)\right].$$

As in (10.3.15) we modify the function f_ξ linearly near zero in order to use Sobolev's embedding. Define

$$\mathfrak{f}_\xi(t) := \begin{cases} 0 & t = 0, \\ f_\xi(1)t & \text{for } t \in (0,1), \\ f_\xi(t) & \text{for } t \in [1,\infty). \end{cases} \tag{10.3.40}$$

We denote also here the Sobolev's conjugate function $(\mathfrak{f}_\xi)_n := \mathfrak{f}_\xi \circ H_n^{-1}$, with H_n given by $(10.1.16)_1$ with the choice $A \equiv \mathfrak{f}_\xi$. We infer as in (10.3.25)

$$|D_k| \leq \frac{1}{(f_\xi)_n(k)} \int_{D_k} (\mathfrak{f}_\xi)_n \left(\frac{|u-v|}{c_S(n)\mathcal{F}} \right) dx$$

Note that

$$\int_0 \left(\frac{s}{\mathfrak{f}_\xi(s)} \right)^{\frac{1}{n-1}} ds < \infty \quad \text{and} \quad \int^\infty \left(\frac{s}{\mathfrak{f}_\xi(s)} \right)^{\frac{1}{n-1}} ds = +\infty;$$

the first since f_ξ is linear near zero and the second by (10.3.36), since

$$\mathfrak{f}_\chi(s) \lesssim G(s) \frac{[g(s)]^{\xi-1}}{s} \lesssim_\chi G(s)\, s^{g_0(\xi-1)-1} \leq G(s)$$

when $\sigma \geq 1$, since $\chi < \frac{1+g_0}{g_0}$. At this point, using Young's inequality with conjugate functions H_ξ, $\widetilde{H_\xi}$ and with $\varepsilon \in (0,1)$ to be chosen, estimating exactly as in (10.3.26)

$$\sum_{k=1}^\infty H_\xi^{-1}\left(\frac{|D_k|}{\mathcal{F}} \right) \leq \frac{\varepsilon}{\mathcal{F}} \int_{B_1} f_\xi\big(|Du - Dv|\big)\, dx + \frac{c(\delta, g_0)\varepsilon}{\mathcal{F}} |B_1|$$
$$+ c(\delta, \varepsilon) \sum_{k=1}^\infty H_\xi^{-1}\left(\widetilde{H_\xi}\Big(\frac{1}{(f_\xi)_n(k)} \Big) \right).$$

At this point in order to estimate the summation on the right-hand side we deduce the following chain of up-to-constants equivalences: for $\alpha > 0$

$$[\widetilde{H_\xi}]^{-1}\left(\frac{1}{G(\alpha)} \right) \approx \frac{1}{G(\alpha)} \left[H_\xi^{-1}\left(\frac{1}{G(\alpha)} \right) \right]^{-1} = \big[f_\xi(\alpha)\big]^{-1};$$

for the first one we used (10.1.13) and for the second one just the definition of H_ξ^{-1}. At this point with $\alpha = H_n^{-1}(k)$, using again the definition of H_ξ^{-1}

$$H_\xi^{-1}\left(\widetilde{H_\xi}\Big(\frac{1}{(f_\xi)_n(k)} \Big) \right) \lesssim \frac{\big[g\big(H_n^{-1}(k)\big)\big]^{\xi-1}}{H_n^{-1}(k)}. \tag{10.3.41}$$

Also here we used that $H_n^{-1}(k) \geq 1$ for $k \in \mathbb{N}$ and $\mathfrak{f}_\xi(t) \equiv f_\xi(t)$ for $t \geq 1$. The convergence of the series is hence equivalent to fact

$$\int_1^\infty \frac{\big[g\big(H_n^{-1}(\sigma)\big)\big]^{\xi-1}}{H_n^{-1}(\sigma)}\, d\sigma < \infty$$

– again a long calculation shows that the function in (10.3.41) is decreasing. Again the change of variable $s = H_n^{-1}(\sigma)$ is allowed; estimating in a completely similar way as in (10.3.28) and subsequent line we have

$$\int_1^\infty \frac{\big[g\big(H_n^{-1}(\sigma)\big)\big]^{\xi-1}}{H_n^{-1}(\sigma)}\, d\sigma \lesssim_{n,\xi} \int_1^\infty [g(s)]^{\xi(1-\frac{1}{n})} \frac{ds}{sg(s)}.$$

and the integral is finite since the exponent of $g(s)$ is negative, i.e.

$$\xi\big(1 - \frac{1}{n}\big) - 1 < 0$$

since $\xi < \frac{n}{n-1}$. Coming then back to (10.3.26) and then to (10.3.22)

$$\int_{B_1} f_\xi\big(|Du - Dv|\big)\, dx \le \tilde{c}\, \mathcal{F} H_\xi^{-1}\left(\frac{|B_1|}{\mathcal{F}}\right) \\ + \varepsilon \tilde{c} \int_{B_1} f_\xi\big(|Du - Dv|\big)\, dx + c\,\varepsilon + c(n, \delta, g_0, \varepsilon, \chi)\mathcal{F}.$$

After reabsorbing the second and the fourth term first by an appropriate choice of ε and then by the use of Young's inequality, we estimate

$$\mathcal{F} H_\xi^{-1}\left(\frac{|B_1|}{\mathcal{F}}\right) \lesssim_{n,\xi} \mathcal{F}^\xi \Big[G^{-1}(\mathcal{F})\Big]^{-\xi}$$

by the definition of $H_\xi{}^{-1}$. Again using (10.1.8) and recalling that $\mathcal{F} \ge 1$ gives

$$\mathcal{F} H_\xi^{-1}\left(\frac{|B_1|}{\mathcal{F}}\right) \lesssim_{n,\xi} \mathcal{F}^{\xi(1 - \frac{1}{1+g_0})} = \mathcal{F}^{\xi \frac{g_0}{1+g_0}}$$

and since the exponent is strictly smaller than one by $\xi < \frac{1+g_0}{g_0}$, we use again Young's inequality to finally get (10.3.35).

The fast growth case. The case where

$$\int^\infty \left(\frac{s}{G(s)}\right)^{\frac{1}{n-1}} ds < \infty.$$

is again much simpler. Directly testing (10.3.13) with $\varphi = u - v \in W_0^{1,G}(B_1) \cap L^\infty(B_1)$ yields

$$\int_{B_1} G\big(|Du - Dv|\big)\, dx \le c(n, \delta, g_0)\, \|Du - Dv\|_{L^G(B_1)} \\ \le \frac{1}{2} \int_{B_1} G(|Du - Dv|)\, dx + c(n, \delta, g_0),$$

by the fact that $|\mu|(B_1) = 1$ and using Young's inequality (10.3.30). Finally, using Jensen's inequality as in (10.3.39) and the monotonicity of $t \mapsto \big(f_\chi \circ G^{-1}\big)(t)$

$$\fint_{B_1} f_\xi\big(|Du - Dv|\big)\, dx \le c\left(f_\xi \circ G^{-1}\right)\left(\fint_{B_1} G\big(|Du - Dv|\big)\, dx\right) \le c$$

and the proof is concluded. □

LEMMA 10.12. *Let u and v as above. Then there exists a constant $c \equiv c(n, \delta, g_0, \nu)$ such that*

$$\fint_{B_R} \frac{|V_g(Du) - V_g(Dv)|^2}{\big(\alpha + |u - v|\big)^\xi}\, dx \le \xi c\, \frac{\alpha^{1-\xi}}{\xi - 1} \frac{|\mu|(B_R)}{R^n} \tag{10.3.42}$$

holds whenever $\alpha > 0$ and $\xi > 1$.

PROOF. The proof is exactly the same given in [**106**] for the standard case, once we replace the monotonicity condition therein considered with (10.0.31). Note that our monotonicity condition (10.0.31) reads exactly as the one in [**106**], once we replace the standard V_s function with the one defined in (10.0.29). □

Now a couple of technical Lemmata. Their proof is only sketched, since they are very similar to those in [**106**], which are already almost trivial once having at hand the previous results. From now on, u and v will be the functions of Lemma 10.9 and B_R the ball therein appearing.

LEMMA 10.13. *Suppose that*

$$g^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right) \le \lambda \qquad \textit{and} \qquad \fint_{B_R} |Du|\,dx \le \lambda$$

hold; then for a constant c_2 depending on n, ν, L, δ, g_0 there holds

$$\sup_{B_{R/4}} |Dv| \le c_2\lambda. \tag{10.3.43}$$

PROOF. Use Lemma 10.6 and then Lemma 10.9 with $\chi = -1$ to estimate the left-hand side of (10.3.43):

$$\begin{aligned}\sup_{B_{R/4}} |Dv| &\le \fint_{B_R} |Du - Dv|\,dx + \fint_{B_R} |Du|\,dx \\ &\le c_1 g^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right) + \fint_{B_R} |Du|\,dx \le (c_1+1)\lambda.\end{aligned}$$

□

LEMMA 10.14. *Let $\tilde\eta, \vartheta \in (0,1]$, and suppose that*

$$g^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right) \le \frac{\tilde\eta^n}{c_1}\vartheta\lambda,$$

where c_1 is the constant appearing in Lemma 10.9 for $\chi = -1$. Then the lower bound

$$\fint_{\tilde\eta B_R} |Du|\,dx - \vartheta\lambda \le \fint_{\tilde\eta B_R} |Dv|\,dx \tag{10.3.44}$$

holds.

PROOF. Use triangle's inequality and Lemma 10.9 for $\chi = -1$. □

10.4. Proof of Theorem 4.25

Define the scaling parameter $\eta \in (0, \frac{1}{2})$ in the following way:

$$\eta := \left(\frac{1}{10\,2^{3\alpha+10}c_o c_2^2 c_h}\right)^{\frac{1}{\alpha}} \le \min\left\{\left(\frac{1}{2^4 c_h}\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2^{3\alpha+2}c_o\,(48c_2)^2}\right)^{\frac{1}{\alpha}}\right\}. \tag{10.4.1}$$

Here α is the exponent and c_o, c_h are the constants appearing in Lemma 10.6 and c_2 appears in Lemma 10.13. All these quantities are a priori defined, depending only on n, ν, L, δ, g_0 and therefore also η is a universal constant depending only on n, ν, L, δ, g_0.

For a fixed ball $B_R \equiv B_R(x)$ such that $B_{2R} \subset \Omega$ as in the statement of Theorem 4.25, build the sequence of shrinking balls $\{B_i\}_{i=0,1,\ldots}$ defined by

$$B_i := B_{R_i}(x) \qquad \text{where} \qquad R_i := \eta^i R, \tag{10.4.2}$$

and subsequently the sequence of functions v_i solutions to the homogeneous problem (10.3.1) in the ball $B_R \equiv B_i$:

$$\begin{cases} \operatorname{div} a(Dv_i) = 0 & \text{in } B_i, \\ v_i = u & \text{on } \partial B_i. \end{cases} \tag{10.4.3}$$

LEMMA 10.15. *Suppose that for a certain index $i \in \mathbb{N}$*

$$g^{-1}\left(\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right) + g^{-1}\left(\frac{|\mu|(B_i)}{r_i^{n-1}}\right) \le \lambda \tag{10.4.4}$$

hold for a number $\lambda > 0$ *and moreover*

$$\frac{\lambda}{H} \leq |Dv_i| \leq H\lambda \quad \textit{in } B_{i+1}, \qquad \frac{\lambda}{H} \leq |Dv_{i-1}| \leq H\lambda \quad \textit{in } B_i \tag{10.4.5}$$

for a constant $H \geq 1$*. Then there exists a constant* $c_H \equiv c_H(n, \nu, L, \delta, g_0, H)$ *such that*

$$\fint_{B_{i+1}} |Du - Dv_i|\, dx \leq c_H \frac{\lambda}{g(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]. \tag{10.4.6}$$

PROOF. Define the parameter $\chi > 0$ in the following way:

$$2\chi := \frac{1}{2} \min\left\{\frac{1}{g_0 - 1}, \frac{g_0}{(g_0 - 1)(n-1)}, \frac{1}{n-1}\right\} \tag{10.4.7}$$

and let $\xi := 1 + 2\chi$. Note that $\xi < 1^* = n/(n-1)$ and $\chi, \xi \equiv \chi, \xi(n, g_0)$. By (10.0.30) and by monotonicity (10.1.1) it follows that

$$\left[\frac{g(|Dv_i|)}{|Dv_i|}\right]^{1+\chi} |Du - Dv_i| \lesssim \left[\frac{g(|Dv_i|)}{|Dv_i|}\right]^{\frac{1+2\chi}{2}} \big|V_g(Du) - V_g(Dv_i)\big|.$$

Recalling the definition of h_χ, taking averages over B_i and using Schwarz-Hölder's inequality yields, for $\alpha > 0$:

$$\begin{aligned} &\fint_{B_i} h_\chi(|Dv_i|)|Du - Dv_i|\, dx \qquad (10.4.8)\\ &\lesssim \fint_{B_i} \left[\frac{\big|V_g(Du) - V_g(Dv_i)\big|^2}{(\alpha + |u - v_i|)^\xi}\right]^{\frac{1}{2}} \left[h_{2\chi}(|Dv_i|)(\alpha + |u - v_i|)^\xi\right]^{\frac{1}{2}} dx\\ &\lesssim \left[\fint_{B_i} \frac{\big|V_g(Du) - V_g(Dv_i)\big|^2}{(\alpha + |u - v_i|)^\xi}\, dx\right]^{\frac{1}{2}} \left[\fint_{B_i} h_{2\chi}(|Dv_i|)(\alpha + |u - v_i|)^\xi\, dx\right]^{\frac{1}{2}}. \end{aligned}$$

Now we use (10.3.42) to bound the first term of the right-hand side and we choose α such that

$$\fint_{B_i} h_{2\chi}(|Dv_i|)|u - v_i|^\xi\, dx = \alpha^\xi \fint_{B_i} h_{2\chi}(|Dv_i|)\, dx.$$

Note that this definition of α makes sense, see (10.4.12); moreover the integral on the left-hand side is finite, see the calculations after (10.4.13). With these actions (10.4.8) takes the form

$$\begin{aligned} \fint_{B_i} h_\chi(|Dv_i|)|Du - Dv_i|\, dx &\leq c \left[\alpha^{1-\xi} \frac{|\mu|(B_i)}{r_i^n}\right]^{\frac{1}{2}} \left[\alpha^\xi \fint_{B_i} h_{2\chi}(|Dv_i|)\, dx\right]^{\frac{1}{2}}\\ &= c \left[\frac{\alpha}{r_i} \frac{|\mu|(B_i)}{r_i^{n-1}} \fint_{B_i} h_{2\chi}(|Dv_i|)\, dx\right]^{\frac{1}{2}}. \qquad (10.4.9) \end{aligned}$$

with $c \equiv c(n, \nu, \delta, g_0)$. Note that since $t \mapsto g(t)/t$ is increasing and satisfies a doubling property, then Remark 10.2 applies; therefore for h_χ a sort of triangle's inequality, as (10.1.9), holds true, with a constant depending on n, g_0. Hence

$$\begin{aligned} \fint_{B_i} h_{2\chi}(|Dv_i|)\, dx &\lesssim_n \fint_{B_i} h_{2\chi}(|Du - Dv_{i-1}|)\, dx\\ &\quad + \fint_{B_i} h_{2\chi}(|Du - Dv_i|)\, dx + \fint_{B_i} h_{2\chi}(|Dv_{i-1}|)\, dx =: I_1 + I_2 + I_3. \end{aligned}$$

Before estimating term by term, we introduce for ease of notation the following quantities:

$$A_{i-1} := g^{-1}\left(\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right), \qquad A_i := g^{-1}\left(\frac{|\mu|(B_i)}{r_i^{n-1}}\right); \tag{10.4.10}$$

note that $A_i + A_{i-1} \le \lambda$ by (10.4.4) and therefore by monotonicity $h_\chi(A_i) \le h_\chi(\lambda)$ and analogously for A_{i-1}. Now by the pointwise estimate $(10.4.5)_2$ and the definition of $h_{2\chi}$ we have $I_3 \lesssim c(n, g_0, H)\big[g(\lambda)/\lambda\big]^\xi$. We estimate I_1 using Corollary 10.10, due to (10.4.7):

$$I_1 \le \eta^{-n} \fint_{B_{i-1}} h_{2\chi}\big(|Du - Dv_{i-1}|\big)\, dx \le c\, h_{2\chi}(A_{i-1}) \le c\, h_{2\chi}(\lambda),$$

with $c \equiv c(n, \nu, L, \delta, g_0, H)$. The estimate for I_2 is analogous and even more direct. Hence we have, using Young's inequality with $\varepsilon \in (0, 1)$ to be chosen and the definition of $h_{2\chi}(\lambda)$

$$\begin{aligned}\fint_{B_i} h_\chi\big(|Dv_i|\big)|Du - Dv_i|\, dx &\le c\left[\frac{\alpha}{r_i}\frac{|\mu|(B_i)}{r_i^{n-1}}\left[\frac{g(\lambda)}{\lambda}\right]^{1+2\chi}\right]^{\frac12} \\ &\le \varepsilon\frac{\alpha}{r_i}\left[\frac{g(\lambda)}{\lambda}\right]^{1+\chi} + c(\varepsilon)\frac{|\mu|(B_i)}{r_i^{n-1}}\left[\frac{g(\lambda)}{\lambda}\right]^{\chi}. \qquad (10.4.11)\end{aligned}$$

To conclude the proof, we need to estimate α. As a first step we bound from below

$$\fint_{B_i} h_{2\chi}\big(|Dv_i|\big)\, dx \ge \eta^n \fint_{B_{i+1}} h_{2\chi}\big(|Dv_i|\big)\, dx \ge c\left[\frac{g(\lambda)}{\lambda}\right]^{1+2\chi} \qquad (10.4.12)$$

and therefore

$$\alpha^\xi \le c\left[\frac{\lambda}{g(\lambda)}\right]^\xi \fint_{B_i} h_{2\chi}\big(|Dv_i|\big)|u - v_i|^\xi\, dx, \qquad (10.4.13)$$

with $c \equiv c(n, \nu, L, \delta_0, H)$. We split the latter averaged integral in the following way

$$\begin{aligned}\fint_{B_i} h_{2\chi}\big(|Dv_i|\big)|u - v_i|^\xi\, dx &\le c\fint_{B_i} h_{2\chi}\big(|Dv_i - Dv_{i-1}|\big)|u - v_i|^\xi\, dx \\ &\quad + c\fint_{B_i} h_{2\chi}\big(|Dv_{i-1}|\big)|u - v_i|^\xi\, dx =: c\,(II_1) + c\,(II_2).\end{aligned}$$

We begin with the easier (II_2): since we have the pointwise estimate $h_{2\chi}(|Dv_{i-1}|) \approx_{n,H} h_{2\chi}(\lambda) \approx_{n,H} \big[g(\lambda)/\lambda\big]^{1+2\chi}$ on B_i, using standard Sobolev's embedding by (10.4.7), "triangle's inequality" for h_χ and recalling that $\xi = 1 + 2\chi$, we infer

$$\begin{aligned}\frac{(II_2)^{\frac1\xi}}{r_i} &\le c\frac{g(\lambda)}{\lambda}\fint_{B_i} |Du - Dv_i|\, dx = c\left[\frac{\lambda}{g(\lambda)}\right]^\chi h_\chi(\lambda)\fint_{B_i} |Du - Dv_i|\, dx \\ &\le c\left[\frac{\lambda}{g(\lambda)}\right]^\chi \fint_{B_i} h_\chi\big(|Dv_{i-1}|\big)|Du - Dv_i|\, dx \\ &\le c\left[\frac{\lambda}{g(\lambda)}\right]^\chi \fint_{B_i} h_\chi\big(|Dv_i|\big)|Du - Dv_i|\, dx + c\left[\frac{\lambda}{g(\lambda)}\right]^\chi (III)\end{aligned} \qquad (10.4.14)$$

see next estimate for the definition of (III), with $c \equiv c(n, \delta, g_0, H)$. Moreover, again "triangle's inequality" for h_χ gives

$$\begin{aligned}(III) &:= \fint_{B_i} h_\chi\big(|Dv_i - Dv_{i-1}|\big)|Du - Dv_i|\, dx \\ &\lesssim_n \fint_{B_i} g_\chi\big(|Du - Dv_i|\big)\, dx + \fint_{B_i} h_\chi\big(|Du - Dv_{i-1}|\big)|Du - Dv_i|\, dx\end{aligned} \qquad (10.4.15)$$

While the first integral is less or equal than $c_1\, g_\chi(A_{i-1})$, with c_1 depending on $data$, by (10.3.7), for the second one we need the pointwise estimate

$$\widetilde{g_\chi}\big(h_\chi(t)\big) = \widetilde{g_\chi}\bigg(\frac{g_\chi(t)}{t}\bigg) \le g_\chi(t)$$

see (10.1.12). Therefore Young's inequality with conjugate functions g_χ and $\widetilde{g_\chi}$ gives

$$\begin{aligned}\fint_{B_i} h_\chi\big(|Du - Dv_{i-1}|\big)|Du - Dv_i|\,dx &\le c\fint_{B_i} g_\chi\big(|Du - Dv_i|\big)\,dx\\ &\quad + c\fint_{B_i} g_\chi\big(|Du - Dv_{i-1}|\big)\,dx \le c\, g_\chi(A_{i-1}),\end{aligned}$$

as for the first term in the second line of (10.4.15). Note that here we used $g_\chi(A_i) + g_\chi(A_{i-1}) \le c(n,\nu,L,\delta,g_0)\, g_\chi(A_{i-1})$, following from (10.4.2) and the monotonicities of both the measure $|\mu|$ and g_χ. We here have the following algebraic manipulation:

$$\left[\frac{\lambda}{g(\lambda)}\right]^\chi g_\chi\big(A_{i-1}\big) = \left[\frac{\lambda}{g(\lambda)}\right]^\chi \left[\frac{g\big(A_{i-1}\big)}{A_{i-1}}\right]^\chi g\big(A_{i-1}\big) \le \frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}, \quad (10.4.16)$$

since $t \mapsto g(t)/t$ is monotone and $A_{i-1} \le \lambda$ by (10.4.4). The reader here may need to recall also the definition of A_{i-1} in (10.4.10). Therefore, taking into account (10.4.16), we have proved

$$\left[\frac{\lambda}{g(\lambda)}\right]^\chi (III) \le c\left[\frac{\lambda}{g(\lambda)}\right]^\chi g_\chi\big(A_{i-1}\big) \le c\,\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}.$$

Hence, putting the last estimate into (10.4.14), all in all we have

$$(II_2) \le c\, r_i^\xi\left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^\xi + c\, r_i^\xi\left[\frac{\lambda}{g(\lambda)}\right]^{\chi\xi}\left[\fint_{B_i} h_\chi\big(|Dv_i|\big)|Du - Dv_i|\,dx\right]^\xi. \quad (10.4.17)$$

Now we come to the estimate of (II_1): we use Young's inequality with $k(t) := [2g(t^{\frac{1}{\xi}})]^\xi$ and $\tilde{k}(t)$ and then we estimate the first term with Hölder's inequality and Proposition 10.5:

$$\begin{aligned}(II_1) &= \fint_{B_i}\left[\frac{g\big(|Dv_i - Dv_{i-1}|\big)}{|Dv_i - Dv_{i-1}|}\right]^\xi |u - v_i|^\xi\,dx \quad (10.4.18)\\ &\le (2r_i)^\xi\fint_{B_i}\left[g\bigg(\frac{|u - v_i|}{r_i}\bigg)\right]^\xi dx + r_i^\xi\fint_{B_i}\tilde{k}\left(\left[\frac{g\big(|Dv_i - Dv_{i-1}|\big)}{|Dv_i - Dv_{i-1}|}\right]^\xi\right)dx\\ &\le c\, r_i^\xi\left[\fint_{B_i} g\big(|Du - Dv_i|\big)\,dx\right]^\xi + r_i^\xi\fint_{B_i}\tilde{k}\left(\left[\frac{g\big(|Dv_i - Dv_{i-1}|\big)}{|Dv_i - Dv_{i-1}|}\right]^\xi\right)dx.\end{aligned}$$

While for the first term we then have

$$r_i^\xi\left[\fint_{B_i} g\big(|Du - Dv_i|\big)\,dx\right]^\xi \le c\, r_i^\xi\big[g\big(A_i\big)\big]^\xi \le c\, r_i^\xi\left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^\xi,$$

from Lemma 10.9 or 10.11, for the second one we need, for $\alpha \ge 0$, the following estimate:

$$\begin{aligned}\tilde{k}(\alpha^\xi) &= \sup_{s>0}\Big\{\alpha^\xi s - \big[2g(s^{\frac{1}{\xi}})\big]^\xi\Big\} = \sup_{\sigma>0}\Big\{\alpha^\xi\sigma^\xi - 2^\xi\big[g(\sigma)\big]^\xi\Big\}\\ &\le 2^\xi\left[\sup_{\sigma>0}\Big\{\alpha\sigma - g(\sigma)\Big\}\right]^\xi = c(n,g_0)\,\big[\tilde{g}(\alpha)\big]^\xi\end{aligned}$$

since $\xi \geq 1$. Therefore with $\alpha = g\big(|Dv_i - Dv_{i-1}|\big)/|Dv_i - Dv_{i-1}|$, using (10.1.12) and Lemma 10.11

$$\begin{aligned}
&\fint_{B_i} \tilde{k}\left(\left[\frac{g\big(|Dv_i - Dv_{i-1}|\big)}{|Dv_i - Dv_{i-1}|}\right]^{\xi}\right) dx \leq \fint_{B_i} \left[\tilde{g}\left(\frac{g\big(|Dv_i - Dv_{i-1}|\big)}{|Dv_i - Dv_{i-1}|}\right)\right]^{\xi} dx \\
&\leq \fint_{B_i} \big[g\big(|Dv_i - Dv_{i-1}|\big)\big]^{\xi} dx \\
&\lesssim_n \fint_{B_i} \big[g\big(|Du - Dv_i|\big)\big]^{\xi} dx + \eta^{-n} \fint_{B_{i-1}} \big[g\big(|Du - Dv_{i-1}|\big)\big]^{\xi} dx \\
&\leq c\left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^{\xi}.
\end{aligned}$$

Plugging these two estimates into (10.4.18) and also taking into account (10.4.17) yields

$$\begin{aligned}
(II_1) + (II_2) \leq c\, r_i^{\xi}\left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^{\xi} \\
+ c\, r_i^{\xi}\left[\frac{\lambda}{g(\lambda)}\right]^{\chi\xi}\left[\fint_{B_i} h_{\chi}\big(|Dv_i|\big)|Du - Dv_i|\, dx\right]^{\xi};
\end{aligned}$$

therefore finally we estimate α as follows: from (10.4.13)

$$\begin{aligned}
\frac{\alpha}{r_i} &\leq \frac{c}{r_i}\frac{\lambda}{g(\lambda)}\left[\fint_{B_i} h_{2\chi}\big(|Dv_i|\big)|u - v_i|^{\xi}\, dx\right]^{\frac{1}{\xi}} \\
&\leq c\,\frac{\lambda}{g(\lambda)}\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}} + \tilde{c}\left[\frac{\lambda}{g(\lambda)}\right]^{1+\chi}\fint_{B_i} h_{\chi}\big(|Dv_i|\big)|Du - Dv_i|\, dx.
\end{aligned}$$

Inserting this estimate into (10.4.11) and choosing $\varepsilon \equiv \varepsilon(n,\nu,L,\delta,g_0,H) \in (0,1)$ small enough - i.e. $\varepsilon = 1/(2\tilde{c})$ leads to

$$\frac{1}{2}\fint_{B_i} h_{\chi}\big(|Dv_i|\big)|Du - Dv_i|\, dx \leq c\,\frac{|\mu|(B_i)}{r_i^{n-1}}\left[\frac{g(\lambda)}{\lambda}\right]^{\chi}.$$

Now (10.4.6) plainly follows taking into account that

$$\fint_{B_i} h_{\chi}\big(|Dv_i|\big)|Du - Dv_i|\, dx \gtrsim_{n,H} \eta^n\left[\frac{g(\lambda)}{\lambda}\right]^{1+\chi}\fint_{B_{i+1}} |Du - Dv_i|\, dx.$$

□

Let $x \in \Omega$ be a Lebesgue's point of Du and let $B_{2R}(x) \subset \Omega$. Define the quantity

$$\lambda := g^{-1}\left(H_1\, g\left(\fint_{B_R} |Du|\, dx\right) + H_2\, \mathbf{I}_1^{|\mu|}(x, 2R)\right), \tag{10.4.19}$$

where the constants H_1, H_2 will be fixed in a few lines, in a way making them depending only on n,ν,L,δ,g_0. We want to prove that

$$|Du(x)| \leq \lambda, \tag{10.4.20}$$

and (4.7.6) will follow simply taking $c := \max\{H_1, H_2\}$. Without loss of generality we can clearly assume $\lambda > 0$, whether this were not the case (10.4.20) would trivially follow by the monotonicity of the vector field.

Step 1: the choice of the constants. With $i \in \mathbb{N}+1$ define the quantity

$$C_i := \sum_{j=i-2}^{i} \fint_{B_j} |Du|\, dx + \eta^{-n} E(Du, B_i) \leq 5\eta^{-3n} \fint_{B_{i-2}} |Du|\, dx. \quad (10.4.21)$$

Note that the inclusions $B_{i+1} = \eta B_i \subset \frac{1}{4} B_i \subset B_i$ hold. Take $k \in \mathbb{N}$, $k \geq 3$ as the smallest integer such that

$$(8\eta^k)^\alpha \leq \eta^n \frac{1}{128 c_o c_2}; \quad (10.4.22)$$

here c_o is the constant of Lemma 10.6 and c_2 is the one appearing in Lemma 10.13. Once fixed $k \equiv k(n, \nu, L, \delta, g_0)$ in such way, fix the constant H_1 and H_2 as follows

$$H_1 = \left(10\eta^{-4n}\right)^{g_0}, \qquad H_2 := 2^{7g_0} c_1^{g_0} \eta^{-ng_0(k+1+\frac{1}{\delta})} c_{200c_2}. \quad (10.4.23)$$

here c_1 is the constant appearing in Lemma 10.9 for $\chi = -1$ and c_{200c_2} is the constant c_H appearing in Lemma 10.15 for the choice $H = 200c_2$. Note that the dependences of k and η yield that both H_1 and H_2 are *a priori* constants depending only on n, ν, L, δ, g_0; moreover with this choice there holds – recall that $k \geq 3$

$$\eta^{-\frac{n}{\delta}} H_2^{-\frac{1}{g_0}} \leq \frac{1}{2}, \qquad \eta^{-\frac{n}{\delta}} H_2^{-\frac{1}{g_0}} \leq \frac{\eta^{nk}}{96 c_1}, \qquad c_1 \eta^{-n(k+\frac{1}{\delta})} H_2^{-\frac{1}{g_0}} \leq \frac{\eta^n}{128},$$
$$4c_{200c_2} \frac{\eta^{-2n}}{H_2} \leq \frac{1}{4}, \qquad 8c_{200c_2} \frac{\eta^{-3n}}{H_2} \leq \frac{1}{4}, \quad (10.4.24)$$

which we shall use in this order. Moreover the choice of H_1 implies

$$C_2 + C_3 \leq 10\eta^{-4n} \fint_{B_0} |Du|\, dx \leq 10\eta^{-4n} H_1^{-\frac{1}{g_0}} \lambda \leq \frac{\lambda}{8} \leq \lambda. \quad (10.4.25)$$

We recall the dyadic decomposition

$$\eta^n \sum_{j=0}^{\infty} \frac{|\mu|(B_j)}{R_j^{n-1}} \leq \int_0^{2R} \frac{|\mu|(B_\rho(x))}{\rho^{n-1}} \frac{d\rho}{\rho} = \mathbf{I}_1^{|\mu|}(x, 2R),$$

see [69, 106]. Hence for every $i \in \mathbb{N}_0$ using (10.4.24) we have

$$g^{-1}\left(\frac{|\mu|(B_i)}{R_i^{n-1}}\right) \leq g^{-1}\left(\sum_{j=0}^{\infty} \frac{|\mu|(B_j)}{R_j^{n-1}}\right) \leq g^{-1}\left(\eta^{-n} \mathbf{I}_1^{|\mu|}(x, 2R)\right)$$
$$\leq \eta^{-\frac{n}{\delta}} H_2^{-\frac{1}{g_0}} \lambda \leq \frac{\lambda}{2}. \quad (10.4.26)$$

Step 2: the exit time and after the exit time. Now we state that we can suppose that there exists an "exit time" index $i_e \geq 3$, see (10.4.25), such that

$$C_{i_e} \leq \frac{\lambda}{8} \qquad \text{but} \qquad C_j > \frac{\lambda}{8} \qquad \text{for every } j > i_e. \quad (10.4.27)$$

Indeed, on the contrary, we would have $C_{i_h} \leq \lambda/8$ for an increasing subsequence $\{i_h\}$ and then, being x a Lebesgue point for Du,

$$|Du(x)| \leq \lim_{h\to\infty} \fint_{B_{i_h}} |Du|\, dx \leq \frac{\lambda}{8}$$

and the proof would be finished. Now an important Lemma which asserts that after the exit time the gradient Dv_i is far away from zero; this finally gives meaning to the assumption (10.4.5) we imposed on the Dv_is.

LEMMA 10.16. *Suppose that*

$$\fint_{B_i} |Du|\, dx \le \lambda \tag{10.4.28}$$

holds for a certain index $i \in \mathbb{N}$, $i \ge i_e - 2$, *for* $\lambda > 0$ *defined in* (10.4.19). *Then*

$$\frac{\lambda}{200c_2} \le |Dv_i| \le c_2\lambda \qquad \text{in } B_{i+1}, \tag{10.4.29}$$

where c_2 *is the constant appearing in Lemma 10.13.*

PROOF. The right-hand side estimate in (10.4.29) is a consequence of Lemma 10.13 applied with $B_R \equiv B_i$, which gives

$$\sup_{B_i/4} |Dv_i| \le c_2\lambda. \tag{10.4.30}$$

Note that the assumptions of the Lemma are satisfied since (10.4.28) and (10.4.26) hold, and moreover $B_{i+1} \subset B_i/4$. In order to prove the left-hand side inequality we want to use Lemma 10.8 with $\sigma \equiv \eta$, $B \equiv B_i$ so that $\sigma B = B_{i+1}$; since we already have (10.4.30) we just need to prove

$$\frac{\lambda}{C} \le \fint_{B_{i+m}} |Dv_i|\, dx$$

for some $m \in \mathbb{N}$ and some $C \ge 1$. We start by proving that

$$\frac{\lambda}{16} \le \sum_{j=i-2+k}^{i+k} \fint_{B_j} |Dv_i|\, dx, \tag{10.4.31}$$

where $k \ge 3$ is the number defined in (10.4.22). By (10.4.26) we can apply Lemma 10.14 three times, with $B_R \equiv B_i$, respectively $\tilde{\eta} \equiv \eta^k, \eta^{k-1}, \eta^{k-2}$ and $\vartheta \equiv 1/96$. Note indeed that from (10.4.26) and condition (10.4.24) follows

$$g^{-1}\left(\frac{|\mu|(B_i)}{R_i^{n-1}}\right) \le \eta^{-\frac{n}{\delta}} H_2^{-\frac{1}{90}} \lambda \le \frac{\eta^{nk}}{c_1}\frac{1}{96}\lambda \le \frac{\eta^{nj}}{c_1}\vartheta\lambda$$

for $j = k-2, k-1, k$. Summing up the resulting inequalities gives

$$\begin{aligned} C_{i+k} - \eta^{-n}E(Du, B_{i+k}) - 3\lambda\vartheta &= \sum_{j=i-2+k}^{i+k} \fint_{B_j} |Du|\, dx - 3\lambda\vartheta \\ &\le \sum_{j=i-2+k}^{i+k} \fint_{B_j} |Dv_j|\, dx. \end{aligned}$$

Since $i \ge i_e - 2$ and $k \ge 3$, we have $i + k > i_e$ and subsequently by the definition of the exit time index $C_{i+k} \ge \lambda/8$. Using this fact and the value of ϑ in the inequality above gives

$$\frac{\lambda}{8} - \eta^{-n}E(Du, B_{i+k}) - \frac{\lambda}{32} \le \sum_{j=i-2+k}^{i+k} \fint_{B_j} |Dv_j|\, dx. \tag{10.4.32}$$

In order to estimate the excess term first we note that enlarging the domain of integration from B_{i+k} to B_i and using (10.4.26) gives

$$\begin{aligned} \fint_{B_{i+k}} |Du - Dv_i|\, dx &\le \frac{|B_i|}{|B_{i+k}|} c_1 g^{-1}\left(\frac{|\mu|(B_i)}{R_i^{n-1}}\right) \\ &\le c_1 \eta^{-kn-\frac{n}{\delta}} H_2^{-\frac{1}{90}} \lambda \le \eta^n \frac{\lambda}{128} \end{aligned} \tag{10.4.33}$$

where we used (10.3.7) with $\chi = -1$. Lemma 10.6 applied with $B_R \equiv B_i/4$, $B_{r/2} \equiv B_{i+k}$ using the just proved right-hand side inequality of (10.4.29) and the definition of k gives

$$2 \operatorname*{osc}_{B_{i+k}} Dv_i \leq 2c_o(8\eta^k)^\alpha c_2\, \lambda \leq \eta^n \frac{\lambda}{64}$$

using (10.4.22), so that, with the help of (3.1.3) and then of (10.4.33) we infer

$$\begin{aligned} E(Du, B_{i+k}) &\leq 2 \fint_{B_{i+k}} \left|Du - (Dv_i)_{B_{i+k}}\right| dx \\ &\leq 2 \fint_{B_{i+k}} \left|Dv_i - (Dv_i)_{B_{i+k}}\right| dx + 2 \fint_{B_{i+k}} |Du - Dv_i|\, dx \\ &\leq 2 \operatorname*{osc}_{B_{i+k}} Dv_i + 2\eta^n \frac{\lambda}{128} \leq \eta^n \frac{\lambda}{32}. \end{aligned} \tag{10.4.34}$$

Inserting this last estimate into (10.4.32) gives (10.4.31). (10.4.31) in turn implies that there exists an index $m \in \{k-2, k-1, k\}$ such that

$$\fint_{B_m} |Dv_i|\, dx \geq \frac{1}{3}\frac{\lambda}{16} \geq \frac{\lambda}{48c_2}.$$

Hence now we can apply Lemma 10.8 with the choices listed just above and with $C \equiv 48c_2$. Note that the condition on σ (10.2.19) holds true by (10.4.1); then the choice $\sigma = \eta$ is allowed. Therefore we can conclude with

$$\frac{\lambda}{200c_2} \leq |Dv_i| \qquad \text{in } B_{i+1}$$

which coupled with (10.4.30) gives (10.4.29). □

Step 3: Iteration.

LEMMA 10.17. *Suppose that for some $i \in \mathbb{N}$, $i \geq i_e - 1$ there holds*

$$\fint_{B_{i-1}} |Du|\, dx \leq \lambda \qquad \textit{and} \qquad \fint_{B_i} |Du|\, dx \leq \lambda. \tag{10.4.35}$$

Then there exists a constant c_3 depending on n, ν, L, δ, g_0 such that

$$E(Du, B_{i+2}) \leq \frac{1}{4} E(Du, B_{i+1}) + c_3 \frac{\lambda}{g(\lambda)} \left[\frac{|\mu|(B_{i-1})}{R_{i-1}^{n-1}}\right]$$

holds true. c_3 has the expression $4\eta^{-n}c_{200c_2}$, where c_{200c_2} is the constant of Lemma 10.15 for $H = 200c_2$.

PROOF. We clearly want to apply Lemma 10.15. Assumption (10.4.4) is satisfied as a consequence of (10.4.26), while for (10.4.5) we appeal to Lemma 10.16: since $i \geq i_e - 1$, obviously $i - 1 \geq i_e - 2$ and then we can use estimate (10.4.29) both for Dv_i in B_{i+1} and Dv_{i-1} in B_i, i.e.

$$\frac{\lambda}{200c_2} \leq |Dv_{i-1}| \leq c_2\lambda \qquad \text{in } B_i, \qquad \frac{\lambda}{200c_2} \leq |Dv_i| \leq c_2\lambda \qquad \text{in } B_{i+1}.$$

Hence assumptions (10.4.5) are satisfied with $H \equiv 200c_2$, so we have

$$\fint_{B_{i+1}} |Du - Dv_i|\, dx \leq c_{200c_2} \frac{\lambda}{g(\lambda)} \left[\frac{|\mu|(B_{i-1})}{R_{i-1}^{n-1}}\right], \tag{10.4.36}$$

where c_{200c_2} is a constant depending on n, ν, L, δ, g_0. Estimate (10.2.3) applied with $B_r, B_R \equiv B_{i+2}, B_{i+1}$ gives using (10.4.1)

$$E(Dv_i, B_{i+2}) \leq 2^{-4} E(Dv_i, B_{i+1})$$

so we get the thesis performing a computation similar to (10.4.34):

$$\begin{aligned} E(Du, B_{i+2}) &\leq 2E(Dv_i, B_{i+2}) + 2 \fint_{B_{i+2}} |Du - Dv_i| \, dx \\ &\leq 2^{-3} E(Dv_i, B_{i+1}) + 2\eta^{-n} \fint_{B_{i+i}} |Du - Dv_i| \, dx \\ &\leq 2^{-2} E(Du, B_{i+1}) + 2(\eta^{-n} + 1) \fint_{B_{i+i}} |Du - Dv_i| \, dx \\ &\leq 2^{-2} E(Du, B_{i+1}) + c_3 \frac{\lambda}{g(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}} \right], \end{aligned} \tag{10.4.37}$$

where we used (10.4.36). The proof is concluded. □

Now we proceed with the proof of Theorem 4.25 and we define

$$A_i := E(Du, B_i) \qquad \text{and} \qquad m_i := \left|(Du)_{B_i}\right|.$$

Recalling the definition in (10.4.21) and (10.4.27), we have

$$\sum_{j=i_e-2}^{i_e} m_j + \eta^{-n} A_{i_e} \leq C_{i_e} \leq \frac{\lambda}{8}. \tag{10.4.38}$$

Our goal is to prove by induction that

$$m_j + A_j \leq \lambda \qquad \text{for all } j \geq i_e. \tag{10.4.39}$$

The case $j = i_e$ holds true from the definition of C_{i_e} and the exit time (10.4.27):

$$m_{i_e} + A_{i_e} \leq 3 \fint_{B_{i_e}} |Du| \, dx \leq 3\frac{\lambda}{8}.$$

Assume now that (10.4.39) holds true for $j = i_e, \ldots, i$. By (10.4.38) for $j = i_e, \ldots, i$ and directly from the definition of C_{i_e} and from (10.4.27) for $j = i_e - 2, i_e - 1$ – indeed for these two exponents $A_j \leq 2 \fint_{B_j} |Du| \, dx$ – we have in particular that

$$\fint_{B_j} |Du| \, dx \leq \lambda \qquad \text{for } j = i_e - 2, \ldots, i.$$

Since assumption (10.4.35) is satisfied, we can apply the excess decay Lemma 10.17 that gives

$$A_{j+2} \leq \frac{1}{4} A_{j+1} + c_3 \frac{\lambda}{g(\lambda)} \left[\frac{|\mu|(B_{j-1})}{R_{j-1}^{n-1}} \right] \qquad \text{for } j = i_e - 1, \ldots, i. \tag{10.4.40}$$

When $j = i - 1$ the previous inequality in particular gives

$$A_{i+1} \leq \frac{1}{4} A_i + c_3 \frac{\lambda}{g(\lambda)} \left[\frac{|\mu|(B_{i-2})}{R_{i-2}^{n-1}} \right] \leq \frac{\lambda}{4} + \frac{1}{4} \frac{\lambda}{g(\lambda)} g(\lambda) \leq \frac{\lambda}{2}. \tag{10.4.41}$$

since by inductive hypothesis $A_i \leq \lambda$ and since the following inequality holds true for all i (recall $i \geq i_e \geq 3$):

$$c_3 \frac{|\mu|(B_{i-2})}{R_{i-2}^{n-1}} \leq c_3 \eta^{-n} \mathbf{I}_1^{|\mu|}(x, 2R) \leq c_3 \frac{\eta^{-n}}{H_2} g(\lambda) \leq \frac{1}{4} g(\lambda),$$

see (10.4.24) and recall that $c_3 = 4\eta^{-n} c_{200c_2}$. Moreover, summing (10.4.40) for $i \in \{i_e - 1, i - 2\}$ and performing some algebraic manipulations leads to

$$\sum_{j=i_e}^{i} A_j \leq A_{i_e} + \frac{1}{4} \sum_{j=i_e}^{i-1} A_j + c_3 \frac{\lambda}{g(\lambda)} \sum_{j=0}^{\infty} \frac{|\mu|(B_j)}{R_j^{n-1}}$$

which gives, after reabsorption,

$$\sum_{j=i_e}^{i} A_j \leq 2A_{i_e} + 2c_3 \frac{\lambda}{g(\lambda)} \sum_{j=0}^{\infty} \frac{|\mu|(B_j)}{R_j^{n-1}}. \tag{10.4.42}$$

On the other hand,

$$\begin{aligned} m_{i+1} - m_{i_e} = \sum_{j=i_e}^{i} (m_{j+1} - m_j) &\leq \sum_{j=i_e}^{i} \fint_{B_{j+1}} \left|Du - (Du)_{B_j}\right| dx \\ &\leq \sum_{j=i_e}^{i} \frac{|B_j|}{|B_{j+1}|} E(Du, B_j) \end{aligned}$$

and therefore, using (10.4.42), (10.4.38) and (10.4.24),

$$\begin{aligned} m_{i+1} &\leq m_{i_e} + \eta^{-n} \sum_{j=i_e}^{i} A_j \\ &\leq m_{i_e} + 2\eta^{-n} A_{i_e} + 2\eta^{-n} c_3 \frac{\lambda}{g(\lambda)} \sum_{j=0}^{\infty} \frac{|\mu|(B_j)}{r_j^{n-1}} \leq 2\frac{\lambda}{8} + \frac{\lambda}{4} \leq \frac{\lambda}{2}. \end{aligned} \tag{10.4.43}$$

Merging together (10.4.41) and (10.4.43) gives $m_{i+1} + A_{i+1} \leq \lambda$. Being $x \in \Omega$ a Lebesgue point of Du then we have

$$|Du(x)| = \lim_{i \to \infty} m_i \leq \lambda.$$

10.5. Very weak solutions and sketches of other proofs

First we quickly show how to extend the potential estimate Theorem 4.27 to the so-called *very weak solutions*. Being a the vector field considered in (4.7.3), for Dirichlet problems of the type

$$\begin{cases} -\operatorname{div} a(Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10.5.1}$$

we give the following definition:

DEFINITION 10.18. *A very weak solution to* (10.5.1) *is a function* $u \in W_0^{1,g}(\Omega)$ *such that*

$$\int_\Omega \langle a(Du), D\varphi \rangle \, dx = \int_\Omega \varphi \, d\mu \tag{10.5.2}$$

for every $\varphi \in C_c^\infty(\Omega)$.

Note that the requirement $u \in W^{1,g}(\Omega)$ is the minimal assumption which gives meaning to the distributional solution (10.5.2) when the vector field a satisfies assumptions (4.7.4). Existence (but not uniqueness) of very weak solutions can be deduced by an adaption of the method developed in [**26, 25**]. We briefly recall it. Consider a sequence of approximating regular functions $f_k \in L^\infty(\Omega)$ weakly-$*$ converging to μ and such that

$$|f_k|(B_{R+1/k}) = \int_{B_{R+1/k}} |f_k| \, dx \leq |\mu|(B_R). \tag{10.5.3}$$

Solve the weak formulation (10.5.2) with datum f_k, i.e with right-hand side $\int_\Omega f_k \varphi \, dx$, and by monotonicity methods get regular solutions $u_k \in W^{1,G}(\Omega)$. By the compactness and truncation arguments in [**26, 25**] we get pointwise convergence of both u_k and Du_k to a limit function belonging at least to $W^{1,g}(\Omega)$. Therefore also using $f_k \rightharpoonup \mu$ we get that the limit function solves the distributional formulation (10.5.2). Note that due to (4.7.2), $L^g(\Omega)$ turns out to be reflexive.

After this introduction we can state the variation of Theorem 4.27:

THEOREM 10.19. *Let $u \in W^{1,g}(\Omega)$ be a SOLA of the Dirichlet problem (10.5.1), with μ and a as in Theorem 4.27. Then there exists a constant c, depending on n, ν, L, δ, g_0, such that the pointwise estimate*

$$g(|Du(x)|) \leq c\,\mathbf{I}_1^{|\mu|}(x, 2R) + c\,g\left(\fint_{B_R(x)} |Du|\,d\xi\right)$$

holds for every $x \in \Omega$ and for every ball $B_{2R}(x) \subset \Omega$.

The proof of the previous Theorem is very simple: consider the approximating problems just described, apply Theorem 4.27 to the approximating functions u_k and then pass to the limit for almost every point using also (10.5.3).

Now we come to the proof of Proposition 4.29. More precisely, we will just sketch the proof of the following Proposition, since once having the potential estimate (4.7.6), the Lemmata proved in the previous sections and the Proposition 10.20 at hand, using simple tricks extensively used in the preceding pages, as, for example, triangle's inequality of Remark 10.2, this proof is just an adaption of the one in [**106**]. Actually it would be enough to prove the Proposition using assumption (1), since (1) is obviously implied by (3) and also by (2), see [**68**, Lemmata 1 & 3].

PROPOSITION 10.20. *Let $u \in W^{1,G}(\Omega)$ be as in Corollary 4.28. If $x \mapsto |\mu|(x, R)$ is locally bounded in Ω for some $R > 0$ and if*

$$\lim_{R\to 0} \frac{|\mu|\big(B_R(x)\big)}{R^{n-1}} = 0 \quad \text{locally uniformly in } \Omega \text{ w.r.t. } x, \tag{10.5.4}$$

then Du is locally VMO-regular in Ω.

Again we recall that the stated regularity of Du means that for every $\Omega' \Subset \Omega$

$$\lim_{R\to 0} \omega(R) = 0 \quad \text{where} \quad \omega(R) \equiv \omega_{\Omega'}(R) := \sup_{\substack{B_r \Subset \Omega' \\ r \leq R}} \fint_{B_r} \big|Du - (Du)_{B_r}\big|\,dx.$$

PROOF. Consider an intermediate open set Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$. Since by Corollary (4.26) and by our assumptions Du is locally bounded, it makes sense to prove that for every $\varepsilon \in (0,1)$, there exists a positive radius $r_\varepsilon < \text{dist}(\Omega', \partial\Omega'')$, depending on $n, \nu, L, \delta, g_0, \mu(\cdot), \|Du\|_{L^\infty(\Omega'')}, \text{dist}(\Omega', \partial\Omega), \varepsilon$, such that

$$\fint_{B_\rho(y)} \big|Du - (Du)_{B_\rho(y)}\big|\,dx \leq \varepsilon\lambda, \qquad \lambda := \|Du\|_{L^\infty(\Omega'')} \tag{10.5.5}$$

holds whenever $\rho \in (0, r_\varepsilon)$ and $y \in \Omega'$. This would give the local VMO regularity of Du. For ε given as in the statement and fixed, consider the quantity

$$\eta := \left(\frac{\varepsilon^2}{2^{3\alpha+10} c_o c_2^2 c_h}\right)^{\frac{1}{\alpha}} \leq \frac{1}{(2^3 c_h)^{1/\alpha}}$$

where the involved quantities are the ones appearing also in (10.4.1). This gives that η is a constant depending only on $n, \nu, L, \delta, g_0, \varepsilon$. Then take the constant $c_{16c_2/\varepsilon}$ as the constant c_H appearing in Lemma 10.15 with the choice $H = 16c_2/\varepsilon$. Then it also depends upon $n, \nu, L, \delta, g_0, \varepsilon$. Finally chose a radius $R_0 < \text{dist}(\Omega'', \partial\Omega')$ depending on

$n, \nu, L, \delta, g_0, \mu(\cdot), \|Du\|_{L^\infty(\Omega'')}, \mathrm{dist}(\Omega', \partial\Omega), \varepsilon$ such that

$$\sup_{0<\rho\le R_0}\sup_{x\in\Omega'}\frac{|\mu|\big(B_\rho(x)\big)}{\rho^{n-1}} \le g\left(\varepsilon\frac{\eta^{2n}}{4c_1}\left(\frac{\eta^n}{16c_1c_{16c_2/\varepsilon}}\right)^{1/\delta}\lambda\right); \tag{10.5.6}$$

this is allowed by (10.5.4). Finally for $x \in \Omega'$ fixed define for $i \in \mathbb{N}_0$ the chain of ball $B_i \equiv B_{r_i}$ as in (10.4.2), with radius $r_i := \eta^i r$ and where $r \in (\eta R_0, R_0]$ is fixed. Define subsequently the comparison functions v_i over B_i as in (10.4.3). Note that by the definition of λ and the fact that $B_i \subset \Omega''$ then

$$⨍_{B_i} |Du|\,dx \le \lambda \tag{10.5.7}$$

for all $i \in \mathbb{N}_0$. What we want to prove is

$$E(Du, B_{i+2}) \le \varepsilon\lambda \qquad i \in \mathbb{N}. \tag{10.5.8}$$

Fix hence an index $i \in \mathbb{N}$ and suppose without loss of generality

$$⨍_{B_{i+2}} |Du|\,dx \ge \varepsilon\frac{\lambda}{2}. \tag{10.5.9}$$

On the contrary (10.5.8) would plainly follow by triangle's inequality. Now using an approach similar to Lemmata 10.16 and 10.17 we will prove that

$$\begin{aligned} E(Du, B_{i+2}) &\le \frac{\varepsilon}{4}E(Du, B_{i+1}) + 4c_{16c_2/\varepsilon}\eta^{-n}\frac{\lambda}{g(\lambda)}\left[\frac{|\mu|(B_{i-1})}{R_{i-1}^{n-1}}\right] \\ &\le \frac{\varepsilon}{4}E(Du, B_{i+1}) + \frac{\varepsilon}{4}\lambda \end{aligned} \tag{10.5.10}$$

by (10.5.6). This would be enough to get (10.5.8) by induction. First apply Lemma 10.14 with $B_R \equiv B_i$, $\tilde\eta \equiv \eta^2$, $\vartheta \equiv \varepsilon/4$ which, together with (10.5.9) gives

$$\varepsilon\frac{\lambda}{4} \le ⨍_{B_{i+2}} |Du|\,dx - \varepsilon\frac{\lambda}{4} \le ⨍_{B_{i+2}} |Dv_i|\,dx.$$

Note now that Lemma 10.13 with $B_R \equiv B_i$ yields in particular $\sup_{B_i/4}|Dv_i| \le c_2\lambda$ by (10.5.7); therefore Lemma 10.8 with $B \equiv B_i$, $\sigma \equiv \eta$, $m = 2$ and $C = 4c_2/\varepsilon$ gives

$$\varepsilon\frac{\lambda}{16c_2} \le |Dv_i| \qquad \text{in } B_{i+1}.$$

A similar reasoning yields $\varepsilon\lambda/16c_2 \le |Dv_{i-1}| \le c_2\lambda$ in B_i. Then we apply Lemma 10.15 for $H = 16c_2/\varepsilon$ and we have

$$⨍_{B_{i+1}} |Du - Dv_i|\,dx \le c_{16c_2/\varepsilon}\frac{\lambda}{g(\lambda)}\left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]$$

At this point since the choice of η gives $E(Dv_i, B_{i+2}) \le 2^{-3}E(Dv_i, B_{i+1})$ by (10.2.3), performing a calculation completely similar to (10.4.37) brings us to (10.5.10).

A brief argument similar to the one in [**106**] concludes the proof. Since all these estimates are uniform with respect the choice of $x \in \Omega'$ and the radius $r \in (\eta R_0, R_0]$, then we obtain (10.5.5) with $r_\varepsilon := \eta^3 R_0$. Indeed let $\rho \le \eta^3 R_0$: then exists an integer $m \ge 3$ such that $\eta^{m+1}R_0 < \rho \le \eta^m R_0$. Therefore $\rho = \eta^m r$ for some $r \in (\eta R_0, R_0]$ and (10.5.5) follows from (10.5.8). □

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